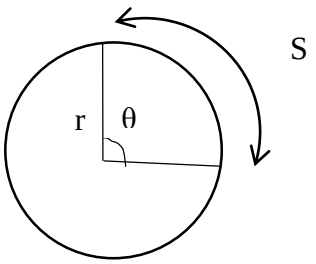
UNIT 6: ROTATIONAL MOTION

Angular displacement

If a particle is moving around a circular path, the angle through which it rotates is given by

$$\theta = \frac{Arc\ length}{Radius} \qquad \theta \text{ is in radian}$$



$$\theta = \frac{S}{r}$$

When $S = r$, $\theta = 1\text{rad} = 57.3^0$

$1\text{rev} = 2\pi \text{ radian} = 360^0$

The arc length S is related to the angle θ through the relationship

$$S = \theta r$$

The direction of θ is determined using the right hand rule. "*Curl your fingers of the right hand in the direction of the rotation, then the extended thumb points in the direction of* θ."

Example

1. Find θ if S = 60*cm* and *r* = 120 *cm*. **Ans**. θ = 0.5rad
2. The minute hand of a wrist watch turns 4/5 rev in 48 minutes. Express its angular displacement in rad and degrees. **Ans**. $\theta = 1.6\pi\ rad = 288^0$
3. A bicycle has a wheel of radius 30cm. Calculate the magnitude of the angular displacement of the wheel, when it is rotated through
   a)120*cm*. **Ans**. $\theta = 4\text{ rad} = 229.3^0$ b) 24 *cm*. **Ans**. $\theta = 0.8\text{ rad} = 45.86^0$
4. Calculate how far the bicycle in the above example move along the ground when its angular displacement is 500rev. **Ans**. S = 942m
5. What is the angular displacement of a second hand of a clock in a time interval of 25 min?
   Ans. $\theta = 2.62\text{rad} = 150^0 = 0.417\text{rev}$
   For a clockwise motion, the direction of the angular displacement is along the negative z-axis

$$\vec{\theta} = -2.62rad\,\hat{k} = -150^0\,\hat{k} = -0.417rev\,\hat{k}$$

Angular velocity

Angular velocity is the rate of change of angular displacement. $\omega = \frac{\Delta\theta}{\Delta t}$

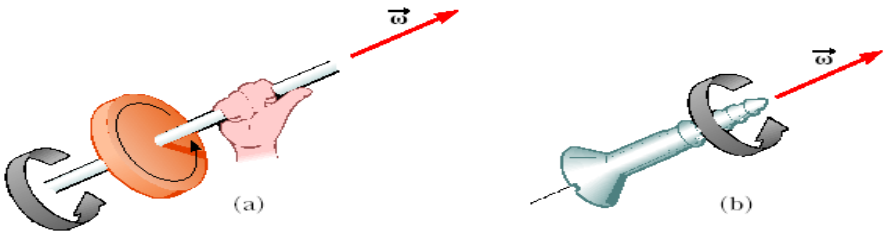
In analogy to linear speed, the instantaneous angular speed ω is defined as the limit of the ratio $\frac{\Delta\theta}{\Delta t}$ as Δt approaches zero: $\omega_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$

The right-hand rule for determining the direction of the angular velocity vector.

The direction of the angular velocity vector $\vec{\omega}$ can be found with the **right-hand rule**, as illustrated in figure (a). Grasp the axis of rotation with your right hand so that your fingers wrap in the direction of rotation. Your extended thumb then points in the direction of $\vec{\omega}$.

Figure (b) shows that is also in the direction of advance of a rotating right-handed screw.

Finally, the directions of the angular acceleration $\vec{\alpha}$ and the angular velocity $\vec{\omega}$ are the same if the magnitude of $\vec{\omega}$ is increasing with time, and opposite each other if the angular speed is decreasing with time.



(a) The right-hand rule for determining the direction of the angular velocity vector $\vec{\omega}$.

(b) The direction of $\vec{\omega}$ *is in the direction of advance of a right-handed screw.*

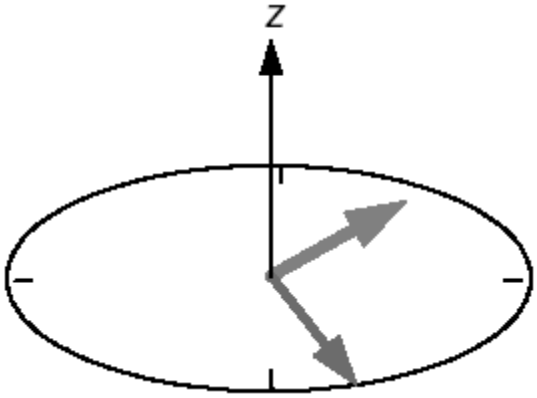
Example

1. What is ω for a wheel that rotates through 100rev in 50sec? **Ans**. ω = 2rev/sec = 12.56rad/sec
2. A wheel turns through 180rev in 1min. Calculate its angular velocity.
   Ans. ω = 3rev/s = 18.84rad/sec
3. Calculate the angular velocity of the hour, minute and second hands of a clock.

Ans. $\vec{\omega}_h = 1.4537\times10^{-4} rad/\sec(-\hat{k})$

$\vec{\omega}_{\min} = 1.744\times10^{-3} rad/\sec(-\hat{k})$

$\vec{\omega}_{\sec} = 0.105 rad/\sec(-\hat{k})$



4. Calculate the time it takes for a cyclist to travel 1.2km if ω_{cyc} = 4rev/sec. r = 30cm.
   Ans. t = 159sec
5. Calculate the linear speed of a point on the surface of the earth at the equator.
   $R_E = 6.37\times10^6$m. **Ans**. V = 463m/sec

Angular acceleration:- is the rate of change of angular velocity.

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

The instantaneous angular acceleration is

$$\alpha_{inst} = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t}$$

The direction of angular acceleration is:-

- parallel to the angular velocity for increasing velocity.
- anti-parallel to the angular velocity for decreasing velocity.

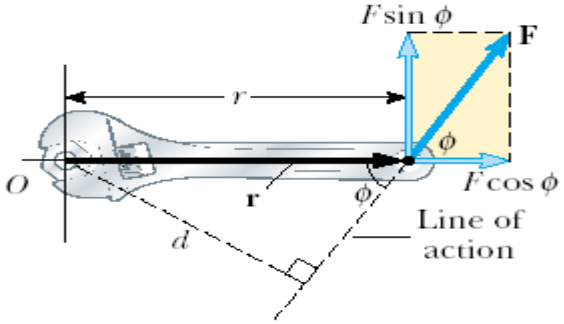
*The angular position of a swinging door is described by $\theta = 5 + 10t + 2t^2$. What is the angular speed and acceleration of the door at t = 3sec? **Ans**. ω = 22rad/sec, α = 4rad/s^2

Torque and angular acceleration

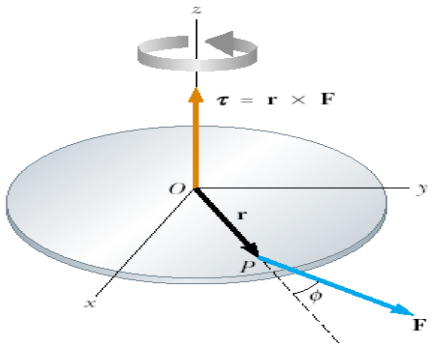
Torque about a point is the turning effect of force about the point and is measured by the product of the force and the perpendicular distance of this point from the line of action of the force.

$$\vec{\tau} = \vec{r}\times\vec{F}$$

Or $$\tau = rF\sin\phi$$



The direction of the torque is found using the right hand rule. "Curl the fingers of your right hand from the first vector, $\vec{r}$, to the second vector, $\vec{F}$. Then the extended thumb points in the direction of the torque."



