

ANSWER KEY



ADDIS ABABA SCIENCE AND TECHNOLOGY UNIVERSITY

COLLEGE OF NATURAL AND SOCIAL SCIENCES

MATHEMATICS DEPARTMENT

Final Exam (50%) - Jan 17, 2022

Applied Mathematics IB Course Code: Math 1014

Program: Regular

Total pages: 07

Time Allowed: 2 hours and 45 minutes

General Directions:

- Check that there are **seven** pages including the cover page in this closed book examination booklet
- Check that there are **seven** multiple choice, **eight** fill the blank space and **four** work-out questions.
- Attempt all the questions according to the instruction of each part.
- Use the backside of the pages for any rough work
- Strictly forbidden materials to use: **Any Calculators, phone, extra papers, red pen and pencil**

Name: _____ ID No: _____

Department: _____ section: _____

If you are adding this course, please tick the box ☐

FOR INSTRUCTORS USE ONLY

Multiple choices
[14pts]

☐

Blank space [16pts]

☐

Work out [20pts]

☐☐☐☐

Total [50]

☐

ion 1. Choose the choice with the right answer and write the letter of your choice
on the shaded space provided ONLY (2 points each)

B 1. The angle between the vectors $u = (1, 0, -1)$ and $v = (1, 1, 0)$ is

A. $\frac{\pi}{2}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{4}$

D. π

C 2. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 1} + x - x$ equals

A. 3.5

B. 2.5

C. 0.5

D. 0

B 3. Which one of the following is not always true about a function f ?

A. A critical number c of f is a number in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist

B. If c is a critical number of f , then $f(c)$ is local extreme value of f

C. If f has local extreme value at c and $f'(c)$ exists then $f'(c) = 0$

D. None of the above

C 4. If $\int_0^{10} f(x) dx = 17$ and $\int_0^8 f(x) dx = 12$ for function f , then the value of $\int_8^{10} f(x) dx$ is

A. 29

B. 204

C. 5

D. -5

A 5. The equation of a plane containing the point $(1, 2, -2)$ and perpendicular to the line

$$\ell: \frac{x-1}{2} = \frac{y+4}{3} = z-2 \text{ is}$$

A. $2x + 3y + z = 6$

B. $x + 2y - z = 7$

C. $2x - 3y - 4z = 4$

D. $x + y + z = 1$

D 6. Given $y = \cot^{-1}(\tan x^3)$ then $\frac{dy}{dx}\bigg|_{x=0}$ is

A. 4

B. -2

C. -3

D. 0

C 7. $\int_1^e \ln(x) dx$ equals to

A. $-e$

B. -1

C. 1

D. e

on it. Fill the following blank spaces by the most simplified and finalized answers on the given shaded spaces ONLY. (2 points each)

1. The unit vector parallel to xy plane and orthogonal to the vector $2i - 4j + 2k$ is $u = \frac{1}{\sqrt{2}}(2i + j)$
2. If $f(x) = 3e^x$ and $f(0) = 3$ then $\int_0^1 e^{-x} |f(x) - f'(x)| dx$ is -8
3. The set of integers is a vector space over the set of rational numbers under the operations of addition and multiplication. True/False **False**
4. If $\lim_{x \rightarrow \infty} \left(\frac{x+k}{x-3} \right)^{2x} = e^{16}$, $k \in \mathbb{R}$ then $k = 13$
5. The slope of the tangent line to $xe^y + ye^x = 1$ at $(0,1)$ is $-1 - e$
6. For function f , let $f'(a) = 10$, then $\lim_{h \rightarrow 0} \frac{f(a+3h) - f(a)}{h}$ is 30
7. The value of m such that $f(x) = \begin{cases} \sin x & , x \neq 0 \\ x & , x = 0 \end{cases}$ can be continuous at $x = 0$ is -3
8. If $f(x) = \int_1^x \cos(2t + t^2) dt$ then $f'(0) = 1$

on III: Work out the following questions showing all the necessary steps clearly on the given space provided (5 points each)

1. Show that $w = \{1, x, x+x^2\}$ is a basis of vector space of the set of polynomial functions of degree less than or equal to two under the operations of addition and multiplication over the field of set of real numbers.

