Trends in Mathematics

Fabrizio Colombo
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Editors

Advances in Complex Analysis and Operator Theory

Festschrift in Honor of Daniel Alpay's 60th Birthday

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Preface

Daniel Alpay graduated in 1986 from the Weizmann Institute of Science with the Ph.D. thesis: "Reproducing kernel Krein spaces of analytic functions and inverse scattering" under the supervision of Prof. Harry Dym. After graduating, he occupied positions at the University of Tel Aviv, at the University of Groningen, at the Virginia Polytechnic Institute in Blacksburg, at the Weizmann Institute of Science in Rehovot, and at INRIA in Valbonne. In 1991 he moved to Ben Gurion University in Beer-Sheva, Israel, where he joined the Department of Mathematics. There he became a tenured member in 1995, and his mathematical career took off from there; later, in December 2005, Daniel was awarded the Earl Katz family chair in algebraic system theory. In August 2016 Daniel was offered the Foster G. and Mary McGaw Professorship in Mathematical Sciences at Chapman University where he was received with open arms.

Apart from his own extensive research, Daniel is the initiator and Editor-in-Chief of the journal "Complex Analysis and Operator Theory" published by Birkhäuser whose first issue appeared in January 2007. Since 2005 he is also a co-editor of a sub-series entitled "Linear Operators and Linear Systems" of the book series "Operator Theory: Advances and Applications" also published by Birkhäuser, as well as a member of the editorial boards of four other journals. His prolific career has produced 230 papers and 6 books and counting, with over 70 collaborators all over the world.

Daniel's current research interests include: Schur analysis in the setting of slice-hyperholomorphic functions, infinite dimensional analysis in the white noise space setting, free analysis, rational functions and applications to linear system theory and wavelets, Schur analysis in the setting of bicomplex functions, operator theory and Riemann surfaces, interpolation theory, reproducing kernel methods in one and several complex variables. This, by any means, is not an exhaustive list.

This volume collects contributions written by Daniel's friends and collaborators. Several of them have participated in the conference *International Conference on Complex Analysis and Operator Theory* held in honor of Daniel's 60th birthday at Chapman University in November 2016. We are grateful to all the authors and to the referees who helped us to form this volume.

Fabrizio Colombo
Irene Sabadini
Daniele C. Struppa
Mihaela B. Vajiac

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Classes of de Branges Matrices and Corresponding de Branges Spaces

Damir Z. Arov and Harry Dym

To Daniel on the occasion of his sixtieth birthday, with our best wishes

Abstract. Reproducing kernel Hilbert spaces $\mathcal{B}(\mathfrak{E})$ of vector-valued entire functions and reproducing kernel

$$K_\omega^{\mathfrak{E}}(\lambda) = \frac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_+(\omega)^*}{-2\pi i(\lambda - \overline{\omega})}$$

based on an entire matrix-valued function $\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix}$ with $p \times p$ blocks $E_\pm(\lambda)$ were introduced and extensively studied by Louis de Branges. In this paper a new subclass of the matrices $\mathfrak{E}(\lambda)$ is introduced and its relation to other subclasses that were presented earlier is discussed.

Mathematics Subject Classification (2000). 46E22, 47B32, 30H99.

Keywords. de Branges spaces, de Branges matrices, reproducing kernels, entire matrix-valued inner functions, reproducing kernel Hilbert spaces.

1. Introduction

An entire $p \times 2p$ mvf (matrix-valued function) $\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda & E_+(\lambda) \end{bmatrix}$ with $p \times p$ blocks $E_\pm(\lambda)$ will be called an **entire dB (de Branges) matrix** if

(1) $\det E_+(\lambda) \not\equiv 0$ in the complex plane $\mathbb{C}$ and
(2) the mvf $\chi(\lambda) = (E_+^{-1}E_-)(\lambda)$ is holomorphic and contractive in the open upper half plane $\mathbb{C}_+$ and unitary on the line $\mathbb{R}$.

Since $E_\pm(\lambda)$ are entire mvf's, the condition in (2) ensures that

$$E_+(\lambda)E_+^\#(\lambda) - E_-(\lambda)E_-^\#(\lambda) = 0 \quad \text{for every point } \lambda \in \mathbb{C},$$

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where $f^{\#}(\lambda) = f(\overline{\lambda})^*$. Moreover, the kernel

$$K_\omega^{\mathfrak{E}}(\lambda) = \frac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^*}{-2\pi i(\lambda - \overline{\omega})}$$

is positive in the sense of (P3) in Section 3 and hence there exists exactly one RKHS (reproducing kernel Hilbert space) $\mathcal{B}(\mathfrak{E})$ of $p \times 1$ entire vvf's (vector-valued functions) with RK (reproducing kernel) $K_\omega^{\mathfrak{E}}(\lambda)$, i.e., for every point $\omega \in \mathbb{C}$, every u in the space $\mathbb{C}^p$ of complex $p \times 1$ vectors and every $f \in \mathcal{B}(\mathfrak{E})$

(1) $K_\omega^{\mathfrak{E}} u \in \mathcal{B}(\mathfrak{E})$ (as a function of λ) and
(2) $\langle f, K_\omega^{\mathfrak{E}} u\rangle_{\mathcal{B}(\mathfrak{E})} = u^* f(\omega)$.

The space $\mathcal{B}(\mathfrak{E})$ will be called the **dB (de Branges) space** corresponding to the dB matrix $\mathfrak{E}$.

The spaces $\mathcal{B}(\mathfrak{E})$ were first introduced by L. de Branges for the case $p = 1$ in an extensive series of papers on Hilbert spaces of entire functions in the late fifties and early sixties that culminated in the monograph [10]. De Branges also considered generalizations to vector-valued spaces of holomorphic functions in [7], [8], [9] and [11].

The spaces $\mathcal{B}(\mathfrak{E})$ are of fundamental importance in the study of direct and inverse problems for canonical systems of differential and integral equations and numerous other problems of analysis; see, e.g., [2], [3] and [6] and the references cited therein; the review papers [4] and [5] may also be helpful for the reader.

In this paper attention is focused on subclasses of a class $\mathcal{I}(j_p)$ of entire dB matrices $\mathfrak{E}$ for which the corresponding RKHS $\mathcal{B}(\mathfrak{E})$ is invariant under the generalized backwards shift operator. An essential role is played by a pair of entire inner mvf's $\{b_3, b_4\}$ which are characterized by the condition $E_-(\overline{\lambda})^* b_3(\lambda)$ and $b_4(\lambda)E_+(\lambda)$ are outer mvf's. A new subclass $\mathcal{I}_{sR}(j_p)$ of **strongly regular dB matrices** of $\mathcal{I}(j_p)$ is introduced and some of its properties, including its relation to other subclasses of $\mathcal{B}(\mathfrak{E})$ that have been presented earlier are discussed. The main new results are presented in Section 6 and in Theorem 5.3.

The rest of this paper is organized as follows: Supplementary notation is presented in Section 2; RKHS's are reviewed briefly in Section 3;the specific RKHS's $\mathcal{B}(\mathfrak{E})$ based on dB matrices $\mathfrak{E}$ and some of their main properties are discussed in Section 4. Sections 5 and 6 focus on subclasses of $\mathcal{B}(\mathfrak{E})$ and finally Section 7 discusses some connections with a class of entire J_p-inner mvf's $A(\lambda)$ and the corresponding RKHS's $\mathcal{H}(A)$ that were not pursued in this paper and references to other related work that was not treated here.

2. Notation

To proceed further we will need some notation. Here and below $\mathbb{C}$ denotes the complex plane, $\mathbb{C}_+$ (resp., $\mathbb{C}_-$) the open upper (resp., lower) half plane, $\mathbb{R}$ the real axis; mvf (resp., vvf)is an acronym for matrix-valued function (resp., vector-valued

function); $M \succeq 0$ (resp., $M \succ 0$) means that M is a positive semi-definite (resp., positive definite) matrix, and cls is an acronym for closed linear span.

The symbols

$$\rho_\omega(\lambda) = -2\pi i(\lambda - \overline{\omega}), \quad f^{\#}(\lambda) = f(\overline{\lambda})^*,$$

and, for mvf's that are holomorphic in a neighborhood of $\alpha \in \mathbb{C}$,

$$(R_\alpha f)(\lambda) = \begin{cases} \dfrac{f(\lambda) - f(\alpha)}{\lambda - \alpha} & \text{for } \lambda \neq \alpha, \\ f'(\alpha) & \text{for } \lambda = \alpha, \end{cases}$$

will be used, as will the signature matrices

$$j_p = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix} \quad \text{and} \quad J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}; \tag{2.1}$$

they are unitarily equivalent:

$$J_p = \mathfrak{V} j_p \mathfrak{V}^*, \quad \text{where} \quad \mathfrak{V} = \mathfrak{V}^* = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix}. \tag{2.2}$$

A $p \times q$ mvf $f(\lambda)$ belongs to the class:

- $L_2^{p\times q}$ if f is measurable on $\mathbb{R}$ and
$$\|f\|_{\rm st}^2 = \int_{-\infty}^{\infty} \operatorname{trace}\{f(\mu)^* f(\mu)\} d\mu < \infty;$$
- $H_2^{p\times q}$ (the **Hardy class**) if it is holomorphic in $\mathbb{C}_+$ and if
$$\|f\|_2^2 = \sup_{\nu>0} \int_{-\infty}^{\infty} \operatorname{trace}\{f(\mu+i\nu)^* f(\mu+i\nu)\} d\mu \ < \ \infty;$$
- $(H_2^{p\times q})^\perp$ if $f^{\#} \in H_2^{q\times p}$ (the superscript $\perp$ is in the notation because $H_2^{p\times q}$ and $(H_2^{p\times q})^\perp$ are orthogonal to each other when regarded as subspaces of $L_2^{p\times q}$);
- $H_\infty^{p\times q}$ if it is holomorphic in $\mathbb{C}_+$ and if
$$\|f\|_\infty = \sup\{\|f(\lambda)\| : \lambda \in \mathbb{C}_+\} < \infty;$$
- $\mathcal{S}^{p\times q}$ (the **Schur class**) if it is in $H_\infty^{p\times q}$ and $\|f\|_\infty \leq 1$;
- $\mathcal{S}_{\rm in}^{p\times q}$ (the class of **inner** $p \times q$ mvf's) if it is in $\mathcal{S}^{p\times q}$ and the limit $f(\mu) = \lim_{\nu\downarrow 0} f(\mu+i\nu)$ (which exists a.e. by a lemma of Fatou) meets the constraint $f(\mu)^* f(\mu) = I_q$ a.e. on $\mathbb{R}$;
- $\mathcal{S}_{\rm out}^{p\times q}$ (the class of **outer** contractive $p \times q$ mvf's) if it is in $\mathcal{S}^{p\times q}$ and fH_2^q is dense in H_2^p;
- $\mathcal{C}^{p\times p}$ (the **Carathéodory class**) if $q = p$, f is holomorphic in $\mathbb{C}_+$ and
$$(\Re f)(\lambda) \ = \ \frac{f(\lambda) + f(\lambda)^*}{2} \ \succeq \ 0$$
for every point $\lambda \in \mathbb{C}_+$;

- $\mathcal{N}^{p\times q}$ (the **Nevanlinna class** of mvf's with bounded Nevanlinna characteristic) if it can be expressed in the form $f = h^{-1}g$, where $g \in \mathcal{S}^{p\times q}$ and $h \in \mathcal{S} \stackrel{\text{def}}{=} \mathcal{S}^{1\times 1}$;
- $\mathcal{N}_+^{p\times q}$ (the **Smirnov class**) if it can be expressed in the form $f = h^{-1}g$, where $g \in \mathcal{S}^{p\times q}$ and $h \in \mathcal{S}_{\text{out}} \stackrel{\text{def}}{=} \mathcal{S}_{\text{out}}^{1\times 1}$;
- $\mathcal{N}_{\text{out}}^{p\times q}$ (the class of **outer** mvf's in $\mathcal{N}^{p\times q}$) if it can be expressed in the form $f = h^{-1}g$, where $g \in \mathcal{S}_{\text{out}}^{p\times q}$ and $h \in \mathcal{S}_{\text{out}}$;
- $\Pi^{p\times q}$ if it belongs to $\mathcal{N}^{p\times q}$ and there exists a $p\times q$ mvf f_- that is meromorphic in $\mathbb{C}_-$ such that $f_-^{\#} \in \mathcal{N}^{q\times p}$ and $\lim_{\nu\downarrow 0} f(\mu+i\nu) = \lim_{\nu\downarrow 0} f_-(\mu - i\nu)$ a.e. on $\mathbb{R}$.
- $\mathcal{E}^{p\times q}$ if it is an entire $p \times q$ mvf;

For each class of $p \times q$ mvf's $\mathcal{X}^{p\times q}$ we shall use the symbols

$$\mathcal{X} \quad \text{instead of } \mathcal{X}^{1\times 1} \quad \text{and} \quad \mathcal{X}^p \quad \text{instead of } \mathcal{X}^{p\times 1}; \tag{2.3}$$

$$\mathcal{X}_{\text{const}}^{p\times q} \quad \text{for the set of mvf's in } \mathcal{X}^{p\times q} \text{ that are constant;}$$

$$\mathcal{E}\cap\mathcal{X}^{p\times q} \quad \text{for the class of entire mvf's in } \mathcal{X}^{p\times q}.$$

3. Reproducing kernel Hilbert spaces

A Hilbert space $\mathcal{H}$ of $n \times 1$ vvf's (vector-valued functions) on a set $\Omega \subseteq \mathbb{C}$ is a RKHS if there exists an $n \times n$ mvf $K_\omega(\lambda)$ on $\Omega \times \Omega$ such that for every choice of λ, $\omega \in \Omega$, $\xi \in \mathbb{C}^n$ and $f \in \mathcal{H}$

(1) The vvf $K_\omega\xi \in \mathcal{H}$.
(2) $\langle f, K_\omega\xi\rangle_{\mathcal{H}} = \xi^* f(\omega)$.

The mvf $K_\omega(\lambda)$ is called a RK (reproducing kernel) for $\mathcal{H}$. The following properties of a RKHS are well known and easily checked:

(P1) $K_\alpha(\beta)^* = K_\beta(\alpha)$.
(P2) A RKHS has exactly one RK.
(P3) A RK is positive in the sense that

$$\sum_{i,j=1}^{n} v_j^* K_{\omega_i}(\omega_j) v_i \geq 0$$

for every choice of points $\omega_1, \dots, \omega_n \in \mathbb{C}$ and vectors $v_1, \dots, v_n \in \mathbb{C}^p$ and every positive integer n.
(P4) $\|f(\omega)\| \leq \|K_\omega(\omega)\|^{1/2}\|f\|_{\mathcal{H}}$.

