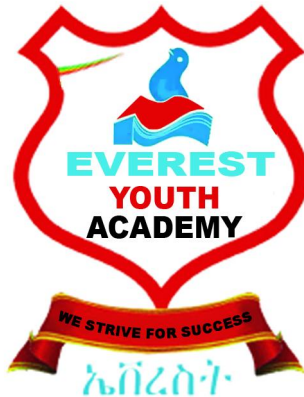


EVEREST YOUTH ACADEMY



GRADE 11 MATHEMATICS SHORT NOTES AND PRACTICE QUESTIONS FOR NATURAL SCIENCES (Units 5-9)

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UNIT 5 STATISTICS AND PROBABILITY

Unit outcomes:

After completing this unit, you should be able to:

- knows specific facts about types of data.
- know basic concepts about grouped data.
- know principles of counting.
- apply facts and principles in computation of probability.

STATISTICS

Statistics is the science of collecting, organizing, presenting, analyzing and interpreting of data (qualitative information) in order to draw conclusion.

Types of data:

1. **Qualitative data;** is obtained when a given population or sample is classified in accordance with an attribute that cannot be measured or expressed in numbers

Examples: color, sex, religion, nationality, interest, beauty, honesty, intelligence, educational qualifications etc.

2. **Quantitative data:** is obtained by assigning a real number to each member of the population, under study. It is measurable; we can express in terms of numbers.

Examples: Age, Weight, Height, Income, number of population, size, score in test, pressure and distance are quantitative data

Definition:

A number, which is used to describe the attribute and which can take different values is called a variable

Note: Variables are denoted by letters such as x, y, z, ... and it may be either discrete or continuous

Definition:

A discrete variable is one which is measured or expressed only in terms of whole number values.

Examples: Number of students, number of chairs, number of cars, and etc.

Continuous variable: It is measured or expressed by fraction or decimal.

Examples: Height, weight, age, size of class, temperature of room, distance of road are examples of continuous variables.

Introduction to Grouped Data

Discrete frequency distribution

List of all data values obtained with their respective frequencies

Grouped frequency distribution- used to present large amount of data in class.

Introduction to ungrouped data

These are data in their original form without being arranged or summarized.

Grouped data: data organized and summarized in frequency distribution are grouped data.

The following are basic concepts that are used in frequency distribution.

Cumulative frequencies- are the sum of frequency of all the classes from the first to up class.

Example: If f_1, f_2, f_3, f_4 are frequency of 4 classes then the cumulative frequency of these classes are $f_1, f_1 + f_2, f_1 + f_2 + f_3$ and $f_1 + f_2 + f_3 + f_4$ respectively these are called less than cumulative frequency if the process is done from the total frequency to back ward is called more than frequency.

class interval =
$$\frac{\text{maximum value} - \text{minimum value}}{\text{number of class required}}$$

Class limits: Is the lower and the upper elements in a given class.

- ✓ Lower class limits: is the first no in given class.
- ✓ Upper class limits: is the second no in given class.

class mark or midpoint =
$$\frac{\text{lower class limit} + \text{upper class limit}}{2}$$

Statistical Terms

1. **Raw data:** Data that has not been organized numerically.
2. **Arrays:** An arrangement of raw data numerical data in ascending order of magnitude.
3. **Range:** The difference between the largest and the smallest numbers in a data.
4. **Class intervals:** In a range of grouped data e.g. 21-30, 31-40 etc., then 21-30 is called the class interval.
5. **Class limits:** In a class interval of 21-30, then 21 and 30 are called class limits.
6. **Lower class limits:** In the class interval 21-30, the lower class limit is 21.
7. **Upper class limit:** in the class interval 21-30, the upper class limit is 30.
8. **Lower and upper class boundaries:** In the class interval 21-30, the lower class boundary is 20.5 and the upper class boundary is 30.5.
These boundaries assume that theoretically measurements for a class interval 21-30 includes all the numbers from 20.5 to 30.5
9. **Class Interval:** In a class 21-30, then the class interval is the difference between the upper class limit and the lower class limit i.e. $30.5 - 20.5 = 10$. The class interval is also known as class width or class size.
10. **Class Mark or Mid-point:** In a class interval 21-30, the class mark is the average of 21 and 30 i.e.
$$\frac{21 + 30}{2} = 15.5$$
11. **Frequency Distributions:** large masses of raw data may be arranged in classes in tabular form with their corresponding frequencies.

Example:

Mass(kg)	10 – 19	20 – 29	30 – 39	40 – 49
N ^o of population	5	7	10	6

This tabular arrangement is called a **frequency distribution** or frequency table.

12. **Cumulative Frequency:** For the following frequency distribution, the cumulative frequencies are calculated as additions of individual

Frequencies mass(x)	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44
Frequency(f)	4	10	16	8	2
Cumulative frequency	4	14	30	38	40

The correction factor is half the difference between the upper class limit of a class and the lower class limit of the subsequent.



Example: The following are scores of 40 students in mathematics exam.

50 72 56 31 48 33 56 54 41 35
22 76 32 66 56 38 48 36 44 46
36 49 51 59 62 41 36 50 41 42
50 50 49 60 36 46 42 42 47 62

Then;

- a. prepare a grouped frequency distribution using 7 classes.
- b. find class interval
- c. determine the cumulative frequency distribution.
- d. what is the upper class limit of 3rd class.
- e. what is the lower class limit of 5th class.
- f. find the cumulative frequency of 3rd class.
- g. determine class boundaries.
- h. determine class mark.
- i. determine the correction factor

Solution:

Maximum value=76, Minimum value=22, Number of class=7

class interval= $\frac{\text{maximum value} - \text{minimum value}}{\text{number of class required}} = \frac{76 - 22}{7} = 8$

Score(x)	Number of students(f)	Cumulative frequency(c.f)
22-29	1	1
30-37	8	9
38-45	8	17
46-53	12	29
54-61	6	35
62-69	3	38
70-77	2	40

Score(x)	Number of student	Class mark	cumulative	Class boundary
22-29	1	25.5	1	21.5-29.5
30-37	8	33	9	29.5-37.5
38-45	8	41.5	17	37.5-45.5
46-53	12	49.5	29	45.5-53.5
54-61	6	57.5	35	51.5-60.5
62-69	3	65.5	38	61.5-69.5
70-77	2	73.5	40	69.5-77.5

Correction factor = $\frac{1}{2}(58 - 57) = \frac{1}{2}(30 - 29) = 0.5$

Relative frequency = $\frac{\text{class frequency}}{\text{total frequency}}$

Example:

The following are marks (out of 100) obtained by 50 students in mathematics.

58 60 63 55 72 86 42 38 96 58
42 51 48 86 38 91 85 75 56 60
86 82 56 96 46 78 82 72 68 70
92 78 68 45 45 83 48 86 62 82
88 38 73 56 39 85 78 46 42 94

Construct the frequency distribution of the given data to 6 classes.

Solution:

First class interval = $\frac{\text{highest value} - \text{lowest value}}{\text{class size}} = \frac{96 - 38}{6} = 9.67 \approx 10$

Class limit	number of frequency	class boundary	class mark	less than cum. frequency	more than cumulative frequency
38-47	11	37.5-48.5	42.5	11	50
48-57	7	48.5-57.5	52.5	18	39
58-67	6	57.5-67.5	62.5	24	32
68-77	8	67.5-77.5	71.5	32	26
78-87	12	78.5-87.5	82.5	44	18
88-97	88.55-97.5	92.5	50	6	6

MEASURES OF LOCATION OR MEASURES OF CENTRAL TENDENCY

The most commonly used measures of central tendency are: Mean (arithmetic mean), Median, Mode, Quartiles, Deciles and percentiles

1. The mean

If $x_1, x_2, x_3, \dots, x_n$ are n values, their mean $\bar{x}$ is given by; $\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n}$

If $x_1, x_2, x_3, \dots, x_n$ is a set of data items with frequencies $f_1, f_2, f_3, \dots, f_n$, respectively, then their mean is given by; $\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \dots + f_nx_n}{f_1 + f_2 + f_3 + \dots + f_n}$

Mean of frequencies distribution

Mean($\bar{x}$) = $\frac{f_1m_1 + f_2m_2 + f_3m_3 + \dots + f_nm_n}{f_1 + f_2 + f_3 + \dots + f_n}$, where m is the class mark ($x_c = m$)

Steps to find the mean from a grouped distribution: from a grouped frequency distribution

- 1. Find the class mark (midpoint) x_c , of each class by; $\frac{\text{lower class limit} + \text{upper class limit}}{2}$
- 2. multiply x_c , by its corresponding frequency add
- 3. divide the sum obtained in step 2 by the sum of the frequencies

Mean($\bar{x}$) = $\frac{f_1x_{c1} + f_2x_{c2} + \dots + f_nx_{cn}}{f_1 + f_2 + f_3 + \dots + f_n}$

Example: The following is the age distribution of 20 students in a class. Find the mean age of these students

Age in years	Class Mid-point (X_c)	Number of students (f)	fx_c
14-18	16	2	32
19-23	21	7	147
24-28	26	6	156
29-33	31	5	155

Solution: Mean($\bar{x}$) = $\frac{\sum fx_c}{\sum f} = \frac{490}{20} = 24.5$

Example: In a class where are there are 50 students if they average age of 30 boys is 20 and the average age of 20 girls is 19 years. Calculate the average age of the class?

Solution:

Using the formula given above, the average age of the class is:

$\bar{x} = \frac{30(20) + 20(19)}{50} = 19.6$

a) Mean or Average of a data as follows: Average of a data = Sum total of the data divided by number of items in data.

Examples: Calculate the mean of the following data:

- 1. 1,3,4,4,5,6,3,7

Solution: Mean, $\bar{x} = \frac{1+3+4+4+5+6+3+7}{8} = \frac{33}{8} = 4.125$

- 2. 650,675,700,725,800,900,1050,1125,1200,575

Solution: Mean, $\bar{x} = \frac{650+675+700+725+800+900+1050+1125+1200+575}{10} = \frac{8400}{10} = 840$

Mean Of Discrete Data

Example:

1. Find the mean of the following data:

X	22	24	25	33	36	37	41
f	5	7	8	4	6	9	11

Solution:

Mean, $\bar{x} = \frac{22(5) + 24(7) + 25(8) + 33(4) + 36(6) + 37(9) + 41(11)}{5 + 7 + 8 + 4 + 6 + 9 + 11} = \frac{1628}{50} = 32.56$

2. Find the mean wage of the workers:

Wages in Birr	220	250	300	350	375
No. of workers	12	15	18	20	5

Solution:

Mean, $\bar{x} = \frac{220(12) + 250(15) + 300(18) + 350(20) + 375(5)}{12 + 15 + 18 + 20 + 5} = \frac{20665}{70} = 295.214 \text{ Birr}$

3. The weights of milk deliveries to a processing plant are shown below:

45 49 50 46 48 42 39 47 42 51 48 45 45 41 46 37 46 47 43 33 56 36 42 39 52 46 43
51 46 54 39 47 46 45 35 44 45 46 40 47

Calculate the mean mass of the milk delivered, using class interval of 5

Solution:

Class	Frequency (f)	Mid-point(x)	f_x
33-37	4	$\frac{33 + 37}{2} = 35$	$4 \times 35 = 140$
38-42	8	$\frac{38 + 42}{2} = 40$	$8 \times 40 = 320$
43-47	19	$\frac{43 + 47}{2} = 45$	$19 \times 45 = 855$
48-52	7	$\frac{48 + 52}{2} = 50$	$7 \times 50 = 350$
53-57	2	$\frac{53 + 57}{2} = 55$	$2 \times 55 = 110$
Total	40		1775

Mean($\bar{x}$) = $\frac{\sum f_x}{\sum f} = \frac{1775}{40} = 44.375$

2. Median

The median is the value in the middle of a distribution e.g. in 1, 2, 3, 4, 5, the median is 3 i.e. it comes at exactly in the middle of the distribution. For the data 1, 2, 2, 3, 4, 5, 6, 7, 7, 8; there are 10 terms and no middle number. In such a case, the median is the average of the two numbers bordering the Centre line

Median of ungrouped data: to find the median from ungrouped data, first arrange the data either in ascending order or in descending order. $Median = \left(\frac{n+1}{2}\right)^{th}$ item for odd number of data and for even

number of data $median = \frac{\left(\frac{n}{2}\right)^{th} item + \left(\frac{n}{2} + 1\right)^{th} item}{2}$.

Examples: Find the median for the following data

1. 1, 2, 2, 3, 4, 5, 6, 7, 7, 8

Solution: $median = \frac{4+5}{2} = 4.5$

2. Given 13 observations 1, 1, 2, 3, 4, 4, 5, 6, 8, 10, 14, 15, 17

Solution: $median = \left(\frac{n+1}{2}\right)^{th} item = \left(\frac{13+1}{2}\right)^{th} item = (7)^{th} item = 5$

3. 1, 1, 2, 2, 3, 4, 4, 5, 6, 8, 10, 14, 15, 17

Solution: $median = \frac{\left(\frac{n}{2}\right)^{th} item + \left(\frac{n}{2} + 1\right)^{th} item}{2}$
 $= \frac{\left(\frac{14}{2}\right)^{th} item + \left(\frac{14}{2} + 1\right)^{th} item}{2}$
 $= \frac{(7)^{th} item + (8)^{th} item}{2} = \frac{4+5}{2} = 4.5$

4. Find the median of the following observation

- a. 22, 24, 27, 26, 30, 29, 36
b. 36, 22, 24, 26, 27, 30, 20

Solution:

- a. Arranging the given data in ascending order we get 22, 24, 26, 27, 29, 30, 36.

$Median = \left(\frac{7+1}{2}\right)^{th} item = (4)^{th} item = 27$

- b. Arranging the data in ascending order, we get 20, 22, 24, 26, 27, 29, 30, 36.

$Median = \frac{(4)^{th} item + (5)^{th} item}{2} = \frac{26+27}{2} = 26.5$

Median of grouped data

Steps to find the median of a grouped frequency distribution

- 1. Prepare a cumulative frequency distribution
- 2. Find the class where the median is located.it is the lowest class for which the cumulative frequency equals or exceeds $\frac{n}{2}$

3. Determine the median by the formula; $Md = B_L + \left(\frac{\frac{n}{2} - cf_b}{f_c} \right) i$

Where,

- B_L =Lower boundary of the class containing the median class
- cf_b =the cumulative frequency in the class preceding the class containing the median
- f_c =the number of observation (frequency) in the class containing the median
- i =the size of the class interval (i.e. width of the median class)

3. Mode

The mode is a value which occurs most frequently in asset of observation. The mode may not exist and even if it does not exist it may not unique.

Examples:

1. Find the mode of the following data: 1,3,4,4,5,6,1,3,3,2,2,3,3,5

Solution:

The mode of a data is the item that appears most times. In this data, 3 occur most times or most frequently i.e. 5 times. Therefore the mode is 3.

2. Find the mode of the following data: 22, 24, 25,22, 27, 22, 25, 30, 25, 31

Solution:

22 and 25 occur three times each. Therefore the modes are 22 and 25. this is called a **bimodal** data.

3. Find the mode of the data:

Observation(x)	0	1	2	3	4
Frequency(f)	3	7	10	16	11

Solution: The most occurring observation is 3 i.e. 3 occur 16 times.

4. Find the modal class of the following data

Weight(x)	50-54	55-59	60-64	65-69	70-74	75-79	80-84
Frequency(f)	3	6	8	5	15	9	13

Solution:

The modal class is 70-74 because it has the highest frequency of occurrence.

Mode for grouped data

Mode of grouped data in given uniform class interval can be obtained by the formula

Mode= $M_o = B_L + \left(\frac{d_1}{d_1 + d_2} \right) i$, where

- B_L =lower class boundary of Modal class
- d_1 =the difference between the frequency of the modal class and the frequency of the preceding class
- d_2 =the difference between the frequency of the modal class and the frequency of next class
- i =size of the class interval.

Example: Find the mode;

Class interval	Frequency
4-6	14
7-9	39
10-12	27
13-15	16

Solution: The modal class is the 2nd class because its frequency is the largest.

$B_L=6.5, \quad d_1=39-14=25,$

$d_2=39-27=12, \quad i=9-6=3$ then

$$\text{Mode}=M_o = B_L + \left(\frac{d_1}{d_1 + d_2} \right) i = 6.5 \left(\frac{25}{25 + 12} \right) (3) = 8.52$$

4. Quartiles, deciles and percentiles

1. **Quartiles:** Data arranged in order of magnitude can be subdivided into four equal portions i.e. 25% each. The first portion is the lower quartile occurring at 25%. The middle or Centre occurring at 50% is called the median while the third quarter occurring at 75% is called the upper quartile. We have three quartiles. These are Q_1, Q_2 and Q_3 are obtained after arranging the data in to an increasing order.