Inertia:- is the property of a body to resist (oppose) any change in its state of rest or its linear motion. *Mass is the measure of inertia*.

Moment of inertia (I):- is a measure of the resistance of an object to change in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion.

Consider a particle of mass *m* rotating in a circle of radius *r* under the influence of a tangential force F_t as shown in figure below.

The tangential force provides a tangential acceleration a_t, and

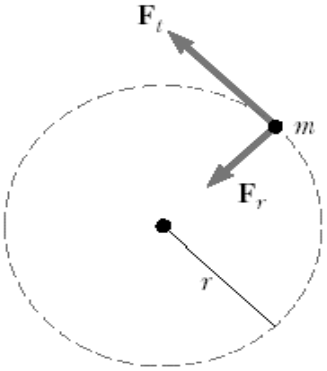
$F_t = ma_t$

The magnitude of the torque about the center of the circle due to F_t is

$\tau = F_t r$
$= (ma_t)\, r$
$= (mr\alpha)\, r$
$= (mr^2)\, \alpha$

Where $mr^2 = I$ is the moment of inertia of the particle, so that

$\tau = I\, \alpha$



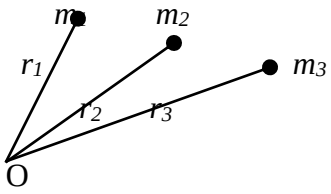
That is, **the torque acting on the particle is proportional to its angular acceleration**, and the proportionality constant, **the moment of inertia**. Note that $\tau = I\, \alpha$ is the rotational analog of Newton's second law of motion, $F = ma$. Moment of inertia, I, is the counterpart of mass *m*.

Total moment of inertia of m_1, m_2 and m_3 rotating about point O is

$$I_{Tot} = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

For *n* particles

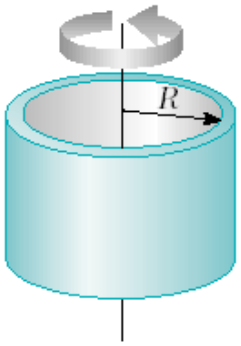
$$I_{Tot} = \sum_{i=1}^{n} m_i r_i^2$$



Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

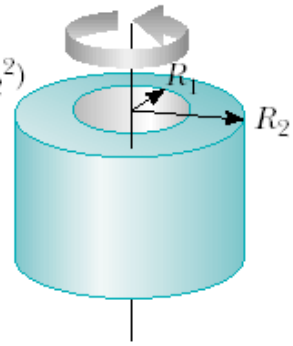
Hoop or thin cylindrical shell

$I_{CM} = MR^2$



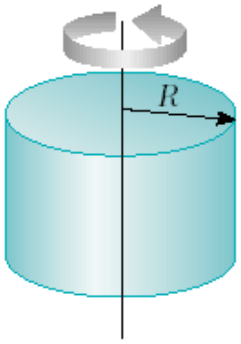
Hollow cylinder

$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



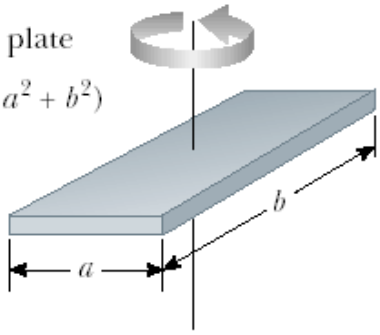
Solid cylinder or disk

$I_{CM} = \frac{1}{2}MR^2$



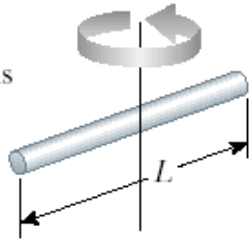
Rectangular plate

$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



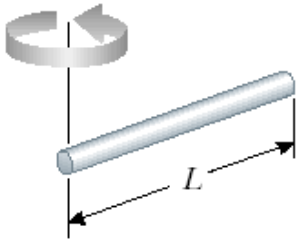
Long thin rod with rotation axis through center

$I_{CM} = \frac{1}{12}ML^2$



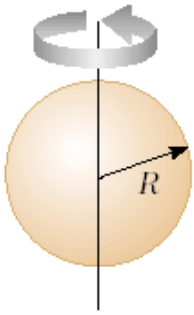
Long thin rod with rotation axis through end

$I = \frac{1}{3}ML^2$



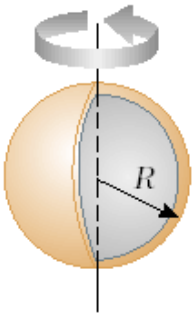
Solid sphere

$I_{CM} = \frac{2}{5}MR^2$



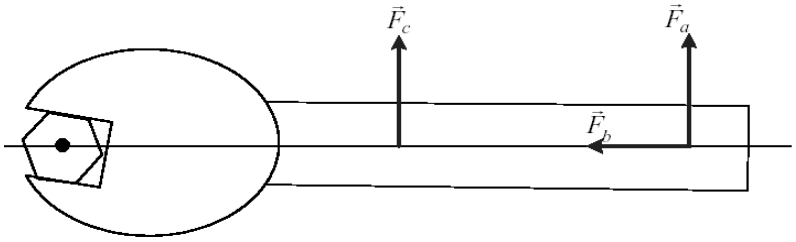
Thin spherical shell

$I_{CM} = \frac{2}{3}MR^2$

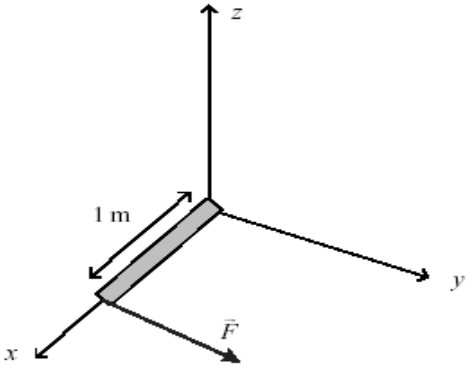


Example

1. Which of these three forces is most likely to loosen the tight bolt?



2. A force $\vec{F} = 5N\,\hat{j}$ is applied to the end of the rod. What is the torque? **Ans**. $\vec{\tau} = 5Nm\,\hat{k}$

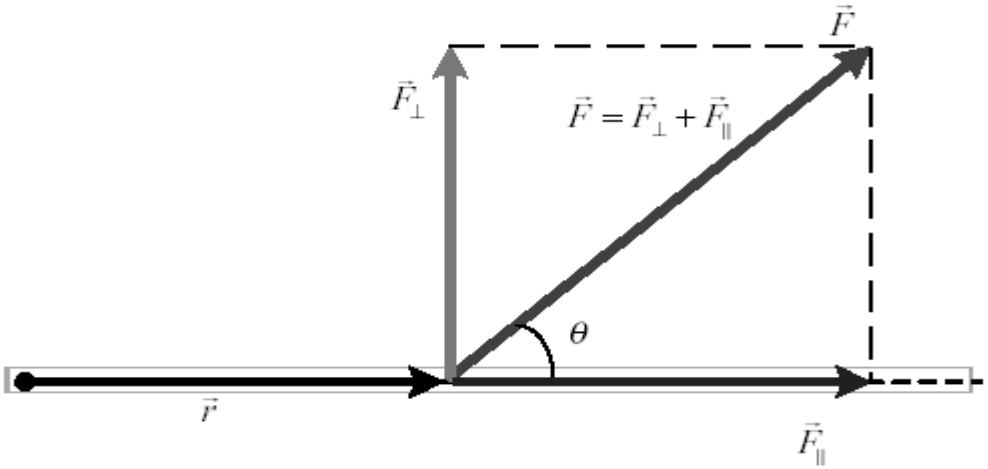


3. A 1kg particle moves in a circle of radius 1m. A constant tangential force of 10N is applied. Calculate *I,* τ and α.**Ans**. I = 1kg m^2, τ= 10Nm and α = 10rad/s^2

4. $\vec{r} = (3\hat{i} + 2\hat{j})m$, $\vec{F} = (3\hat{i} + 5\hat{j})$N. What is $\vec{\tau}$ and what is the direction of rotation?