Conversely, if $K_\omega(\lambda)$ is a positive kernel on $\Omega \times \Omega$ in the sense of (P3), then, by the matrix version of a theorem of Aronszjan (see, e.g., Theorem 5.2 of [2]),

$$\text{there exists exactly one RKHS} \quad \mathcal{H} \quad \text{with } K_\omega(\lambda) \text{ as its RK.} \tag{3.1}$$

In this paper we shall deal primarily with RKHS's of entire vvf's. The next lemma provides useful necessary and sufficient conditions on a kernel in order for it to be the RK of a RKHS of entire vvf's.

Lemma 3.1. *If $\mathcal{H}$ is a RKHS of $m \times 1$ vvf's on some nonempty open subset Ω of $\mathbb{C}$ with RK $K_\omega(\lambda)$ on $\Omega \times \Omega$, then every vvf $f \in \mathcal{H}$ is holomorphic in Ω if and only if the following two conditions are met:*

(1) *$K_\omega(\lambda)$ is a holomorphic function of λ in Ω for every point $\omega \in \Omega$ and*

(2) *the function $K_\omega(\omega)$ is continuous on Ω.*

Proof. See, e.g., Lemma 5.6 in [2]. □

Example 3.2. The Hardy space H_2^p is a RKHS of $p \times 1$ vvf's that are holomorphic in $\mathbb{C}_+$ with RK

$$K_\omega(\lambda) = \frac{I_p}{\rho_\omega(\lambda)} \quad \text{for } \lambda,\, \omega \in \mathbb{C}_+.$$

Example 3.3. The space $(H_2^p)^\perp$ is a RKHS of $p \times 1$ vvf's that are holomorphic in $\mathbb{C}_-$ with RK

$$K_\omega(\lambda) = -\frac{I_p}{\rho_\omega(\lambda)} \quad \text{for } \lambda,\, \omega \in \mathbb{C}_-.$$

Example 3.4. If $b \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, then b is of exponential type, $\det b(\lambda) = e^{i\lambda\tau} b(0)$ for some $\tau \geq 0$ and

$$b^\#(\lambda) b(\lambda) = I_p \quad \text{for every point } \lambda \in \mathbb{C}.$$

Moreover, the space

$$\mathcal{H}(b) = H_2^p \ominus bH_2^p$$

is a RKHS of $p \times 1$ entire vvf's with RK

$$k_\omega^b(\lambda) = \frac{I_p - b(\lambda) b(\omega)^*}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}.$$

Example 3.5. If $b \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, then, in view of the preceding example, the kernel

$$\ell_\omega^b(\lambda) = b^\#(\lambda) k_\omega^b(\lambda) b^\#(\omega)^* = \frac{b^\#(\lambda) b^\#(\omega)^* - I_p}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}$$

is also positive in the sense of (P3) and may be identified as the RK for the space

$$\mathcal{H}_*(b) = (H_2^p)^\perp \ominus b^\#(H_2^p)^\perp = b^\# \mathcal{H}(b)$$

of $p \times 1$ entire vvf's.

Example 3.6. If $b_3,\, b_4 \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, then $\mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3)$ is a RKHS of $p \times 1$ entire vvf's with RK

$$\ell_\omega^{b_4}(\lambda) + k_\omega^{b_3}(\lambda) = \frac{b_4^\#(\lambda) b_4^\#(\omega)^* - b_3(\lambda) b_3(\omega)^*}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}.$$

4. Entire de Branges matrices $\mathfrak{E}$ and de Branges spaces $\mathcal{B}(\mathfrak{E})$

Recall that an entire $p \times 2p$ mvf $\mathcal{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix}$ with $p \times p$ blocks $E_\pm$ will be called an entire dB matrix if

$$\det E_+(\lambda) \not\equiv 0 \quad \text{and} \quad E_+^{-1}E_- \in \mathcal{S}_{\text{in}}^{p\times p}. \tag{4.1}$$

If $\mathfrak{E}$ is a dB matrix, then the set of entire $p \times 1$ vvf's that meet the constraints

$$E_+^{-1}f \in H_2^p \quad \text{and} \quad E_-^{-1}f \in (H_2^p)^\perp \tag{4.2}$$

is a RKHS $\mathcal{B}(\mathfrak{E})$ with RK

$$K_\omega^{\mathcal{E}}(\lambda) = \begin{cases} \dfrac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^*}{\rho_\omega(\lambda)} = -\dfrac{\mathfrak{E}(\lambda)j_p\mathfrak{E}(\omega)^*}{\rho_\omega(\lambda)} & \text{if } \lambda \neq \overline{\omega}, \\ -\frac{1}{2\pi i}\{E_+'(\overline{\omega})E_+(\omega)^* - E_-'(\overline{\omega})E_-(\omega)^*\} & \text{if } \lambda = \overline{\omega}, \end{cases} \tag{4.3}$$

with respect to the inner product

$$\langle f, g\rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(\mu)^*\Delta_{\mathfrak{E}}(\mu)f(\mu)d\mu, \tag{4.4}$$

where

$$\Delta_{\mathfrak{E}}(\mu) = \{E_+(\mu)E_+(\mu)^*\}^{-1} = \{E_-(\mu)E_-(\mu)^*\}^{-1}$$

at points $\mu \in \mathbb{R}$ at which $\det E_\pm(\mu) \neq 0$; see, e.g., Section 4.10 in [3] and Section 3.2 in [6].

Remark 4.1. Since $\mathfrak{E}(\lambda)$ is an entire mvf, the second condition in (4.1) implies that

$$E_+(\lambda)E_+^\#(\lambda) - E_-(\lambda)E_-^\#(\lambda) = 0 \quad \text{for every point } \lambda \in \mathbb{C}. \tag{4.5}$$

Moreover, the mvf $\chi = E_+^{-1}E_-$, which belongs to $\mathcal{S}_{\text{in}}^{p\times p}$ by definition, extends as a meromorphic mvf in $\mathbb{C}$ and the formulas

$$V_1 f = E_+^{-1}f \quad \text{and} \quad V_2 f = E_-^{-1}f$$

define unitary operators from $\mathcal{B}(\mathfrak{E})$ onto the RKHS's $\mathcal{H}(\chi)$ and $\mathcal{H}_*(\chi)$, respectively. The RK's of these spaces are related by the formulas

$$K_\omega^{\mathfrak{E}}(\lambda) = E_+(\lambda)k_\omega^\chi(\lambda)E_+(\omega)^* \quad \text{for } \lambda,\, \omega \in \mathfrak{h}_\chi \tag{4.6}$$

and

$$K_\omega^{\mathfrak{E}}(\lambda) = E_-(\lambda)\ell_\omega^\chi(\lambda)E_-(\omega)^* \quad \text{for } \lambda,\, \omega \in \mathfrak{h}_{\chi^\#}, \tag{4.7}$$

where $\mathfrak{h}_\chi$ (resp., $\mathfrak{h}_{\chi^\#}$) denotes the domain of holomorphy of χ (resp., $\chi^\#$) in $\mathbb{C}$. $\diamond$

An entire dB matrix $\mathfrak{E}$ belongs to the class $\mathcal{I}(j_p)$ if

$$(\rho_i E_-^\#)^{-1} \in H_2^{p\times p} \quad \text{and} \quad (\rho_i E_+)^{-1} \in H_2^{p\times p}. \tag{4.8}$$

Lemma 4.2. *If $\mathfrak{E} \in \mathcal{I}(j_p)$, then*

(1) $\det E_+(\lambda) \neq 0$ *for every point $\lambda \in \overline{\mathbb{C}_+}$ and E_+^{-1} is holomorphic in $\overline{\mathbb{C}_+}$; $\det E_-(\lambda) \neq 0$ for every point $\lambda \in \overline{\mathbb{C}_-}$ and E_-^{-1} is holomorphic in $\overline{\mathbb{C}_-}$. .*
(2) *The dB space $\mathcal{B}(\mathfrak{E})$ is R_α-invariant for every point $\alpha \in \mathbb{C}$.*
(3) *$\mathfrak{E} \in \mathcal{E} \cap \Pi^{p\times 2p}$ and $\mathcal{B}(\mathfrak{E}) \subset \mathcal{E} \cap \Pi^p$.*

(4) $R_\alpha E_+\eta \in \mathcal{B}(\mathfrak{E})$ *and* $R_\alpha E_-\eta \in \mathcal{B}(\mathfrak{E})$ *for every* $\eta \in \mathbb{C}^p$ *and* $\alpha \in \mathbb{R}$.
(5) *The subspaces*

$$\mathcal{N}_\alpha \stackrel{\text{def}}{=} \{u \in \mathbb{C}^p : K_\alpha^{\mathfrak{E}}(\alpha)u = 0\} \quad \text{and} \quad \mathcal{R}_\alpha \stackrel{\text{def}}{=} \{K_\alpha^{\mathfrak{E}}(\alpha)u : u \in \mathbb{C}^p\} \tag{4.9}$$

are independent of α,

$$\mathbb{C}^p = \mathcal{N}_\alpha \oplus \mathcal{R}_\alpha \tag{4.10}$$

and

$$\mathcal{R}_\alpha = \{f(\alpha) : f \in \mathcal{B}(\mathfrak{E})\}. \tag{4.11}$$

(6) $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0 \iff \mathcal{R}_\alpha = \mathbb{C}^p \iff \mathcal{N}_\alpha = \{0\}$.
(7) $K_\omega^{\mathfrak{E}}(\omega) \succ 0$ *for at least one point* $\omega \in \mathbb{C}$ *if and only if* $K_\omega^{\mathfrak{E}}(\omega) \succ 0$ *for every point* $\omega \in \mathbb{C}$.

Moreover, if $\mathfrak{E}$ *is an entire dB matrix and there exists at least one point* $\alpha \in \mathbb{C}$ *such that*

(a) $\mathcal{B}(\mathfrak{E})$ *is* R_α *invariant and* $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$,

or, there exists a pair of points α, $\beta \in \mathbb{C}^p$ *such that*

(b) $E_+(\alpha)$ *is invertible,* $E_-(\beta)$ *is invertible,* $R_\alpha E_+\eta \in \mathcal{B}(\mathfrak{E})$ *and* $R_\beta E_-\eta \in \mathcal{B}(\mathfrak{E})$ *for every* $\eta \in \mathbb{C}^p$,

then $\mathfrak{E} \in \mathcal{I}(j_p)$.

Proof. See Lemma 3.19 in [6]. □

A nondecreasing $p \times p$ mvf $\sigma(\mu)$ on $\mathbb{R}$ is called a **spectral function** for a dB space $\mathcal{B}(\mathfrak{E})$ based on a dB matrix $\mathfrak{E} \in \mathcal{I}(j_p)$ if

$$\langle g, g\rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(\mu)^*\, d\sigma(\mu)\, g(\mu) \quad \text{for every } g \in \mathcal{B}(\mathfrak{E}).$$

The set of spectral functions for $\mathcal{B}(\mathfrak{E})$ will be denoted $(\mathcal{B}(\mathfrak{E}))_{sf}$. If $\sigma \in (\mathcal{B}(\mathfrak{E}))_{sf}$ is locally absolutely continuous and $\Delta(\mu) = \sigma'(\mu)$ a.e. is such that

$$\langle g, g\rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(\mu)^*\, \Delta(\mu)\, g(\mu) d\mu \quad \text{for every } g \in \mathcal{B}(\mathfrak{E}),$$

then Δ will be called a **spectral density** for $\mathcal{B}(\mathfrak{E})$. Since

$$\mathfrak{E} \in \mathcal{I}(j_p) \implies \int_{-\infty}^{\infty} \frac{\Delta_{\mathfrak{E}}(\mu)}{1+\mu^2} d\mu \quad \text{is finite,}$$

the function

$$\sigma_{\mathfrak{E}}(\mu) = \int_0^{\mu} \Delta_{\mathfrak{E}}(\nu) d\nu$$

is a spectral function for $\mathcal{B}(\mathfrak{E})$ and $\Delta_{\mathfrak{E}}$ is a spectral density for $\mathcal{B}(\mathfrak{E})$.

Lemma 4.3. *If* $\mathfrak{E} \in \mathcal{I}(j_p)$ *and* $K_\omega^{\mathfrak{E}}(\omega) \succ 0$ *for at least one point* $\omega \in \mathbb{C}$, *then*

$$\int_{-\infty}^{\infty} \operatorname{trace}\left\{\frac{d\sigma(\mu)}{1+\mu^2}\right\} < \infty$$

for every $\sigma \in (\mathcal{B}(\mathfrak{E}))_{sf}$.

Proof. Let $f \in \mathcal{B}(\mathfrak{E})$. Then, as

$$\frac{f(i)}{\lambda - i} = \frac{f(\lambda)}{\lambda - i} - (R_i f)(\lambda),$$

it is readily checked that

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(i)^* \frac{d\sigma(\mu)}{1+\mu^2} f(i) \\
&\le 2\int_{-\infty}^{\infty} f(\mu)^* \frac{d\sigma(\mu)}{1+\mu^2} f(\mu) + 2\int_{-\infty}^{\infty} (R_i f)(\mu)^* d\sigma(\mu)(R_i f)(\mu) \\
&\le 2\|f\|^2_{\mathcal{B}(\mathfrak{E})} + 2\|R_i f\|^2_{\mathcal{B}(\mathfrak{E})}.
\end{aligned}$$

The asserted bound follows from the fact that $\{f(i) : f \in \mathcal{B}(\mathfrak{E})\} = \mathbb{C}^p$, by Lemma 4.2. □

Example 4.4. If b_1, $b_2 \in \mathcal{E} \cap \mathcal{S}^{p\times p}_{\rm in}$ and $\varphi \in \mathcal{E}^{p\times p}$ with $\det \varphi(\lambda) \not\equiv 0$, then

$$\mathfrak{E} = \begin{bmatrix} \varphi b_1 & \varphi b_2^{-1} \end{bmatrix} \quad \text{is a dB matrix and} \quad \Delta_{\mathfrak{E}}(\mu) = \{\varphi(\mu)\varphi(\mu)^*\}^{-1} \text{ a.e. on } \mathbb{R}.$$

If, for example, $\varphi(\lambda) = \lambda I_p$, then $\Delta_{\mathfrak{E}}(\mu) = \mu^{-2} I_p$ for $\mu \neq 0$. Moreover,

(1) if $\mathfrak{E} = \begin{bmatrix} b_1 & b_2^{-1} \end{bmatrix}$, then $\mathcal{B}(\mathfrak{E}) = \mathcal{H}_*(b_1) \oplus \mathcal{H}(b_2)$.
(2) if $\mathfrak{E} = \begin{bmatrix} b_1 & I_p \end{bmatrix}$, then $\mathcal{B}(\mathfrak{E}) = \mathcal{H}_*(b_1)$.
(3) if $\mathfrak{E} = \begin{bmatrix} I_p & b_2^{-1} \end{bmatrix}$, then $\mathcal{B}(\mathfrak{E}) = \mathcal{H}(b_2)$. ◇

A theorem of M.G. Krein (see e.g., Theorem 3.91 in [2]) guarantees that if $f \in \mathcal{E} \cap \Pi^{p\times q}$, then f has finite exponential type

$$\tau_f = \limsup_{r\uparrow\infty} \left\{ \frac{\ln \|f(\lambda)\|}{r} : |\lambda| < r \right\} < \infty$$

and that

$$\tau_f = \max\{\tau_f^-, \tau_f^+\} \quad \text{where} \quad \tau_f^{\pm} = \limsup_{\nu\uparrow\infty} \left\{ \frac{\ln \|f(\pm i\nu)\|}{\nu} \right\}.$$

The next example demonstrates that the inclusions

$$\mathcal{I}(j_p) \subset \{\text{dB matrices } \mathfrak{E} \in \mathcal{E} \cap \Pi^{p\times 2p}\} \subset \{\text{dB matrices } \mathfrak{E} \in \mathcal{E}^{p\times 2p}\}$$

are proper.