To calculate quartile for un grouped data we have to follow the following steps

Step1:arrange the data in increasing order of magnitude

Step2:if the number of observation is

i) odd, then $Q_k = \frac{(\frac{N+1}{2})^{th}}{N}$ value

ii) even, then $Q_k = \left(\frac{N}{2} \right)^{th}$ value + $\left(\frac{N}{2} + 1 \right) / 2$ value

Examples:

1. Find Q_1 and Q_3 for the following data 15, 38, 9, 18, 25, 48, 32, 16 ,21 ,29, 11, 7, 13, 52, 61

Solution

Arranging in increasing order of magnitude, we get

$7, 9, 11, 13, 15, 16, 18, 21, 25, 29, 32, 38, 48, 52, 61$

$Q_1 = (1(15+1)/4)^{th} = (4^{th})$ value so Q_1 is 4th value which is 13

$Q_2 = (2(15+1)/4)^{th} = 8^{th}$ value so Q_2 is the 8th value which is 38

$Q_3 = (3(15+1)/4)^{th} = 12^{th}$ value so Q_3 is the 12th value which is 38

2. Find Q_1 and Q_2 for the following data 25, 38, 42, 46 ,31 ,29 ,21 ,9 ,5

Solution:

Arranging in increasing order of magnitude we get

$5, 9, 21, 25, 29, 31, 38, 42, 46$

$Q_1 = \frac{(1(9+1)/4)^{th}}{9} = (2.5^{th})$

$Q_2 = 2^{nd} \text{ item} + \frac{1}{2} (3^{rd} \text{ item} - 2^{nd} \text{ item}) = 9 + \frac{1}{2} (21 - 9) = 15$

$Q_2 = \frac{(2(9+1)/4)^{th}}{9} = 5^{th} \text{ value} = 29$



$Q_3 = \frac{(n+1) \cdot \frac{3}{4}}{2} = (7.5^{th}) \text{ item}$

It is half of the way between the 7th and 8th items

$Q_3 = \frac{7^{th} + 8^{th}}{2} = \frac{30 + 40}{2} = 35$

3. Find Q_1, Q_2 and Q_3 for the following data 3 ,6 ,10 ,25 ,19 ,14 ,17 ,22 ,26

Solution:

3 ,6 ,10 ,14 ,17 ,19 ,22 ,25 ,26

$Q_1=8$, $Q_2=17$ = median , $Q_3=23.5$

2. **Deciles** If data arranged in order of magnitude is sub-divided into 10 equal portions (10% each), then each portion constitutes a decile. The deciles are denoted by $D_1, D_2, D_3, \dots, D_9$

Steps to calculate deciles for un grouped

1st) arrange the data in increasing order of magnitude

2nd) if the number of observation is

a) odd, then $D_i = \frac{(n+1) \cdot \frac{i}{10}}{2}$ item

b) even, then $D_i = \frac{\frac{(n+1) \cdot \frac{i}{10}}{2} + \frac{(n+1) \cdot \frac{i}{10} + 1}{2}}{2}$

Example: Find D_3 and D_8 for the following data, 15, 18, 38, 46, 53, 12, 24, 36, 48, 25

Solution:

Arranging in increasing order of magnitude we get 12 ,15 ,18 ,24 ,25 ,36 ,38 ,46 ,48 ,53

$$D_3 = \frac{\left(\frac{3n}{10}\right) \text{value} + \left(\frac{3n}{10} + 1\right) \text{value}}{2} = \frac{\left(\frac{3 \times 10}{10}\right) \text{value} + \left(\frac{3 \times 10}{10} + 1\right) \text{value}}{2}$$
$$= \frac{3^{rd} \text{value} + 4^{th} \text{value}}{2}$$
$$= \frac{18 + 24}{2} = 21$$

$$D_8 = \frac{\left(\frac{8n}{10}\right) \text{value} + \left(\frac{8n}{10} + 1\right) \text{value}}{2} = \frac{\left(\frac{8 \times 10}{10}\right) \text{value} + \left(\frac{8 \times 10}{10} + 1\right) \text{value}}{2}$$
$$= \frac{8^{th} \text{value} + 9^{th} \text{value}}{2}$$
$$= \frac{46 + 48}{2} = 47$$

3. **Percentiles** If data divided arranged in order of magnitude is subdivided into 100 equal portions (1%each), then the portion constitutes a percentile. Percentiles are denoted as $P_1, P_2, P_3, \dots, P_{99}$

Step1: arrange the data in increasing order of magnitude.

Step2: if the number of observation is:

i) Odd, then $P_i = \frac{(n+1) \cdot \frac{i}{100}}{2}$ value

ii) Even, then $P_i = \frac{\frac{(n+1) \cdot \frac{i}{100}}{2} + \frac{(n+1) \cdot \frac{i}{100} + 1}{2}}{2}$ value

Examples: See from students text book page 171

Measures of Dispersion

The degree to which numerical data is spread about an average value is called the variation or dispersion of the data.

The common measures of variation that are going to see are **range, variance and standard deviation**.

Range

Range is the difference between the maximum and the minimum values in a data set.

$$\text{Range} = X_{\text{max}} - X_{\text{min}}$$

Examples:

- 1. Find range of order
- a) 3 ,4 ,6 ,2 ,10 ,18 ,28
- b) 2 ,6 ,4 ,18 ,10 ,2

Solution:

- a) $R=28-2=26$ and
- b) $R=18-2=16$
- 2. 23, 26, 34, 47,63 the range is $63-23=40$ and in 121, 65, 78, 203, 298, 174 the range is $298 - 65= 233$

Range for grouped data

Range for grouped data is defined as the difference between upper class boundary of the highest class

BU (H) and the lower class boundary of the lowest class **BL (L)**.that is **($R=BU(H)-BL(L)$)**

The range of a data is simply the difference between the highest and the lowest score in a data

Variance and standard deviation

The Variance:

The variance of a set of data is defined as the square of the standard deviation / squared deviation from the mean i.e. $\text{variance} = s^2$ We sometimes use s to denote the standard deviation of a sample of a population and σ (Greek letter sigma) to denote the standard deviation of a population . Thus σ^2 can represent the variance of a population and s^2 the variance of sample of a population.

Standard deviation is the most common used measures of dispersion.

The value of standard tells us how closely the value of data set clustered around the mean.

A lower value of the standard deviation for a data set indicates that the values of the set are spread over a relatively small range around the mean.

Large value of the standard deviation for a data set indicates that the values of the data set are spread over a relatively large range around the mean.

Variance for ungrouped data

If $x_1, x_2, x_3, \dots, x_n$ are observed values, then the variance for the t sample data is given by

$$\text{Variance}(s^2) = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \dots + (x_n - \bar{x})^2}{n} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$$

The positive square root of variance is called standard deviation

$$\text{Standard deviation}(sd) = \sqrt{\text{variance}} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}}$$

Steps to calculate variance for ungrouped data

- ☞ calculate the mean of the distribution
- ☞ Find the deviation of each value from the mean and square it
- ☞ Add the square deviations
- ☞ Divide the sum obtained in step 3 by n

Examples

1. Find the variance and standard deviation of the population

- a. 1,2,3,4
b.

X	3	4	5	6	7
f	2	1	3	3	1

Solution:

a. $mean(\bar{x}) = \frac{1+2+6+3}{4} = 3$

x	$x - \bar{x}$	$(x - \bar{x})^2$
1	1-3=-2	4
2	2-3=-1	1
3	3-3=0	0
6	6-3=3	9

$$\sum (x - \bar{x})^2 = 4 + 1 + 0 + 9 = 14$$

$$\text{Variance } (\sigma^2) = \frac{\sum (x - \bar{x})^2}{n} = \frac{14}{4} = 3.5$$

$$\text{Standard deviation } (sd) = \sqrt{\text{variance}} = \sqrt{3.5}$$

b. $mean(\bar{x}) = \frac{2 \times 3 + 1 \times 4 + 3 \times 5 + 3 \times 6 + 1 \times 7}{2 + 1 + 3 + 3 + 1} = \frac{6 + 4 + 15 + 18 + 7}{10} = \frac{50}{10} = 5$

x	F	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
3	2	3-5=-2	4	8
4	1	4-5=-1	1	1
5	3	5-5=0	0	0
6	3	6-5=1	1	3
7	1	7-5=2	4	4

$$\text{Therefore, variance} = \frac{\sum f(x - \bar{x})^2}{\sum f} = \frac{8 + 1 + 0 + 3 + 4}{2 + 1 + 3 + 3 + 1} = \frac{16}{10} = 1.6$$

$$\text{Standard deviation} = \sqrt{1.6}$$

Probability

Probability is a measure of uncertainty involved in the happening of event so that definite value may be assigned to it.

1. Sample Space and Events

- a. *A Probability experiment*: when you toss a coin or pick a card from a deck of playing cards or roll a dice, the act constitutes a probability experiment. In a probability experiment, the chances are well defined with equal chances of occurrence

Example: There are only two possible chances of occurrence in tossing a coin. You get either a head or tail. The head and the tail have equal chances of occurrence.

- b. *An Outcome*: This is defined as the result of a single trial of a probability experiment e.g. When you toss a coin once, you get either head or tail.
- c. *A trial*: This refers to an activity of carrying out an experiment like picking a card from a deck of cards or rolling a die or dices.
- d. *Sample Space*: This refers to all possible outcomes of a probability experiment.

Example: In tossing a coin, the outcomes are either Head (H) or tail (T) i.e. there are only two possible outcomes in tossing a coin. The chances of obtaining a head or a tail are equal.

- e. *A Simple and Compound Events*: In an experimental probability, an event with only one outcome is called a simple event. If an event has two or more outcomes, it is called a compound event.

2. Definition of Probability

Probability can be defined as the mathematics of chance.

$$P(\text{event}) = \frac{\text{Number of favorable outcomes}}{\text{Number of all possible outcomes}} = \frac{\text{Number of expected events}}{\text{Number of all possible outcomes}}$$

There are mainly four approaches to probability;

- The classical or priori approach
- The relative frequency or empirical approach
- The axiomatic approach
- The personalistic approach

The Classical or A Priori Approach Probability is the ratio of the number of favorable cases as compared to the total likely cases. Suppose an event can occur in N ways out of a total of M possible ways. Then the probability of occurrence of the event is denoted by; $P = \Pr(N) = \frac{N}{M}$

Probability refers to the ratio of possible outcomes to all possible outcomes. The probability of non-occurrence of the same event is given by $\{1 - P(\text{occurrence})\}$. The probability of occurrence plus non-occurrence is equal to one.

If $P(O)$ is probability occurrence and $P(O')$ probability of non-occurrence, then $P(O) + P(O') = 1$.

Empirical Probability (Relative Frequency Probability)

Empirical probability arises when frequency distributions are used.

For example:

Observation (X)	0	1	2	3	4
Frequency (f)	3	7	10	16	11

The probability of observation (X) occurring 2 times is given by the formulae

$$P(2) = \frac{\text{frequency of } (2)}{\text{sum of frequencies}} = \frac{f(2)}{\sum f} = \frac{10}{11}$$

Properties of Probability

- ✓ Probability of any event lies between 0 and 1 i.e. It follows that probability cannot be negative nor greater than 1.
- ✓ Probability of an impossible event (an event that cannot occur) is always zero(0)
- ✓ Probability of an event that will certainly occur is 1.
- ✓ The total sum of probabilities of all the possible outcomes in a sample space is always equal to one(1).
- ✓ If the probability of occurrence is $P(O) = A$, then the probability of non-occurrence is $1-A$.

Counting Rules

- i. **Factorials Definition:** Factorial $4! = 4 \times 3 \times 2 \times 1$ and $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
- ii. **Permutation Rules Definition:** $nPr = \frac{n!}{(n-r)!}$

Examples

$$\diamond 5P3 = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$$

$$\diamond 8P5 = \frac{8!}{(8-5)!} = 6720$$

iii. Combinations

Definition: $nCr = \frac{n!}{r!(n-r)!}$

Examples

$$\hat{1} 5C2 = \frac{5!}{(5-2)!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3!2!} = 10$$

$$\hat{1} 10C6 = \frac{10!}{(10-6)!6!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4!6!} = 210$$

Rules of Probability

Addition Rules

Rule 1: When two events A and B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$

Example: When a dice is tossed, find the probability of getting a 3 or 5.

Solution: $P(3) = 1/6$ and $P(5) = 1/6$. Therefore $P(3 \text{ or } 5) = P(3) + P(5) = 1/6 + 1/6 = 2/6 = 1/3$.

Rule 2: If A and B are two events that are NOT mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, where A and B means the number of outcomes that event A and B have in common.

Example: When a card is drawn from a pack of 52 cards, find the probability that the card is a 10 or a heart.

Solution:

$$P(10) = 4/52 \text{ and } P(\text{heart}) = 13/52 \quad P(10 \text{ that is Heart}) = 1/52 \quad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \\ = 4/52 + 13/52 - 1/52 = 16/52.$$

Multiplication Rules

Rule 1: For two independent events A and B, then $P(A \text{ and } B) = P(A) \times P(B)$.

Example: Determine the probability of obtaining a 5 on a die and a tail on a coin in one throw.

Solution: $P(5) = 1/6$ and $P(T) = 1/2$. $P(5 \text{ and } T) = P(5) \times P(T) = 1/6 \times 1/2 = 1/12$.

Rule 2: When two events are dependent, the probability of both events occurring is $P(A \text{ and } B) = P(A) \times P(B|A)$, where $P(B|A)$ is the probability that event B occurs given that event A has already occurred.

Example: Find the probability of obtaining two Aces from a pack of 52 cards without replacement.

Solution: $P(\text{Ace}) = 2/52$ and $P(\text{second Ace if NO replacement}) = 1/51$ Therefore $P(\text{Ace and Ace}) = P(\text{Ace}) \times P(\text{Second Ace}) = 2/52 \times 1/51 = 1/1326$

Conditional probability

Conditional probability of the event A given that the event B has occurred. We write this as $P(A | B)$.

Definition: If A and B are any 2 events with $P(B) > 0$, then $P(A | B) = P(A \cap B) / P(B)$

Note that now we can derive the multiplication law of probability:

$$P(A \cap B) = \frac{P(A \cap B)}{P(B)} \times P(B) = P(A | B) \times P(B)$$

Independence

Two events A and B are said to be independent if $P(A | B) = P(A)$, that is, knowledge of B does not change the probability of A.

So if A and B are independent we have $P(A \cap B) = P(A)P(B)$

The conditional probability of two events A and B is $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$, where $P(A \text{ and } B)$ means the probability of the outcomes that events A and B have in common.

Example: When a die is rolled once, find the probability of getting a 4 given that an even number occurred in an earlier throw.

Solution: $P(4 \text{ and an even number}) = 1/6$ i.e. $P(A \text{ and } B) = 1/6$. $P(\text{even number}) = 3/6 = 1/2$.

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Examples

1. A bag contains 3 orange, 3 yellow and 2 white marbles. Three marbles are selected without replacement. Find the probability of selecting two yellow and a white marble.

Solution: $P(1^{st} Y) = 3/8$, $P(2^{nd} Y) = 2/7$ and

$$P(W) = 2/6 \quad P(Y \text{ and } Y \text{ and } W) = P(Y) \times P(Y) \times P(W) \\ = 3/8 \times 2/7 \times 2/6 = 1/28$$

2. In a class, there are 8 girls and 6 boys. If three students are selected at random for debating, find the probability that all girls.

Solution: $P(G) = 8/14$ and $P(B) = 6/14$. $P(1^{st} G) = 8/14$,

$$P(2^{nd} G) = 7/13 \text{ and } P(3^{rd} G) = 6/12.$$

$$P(\text{three girls}) = 8/14 \times 7/13 \times 6/12 = 2/13$$

3. In how many ways can 3 drama officials be selected from 8 members?

$$\text{Solution: } {}^8C_3 = \frac{8!}{3!5!} = 56 \text{ ways.}$$

4. A box has 12 bulbs, of which 3 are defective. If 4 bulbs are sold, find the probability that exactly one will be defective.