Ans. $\vec{\tau} = 9Nm\,\hat{k}$. The rotation is anti -clockwise around the z –axis.

5. Show that only the component of $\vec{F}$ perpendicular to $\vec{r}$ contributes to $\vec{\tau}$.

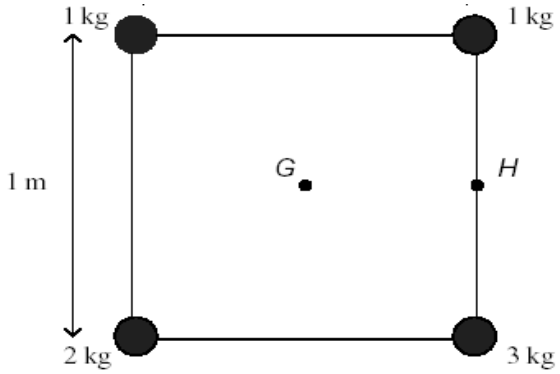


6. A thin rectangular sheet of steel is 0.3m by 0.4m and has mass 24kg. Calculate the moment of inertia about an axis through the center, perpendicular to the plane. **Ans**. I = 0.5kg m^2

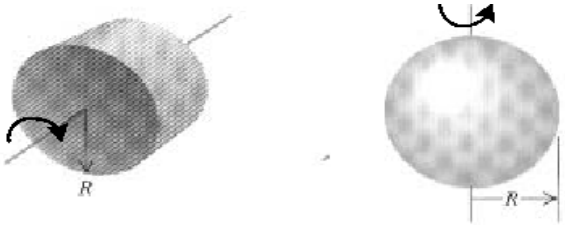
Exercise

1. Four particles at the corners of a square of sides of length 1 m rotate about an axis perpendicular to the plane of the square. What are the moments of inertia about *G* and *H*?

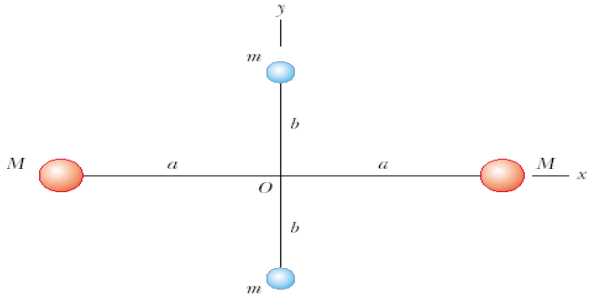
   G is at the center of the square.
   H is 0.5m from the 3kg mass.
   Ans. $I_G = 3.5\text{kg m}^2$
   $I_H = 4.75\text{kg m}^2$



2. Calculate the moment of inertial of four equal masses, 25g each situated at the corners of a square of sides 20cm, about perpendicular axis passing through the center of the square.
   Ans. $I = 0.002\text{kg m}^2$
3. A force of 40N is applied to a spanner at a distance of 20cm from a bolt. If the force makes an angle of 30^0 with the line from the bolt to the point of application. Calculate the torque exerted at the bolt. **Ans**. $\tau = 4\text{Nm}$
4. Equal torques are applied to a solid cylinder and solid sphere. Which of the two will experience the larger angular acceleration if $M_{cyli} = M_{sph}$ and $R_{cyli} = R_{sphe}$?
   Ans. $\alpha_{sphere} > \alpha_{cylinder}$



5. A solid sphere A and a hollow sphere B roll down an incline, starting from rest. Sphere A has the same mass and radius as sphere B. Which one arrives at the bottom first?
   A) Sphere A
   B) Sphere B
   C) Both arrive at the same time.
   D) Impossible to determine
6. Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the *xy* plane. Calculate the total moment of inertia if the system rotates about
   a) the y axis. **Ans**. $I = 2Ma^2$
   b) the *z* axis. **Ans**. $I = 2Ma^2 + 2mb^2$
   c) the x axis. **Ans**. $I = 2mb^2$



7. A rope is wound around the rim of a wheel of radius 12*cm* and moment of inertia 0.02kg m^2. If the rope is pulled with a force of 2.5N, what is its angular acceleration? **Ans**. $\alpha = 15\text{rad/s}^2$
8. A disk of mass 10kg and radius 20cm is free to rotate about an axis through its center perpendicular to the plane of the disk. If a force of 50N is applied tangentially, calculate the angular acceleration. **Ans**. $\alpha = 50\text{rad/s}^2$

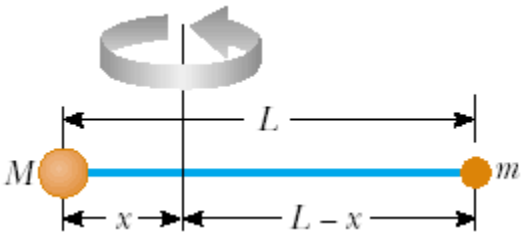
9. Two balls with masses *M* and *m* are connected by a rigid rod of length *L* and negligible mass as in figure below. Show that

   a) the center of mass of the system, from the larger mass M, is given by: $x = \frac{m}{M+m}L$

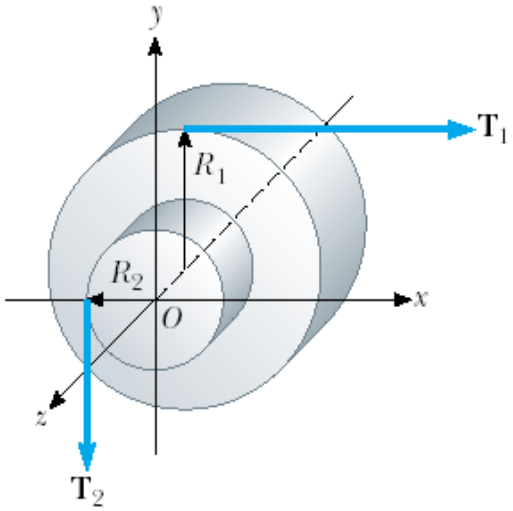
   Hint: Take the origin to be at the axis of rotation.

   b) the moment of inertia of the system through the center of mass, perpendicular to the rod, is given by

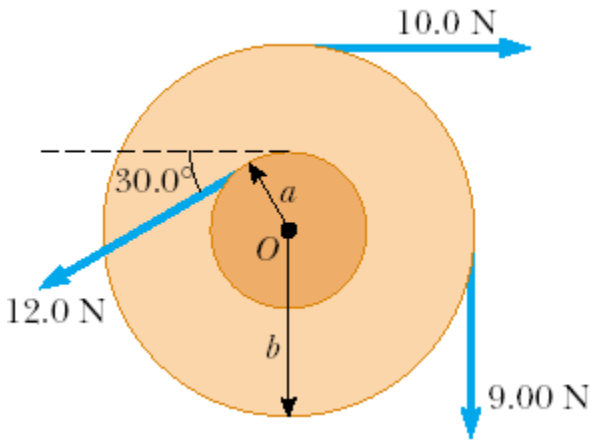
$$I = \frac{mM}{M+m}L^2$$



10. The cylinder in the figure below is free to rotate about the central axis. A rope wrapped around the drum, which has radius R_1 = 1m, exerts a force, T_1=5N to the right on the cylinder. A rope wrapped around the core, which has radius R_2 = 0.5m, exerts a force T_2=15N downward on the cylinder. What is the net torque acting on the cylinder about the rotation axis which is the *z* axis in the figure? **Ans**. τ_{net} = 2.5Nm



11. Find the net torque on the wheel in figure below about the axle through *O* if *a*= 10cm and *b* = *25cm*.**Ans**. τ_{net} = 3.55 N.m



Questions from national exam

1.A pulley 12cm in diameter is free to rotate about a horizontal axle. A block of weight 4N and another block of weight 10N are tied to either end of a mass less string, and the string is hung over the pulley. If the string does not slip, what torque must be applied to stop the pulley from rotating?