Example 4.5. Let $\mathfrak{E} = \begin{bmatrix} \varphi b_1 & \varphi b_2^{-1} \end{bmatrix}$ be the dB matrix with b_1, $b_2 \in \mathcal{E} \cap \mathcal{S}^{p\times p}_{\rm in}$, $\varphi \in \mathcal{E}^{p\times p}$ and $\det \varphi(\lambda) \not\equiv 0$.

$$\text{If } \varphi(\lambda) = \lambda I_p, \text{ then } \mathfrak{E} \in \mathcal{E} \cap \Pi^{p\times 2p} \text{ but } \mathfrak{E} \notin \mathcal{I}(j_p),$$

since $b_2(\varphi\rho_i)^{-1} \notin H_2^{p\times p}$. On the other hand, if

$$\varphi(\lambda) = e^{\lambda^2} I_p \quad \text{then } \mathfrak{E} \text{ is an entire dB matrix but } \mathfrak{E} \notin \Pi^{p\times 2p}$$

because $\mathfrak{E}$ is not of exponential type. ◇

If $\mathfrak{E} \in \mathcal{I}(j_p)$, then, in view of (4.8), there exist a pair of mvf's b_3, $b_4 \in \mathcal{S}_{\text{in}}^{p\times p}$ and $\varphi_3, \varphi_4 \in \mathcal{N}_{\text{out}}^{p\times p}$ such that

$$(E_-^{\#})^{-1} = b_3\varphi_3 \quad \text{and} \quad (E_+)^{-1} = \varphi_4 b_4. \tag{4.12}$$

The pair $\{b_3, b_4\}$ is uniquely determined by (4.12) up to $p \times p$ constant unitary multipliers on the right and left, respectively. The set

$$\{(b_3u, vb_4) : u \text{ and } v \text{ are unitary } p \times p \text{ matrices}\}$$

is called the set of **associated pairs of $\mathfrak{E}$** and is denoted $ap(\mathfrak{E})$;

$$\text{the associated pairs of } \mathfrak{E} \in \mathcal{I}(j_p) \text{ are entire mvf's.} \tag{4.13}$$

The verification of (4.13) follows from [1] and the fact that $ap(\mathfrak{E})$ coincides with the set of associated pairs of the second kind of a mvf A that belongs to the class $\mathcal{E} \cap \mathcal{U}(J_p)$ that is considered briefly in Section 7; see especially (7.2) and (7.1) and, for additional discussion, Theorem 4.54 in [2]).

Lemma 4.6. *If $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ belongs to the class $\mathcal{I}(j_p)$, then*

(1) $\tau_{\mathfrak{E}}^+ = \max\{\tau_f^+ : f \in \mathcal{B}(\mathfrak{E})\}$ *and* $\tau_{\mathfrak{E}}^- = \max\{\tau_f^- : f \in \mathcal{B}(\mathfrak{E})\}$*;*

(2) $\tau_{\mathfrak{E}}^+ = \tau_{E_+}^+ = \tau_{b_4}$ *and* $\tau_{\mathfrak{E}}^- = \tau_{E_-}^- = \tau_{b_3}$.

Proof. See Lemma 3.41 in[1] [6]. □

Let $\mathcal{U}_{\text{const}}(j_p)$ denote the set of $V \in \mathbb{C}^{2p\times 2p}$ such that $V^* j_p V = j_p$.

Lemma 4.7. *If $\mathfrak{E}$ and $\widetilde{\mathfrak{E}}$ are entire dB matrices and $K_0^{\mathfrak{E}}(0) \succ 0$, then*

$$\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\widetilde{\mathfrak{E}}) \iff \widetilde{\mathfrak{E}}(\lambda) = \mathfrak{E}(\lambda)V \quad \textit{for some } V \in \mathcal{U}_{\text{const}}(j_p). \tag{4.14}$$

Proof. The implication $\Longleftarrow$ is obvious; a proof of the opposite implication $\Longrightarrow$ is furnished in Theorem 3.22 of [6]. □

The set of matrices

$$V_\alpha = \begin{bmatrix} I_p + i\alpha & i\alpha \\ -i\alpha & I_p - i\alpha \end{bmatrix} \quad \text{with } \alpha = \alpha^* \in \mathbb{C}^{p\times p} \tag{4.15}$$

is a subgroup of the multiplicative group $\mathcal{U}_{\text{const}}(j_p)$ with the property

$$V_\alpha\, V_\beta = V_{\alpha+\beta}. \tag{4.16}$$

Lemma 4.8. *If $V \in \mathcal{U}_{\text{const}}(j_p)$, then*

$$\begin{bmatrix} I_p & I_p \end{bmatrix} V = \begin{bmatrix} I_p & I_p \end{bmatrix} \Longleftrightarrow V = V_\alpha$$

for some Hermitian matrix $\alpha \in \mathbb{C}^{p\times p}$.

Proof. See Lemma 3.11 in [6]. □

Let

$$\mathcal{I}^\circ(j_p) = \{\mathfrak{E} \in \mathcal{I}(j_p) : \mathfrak{E}(0) = \begin{bmatrix} I_p & I_p \end{bmatrix}\}.$$

[1]The enumeration of some of the items referred to in [6] might shift a little in the final edition.

Lemma 4.9. *If $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ is an entire dB matrix, then there exists a matrix $V \in \mathcal{U}_{\rm const}(j_p)$ such that $(\mathfrak{E}\, V)(0) = \begin{bmatrix} I_p & I_p \end{bmatrix}$ if and only if $E_+(0)$ is invertible.*

Proof. See Theorem 3.12 in [6]. □

Lemma 4.10. *If $\mathcal{E}$, $\mathfrak{E}_1 \in \mathcal{I}(j_p)$ and $\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\mathfrak{E}_1)$, then $ap(\mathfrak{E}) = ap(\mathfrak{E}_1)$.*

Proof. This is Corollary 3.17 in [6]. □

5. Subclasses of $\mathcal{I}_R(j_p)$

A Hilbert space $\mathcal{H}_1$ is said to be **contractively included** in a Hilbert space $\mathcal{H}_2$ if

$$f \in \mathcal{H}_1 \implies f \in \mathcal{H}_2 \quad \text{and} \quad \|f\|_{\mathcal{H}_2} \le \|f\|_{\mathcal{H}_1};$$

the indicated inclusion is said to be **isometric** if

$$\|f\|_{\mathcal{H}_2} = \|f\|_{\mathcal{H}_1} \quad \text{for every } f \in \mathcal{H}_1.$$

The notation $\mathcal{H}_1 \sim \mathcal{H}_2$ means that the two Hilbert spaces coincide as vector spaces and have equivalent norms, i.e., there exist a pair of constants $\gamma_1 > 0$ and $\gamma_2 \ge \gamma_1$ such that

$$\gamma_1 \|f\|_{\mathcal{H}_1} \le \|f\|_{\mathcal{H}_2} \le \gamma_2 \|f\|_{\mathcal{H}_1}.$$

If $\mathfrak{E} \in \mathcal{I}(j_p)$ and $\{b_3, b_4\} \in ap(\mathfrak{E})$, then $\mathfrak{E}$ belongs to the subclass

$\mathcal{I}_S(j_p)$ if b_3 and b_4 are constant unitary matrices;

$\mathcal{I}_R(j_p)$ if $\mathfrak{E}_1 \in \mathcal{I}(j_p)$, $\{b_3, b_4\} \in ap(\mathfrak{E}_1)$ and $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is included isometrically in $\mathcal{B}(\mathfrak{E})$, then $\mathcal{B}(\mathfrak{E}_1) = \mathcal{B}(\mathfrak{E})$;

$\mathcal{I}_{AR}(j_p)$ if every mvf $\mathfrak{E}_1$ for which $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is isometric belongs to the class $\mathcal{I}_R(j_p)$.

There exist other characterizations of these subclasses. Thus, for example,

$$\mathcal{I}_S(j_p) = \{\mathfrak{E} \in \mathcal{I}(j_p) : \tau_{\mathfrak{E}} = 0\} = \{\begin{bmatrix} E_- & E_+ \end{bmatrix} \in \mathcal{I}(j_p) : E_-^{\#}, E_+ \in \mathcal{N}_{\rm out}^{p\times p}\}.$$

Example 5.1. If $p(\lambda)$ is a polynomial of degree ≥ 1 with all its zeros in $\mathbb{C}_-$, then the mvf $\mathfrak{E}(\lambda) = \begin{bmatrix} p^{\#}(\lambda) I_p & p(\lambda) I_p \end{bmatrix}$ belongs to the class $\mathcal{E} \cap \mathcal{I}_S(j_p)$.

If b_3, $b_4 \in \mathcal{E} \cap \mathcal{S}_{\rm in}^{p\times p}$ and $b_4(\lambda) b_3(\lambda)$ is not constant and $p(\lambda)$ is as above, then the mvf $\mathfrak{E}(\lambda) = \begin{bmatrix} p^{\#}(\lambda) b_3(\lambda) & p(\lambda) b_4(\lambda)^{-1} \end{bmatrix}$ belongs to the class $\mathcal{E} \cap \mathcal{I}(j_p)$ but does not belong to the class $\mathcal{I}_S(j_p)$ or to the class $\mathcal{I}_R(j_p)$. ◇

The next theorem will be used to help establish Theorem 5.3. It follows from the connections between dB matrices $\mathfrak{E}$ and J_p-inner mvf's $A(\lambda)$ that will be discussed briefly in Section 7; complete proofs will be presented in [6].

Theorem 5.2. *If $\mathfrak{E} = \mathcal{I}(j_p)$ and $\{b_3, b_4\} \in ap(\mathfrak{E})$, then:*

(1) *Every closed R_0-invariant subspace $\mathcal{L}$ of $\mathcal{B}(\mathfrak{E})$ is equal to a dB space $\mathcal{B}(\mathfrak{E}_1)$ for some dB matrix $\mathfrak{E}_1 \in \mathcal{I}(j_p)$ (and hence $\|f\|_{\mathcal{B}(\mathfrak{E}_1)} = \|f\|_{\mathcal{B}(\mathfrak{E})}$ for every $f \in \mathcal{B}(\mathfrak{E}_1)$).*

(2) *If* $\mathfrak{E}_1 \in \mathcal{I}(j_p)$, $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$ *and* $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$, *then*

$$(b_3^{(1)})^{-1} b_3 \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p} \quad \text{and} \quad b_4 (b_4^{(1)})^{-1} \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}.$$

(3) *If the mvf's* $b_3^{(1)}$, $b_4^{(1)}$, $(b_3^{(1)})^{-1} b_3$ *and* $b_4 (b_4^{(1)})^{-1}$ *all belong to the class* $\mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, *then there exists a dB matrix* $\mathfrak{E}_1 \in \mathcal{I}_R(j_p)$ *such that* $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ *isometrically and* $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$. *Moreover, this dB matrix is unique up to a constant* j_p*-unitary factor* V *on the right.*

Theorem 5.3. *If* $\mathfrak{E} \in \mathcal{I}(j_p)$, $\{b_3, b_4\} \in ap(\mathfrak{E})$ *and the four mvf's* $b_3^{(1)}$, $(b_3^{(1)})^{-1} b_3$, $b_4^{(1)}$ *and* $b_4 (b_4^{(1)})^{-1}$ *all belong to the class* $\mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, *then:*

(1) $k_\omega^{b_3^{(1)}} \xi \in \mathcal{B}(\mathfrak{E})$ *and* $\ell_\omega^{b_4^{(1)}} \xi \in \mathcal{B}(\mathfrak{E})$ *for every* $\xi \in \mathbb{C}^p$ *and* $\omega \in \mathbb{C}$.

(2) *The space*

$$\mathcal{L} = \operatorname{cls}\{k_\alpha^{b_3^{(1)}} \xi + \ell_\beta^{b_4^{(1)}} \eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \quad \text{in } \mathcal{B}(\mathfrak{E}) \tag{5.1}$$

is a dB space $\mathcal{B}(\mathfrak{E}_1)$ *based on a dB matrix* $\mathfrak{E}_1 \in \mathcal{I}(j_p)$ *and hence* $\|f\|_{\mathcal{B}(\mathfrak{E}_1)} = \|f\|_{\mathcal{B}(\mathfrak{E})}$ *for every* $f \in \mathcal{B}(\mathfrak{E}_1)$, *i.e.,* $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ *and the inclusion is isometric.*

(3) $\mathfrak{E}_1 \in \mathcal{I}_R(j_p)$ *and* $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$.

(4) $\mathfrak{E} \in \mathcal{I}_R(j_p)$ *if and only if*

$$\mathcal{B}(\mathfrak{E}) = \operatorname{cls}\{k_\alpha^{b_3} \xi + \ell_\beta^{b_4} \eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \quad \text{in } \mathcal{B}(\mathfrak{E}). \tag{5.2}$$

Proof. The proof is divided into steps.

1. *If* $b \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, *then* $\rho_i k_\omega^b \in H_\infty^{p\times p}$ *and* $\rho_i (\ell_\omega^b)^\# \in H_\infty^{p\times p}$ *for each choice of* $\omega \in \mathbb{C}$.

