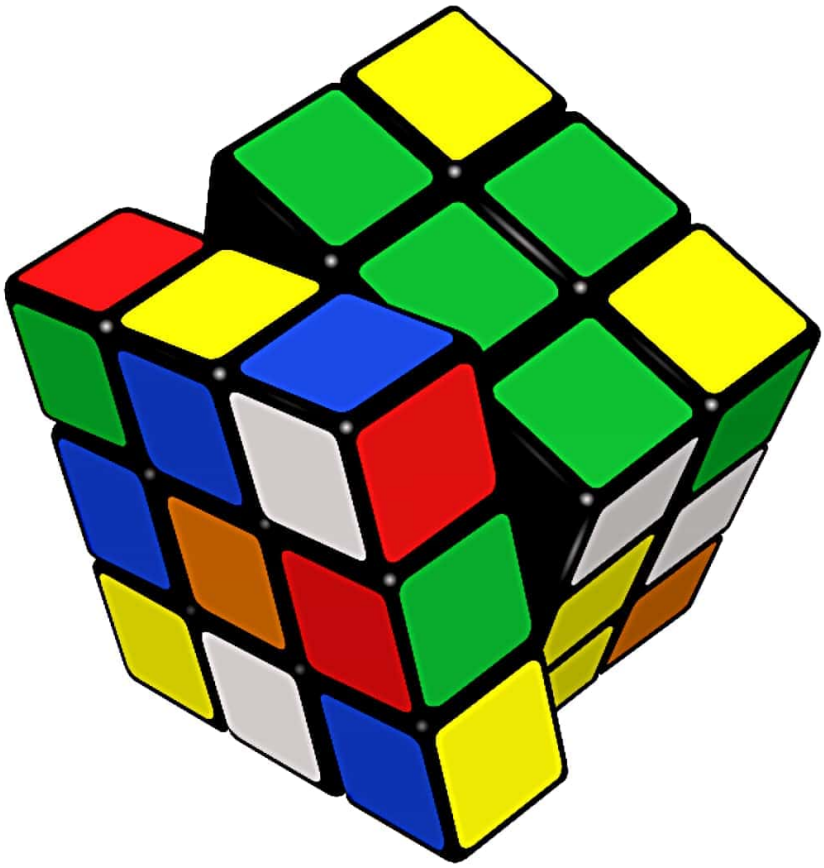
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GATE HIMANSHOO T. 9029359640

MODERN ALGEBRA

IIT-Roorkee ✓ G-2009.

Q.14. $\frac{Z_2[x]}{\langle x^3+x^2+1\rangle}$ is field of order 8.

Soln → If F is field and $f(x) \in F[x]$ is irr. over F then $I = \langle f(x)\rangle$ is maximal and Prime Ideal.

Now, $f(x) = x^3+x^2+1$ is irreducible over Z_2

{ $\because$ $f(0)=1 \neq f(1)=1$ }

$\Rightarrow f(a) \neq 0 \quad \forall\ a \in Z_2$ }

$\Rightarrow \frac{Z_2[x]}{\langle x^3+x^2+1\rangle}$ is Galois field of order 8.

Note → If F is Galois Field then F is seperable and Normal Extension then $\frac{Z_2[x]}{\langle x^3+x^2+1\rangle}$ is seperable and Normal Extension.

Q.15. Which of the following is possible order of P.I.D.

(I) ~~15~~ (II) 25 (III) ~~35~~ (IV) 36.

Soln → W.K.T. that F is finite P.I.D. then F is finite Field

$\Rightarrow F \approx GF(p^n)$

$\Rightarrow O(F) = p^n$

Q.38. Which of the following ideals of the ring Z[i] of Gausian Integer is not maximal?

~~Sol~~ (I) $\langle 1+i\rangle$ (II) $\langle 1-i\rangle$

(III) $\langle 2+i\rangle$ (IV) ~~$\langle 3+i\rangle$~~

Soln → $\frac{Z[i]}{\langle a+ib\rangle} \approx Z_{a^2+b^2}$ if $\gcd(a,b)=1$

$\because \frac{Z[i]}{\langle 1+i\rangle} \approx Z_{1^2+1^2} = Z_2$ is field

$\Rightarrow \langle 1+i\rangle$ is maximal.

Similarly, $\langle 1-i\rangle$, $\langle 2+i\rangle$ are maximal ideal.

Now, $\frac{Z[i]}{\langle 3+i\rangle} \approx Z_{10}$ is not I.D.

$\Rightarrow \langle 3+i\rangle$ is not maximal & Prime ideal.