Solution: $P(\text{defective bulb}) = {}^3C_1$ and $P(\text{non-defective bulbs}) = {}^9C_3$

$${}^3C_1 \times {}^9C_3 = \frac{3!}{(3-1)!1!} \times \frac{9!}{(9-3)!3!} = 252$$

$$P(4 \text{ bulbs from } 12) = {}^{12}C_4 = 495.$$

$$P(1 \text{ defective bulb and } 3 \text{ okay bulbs}) = 252/495 = 0.509.$$



Exercises

I. Choose the correct answer for each of the following questions

- 1. When a die is rolled, the probability of getting a number greater than 4 is:
A. $\frac{1}{6}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. 1
- 2. A single card is drawn at random from a standard deck of cards. Find the probability that is a queen.
A. $\frac{1}{13}$ B. $\frac{1}{52}$ C. $\frac{4}{13}$ D. $\frac{1}{2}$
- 3. Out of 100 numbers, 20 were 4's, 40 were 5's, 30 were 6's and the remainder were 7's. Find the arithmetic mean of the numbers.
A. 0.22 B. 0.53 C. 2.20 D. 5.30
- 4. Calculate the mean of the following data.

Height (cm)	Class mark (x)
60-62	61
63-65	64
66-68	67
69-71	70
72-74	73

- A. 57.40 B. 62.00 C. 67.45 D. 72.25
- 5. Find the mode of the following data: 5, 3, 6, 5, 4, 5, 2, 8, 6, 5, 4, 8, 3, 4, 5, 4, 8, 2, 5, and 4.
A. 4 B. 5 C. 6 D. 8
- 6. The range of the values a probability can assume is:
A. From 0 to 1 B. From -1 to +1 C. From 1 to 100 D. From 0 to 21
- 7. Find the median of the following data 8, 7, 11, 5, 6, 4, 3, 12, 10, 8, 2, 5, 1, 6, 4 .
A. 12 B. 5 C. 8 D. 6
- 8. Find the range of the set of numbers 7, 4, 10, 9, 15, 12, 7, 9 .
A. 9 B. 11 C. 7 D. 8.88
- 9. When two coins are tossed, the sample space is
A. H, T and HT B. HH, HT, TH, TT C. HH, HT, TT D. H, T
- 10. If a letter is selected at random from the word "Mississippi", find the probability that it is an "i"
A. $\frac{8}{11}$ B. $\frac{2}{11}$ C. $\frac{11}{3}$ D. $\frac{11}{4}$
- 11. In a Class where there are a re 40 students if the average age of 25 boys is 20 and the average age of 15 girls is 16 years, then what is the average age of the class?
A. 16.5 B. 18.5 C. 17.5 D. 19.5
- 12. A school has three classrooms for grade 11, namely, 11A, 11B and 11C . The number of students in these classrooms is 28, 20 and 22, respectively. All the students took examination and the average score of the students of 11A, 11B and 11C is 60, 70 and 70, respectively. What is the average score in this examination for all grade 11 students?
A. 66 B. 66.67 C. 65 D. 65.67
- 13. A three-digit library identification card is to be printed from the numbers 0, 1, 2, 3, 4, 5 in such a way that the first is non-zero and no number is to be repeated. How many such card can be printed?
A. 100 B. 120 C. 150 D. 180



14. A student needs to be select 3 books from 3 different mathematics, 3 different physics and 1 history book. What is the probability that one of them is mathematics and the other two is either physics or history books?
- A. $\frac{3}{35}$ B. $\frac{9}{35}$ C. $\frac{15}{35}$ D. $\frac{18}{35}$
15. Items produced by a certain company are subjected to two kinds of defects D_1 and D_2 . Out of the total production, if 5% have defect D_1 , 10% have defect D_2 and 2% have both defects, then what is the probability for an item to have defect D_2 , given that it has defect D_1 .
- A. 0.2 B. 0.05 C. 0.1 D. 0.4
16. Among students who took a quiz, 15 students scored 6, 20 students scored 7, 10 students scored 8 and 5 students scored 10. What is the average score of the students?
- A. 7.8 B. 7.5 C. 7.2 D. 7.0
17. A committee consisting of 3 students is to be selected from 10 candidates among which 4 are girls. What is the probability that at least one girl is selected?
- A. $\frac{5}{6}$ B. $\frac{2}{3}$ C. $\frac{1}{3}$ D. $\frac{1}{6}$
18. A pair of dice is rolled and the sum of the numbers on the up turned faces is recorded. What is the probability of obtaining a sum 5?
- A. 1/9 B. 1/3 C. 5/6 D. 1/5
19. An examination paper has 6 questions but only 4 are to be answered.in how many different ways can the answers be arranged?
- A. 420 B.360 C. 15 D. 25
20. Which of the following can be valid assignments of probabilities for out comes of sample space $S=\{w_1, w_2, w_3, w_4\}$ if the out comes are independent.

	W_1	W_2	W_3	W_4
a	0.1	0.001	0.05	0.03
b	0.1	0.4	0.3	0.2
c	0.4	0.3	0.5	0.1
d	0.3	0.4	-0.2	0.5

- A. a B. c C. d D. b
21. A box contains 12 bulbs with 3 defective ones. If two bulbs are drawn from the box together, what is the probability that both are non- defective?
- A. 3/66 B.18/66 C.18/33 D.27/66
22. In how many ways can 7 dresses be displayed in a row on a shelf?
- A. 5000 B.5020 C.5040 D.5060
23. In how many ways can 3 pens be selected from 12 pens?
- A. 220 B. 330 C. 420 D.225
24. From a pack of 52 cards, 3 cards are selected. What is the probability that they will all be diamonds?
- A. 0.012 B. 0.013 C. 0.0013 D.0.0012





II. Work out

25. Find the mean of the following data

Weight (x)	30.5-36.5	36.5-42.5	42.5-48.5	48.5-54.5	54.5-60.5
Frequency(f)	4	10	14	27	45

26. Work out the modes or modal classes of the following data;
20.4, 20.8, 22.1, 23.4, 19.7, 31.2, 23.4, 20.8, 25.5, 23.4

27. Work out the modes or modal classes of the following data;

Weight (x)	30.5-36.5	36.5-42.5	42.5-48.5	48.5-54.5	54.5-60.5
Frequency (f)	4	10	14	27	45

UNIT 6
MATRICES AND DETERMINANTS

Unit Outcomes:

After completing this unit, you should be able to:

- Í know basic concepts about matrices.
- Í know specific ideas, methods and principles concerning matrices.
- Í perform operation on matrices.
- Í apply principles of matrices to solve problems.

MATRICES

Definition:

Let $\mathbb{R}$ be the set of real numbers and m and n be positive integers. A rectangular array of numbers in $\mathbb{R}$ of the form,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is called an m by n ($m \times n$) matrix in $\mathbb{R}$. We denote matrix by A, B, C etc.

$$\begin{bmatrix} 1 & 5 \\ 3 & 0 \end{bmatrix} \text{ is a matrix of order } 2 \times 2.$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 3 & 2 & 5 \end{bmatrix} \text{ is a matrix of order } 2 \times 3.$$

Read types of matrices and their examples on your textbook page 223

É **Types of Matrices:** Let $A = [a_{ij}]_{m \times n}$

(1) **Row matrix:** A $1 \times n$ matrix $[a_{11} \ a_{12} \ a_{13} \ \dots \ a_{1n}]$ is called a row matrix (row vector).

(2) **Column matrix:** A $m \times 1$ matrix $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$ is called a column matrix (column vector).

(3) **Square matrix:** A $n \times n$ matrix is called a square matrix.

(4) **Diagonal matrix:** If in a square matrix $A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0$ whenever $i \neq j$, then A is called a diagonal matrix.

(5) **Zero (null) matrix:** A matrix with all elements are zero is called zero (null) matrix. It is denoted by $[0]_{m \times n}$ or $O_{m \times n}$ or O .

(6) **Identity (unit) matrix:** In a diagonal matrix all elements of principal diagonal are 1 is called Identity (unit) Matrix and is denoted by I or I_n or $I_{n \times n}$.



(7) **Scalar Matrix:** If $k \in \mathbb{R}$, then kI called a scalar matrix.

(8) **Transpose of a Matrix:** If all the rows of matrix $A = [a_{ij}]_{m \times n}$ are converted into corresponding column, the matrix so obtained is called the transpose of A . It is denoted by A^T or A' . $A^T = [a_{ji}]_{n \times m}$.

Properties of Transpose

- (i) $(A^T)^T = A$
 - (ii) $(A + B)^T = A^T + B^T$
 - (iii) $(kA)^T = kA^T, k \in \mathbb{R}$
 - (iv) $(AB)^T = B^T A^T$
- (9) **Symmetric matrix:** For a square matrix A , if $A^T = A$, then A is called a symmetric matrix. Here $a_{ij} = a_{ji}$ for all i and j .
- (10) **Skew-symmetric matrix:** For a square matrix A , if $A^T = -A$, then A is called a skew-symmetric matrix. Here $a_{ij} = -a_{ji}$ for all i and j .
For square matrix A , $A + A^T$ is symmetric and $A - A^T$ is skew - symmetric matrix.

(11) Triangular matrices:

(i) **Upper Triangular matrix:** A square matrix whose element $a_{ij} = 0$ for $i > j$ is called an upper triangular matrix.

Example: $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

(ii) **Lower Triangular matrix:** A square matrix whose element $a_{ij} = 0$ for $i < j$ is called a lower triangular matrix.

Example: $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$

Algebra of Matrices

Equality of matrices: $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are said to be equal i.e. $A=B$ if

- i. $a_{ij} = b_{ij} \forall i \& j$
- ii. Order of A = order of B i.e. $m = p$ and $n = q$

Sum and Difference : If A and B are of same order i.e. $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}, C = [c_{ij}]_{m \times n}$, then
 $A+B=C \Rightarrow [a_{ij} + b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$ and $A-B=C \Rightarrow [a_{ij} - b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$

Properties of addition

If matrices A, B, C and O are of same order, then

- (i) $A+B=B+A \quad A + B = B + A$ (Commutative law)
- (ii) $A+(B+C)=(A+B)+C$ (Associative law)



- (iii) $A + O = O + A$

(Existence of Identity)
- (iv) $(-A) + A = A + (-A) = O$

(Existence of Inverse)

É **Matrix Multiplication:** Let $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times p}$. Then $AB = C$,

where $C_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$ = scalar product of i^{th} row of A and j^{th} Column of B.

Note:

- Í Product AB defined if and only if number of column of A = number of rows of B.
- Í If A is $m \times n$ matrix and B is $n \times p$ matrix then AB is $m \times p$ matrix.

Properties of multiplication of matrices

Let the matrices A, B, C and O have order computable for the operations involved;

- (i) $A(B + C) = AB + AC$
- (ii) $(A + B)C = AC + BC$
- (iii) $A(BC) = (AB)C$
- (iv) $AO = O = OA$
- (v) $AB \neq BA$, generally
- (vi) $AB = O$ need not imply $A = O$ or $B = O$
- (vii) $AB = AC$ need not imply $B = C$

É **The determinant of a square matrix:**

If all entries of a square matrix are kept in their respective places and the determinant of this array is taken, then the determinant so obtained is called the determinant of the given square matrix. If A is a square matrix, then determinant of A is denoted by $|A|$ or $\det A$.

Evaluation of Determinants (Expansion)

Í Second order determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Í Third order determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$

Note:

- (i) $|A^T| = |A|$
- (ii) $|AB| = |A||B|$ and $|ABC| = |A||B||C|$
- (iii) $|I| = 1$



Properties of determinants:

- (i) If a row (column) is a zero vector (i.e. all elements of a row or a column are zero), then $D = 0$.
- (ii) If two rows (columns) are identical, then $D = 0$.
- (iii) If any two rows (columns) are interchanged, D is unchanged = $|A^T| = |A|$.

(iv)
$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(v)
$$\begin{vmatrix} a_1 + d_1 & b_1 + e_1 & c_1 + f_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- (vi) If any rows (columns) is multiplied by $k \in \mathbb{R} (k \neq 0)$ and added to another rows (columns), then D is unchanged.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- (vii) If all rows of a determinant are converted into corresponding column, D is unchanged.

➤ Minor and cofactor

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The minor of the element $a_{ij} (i, j \in 1, 2, 3)$ in $A = M_{ij}$ is the determinant obtained from A on deleting the row and the column in which a_{ij} occurs.

The cofactor of the element $a_{ij} (i, j \in 1, 2, 3)$ in $A = A_{ij} = (-1)^{i+j} M_{ij}$. The value of any third order determinant can be obtained by adding the products of the elements of any of its rows (columns) by their corresponding co-factor.

If we multiply all the elements of any rows (columns) of any third order determinant by the cofactors of the corresponding elements of another row (column) and add the products, then the sum is zero.

➤ Adjoint of Matrix

Adjoint Matrix of $A = \text{adj} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \text{Transpose of the matrix of cofactor } \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

To obtain the adjoint of 2×2 matrix, interchange the elements on the principal diagonal and change the sign of the elements on the secondary diagonal.

➤ Inverse of a Matrix

A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$. If A is a square matrix of order n, if there exists another square matrix of order n such that $AB = I_n$ & $BA = I_n$. If inverse of matrix A exists, then it is unique.

Note:

1. $|A^{-1}| = |A|^{-1}$

2. $(A^T)^{-1} = (A^{-1})^T$

3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^k)^{-1} = (A^{-1})^k, k \in \mathbb{Z}$

5. Inverse of a symmetric matrix is symmetric.

Systems of equations with two or three Variables

Let three equations
$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$
 be given.

Equations can be expressed as a matrix form $AX = B$ where,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$
 If $|A| \neq 0$ (A is non singular), A^{-1} exists, then the solution is $X = A^{-1}B$.

Cramer’s Rule

The value of the variables is:

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D},$$

where $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Examples: Solve the following systems of equations using Cramer’s rule

1.
$$\begin{cases} 4x + 6y = 3 \\ 8x + 3y = 1 \end{cases}$$

Solution:

$$D = \begin{vmatrix} 4 & 6 \\ 8 & 3 \end{vmatrix} = 12 - 48 = -36, D_x = \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} = 9 - 6 = 3 \text{ and } D_y = \begin{vmatrix} 4 & 3 \\ 8 & 1 \end{vmatrix} = 4 - 24 = -20$$

So $x = \frac{D_x}{D} = \frac{3}{-36} = -\frac{1}{12}, y = \frac{D_y}{D} = \frac{-20}{-36} = \frac{5}{9}$

Therefore $x = -\frac{1}{12}, y = \frac{5}{9}$

2.
$$\begin{cases} 2x + y + z = 1 \\ x + 3y + 2z = 2 \\ 3x + y + 3z = 4 \end{cases}$$

Solution:

$$DZ \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{vmatrix} Z 23, D_x Z \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 1 & 3 \end{vmatrix} Z J1, D_y Z \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 4 & 3 \end{vmatrix} Z 5 \text{ and } D_z Z \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 4 \end{vmatrix} Z 30$$

Therefore the solution set is $\left\{ J \frac{1}{23}, \frac{5}{23}, \frac{30}{23} \right\}$

Exercises

Choose the correct answer for each of the following questions

1. If $M = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 7 & 3 \\ 2 & 9 & 4 \end{bmatrix}$, then which one of the following is equal to the determinant of M ?
- A. $J31$ B. $J8$ C. $J1$ D. 14
2. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, then which of the following is equal to $A^T + 2B$?
- A. $\begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$ C. $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ D. $\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$
3. If $\begin{bmatrix} x & 3 \\ 1 & y \end{bmatrix}$ is the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then the values of x and y respectively are:
- A. $1, 3$ B. $2, 2$ C. $4, 1$ D. $\frac{1}{2}, J\frac{3}{2}$
4. What would be the value of k so that the system $\begin{cases} 2x + y = 4 \\ 4x + ky = 1 \end{cases}$ has a unique solution?
- A. $k \div 0$ B. $k \geq 4$ C. $k \div 2$ D. $k \geq 1$
5. Let $A = \begin{bmatrix} 2 & 0 & x \\ 2y & x+y & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & y \\ 0 & 3 \\ 1 & x & 2 \end{bmatrix}$ such that $A + 2B^T = 0$. Then which of the following is the value of y ?
- A. 0 B. $J\frac{13}{2}$ C. $J8$ D. any real number
6. Let A and B be 3×3 matrices such that $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 5 & 0 \\ 0 & 1 & 0.5 \end{bmatrix}$ and $|B| = \frac{1}{10}$. Which one of the following is equal to $|2AB^T|$?
- A. 1 B. 4 C. 100 D. 400

- $$\begin{cases} x + y + 2z = 1 \\ x + 2y + z = 2 \\ Jx + 2y + 4z = J2 \end{cases}$$

7. What is the solution set of the following system of equations ?

A. $\{(0, 1, 0)\}$

B. $(J\tilde{a}, \tilde{a})$

C. $\{(J3k, kH1, k) \mid k \in (J\tilde{a}, \tilde{a})\}$

D. $\{(3k, kJ1, k) \mid k \in (J\tilde{a}, \tilde{a})\}$
8. If $\begin{bmatrix} \alpha & 2 & \beta \\ 2 & 1 & 3 \\ J1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} J5 \\ 5 \\ 0 \end{bmatrix}$ and the determinant of the coefficient matrix is J5, then the value of x is equal to:

A. 3

B. $\alpha H \beta$

C. $J5\alpha$

D. 5

9. The inverse of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ is:

A. $\begin{bmatrix} 1 & 1 & J1 \\ \frac{1}{2} & 1 & J1 \\ J2 & 2 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 2 \\ \frac{1}{J2} & J1 & 2 \\ H & 1 & J1 \end{bmatrix}$

C. $\begin{bmatrix} J1 & J1 & 2 \\ \frac{1}{2} & J1 & J2 \\ H & 1 & J1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 1 & 2 \\ \frac{1}{2} & J1 & 2 \\ H & 1 & J1 \end{bmatrix}$

10. Find the inverse matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & J1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$.