Since $\rho_i(\lambda) k_\omega^b(\lambda)$ is an entire mvf of λ for each choice of $\omega \in \mathbb{C}$,

$$\|\rho_i k_\omega^b\| \le c_1 < \infty \quad \text{for } |\lambda - \overline{\omega}| \le 1.$$

On the other hand, if $\lambda = \overline{\omega} + \beta$ with $|\beta| > 1$, then

$$\left| \frac{\rho_i(\lambda)}{\rho_\omega(\lambda)} \right| = \left| \frac{\lambda + i}{\lambda - \overline{\omega}} \right| = \frac{|\overline{\omega} + \beta + i|}{|\beta|} \le 1 + |\overline{\omega} + i|.$$

Therefore,

$$\rho_i k_\omega^b = \frac{\rho_i}{\rho_\omega} \{I_p - b b(\omega)^*\} \in H_\infty^{p\times p}.$$

The verification of the second assertion is similar.

2. $(b_3^{(1)})^\# (\rho_i E_-^\#)^{-1} \in H_2^{p\times p}$ *and* $(\rho_i E_+)^{-1} (b_4^{(1)})^\# \in H_2^{p\times p}$. In view of (4.12),

$$E_-^\# = \varphi_3^{-1} b_3^\# \quad \text{and} \quad E_+ = b_4^\# \varphi_4^{-1}$$

with φ_3 and φ_4 in $\mathcal{N}_{\text{out}}^{p\times p}$. Therefore, since

$$(\rho_i E_+)^{-1} \in H_2^{p\times p} \quad \text{and} \quad (\rho_i E_-^\#)^{-1} \in H_2^{p\times p},$$

the mvf's

$$(\rho_i E_+)^{-1} b_4^{\#} = \rho_i^{-1}\varphi_4 \quad \text{and} \quad b_3^{\#}(\rho_i E_-^{\#})^{-1} = \rho_i^{-1}\varphi_3$$

belong to $L_2^{p\times p} \cap \mathcal{N}_+^{p\times p}$. Thus, the Smirnov maximum principle guarantees that

$$\rho_i^{-1}\varphi_3 \in H_2^{p\times p} \quad \text{and} \quad \rho_i^{-1}\varphi_4 \in H_2^{p\times p} \tag{5.3}$$

and hence that $b_3^{\#}(\rho_i E_-^{\#})^{-1} \in H_2^{p\times p}$ and $(\rho_i E_+)^{-1} b_4^{\#} \in H_2^{p\times p}$. But this implies the assertion of Step 2, since $(b_3^{(1)})^{-1} b_3 \in \mathcal{E} \cap \mathcal{S}_{\rm in}^{p\times p}$ and $b_4 (b_4^{(1)})^{-1} \in \mathcal{E} \cap \mathcal{S}_{\rm in}^{p\times p}$.

3. *Verification of* (1). The proof is based on the fact that

$$f \in \mathcal{B}(\mathfrak{E}) \iff E_+^{-1} f \in H_2^p \quad \text{and} \quad E_-^{-1} f \in (H_2^p)^{\perp}.$$

In view of Steps 1 and 2, it is easily seen that

$$E_+^{-1} k_\omega^{b_3^{(1)}} \xi = (\rho_i E_+)^{-1} \rho_i k_\omega^{b_3^{(1)}} \xi \in H_2^p \tag{5.4}$$

for every choice of $\omega \in \mathbb{C}$ and $\xi \in \mathbb{C}^p$. Moreover, by a self-evident variant of Step 1 and Step 2,

$$E_-^{-1} k_\omega^{b_3^{(1)}} \xi = \varphi_3^{\#} b_3^{-1} k_\omega^{b_3^{(1)}} \xi = \{\rho_{-i}^{-1}\varphi_3^{\#}\}\,\{\rho_{-i} b_3^{-1} k_\omega^{b_3^{(1)}} \xi\} \in (H_2^p)^{\perp}$$

for every $\omega \in \mathbb{C}$ and $\xi \in \mathbb{C}^p$ (keep (5.3) in mind). This completes the proof of the first assertion in (1); the proof of the second is similar.

4. *Verification of* (2). We shall assume that $b_3^{(1)}(0) = b_4^{(1)}(0) = I_p$. In view of Step 1, $\mathcal{L}$ is a closed subspace of $\mathcal{B}(\mathfrak{E})$ and hence is automatically isometrically included in $\mathcal{B}(\mathfrak{E})$. Moreover, since

$$k_\omega^{b_3^{(1)}}(\lambda) = \frac{I_p - b_3^{(1)}(\lambda) b_3^{(1)}(\omega)^*}{\rho_\omega(\lambda)} = \frac{1}{2\pi i}(R_{\overline{\omega}} b_3^{(1)})(\lambda) b_3^{(1)}(\omega)^*,$$

the resolvent identity $R_\alpha - R_\beta = (\alpha - \beta) R_\alpha R_\beta$ with $\alpha = 0$ and $\beta = \overline{\omega}$ implies that

$$\begin{aligned}(R_0 k_\omega^{b_3^{(1)}})(\lambda) &= \frac{i}{2\pi\overline{\omega}}\{(R_0 b_3^{(1)})(\lambda) - (R_{\overline{\omega}} b_3^{(1)})(\lambda)\} b_3^{(1)}(\omega)^* \\ &= \frac{k_0^{b_3^{(1)}}(\lambda) b_3^{(1)}(\omega)^* - k_\omega^{b_3^{(1)}}(\lambda)}{-\overline{\omega}} \quad \text{if } \omega \neq 0,\end{aligned}$$

and hence that R_0 maps finite sums of the form $\sum k_{\omega_j}^{b_3^{(1)}} \xi_j$ (with $\omega_j \neq 0$) into finite sums of the same form. Similarly, since

$$\ell_\omega^{b_4^{(1)}}(\lambda) = \frac{(b_4^{(1)})^{\#}(\lambda)(b_4^{(1)})^{\#}(\omega)^* - I_p}{\rho_\omega(\lambda)} = -\frac{1}{2\pi i}(R_{\overline{\omega}}(b_4^{(1)})^{\#})(\lambda)(b_4^{(1)})^{\#}(\omega)^*$$

$$\begin{aligned}R_0 \ell_\omega^{b_4^{(1)}} &= -\frac{1}{2\pi i}(R_0 R_{\overline{\omega}}(b_4^{(1)})^{\#})(\lambda)(b_4^{(1)})^{\#}(\omega)^* \\ &= -\frac{1}{2\pi i\overline{\omega}}\left\{(R_{\overline{\omega}}(b_4^{(1)})^{\#})(\lambda) - (R_0(b_4^{(1)})^{\#})(\lambda)\right\} b_4^{(1)}(\overline{\omega}),\end{aligned}$$

R_0 maps finite sums of the form $\sum \ell_{\omega_j}^{b_4^{(1)}} \eta_j$ (with $\omega_j \neq 0$) into finite sums of the same form. Thus, as such sums are dense in $\mathcal{L}$ and R_0 is a bounded operator on $\mathcal{B}(\mathfrak{E})$, $\mathcal{L}$ is invariant under R_0. Therefore, by Theorem 5.2, $\mathcal{L} = \mathcal{B}(\mathfrak{E}_1)$ for some dB matrix $\mathfrak{E}_1 \in \mathcal{I}(j_p)$.

5. *Verification of* (3). Since the blocks in $\mathfrak{E}_1 = \begin{bmatrix} E_-^{(1)} & E_+^{(1)} \end{bmatrix}$ admit factorizations of the form

$$E_-^{(1)} = \widetilde{b}_3 \widetilde{\varphi}_3^{\#} \quad \text{and} \quad E_+^{(1)} = (\widetilde{b}_4)^{-1} \widetilde{\varphi}_4 \quad \text{with } \widetilde{\varphi}_3,\ \widetilde{\varphi}_4 \in \mathcal{N}_{\text{out}}^{p\times p}$$

and $k_\omega^{b_3^{(1)}} \xi$, $\ell_\omega^{b_4^{(1)}} \xi \in \mathcal{B}(\mathfrak{E}_1)$ for every choice of $\omega \in \mathbb{C}$ and $\xi \in \mathbb{C}^p$,

$$(E_+^{(1)})^{-1} \ell_\omega^{b_4^{(1)}} \xi = (\widetilde{\varphi}_4)^{-1} \widetilde{b}_4 \ell_\omega^{b_4^{(1)}} \xi \in H_2^p \subset \mathcal{N}_+^{p\times p} \tag{5.5}$$

and

$$(E_-^{(1)})^{-1} k_\omega^{b_3^{(1)}} \xi = (\widetilde{\varphi}_3)^{-\#} (\widetilde{b}_3)^{-1} k_\omega^{b_3^{(1)}} \xi \in (H_2^p)^\perp \tag{5.6}$$

for every $\omega \in \mathbb{C}$ and $\xi \in \mathbb{C}^p$. In particular, (5.5) implies that $\widetilde{b}_4 \ell_\omega^{b_4^{(1)}} \xi \in L_2^p \cap \mathcal{N}_+^p = H_2^p$ and hence that

$$\ell_\omega^{b_4^{(1)}} \xi \in (\widetilde{b}_4^{\#} H_2^p) \cap (H_2^p)^\perp = \mathcal{H}_*(\widetilde{b}_4)$$

and, by a similar argument based on (5.6), that

$$k_\omega^{b_3^{(1)}} \xi \in \mathcal{H}(\widetilde{b}_3) \quad \text{for every } \omega \in \mathbb{C} \text{ and } \xi \in \mathbb{C}^p.$$

Thus,

$$\ell_\omega^{b_4^{(1)}}(\omega) \preceq \ell_\omega^{\widetilde{b}_4}(\omega) \quad \text{and} \quad k_\omega^{b_3^{(1)}}(\omega) \preceq k_\omega^{\widetilde{b}_3}(\omega) \quad \text{for every point } \omega \in \mathbb{C}.$$

But this in turn leads easily to the inequalities

$$(b_4^{(1)})^{\#}(\omega)(b_4^{(1)})^{\#}(\omega)^* \preceq \widetilde{b}_4^{\#}(\omega)\widetilde{b}_4^{\#}(\omega)^* \quad \text{and} \quad \widetilde{b}_3(\omega)\widetilde{b}_3(\omega)^* \preceq b_3^{(1)}(\omega) b_3^{(1)}(\omega)^*$$

for $\omega \in \mathbb{C}_+$. Therefore,

$$\widetilde{b}_4 (b_4^{(1)})^{-1} \in \mathcal{S}_{\text{in}}^{p\times p} \quad \text{and} \quad (b_3^{(1)})^{-1} \widetilde{b}_3 \in \mathcal{S}_{\text{in}}^{p\times p}. \tag{5.7}$$

In view of Theorem 5.2, there exists an essentially unique dB matrix $\mathfrak{E}_2 \in \mathcal{I}_R(j_p)$ such that $\mathcal{B}(\mathfrak{E}_2) \subseteq \mathcal{B}(\mathfrak{E}_1)$ isometrically, $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_2)$ and $\mathcal{B}(\mathfrak{E}_2) \subseteq \mathcal{B}(\mathfrak{E})$ isometrically. Therefore, by (2) applied to $\mathfrak{E}_2$ instead of $\mathfrak{E}$, it follows that $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E}_2)$ isometrically. Therefore, $\mathcal{B}(\mathfrak{E}_2) = \mathcal{B}(\mathfrak{E}_1)$, as needed.

6. *Verification of* (4). This follows from (2) and (3) by setting $b_3^{(1)} = b_3$ and $b_4^{(1)} = b_4$, since in this case $\mathfrak{E} \in \mathcal{I}_R(j_p)$ if and only if $\mathcal{B}(\mathfrak{E}_1) = \mathcal{B}(\mathfrak{E})$. □

6. The class of strongly regular entire de Branges matrices

A mvf $\mathfrak{E} \in \mathcal{I}(j_p)$ will be called a **strongly regular entire dB matrix** if $\mathcal{B}(\mathfrak{E}) \subset L_2^p$ and there exist a pair of constants $\gamma_1 > 0$ and $\gamma_2 \geq \gamma_1$ such that

$$\gamma_1 \|f\|_{\rm st} \leq \|f\|_{\mathcal{B}(\mathfrak{E})} \leq \gamma_2 \|f\|_{\rm st} \quad \text{for every } f \in \mathcal{B}(\mathfrak{E}). \tag{6.1}$$

The class of strongly regular entire dB matrices will be denoted $\mathcal{I}_{sR}(j_p)$.

Lemma 6.1. *If $\mathfrak{E} \in \mathcal{I}(j_p)$, $\{b_3, b_4\} \in ap(\mathfrak{E})$ and $f \in \mathcal{B}(\mathfrak{E}) \cap L_2^p$, then*

$$f \in \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3). \tag{6.2}$$

Moreover, $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$ if and only if

$$\mathcal{B}(\mathfrak{E}) \sim \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3). \tag{6.3}$$

Proof. If $f \in \mathcal{B}(\mathfrak{E})$, then $E_+^{-1} f \in H_2^p$ and $E_-^{-1} f \in (H_2^p)^\perp$. Thus, in view of the factorizations in (4.12), there exist a pair of mvf's φ_3, $\varphi_4 \in \mathcal{N}_{\rm out}^{p\times p}$ such that

$$\varphi_4 b_4 f \in H_2^p \quad \text{and} \quad \varphi_3^{\#} b_3^{\#} f \in (H_2^p)^\perp.$$

Therefore,

$$b_4 f \in \mathcal{N}_+^{p\times 1} \quad \text{and} \quad (b_3^{\#} f)^{\#} \in \mathcal{N}_+^{1\times p}.$$

Under the extra assumption that $f \in L_2^p$ (as well as to $\mathcal{B}(\mathfrak{E})$) it follows from the Smirnov maximum principle that $b_4 f \in H_2^p$ and $b_3^{\#} f \in (H_2^p)^\perp$. Therefore, f is orthogonal to $b_4^{\#}(H_2^p)^\perp$ and to $b_3 H_2^p$, i.e., (6.2) holds.

Suppose next that $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$. Then $\mathcal{B}(\mathfrak{E}) \subset L_2^p$ and hence

$$\mathcal{B}(\mathfrak{E}) \subseteq \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3),$$

since the inclusion (6.2) is in force. At the same time, Theorem 5.3 guarantees that

$$\{k_\alpha^{b_3}\xi + \ell_\beta^{b_4}\eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \subseteq \mathcal{B}(\mathfrak{E}). \tag{6.4}$$

Let $f \in \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3)$. Then, since

$$\mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3) = \operatorname{cls}\{k_\alpha^{b_3}\xi + \ell_\beta^{b_4}\eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \quad \text{in } L_2^p,$$

there exists a Cauchy sequence $\{f_n\}$ of finite linear combinations of vvf's in the set on the left in (6.4) such that $\|f_n - f\|_{\rm st} \to 0$ as $n \uparrow \infty$. In view of (6.1), $\{f_n\}$ is also a Cauchy sequence in $\mathcal{B}(\mathfrak{E})$. Therefore, $\|f_n - g\|_{\mathcal{B}(\mathfrak{E})} \to 0$ as $n \uparrow \infty$ for some vvf $g \in \mathcal{B}(\mathfrak{E})$. But $f = g$, since norm convergence implies pointwise convergence in a RKHS. Thus, (6.3) holds when $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$.