A) 0.36Nm B) 0.72Nm C) 0.84Nm D) 1.68Nm

2. A 50Nm torque acts on a wheel of moment of inertia 200kg m^2. If the wheel starts from rest, how long will it take the wheel to reach an angular speed of 1rad/sec?

A**)** 4*s* B) 2*s* C) 1*s* D) 0.25*s*

3. A 2kg uniform disk has a radius of 10cm.Starting from rest, it rotates about an axis under the action of a constant torque of 0.5Nm. How long does the disk take to reach an angular speed of 150rad/s?

A) 1*s* B) 2*s* C**)** 3*s* D) 4*s*

4. The moment of inertia of an object depends on which of the following?

A) The mass of an object. C) The axis of rotation.
B) The shape of the object. D) All of the above.

5. What is the torque due to a force $\vec{F} = \left(2\hat{i} - 3\hat{j} + 4\hat{k}\right)N$ acting at a point $\vec{r} = \left(3\hat{i} + 2\hat{j} + 3\hat{k}\right)m$ about the origin? **Ans.** $\vec{\tau} = \left(17\hat{i} - 6\hat{j} - 13\hat{k}\right)Nm$

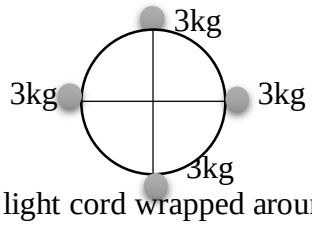
A) (6*i*–6*j*+12*k*)Nm C) (–17*i*+6*j*+13*k*)Nm
B) (–6*i*+6*j*+12*k*)Nm D) (17*i*+6*j*–13*k*)Nm

6. Which of the following objects has the largest moment of inertia assuming they all have the same mass and radius?

A) A solid disk. C) A circular loop.
B) A solid sphere. D) A solid cylinder.

7. The radius of the wheel shown below is 0.75m. What is its moment of inertia?

A) 2.25kg m^2 C**)** 6.75kg m^2
B) 3.38kg m^2 D) 9.00kg m^2

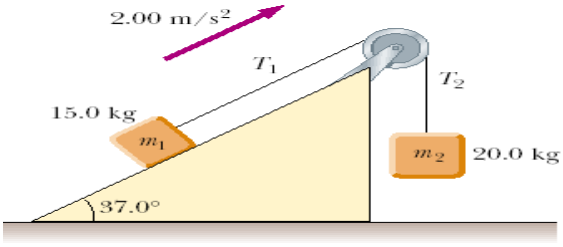


8. A 2kg disk is mounted on a frictionless and horizontal axle. A light cord wrapped around the disk supports 1kg object. What is the acceleration of the suspended object? (g=10m/s^2)

A)8m/s^2 B) 10m/s^2 C) 4m/s^2 D**)** 5m/s^2

9. As shown in the figure, two blocks are connected by a string of negligible mass passing over a pulley of radius *r* = 0.2m and moment of inertia *I*. The block on the frictionless incline is moving with a constant acceleration *a* = 2m/s^2 up along the incline. What is the moment of inertia of the pulley?

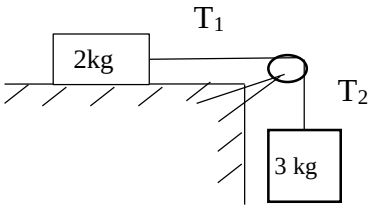
A**)** 0.8 kg m^2 C) 4kg m^2
B) 1.25kg m^2 D) 40kg m^2



10. Two blocks are connected by a string of negligible mass passing over a pulley of radius r = 0.2m and moment of inertia I as shown in figure below. The block on the horizontal frictionless surface is moving with a constant acceleration a = 2.0m/s^2. If the pulley axis is frictionless, the string is inextensible and does not slip on the pulley, what should be the moment of inertia of the pulley?

A**)** 0.40kg m^2 C) 4.00kg m^2

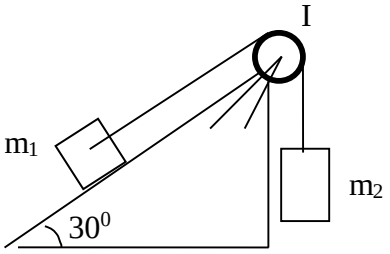
B) 0.80kg m^2 D) 8.00kg m^2



11. A block of mass m_1 = 10 kg is placed on a frictionless ineclined plane connected to mass m_2 = 15 kg using a string of negligible mass passing over a pulley of moment of inertia I = 0.25 kg. m^2 and radius R = 10 cm as shown in the diagram. What is the torque on the pulley**?**

A) 2.5 Nm C**)** 5 Nm

B) 4 Nm D) 12 Nm



Radius of Gyration (γ):-is the radial distance from the axis of rotation at which the concentration of the entire mass M of the object would produce the same rotational inertia as that of the extended object itself.

$\gamma = \sqrt{\frac{I}{M}}$ Where I is the moment of inertia of the body about the axis of rotation.

$I = M\gamma^2$

Example

1. Show that radius of gyration of

a) solid sphere is $\gamma = \sqrt{\frac{2}{5}}$ R b) disk is $\gamma = \sqrt{\frac{1}{2}}$ R

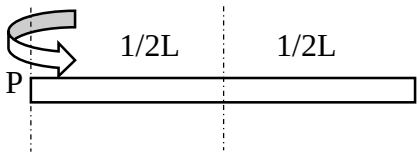
2. An 80kg uniform sphere has a radius of gyration of 0.25m. Calculate its moment of inertia.

Ans. I = 5kg. m^2

Parallel axis theorem

The moment of inertia I_p about an axis through an arbitrary point P, parallel to an axis through the center of mass is

$I_P = I_{CM} + Md^2$



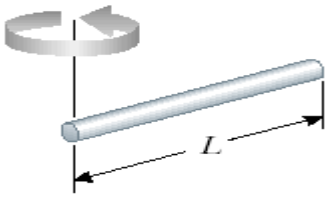
Where I_{CM} is moment of inertia about an axis through its center of mass.

d is perpendicular distance between the two parallel axes.

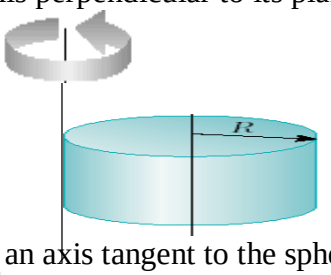
M is mass of the rod.