A. $\begin{bmatrix} J1 & 0 & J2 \\ J2 & 1 & J3 \\ H4 & J1 & J8 \end{bmatrix}$

B. $\begin{bmatrix} J11 & 2 & 2 \\ J4 & 0 & 1 \\ 6 & J1 & J1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & J1 & \frac{1}{3} \\ \frac{1}{4} & 1 & \frac{1}{8} \end{bmatrix}$

11. Suppose that A and B are 3×3 matrices, I is the identity matrix of order 3 such that $AB = 2I$. If $\det B = 6$ then what is $\det(A^T)$?

A. $\frac{1}{3}$

B. $\frac{4}{3}$

C. 12

D. 48

12. If $\begin{bmatrix} 0 & 4 & 1 \\ 2 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and B is a 3×3 matrix such that $\det(2B) = 40$, then $\det(AB)$ is equal to:

A. 200

B. J200

C. 50

D. J50

13. Consider the system $\begin{cases} \alpha x + y + z = 1 \\ x + 2y + 4z = 0 \\ 5x + y + z = 0 \end{cases}$. If the determinant of the coefficient matrix is 2, then what is the solution of the system of equations?

A. $3\alpha, \frac{19\alpha}{2}, \frac{J11\alpha}{2}$

B. $3, \frac{19}{2}, \frac{J11}{2}$

C. $\frac{3}{\alpha}, \frac{J19}{2}, \frac{11}{2}$

D. $\frac{3}{2}, \frac{19}{2}, \frac{J9}{2}$

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26

21. Given two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$, then which one of the following is **NOT** true about the given matrices?

- A. $(A^T)^T = A$

B. $(AB)^T = A^T B^T$, provided that AB is defined
- C. $(rB)^T = rB^T$, if r is any scalar

D. $(A \cup B)^T = A^T \cup B^T$

22. Which of the following square matrices is skew-symmetric?

- A. $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 7 \\ 4 & 7 & 0 \end{bmatrix}$
- C. $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 7 \\ 9 & 6 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 10 & 11 & 21 \\ 21 & 10 & 11 \\ 11 & 21 & 10 \end{bmatrix}$

23. Let $\begin{bmatrix} 0 & 3 & 4 \\ k & 0 & 8 \\ 4 & 8 & 0 \end{bmatrix}$ be skew-symmetric matrix. Then what is the value of k ?

- A. 3

B. $\frac{3}{2}$
- C. 3

D. 4

24. Which of the following statements is **NOT** true?

- A. If B is an inverse of an invertible matrix A , then $(B^T)^{-1} = A^T$.
- B. If A is any square matrix, then its inverse A^{-1} is given by $A^{-1} = \frac{1}{|A|} \text{adj} A$.
- C. Not every square matrix is non-singular.
- D. If a matrix is non-singular, then its inverse is necessarily being unique.

25. If $A = \begin{bmatrix} 0 & 4 & 1 \\ 2 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and B is a 3×3 matrix such that $\det(2B) = 40$, then $\det(AB)$ is equal to:

- A. 50

B. 150
- C. 1260

D. 1200

26. Let $A = \begin{bmatrix} 0 & \alpha & \beta \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$, and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. If $\det(A) = 3$, then what is the solution set of the system $AX = b$?

- A. $\{(6, 2, 8)^T\}$

B. $\{(3, 1, 4)^T\}$
- C. $\left\{0, \frac{1}{a}, \frac{5}{b}\right\}$

D. $\emptyset$

27. Let matrix $A = \begin{bmatrix} 4 & 4k & k \\ 0 & k & 4k \\ 0 & 0 & 4 \end{bmatrix}$. If $\det(A^2) = 16$, then find the value of $|k|$.

- A. $\frac{1}{4}$

B. 4
- C. $\frac{1}{16}$

D. 1

28. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, then which of the following is equal to $A + A^T$?

A. $\begin{bmatrix} 4 & 6 & 4 \\ 4 & 10 & 8 \\ 10 & 4 & 12 \end{bmatrix}$

B. $\begin{bmatrix} 6 & 4 & 4 \\ 4 & 8 & 10 \\ 4 & 10 & 12 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 8 & 10 \\ 6 & 4 & 2 \\ 6 & 10 & 12 \end{bmatrix}$

D. $\begin{bmatrix} 6 & 4 & 2 \\ 4 & 8 & 10 \\ 6 & 10 & 12 \end{bmatrix}$

29. If matrix $B = \begin{bmatrix} a & 3 \\ 3 & a \end{bmatrix}$ and $|B^3| = 343$, then which of the following is the value of a ?

A. -4

B. -2

C. -3

D. -1

30. If M is a 3×3 matrix, where $M^T M = I$ and $\det(M) = 1$, then find $\det(M - I)$.

A. -1

B. -3

C. 4

D. 0

31. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, where $a, b, c \in \mathbb{R}^H$, $abc \neq 1$ and $|A| \neq 0$, $A^T A = I$, then $a^3 + b^3 + c^3$ is equal to:

A. 4

B. -8

C. 12

D. 28

32. Let $A = \begin{bmatrix} 5 & 5 & 2 \\ 1 & 0 & 6 \\ 3 & 3 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 1 & 6 \\ 9 & 4 & 4 \\ 2 & 10 & 9 \end{bmatrix}$, $C = (3 \ 1 \ 2)$ and $D = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$. Then which is the following is the value of $AD + (CB)^T$?

A. $(55 \ 22 \ 67)^T$

B. $(89 \ 5 \ 71)^T$

C. $(34 \ 27 \ 4)^T$

D. $(89 \ 5 \ 67)^T$

UNIT 7
THE SET OF COMPLEX NUMBERS

Unit Outcomes:

After completing this unit, you should be able to:

- c know basic concepts about complex numbers.
- c know general principles of performing operations on complex numbers.
- c understand facts and procedures in simplifying complex numbers.
- c show the geometric representation of complex numbers on the Argand plane.

THE CONCEPT OF COMPLEX NUMBERS

Complex number: A number of the form $x + iy$ where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ is called a complex number.

A complex number can also be defined as an ordered pair $z = (x, y)$ of real number. So $z = (x, y) = x + iy$.

The set of complex numbers denoted by $\mathbb{C}$ is given by

$$\mathbb{C} = \{z \mid z = x + iy, \text{ where } x \text{ and } y \text{ are real numbers and } i = \sqrt{-1}\}.$$

Example: $i^2 = -1, i^3 = i^2 \cdot i = -1 \cdot i = -i, i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$

Note:

- a. $i^{2n} = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$
- b. $i^{2n+1} = \begin{cases} i, & \text{if } n \text{ is even} \\ -i, & \text{if } n \text{ is odd} \end{cases}$
- c. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
- d. $z = x + iy$ where x is called real part of $z = \text{Re}(z)$ and y is called imaginary part of $z = \text{Im}(z)$.
 $\hat{G} \ z = x + iy = \text{Re}(z) + i \text{Im}(z)$.
- e. Two complex numbers are equal, $z_1 = z_2$ i.e. $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2, y_1 = y_2$.

Operations on Complex Numbers

Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$, then

1. Addition:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

2. Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$z_2 - z_1 = (x_2 - x_1) + i(y_2 - y_1)$$

$$\hat{G} \ z_1 - z_2 = z_2 - z_1$$

3. Multiplication:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

4. Division:

$$\frac{z_1}{z_2} Z \frac{x_1 x_2}{x_2^2} H \frac{y_1 y_2}{y_2^2} H \frac{i(x_2 y_1 J x_1 y_2)}{x_2^2 H y_2^2}, x_2^2 H y_2^2 \div 0$$

Examples:

1. Perform the following operations and write the result in the form of $a + bi$.

- a. $(2 + 3i)(4 + 5i)$

b. $\sqrt{25} + \sqrt{36}$

c. $(4 + 2i)J(2 + 3i)$

d. $(2 + \sqrt{25})J(3 + \sqrt{256})H(1 + \sqrt{9})$
- e. $\frac{i^{20} H i^{11} H i^2}{1 H i}$

f. $\frac{4 H i}{2 J 3i}$

g. $\frac{\hat{H} 2 J 3i \hat{J} \hat{H} 1 H i \hat{J}}{\hat{H} 1 J i \hat{J} \hat{H} 2 H 3i \hat{J}}$

Solutions:

- a. $(2 + 3i)(4 + 5i) Z (2)(4)H(2)(5i)H(3i)(4)H(3i)(5i)$
 $Z 8 + 22i + 15i^2 \text{ but } i^2 Z -1$
 $Z 8 + 22i - 15$
 $Z 7 + 22i$

b. $\sqrt{25} + \sqrt{36} Z \sqrt{(1)(25)} + \sqrt{(1)(36)}$
 $Z \sqrt{1} + \sqrt{25} + \sqrt{1} + \sqrt{36}$
 $Z 1 + 5 + 1 + 6$
 $Z 13$

c. $(4 + 2i)J(2 + 3i) Z (4 + 2i)H(J2i + J3i)$
 $Z 2H(J5i)$
 $Z 2J5i$

d. $(2 + \sqrt{25})J(3 + \sqrt{256})H(1 + \sqrt{9}) Z (2 + 5i)J(3 + 16i)H(1 + 3i)$
 $Z 2 + 5i + J3 + 16i + H1 + 3i$
 $Z 2 + 5J3 + 1H5i + 16i + H3i$
 $Z 0 + 24i + 24i \text{ which is called purely imaginary.}$
- e. $\frac{i^{20} H i^{11} H i^2}{1 H i} Z \frac{i^{20+11} H i^{(20+11)} H i^2}{1 H i}$
 $Z \frac{1 H J i H J 1}{1 H i} Z \frac{J i}{1 H i}$
 $Z \frac{J i (1 J i)}{(1 H i)(1 J i)}$
 $Z \frac{J i H i^2}{1 J i^2}$
 $Z J \frac{1}{2} J \frac{i}{2} \text{ Why?}$

f. $\frac{4Hi}{2J3i} \times \frac{(4Hi)(2H3i)}{(2J3i)(2H3i)}$
 $\times \frac{(4)(2)H(4)(3i)H(i)(2)H(i)(3i)}{(2)^2 J(3i)^2}$
 $\times \frac{8H12iH2iH3i^2}{4H9i^2}$
 $\times J \frac{5H14i}{5}$ **How?**

g. $\frac{2J3i}{1Ji} \times \frac{1Hi}{2H3i} \times \frac{5Ji}{5Hi}$ **How?**
 $\times \frac{(5Ji)(5Ji)}{(5Hi)(5Ji)}$
 $\times \frac{25J10iHi^2}{25Ji^2}$
 $\times \frac{24J10i}{26}$
 $\times \frac{12J5i}{13}$

2. Solve the following equations
- a. $(xHyi)H2(3xJy)H4iZ0$
 - b. $x^2H4xH5Z0$
 - c. $(xH3)iH4(yH4i)H16Z0$
 - d. $\frac{(1Hi)xJ2i}{3Hi}H\frac{(2J3i)yHi}{3Ji}Zi$

Solutions:

a. $(xHyi)H2(3xJy)H4iZ0$
 $\hat{G} xHyiH6xJ2yH4iZ0$
 $\hat{G} (7xJ2y)H(yH4)iZ0H0i$
 $\Leftarrow \frac{7xJ2yZ0}{(yH4)iZ0i}$
 $= \frac{7xJ2yZ0}{yH4Z0}$
 $\hat{G} yZJ4,xZJ\frac{8}{7}$

b. $x^2 + 4x + 5 = 0$

$$\hat{G} \ x \ Z \ \frac{J4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$$
$$\hat{G} \ x \ Z \ \frac{J4 \pm \sqrt{16 - 20}}{2}$$
$$\hat{G} \ x \ Z \ \frac{J4 \pm \sqrt{J4}}{2} \text{ but } \sqrt{J4} \text{ is not defined in the set of real numbers, } \sqrt{J4} = 2i \text{ in the set of } \mathbb{C}.$$
$$\hat{G} \ x \ Z \ \frac{J4 \pm 2i}{2} = J2 \pm 1i$$

Therefore the solution set $Z \{J2 + 1i, J2 - 1i\}$.

c. $(x + 3) + i(4y + 16) = 0$

$$\hat{G} \ (x + 19) + i(4y + 16) = 0 + 0i$$
$$\Leftrightarrow \begin{cases} 4y + 16 = 0 \\ x + 19 = 0 \end{cases}$$
$$\hat{G} \ x = J19, y = J4$$

d. $\frac{(1 + i)x + 2i}{3 + i} + \frac{(2 + 3i)y + i}{3 + i} = i$

$$\hat{G} \ \frac{x + ix + 2i}{3 + i} + \frac{2y + 3yi + i}{3 + i} = i$$
$$\hat{G} \ \frac{(3 + i)(x + ix + 2i) + (3 + i)(2y + 3yi + i)}{(3 + i)(3 + i)} = i$$
$$\hat{G} \ \frac{(3 + i)(x + ix + 2i) + (3 + i)(2y + 3yi + i)}{9 - 1} = i$$
$$\hat{G} \ (4x + 9y + 3) + i(2x + 7y + 3) = 0 + 10i$$
$$= \begin{cases} 4x + 9y + 3 = 0 \\ 2x + 7y + 3 = 10 \end{cases}$$
$$\hat{G} \ x = 3, y = J1$$
$$y(x, y) = Z(3, J1)$$

COMPLEX CONJUGATE AND MODULUS

Definition: The complex conjugate (or conjugate) of a complex number $z = x + iy$, denoted by $\bar{z}$, is given by $\bar{z} = x - iy$.

Examples:

- a. If $z = 5 + 3i$, then $\bar{z} = 5 - 3i$
- b. If $z = 1 + \frac{i}{2}$, then $\bar{z} = 1 - \frac{i}{2}$
- c. If $z = 7$, then $\bar{z} = 7$
- d. If $z = J14i$, then $\bar{z} = J14i$

Note: Properties of Complex Conjugates

- 1. If $z = x + iy$ then;
 - a. $\overline{\overline{z}} = z = x + iy$
 - b. $z + \overline{z} = 2x = 2\operatorname{Re}(z)$
 - c. $z - \overline{z} = 2iy = 2i\operatorname{Im}(z)$
 - d. $z\overline{z} = x^2 + y^2$
- 2. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then;
 - a. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
 - b. $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$
 - c. $\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}, \text{ if } z_2 \neq 0$

Definition: The absolute value (or modulus) of a complex number $z = x + iy$, denoted by $|z|$, is defined to be $|z| = \sqrt{x^2 + y^2}$.