The converse implication is immediate from the definition of the class $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$. □

Lemma 6.2. *$\mathcal{I}_{sR}(j_p) \subset \mathcal{I}_R(j_p)$ and the inclusion is proper.*

Proof. If $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$ and $\{b_3, b_4\} \in ap(\mathfrak{E})$, then (6.3) holds. Assertion (4) of Theorem 5.3 and the relation

$$\mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3) = \operatorname{cls}\{k_\alpha^{b_3}\xi + \ell_\beta^{b_4}\eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \quad \text{in } L_2^p$$

imply that $\mathfrak{E} \in \mathcal{I}_R(j_p)$. Thus, $\mathcal{I}_{sR}(j_p) \subseteq \mathcal{I}_R(j_p)$. However, the inclusion is proper because the mvf

$$\mathfrak{E}_1(\lambda) = \begin{bmatrix}(1+i\lambda)b_3(\lambda) & (1-i\lambda)b_4(\lambda)^{-1}\end{bmatrix}$$

belongs to the class $\mathcal{I}_R(j_p)$ but not to the class $\mathcal{I}_{sR}(j_p)$. (This will be discussed in more detail in[6].) □

Theorem 6.3. *If $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$, $\mathfrak{E}_1 \in \mathcal{I}(j_p)$, $\mathcal{B}(\mathfrak{E}_1)$ is a closed subspace of $\mathcal{B}(\mathfrak{E})$ and the inclusion $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is contractive, then $\mathfrak{E}_1 \in \mathcal{I}_{sR}(j_p)$.*

Proof. Let $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$. Then, since $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{E}) \subset L_2^p$, Lemma 6.1 ensures that $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3^{(1)})$.

On the other hand, if $f \in \mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3^{(1)})$, then there exists a sequence $\{f_n\}$, $n = 1, 2, \ldots$ of linear combinations of vvf's of the form $k_\alpha^{b_3^{(1)}}\xi + \ell_\beta^{b_4^{(1)}}\eta$ with $\xi, \eta \in \mathbb{C}^p$ and $\alpha, \beta \in \mathbb{C}$ such that $f_n \to f$ in L_2^p as $n \uparrow \infty$. In view of Theorem 5.3, $f_n \in \mathcal{B}(\mathfrak{E}_1)$ and hence, as $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$, $f_n \in \mathcal{B}(\mathfrak{E})$ for $n = 1, 2, \ldots$. Thus, as $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$, $f \in \mathcal{B}(\mathfrak{E})$ and hence, as $\mathcal{B}(\mathfrak{E}_1)$ is closed in $\mathcal{B}(\mathfrak{E})$, $f \in \mathcal{B}(\mathfrak{E}_1)$. Consequently, $\mathcal{B}(\mathfrak{E}_1) = \mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3)$ as vector spaces. Since $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$ and the inclusion $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is contractive, there exists a constant $\gamma_1 > 0$ such that

$$\gamma_1\|f\|_{st} \le \|f\|_{\mathcal{B}(\mathfrak{E})} \le \|f\|_{\mathcal{B}(\mathfrak{E}_1)} \quad \text{for every } f \in \mathcal{B}(\mathfrak{E}_1).$$

Thus, the identity map T from $\mathcal{B}(\mathfrak{E}_1)$ onto $\mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3^{(1)})$ is subject to the bounds

$$\|Tf\|_{st} = \|f\|_{st} \le \gamma_1^{-1}\|f\|_{\mathcal{B}(\mathfrak{E}_1)}.$$

Therefore, as T is invertible, a well-known theorem of Banach guarantees that T^{-1} is also bounded, i.e., there exists a constant $\gamma > 0$ such that

$$\|f\|_{\mathcal{B}(\mathfrak{E}_1)} = \|T^{-1}f\|_{\mathcal{B}(\mathfrak{E}_1)} \le \gamma\|f\|_{st} \quad \text{for every } f \in \mathcal{B}(\mathfrak{E}_1).$$

Consequently, $\mathfrak{E}_1 \in \mathcal{I}_{sR}(j_p)$. □

7. Other directions

In this paper we have not discussed spectral functions nor the role of vector-valued dB spaces as a model for symmetric operators; see e.g., [3] for the former and [12] and the references cited therein for the latter.

The theory of dB spaces $\mathcal{B}(\mathfrak{E})$ based on entire dB matrices $\mathfrak{E}$ is intimately connected with the theory of entire J_p-inner mvfs A and the corresponding RKHS's $\mathcal{H}(A)$.

Let $\mathcal{E} \cap \mathcal{U}(J_p)$ denote the class of $m \times m$ entire J_p-inner mvf's $A(\lambda)$, i.e., entire mvf's such that

$$J_p - A(\lambda)J_pA(\lambda)^* \succeq 0 \quad \text{for } \lambda \in \overline{\mathbb{C}_+} \text{ with equality on } \mathbb{R}.$$

If $A \in \mathcal{E} \cap \mathcal{U}(J_p)$, then the kernel

$$K_\omega^A(\lambda) = \frac{J_p - A(\lambda)J_pA(\omega)^*}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}$$

is positive in the sense of (P3) in Section 3. Therefore, $K_\omega^A(\lambda)$ can be identified as the RK of a RKHS of $2p \times 1$ vvf's that will be denoted $\mathcal{H}(A)$. Moreover, every vvf $f \in \mathcal{H}(A)$ is entire, the mvf

$$\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix} = \mathfrak{E}_A(\lambda) \stackrel{\text{def}}{=} \sqrt{2} \begin{bmatrix} 0 & I_p \end{bmatrix} A(\lambda)\mathfrak{V} \tag{7.1}$$

is a dB matrix and

$$K_\omega^{\mathfrak{E}}(\lambda) = \sqrt{2} \begin{bmatrix} 0 & I_p \end{bmatrix} \frac{J_p - A(\lambda)J_pA(\omega)^*}{\rho_\omega(\lambda)} \sqrt{2} \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

Furthermore,

$$\mathfrak{E} \in \mathcal{I}(j_p) \quad \text{if and only if} \quad \mathfrak{E} = \mathfrak{E}_A \text{ for some } A \in \mathcal{E} \cap \mathcal{U}(J_p). \tag{7.2}$$

We shall say that a mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ belongs to the class

$\mathcal{E} \cap \mathcal{U}_{rR}(J_p)$ if $\mathcal{H}(A) \cap L_2^{m\times m}$ is dense in $\mathcal{H}(A)$,
$\mathcal{E} \cap \mathcal{U}_{rsR}(J_p)$ if $\mathcal{H}(A) \subset L_2^{m\times m}$,
$\mathcal{E} \cap \mathcal{U}_S(J_p)$ if $\mathcal{H}(A) \cap L_2^{m\times m} = \{0\}$,
$\mathcal{E} \cap \mathcal{U}_{AR}(J_p)$ if $A_1 \in \mathcal{E} \cap \mathcal{U}(J_p)$ and $A_1^{-1}A \in \mathcal{E} \cap \mathcal{U}(J_p)$, then $A_1 \in \mathcal{U}_{rR}(J_p)$.

A number of other characterizations of these classes are presented in [2].

If $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ is written in block form $A = \begin{bmatrix} a_{ij}(\lambda) \end{bmatrix}$ with $p \times p$ blocks $a_{ij}(\lambda)$ for $i, j = 1, 2$, then the limit

$$\beta = \lim_{\nu \uparrow \infty} \frac{1}{\nu} \Re (a_{11}(i\nu) + a_{12}(i\nu))(a_{21}(i\nu) + a_{22}(i\nu))^{-1}$$

exists and $\beta \succeq 0$. A mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ is said to be **perfect** if $\beta = 0$.

If $\mathfrak{E} \in \mathcal{I}(j_p)$, then there exists a perfect mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ such that $\mathfrak{E} = \mathfrak{E}_A$. Moreover, in this case the operator

$$U_2 : f \in \mathcal{H}(A) \mapsto \sqrt{2} \begin{bmatrix} 0 & I_p \end{bmatrix} f \in \mathcal{B}(\mathfrak{E})$$

is unitary.

It also turns out that

(1) $\mathfrak{E} \in \mathcal{I}(j_p) \iff \mathfrak{E} = \mathfrak{E}_A$ for a perfect mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$

and, if $\mathfrak{E} = \mathfrak{E}_A$ for some perfect mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$, then

(2) $\mathfrak{E}_A \in \mathcal{I}_S(j_p) \iff A \in \mathcal{E} \cap \mathcal{U}_S(J_p)$,
(3) $\mathfrak{E}_A \in \mathcal{I}_R(j_p) \iff A \in \mathcal{E} \cap \mathcal{U}_{rR}(J_p)$,
(4) $\mathfrak{E}_A \in \mathcal{I}_{sR}(j_p) \impliedby A \in \mathcal{E} \cap \mathcal{U}_{rsR}(J_p)$,

These connections will be discussed in detail in [6].

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Divided Differences and Two-sided Polynomial Interpolation Over Quaternions

Vladimir Bolotnikov

Dedicated to Professor Daniel Alpay on the occasion of his 60th birthday

Abstract. We consider a two-sided interpolation problem of Lagrange–Hermite type for polynomials over quaternions. Necessary and sufficient condition for the problem to have a solution is given and a particular low-degree solution is constructed in terms of a certain Sylvester equation.

1. Introduction

Let $\mathbb{H}$ be the skew field of real quaternions with imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ commuting with $\mathbb{R}$ and satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. For $\alpha = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ ($x_i \in \mathbb{R}$), its real and imaginary parts, the quaternionic conjugate and the absolute value are defined as $\Re(\alpha) = x_0$, $\Im(\alpha) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, $\overline{\alpha} = \Re(\alpha) - \Im(\alpha)$ and $|\alpha| = \sqrt{\alpha\overline{\alpha}} = \sqrt{|\Re(\alpha)|^2 + |\Im(\alpha)|^2}$.

Let $\mathbb{H}[z]$ denote the ring of polynomials in one formal variable z which commutes with quaternionic coefficients. The division algorithm holds in $\mathbb{H}[z]$ on either side, and hence, any (left or right) ideal in $\mathbb{H}[z]$ is principal. We denote by $\langle G\rangle_{\mathbf{r}}$ and $\langle G\rangle_{\boldsymbol{\ell}}$ the right and left ideals generated by $G \in \mathbb{H}[z]$ dropping the subscript if the ideal is two-sided. To exclude non-uniqueness, all generators will be assumed to be monic. We recall that the ring $\mathbb{R}[z]$ of polynomials with real coefficients is the center of $\mathbb{H}[z]$ and that any two-sided ideal in $\mathbb{H}[z]$ is generated by an element from $\mathbb{R}[z]$.

Adapting the main concept from [6] to the current single-variable noncommutative setting, let us say that a finite collection $\{\Phi_i\}_{i=1}^n$ of left (right) evaluation functionals is a *left (right) ideal interpolation scheme* if

1. the solution set of the interpolation problem with homogeneous conditions $\Phi_i(f) = 0$ for $i = 1, \dots, n$ is a right (left) ideal $\mathbb{I} \subset \mathbb{H}[z]$, and
2. any non-homogeneous problem has a unique solution f_0 modulo $\mathbb{I}$.

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The basic example of the ideal interpolation scheme in $\mathbb{H}[z]$ is the Lagrange left (or right) interpolation problem with interpolation nodes such that none three of them belong to the same similarity (conjugacy) class [7, 8, 14, 15]. To solve a specific problem in $\mathbb{H}[z]$ based on an ideal interpolation scheme, it suffices to find the generator G for the solution set of the homogeneous problem (this usually amounts to computing the least left/right common multiple of several given polynomials) and a particular (low-degree) solution f_0 (which can be done, e.g., using Vandermonde or confluent Vandermonde matrices). Then, by linearity, the solution set for the original problem can be written as $f_0 + \langle G\rangle_{\mathbf{r}}$ (or as $f_0 + \langle G\rangle_{\boldsymbol{\ell}}$) for the left (right) interpolation scheme. In this paper, we make the next step considering the combination of left and right interpolation schemes. Disregarding specific evaluation functionals we may start directly with solution sets for the one-sided parts of the problem. In other words, we will be concerned with the following problem:

(**P**): *Given polynomials the* G, H, $f_{\boldsymbol{\ell}}$, $f_{\mathbf{r}} \in \mathbb{H}[z]$, *find a polynomial* $f \in \mathbb{H}[z]$ *such that*

$$f = f_{\boldsymbol{\ell}} + Gp \quad \text{and} \quad f = f_{\mathbf{r}} + \widetilde{p}H \quad \text{for some} \quad p, \widetilde{p} \in \mathbb{H}[z]. \tag{1.1}$$

The solvability criterion and a recipe for constructing a particular solution of the problem (**P**) by means of certain Sylvester equation are given in Theorem 1.1 below. In what follows, we use the notation

$$E_n = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{H}^{1\times n} \quad \text{and} \quad \boldsymbol{\rho}_\alpha(z) := z - \alpha \quad \text{for a fixed } \alpha \in \mathbb{H}. \tag{1.2}$$

Without loss of generality we may (and will) assume that the polynomials G and H are monic and that $\deg f_{\boldsymbol{\ell}} < \deg G$ and $\deg f_{\mathbf{r}} < \deg H$. Since any quaternionic polynomial can be factored into the product of linear factors, we may take G and H in the form

$$G = \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_k}, \quad H = \boldsymbol{\rho}_{\beta_m} \cdots \boldsymbol{\rho}_{\beta_1}, \quad \alpha_i, \beta_j \in \mathbb{H}. \tag{1.3}$$

With factorizations (1.3), we associate the matrices

$$\mathcal{J}_{\boldsymbol{\alpha}} = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 1 & \alpha_2 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & \alpha_k \end{bmatrix} \quad \text{and} \quad \mathcal{J}_{\boldsymbol{\beta}} = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 1 & \beta_2 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & \beta_m \end{bmatrix}. \tag{1.4}$$

For a polynomial $g(z) = \sum_{j=0}^{N} g_j z^j$, let us define the column $\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g) \in \mathbb{H}^{k\times 1}$ and the row $\Delta_{\mathbf{r}}(g; \boldsymbol{\beta}) \in \mathbb{H}^{1\times m}$ by the formulas (with E_k and E_m as in (1.2))

$$\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g) = \sum_{j=0}^{N} \mathcal{J}_{\boldsymbol{\alpha}}^j E_k^\top g_j \quad \text{and} \quad \Delta_{\mathbf{r}}(g; \boldsymbol{\beta}) = \sum_{j=0}^{N} g_j E_m (\mathcal{J}_{\boldsymbol{\beta}}^\top)^j. \tag{1.5}$$