Solution

$$\text{Let } w = \{1, x, x+x^2\} \subseteq p^2 \text{ and } w \neq \emptyset$$

where $p^2 =$ set of polynomials with degree less than or equal to 2.

i. Check for linear independency

$$\text{let } a, b, c \in \mathbb{R}$$

$$a(1) + b(x) + c(x+x^2) = 0 \quad \leftarrow (1 \text{ point})$$

$$\Rightarrow a + bx + cx + cx^2 = 0 \Rightarrow a + (b+c)x + cx^2 = 0$$

$$(1 \text{ point}) \rightarrow \begin{cases} a = 0 \\ b+c = 0 \\ c = 0 \end{cases} \Rightarrow b = -c \Rightarrow b = 0$$

$$\therefore a = 0, b = 0 \text{ and } c = 0$$

$\Rightarrow w$ is set of linear independent functions

(0.5 point)

ii. Check spanning condition.

let $ax^2 + bx + c \in p^2$ and α_1, α_2 and α_3 be constants

$$\text{then } \alpha_1(1) + \alpha_2(x) + \alpha_3(x+x^2) = ax^2 + bx + c \quad \leftarrow (1 \text{ point})$$

$$\Rightarrow \alpha_1 + (\alpha_2 + \alpha_3)x + \alpha_3x^2 = ax^2 + bx + c$$

$$(1 \text{ point}) \Rightarrow \begin{cases} \alpha_1 = c \\ \alpha_3 = a \\ \alpha_2 + \alpha_3 = b \end{cases} \Rightarrow \alpha_2 = b - \alpha_3 = b - a$$

(0.5 point)

Thus for values $\alpha_1 = c$, $\alpha_2 = b - a$ and $\alpha_3 = a$ any polynomial with degree less than or equal to two can be spanned by set of elements of w .

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Using i and ii above, w is basis.

Find $\lim_{x \rightarrow \infty} (3^x + 4^x)^{\frac{1}{x}}$

Solution

$$\text{let } y = (3^x + 4^x)^{\frac{1}{x}}$$

$$\Rightarrow \ln y = \ln (3^x + 4^x)^{\frac{1}{x}} = \frac{\ln (3^x + 4^x)}{x} \quad \leftarrow (1 \text{ point})$$

Thus,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln (3^x + 4^x)}{x} \quad \left(\frac{\infty}{\infty} \text{ type} \right) \leftarrow (1 \text{ point})$$

Applying L'Hopital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \frac{3^x \ln 3 + 4^x \ln 4}{3^x + 4^x} \\ &= \lim_{x \rightarrow \infty} \frac{4^x \left(\frac{3^x \ln 3}{4^x} + \ln 4 \right)}{4^x \left(\frac{3^x}{4^x} + 1 \right)} \end{aligned} \quad \leftarrow (1 \text{ point})$$

$$= \lim_{x \rightarrow \infty} \frac{\left(\frac{3}{4}\right)^x \ln 3 + \ln 4}{\left(\frac{3}{4}\right)^x + 1} = \ln 4 \quad \leftarrow (1 \text{ point})$$

$$\therefore \lim_{x \rightarrow \infty} \ln y = \ln 4$$

$$y = e^{\ln y} \Rightarrow \lim_{x \rightarrow \infty} (3^x + 4^x)^{\frac{1}{x}} = \lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} e^{\ln y}$$

$$= \lim_{x \rightarrow \infty} e^{\ln y}$$

$$= e^{\ln 4}$$

$$= \underline{\underline{4}} \quad \leftarrow (1 \text{ point})$$

Find the largest interval of values of n for which the improper integral $\int_1^{\infty} \frac{1}{x^n} dx$