Examples:

- 1. simplify the following and write in the form of $z = x + iy$
 - a. $\frac{2 + 3i}{3 + 4i}$
 - b. $\frac{4 + 3i}{3 + 4i} \cdot \frac{1 + 2i}{1 + 2i}$
 - c. $\frac{(2 + 3i)(4 + i)}{(3 + i)(2 + 5i)}$

Solutions:

- a. $\frac{2 + 3i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{2(3 - 4i) + 3i(3 - 4i)}{9 - 16i^2} = \frac{6 - 8i + 9i - 12i^2}{9 + 16} = \frac{18 + i}{25} = \frac{18}{25} + \frac{1}{25}i$
 - b. $\frac{4 + 3i}{3 + 4i} \cdot \frac{1 + 2i}{1 + 2i} = \frac{(4 + 3i)(1 + 2i)}{1 + 4i^2} = \frac{4 + 8i + 3i + 6i^2}{1 - 4} = \frac{-2 + 11i}{-3} = \frac{2}{3} - \frac{11}{3}i$
 - c. $\frac{(2 + 3i)(4 + i)}{(3 + i)(2 + 5i)} = \frac{(2 + 3i)(4 + i)}{(3 + i)(2 + 5i)} \cdot \frac{(2 - 5i)(3 - i)}{(2 - 5i)(3 - i)} = \frac{(2 + 3i)(4 + i)(2 - 5i)(3 - i)}{(3 + i)(2 + 5i)(2 - 5i)(3 - i)} = \frac{11 + 10i}{17 + i} \cdot \frac{159 - 197i}{290} = \frac{(11 + 10i)(159 - 197i)}{(17 + i)(290)}$ (Check!)
- 2. Find the modulus of each of the following complex numbers
 - a. $1 + 2i$
 - b. $\frac{1}{i^3} \cdot \frac{3 + 5i}{12 + 5i}$

Solutions:

- a. The modulus of $1 + 2i$, $|1 + 2i| = \sqrt{1^2 + 2^2} = \sqrt{5}$
- b. $\frac{1}{i^3} \cdot \frac{3 + 5i}{12 + 5i} = \frac{1}{-i} \cdot \frac{3 + 5i}{12 + 5i} = \frac{11 + 75i}{169} = \frac{11}{169} + \frac{75}{169}i$
 $\frac{11 + 75i}{169} \cdot \frac{158 - 75i}{158 - 75i} = \frac{158 + 75i}{169}$
So the modulus of $\frac{1}{i^3} \cdot \frac{3 + 5i}{12 + 5i}$, $\left| \frac{1}{i^3} \cdot \frac{3 + 5i}{12 + 5i} \right| = \left| \frac{158 + 75i}{169} \right| = \sqrt{\left(\frac{158}{169}\right)^2 + \left(\frac{75}{169}\right)^2}$

Theorems: For any complex numbers z_1 and z_2 the following properties hold true

- a. $z_1 \overline{z_1} = |z_1|^2$

b. $|z_1| = |\overline{z_1}|$

c. $|\operatorname{Re}(z_1)| \leq |z_1|$

d. $|\operatorname{Im}(z_1)| \leq |z_1|$
- e. $|z_1 z_2| = |z_1| |z_2|$

f. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

g. Triangular inequality $|z_1 + z_2| \leq |z_1| + |z_2|$

h. $|z_1 - z_2| \leq |z_1| + |z_2|$

Proof: Exercise

SIMPLIFICATION OF COMPLEX NUMBERS

Theorem: If w is a non-zero complex number, then the equation $z^2 = w$ has a solution $z \in \mathbb{C}$.

Proof: Read your textbook pages 280-281.

Examples: Solve the following equations

- a. $z^2 = 1 + i$
- b. $3z^2 + 2z + 1 = 0$

Solutions:

- a. put $z = x + iy$

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi = 1 + i$$

$$\Rightarrow x^2 - y^2 = 1 \text{ and } 2xy = 1 \Rightarrow y = \frac{1}{2x}$$

So $x^2 + \frac{1}{4x^2} = 1 \Rightarrow 4x^4 - 4x^2 + 1 = 0$. Let $t = x^2$, then solve for x .

$$\Rightarrow x = \pm \sqrt{\frac{1 + i\sqrt{2}}{2}}, y = \frac{1}{2x} = \pm \frac{1}{\sqrt{2} \sqrt{1 + i\sqrt{2}}}$$

$$y = \pm \sqrt{\frac{1 + i\sqrt{2}}{2}} \cdot \frac{i}{\sqrt{2} \sqrt{1 + i\sqrt{2}}} = \pm \frac{i}{\sqrt{2}}$$

- b. put $z = x + iy$

$$3z^2 + 2z + 1 = 3(x + iy)^2 + 2(x + iy) + 1 = 0$$

$$\Rightarrow 3x^2 - 3y^2 + 2x + 1 = 0 \text{ and } 4xy = 0 \Rightarrow y = 0$$

$$\Rightarrow 3x^2 + 2x + 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm i\sqrt{2}}{3}$$

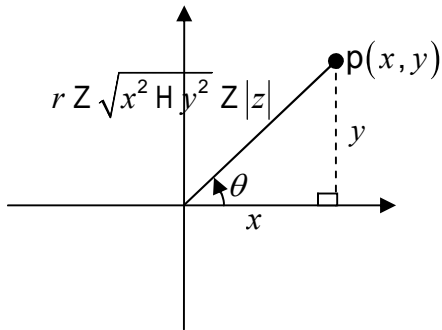
$$y = \frac{-1 \pm i\sqrt{2}}{3}$$



ARGAND DIAGRAM AND POLAR REPRESENTATION OF COMPLEX NUMBERS

Polar Representation of Complex Numbers

Let $z = x + iy \in (x, y)$



$x = r \cos \theta, y = r \sin \theta$
 $z = x + iy$
 $= r \cos \theta + i r \sin \theta$ -Polar representation of z
 $= r(\cos \theta + i \sin \theta)$

When a complex number is written in the form $z = r(\cos \theta + i \sin \theta)$, θ is called an argument of z and is denoted by $\arg z$. The particular argument of z lying in the range $-\pi < \theta \leq \pi$ is called the principal argument of z and is denoted by $\text{Arg} z$.

Examples:

1. Express the following complex numbers in polar form

a. $z = 2 + 2i\sqrt{3}$

b. $z = 2i$

c. $z = -2\sqrt{6} - 2i\sqrt{2}$

Solutions:

a. $r = \sqrt{x^2 + y^2} = \sqrt{2^2 + (2\sqrt{3})^2} = 4$ and $\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1} \left(\frac{2\sqrt{3}}{2} \right) = \frac{\pi}{3} = \text{Arg} z$

Therefore $z = 2 + 2i\sqrt{3} = 4 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

b. $r = \sqrt{0^2 + (2)^2} = 2$ and $\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1} \left(\frac{2}{0} \right)$ undefined, this means $\theta = \frac{\pi}{2}$

$z = 2i = 2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$

c. $r = \sqrt{(-2\sqrt{6})^2 + (-2\sqrt{2})^2} = 4\sqrt{2}$ and $\theta = \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) = \frac{5\pi}{6}$ because z is in the 3rd quadrant.

$z = -2\sqrt{6} - 2i\sqrt{2} = 4\sqrt{2} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$

Read the note on your textbook page 288

2. Find the principal argument of the following complex number $z = \frac{\sqrt{3} + i}{1 + i}$.

Solutions:

$$r = |z| = \frac{|\sqrt{3} + i|}{|1 + i|} = \frac{(\sqrt{3})^2 + 1^2}{1^2 + 1^2} = \frac{4}{2} = 2, \theta = \arg z = \frac{17\pi}{12}$$

$$\text{So Arg } z = \frac{17\pi}{12} + 2\pi k, k \in \mathbb{Z}$$

$$z = 2 \left[\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right], k \in \mathbb{Z}$$

Exercises

Choose the correct answer for each of the following questions

- If $z_1 = 2 + i$ and $z_2 = 1 + i$, then find $\left| \frac{z_1 + z_2}{z_1 - z_2} \right|$.
A. $\frac{2\sqrt{5}}{5}$ B. $\frac{4}{5}$ C. $\frac{\sqrt{5}}{2}$ D. $\sqrt{\frac{3}{5}}$
- Given that $z = (1 + 6i)$ and $w = 3 + i$. What is the value of u so that $\frac{1}{10u} + \frac{3}{z} = \frac{1}{2w}$?
A. $u = \frac{13 + 11i}{10}$ B. $u = 9 + 7i$ C. $u = 9 + 7i$ D. $u = 9 + 7i$
- If $\begin{vmatrix} 6i & 3i & 1 \\ 4 & 3i & 1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then which of the following is the value of x and y respectively?
A. 3, 1 B. 1, 3 C. 0, 1 D. 0, 0
- If $z = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$, then what is the value of z^2 ?
A. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ B. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ C. $\frac{1}{2} + \frac{1}{2}i$ D. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
- If $z = \frac{1 + 7i}{(2 + i)^2}$, then the polar form of z is equal to:
A. $\sqrt{2} \left[\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right]$ C. $\sqrt{2} \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]$
B. $\sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]$ D. $\sqrt{2} \left[\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right]$
- The complex numbers $z_1 = 2 + 2i$, $z_2 = a + bi$ where $a \in \mathbb{R}$, $b \in \mathbb{R}$ are given. If $|z_1 z_2| = 16$, then which of the following is the modulus of z_2 ?
A. 32 B. $4\sqrt{2}$ C. $8\sqrt{2}$ D. $\sqrt{2}$

7. Which of the following is the solution set of a quadratic function $f(x) = ax^2 + bx + c$ that contains the points (1, 9), (4, 6) and (6, 14)?
- A. $\{3 - 2i\sqrt{5}, 3 + 2i\sqrt{5}\}$ C. $\{3 - i\sqrt{5}, 3 + i\sqrt{5}\}$
B. $\{1, 6, 14\}$ D. $\{3 - 2\sqrt{5}, 3 + 2\sqrt{5}\}$
8. When $z = \sqrt{2} \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]$ is converted to Cartesian we get:
- A. (2, 2) B. (1, 1) C. ($\sqrt{2}$, 1) D. ($\sqrt{2}$, $\sqrt{2}$)
9. The values of x and y that solves the equation $(2x + yi)(3 + i) = 20i$ are:
- A. $x = 0, y = 20$ C. $x = 1, y = 6$
B. $x = 3, y = 7$ D. $x = 5, y = 3$
10. In the set of complex numbers, the solution set of $x^2 + 2x + 5 = 0$ is:
- A. $\emptyset$ C. $\{1 + 2i, 1 - 2i\}$
B. $\{2 + i, 2 - i\}$ D. $\{2 + 4i, 2 - 4i\}$
11. Let z be complex number with modulus 2 and argument $\frac{2\pi}{3}$, then $z =$ _____.
- A. $\frac{1 + i\sqrt{3}}{2}$ B. $1 + i\sqrt{3}$ C. $\frac{1 + i\sqrt{3}}{2}$ D. $1 + i\sqrt{3}$
12. Which of the following is the modulus of $\frac{4 + 3i}{3 + 4i} + \overline{1 + 2i}$?
- A. $\sqrt{2}$ B. 2 C. $\sqrt{3}$ D. 1
13. If $z = \frac{1 + 3i}{2 + i}$, then what is the modulus (or the absolute values) of z ?
- A. 2 B. $1 + i$ C. $1 + i$ D. $\sqrt{2}$
14. When $z = 2\sqrt{6} + 2\sqrt{2}i$ expressed in the polar form, we get:
- A. $z = 2\sqrt{2} \left[\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right]$ C. $z = 4\sqrt{2} \left[\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right]$
B. $z = \sqrt{2} \left[\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right]$ D. $z = 4\sqrt{2} \left[\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right]$
15. Evaluate $i^{19} + \frac{1}{i} + \frac{1}{i^2} + \frac{1}{i^3} + \frac{1}{i^4}$.
- A. 4 B. 5 C. 4 D. 5
16. $i + i^2 + i^3 + i^4 + \dots + i^{1000} =$ _____
- A. 0 B. 2 C. 1 D. $\sqrt{2}$
17. If $z = x + iy$ and $|3z| = |z + 4|$, then find $x^2 + y^2$.
- A. 2 B. 1 C. 2 D. 1
18. Let z complex number such that $|z + 3| \leq 8$. Then the value of $|z - 2|$ lies in _____
- A. $[2, 13]$ B. $[2, 13]$ C. $[0, 13]$ D. $[13, 2]$

19. If $z = r(\cos \theta + i \sin \theta)$, then find $\arg(z^2 + z)$.

- A. $\frac{3\theta}{2}$ B. θ C. $\frac{\theta}{2}$ D. 3θ

20. The simplified form of $\frac{\hat{H}}{H} \frac{1 \cos \frac{\pi}{12} H i \sin \frac{\pi}{12} J}{1 \cos \frac{\pi}{12} J i \sin \frac{\pi}{12} K} \hat{J}^{36}$ Z _____

- A. J1 B. 0 C. 1 D. $\frac{1}{2}$

21. If $\frac{1}{2} \frac{H^2 i}{H i} Z_r(\cos \theta H i \sin \theta)$, then;

- A. $r \geq 1, \theta \geq \tan^{-1} \frac{4}{3}$
- B. $r \geq \sqrt{5}, \theta \geq \tan^{-1} \frac{5}{4}$
- C. $r \geq 1, \theta \geq \tan^{-1} \frac{3}{4}$
- D. $r \geq 2, \theta \geq \tan^{-1} \frac{3}{4}$

22. Let the complex number $z = \frac{3+4i}{2-3i}$. Find the complex number w which satisfies $zw = 1$.

- A. $\frac{6H17i}{25}$ B. $\frac{J6J17i}{25}$ C. $\frac{6H17i}{5}$ D. $\frac{J6J17i}{5}$

23. If $z = \frac{1 + 7i}{(2 + i)^2}$, then which of the following is the polar form of z ?

- A. $\sqrt{2} \hat{i} \cos \frac{5\pi}{4} + \hat{j} \sin \frac{5\pi}{4}$
- B. $\sqrt{2} \hat{i} \cos \frac{7\pi}{4} + \hat{j} \sin \frac{7\pi}{4}$
- C. $\sqrt{2} \hat{i} \cos \frac{3\pi}{4} + \hat{j} \sin \frac{3\pi}{4}$
- D. $\sqrt{2} \hat{i} \cos \frac{\pi}{4} + \hat{j} \sin \frac{\pi}{4}$

UNIT 8

VECTORS AND TRANSFORMATION OF THE PLANE

Unit Outcomes:

After completing this unit, you should be able to:

- know basic concepts and procedures about vectors and operation on vectors.
- know specific facts about vectors.
- apply principles and theorems about vectors in solving problems involving vectors.
- apply methods and procedures in transforming plane figures.

REVISION ON VECTORS AND SCALARS

Definitions:

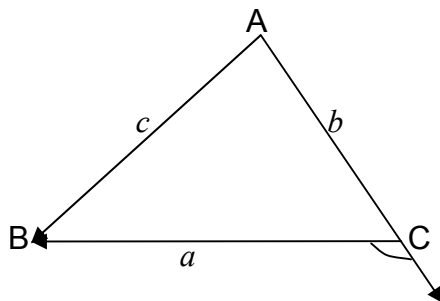
1. A quantity which can be completely described by its magnitude expressed in some particular unit is called a scalar quantity.
Examples of scalar quantities are mass, time, temperature, etc.
2. A quantity which can be completely described by stating both its magnitude expressed in some particular unit and its direction is called a vector quantity.
Examples of vector quantities are velocity, acceleration, etc.
3. if $\vec{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is (x, y) , then the coordinate form of $\vec{v}$ is given by $\vec{v} = (x, y)$ or $\begin{pmatrix} x \\ y \end{pmatrix}$ where x and y are components (or coordinates) of $\vec{v}$.
4. Two vectors $\vec{a}$ and $\vec{b}$ are equal, $\vec{a} = \vec{b}$ and $\vec{b} = \vec{a}$ if they have the same direction and equal magnitude i.e. $\vec{a} = \vec{b}$ implies $|\vec{a}| = |\vec{b}|$ and the direction of $\vec{a}$ and $\vec{b}$ are the same.
5. The cosine angle between two vectors $\vec{a}$ and $\vec{b}$ is given by $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$.

APPLICATION OF VECTOR

Examples

1. In $\triangle ABC$, if a, b and c are lengths of the sides opposite to $\hat{A}, \hat{B}$ and $\hat{C}$ respectively, then show that $c^2 = a^2 + b^2 - 2ab \cos \hat{C}$.

Solution: Consider the following figure

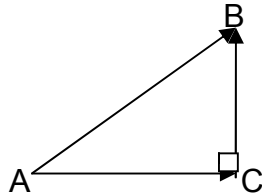


$$\begin{aligned}
 &\text{As shown in the figure } \vec{AC} + \vec{CB} = \vec{AB} \\
 &|\vec{AB}|^2 = |\vec{AC} + \vec{CB}|^2 = (\vec{AC} + \vec{CB}) \cdot (\vec{AC} + \vec{CB}) \\
 &= \vec{AC} \cdot \vec{AC} + \vec{CB} \cdot \vec{CB} + 2\vec{AC} \cdot \vec{CB} \\
 &= |\vec{AC}|^2 + |\vec{CB}|^2 + 2|\vec{AC}||\vec{CB}|\cos(\pi - \hat{C}) \\
 &= b^2 + a^2 - 2ab \cos \hat{C} \\
 &\therefore |\vec{AB}|^2 = c^2 = a^2 + b^2 - 2ab \cos \hat{C}
 \end{aligned}$$

2. Show that in a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Solution:

Let $\triangle ABC$ be a given right-angled triangle with $\hat{C}=90^\circ$. Now consider vectors $\vec{AC}$, $\vec{CB}$ and $\vec{AB}$ as shown below.