Theorem 1.1. *Given* G *and* H *as in* (1.3) *and given polynomials* $f_{\boldsymbol{\ell}}$, $f_{\mathbf{r}}$, *let* $\mathcal{J}_{\boldsymbol{\alpha}}$, $\mathcal{J}_{\boldsymbol{\beta}}$, $\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}})$ *and* $\Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta})$ *be defined via formulas* (1.4), (1.5). *There is an* $f \in \mathbb{H}[z]$

subject to conditions (1.1) *if and only if the equation*

$$\mathcal{J}_{\boldsymbol{\alpha}} X - X \mathcal{J}_{\boldsymbol{\beta}}^{\top} = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}}) E_m - E_k^{\top} \Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta}) \tag{1.6}$$

has a solution $X \in \mathbb{H}^{k\times m}$. *If this is the case, then the formula*

$$f_X = f_{\boldsymbol{\ell}} + G \cdot \left(X_{k,1} + X_{k,2} \boldsymbol{\rho}_{\beta_1} + X_{k,3} \boldsymbol{\rho}_{\beta_2} \boldsymbol{\rho}_{\beta_1} + \ldots + X_{k,m} \boldsymbol{\rho}_{\beta_{m-1}} \cdots \boldsymbol{\rho}_{\beta_1} \right) \tag{1.7}$$

establishes a one-to-one correspondence between solutions $X = [X_{i,j}] \in \mathbb{H}^{k\times m}$ *to the equation* (1.6) *and solutions* f_X *to the problem* (**P**) *of degree less than* $\deg G + \deg H$ *(low-degree solutions).*

Remark 1.2. The entries in the equation (1.6) are based on factorizations (1.3) of G and H which in general are not unique. It follows from Theorem 1.1 that if $G = \boldsymbol{\rho}_{\gamma_1} \cdots \boldsymbol{\rho}_{\gamma_k}$ and $H = \boldsymbol{\rho}_{\eta_m} \cdots \boldsymbol{\rho}_{\eta_1}$ are two other factorizations of G and H, then the equation

$$\mathcal{J}_{\boldsymbol{\gamma}} Y - Y \mathcal{J}_{\boldsymbol{\eta}}^{\top} = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\gamma}; f_{\boldsymbol{\ell}}) E_m - E_k^{\top} \Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\eta}) \tag{1.8}$$

has a solution if and only if the equation (1.6) does. Actually, it can be shown that for matrices $\mathcal{J}_{\boldsymbol{\alpha}}$ and $\mathcal{J}_{\boldsymbol{\gamma}}$ constructed from different factorizations of G, there is an invertible matrix T such that $T\mathcal{J}_{\boldsymbol{\gamma}} = \mathcal{J}_{\boldsymbol{\alpha}} T$ and $TE_k^{\top} = E_k^{\top}$. Similarly, there is an invertible matrix $\widetilde{T}$ such that $\mathcal{J}_{\boldsymbol{\eta}}^{\top} \widetilde{T} = \widetilde{T} \mathcal{J}_{\boldsymbol{\beta}}^{\top}$ and $E_m \widetilde{T} = E_m$. Then it follows from (1.5) that

$$T \Delta_{\boldsymbol{\ell}}(\boldsymbol{\gamma}; g) = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g) \quad \text{and} \quad \Delta_{\mathbf{r}}(g; \boldsymbol{\eta}) \widetilde{T} = \Delta_{\mathbf{r}}(g; \boldsymbol{\beta}) \quad \text{for all } g \in \mathbb{H}[z],$$

from which it is seen that Y solves (1.8) if and only if $X = TY\widetilde{T}$ solves (1.6).

The solvability criterion from Theorem 1.1 is fairly satisfactory due to existing procedures verifying whether or not the Sylvester equation (1.6) has a solution. One such procedure is based on the *complex representation* $\varphi(M)$ of a quaternionic matrix M suggested in [17]:

$$\varphi(M) = \begin{bmatrix} M_1 & M_2 \\ -\overline{M}_2 & \overline{M}_1 \end{bmatrix}, \quad \text{where } M = M_1 + M_2 \mathbf{j}, \;\; M_1, M_2 \in \mathbb{C}^{k\times m}. \tag{1.9}$$

The map $M \mapsto \phi(M)$ is additive and multiplicative and hence, for any solution X of (1.6), the matrix $Y = \varphi(X)$ solves the complex Sylvester equation

$$\varphi(\mathcal{J}_{\boldsymbol{\alpha}}) Y - Y \varphi(J_{\boldsymbol{\beta}}^{\top}) = \varphi(\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}}) E_m - E_k^{\top} \Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta})). \tag{1.10}$$

On the other hand, if $Y = \left[\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right]$ ($Y_{ij} \in \mathbb{C}^{k\times m}$) satisfies (1.10), then the matrix

$$X = \frac{1}{2} \left(Y_{11} + \overline{Y}_{22} + (Y_{12} - \overline{Y}_{21}) \, \mathbf{j} \right) \tag{1.11}$$

is a solution to (1.6) (see [13]). Thus, the quaternion Sylvester equation (1.6) has a solution if and only if the complex equation (1.10) does. Moreover, from each solution Y to the equation (1.10), one can combine formulas (1.11) and (1.7) to get a particular solution f to the problem (**P**). The results on complex Sylvester equations are classic and we refer to the survey [16] rather than recall them here. Theorem 1.1 and the complex representation approach are illustrated in Example

3.4 below. We note that combining the formula (1.7) with complex representation (1.9) is particularly efficient in case the problem (**P**) has a unique low-degree solution (see Section 3.1). Otherwise, instead of finding all low-degree solutions via this approach, it seems to be more practical to combine one particular low-degree solution f_X with the general low-degree solution of the homogeneous problem. This second approach will be discussed in Section 3.2. The proof of Theorem 1.1 is presented in Section 2.

2. Divided differences and related Sylvester identity

Divided differences proved to be useful tools in classical polynomial interpolation theory. In the noncommutative setting, divided differences appear in two (left and right) versions which serve to construct low-degree solutions for respectively, left and right Lagrange-Hermite interpolation problems. We will recall these one-sided divided differences in formulas (2.6), (2.7) below. A straightforward computation verifies that for any $\alpha \in \mathbb{H}$ and $f \in \mathbb{H}[z]$,

$$f(z) = f^{\boldsymbol{e}_\ell}(\alpha) + (z-\alpha)\cdot(L_\alpha f)(z) = f^{\boldsymbol{e}_r}(\alpha) + (R_\alpha f)(z)\cdot(z-\alpha), \tag{2.1}$$

where $f^{\boldsymbol{e}_\ell}(\alpha)$ and $f^{\boldsymbol{e}_r}(\alpha)$ are left and right evaluation of f at α given by

$$f^{\boldsymbol{e}_\ell}(\alpha) = \sum_{k=0}^{m} \alpha^k f_k \quad \text{and} \quad f^{\boldsymbol{e}_r}(\alpha) = \sum_{k=0}^{m} f_k \alpha^k \quad \text{if } f(z) = \sum_{j=0}^{m} z^j f_j, \tag{2.2}$$

and where $L_\alpha f$ and $R_\alpha f$ are polynomials of degree $m-1$ given by

$$(L_\alpha f)(z) = \sum_{j+k=0}^{m-1} \alpha^j f_{k+j+1} z^k, \quad (R_\alpha f)(z) = \sum_{j+k=0}^{m-1} f_{k+j+1}\alpha^j z^k. \tag{2.3}$$

Observe that the mappings $f \mapsto L_\alpha f$ and $f \mapsto R_\alpha f$ define a right linear operator L_α and a left linear operator R_α acting on $\mathbb{H}[z]$ (interpreted a a vector space over $\mathbb{H}$). We next observe that for any $\alpha, \beta \in \mathbb{H}$ and $f \in \mathbb{H}[z]$,

$$L_\alpha R_\beta f = R_\beta L_\alpha f \quad \text{and} \quad (L_\alpha f)^{\boldsymbol{e}_r}(\beta) = (R_\beta f)^{\boldsymbol{e}_\ell}(\alpha). \tag{2.4}$$

Making use of (2.3), one may first verify equalities (2.4) for $f(z) = cz^n$ as follows

$$L_\alpha R_\beta f = \sum_{k=0}^{n-2} \Big(\sum_{i+j=n-k-2} \alpha^i c \beta^j \Big) z^k = R_\beta L_\alpha f,$$
$$(L_\alpha f)^{\boldsymbol{e}_r}(\beta) = \sum_{i+j=n-1} \alpha^i c \beta^j = (R_\beta f)^{\boldsymbol{e}_\ell}(\alpha),$$

and then get the general case by linearity. Now we will use operators (2.3) to introduce divided differences.

Given an $f \in \mathbb{H}[z]$, the successive application of the first formula in (2.1) to the elements $\alpha_1, \alpha_2, \ldots \in \mathbb{H}$ and polynomials $f,\ L_{\alpha_1} f,\ L_{\alpha_2} L_{\alpha_1} f, \ldots$ and the second

formula in (2.1) to elements $\beta_1, \beta_2, \ldots$ and polynomials $f,\ R_{\beta_1} f,\ R_{\beta_2} R_{\beta_1} f, \ldots$, lead us, respectively, to representations

$$\begin{aligned} f &= f^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha_1) + \sum_{j=1}^{\deg f - 1} \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_j} \cdot (L_{\alpha_j} \cdots L_{\alpha_1} f)^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha_{j+1}), \\ f &= f^{\boldsymbol{e}_{\boldsymbol{r}}}(\beta_1) + \sum_{j=1}^{\deg f - 1} (R_{\beta_j} \cdots R_{\beta_1} f)^{\boldsymbol{e}_{\boldsymbol{r}}}(\beta_{j+1}) \cdot \boldsymbol{\rho}_{\beta_j} \cdots \boldsymbol{\rho}_{\beta_1}, \end{aligned} \tag{2.5}$$

which, being (respectively, left and right) quaternionic analogs of the Newton interpolation formula, suggest to introduce *left* and *right divided differences*

$$\begin{aligned} [\alpha_1; f]_{\boldsymbol{\ell}} &= f^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha_1), \\ [\alpha_1, \ldots, \alpha_i; f]_{\boldsymbol{\ell}} &= (L_{\alpha_{i-1}} \cdots L_{\alpha_1} f)^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha_i) \quad \text{for} \quad i \geq 2, \end{aligned} \tag{2.6}$$

$$\begin{aligned} [f; \beta_1]_{\mathbf{r}} &= f^{\boldsymbol{e}_{\boldsymbol{r}}}(\beta_1), \\ [f; \beta_1, \ldots, \beta_j]_{\mathbf{r}} &= (R_{\beta_{j-1}} \cdots R_{\beta_1} f)^{\boldsymbol{e}_{\boldsymbol{r}}}(\beta_j) \quad \text{for} \quad j \geq 2, \end{aligned} \tag{2.7}$$

based on given tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$.

Remark 2.1. It follows from (2.5)–(2.7) that for G and H of the form (1.3),

1. $f \in \langle G \rangle_{\mathbf{r}}$ if and only if $[\alpha_1, \ldots, \alpha_i; f]_{\boldsymbol{\ell}} = 0$ for $i = 1, \ldots, k$;
2. $f \in \langle H \rangle_{\boldsymbol{\ell}}$ if and only if $[f; \beta_1, \ldots, \beta_j]_{\mathbf{r}} = 0$ for $j = 1, \ldots, m$.

It turns out that in the context of a two-sided interpolation problem, one needs "two-sided" divided differences which we will refer to as to *mixed divided differences* and which we define as follows:

$$\begin{aligned} [\alpha_1, \ldots, \alpha_i; f; \beta_1, \ldots, \beta_j] &:= (L_{\alpha_i} \cdots L_{\alpha_1} R_{\beta_{j-1}} \cdots R_{\beta_1} f)^{\boldsymbol{e}_{\boldsymbol{r}}}(\beta_j) \\ &= (L_{\alpha_{i-1}} \cdots L_{\alpha_1} R_{\beta_j} \cdots R_{\beta_1} f)^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha_i), \end{aligned} \tag{2.8}$$

where the second equality holds due to (2.4). For $j = 1$ and for $i = 1$ the first and the second formulas in (2.8) take the form

$$\begin{aligned} [\alpha_1, \ldots, \alpha_i; f; \beta_1] &= (L_{\alpha_i} \cdots L_{\alpha_1} f)^{\boldsymbol{e}_{\boldsymbol{r}}}(\beta_1), \\ [\alpha_1; f; \beta_1, \ldots, \beta_j] &= (R_{\beta_j} \cdots R_{\beta_1} f)^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha_1). \end{aligned} \tag{2.9}$$

In the next lemma, the column $\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f)$ and the row $\Delta_{\mathbf{r}}(f; \boldsymbol{\beta})$ defined in (2.10), (2.11) as certain sums (in accordance to (1.5)), turn out to be the column and the row of left and right divided differences of f based on the tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. We refer to [8, Remark 2.5] for the proof of the second equalities in (2.10) and (2.11).