NOTE → (I) $\langle 3+i\rangle$ is not maximal ideal in Z[i] then $\exists$ ideal I of Z[i] such that

$\langle 3+i\rangle \subseteq I \subseteq Z[i]$

But $\langle 3+i\rangle \neq I$ and $I \neq Z[i]$

Now, $3+i = (2-i)(1+i)$

Now, $\langle 3+i\rangle \subseteq \langle 1+i\rangle \subseteq Z[i]$

But $\langle 3+i\rangle \neq \langle 1+i\rangle \neq \langle 1+i\rangle \neq Z[i]$

Then, $I = \langle 3+i\rangle$ is not maximal.

(II) $a = 2+i \in Z[i]$

$b = 1+i \in Z[i]$

s.t. $a \cdot b = (2+i)(1+i) = (3+i)$

$\in \langle 3+i\rangle$

But $2+i \notin \langle 3+i\rangle$

& $1+i \notin \langle 3+i\rangle$

Then ideal generated by $I = \langle (3+i)\rangle$ is not Prime Ideal.

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Q.39. Let $z(G)$ be center of G. Then $o\left(\frac{G}{z(G)}\right)$ cann't be:

(i) 4 (ii) 6 (iii) 15 (iv) 25

Soln $o\left(\frac{G}{z(G)}\right) > 1$ and unique upto iso. is not possible.

(i) $o\left(\frac{G}{z(G)}\right) = 4 \Rightarrow Z_4$ or $Z_2 \times Z_2$ Then it is possible.

(ii) $o\left(\frac{G}{z(G)}\right) = 6 \Rightarrow Z_6$ or S_3 "

(iii) $o\left(\frac{G}{z(G)}\right) = 15 \approx Z_{15}$ Not Possible

(iv) $o\left(\frac{G}{z(G)}\right) = 25 \Rightarrow Z_{25}$ or $Z_5 \times Z_5$ Possible

Q.40. Led $Aut(G)$ denote the group of Automorphism of a group G. Which of the following is not a cyclic group?

(i) $Aut(Z_4)$ (ii) $Aut(Z_6)$
(iii) $Aut(Z_8)$ (iv) $Aut(Z_{10})$

Soln $Aut(Z_n) \approx U(n)$

G.2010.

Q.16. Which of the following groups is simple?

a) S_3 b) $GL(2,R)$
c) $Z_2 \times Z_2$ d) A_5

Soln a) $\because A_n$ is normal subgroup of S_n, $n \geq 3$, other than I and S_n.
$\therefore S_3$ is not simple.

b) $SL_2(R)$ is normal subgroup of $GL_2(R)$.
$\therefore GL_2(R)$ is not simple.

c) $Z_2 \times Z_2$ has five normal subgp.
$\therefore Z_2 \times Z_2$ is not simple.

NOTE> (i) $GL_2(R) \mid SL_2(R)$ has elts of order $1, 2, 3, \ldots$ and infinity then $GL_2(R) \mid SL_2(R)$ has cyclic subgroup of order $1, 2, 3, \ldots$ and ∞. ($F = \mathbb{C}$, $\forall (n \geq 2)$)

(ii) Every Finite group is isomorphic to subgroup subgroup of S_n.
S_n isomorphic to $GL_n(R)$

Q. $S_n \not\approx GL_2(R)$?

Q. $E = Q(\sqrt{2}, \sqrt{3}, \sqrt{5})$
Then: $[E:Q] = ?$

Soln $E = Q(\sqrt{2}, \sqrt{3}, \sqrt{5})$

$f(x) = x^2 - 2$ is irr poly of deg(2) then $\beta_1 = \{1, \sqrt{2}\}$

$f(x) = x^2 - 3$ " " $\beta_2 = \{1, \sqrt{3}\}$

$f(x) = x^2 - 5$ " " $\beta_3 = \{1, \sqrt{5}\}$

$\beta_1, \beta_2, \beta_3$ are L.I

$\therefore [E:Q] = 2 \cdot 2 \cdot 2 = 8$

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GATE-19

Q.21: Find degree of extension of $\mathbb{Q}[i, \sqrt{3}, \omega]$ over $\mathbb{Q}$.