Example

1. Consider a uniform rigid rod of mass *M* and length *L* shown in figure below. Find the moment of inertia of the rod about an axis perpendicular to the rod through one end. **Ans.** $I = \frac{1}{3}ML^2$

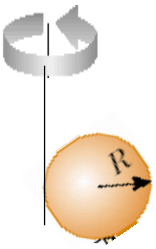


2. Find the moment of inertia of a uniform disk about an axis perpendicular to its plane at the edge. **Ans.** $I = \frac{3}{2}MR^2$



3. Determine the moment of inertia of a solid sphere about an axis tangent to the sphere.

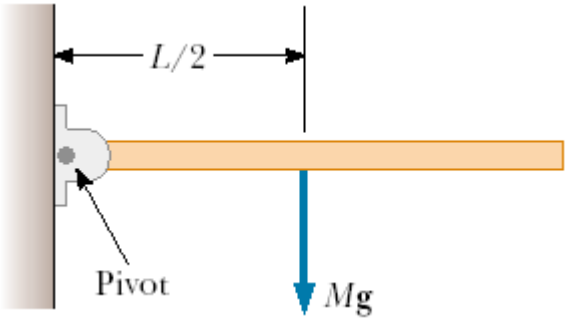
Ans. $I = \frac{7}{5}MR^2$



4. A uniform rod of length *L* and mass *M* is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane, as shown in figure below. The rod is released from rest in the horizontal position. What is

   a) the initial angular acceleration of the rod? **Ans.** $\alpha = \frac{3g}{2L}$

   b) the initial linear acceleration of its right end? **Ans.** $a_t = \frac{3}{2}g$



Rotational Kinetic Energy

When an object is performing pure rotation about a fixed axis, its rotational energy is given by

$K.E_{rot} = \frac{1}{2} I \omega^2$ Where I is moment of ineertia

ω is angular speed

The quantities *I* and ω in rotational motion are analogous to *m* and *v* in linear motion.

If the body is rotating and moving linearly at the same time, its total kinetic energy is the sum of rotational and translational kinetic energies.

$$K.E_{Tot} = K.E_{rot} + K.E_{tran}$$
$$= \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$
$$= \frac{1}{2} I\omega^2 + \frac{1}{2} m(\omega r)^2$$
$$= \frac{1}{2} I\omega^2 + \frac{1}{2} m\omega^2 r^2$$

In linear motion, work done by a force is

$$W = \vec{F} \bullet \vec{S}$$

The angular equivalent of force is torque $\vec{\tau}$ and the rotational equivalent of linear displacement is angular displacement $\vec{\theta}$

The rotational work done is

$$W = \vec{\tau} \bullet \vec{\theta}$$

The rotational power is

$$P = \vec{\tau} \bullet \vec{\omega}$$

The net work done by external forces in rotating a symmetric rigid object about a fixed axis equals the change in the object's rotational energy.

$$W = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$

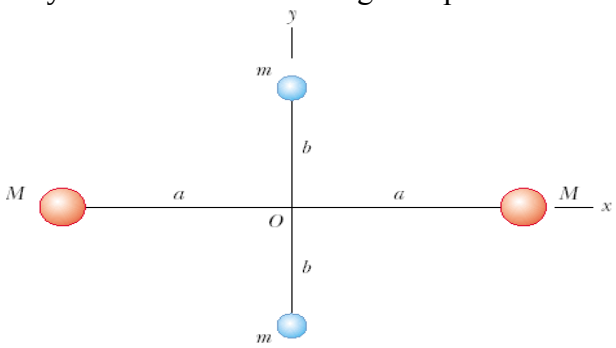
Example

1. What is the kinetic energy of a body with I = 0.004kg.m^2 rotating at ω = 0.5rad/s?
   Ans. $K.E_{rot} = 5\times10^{-4}$J
2. Find the kinetic energy of a body having I = 0.003kg.m^2 rotating at ω = 0.6rad/s.
   Ans. $K.E_{rot} = 5.4\times10^{-4}$J
3. Calculate the work done by a torque of 3.5Nm when it rotates an object through an angle of π/2 rad. **Ans**. W_{rot} = 5.5J
4. What is the work done by a torque of 5Nm when it turns a body though an angle of πrad?
   Ans. W_{rot} = 15.7J
5. A 270N force is applied at the end of a 50*cm* long spanner to tighten a bolt. If the bolt turns through half a turn, how much work is done? **Ans**. W = 424J

Exercise

1. Calculate the rotational kinetic energy of a solid sphere with *m* = 20kg and R = 2m if it completes 1rev every 2 *sec*. **Ans**. $K.E_{rot}$ = 158J
2. The earth is considered as a sphere of mass 6×10^{24}kg and $R_E = 6.4\times10^6$m. Calculate the rotational kinetic energy of the earth with respect to its own axis of rotation.
   Ans. $K.E = 2.6\times10^{29}$J
3. A constant torque of 100 Nm rotates a door by 90^0 about its hinges. What is the work done on the door?
   a) 9000 J b) 0.9 J c) 10 J d) 157 J

4. Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the *xy* plane. Calculate the total rotational kinetic energy if the system rotates with an angular speed ω about
   a) the y axis. **Ans**. $KE_{rot} = Ma^2\omega^2$
   b) the *z* axis. **Ans**. $KE_{rot} = (Ma^2 + mb^2)\omega^2$
   c) the x axis. **Ans**. $KE_{rot} = mb^2\omega^2$

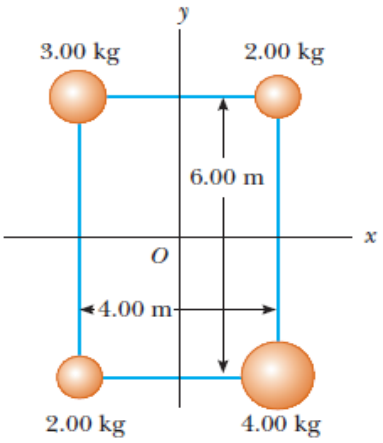


5. A 1m long starter cord for a generator is wound on a drum with a diameter of 10cm. The cord is pulled with a force of 100N. What is
   a) the torque applied? **Ans**. τ = 5Nm. c) the work done? **Ans**. W = 100J
   b) the angle through which it rotates? **Ans**. θ = 20rad
6. Show that the total kinetic energy of
   a) a solid cylinder, rolling with a speed v without slipping down the plane is $K.E_{Tot} = ¾\ mv^2$
   b) a solid sphere, rolling with a speed v without slipping down the plane is $K.E_{Tot} = 7/10\ mv^2$
   c) a hollow sphere, rolling with a speed v without slipping down the plane is $K.E_{Tot} = 5/6mv^2$

Questions from national exam

1. A ring, a coin and a solid sphere all have the same radius and mass start from rest and roll down without slipping from the top of an inclined plane at the same time. What is the order in which they reach the bottom?
   A) Ring, coin, then sphere. C) Sphere, ring, then coin.
   B) Coin, sphere, then ring. D) Sphere, coin, then ring.
2. What is the moment of inertia of a wheel that has a kinetic energy of 250J when it is rotating at 10rad/s?
   A) $2.5kg.m^2$ B) $5kg.m^2$ C) $7.5kg.m^2$ D) $10kg.m^2$
3. A solid cylinder with mass *m* and radius *r* rolls along a level surface without slipping with a linear velocity v. What is the ratio of its rotational kinetic energy to its linear kinetic energy?
   A) ¼ B) ½ C) 1/1 D) 2/1
4. A uniform ball of radius *r* and mass *m* starts from rest and rolls down an incline of height *h*. When the ball reaches the ground, how fast is it going? **Ans**. $v = \sqrt{\frac{6}{5}gh}$
   A) $\sqrt{\frac{10}{7}gh}$ B) $\sqrt{\frac{9}{10}gh}$ C) $\sqrt{\frac{10}{9}gh}$ D) $\sqrt{\frac{7}{10}gh}$
5. A disk of radius R with uniform distribution of mass M, rotates about an axis perpendicular to its plane at the rim with angular speed ω. The moment of inertia of the disk is ½ MR^2. What is the kinetic energy of the disk**?**
   A) ¼ $MR^2\ \omega^2$ B) 1/3 $MR^2\omega^2$ C) ½ $MR^2\omega^2$ D) ¾ $MR^2\omega^2$

6. The particles shown in figure below are connected by rectangular rigid rods of negligible mass. The origin of the *xy* plane is at the center of the rectangle. The system rotates in the *xy* plane about the *z* axis with an angular speed of 6rad/s. What is the rotational kinetic energy of the system?