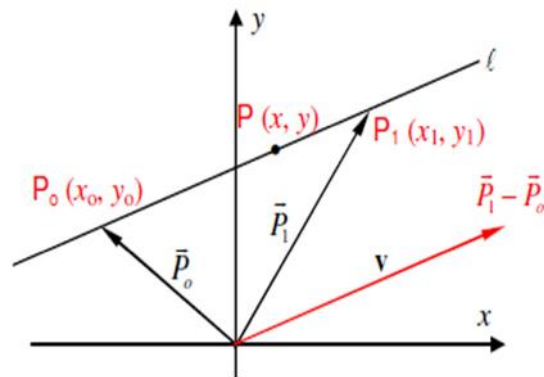
Since $\hat{C}=90^\circ$, $\vec{AC} \cdot \vec{CB} = 0$. By vector addition we have $\vec{AC} + \vec{CB} = \vec{AB}$.

$$\begin{aligned} \text{Now } (\vec{AB})^2 &= \vec{AB} \cdot \vec{AB} = (\vec{AC} + \vec{CB}) \cdot (\vec{AC} + \vec{CB}) \\ &= \vec{AC} \cdot \vec{AC} + 2\vec{AC} \cdot \vec{CB} + \vec{CB} \cdot \vec{CB} \\ &= \vec{AC}^2 + \vec{CB}^2 \text{ since } \vec{AC} \cdot \vec{CB} = 0 \end{aligned}$$

$$(\vec{AB})^2 = \vec{AC}^2 + \vec{CB}^2$$

Vectors and Lines

Consider the following figure



c $\vec{P_1} - \vec{P_0} = (x_1 - x_0, y_1 - y_0)$ is direction vector $\vec{v}$.

c The line ℓ through P_0 and P_1 is parallel to the vector $\vec{P_1} - \vec{P_0} = (x_1 - x_0, y_1 - y_0)$.

c The vector equation of the line determined by P_0 and $\vec{v}$ is given by $\vec{r} = \vec{P_0} + \lambda \vec{v}$, $\lambda \in \mathbb{R}$, $\vec{v} \neq \mathbf{0}$.

Let $\vec{v} = (a, b)$, $P(x, y)$ and $P_0(x_0, y_0)$, then;

a. $(x, y) = (x_0, y_0) + \lambda(a, b)$ or $\begin{cases} x - x_0 = \lambda a \\ y - y_0 = \lambda b \end{cases}$, $\lambda \in \mathbb{R}$, $(a, b) \neq \mathbf{0}$ is called the parametric equation of the line ℓ ,

b. $\lambda = \frac{x - x_0}{a} = \frac{y - y_0}{b} \Rightarrow Ax + By + C = 0$, where $A = \frac{1}{a}$, $B = -\frac{1}{b}$, $C = \frac{1}{b}y_0 - \frac{1}{a}x_0$ is called the standard equation of the line.

Examples:

1. Find the vector equation of the line that passes through the point P_0 and parallel to the vector $\vec{v}$ where $P_0 \in \mathbb{R}^2, \vec{v} \in \mathbb{R}^2$.

Solution:

The vector equation of the line is $P \in \mathbb{R}^2 + \lambda \vec{v}$
 $\vec{r} \in \vec{r}_0 + \lambda \vec{v}$

The parametric equations are $x \in x_0 + \lambda v_x$ and $y \in y_0 + \lambda v_y$.

The standard equation $\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} \iff (x - x_0)v_y - (y - y_0)v_x = 0$

2. What is the standard equation of the line passing through the point $(2, 3)$ and parallel to the line given by $\begin{cases} x \in 1 + 2\lambda \\ y \in 2 + \lambda \end{cases}, \lambda \in \mathbb{R}$?

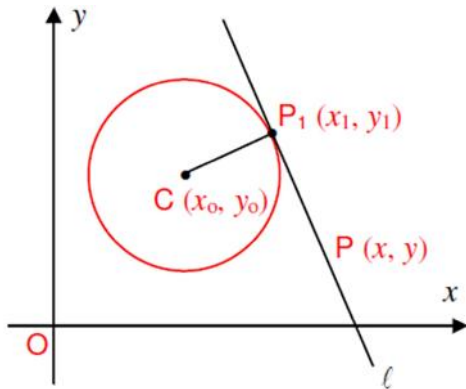
Solution:

The slope of the required line is $\frac{1}{2}$ because it is parallel to the line $\begin{cases} x \in 1 + 2\lambda \\ y \in 2 + \lambda \end{cases}, \lambda \in \mathbb{R}$.

Then the standard equation of the line is $x - 2y + 8 = 0$.

Vectors and Circles

Consider the following figure



$\vec{CP_1} \cdot \vec{P_1P} = 0$ because $\vec{CP_1}$ is perpendicular to $\vec{P_1P}$.

The equation of the line ℓ is: $(x - x_0)(x_1 - x_0) + (y - y_0)(y_1 - y_0) = r^2$ is equation of the tangent line.

Example: A line given by vector equation $\vec{r}(t) \in (0, 3) + t(1, 1)$ is tangent to a circle at a point $P(0, 3)$. If the radius of the circle is $\sqrt{2}$, then find the center of the circle.

Solution:

The line $\vec{r}(t) \in (0, 3) + t(1, 1) \iff y \in x + 3$ is tangent to the circle at $(0, 3)$.

Let $C(h, k)$ be the center of the circle.

Then the slope of the line passing through $C(h,k)$ and $P(0,3)$ is $\frac{k-3}{h-0}$

$$\hat{=}\frac{k-3}{h-0}$$

The distance between $C(h,k)$ and $P(0,3)$ is $\sqrt{h^2+(k-3)^2}$

$$\hat{=}\sqrt{(h-0)^2+(k-3)^2}$$
$$\hat{=}\sqrt{h^2+(k-3)^2}$$
$$\hat{=}\sqrt{h^2+k^2-6k+9}$$

Since $C(1,2)$ is the center of the circle.

TRANSFORMATION OF THE PLANE

In this section you are going to consider three types of transformations namely translation, reflection and rotation.

1. Translations:

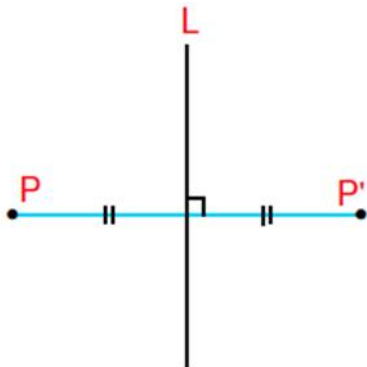
- Translations are a slide or shift.
- Translations can be achieved by performing two composite reflections over parallel lines.
- Translations are isometric, and preserve orientation.
- $T(x,y) = (x+h,y+k)$ where h and k are the horizontal and vertical shifts.

Note: If movement is left, then h is negative. If movement is down, then k is negative.

2. Reflections:

- Reflections are a flip.
- The flip is performed over the “line of reflection.” Lines of symmetry are examples of lines of reflection.
- Reflections are isometric, but do not preserve orientation.

The line L is said to be the line of reflection or the axis of reflection.



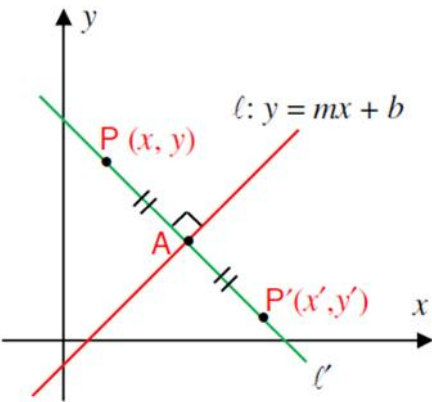
- $\overline{PP'}$ perpendicular to L .
- L is perpendicular bisector of $\overline{PP'}$.
- If P is on L , its image P' in L is P .



Coordinate plane rules:

- C Over the x axis: $M(x, y) \rightarrow (x, -y)$
- C Over the y axis: $M(x, y) \rightarrow (-x, y)$
- C Over the line $y = x$: $M(x, y) \rightarrow (y, x)$
- C Through the origin: $M(x, y) \rightarrow (-x, -y)$

In general to find the image of the point after reflection in the line $y = mx + b$ follow the following figure.



3. Rotations:

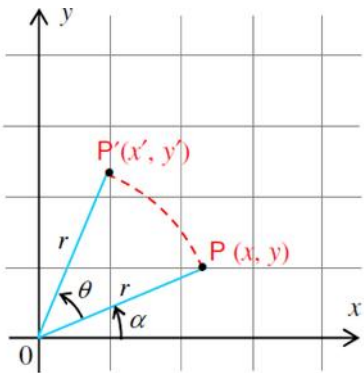
- ❖ Rotations are a turn.
- ❖ Rotations can be achieved by performing two composite reflections over intersecting lines. The resulting rotation will be double the amount of the angle formed by the intersecting lines.
- ❖ Rotations are isometric, and do not preserve orientation unless the rotation is 360° or exhibit rotational symmetry back onto itself.
- ❖ Rotations of 180° are equivalent to a reflection through the origin.

Coordinate plane rules:

Counter-clockwise	Clockwise	Rule:
90°	270°	$R_\theta(x, y) \rightarrow (-y, x)$
180°	180°	$R_\theta(x, y) \rightarrow (-x, -y)$
270°	90°	$R_\theta(x, y) \rightarrow (y, x)$



Theorem: Let R be a rotation through angle θ about the origin. If $R_\theta(x, y) = (x', y')$, then $x' = x \cos \theta - y \sin \theta$ and $y' = x \sin \theta + y \cos \theta$



But if the center of rotation is at (x_0, y_0) , then;
 $x' = (x - x_0) \cos \theta - (y - y_0) \sin \theta + x_0$ and $y' = (x - x_0) \sin \theta + (y - y_0) \cos \theta + y_0$

Examples:

1. If the translation T takes the origin to the point $(1, 3)$, then find;
a. the image of the $A(3, 1)$, $B(6, 3)$ and $C(4, 1)$

Solution:

$$T(0, 0) = (1, 3)$$

So $T(A) = T(3, 1) = (3 + 1, 1 + 3) = (4, 4)$,
 $T(B) = T(6, 3) = (6 + 1, 3 + 3) = (7, 6)$ and
 $T(C) = T(4, 1) = (4 + 1, 1 + 3) = (5, 4)$

- b. the equation of the image of the circle whose equation is $(x - 2)^2 + (y - 7)^2 = 7$.

Solution:

Note that the image of a circle is always a circle with the same radius and different center.

Now find the image of the center of the circle.

$T(2, 7) = (2 + 1, 7 + 3) = (3, 10)$ which is the center of the image of the circle $(x - 2)^2 + (y - 7)^2 = 7$.

$(x - 1)^2 + (y - 10)^2 = 7$ is the image of $(x - 2)^2 + (y - 7)^2 = 7$.

2. Find images of the following lines and circles after reflection in the line $L: y = 2x + 3$.
a. $\ell: y = 2x + 1$
b. $\ell: y = 3x + 2$
c. $(x - 2)^2 + (y - 3)^2 = 25$



Solutions:

- a. Let ℓ' be the image of ℓ . Since ℓ is parallel to line of reflection L , ℓ' is also parallel to ℓ .
= The slope of ℓ' is 2.

Now select any point on ℓ , let $P(0,1)$. Let the image of $P(0,1)$ be $Q(a,b)$ which is ℓ' .

The slope of the line passing through $P(0,1)$ to $Q(a,b)$ is $\frac{1}{2}$

$$\frac{b-1}{a-0} = \frac{1}{2} \Rightarrow a = 2(b-1) \dots\dots\dots (*)$$

The midpoint of $P(0,1)$ and $Q(a,b)$ is on L

$$\left(\frac{a+0}{2}, \frac{b+1}{2}\right) \text{ is on } L: y = 2x + 3$$

$$\frac{b+1}{2} = 2\left(\frac{a+0}{2}\right) + 3 \Rightarrow b+1 = 2a+6 \Rightarrow b = 2a+5$$

$$a = \frac{b-5}{2} \text{ and using } (*), \text{ we get } b = \frac{3}{5}, a = \frac{16}{5}$$

= The image of $P(0,1)$ is $P'\left(\frac{16}{5}, \frac{3}{5}\right)$

$y = \ell': y = 2x + \frac{7}{5}$

- b. $\ell: y = 3x + 2$ is not parallel to the line of reflection $L: y = 2x + 3$.

Find the intersection point of $\ell: y = 3x + 2$ and $L: y = 2x + 3$.

$$y = 3x + 2 = 2x + 3$$

$$x = \frac{1}{5}, y = \frac{13}{5} \text{ this point is also on } \ell'.$$

Next select any point on ℓ , let $P(0,2)$. Let the image of $P(0,2)$ be $Q(a,b)$ which is ℓ' .

The slope of the line passing through $P(0,2)$ to $Q(a,b)$ is $\frac{1}{2}$

$$\frac{b-2}{a-0} = \frac{1}{2} \Rightarrow a = 2(b-2) \dots\dots\dots (*)$$

The midpoint of $P(0,2)$ and $Q(a,b)$ is on L

$$\left(\frac{a+0}{2}, \frac{b+2}{2}\right) \text{ is on } L: y = 2x + 3$$

$$\frac{b+2}{2} = 2\left(\frac{a+0}{2}\right) + 3 \Rightarrow b+2 = 2a+6 \Rightarrow b = 2a+4$$

$$a = \frac{b-4}{2} \text{ and using } (*), \text{ we get } b = 0, a = 4$$

= The image of $P(0,2)$ is $P'(4, 0)$

Now find the equation of the line passing through $P\left(\frac{1}{5}, \frac{13}{5}\right)$ and $Q(4,0)$ which is the image is of

$\ell: y = 3x + 2$

$y = \ell': y = \frac{13}{9}x + \frac{52}{9}$

c. Find the image of the center $C(2,3)$. Let the image of $C(2,3)$ be $Q(a,b)$

The slope of the line passing through $C(2,3)$ to $Q(a,b)$ is $\frac{1}{2}$

$$\frac{b-3}{a-2} = \frac{1}{2} \quad \text{.....(*)}$$

The midpoint of $C(2,3)$ and $Q(a,b)$ is on L

$$\left(\frac{a+2}{2}, \frac{b+3}{2}\right) \text{ is on } L: y = 2x + 3$$

$$\frac{b+3}{2} = 2\left(\frac{a+2}{2}\right) + 3$$

$$b + 3 = 2(a + 2) + 6 \text{ and using (*), we get } b = \frac{11}{5}, a = \frac{18}{5}$$

$$\therefore \text{The image of } C(2,3) \text{ is } Q\left(\frac{18}{5}, \frac{11}{5}\right)$$

Now find the equation of the circle whose center is at $Q\left(\frac{18}{5}, \frac{11}{5}\right)$

$$\left(x - \frac{18}{5}\right)^2 + \left(y - \frac{11}{5}\right)^2 = 25 \text{ which is the image of } (x-2)^2 + (y-3)^2 = 25$$

3. Find the image of the circle $(x-1)^2 + (y-2)^2 = 9$ after it has been rotated about the origin through angle $\theta = \frac{\pi}{4}$.

Solution:

The center of the circle is at $C(1,2)$

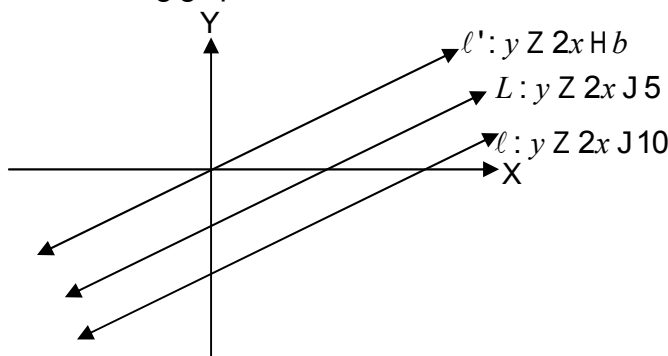
$$\begin{aligned} R_{\frac{\pi}{4}}(1,2) &= \left(1\cos\frac{\pi}{4} - 2\sin\frac{\pi}{4}, 1\sin\frac{\pi}{4} + 2\cos\frac{\pi}{4}\right) \\ &= \left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \end{aligned}$$

$$\therefore \text{Therefore the image of } (x-1)^2 + (y-2)^2 = 9 \text{ is } \left(x - \frac{3\sqrt{2}}{2}\right)^2 + \left(y - \frac{\sqrt{2}}{2}\right)^2 = 9.$$

4. Let ℓ be the line whose equation is $2x - y - 10 = 0$. Find the equation of the image of ℓ after a reflection in the line $L: y = 2x + 5$ followed by a rotation through the angle of 90° about the origin.