Soln $[\mathbb{Q}(i, \sqrt{3}, \omega) : \mathbb{Q}]$

$f(x) = x^2 + 1$ is irr. poly of deg 2. $\beta_1 = \{1, i\}$

$f(x) = x^2 - 3$ is irr. poly of deg 2. $\beta_2 = \{1, \sqrt{3}\}$

$f(x) = x^3 - 1 = (x-1)(x^2+x+1)$ has irr poly of deg 2, then

$\beta_3 = \{1, \frac{-1+\sqrt{3}i}{2}\} = \{1, \omega\}$

$\beta' = \beta_1 \cdot \beta_2 = \{1, i, \sqrt{3}, \sqrt{3}i\}$

Now, $\omega = \frac{-1+\sqrt{3}i}{2}$ can be written as Linear combination of elements of β' over $\mathbb{Q}$.

Then $\dim [\mathbb{Q}(i, \sqrt{3}, \omega) : \mathbb{Q}] = 4$

Q.37. P_j: G_j has unique subgroup of order 2.

where: G_1 be abelian group of order 6.

$G_2 = S_3$

Soln $P_j \longrightarrow G_j$, $j = 1, 2$

where, G_1 is abelian group of order 6 then $G_1 \approx Z_6$ and $G_2 \approx S_3$

$G_1 = Z_6$ has uniques subgroup of order 2.

& $G_2 = S_3$ has 3 subgroup of order 2.

Q.38. Let G be the group of all symetries of the square. Then the number of conjugate classes in G is

(i) 4 (ii) 5

(iii) 6 (iv) 7

Soln G = set of all Rotation & Reflection



square then no. of conjugate class in G.

Here: $G \approx D_4$

We know that number of conjugate class in $D_n = \begin{cases} \frac{n+6}{2}, & \text{if } n \text{ is even} \\ \frac{n+3}{2}, & \text{if } n \text{ is odd.} \end{cases}$

$\therefore$ No. of cojugate class of G $= \frac{4+6}{2} = 5$

H.W.

Q. Let G be a group of all Rotation and reflection of  then find number of conjugate class in $G = ?$

Soln $G = \{R_0, R_{180}, H, V\} \approx Z_2 \times Z_2$ is an abelian group then no. of conjugate class in $G = o(G) = 4$

Q. Same Q for  Rhombus.

Soln Same ans.

39. Ideal of $\mathbb{Q}[x]$ generated by $x^2 - 3$ is:

a) maximal but not prime

b) prime but not maximal

c) both maximal and prime

d) neither maximal nor prime.

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Soln $R = Q[x]$

$\because f(x) = x^2 - 3$ is irr. poly over Q then $I = \langle x^2-3 \rangle$ is maximal and Prime Ideal in $Q[x]$

$\Rightarrow \frac{Q[x]}{\langle x^2-3 \rangle}$ is field

$\Rightarrow \frac{Q[x]}{\langle x^2-3 \rangle} \approx Q[\sqrt{3}]$

G-2011

39. $Z[i]/\langle (3-i) \rangle \approx Z_{10}$

40. For the rings : $L = \frac{R[x]}{\langle x^2-x+1 \rangle}$

$M = \frac{R[x]}{\langle x^2+x+1 \rangle}$

$N = \frac{R[x]}{\langle x^2+2x+1 \rangle}$

Soln: $\because x^2-x+1$ & x^2+x+1 are irr. poly over R.

$\therefore L \approx \phi$, $M \approx \phi$.

x^2+2x+1 is reducible over R.

$\Rightarrow N \approx R \times R$

$\therefore L \approx M \not\approx N$

16. The number of irr. quadratic poly. over the field of two elements F_2 is :

(i) 0 (ii) 1

(iii) 2 (iv) 4

Soln → (i) No. of irr. monic poly of degree 2 over $Z_p = \frac{p(p-1)}{2}$

(ii) No. of reducible monic poly of deg 2 over $Z_p = p^2 - \frac{p(p-1)}{2}$

$= \frac{p(p+1)}{2}$

Z_2 के ऊपर ~~degree 2~~ irr poly bole तो वो Automatically monic हो हो जाएगा.

$a_0 + a_1x + a_2x^2$

$f(x) = a_0 + a_1x + a_2x^2$

$\therefore f(x) = x^2$ ✗

$f(x) = 1+x^2$ ✗ } reducible

$f(x) = x+x^2$ ✗

$f(x) = 1+x+x^2$ ✓

Q. The number of elts in conjugacy class of the 3-cycle (2 3 4) in the symetric group S_6 is,

(i) 20 ~~(ii)~~ 40

(iii) 120 (iv) 216

Soln $\alpha = (2\ 3\ 4)$ is the member of $3+1+1+1$ class of S_6

$\therefore O(Cl(2\ 3\ 4)) = \frac{6!}{1^3 3^1 \cdot 3!(1)!}$

$= 40$

IIT-D से set huaa!