- a) Converges
- b) Diverges

Solution

$$\int_1^{\infty} \frac{1}{x^n} dx = \int_1^{\infty} x^{-n} dx$$

$$\int_1^{\infty} x^{-n} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-n} dx$$

$$= \lim_{b \rightarrow \infty} \left. \frac{x^{-n+1}}{-n+1} \right|_1^b$$

$$= \lim_{b \rightarrow \infty} \frac{b^{1-n} - 1}{1-n}$$

$$= \frac{1}{1-n} \left[\lim_{b \rightarrow \infty} b^{1-n} - \lim_{b \rightarrow \infty} 1 \right]$$

$$= \frac{1}{1-n} \left[\lim_{b \rightarrow \infty} b^{1-n} - 1 \right]$$

$$= \frac{1}{1-n} \left[-1 + \lim_{b \rightarrow \infty} b^{1-n} \right]$$

a) For $\int_1^{\infty} \frac{1}{x^n} dx$ to be convergent, $\underline{1-n < 0} \Rightarrow n > 1$

Hence, the largest interval of values of n is $\{n \in \mathbb{R} : n > 1\} = \underline{(1, \infty)}$

b) For $\int_1^{\infty} \frac{1}{x^n} dx$ to be divergent, $\underline{1-n \geq 0} \Rightarrow n \leq 1$

Hence, the largest interval of values of n is $\{n \in \mathbb{R} : n \leq 1\} = \underline{(-\infty, 1]}$

$$f(x) = \frac{(x+1)^2}{x^2+1} \text{ then find}$$

critical number(s)

Interval of increasing and decreasing

Relative extreme values if any

Intervals of concave upward and downward

Inflection point(s) if any

Solution.

$$f(x) = \frac{(x+1)^2}{x^2+1} \Rightarrow f'(x) = \frac{2(x+1)(1+x^2) - 2x(1+x)^2}{(1+x^2)^2}$$

$$f'(x) = \frac{(x+1)(2-2x)}{(1+x^2)^2}$$

$$\Rightarrow f'(x) = \frac{2(1-x)(1+x)}{(1+x^2)^2}$$

$$f'(x) = 0 \Rightarrow x = 1, x = -1$$

Thus 1 and -1 are critical numbers

b) $(1+x^2)^2 > 0$ for all $x \in \mathbb{R}$

$1+x$	$-\infty$	-1	1	∞
$1-x$	$+$	$+$	$-$	$-$
$f'(x)$	$-$	$+$	$-$	$-$

$\Rightarrow f$ is increasing on $(-1, 1)$

and f is decreasing on $(-\infty, -1) \cup (1, \infty)$

c) From the table done above,

f' changes from -ve to +ve at -1

$$\Rightarrow f(-1) = 0 \Rightarrow 0 \text{ is local min value}$$

f' changes from +ve to -ve at 1

$$\Rightarrow f(1) = 2 \Rightarrow 2 \text{ is local max value}$$

$$d) f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow f''(x) = \frac{-4x(1+x^2) - 8x(1-x^2)}{(1+x^2)^3}$$

$$= \frac{-4x(3-x^2)}{(1+x^2)^3}$$

$$f''(x) = \frac{-4x(\sqrt{3}-x)(\sqrt{3}+x)}{(1+x^2)^3}$$

Here $(1+x^2)^3 > 0$ for all $x \in \mathbb{R}$

$-4x$	$-\infty$	$-\sqrt{3}$	0	$\sqrt{3}$	∞
$\sqrt{3}-x$	$+$	$+$	$+$	$-$	$-$
$\sqrt{3}+x$	$-$	$-$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$+$	$-$

$$f''(x) > 0 \text{ on } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$$

$\Rightarrow f$ is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

$$f''(x) < 0 \text{ on } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

$\Rightarrow f$ is concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

$$e) f''(x) = 0 \text{ if } x = 0, \sqrt{3} \text{ and } -\sqrt{3}$$

Thus, $(0, f(0))$, $(\sqrt{3}, f(\sqrt{3}))$ and $(-\sqrt{3}, f(-\sqrt{3}))$ are inflection points

Hence

$$\left\{ \begin{array}{l} (0, 1) \\ (\sqrt{3}, 1+\frac{\sqrt{3}}{2}) \\ (-\sqrt{3}, 1-\frac{\sqrt{3}}{2}) \end{array} \right\} \text{ are inflection points.}$$

