Game Theory

Theory of Games

- It refers to the general situation of conflict and competition in which two or more competitors (or participants) are involved in decision-making activities in anticipation of certain outcomes over a period of time

- In the automobile industry, for example, the strategies of competitors to introduce certain models with certain features can dramatically affect the profitability of other carmakers
- Today, businesses cannot make important decisions without considering what other organizations or individuals are doing or might do

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- Game theory is one way to consider the impact of the strategies of others on our strategies and outcomes
- Game theory is the study of how optimal strategies are formulated in conflict
- Game models are classified by the number of players, the sum of all payoffs, and the number of strategies employed
- It deals with mathematical analysis of competitive problems and is based on the Minimax principle put forward by Von Neumann which implies that each competitor act so as to minimize his maximum loss (or maximize his minimum gain)

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- The competitors are referred to as players that may be an individual, a group of individuals, or an organization

- For instance, pricing of products, where the ultimate sales are determined not only by the price levels it selects but also by the prices its competitors set

- Success of an advertising/marketing campaign depends largely on various types of services offered to the customers, etc.

Properties of Competitive Game

- There are finite number of participants (n)
  - two person game if n = 2 & n-person game if n ≥ 2
- Each participant has a finite number of possible course of actions
- Each participant must know all the courses of action available to others but must not know which of these will be chosen

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- A play of the game is said to be occur when each player chooses one of his courses of action
- The choices are assumed to be made simultaneously, so that no participant knows the choice of other unit he has decided his own
- The gain of the participant depends up on his own actions as well as those of others

Basic Terminology

- Each participant is called a player
- After each play of the game, one player pays the other an amount determined by the courses of action chosen
- The decision rule by which a player determines his course of action is called a strategy
- If a player uses only one particular strategy, it is said pure strategy

...continued ...Basic terminologies

- If a player needs to use all or some of his available strategies in some fixed proportion, it is said to be mixed strategy

- Two-person zero-sum game & n-person zero-sum game – if their net gain is zero in both cases

- Payoff is the outcome of playing the game.
  - A payoff matrix is a table showing the amount received by the player name at the left hand side after all possible plays of the game where as the payment is made by the player named at the top of the table

Two person Zero-sum game

- Example: Illustrate the following game, where competitors A and B are assumed to be equal in ability and intelligence. A has a choice to strategy 1 or strategy 2, while B can select strategy 3 or 4

	Competitor B		
Competitor A		Strategy 3	Strategy 4
	Strategy 1	+4	+6
	Strategy 2	+3	+5

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- How much is the value of the game? Minimax for the column & Maximin for the row

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Solution:

- A maximize his gains where as B minimize his losses since there is no negative flows
- B loses the same amount as A gains which results a zero-sum game
- The strategy that A uses is to ensure that his average gain per play is at least equal to the value of the game, where B ensures that his average loss per play is no more than the game value

…Solution

	Competitor B			
Competitor A		Strategy 3	Strategy 4	Maximin
	Strategy 1	+4	+6	+4
	Strategy 2	+3	+5	+3
	Minimax	+4	+6	

Pure Strategy Games

- It is the same strategies each player always follows regardless of the other player's strategy
- A dominant strategy = Pure strategy
- A saddle point is a situation where both players are facing pure strategies
- Use Minimax rule for the column and Maximin for the row to identify the saddle point as well as the game outcome

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Pure Strategy with saddle point (2-person zero sum game)

- Example: consider the following game

		Player B	
		Plan P	Plan Q
Player A	Choice L	-3	3
	Choice M	-2	4
	Choice N	2	3

…continued

- The value of the game and the pure strategies for both player A and Player B are?
- Does it have a saddle point?