Solution:

Consider the following graph



When we reflect $\ell: y = 2x + 10$ about the line $L: y = 2x + 5$, the equation of $\ell': y = 2x + 6$. (How?)

Now to rotate the line $\ell': y = 2x + 6$, select any two points on it. Let $P(0, 6)$ and $Q(1, 8)$

$$\hat{G} R_{90^\circ} (0, 6) \rightarrow (0, 0) \text{ and } R_{90^\circ} (1, 8) \rightarrow (-2, 1)$$

Then find the equation of the line passing through $(0, 0)$ and $(-2, 1)$

$$\hat{G} 2y + x = 0$$

Therefore the image of the line $2x - y - 10 = 0$ after a reflection in the line $L: y = 2x + 5$ followed by a rotation through the angle of 90° about the origin is $2y + x = 0$

Exercises

Choose the correct answer for each of the following questions

- The image of a figure with vertices $A(1, 2)$, $B(3, 6)$, $C(-1, 2)$, and $D(-2, 2)$ after reflection across the x -axis is:
 - $A'(1, 2)$, $B'(3, 6)$, $C'(-1, 2)$, and $D'(-2, 2)$
 - $A'(-1, 2)$, $B'(-3, 6)$, $C'(1, 2)$, and $D'(2, 2)$
 - $A'(1, -2)$, $B'(3, -6)$, $C'(-1, -2)$, and $D'(-2, -2)$
 - $A'(1, -2)$, $B'(3, -6)$, $C'(-1, -2)$, and $D'(-2, -2)$
- Consider the circle given by $x^2 + y^2 = 2$ and the line l given by parametric vector equation $(x, y) = (2, 0) + t(1, 1)$. Which of the following is true?
 - l is tangent to the circle at $\left(\frac{1}{2}, \frac{\sqrt{7}}{2}\right)$.
 - l intersects with the circle at two distinct points.
 - l and the circle have no common point.
 - The distance from the center of the circle to l is $\sqrt{2}$.

- | | |
|--|--|
| | |
|--|--|

- A. $y \geq \frac{13}{9}(x + 4)$
- B. $y \geq \frac{13}{9}(x - 4)$
- C. $y \geq \frac{1}{9}(9x + 105)$
- D. $y \geq \frac{13}{9}(x + 4)$

- A. $T(5, 6)Z(7, 8)$ C. $T^{J1}(0, 0)Z(J2, J2)$
 B. $T(0, 0)Z(2, 2)$ D. $T(J3, 2)Z(J2, 4)$

- A. $x^2 H 4(y J 1)^2 Z 1$ C. $x^2 H 4(y H 1)^2 Z 1$
B. $4x^2 H (y J 1)^2 Z 1$ D. $4x^2 H (y H 1)^2 Z 1$

- A. (J1,1) B. (J3,3) C. (J7, J1) D. (1, 7)

UNIT 9
FURTHER ON TRIGONOMETRIC FUNCTIONS

Unit Outcomes:

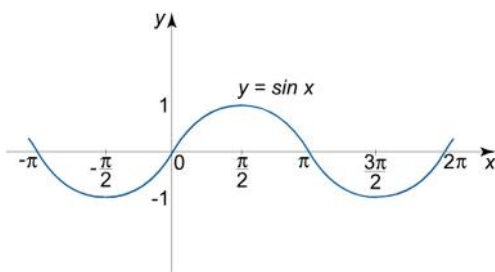
After completing this unit, you should be able to:

- ❖ know basic concepts about reciprocal functions.
- ❖ sketch graphs of some trigonometric functions.
- ❖ apply trigonometric functions to solve related problems.

The functions $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$

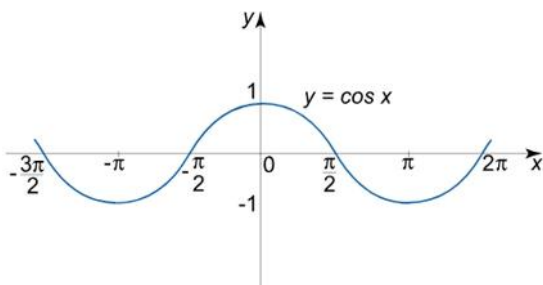
1. The graph of $y = \sin x$

- Domain: $x \in \mathbb{R}$
- Range: $-1 \leq \sin x \leq 1$
- Period: 2π



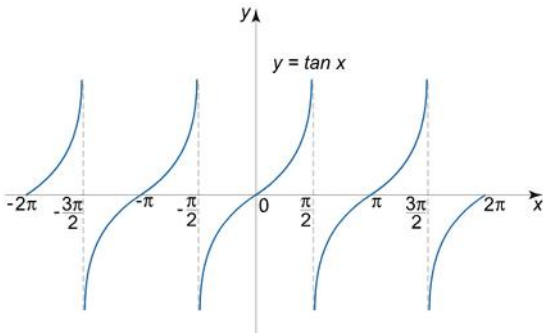
2. The graph of $y = \cos x$

- Domain: $x \in \mathbb{R}$
- Range: $-1 \leq \cos x \leq 1$
- Period: 2π



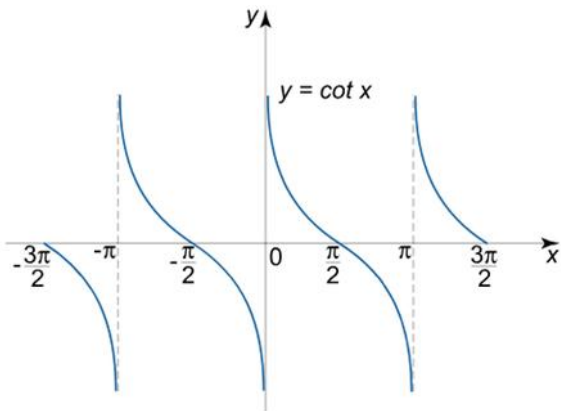
3. The graph of $y = \tan x$

- Domain: $x \in \mathbb{R}, x \neq \frac{\pi}{2} + k\pi \text{ where } k \in \mathbb{Z}$
- Range: $\mathbb{R}$ or $(-\infty, \infty)$
- Period: π



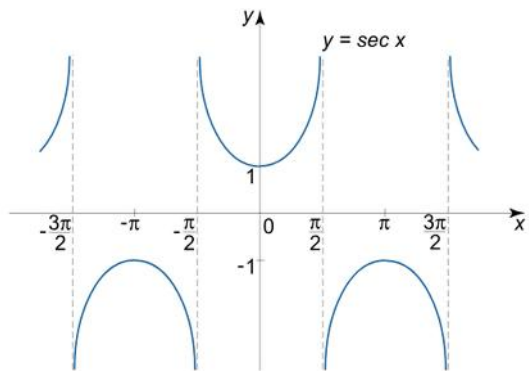
4. The graph of $y = \cot x$

- Domain: $x \in \mathbb{R}, x \neq k\pi, k \in \mathbb{Z}$
- Range: $\mathbb{R}$ or $(-\infty, \infty)$
- Period: π



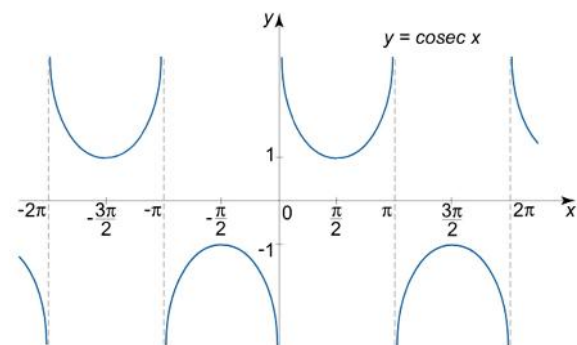
5. The graph of $y = \sec x$

- Domain: $x \in \mathbb{R}, x \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$
- Range: $(-\infty, -1] \cup [1, \infty)$
- Period: 2π



6. The graph of $y = \csc x$

- Domain: $x \in \mathbb{R}, x \neq k\pi, k \in \mathbb{Z}$
- Range: $(-\infty, -1] \cup [1, \infty)$
- Period: 2π



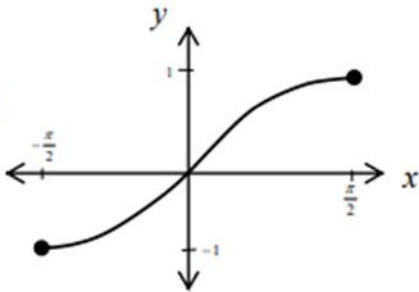
Inverse of trigonometric functions

Each trigonometric function has a restricted domain for which an inverse function is defined. The restricted domains are determined so the trig functions are one-to-one.

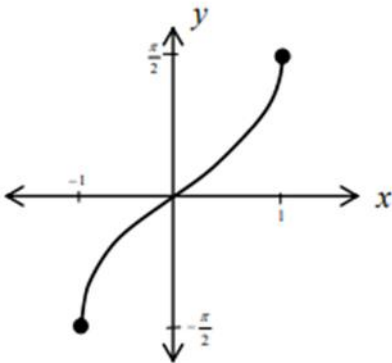
Trig function	Restricted domain	Inverse trig function	Principle value range
$y = \sin x$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	$y = \arcsin x$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos x$	$0 \leq x \leq \pi$	$y = \arccos x$	$0 \leq y \leq \pi$
$y = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$y = \arctan x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Graphs:

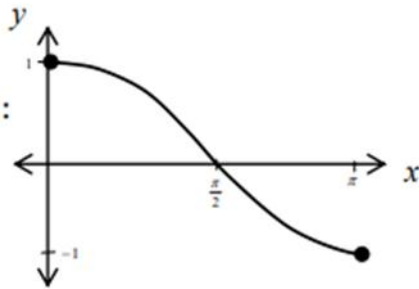
$y = \sin x :$



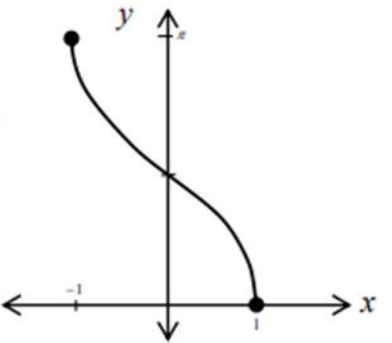
$y = \arcsin x = \sin^{-1} x :$



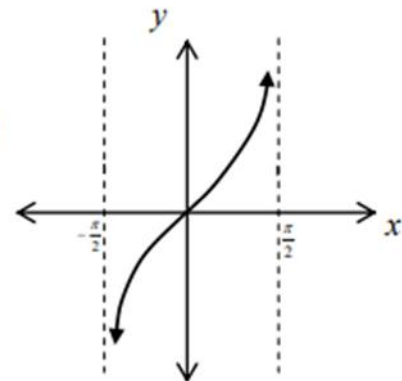
$y = \cos x :$



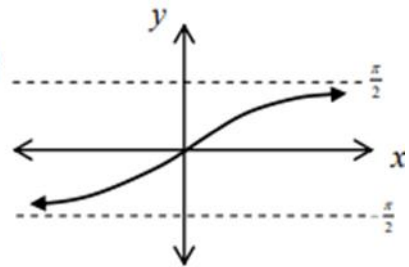
$y = \arccos x = \cos^{-1} x :$



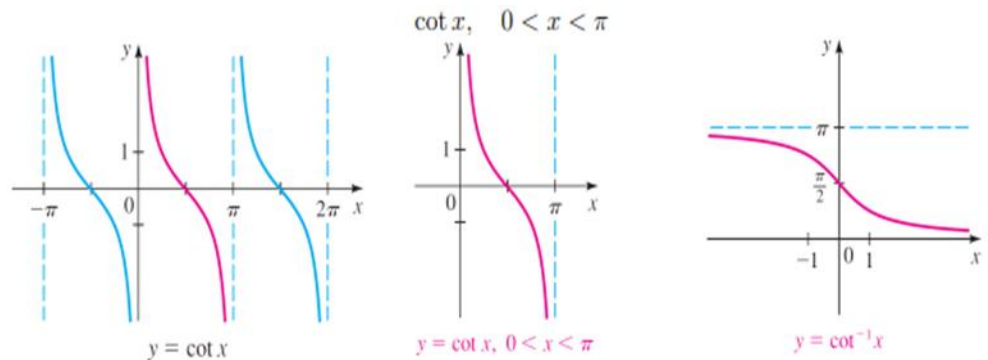
$y = \tan x :$



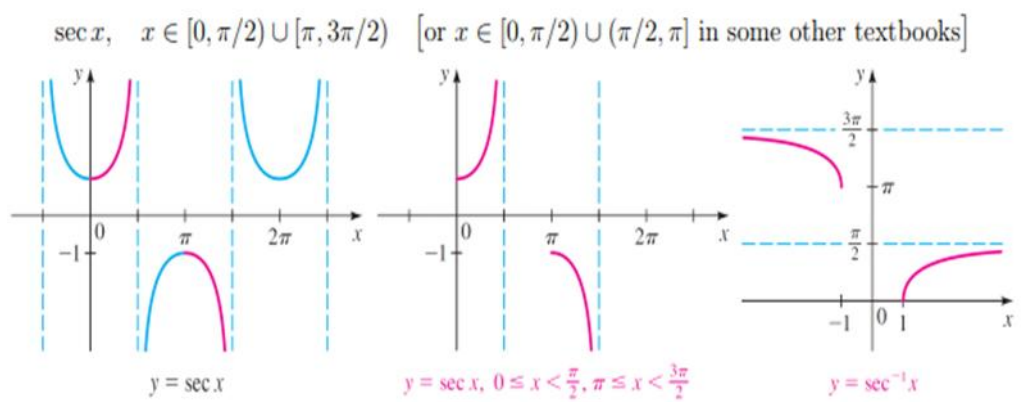
$y = \arctan x = \tan^{-1} x :$



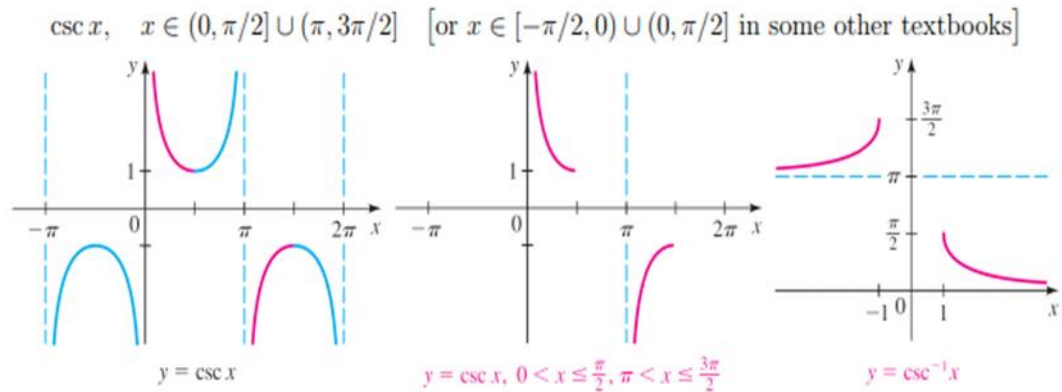
Definition: The inverse cotangent function, denoted by $\cot^{-1}x$ (or $\operatorname{arccot}x$), is defined to be the inverse of the restricted cotangent function



Definition: The inverse secant function, denoted by $\sec^{-1}x$ (or $\operatorname{arcsec}x$), is defined to be the inverse of the restricted secant function



Definition: The inverse cosecant function, denoted by $\csc^{-1}x$ (or $\operatorname{arccsc}x$), is defined to be the inverse of the restricted cosecant function



TRIGONOMETRIC IDENTITIES

Theorems:

- 1. $\sin^2 \theta + \cos^2 \theta = 1$
 - i. $1 + \tan^2 \theta = \sec^2 \theta$
 - ii. $1 + \cot^2 \theta = \csc^2 \theta$
- 2. **Sum and difference identities**
 - a. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
 - b. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
 - c. $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, x, y \neq (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}$
- 3. **Double angle formula**
 - a. $\sin 2\theta = 2\sin \theta \cos \theta$
 - b. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
 $= 2\cos^2 \theta - 1$
 $= 1 - 2\sin^2 \theta$
 - c. $\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}$
- 4. **Half angles**
 - a. $\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$
 - b. $\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$
 - c. $\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$

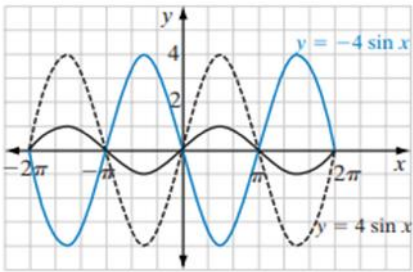
GRAPHS OF SOME TRIGONOMETRIC FUNCTIONS

Examples:

- 1. Describe the relationship between the graph of $y = 4\sin x$ and the graph of $y = \sin x$ and sketch the graph.