GATE-12

14. $R = Z \times Z \times Z$

& $I = Z \times Z \times \{0\}$

Then I is maximal but prime Ideal.

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Sol^n $\frac{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\times\mathbb{Z}\times\{0\}} \approx \mathbb{Z}$

$\mathbb{Z}$ is an I.D But not field

Then, $I = \mathbb{Z}\times\mathbb{Z}\times\{0\}$ is Prime Ideal of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ but not maximal ideal.

Q. 31. The order of the smallest possible non-trivial group containing elements x and y such that $x^7 = y^2 = e$ and $yx = x^4y$ is :

(i) 1 (ii) 2

(iii) 7 (iv) 14

Sol^n $x^7 = y^2 = e$, $yx = x^4y$

$G = \{x^i y^j \mid x^7 = e, y^2 = e, yx = x^4y\}$

Then $O(G) = ?$

$yx = x^4y$ (दोनों side में power लगाओ ताकि x को free करो।)

$\Rightarrow x^4 = yxy^{-1}$ ...(i)

$\Rightarrow (x^4)^2 = yx^2y^{-1}$

$\Rightarrow x^8 = yx^2y^{-1}$

$\Rightarrow x^{16} = (yx^2y^{-1})(yx^2y^{-1})$

$= yx^4y^{-1}$ ...(ii)

From (i) and (ii),

$x^{16} = y^2xy^{-2} = exe^{-1}$

$\Rightarrow x^{15} = e$ ...(iii)

$\because x^7 = e \Rightarrow o(x) \mid 7$

& $x^{15} = e \Rightarrow o(x) \mid 15$

$\because o(x) \mid 7$ & $o(x) \mid 15$

$\Rightarrow o(x) \mid (7, 15) \Rightarrow o(x) \mid 1$

$\Rightarrow o(x) = 1$

$\Rightarrow x$ = identity.

$\therefore G = \{y^j \mid y^2 = e\}$

$\Rightarrow O(G) = 2$ {$\because$ We want non-trivial group $\therefore y \neq e$}

Q. No. of 5-sylow subgroup in group of order 45 is :

a) 1 b) 2

c) 3 d) 4

Sol^n $O(G) = 45 = 3^2 \times 5^1$

$5^1 \mid O(G)$ But $5^{1+1} \nmid O(G)$

Then, G has 5-SSG of order 5

$\therefore n_5 = 1 + 5k$

Positive divisor of 45 are, 1, 3, 5, 9, 15, 45 (✓ ✗ ✗ ✗ ✓ ✗)

Then, & $n_5 = 1 + 5k = 1$ is possible then G has unique 5-SSG.

NOTE→ If $O(G) = p^2q$, $p < q$

& $p \nmid q-1$ and $q \nmid p^2-1$ then G is always abelian

e.g. (i) $O(G) = 99 = 3^2 \times 11$, $3 < 11$

& $3 \nmid 11-1$ & $11 \nmid 3^2-1$

$\Rightarrow$ G is always abelian

Then G has unique subgroup of order 11 and 9.

NOTE→ (i) If $O(G) = p^2q^2$, $p < q$

If $p \nmid q^2-1$ & $q \nmid p^2-1$

Then G is always abelian.

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(iv) If $o(G) = p^2q^2$, $p<q$ and $p \nmid q^2-1$ & $q^2 \nmid p^2-1$ then No. of non-isomorphic group

$= p(2)\,p(2)$

$= 4$

GATE-13

6. Suppose R is U.F.D. & $a, b \in R$ are distinct irreducible elts. Which of following statement is True?

(i) $\langle 1+a \rangle$ is P.I. $\to \langle 1+3 \rangle = \langle 4 \rangle \to$ Not P.I. of Z.

(ii) $\langle a+b \rangle$ is P.I. $\to \langle 3+5 \rangle = \langle 8 \rangle$ "

(iii) $\langle 1+ab \rangle$ is P.I. $\to \langle 1+3\cdot 5 \rangle = \langle 16 \rangle$ " ...

(iv) $\langle a \rangle$ is not necessarily max. Ideal.

Soln Z → U.F.D. Take irr. elts 3 and 5

9. Which of the following groups has a proper subgroup that is not cyclic?

(i) $Z_{15} \times Z_{11}$ (ii) S_3

(iii) $(Z, +)$ (iv) $(Q, +)$

Soln $Z_{15} \times Z_{11}$ → cyclic Group ⇒ all subgroup are cyclic

S_3 → order of proper subgrp of S_3 is $1, 2 \& 3$. ⇒ All proper subgroup are cyclic

Z → cyclic ⇒ All subgroups of Z is cyclic.