A) 2574J C) 1287J

B) 2340J D) 1170J

7. A thin rod when rotated about an axis about the end of the rod has a moment of inertia $\frac{1}{3}ML^2$ The rod has a mass of 1.5 kg and length of 1 m. If the rod initially at rest gains a rotational kinetic energy of 1000 J, at what speed does it rotate?

(A) $10\sqrt{5}\,rad/\sec$ (C) $20\sqrt{10}\,rad/\sec$

(B) 10 *rad/sec* (D) $20\sqrt{10}\,rad/\sec$

8. A ring, a solid cylinder and a solid sphere, all having the same mass and the same radius, are held at the top of an inclined plane. If the objects roll without sliping , which one of the following shows the order in which they get to the bottom of the inclined plane?
   (A) All will reach the bottom at the same time.
   (B) Ring, solid cylinder, sphere.
   (C) Solid cylinder, sphere, ring.
   (D) Sphere, solid cylinder, ring.
9. Five beads of equal mass are fixed on a rigid light rod that is free to rotate about a frictionless vertical bearing. The arrangement is such that, four of them are placed at 0.25 m and the fifth one is at 0.5 m from the axis of rotation. The rotational kinetic energy is 144 J when the beads were rotating with angular speed of 48 rad/s. What is the mass of each bead?

A) 1 kg B) 0.5 kg C) 0.75 kg D) 0.25 kg

Angular momentum and angular impulse

Linear momentum $\vec{P}$ is defined as the product of mass *m* and velocity $\vec{v}$.

$$\vec{P} = m\vec{v}$$

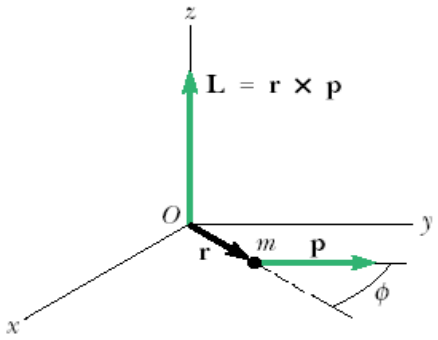
Angular momentum $\vec{L}$ is defined as the product of moment of inertia *I* and angular velocity $\vec{\omega}$

$$\vec{L} = I\vec{\omega} \quad \text{or}$$

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times (m\vec{v})$$

$$= m\,\vec{r} \times \vec{v}$$

$$|\vec{L}| = m|\vec{r}||\vec{v}| \sin\phi$$



Where $\vec{r}$ is the position vector with respect to the axis of rotation.

ϕ is the angle between $\vec{r}$ and $\vec{v}$

The rotational analogy to Newton's second law is

$$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t} \longrightarrow \vec{F} = \frac{\Delta\vec{P}}{\Delta t}$$

The angular impulse $\vec{J}$ is given by

$$\vec{J} = \vec{\tau}\,\Delta t \qquad \vec{I} = \vec{F}\Delta t$$

$$\vec{J} = \Delta\vec{L} \qquad \vec{I} = \Delta\vec{P}$$

$$\vec{J} = \vec{L} - \vec{L}_0 \qquad \vec{I} = \vec{P}_f - \vec{P}_i$$

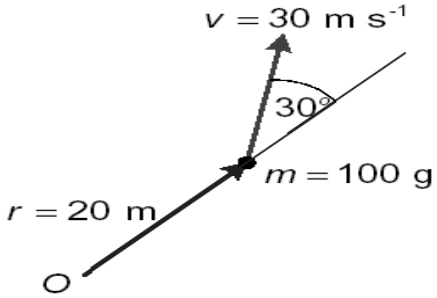
$$\vec{J} = I\vec{\omega} - I\vec{\omega}_o \qquad \vec{I} = m\vec{v} - m\vec{u}$$

Moment of inertia — linear impulse

Example

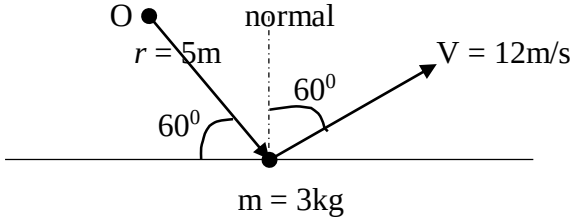
1. A flywheel with I = 50kg.m^2 is rotated at 1200rev/min. What is the magnitude of the angular momentum of the wheel? **Ans**. L = 6280kg m^2/s
2. Find the magnitude of the angular momentum, *L*, of this particle with respect to point O:

Ans. L = 30Js

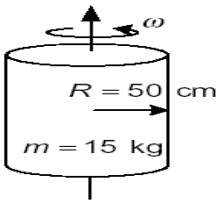


3. What is the angular momentum of the 3kg object about point O in the following figure?

Ans. L = 180J s



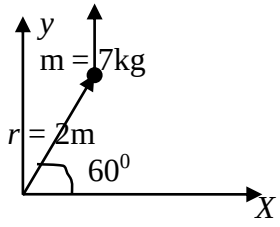
4. If the mass, position vector and velocity are given by m = 8kg, $\vec{r} = \left(12\hat{i} - 18\hat{j}\right)m$ and $\vec{V} = \left(-3\hat{i} + 7\hat{j}\right)m/s$, what is $\vec{L}$? **Ans**. $\vec{L}$ = 240Js $\hat{k}$
5. Calculate the angular impulse of the solid cylinder in fig below after 30 min, if ω_o = 100rev/min and $\alpha = 0.05 rad/s^2$. **Ans**. J = 168.75Js



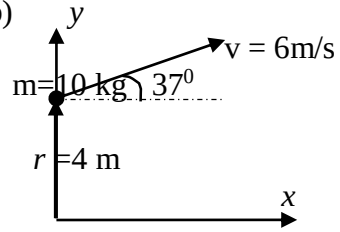
Exercise

1. Calculate the magnitude of the angular momentum of the earth with respect to
   a) its own axis of rotation. $M_E = 5.98\times10^{24}$kg, $R_E = 6.37\times10^{6}$m. **Ans**. L = 7.06×10^{33}Js
   b) the sun. ($R_{ES} = 1.5\times10^{11}$m) **Ans**. L= 2.68×10^{40}Js
2. If *m* = 8kg, $\vec{r} = \left(15\,\hat{i} - 8\hat{j}\right)m$ and $\vec{V} = 60km/h$ at 50^0 above *x*-axis, what is $\vec{L}$?
   (Use cos 50^0 = 0.643, sin 50^0 = 0.766) **Ans**. $\vec{L} = 2.22\times10^3\ Js\hat{k}$
3. A rod of length 1.2*m* and mass 3kg is rotating about an axis through its center. If ω_0 = 2rad/s and $\alpha = 0.1 rad/s^2$, what is its angular impulse after 20 rotations? **Ans**. J = 1.22Js
4. What torque must be applied on a wheel to change its angular speed from rest to 30rev/min in 2min? (M = 30kg, R = 2m) **Ans**. τ = 1.57Nm
5. A 3kg hollow sphere with R = 60cm is rotating about an axis through its center. If its angular speed decreases from 50rad/s to 5rad/s in 30sec, what torque is exerted on it?
   Ans. τ = –1.08Nm
6. Calculate the angular momentum of