Lemma 2.2. *Given tuples $\boldsymbol{\alpha} = (\alpha_1, \dots \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$, let $\mathcal{J}_{\boldsymbol{\alpha}}$ and $\mathcal{J}_{\boldsymbol{\beta}}$ be defined as in* (1.4)*. For an $f(z) = \sum f_j z^j \in \mathbb{H}[z]$, let*

$$\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f) := \sum_{j=0}^{\deg f} \mathcal{J}_{\boldsymbol{\alpha}}^j E_k^\top f_j = \begin{bmatrix} [\alpha_1; f]_{\boldsymbol{\ell}} \\ \vdots \\ [\alpha_1, \dots, \alpha_k; f]_{\boldsymbol{\ell}} \end{bmatrix}, \tag{2.10}$$

$$\Delta_{\mathbf{r}}(f; \boldsymbol{\beta}) := \sum_{j=0}^{\deg f} f_j E_m (\mathcal{J}_{\boldsymbol{\beta}}^\top)^j = \begin{bmatrix} [f; \beta_1]_{\mathbf{r}} & \dots & [f; \beta_1, \dots, \beta_m]_{\mathbf{r}} \end{bmatrix}, \tag{2.11}$$

and let $\Gamma(\boldsymbol{\alpha}; f; \boldsymbol{\beta}) \in \mathbb{H}^{k \times m}$ be the matrix of the mixed divided differences:

$$\Gamma(\boldsymbol{\alpha}; f; \boldsymbol{\beta}) = [\Gamma_{ij}], \quad \text{where } \Gamma_{ij} = [\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_j]. \tag{2.12}$$

Then the following Sylvester equality holds:

$$\mathcal{J}_{\boldsymbol{\alpha}} \Gamma(\boldsymbol{\alpha}; f; \boldsymbol{\beta}) - \Gamma(\boldsymbol{\alpha}; f; \boldsymbol{\beta}) \mathcal{J}_{\boldsymbol{\beta}}^\top = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f)\, E_m - E_k^\top \Delta_{\mathbf{r}}(f; \boldsymbol{\beta}). \tag{2.13}$$

Proof. We will verify the equality of the corresponding entries in (2.13) separately for $(1,1)$, $(1,j)$, $(i,1)$, and all other entries. Due to formulas (1.4) and (2.10)–(2.12), the equalities to be verified are:

$$\alpha_1 \Gamma_{1,1} - \Gamma_{1,1} \beta_1 = [\alpha_1; f]_{\boldsymbol{\ell}} - [f; \beta_1]_{\mathbf{r}}, \tag{2.14}$$

$$\alpha_i \Gamma_{i,1} + \Gamma_{i-1,1} - \Gamma_{i,1} \beta_1 = [\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}} \qquad (i \geq 2), \tag{2.15}$$

$$\alpha_1 \Gamma_{1,j} - \Gamma_{1,j} \beta_j - \Gamma_{1,j-1} = -[f; \beta_1, \dots, \beta_j]_{\mathbf{r}} \qquad (j \geq 2), \tag{2.16}$$

$$\alpha_i \Gamma_{i,j} - \Gamma_{i,j} \beta_j - \Gamma_{i,j-1} + \Gamma_{i-1,j} = 0 \qquad (i, j \geq 2). \tag{2.17}$$

As was observed in [7, Remark 4.1], for any $f \in \mathbb{H}[z]$ and $\alpha, \beta \in \mathbb{H}$,

$$\alpha (L_\alpha f)^{\boldsymbol{e_r}}(\beta) - (L_\alpha f)^{\boldsymbol{e_r}}(\beta) \beta = f^{\boldsymbol{e_\ell}}(\alpha) - f^{\boldsymbol{e_r}}(\beta), \tag{2.18}$$

which is the same as (2.14) (on account of (2.6), (2.7), (2.8) and (2.12)). Letting $\beta = \beta_1$, $\alpha = \alpha_i$ $(2 \leq i \leq n)$ and replacing f by $L_{\alpha_{i-1}} \dots L_{\alpha_1} f$ in (2.18), we get

$$\begin{aligned} \alpha_i (L_{\alpha_i} L_{\alpha_{i-1}} \dots L_{\alpha_1} f)^{\boldsymbol{e_r}}(\beta_1) - (L_{\alpha_i} L_{\alpha_{i-1}} \dots L_{\alpha_1} f)^{\boldsymbol{e_r}}(\beta_1) \beta_1 \\ = (L_{\alpha_{i-1}} \dots L_{\alpha_1} f)^{\boldsymbol{e_\ell}}(\alpha_i) - (L_{\alpha_{i-1}} \dots L_{\alpha_1} f)^{\boldsymbol{e_r}}(\beta_1) \end{aligned}$$

which can be written in terms of notation (2.6), (2.8) as

$$\alpha_i [\alpha_1, \dots, \alpha_i; f; \beta_1] - [\alpha_1, \dots, \alpha_i; f; \beta_1] \beta_1 = [\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}} - [\alpha_1, \dots, \alpha_{i-1}; f; \beta_1].$$

The latter is equivalent to (2.15), due to (2.12). Making use of the second equality in (2.4), we write (2.18) in the form

$$\alpha (R_\beta f)^{\boldsymbol{e_\ell}}(\alpha) - (R_\beta f)^{\boldsymbol{e_\ell}}(\alpha) \beta = f^{\boldsymbol{e_\ell}}(\alpha) - f^{\boldsymbol{e_r}}(\beta)$$

and then let $\alpha = \alpha_1$, $\beta = \beta_j$ $(2 \leq j \leq m)$ and replace f by $R_{\beta_{j-1}} \cdots R_{\beta_1} f$ to get

$$\alpha_1 [\alpha_1; f; \beta_1, \dots, \beta_j] - [\alpha_1; f; \beta_1, \dots, \beta_j] \beta_j = [\alpha_1; f; \beta_1, \dots, \beta_{j-1}] - [f; \beta_1, \dots, \beta_j]_{\mathbf{r}},$$

which, is equivalent to (2.16). Finally, letting $\alpha = \alpha_i$ $(2 \le i \le n)$, $\beta = \beta_j$ $(2 \le j \le m)$ and replacing f by $L_{\alpha_{i-1}} \dots L_{\alpha_1} R_{\beta_{j-1}} \cdots R_{\beta_1} f$ in (2.18), we get

$$\begin{aligned}&\alpha_i (L_{\alpha_i} \dots L_{\alpha_1} R_{\beta_{j-1}} \cdots R_{\beta_1} f)^{\boldsymbol{e_r}}(\beta_j) - (L_{\alpha_i} \dots L_{\alpha_1} R_{\beta_{j-1}} \cdots R_{\beta_1} f)^{\boldsymbol{e_r}}(\beta_j)\beta_j \\ &= (L_{\alpha_{i-1}} \dots L_{\alpha_1} R_{\beta_{j-1}} \cdots R_{\beta_1} f)^{\boldsymbol{e_\ell}}(\alpha_i) - (L_{\alpha_{i-1}} \dots L_{\alpha_1} R_{\beta_{j-1}} \cdots R_{\beta_1} f)^{\boldsymbol{e_r}}(\beta_j)\end{aligned}$$

which is the same as (2.17). □

Corollary 2.3. *Left divided differences* $[\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}}$ $(1 \le i \le k)$ *and mixed divided differences* $[\alpha_1, \dots \alpha_k; f; \beta_1, \dots, \beta_j]$ $(1 \le j \le m)$ *uniquely determine mixed divided differences* $[\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_j]$ *and right divided differences* $[f; \beta_1, \dots, \beta_j]_{\mathbf{r}}$ *for all* $1 \le i < k$ *and* $1 \le j \le m$.

Proof. Let us denote by Γ_i the i-th row of the matrix (2.12):

$$\Gamma_i = \big[[\alpha_1, \dots, \alpha_i; f; \beta_1] \quad [\alpha_1, \dots, \alpha_i; f; \beta_1, \beta_2] \quad \dots \quad [\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_m]\big] .$$

Comparing the i-th rows in the matrix equality (2.13) for $i = k, \dots, 1$, we conclude

$$\Gamma_{i-1} = \Gamma_i \mathcal{J}_{\boldsymbol{\beta}}^{\top} - \alpha_i \Gamma_i + [\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}} E_m \quad (i = k, \dots, 2), \tag{2.19}$$

$$\Delta_{\mathbf{r}}(f; \boldsymbol{\beta}) = \Gamma_1 \mathcal{J}_{\boldsymbol{\beta}}^{\top} - \alpha_1 \Gamma_1 + [\alpha_1; f]_{\boldsymbol{\ell}} E_m. \tag{2.20}$$

Since the row Γ_k is given, we recursively recover all other rows in Γ by formulas (2.19) and then we recover all right divided differences by formula (2.20). □

Proof of Theorem 1.1. Conditions (1.1) mean that $(f - f_{\boldsymbol{\ell}}) \in \langle G \rangle_{\mathbf{r}}$ and $(f - f_{\mathbf{r}}) \in \langle H \rangle_{\boldsymbol{\ell}}$. Since left (right) divided differences are right (left) linear with respect to f, we conclude by Remark 2.1 that the first and the second equalities in (1.1) are equivalent, respectively, to conditions

$$[\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}} = [\alpha_1, \dots, \alpha_i; f_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}} \quad (i = 1, \dots, k), \tag{2.21}$$

$$[f; \beta_1, \dots, \beta_j]_{\mathbf{r}} = [f_{\mathbf{r}}; \beta_1, \dots, \beta_j]_{\mathbf{r}} \quad (j = 1, \dots, m), \tag{2.22}$$

which in turn, can be written in the vector form as

$$\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f) = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}}) \quad \text{and} \quad \Delta_{\mathbf{r}}(f; \boldsymbol{\beta}) = \Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta}). \tag{2.23}$$

If $f \in \mathbb{H}[z]$ satisfies conditions (2.23), then the matrix $X = \Gamma(\boldsymbol{\alpha}; f; \boldsymbol{\beta})$ of its mixed divided differences is a solution to the equation (1.6), by Lemma 2.2. This proves the "only if" part in Theorem 1.1. Moreover, if $\deg f < m+k$, then f is necessarily of the form (1.7), where

$$X_{k,j} = [\alpha_1, \dots, \alpha_k; f; \beta_1, \dots, \beta_j] \quad \text{for } j = 1, \dots, m. \tag{2.24}$$

Conversely, let us assume that the matrix $X = [X_{i,j}]$ solves (1.6). Comparing the corresponding rows in (1.6) we conclude as in the proof of Corollary 2.3 that the rows $X_1, \dots, X_k$ of X satisfy equalities

$$X_{i-1} = X_i \mathcal{J}_{\boldsymbol{\beta}}^{\top} - \alpha_i X_i + [\alpha_1, \dots, \alpha_i; f_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}} E_m \quad (i = k, \dots, 2), \tag{2.25}$$

$$\Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta}) = X_1 \mathcal{J}_{\boldsymbol{\beta}}^{\top} - \alpha_1 X_1 + [\alpha_1; f_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}} E_m. \tag{2.26}$$

The polynomial $f = f_X$ defined in (1.7) clearly satisfies conditions (2.21) (i.e., the first condition in (2.23)). Besides, equalities (2.24) hold by construction. With left

divided differences (2.21) and mixed divided differences (2.24) in hands, we can recover the whole matrix $\Gamma(\boldsymbol{\alpha}; f; \boldsymbol{\beta})$ by formulas (2.19). Since $\Gamma_k = X_k$ and due to equalities (2.21), we conclude from (2.19) and (2.25) that $\Gamma_i = X_i$ for $i = 1, \ldots, k$. We now see from (2.26) and (2.20) that

$$\begin{aligned}\Delta_{\mathbf{r}}(f; \boldsymbol{\beta}) &= \Gamma_1 \mathcal{J}_{\boldsymbol{\beta}}^{\top} - \alpha_1 \Gamma_1 + [\alpha_1; f]_{\boldsymbol{\ell}} E_m \\ &= X_1 \mathcal{J}_{\boldsymbol{\beta}}^{\top} - \alpha_1 X_1 + [\alpha_1; f_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}} E_m = \Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta}).\end{aligned}$$

Therefore, f also satisfies the second condition in (2.23) being, therefore, a solution to the problem (**P**).

It remains to show that the correspondence $X \mapsto f_X$ in (1.7) is one-to-one. Indeed, if $X \neq \widetilde{X}$ are two solutions to the equation (1.6), then (since X and $\widetilde{X}$ are completely determined from the entries from their bottom rows) we have $X_{k,j} \neq \widetilde{X}_{k,j}$ for some $j \in \{1, \ldots, m\}$. Then the polynomials f_X and $f_{\widetilde{X}}$ produced by formula (1.6) are not the same, since

$$[\alpha_1, \ldots, \alpha_k; f_X; \beta_1, \ldots, \beta_j] = X_{k,j} \neq \widetilde{X}_{k,j} = [\alpha_1, \ldots, \alpha_k; f_{\widetilde{X}}; \beta_1, \ldots, \beta_j]. \qquad \square$$

Remark 2.4. The polynomial f_X associated with a solution $X = [X_{i,j}] \in \mathbb{H}^{k \times m}$ to the equation (1.6) via formula (1.7) can be alternatively written as

$$f_X = f_{\mathbf{r}} + \big(X_{1,m} + \boldsymbol{\rho}_{\alpha_1} X_{2,m} + \boldsymbol{\rho}_{\alpha_1}\boldsymbol{\rho}_{\alpha_2} X_{3,m} + \ldots + \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_{k-1}} X_{k,m}\big) \cdot H. \quad (2.27)$$

Proof. Formula (1.7) specifies left and mixed divided differences of f_X as follows:

$$[\alpha_1, \ldots, \alpha_i; f_X]_{\boldsymbol{\ell}} = [\alpha_1, \ldots, \alpha_i; f_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}} \quad \text{and} \quad [\alpha_1, \ldots \alpha_k; f_X; \beta_1, \ldots, \beta_j] = X_{k,j}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, m$. Since X is a solution of (1.6), we have $\Gamma(\boldsymbol{\alpha}; f_X; \boldsymbol{\beta}) = X$, by Corollary 2.3. Comparing the rightmost columns in the latter equality gives

$$[\alpha_1, \ldots \alpha_i; f_X; \beta_1, \ldots, \beta_m] = X_{i,m} \quad \text{for} \quad i = 1, \ldots, k. \quad (2.28)$$

Since f_X satisfies the second condition in (1.1) and since $\deg f_{\mathbf{r}} < m$, we have

$$f_X = f_{\mathbf{r}} + \widetilde{p} H, \quad \text{where } \widetilde{p} = R_{\beta_m} \cdots R_{\beta_1}(f_X - f_{\mathbf{r}}) = R_{\beta_m} \cdots R_{\beta_1} f_X. \quad (2.29)$$

Due to the latter formula for $\widetilde{p}$ and on account of (2.28),

$$[\alpha_1, \ldots, \alpha_i; \widetilde{p}]_{\boldsymbol{\ell}} = [\alpha_1, \ldots, \alpha_i; f_X; \beta_1, \ldots, \beta_m] = X_{i,m} \quad (i = 1, \ldots, k). \quad (2.30)$$

Since $\deg \widetilde{p} = \deg f_X - m < k$, we can represent $\widetilde{p}$ as in the first formula in (2.5) and then invoke equalities (2.30) to get

$$\widetilde{p} = [\alpha_1; \widetilde{p}]_{\boldsymbol{\ell}} + \sum_{i=1}^{k-1} \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_i} [\alpha_1, \ldots, \alpha_{i+1}; \widetilde{p}]_{\boldsymbol{\ell}} = X_{1,m} + \sum_{i=1}^{k-1} \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_i} X_{i+1,m}.$$

Substituting the latter representation for $\widetilde{p}$ into (2.29) gives (2.27). $\square$

3. Related questions

In this section, we discuss several questions related to Theorem 1.1. Recall that for a quaternionic polynomial $f(z) = \sum f_k z^k$, its polynomial conjugate is defined as $f^\sharp(z) = \sum \overline{f}_k z^k$. It is readily verified that $(fg)^\sharp = g^\sharp f^\sharp$ for all $f, g \in \mathbb{H}[z]$ and that $ff^\sharp = f^\sharp f$ is a real polynomial for any $f \in \mathbb{H}[z]$.