Solution

- The negative values are a gain for player B where as the positive values are a gain for player A

- When Minimax value = Maximin value – the corresponding pure strategies are called optimal strategies and the game is said to have a saddle point or equilibrium point

- If minimax for the column ≠ maximin for the row – there is no saddle point and the value of the game lies between them

…Solution

		Player B		
		Plan P	Plan Q	Maximin
Player A	Choice L	-3	3	-3
	Choice M	-2	4	-2
	Choice N	2	3	2
	Minimax	2	4	

…Continued

- In reality, player A and B may not see the saddle point at first. After the game is played for some time, however, each player will realize that there is only one strategy that should be played. From then on, these players will play only one strategy, which corresponds to the saddle point

Example: Consider the following game

	B				
A	16	4	0	14	-2
	10	8	6	10	12
	2	6	4	8	14
	8	10	2	2	0

Find

- The saddle point:
- Strategy of A and B:
- The Game value:

Solution:

- Use Maximin rule for the Player A and Minimax rule for the player B
- The saddle point = row 2 and column 3
- Strategy A = playing row 2 and Strategy B = playing column 3
- The game value = +6

The Rule of Dominance

- If no pure strategies exist, the next step is to eliminate certain strategies (rows and/or columns) by dominance. The resulting game can be solved by some mixed strategy

- The principle of dominance can be used to reduce the size of games by eliminating strategies that would never be played

- A strategy can be eliminated if all its game's outcomes are the same or worse than the corresponding game outcomes of another strategy

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- Dominance rule for column(s): every value in the dominating column must be greater than or equal to the corresponding value of the dominated column(s). So, the dominant column must be deleted because players will never play that strategy since it results a heaviest losses to one player and a heaviest gains to the other player

- Rule of dominance for row(s): every value in the dominating row must be less than or equal to the corresponding value of the dominated row(s). So as to delete the dominant row in the same fashion

Example: Does it have a saddle point?

		Player Q		
		W	*B*	*R*
Player P	*W*	0	-4	7
	B	2	3	6
	R	3	-3	8

…continued…Solution

- This matrix has no saddle point, thus, the matrix should be reduced
- Therefore, apply the rule of dominance to identify the mixed strategy but there is no pure strategy since there is no saddle point

...Solution

- Hence, Player q will not play strategy R since this will result in heaviest losses to him and highest gains to player P

- Each elements in column R is greater than or equal to the corresponding elements in the other column(s). So, it has to be deleted since this column is dominant and will never be played by player q

...Solution

- For the same example, Each element in row W is less than or equal to the corresponding elements in the other row(s). So, it has to be deleted. Thus, obtain:

	W	B
B	2	3
R	3	-3

Important Note

- A given strategy can be dominated if it is inferior/superior to an average of two or more other pure strategies
  - i.e., if a given column is greater than the average of the other columns, that column is dominant and will never be played. Hence, it has to be deleted
  - If a given row is less than the average of the other rows, that row is dominant and will never be played. Hence, it has to be deleted so that you will obtain a reduced matrix game

Mixed strategy

- When there is no saddle point, players will play each strategy for a certain percentage of the time
  - Hence, one of the players must determine what proportion of time to play each row while the other must know what proportion of the time to play each column

- In case where there is no saddle point and dominance has been used to reduce the game matrix, players will resort to mixed strategies

- The payoffs obtained will be the expected payoffs and the value of the game will be the expected value of the game

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- Pseudo-saddle point – a saddle point obtained after the original game reduced by dominance rule and it is not necessarily a true saddle point, thus, it will not be considered as a saddle point of a pure strategy

Mixed strategies (2 X 2 Games)

- Arithmetic and algebraic methods are used for finding optimum strategies as well as game value for a 2 X 2 game

Arithmetic method for finding optimum strategies and game value

steps:

(i) subtract the two digits in column 1 and write them under column 2 ignoring sign

(ii) subtract the two digits in column 2 and write them under column 1 ignoring sign

(iii) similarly proceed for the two rows

- These value are called Oddments – they are the frequencies with the players must use their courses of action in their optimum strategies

Example:
Two person zero-sum Game without saddle point

- EX: In a game of matching coins, player A wins br2 if there are two heads, wins nothing if there are two tails and loses br1 when there are one head and one tail. Determine the payoff matrix, best strategies for each player and the value of game to A