17. Let G be a group of order 231. Then no. of elts of order 11 in G is = ?

Soln $o(G) = 3 \times 7 \times 11$

$11^1 \mid o(G)$ But $11^2 \nmid o(G)$

⇒ G has 11-SSG of order 11

Then,

$n_{11} = 1 + 11k$ ... (1)

✓ ✗ ✗ ✗ ✗ ✗ ✗ ✓

1, 3, 7, 11, 21, 33, 77, 231

$\therefore n_{11} = 1$ is possible only.

Then G has unique subgroup of order 11 then,

No. of elts of order $11 = \phi(11) = 10$

19. Which of following is field:

a) $\frac{C[x]}{\langle x^2+2 \rangle}$ → reducible

b) $\frac{Z[x]}{\langle x^2+2 \rangle} \approx Z[\sqrt{2}i]$ Not Field

c) $\frac{Q[x]}{\langle x^2-2 \rangle} \approx Q[\sqrt{2}]$

d) $\frac{R[x]}{\langle x^2-2 \rangle}$ → Reducible.

25. No. of Non-iso abelian group of order 24 is ____.

Soln $o(G) = 24 = 2^3 \times 3$

∴ No. of Non-iso abelian group

$= p(3)\,p(1)$

$= 3$

Q. Which of following grp. of contains a unique normal subgrp of order 4.

(i) $Z_2 \oplus Z_4$

(ii) D_4

(iii) Q_8 means Q_4 here

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Sol^n → (i) $Z_2 \times Z_4$ is abelian

⇒ All subgroups will be Normal

Its has 3 subgroup of order 4

⇒ It has 3 Normal subgroup of order 4.

(ii) $H_1 = \{R_0, R_{90}, R_{180}, R_{270}\}$

$H_2 = \{R_0, R_{180}, H, V\}$

$H_3 = \{R_0, R_{180}, D, D'\}$ are

Normal subgroup of D_4 of order 4.

(iii) $\{1, -1, i, -i\}$

$\{1, -1, j, -j\}$

$\{1, -1, k, -k\}$ are Normal subgroups of Q_4

(iv) $Z_2 \oplus Z_2 \oplus Z_2$

$\{0\} \times Z_2 \times Z_2$

$Z_2 \times \{0\} \times Z_2$

$Z_2 \times Z_2 \times \{0\}$ are more than one normal subgroups of $Z_2 \oplus Z_2 \oplus Z_2$

(और भी है But likh नहीं रहे हैं)

∴ None is correct.

GATE-14.

13. Let a, b, c, d be real number with $a < c < d < b$

$S = \{f \in C[a,b] : f(x) = 0 \text{ for all } x \in [c,d]\}$

Then (i) S is ideal

(ii) S is prime ideal

(iii) S is maximal ideal.

Sol^n S vanishes more than one point

⇒ S is neither maximal nor Prime ideal.

S is ideal.

14. Let R is Ring. If $R[x]$ is P.I.D. then R is necessarily a

(i) U.F.D.

(ii) P.I.D.

(iii) E.D.

(iv) Field.

Sol^n $F[x]$ is P.I.D / E.D. iff F is field

15. $\phi : M_2(R) \to R$ given by $\phi(A) = tr(A)$.

Then $Ker(\phi) \approx$?

Sol^n $\phi : M_2(R) \to R$

$\phi(A) = tr(A)$ is onto homo.

Then, $\frac{M_2(R)}{Ker(\phi)} \approx R$

$\Rightarrow \dim\left(\frac{M_2(R)}{Ker(\phi)}\right) = 1$

$\Rightarrow \dim(M_2(R)) - \dim(Ker(\phi)) = 1$

$\Rightarrow 4 - 1 = \dim(Ker(\phi))$

$\Rightarrow \dim(Ker(\phi)) = 3$

$\Rightarrow Ker(\phi) \approx R^3$ over R.

OR

$Ker(\phi) = \{A \in M_2(R) \mid Tr(A) = 0\}$

$= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0\right\}$

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$\Rightarrow \dim(\text{Ker}(\phi)) = 3$

$\Rightarrow \text{Ker}(\phi) = R^3$

Q9. Let F_{125} be field. The number of non-zero elts of $\alpha \in F_{125}$ s.t. $\alpha^5 = \alpha$ is $= ?$

Soln $F = F_{125}$

$\Rightarrow F^*_{125} \approx Z_{124}$

Now, $\alpha^5 = \alpha \Rightarrow \alpha^4 = 1$

$\Rightarrow \phi \; O(\alpha) \mid 4$

$\Rightarrow O(\alpha) = 1, 2, 4$

No. of elts of order $1 = \phi(1) = 1$

" " $2 = \phi(2) = 1$

" " $4 = \phi(4) = 2$

Total $= 4$

GATE-15.