$f(x) = \frac{(x+1)^2}{x^2+1}$ then find

- Critical number(s)
- Interval of increasing and decreasing
- Relative extreme values if any
- Intervals of concave upward and downward
- Inflection point(s) if any

Solution:

$f(x) = \frac{(x+1)^2}{x^2+1} \Rightarrow f'(x) = \frac{2(x+1)(1+x^2) - 2x(1+x)^2}{(1+x^2)^2}$

$\Rightarrow f'(x) = \frac{(x+1)(2-2x)}{(1+x^2)^2}$ (0.5 point)

$\Rightarrow f'(x) = \frac{2(1-x)(1+x)}{(1+x^2)^2}$ (0.25 point)

$f'(x) = 0 \Rightarrow x = 1, x = -1$ (0.25 point)
Thus 1 and -1 are critical numbers

b) $(1+x^2)^2 > 0$ for all $x \in \mathbb{R}$

$1+x$	$-\infty$	-1	1	∞	
$1-x$		$+$	$+$	$-$	
$f'(x)$		$-$	$+$	$-$	

$\Rightarrow f$ is increasing on $(-1, 1)$ (0.25 point)
and f is decreasing on $(-\infty, -1) \cup (1, \infty)$ (0.25 point)

c) From the table done above,
 f' changes from -ve to +ve at -1
 $\Rightarrow f(-1) = 0 \Rightarrow 0$ is local min value (0.5 point)
 f' changes from +ve to -ve at 1
 $\Rightarrow f(1) = 2 \Rightarrow 2$ is local max value

2 (0.5 point)

d) $f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$

$\Rightarrow f''(x) = \frac{-4x(1+x^2) - 8x(1-x^2)}{(1+x^2)^3}$

$= \frac{-4x(3-x^2)}{(1+x^2)^3}$ (0.25 pt)

$f''(x) = \frac{-4x(\sqrt{3}-x)(\sqrt{3}+x)}{(1+x^2)^3}$

Here $(1+x^2)^3 > 0$ for all $x \in \mathbb{R}$

$-4x$	$-\infty$	$-\sqrt{3}$	0	$\sqrt{3}$	∞	
$\sqrt{3}-x$		$+$	$+$	$+$	$-$	
$\sqrt{3}+x$		$-$	$+$	$+$	$+$	
$f''(x)$		$-$	$+$	$-$	$+$	

$f''(x) > 0$ on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ (0.25)
 $\Rightarrow f$ is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

$f''(x) < 0$ on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$
 $\Rightarrow f$ is concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$ (0.25 pt)

e) $f''(x) = 0$ if $x = 0, \sqrt{3}$ and $-\sqrt{3}$ (0.25 pt)

Thus, $(0, f(0))$, $(\sqrt{3}, f(\sqrt{3}))$ and $(-\sqrt{3}, f(-\sqrt{3}))$ are inflection points

Hence

$\left. \begin{array}{l} (0, 1) \\ (\sqrt{3}, 1+\frac{\sqrt{3}}{2}) \\ (-\sqrt{3}, 1-\frac{\sqrt{3}}{2}) \end{array} \right\}$ are inflection points.

largest interval of values of n for which the improper integral $\int_1^{\infty} \frac{1}{x^n} dx$

converges
Diverges

solution

$$\int_1^{\infty} \frac{1}{x^n} dx = \int_1^{\infty} x^{-n} dx$$

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where $y = (3^x + 4^x)^{\frac{1}{x}}$

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$$= \lim_{x \rightarrow \infty} \frac{\left(\frac{3}{4} \right)^x \ln 3 + \ln 4}{\left(\frac{3}{4} \right)^x + 1} = \ln 4 \quad \leftarrow (1 \text{ point})$$

$$\therefore \lim_{x \rightarrow \infty} \ln y = \ln 4$$

$$y = e^{\ln y} \Rightarrow \lim_{x \rightarrow \infty} (3^x + 4^x)^{\frac{1}{x}} = \lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} e^{\ln y}$$

$$= e^{\lim_{x \rightarrow \infty} \ln y}$$

$$= e^{\ln 4}$$

$$= 4 \quad \leftarrow (1 \text{ point})$$