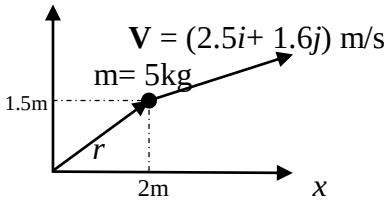
a)



b)



c)



Ans. $\vec{L} = 210\,kg \bullet m^2/s\,\hat{k}$ **Ans**. $\vec{L} = 192kg \bullet m^2/s\left(-\hat{k}\right)$ **Ans**. $\vec{L} = -2.75kg \bullet m^2/s\hat{k}$

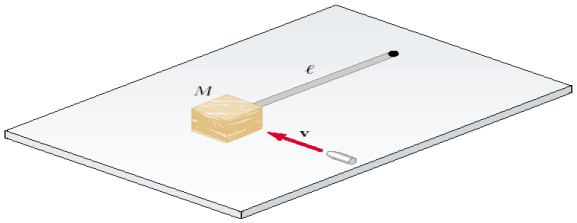
7. A *solid sphere* and a *hollow sphere* have the same mass and radius. They are rotating with the same angular speed. The one with the larger angular momentum is
   A) the solid sphere
   B) they both have the same angular momentum
   C) the hollow sphere
   D) impossible to determine.

8. A wooden block of mass *M* resting on a frictionless horizontal surface is attached to a rigid rod of length ℓ and of negligible mass as in figure below. The rod is pivoted at the other end. A bullet of mass *m* traveling parallel to the horizontal surface and normal to the rod with speed *v* hits the block and becomes embedded in it. Show that

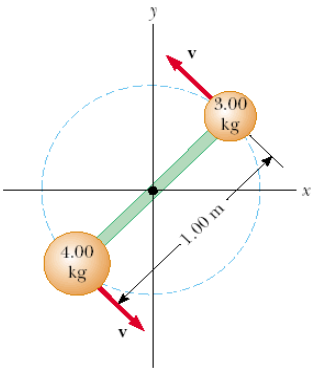
a) the angular momentum of the bullet–block system is given by: $\vec{L} = mv\ell$ down

b) the fraction of the original kinetic energy lost in the collision is given by:

$$\frac{K.E_{lost}}{K.E_{initial}} = \frac{M}{M+m}$$



9. A light rigid rod 1m in length joins two particles, with masses 4kg and 3kg, at its ends. The combination rotates in the *xy* plane about a pivot through the center of the rod as in figure below. Determine the angular momentum of the system about the origin when the speed of each particle is 5m/s. **Ans**. **L** = 17.5kg m^2/s $\hat{k}$



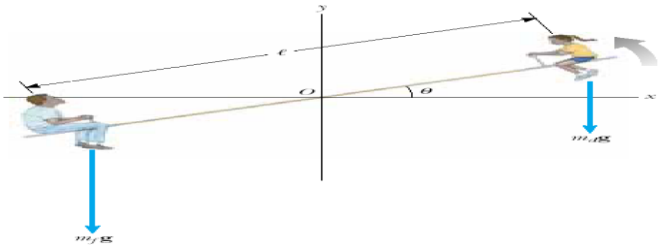
10. A father of mass m_f and his daughter of mass m_d sit on opposite ends of a seesaw at equal distances from the pivot at the center as in the figure below. The seesaw is a rigid rod of mass *M* and length ℓ and is pivoted without friction. At a given moment, the combination rotates in a vertical plane with an angular speed ω. Show that the magnitude of the system's

a) angular momentum is given by:

$$L = \frac{\ell^2}{4}\left(\frac{M}{3} + m_f + m_d\right)\omega$$

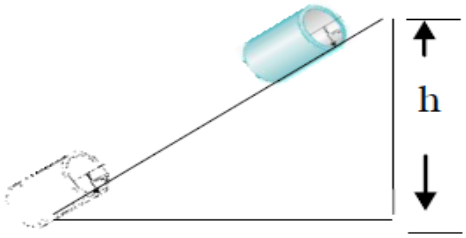
b) angular acceleration when the seesaw makes an angle θ with the horizontal is given by:

$$\alpha = \frac{2(m_f - m_d)g\cos\theta}{\ell\left(\frac{M}{3} + m_f + m_d\right)}$$

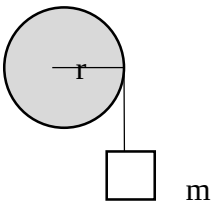


Questions from national exam

1. If an impulsive torque of 100Nm is exerted for 0.01*sec* on a rigid body free to rotate about a given axis, what is the change in angular momentum produced?
   A) 10000Nms B) 100Nms C) 10Nms D**)** 1Nms
2. An impulsive torque of 10Nm acts for a short time, 0.01sec, on a rotating disk. What is the angular impulse imparted on the disk?
   A) 1kg m^2/s B**)** 0.1kgm^2/s C) 10kgm^2/s D) 1000kg m^2/s
3. When a torque acting upon a system is zero, which one of the following will be constant?
   A) Force. C) Angular momentum.
   B) Linear momentum. D) Linear impulse.
4. A hoop of mass M and radius R is at rest at the top of an incline as shown. The hoop rolls down without slipping. If it reaches bottom, then what is its angular momentum about its center of mass?
   A**)** MR $\sqrt{gh}$ C) MR $\sqrt{2gh}$
   B) $\frac{MR}{2}\sqrt{gh}$ D) Mgh



5. The angular momentum of a particle is:
   A) independent of the specific origin of coordinates.
   B) zero when its position and momentum vectors are parallel.
   C) zero when its position and momentum vectors are perpendicular.
   D) not zero for non-zero position and momentum vector.
6. A 1.5kg particle moves in the *xy*-plane with velocity $\vec{V} = \left(4.2\hat{i} - 3.6\hat{j}\right) m/s$. What is its angular momentum when its position vector is $\vec{r} = \left(1.5\hat{i} + 2.2\hat{j}\right) m$?
   A**)** –22kg m^2/s *k* B) 14.6kg m^2/s *k* C) –14.6kg m^2/s *k* D) 22kg m^2/s *k*
7. A hoop of mass M and radius R is at rest at the top of an inclined plane of height *h*. The hoop rolls down the plane without slipping. When the hoop reaches bottom, its angular momentum about its center of mass is:
   A) Mg*h* B) $MR\sqrt{2gh}$ C) $\frac{MR}{2}\sqrt{gh}$ D) $MR\sqrt{gh}$
8. An object with a mass of *m* = 3.0kg is attached to the free end of a light string wrapped around a pulley of radius r = 0.50m and moment of inertia I = 0.75kg m^2. The pulley is free to rotate in a vertical plane about a fixed frictionless axis passing through its center as shown in figure below. The suspended object is released from rest at 3.60m above the floor. What is the angular momentum of the pulley at the time the object strikes the floor?
   A) 4.5kg m^2/s C) 12.3 kg m^2/s
   B**)** 9.0kg m^2/s D) 18 kg m^2/s