Q4. Let $c \in Z_3$ be such that $\frac{Z_3[x]}{x^3+cx+1}$ is a field. Then $c = ?$

Soln $\because \frac{Z_3[x]}{\langle x^3+cx+1\rangle}$ is field

if $c=0$ then x^2+1 is reducible over $Z_3 \Rightarrow \frac{Z_3[x]}{\langle x^3+cx+1\rangle}$ is not field

$\therefore c=0$ not possible.

Similarly, $c=1$ not possible.

$\therefore c = 2$

Q8. Let $G = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$ with $o(x) = 4$, $o(y) = 2$ & $xy = yx^3$.

Then $|Z(G)| = ?$

Soln $\because o(x) = 4$, $o(y) = 2$

& $xy = yx^3 \Rightarrow yx^{-1} \neq yx$

$\Rightarrow G$ is non-abelian group of order 8.

Then, $G \approx D_4$ or Q_4

$\because o(x^2) = 2$ and $o(y) = 2$

Then G has more than one elt of order 2.

$\Rightarrow G \not\approx Q_4$

$\therefore G \approx D_4$

$\Rightarrow Z(G) \approx Z(D_4) = \{e, x^2\}$

$\Rightarrow o(Z(G)) = 2$.

Q9. No. of Ring Homo from $Z_2 \times Z_2$ to Z_4 is equal to $= 1$

$\because$ Idempotent elt of Z_4 are $0 \& 1$

$\therefore T(x,y) = 0$ ✓

$T(x,y) = 1 \cdot x$ ✗

$T(x,y) = 1 \cdot y$ ✗

$T(1,0) = 1 \Rightarrow o(T(1,0)) = o(1) = 4$

$\Rightarrow Z_2 \times Z_2$ has elt of order 4 or multiple of 4.

But $Z_2 \times Z_2$ has no elt of order > 2

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50. $p(x) = 9x^5 + 10x^3 + 5x + 15$

$q(x) = x^3 - x^2 - x - 2$

Comclu

Discuss irreducibily over Q.

Soln⇒ Take $p = 5$,

$5 \mid 15, 5 \mid 5, 5 \mid 10$,

$5 \nmid 9$

$5^2 \nmid 15$

⇒ $p(x)$ is irr. over Q.

$q(x) = x^3 - x^2 - x - 2$

$= (x-2)\, g(x)$

where $g(x) \in Q(x)$

& $\deg(q(x)) = 2$

Also, Neither $(x-2)$ not $g(x)$ unit in $Q(x)$

⇒ $q(x)$ is not irr.

GATE-16.

24. P: H is normal subgroup of order 4 of S_4 then $\frac{S_4}{H}$ is abelian. F.

Q: $\frac{Q_4}{\{-1, 1\}}$ is abelian. T.

Soln⇒ P: $\frac{S}{H} \approx S_3$ → Non-abelian

Q: $\frac{Q_4}{\{-1,1\}} \approx Z_2 \times Z_2$ → abelian.

25. F is field of order 32. Then no. of non-zero soln $(a,b) \in F \times F$ s.t. $x^2 + xy + y^2 = 0$ = ?

Soln⇒ ∵ F is field ⇒ F is I.D.

$o(F) = 32$

Then, $F = \frac{Z_2[x]}{\langle f(x) \rangle}$

$(a,b) \in F \times F$ s.t.

$x^2 + xy + y^2 = 0$

$(0,0) \in F \times F$ s.t. $0^2 + 0.0 + 0^2 = 0$

Then only $(0,0)$ soln.

Then, there is no non-zero soln.

56. $G = \{ x^i y^j \mid x^5 = y^2 = e,\ x^2 y = yx \}$

Then $G \approx$?

✗ (i) Z_5 ✗ (ii) Z_{10}

(iii) Z_2 ✗ (iv) Z_{30}

Soln⇒ Short method by observing options.

∵ All option are abelian grp.

∴ We don't look about non-abelian group.

∴ Given G also be abelian.