Solution:

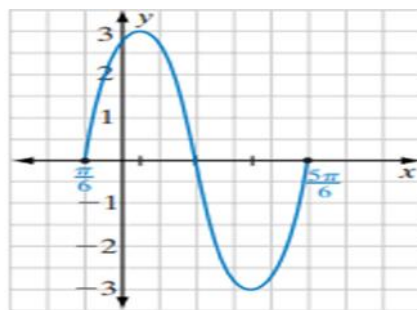
- The function $g(x) = 4f(x)$ is the function $f(x)$ stretched by the factor 4 in vertical direction.
 - The function $-g(x)$ is the function $g(x)$ reflected in the x -axis.
- Apply these rules to the sine function.
- The function $y = 4\sin x$ is the function $y = \sin x$ stretched by the factor 4 in vertical direction and reflected in the x -axis.
 - The amplitude of $y = 4\sin x$ is 4



2. Sketch the graph of $y = 3 \sin 2\left(x + \frac{\pi}{6}\right)$

Solution: How to proceed

- c The phase shift is $-\frac{\pi}{6}$. Locate a point on the x -axis that is $\frac{\pi}{6}$ to the left of the origin. This point, $\left(-\frac{\pi}{6}, 0\right)$, is the starting point of one cycle of the sine curve:
- c The period is $\frac{2\pi}{2} = \pi$. Locate a point π unit to the right of the point in step above. This point, $\left(\frac{5\pi}{6}, 0\right)$, is the upper endpoint of one cycle of the curve:
- c Divide the interval for one cycle into four parts of equal length along the x -axis :
- c The amplitude of the curve is 3:
- c The y -coordinates of the sine curve increases from 0 to the maximum in the first quarter of the cycle, decreases from the maximum to 0 in the second quarter, decreases from 0 to minimum in the third quarter, and increases from minimum to 0 in the fourth quarter. Then Sketch the curve.



SUMMARY

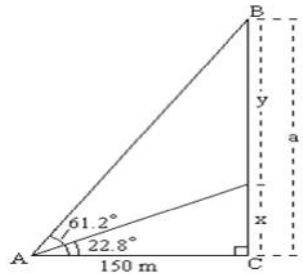
For the graphs $y = a \sin b(x + c)$ and $y = a \cos b(x + c)$

1. The amplitude is $|a|$.
 - o The maximum value of the function is $|a|$ and the minimum value is $-|a|$.
2. The period of the function is $\frac{2\pi}{|b|}$.
 - o There are $|b|$ cycles in a 2π interval.
3. The phase shift is $-c$.
 - o The graph of $y = a \sin b(x + c)$ or $y = a \cos b(x + c)$ is the graph of $y = a \sin x$ or $y = a \cos x$ shifted c units in the horizontal direction.
 - o If c is positive, the graph is shifted $|c|$ units to the left.
 - o If c is negative, the graph is shifted $|c|$ units to the right.
4. The graph of $y = a \sin b(x + c)$ or $y = a \cos b(x + c)$ is a reflection in the x -axis of $y = a \sin b(x + c)$ or $y = a \cos b(x + c)$.
5. The domain of the function is the set of real numbers.
6. The range of the function is the interval $[-|a|, |a|]$.

Applications Trigonometric Functions

Examples:

1. Solve the right triangles shown below for y rounding all measurements to one decimal place (if necessary).



Solution:

In order to find the value of y , we must first solve for a and x . We could then find y by subtracting x from a .

Find a

Side a is the full length of the side between B and C . Since the angle A and its adjacent side are known, we can use the tangent function to find a .

$$\begin{aligned} \tan A &= \frac{a}{b} \hat{=} \tan 61.2^\circ \hat{=} \frac{a}{150} \\ \hat{=} a &\hat{=} 150(\tan 61.2^\circ) \hat{=} 272.8\text{m} \end{aligned}$$

Find x

Side x is also opposite of the angle A so we will once again use the tangent function to find the length of x . However, this time we will use the smaller angle of 22.8° for A .

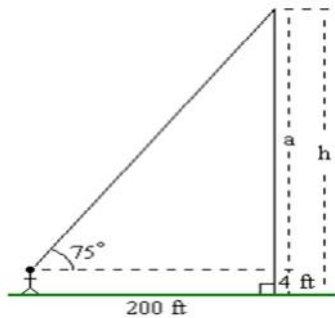
$$\begin{aligned} \tan A &= \frac{x}{b} \hat{=} \tan 22.8^\circ \hat{=} \frac{x}{150} \\ \hat{=} x &\hat{=} 150(\tan 22.8^\circ) \hat{=} 63.1\text{m} \end{aligned}$$

Find y

$$\begin{aligned} x &+ y \hat{=} a \hat{=} 63.1 + y \hat{=} 272.8 \\ \hat{=} y &\hat{=} 209.7\text{m} \end{aligned}$$

2. A child who is 4 feet tall is standing 200 feet away from the center of the base of the tower of Finfinnee. If the child must look up at an angle of 75° to see the top of the tower, how tall is the tower of Finfinnee?

Solution: Draw diagram of the situation



Find a

For the triangle involving a, we know both the measurements of angle A and the length of the side adjacent to it. Therefore, we can use the tangent function to determine a.

$$\tan A = \frac{a}{b} \hat{=} \tan 75^\circ \hat{=} \frac{a}{200}$$
$$\hat{=} a \hat{=} 200(\tan 75^\circ) \hat{=} 746\text{ft}$$

Find h

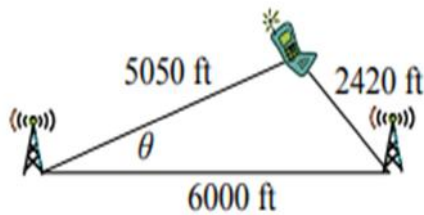
The height of the Tower of Finfinnee is equal to the value of a plus the height of the child.

$$\hat{=} h = 4 + a \hat{=} 750\text{ft}$$

3. On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is done by a process called triangulation, which works by using the distance from two known points. Suppose there are two cell phone towers within range of you, located 6000 feet apart along a straight highway that runs east to west, and you know you are north of the highway. Based on the signal delay, it can be determined you are 5050 feet from the first tower, and 2420 feet from the second. Determine your position relative to the tower to the west and determine how far you are from the highway.

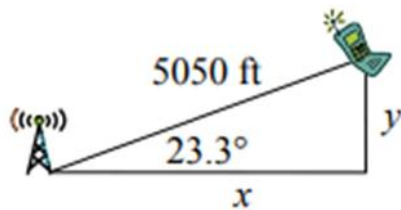
Solution:

For simplicity, we start by drawing a picture and labeling our given information. Using the Law of Cosines, we can solve for the angle θ .



$$2420^2 \hat{=} 6000^2 + 5050^2 - 2(5050)(6000)\cos\theta$$
$$\hat{=} \theta \hat{=} 23.328^\circ$$

Using this angle, we could then use right triangles to find the position of the cell phone relative to the western tower.



$$\cos(23.328^\circ) \hat{=} \frac{x}{5050} \hat{=} x \hat{=} 4637.2\text{ft}$$
$$\sin(23.328^\circ) \hat{=} \frac{y}{5050} \hat{=} y \hat{=} 1999.8\text{ft}$$

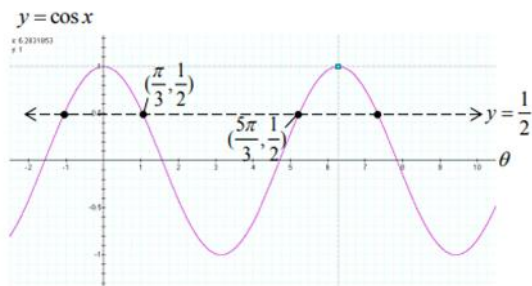
You are 5050ft from the tower and 23.328° North of East. Specifically, you are 4637.2ft East and 1999.8ft North of the western tower.

Note that if you didn't know if you were north of both towers, our calculations would have given two possible locations, one north of the highway and one south. To resolve this ambiguity in real world situations, locating a position using triangulation requires a signal from a third tower.

4. Find all possible values of θ so that $\cos \theta \geq \frac{1}{2}$.

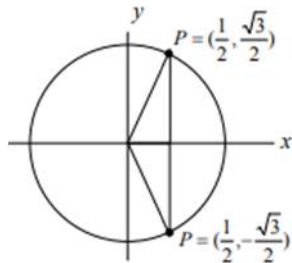
Solution: $\theta \in \left[\frac{\pi}{3}, \frac{5\pi}{3} \right] + 2\pi n$, where n is an integer.

Method 1–Graphically: There are an infinite number of solutions which are represented by the θ value of intersection points of the cosine curve and the constant function $y = \frac{1}{2}$.



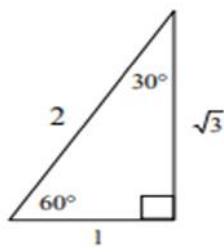
For $0 \leq \theta < 2\pi$, there are two solutions: $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. Generalizing, $\theta \in \left[\frac{\pi}{3}, \frac{5\pi}{3} \right] + 2\pi n$, where n is an integer. Thus all solutions differ from the original two solutions by multiples of the period of the cosine function.

Method 2–Unit Circle Approach: $\cos \theta \geq \frac{1}{2}$ occurs when $x \geq \frac{1}{2}$ for point(s) on the unit circle.



The two points are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right)$. The corresponding angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. Generalizing, $\theta \in \left[\frac{\pi}{3}, \frac{5\pi}{3} \right] + 2\pi n$, where n is an integer.

Method 3–Triangle Approach: $\cos \theta \geq \frac{1}{2}$ is a special case that involves 30°, 60°, 90° triangles.



Since $\cos \theta \geq \frac{\text{adj}}{\text{hyp}} \geq \frac{1}{2}$, this implies θ must be 60° . Generalizing $\cos \theta$ is also positive in Quadrant IV with a reference angle of 60° . Generalizing, $\theta \geq \frac{\pi}{3} + 2\pi n, \theta \geq \frac{5\pi}{3} + 2\pi n$, where n is an integer.

5. Solve for x : $\sin^2 x - \sin x - 2 \geq 0, 0 \leq x < 2\pi$.

Solution: Factor the quadratic expression on the left and set each factor to zero.

$$\sin^2 x - \sin x - 2 \geq 0$$

$$(\sin x - 1)(\sin x + 2) \geq 0$$

$$\sin x - 1 \geq 0 \text{ or } \sin x + 2 \geq 0$$

$$\sin x \geq 1 \text{ or } \sin x \geq -2$$

$$x \geq \frac{\pi}{2} \text{ but } \sin x \geq -2 \text{ has no solution}$$

$$x \geq \frac{\pi}{2}$$

Exercises

Choose the correct answer for each of the following questions

- If $\theta = \arcsin \frac{1}{2}$, then $\tan(2\theta)$ is equal to:
A. $\frac{1}{\sqrt{3}}$ B. 1 C. $\sqrt{3}$ D. $\frac{\sqrt{3}}{2}$
- If $\cot(\theta) = 2$, then which one of the following is equal to $\csc(\theta)$?
A. $\sqrt{5}$ B. $\frac{2}{\sqrt{5}}$ C. $\frac{1}{\sqrt{5}}$ D. $\frac{1}{2}$
- The angle of elevation of a parachute measured from a point at a height of 60m from the surface of the lake is 30° and the angle of depression of reflection of parachute seen in the lake from the same point is 60° . Then find the height of the parachute from the surface of the lake.
A. 150m B. 90m C. 60m D. 120m
- $3\tan 930^\circ + \sin 1200^\circ - \cos 1410^\circ$ equals:
A. $\sqrt{3}$ B. $2\sqrt{3}$ C. $-\sqrt{3}$ D. 0
- What is the value of $\arcsin \frac{\sqrt{2}}{2}$?
A. $\frac{\pi}{4}$ B. $\frac{\pi}{2}$ C. $-\frac{\pi}{4}$ D. $-\frac{\pi}{2}$
- Which one of the following is the simplified form of $(1 + \tan^2 x) \sin x - \frac{3}{2}\pi$?
A. $\sec x$ B. $\cos x$ C. $-\sec x$ D. $-\cos x$

7. What is the amplitude and period respectively, of the graph of $y = 3\cos\left(\frac{\pi}{4}t + \frac{\pi}{2}\right)$?
- A. $2, \frac{\pi}{4}$ B. $3, 8\pi$ C. $3, 8$ D. $3, 8\pi$
8. Which one of the following is a simplified form of $\cos^2 x - \sin^2 x$?
- A. $2\cos x$ B. $2\sin x$ C. $\cos^2 x$ D. $\sin 2x$
9. $\cos(\tan^{-1}(0.5))$ is equal to:
- A. $\frac{1}{\sqrt{5}}$ B. $\frac{1}{\sqrt{3}}$ C. $\frac{2}{\sqrt{5}}$ D. $\frac{2}{\sqrt{3}}$
10. What is the period of the function $f(x) = \cos(3x)$?
- A. 6π B. $\frac{2\pi}{3}$ C. 2π D. $\frac{\pi}{3}$
11. What is $\sin(\tan^{-1} x)$?
- A. $\frac{x}{\sqrt{1+x^2}}$ B. $\frac{1}{\sqrt{1+x^2}}$ C. $\sqrt{1+x^2}$ D. $\frac{x}{\sqrt{1-x^2}}$
12. The simplified form of $\frac{\cot x}{1+\cot^2 x}$ is
- A. $\sin 2x$ B. $\csc 2x$ C. $\frac{1}{2}\sin 2x$ D. $\frac{1}{2}\csc 2x$
13. Which of the following is the domain of $\sec x$?
- A. $\{x \in \mathbb{R} : x \neq n\pi, n \in \mathbb{Z}\}$ C. $\{x \in \mathbb{R} : x \neq 2n\pi, n \in \mathbb{Z}\}$
- B. $\{x \in \mathbb{R} : x \neq \frac{n}{2}\pi, n \in \mathbb{Z}\}$ D. $\{x \in \mathbb{R} : x \neq \frac{(2n+1)}{2}\pi, n \in \mathbb{Z}\}$
14. Which one of the following is equal to $\sin\frac{x}{2}\cos\frac{x}{2}$?
- A. $\frac{1}{2}\sin x$ B. $\frac{1}{2}\sin 2x$ C. $\sin 2x \cos x + \cos 2x \sin x$ D. $\frac{1}{2}\cos x$
15. What is $\sec(\cos^{-1} x)$?
- A. $\frac{x}{\sqrt{1-x^2}}$ B. $\frac{\sqrt{1-x^2}}{x}$ C. $\sqrt{1-x^2}$ D. $\frac{x}{\sqrt{1+x^2}}$
16. What is the possible value of x that solves the equation $2\sin^2 x + \sin x - 1 = 0$?
- A. $\left[\frac{\pi}{2} + 2n\pi, \frac{11\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi; n \in \mathbb{Z}\right]$ C. $\left[\frac{\pi}{2} + 2n\pi, \frac{11\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi; n \in \mathbb{Z}\right]$
- B. $\left[\frac{\pi}{2} + 2n\pi, \frac{11\pi}{6} + 2n\pi; n \in \mathbb{Z}\right]$ D. $\left[\frac{\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi; n \in \mathbb{Z}\right]$



17. A flying airplane is sighted in a line from two observation stations A and B. The angle of elevation of the airplane from A is 30° and from B is 60° . A and B are on the same side of the airplane. If the distance between A and B is 1200m, find the altitude of the airplane.

- A. 1200m B. 1039m C. 309m D. 1050m

18. If $\theta \in \tan^{-1}\left(\frac{3}{4}\right)$, then which of the following is equal to $\cos(2\theta)$?

- A. $\frac{9}{16}$ B. $\frac{9}{25}$ C. $\frac{16}{25}$ D. $\frac{7}{25}$

19. What is the solution set of $\cos(2x) \geq \sin x \geq 0$, in the interval $[0, 2\pi]$?

- A. $\left[0, \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right]$ C. $\left[0, \frac{11\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{2}, \frac{\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{6}\right]$
B. $\left[0, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\right]$ D. $\left[0, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{6}\right]$

20. Which one of the following is equal to $\sec^2\left(\frac{\pi}{2} - x\right) \sin^3 x \cos 2x$?

- A. $2\cos x$ B. $2\sin x$ C. $\cos^2 x$ D. $\sin^2 x$

THE END!

“PRACTICE AND AGAIN PRACTICE, WHICH WILL MAKE YOU PERFECT”!