9. The rotational analogue of mass is:

A) moment of inertia
B) torque
C) angular momentum
D) angular acceleration

Conservation of angular momentum

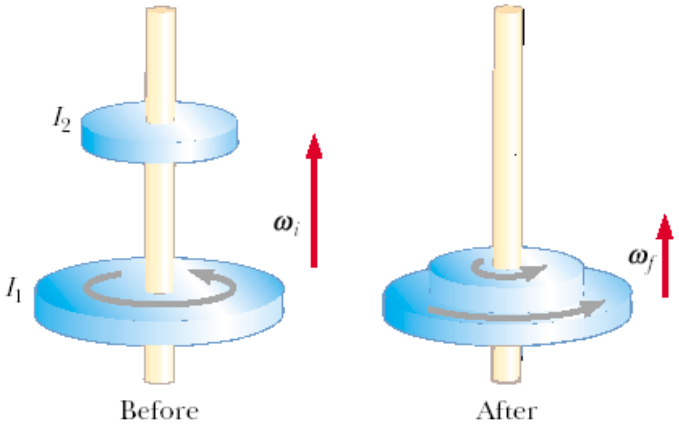
If the net external torque acting on the system is zero, that is, *if the system is isolated the total angular momentum of a system is constant.*

$$\vec{L}_0 = \vec{L}$$

$$I_0\vec{\omega}_0 = I\vec{\omega}$$

Example

1. At the beginning of a spin, moment of inertia of a skater is 60kg m^2 and her angular velocity ω_0 = 0.2rad/sc. She brings in her arms and her moment of inertia decreases to 50kg m^2. Find her final angular velocity. **Ans**. ω = 0.24rad/sec
2. A man, with his arms extended, has rotational speed of 0.25rev/sec. When he draws them in, his speed becomes 0.8rev/sec. Calculate the ratio of his moment of inertia in the first to that in the second. **Ans**. I_0: I = 16: 5
3. If our sun, with $R_0 = 110R_E$ and T_0 = 25days, accidentally shrinks to a sphere of radius R_E, calculate its final period. **Ans**. T_f = 178.5 sec
4. A disk with moment of inertia I_1 rotates about a vertical, frictionless axle with angular speed ω_i. A second disk, having moment of inertia I_2 and initially not rotating, drops onto the first disk as in figure below. The two then rotate with angular speed ω_f .

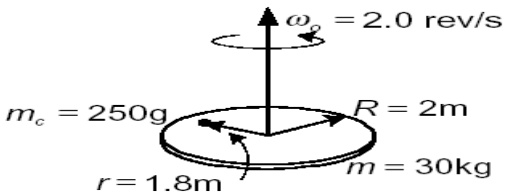


a) Calculate ω_f. **Ans**. $\omega_f = \omega_i \dfrac{I_1}{I_1 + I_2}$

b) Show that the ratio of the final to the initial rotational kinetic energy is: $\dfrac{I_1}{I_1 + I_2}$

c) If I_1 = 1kg m^2, ω_i= 3rad/sec and I_2 = 0.5kg m^{2}calculate ω_f. **Ans**. ω_f = 2rad/sec

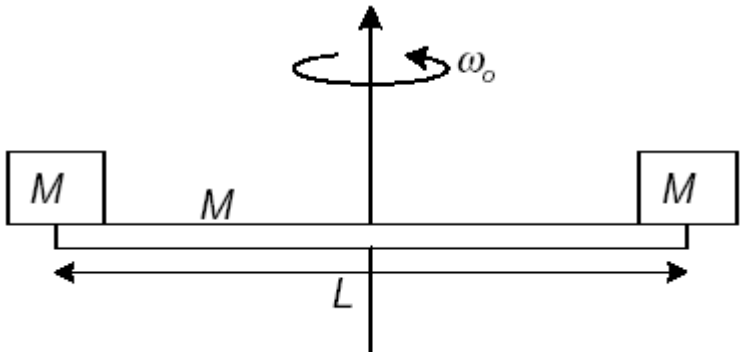
5. Find ω_f of the system after a clay of mass 250g is dropped on the turntable at a distance of r = 1.8m from the axis of rotation. **Ans**. ω_f = 1.97rev/sec



Exercise

1. Calculate the resulting angular speed ω, if the blocks move until they are half as far from the axis of rotation: $M_{rod} = M_{block} = M$

Ans. $\omega = 2.8\ \omega_0$



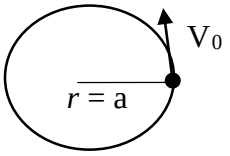
2. A student is holding a dumbbell of mass 4kg in each hand. Find his final angular velocity when he pulls the dumbbells in. $I_{stud} = 5.0 kg\ m^2$, $r_i = 90$ cm, $r_f = 15$cm, $\omega_0 = 0.5$rev/sec.
Ans. ω = 6.96rad/sec.

3. What will happen to the rotational kinetic energy of a spinning ice skater as she folds her horizontally stretched hands?

A) It decreases. C) It does not change.
B**)** It increases. D) It is unpredictable.

Questions from national exam

1. Consider an isolated system shown in the figure below. The particle attached to a cord rotates on the turntable. As the bead rotates the length of the cord shortens. What would be its speed at $r = \frac{1}{2}a$ if its speed at $r = a$ is v_0?



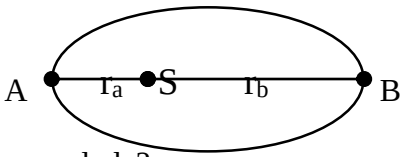
A) $V_0/2$ C) V_0
B**)** $2V_0$ D) $V_0/4$

2. Two discs of moment of inertia I_1 and I_2 about their respective axes (which are normal to the disc and passing through the center), and rotating with angular speeds ω_1 and ω_2 are brought into contact face-to-face with their axes of rotation coincident. Take $\omega_1 \neq \omega_2$. What is the angular speed of the two disc system after contact?

A) $\frac{I_1\omega_1 + 2I_2\omega_2}{I_1 + I_2}$ B**)** $\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$ C) $\frac{I_1\omega_1 + 2I_2\omega_2}{I_1 - 2I_2}$ D) $\frac{I_1\omega_1 + I_2\omega_2}{I_1 - I_2}$

3. A planet of mass *m* is circling a star in an elliptical orbit as shown below. If its velocity at point A is 60,000m/s, and $r_b = 8r_a$, what is its velocity at point B?

A**)** 7,500m/s C) 60,000m/s
B) 8,571m/s D) 488,000m/s



4. If the impulse on a rotating disc is non-zero, what can you conclude?

A) The change in linear momentum is constant.
B) The torque acting on an object is constant.
C) Impulse of the disc equals its moment of inertia.
D) The change in angular momentum is non zero.

5. If in a given rotational system of a body the angular momentum is increased with time, which one of the following statements is **NOT** correct?
   (A) The radius of rotation may increase.
   (B) There is no torque exerted on the system.
   (C) The moment of inertia may increase.
   (D) There is a torque exerted on the system.
6. In a playing room a turn table of radius 2 m has moment of inertia 250 kg m^2 and is rotating about a frictionless vertical axel. When a boy of mass 50 kg stands near the rim, the system rotates with angular speed of 3 rad/s. The boy walks towards the center and stands at 1 m from the axel. What is the final angular speed of the system?
   A)3 rad/s B**)** 4.5 rad/s C) 5.4 rad/s D) 2.5 rad/s
7. A thin circular ring o mass M and radius r and moment of inertia Mr^2 is rotating about its axis with a constant angular velocity ω. Two objects each of mass *m*, are attached gently to the opposite ends of a diameter of the ring. What would be the angular velocity of the wheel?
   A**)** $\frac{\omega M}{(M+2m)}$ B) $\frac{\omega(M+2m)}{(M+2m)}$ C) $\frac{\omega M}{(M+m)}$ D) $\frac{\omega(M+2m)}{M}$