Now, $x^2 y = yx = xy$

⇒ $x^2 y = xy$

⇒ $x^2 = x$

⇒ $\boxed{x = e}$

⇒ $o(G) \leq 2$

∴ $G \approx Z_2$

Proper method: (Gate-12 की कहानी)

Repeat mara है

∵ $x^2 y = yx$

⇒ $x^2 = yxy^{-1}$

⇒ $x^4 = yx^2y^{-1}$

⇒ $x^4 = y^2 x y^{-2}$

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$\Rightarrow x^4 = exe^{-1}$

$\Rightarrow x^4 = x$

$\Rightarrow x^3 = e$

$\Rightarrow o(x) \mid 3 \quad \cdots (ii)$

Given: $x^5 = e$

$\Rightarrow o(x) = 5 \quad \cdots (iii)$

From (ii) and (iii).

$o(x) \mid \gcd(3,5)$

$\Rightarrow o(x) \mid 1$

$\Rightarrow o(x) = 1 \Rightarrow x = e.$

$\Rightarrow o(G) = 2.$

GATE-17.

11. $O(G) = 125 \Leftarrow G$ is non-abelian

Then $|Z(G)| = ?$

Soln $\Rightarrow O(G) = 125 = 5^3 \Leftarrow G$ is non-abelian

Then $|Z(G)| = 5$

12. $|F_1| = 2^9, |F_2| = 2^6$ are sub-fields of F then $|F_1 \cap F_2| = ?$

Soln $\Rightarrow GF(p^m)$ is field of $GF(p^n)$ iff $m \mid n$

$F_1 \cap F_2$ is subfield of F_1 and F_2

Then, $o(F_1 \cap F_2) = 2^K$ and $K \mid 9 \Leftarrow K \mid 6$

$\Rightarrow o(F_1 \cap F_2) = 2^{\gcd(6,9)} = 2^3$

NOTE $\Rightarrow$ If F_1 is subfield of F of order p^m and F_2 is subfield of F of order p^n then,

$o(F_1 \cap F_2) = p^{\gcd(m,n)}$

37. No. of elts of S_9 commute with $z = (123)(4567) = ?$

Soln $\Rightarrow z = (123)(4567)$ is the member of conjugacy class $3+4+1+1$ of S_9.

$o(cl((123)(4567)))$

$= \frac{9!}{1^2 \cdot 3^1 \cdot 4^1 \cdot 2! \, 1! \, 1!} = 24$

(Ise Normalizer भी कहते हैं)

38. No. of max ideal in $\frac{Q[x]}{(x^4-1)} = ?$

Soln $\Rightarrow \frac{Q[x]}{\langle x^4-1 \rangle} = \frac{Q[x]}{\langle (x^2+1)(x+1)(x-1) \rangle}$

$\simeq \frac{Q[x]}{\langle x^2+1 \rangle} \times \frac{Q[x]}{\langle (x+1) \rangle} \times \frac{Q[x]}{\langle (x-1) \rangle}$

$\simeq Q[i] \times Q \times Q$

has exactly 3 maximal and prime ideals.

GATE-18

7. No. of elts of order 8 in $S_6 =$ ___ ?

Soln $\Rightarrow$ If S_6 has elts of order 8 then $\exists \; a_1 + a_2 + \cdots + a_8 = 6$ and $l.c.m.(a_1, a_2, \ldots a_8) = 8$

If $lcm(a_1, a_2, \ldots a_8) = 8$

Then atleast one $a_i = 8$

if $a_i = 8$ then

Then $a_1 + a_2 + \cdots a_8 \neq 6$

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∴ S has no elt of order 8.

20. $|F| = 7^6$ and K is subfield of F & $|K| = 49$

Then: $[F:K] = ?$

Soln → $|F| = 7^6 \Rightarrow F \approx GF(7^6)$

$|K| = 7^2 \Rightarrow K \approx GF(7^2)$

Then: $[GF(p^n) : GF(p^m)] = \frac{n}{m}$

$= \frac{6}{2} = 3$

Good Question.

29. $p(x) = x^4 + 4$ in the ring $\mathbb{Q}[x]$. Then:

a) zeros of $p(x)$ in $\mathbb{C}$ form group under multiplication

b) $p(x)$ is reducible in $\mathbb{Q}[x]$

c) Spliting field of $p(x)$ has degree 3 over $\mathbb{Q}$.

d) " " degree 4 " "

Soln → (b) $f(x) = x^4 + 4$

$= x^4 + 4 + 4x^2 - 4x^2$

$= (x^2+2)^2 - (2x)^2$

$= (x^2 + 2 - 2x)(x^2 + 2 + 2x)$

i.e. reducible over $\mathbb{Q}$.

a) ∵ 1 is not zero of $p(x)$

∴ Not form group.

→ $x = \pm 1 \pm i \in \mathbb{Q}[i]$

Then $\mathbb{Q}[i]$ is splitting field of $p(x)$ over $\mathbb{Q}$.