Solution: Matrix form of the game

- This game doesn’t have a saddle point and hence there will never be a pure strategy to be taken

		Player B	
		H	T
Player A	H	2	-1
	T	-1	0

…continued…Solution

		Player B			
		H	T	Oddment	Proportion
Player A	H	2	-1	1	¼
	T	-1	0	3	¾
	Oddment	1	3	4	
	Proportion	¼	¾		

Solution

- Since there is no saddle point, the optimal strategies will be mixed strategies. Hence:

Using A's oddments:

- B plays H: value of the game = Br[2* 1 + (-1)*3] ÷ (3+1) = Br -1/4

  T: Value of the game = Br[-1*1 + 0*3] ÷ (3+1) = Br -1/4

Using B's oddments:

- A plays H: Value of the game = Br[2*1 + (-1)*3] ÷ (3+1) = Br -1/4

  T: Value of the game = Br [-1*1 + 0*3] ÷(3+1) = Br -1/4

Therefore; A(1/4,3/4), B(1/4,3/4),

The value of the Game, V = Br-1/4

Example 2

- Reduce the following game by dominance and find the game value, the optimal strategy that both player A and Player B chooses to take

		PLAYER B			
		I	II	III	IV
PLAYER A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

...continued ...Solution

- Row I is dominating- each elements in row one is less than or equal to the corresponding value of row III

		PLAYER B			
		I	II	III	IV
PLAYER A	I				
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

...continued...Solution

- column I is dominant- each element in column I is greater than or equal to the corresponding value of column III

		PLAYER B			
		I	II	III	IV
PLAYER A	I				
	II		4	2	4
	III		2	4	0
	IV		4	0	8

...continued ...Solution

- column II is dominant- each element in column II is greater than or equal to the corresponding average value of column III and IV

		PLAYER B			
		I	II	III	IV
PLAYER A	I				
	II			2	4
	III			4	0
	IV			0	8

...continued ...Solution

- Row II is dominating- each element in row II is less or equal the corresponding average value of row III and IV

		PLAYER B			
		I	II	III	IV
PLAYER A	I				
	II				
	III			4	0
	IV			0	8

...continued ...Solution

- What is the proportion of player A and Player B play no...?

		PLAYER B			
		III	IV	oddment	proportion
PLAYER A	III	4	0	8	2/3
	IV	0	8	4	1/3
	oddment	8	4		
	proportion	2/3	1/3		

- Optimal strategy for A : (2/3, 1/3)
- Optimal strategy for B: (2/3, 1/3)
- Value of the game for A, V: 4*2/3 + 0*1/3 = 8/3

Algebraic method of finding the optimal strategy

- Solve the following game using algebraic method and find the game value and the strategy for both players

		Player Y	
		Y_1	Y_2
Payer X	X_1	4	3
	X_2	2	20
	X_3	1	1

Apply Rule of Dominance

- Let the proportion of time that player X will play X1 is X and X2 is 1-X and the proportion of time that player Y will play Y1 is y and Y2 is 1-Y

		Player Y	
		Y_1	Y_2
Player X	X_1	4	3
	X_2	2	20

…continued

- 4* X + 2*(1-X) = 3*X + 20*(1-X)
  - Solve for X and 1-X
    - X = 18/19 1-X = 1/19

- 4 * Y + 3*(1-Y) = 2*Y + 20*(1-Y)
  - Solve for Y and 1-Y
    - Y = 17/19 1-Y = 2/19

…continued

- The game value =
  - 4 * 18/19 + 2 * 1/19 = 74/19
  - 3*18/19 + 20*1/19 = 74/19
  - 4*17/19 + 3*2/19 = 74/19
  - 2*17/19 + 20 *2/19 = 74/19
- The strategy
  - Player Y = Y1 (17/19) and Y2 (2/19)
  - Player X = X1 (18/19), and X2 (1/19)