∴ $[\mathbb{Q}[i] : \mathbb{Q}] = 2$

30. Which of following is true?

a) Every group of order 12 has a non-trivial proper normal subgroup

b) " " not have " "

c) Every group of order 12 has a subgroup of order 6

d) Every group of order 12 has an elt of order 12.

Soln → $|G| = 12$

- Z_{12} → has Normal subgp of order 6
- $Z_2 \times Z_6$ → "
- D_6 → has normal subgp of order 2
- A_4
  - → $H \lhd A_4$, $H = \{I, (12)(34), (13)(24), (14)(34)\}$?
  - (No elt of order 12)
  - → No subgroup of order 6.
- Q_6

Q. $\mathbb{Z}[\sqrt{-p}]$. Then 2 in $\mathbb{Z}[\sqrt{-p}]$ is, p is odd prime:

a) unit b) square

c) prime d) irr.

Soln → (c) $\mathbb{Z}[\sqrt{-p}]$ has exactly two units 1 and -1.

$p = 3$ then, $\mathbb{Z}[\sqrt{-3}]$

$2 \nmid 4$

$2 \mid (1+\sqrt{3}i)(1-\sqrt{3}i)$

But, $2 \nmid 1+\sqrt{3}i$, $2 \nmid (1-3i)$

$\Rightarrow \alpha = 2$ is not prime elt in $\mathbb{Z}[\sqrt{-3}]$

$\alpha = 2$ is irr. elt of $\mathbb{Z}[\sqrt{-p}]$

$2 = (a + i\sqrt{p}\, b)(c + i\sqrt{p}\, d)$

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⇒ Taking conjugate,

$2 = (a - i\sqrt{p}\,b)(c - i\sqrt{p}\,d)$ ---(ii)

From (i) and (ii),

$4 = (a^2 + pb^2)(c^2 + pd^2)$

(i) if $a^2 + pb^2 = 4$ then $(c + i\sqrt{p}\,d)$ is unit.

(ii) if $c^2 + pd^2 = 4$ then $(a + i\sqrt{p}\,d)$ is unit.

(iii) does not exist.

Then $d = 2$ is irr. elts.

GATE-19. 21, 32, 35, 47, 50

(21) Done.

30. $R = Z_{125}$

Then no. of find $n + m = ?$

n = no. of max ideal of Z_{125}

m = no. of non-unit elts of Z_{125}

Soln ⇒ $R = Z_{125}$

m = no. of max ideal of Z_{125}

$= 1$

No. of units in $Z_{125} = \phi(125)$

$= 100$

Then no. of non-unit $= 125 - 100$

$= 25 = n$

$\therefore n + m = 1 + 25 = 26$

47. No. of elts of order 15 in $Z_{60} \times Z_{25} = ?$

Soln

$15 = 15 \cdot 1 \rightarrow 8$

$15 \cdot 5 \rightarrow 8 \cdot 4 = 32$

$3 \cdot 5 \rightarrow \phi(3) \cdot \phi(5) = 8$

Total = 48.

35. (i) $Z[\sqrt{-1}]$ is U.F.D. T

(ii) $Z[\sqrt{-5}]$ is P.I.D. F

(iii) $\langle x^3 + x + 1 \rangle$ is maximal ideal in $Z_2[x]$

(iv) $\langle x^6 + 1 \rangle$ is prime Ideal in $Z_3[x]$

Soln ⇒ (ii) $Z[\sqrt{-5}]$ is not U.F.D.

⇒ Not P.I.D.

(iii) $\langle x^3 + x + 1 \rangle$ is irr over Z_2

⇒ max ideal.

(iv) $x^6 + 1 = (x^2 + 1)(x^4 - x^2 + 1)$

$= (x^2 + 1)(x^4 + 2x^2 + 1)$

is not irr.

⇒ $\frac{Z_3[x]}{\langle x^6 + 1 \rangle}$ is not I.D.

⇒ $\langle x^6 + 1 \rangle$ is not prime ideal.

32. (i) $f: Q \rightarrow Q$ is non-trivial homo then f is an iso.

(ii) Any Quotient Group of a cyclic group is cyclic.

(iii) Every subgrp of G is normal then G is abelian

(iv) Every grp of order 33 is cyclic.

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Soln (i) $T(x) = aDC$ (non-trivial)

(ii) $G = Q_4$

(iv) $O(G) = 33$

$= 3 \times 11$

$3 \nmid 11 - 1$

$\Rightarrow$ G is cyclic.

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