We will write $\alpha \sim \beta$ if the quaternions α and β are *similar* (i.e., $\alpha = h^{-1}\beta h$ for some nonzero $h \in \mathbb{H}$) and we will denote by $[\alpha]$ the *similarity class* of a given $\alpha \in \mathbb{H}$. It turns out that

$$\alpha \sim \beta \quad \Longleftrightarrow \quad \Re(\alpha) = \Re(\beta) \text{ and } |\alpha| = |\beta| \quad \Longleftrightarrow \quad \boldsymbol{\rho}_\alpha \boldsymbol{\rho}_\alpha^\sharp = \boldsymbol{\rho}_\beta \boldsymbol{\rho}_\beta^\sharp \tag{3.1}$$

where the first equivalence was pointed out in [12] and the second is trivial, since

$$\mathcal{X}_{[\alpha]} := \boldsymbol{\rho}_\alpha(z)\boldsymbol{\rho}_\alpha^\sharp(z) = z^2 - 2z\Re(\alpha) + |\alpha|^2. \tag{3.2}$$

The polynomial $\mathcal{X}_{[\alpha]} \in \mathbb{R}[z]$ is called the *characteristic polynomial* of the similarity class $[\alpha]$ which in turn, due to (3.1) is a 2-sphere (of radius $|\Im(\alpha)|$ around $\Re(\alpha)$).

3.1. The determinate case

The problem (**P**) having a unique low-degree solution is of some particular interest. By Theorem 1.1, such a determinacy occurs if and only if the Sylvester equation (1.6) has a unique solution.

Proposition 3.1. *Let G and H be factored as in* (1.3) *and let $\mathcal{J}_{\boldsymbol{\alpha}}$ and $\mathcal{J}_{\boldsymbol{\beta}}$ be defined via formula* (1.4)*. The following are equivalent:*

(1) *Real polynomials $GG^\sharp$ and $H^\sharp H$ are coprime.*
(2) *$\alpha_i \not\sim \beta_j$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m\}$.*
(3) *For any given $C \in \mathbb{H}^{k\times m}$, the Sylvester equation $\mathcal{J}_{\boldsymbol{\alpha}} X - X\mathcal{J}_{\boldsymbol{\beta}}^\top = C$ has a unique solution.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from (1.3) and (3.1), while (1) $\Leftrightarrow$ (2) was established in [13] via the well-known solvability criterion for the complex Sylvester equation (1.10) (with $\varphi(C)$ on the right-hand side). □

Although the next result follows from Theorem 1.1 and Proposition 3.1, we will provide a short alternative proof.

Theorem 3.2. *Let the real polynomials $GG^\sharp$ and $H^\sharp H$ be coprime. Then*

1. *The problem (**P**) admits a unique low-degree solution.*
2. *The solution set to the homogeneous problem (i.e., $f_{\boldsymbol{\ell}} \equiv f_{\mathbf{r}} \equiv 0$) equals*

$$\boldsymbol{\mathcal{B}}_{G,H} = \langle G\rangle_{\mathbf{r}} \cdot \langle H\rangle_{\boldsymbol{\ell}} = G \cdot \mathbb{H}[z] \cdot H = \{GqH : \ q \in \mathbb{H}[z]\}. \tag{3.3}$$

Proof. By the division algorithm, any $f \in \mathbb{H}[z]$ can be uniquely written as

$$f = \widetilde{f} + GqH \quad \text{for some} \quad q, \widetilde{f} \in \mathbb{H}[z], \quad \deg \widetilde{f} < \deg G + \deg H, \tag{3.4}$$

and furthermore, f in (3.4) is a solution to the problem (**P**) if and only if $\widetilde{f}$ is. It remains to show that the problem has a solution. Indeed, since $GG^\sharp$ and $H^\sharp H$ are

coprime, there exist polynomials $A, B \in \mathbb{R}[z]$ such that $AGG^\sharp + BH^\sharp H \equiv 1$. Since real polynomials form the center of $\mathbb{H}[z]$, we have

$$\begin{aligned} f := f_{\boldsymbol{\ell}} + GG^\sharp(f_{\mathbf{r}} - f_{\boldsymbol{\ell}})A &= f_{\boldsymbol{\ell}} + (f_{\mathbf{r}} - f_{\boldsymbol{\ell}})AGG^\sharp \\ &= f_{\boldsymbol{\ell}} + (f_{\mathbf{r}} - f_{\boldsymbol{\ell}})(1 - BH^\sharp H) \\ &= f_{\mathbf{r}} + (f_{\boldsymbol{\ell}} - f_{\mathbf{r}})BH^\sharp H, \end{aligned}$$

and therefore, f satisfies (1.1) with $p = G^\sharp(f_{\mathbf{r}} - f_{\boldsymbol{\ell}})A$ and $\widetilde{p} = (f_{\boldsymbol{\ell}} - f_{\mathbf{r}})BH^\sharp$. □

The latter proof suggests the construction of the unique low degree solution by means of the (real) division algorithm. On the other hand, the explicit formula for this distinguished solution can be derived by combining (1.7) (or (2.27)) and *any* formula for the unique solution of the Sylvester equation (1.6) known from literature (e.g., [13, 18, 9]).

3.2. The structure of the solution set

Given the polynomials $G, H \in \mathbb{H}[z]$, let

$$\boldsymbol{\mathcal{Q}}_{G,H} = \langle G \rangle_{\mathbf{r}} \cap \langle H \rangle_{\boldsymbol{\ell}} \tag{3.5}$$

be the intersection of the left and the right ideal (a quasi-ideal of $\mathbb{H}[z]$, according to the terminology in [19]) and let

$$\boldsymbol{\mathcal{M}}_{G,H} = \{f \in \boldsymbol{\mathcal{Q}}_{G,H} : \deg f < \deg G + \deg H\} \tag{3.6}$$

be the set of all low-degree elements of $\boldsymbol{\mathcal{Q}}_{G,H}$. The set $\boldsymbol{\mathcal{B}}_{G,H}$ defined in (3.3) is another subset of $\boldsymbol{\mathcal{Q}}_{G,H}$. Taking another look at the representation (3.4), we conclude that f in that representation belongs to $\boldsymbol{\mathcal{Q}}_{G,H}$ if and only if $\widetilde{f} \in \boldsymbol{\mathcal{M}}_{G,H}$. Hence,

$$\boldsymbol{\mathcal{Q}}_{G,H} = \boldsymbol{\mathcal{M}}_{G,H} \oplus \boldsymbol{\mathcal{B}}_{G,H}, \tag{3.7}$$

where the direct sum is understood in the sense of real vector spaces. It is clear from (1.1) that $\boldsymbol{\mathcal{Q}}_{G,H}$ is the solution set to the homogeneous problem (**P**) (i.e., with $f_{\boldsymbol{\ell}} \equiv f_{\mathbf{r}} \equiv 0$). Since the (non-homogeneous) problem (**P**) is linear, its solution set can be written as

$$\widetilde{f} + \boldsymbol{\mathcal{Q}}_{G,H} = \widetilde{f} + \boldsymbol{\mathcal{M}}_{G,H} + \boldsymbol{\mathcal{B}}_{G,H} \tag{3.8}$$

where $\widetilde{f}$ is a fixed particular low-degree solution constructed, e.g., as in Theorem 1.1. The two first terms on the right-hand side of (3.8) represent all low-degree solutions to the problem. Observe that in case the polynomials $GG^\sharp$ and $H^\sharp H$ are coprime, $\boldsymbol{\mathcal{M}}_{G,H} = \{0\}$ (due to (3.7), since $\boldsymbol{\mathcal{Q}}_{G,H} = \boldsymbol{\mathcal{B}}_{G,H}$, by Theorem 3.2).

By formula (3.3), the set $\boldsymbol{\mathcal{B}}_{G,H}$ is the one-parameter set (parametrized by the free parameter $q \in \mathbb{H}[z]$). Hence, to make the formula (3.8) more suggestive, it remains to parametrize the set $\boldsymbol{\mathcal{M}}_{G,H}$ (in the case where this set is nontrivial). Although the homogeneous versions of formula (1.7) (with $f_{\boldsymbol{\ell}} = 0$) or formula (2.27) (with $f_{\mathbf{r}} = 0$) provide some parametrizations of $\boldsymbol{\mathcal{M}}_{G,H}$, the parameters $X_{i,j}$ in these parametrizations are neither independent nor free. We briefly recall a more plausible parametrization established in [11].

Let $[\gamma]$ be a non-real similarity class and let $\mathcal{X}_{[\gamma]}$ be its chacracteristic polynomial defined as in (3.2). We say that $[\gamma]$ is a *spherical zero* of $f \in \mathbb{H}[z]$ of *spherical multiplicity* $\mathfrak{s}(f;[\gamma]) = \kappa \geq 0$ if $f \in \langle \mathcal{X}_{[\gamma]}^{\kappa} \rangle \backslash \langle \mathcal{X}_{[\gamma]}^{\kappa+1} \rangle$ (i.e., $\mathcal{X}_{[\gamma]}^{\kappa}$ divides f, while $\mathcal{X}_{[\gamma]}^{\kappa+1}$ does not).

Theorem 3.3. *Given $G, H \in \mathbb{H}[z]$, let $[\gamma_1], \ldots, [\gamma_n]$ be all distinct common spherical zeros of real polynomials $GG^{\sharp}$ and $HH^{\sharp}$ and let $\mathbf{k} = \mathbf{k}_1 + \ldots + \mathbf{k}_n$, where*

$$\begin{aligned}\mathbf{k}_i = &\min\{(\mathfrak{s}(GG^{\sharp};[\gamma_i]) - \mathfrak{s}(G;[\gamma_i]),\ \mathfrak{s}(HH^{\sharp};[\gamma_i]) - \mathfrak{s}(H;[\gamma_i]))\} \\ &- \max\{(\mathfrak{s}(G;[\gamma_i]),\ \mathfrak{s}(H;[\gamma_i]))\} \quad \textit{for} \quad i = 1, \ldots, n. \end{aligned} \tag{3.9}$$

Then $\boldsymbol{\mathcal{M}}_{G,H}$ is the $2\mathbf{k}$-dimensional real vector space, and moreover, there exist monic polynomials P_{ij}, Q_{ij} and quaternions $\eta_{ij}, \nu_{ij} \in [\gamma_i]$ for $j = 1, \ldots, \mathbf{k}_i$ and $i = 1, \ldots, n$, so that

$$\boldsymbol{\mathcal{M}}_{G,H} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\mathbf{k}_i} P_{ij} \Pi_{\eta_{ij},\nu_{ij}} Q_{ij}, \tag{3.10}$$

where $\Pi_{\eta_{ij},\nu_{ij}} := \{\mu \in \mathbb{H} : \eta_{ij}\mu = \mu\nu_{ij}\}$ denotes the two-dimensional real subspace of $\mathbb{H}$ consisting of all elements that intertwine η_{ij} and ν_{ij}.

Formula (3.10) gives rise to the parametrization of $\boldsymbol{\mathcal{M}}_{G,H}$ with $\mathbf{k}$ free independent parameters $\mu_{ij} \in \Pi_{\gamma_{ij},\eta_{ij}}$. We refer to [11] for more details including explicit formulas for P_{ij}, Q_{ij}, η_{ij}, ν_{ij} in terms of the given polynomials G and H. Here, we recall one particular case where all α_i and β_j in (1.3) are similar to each other (i.e., $n = 1$) and in addition,

$$\alpha_{i+1} \neq \overline{\alpha}_i \ \ (1 \leq i < k), \quad \beta_{j+1} \neq \overline{\beta}_j \ \ (1 \leq j < m). \tag{3.11}$$

In this case, $\mathfrak{s}(G;[\alpha_1])) = \mathfrak{s}(H;[\alpha_1])) = 0$, $\mathfrak{s}(GG^{\sharp};[\alpha_1]) = k$, $\mathfrak{s}(HH^{\sharp};[\alpha_1]) = m$, and we conclude from formula (3.10) that $\mathbf{k} = \min\{m, k\}$. Let $\mathcal{X}_{[\alpha_1]}$ be the characteristic polynomial (3.2), let $G_0 = H_0 = 1$ and let

$$G_i = \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_i} \ \ (1 \leq i \leq k), \quad H_j = \boldsymbol{\rho}_{\beta_j} \cdots \boldsymbol{\rho}_{\beta_1} \ \ (1 \leq j \leq m).$$

By [11, Theorem 3.4]), $\boldsymbol{\mathcal{M}}_{G,H}$ admits the direct-sum representation

$$\boldsymbol{\mathcal{M}}_{G,H} = \bigoplus_{j=0}^{\min(k,m)-1} \mathcal{X}_{[\alpha_1]}^{j} G_{k-j} \cdot \Pi_{\alpha_{k-j},\beta_{m-j}} \cdot H_{m-j-1}. \tag{3.12}$$

We next illustrate Theorem 1.1 and Theorem 3.3 by a numerical example.

Example 3.4. *We want to find an $f \in \mathbb{H}[z]$ satisfying conditions* (1.1) *where*

$$G(z) = (z - \mathbf{i})(z - \mathbf{j}), \quad H(z) = (z - \mathbf{k})(z - \mathbf{i})$$

(i.e., $\alpha_1 = \beta_1 = \mathbf{i}$, $\alpha_2 = \mathbf{j}$, $\beta_2 = \mathbf{k}$), and

$$f_{\boldsymbol{\ell}}(z) = z\mathbf{j} + \mathbf{i} - \mathbf{k} \quad \textit{and} \quad f_{\mathbf{r}}(z) = (1 - \mathbf{i} + 2\mathbf{j} - \mathbf{k})z - 1 + \mathbf{j} - 4\mathbf{k}.$$

Thus, we have $\mathcal{J}_{\boldsymbol{\alpha}} = \begin{bmatrix} \mathbf{i} & 0 \\ 1 & \mathbf{j} \end{bmatrix}$, $\mathcal{J}_{\boldsymbol{\beta}}^{\top} = \begin{bmatrix} \mathbf{i} & 1 \\ 0 & \mathbf{k} \end{bmatrix}$ *and, according to formulas* (1.5),

$$\begin{aligned} \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}}) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\mathbf{i} - \mathbf{k}) + \begin{bmatrix} \mathbf{i} \\ 1 \end{bmatrix} \mathbf{j} = \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix}, \\ \Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta}) &= (-1 + \mathbf{j} - 4\mathbf{k}) \begin{bmatrix} 1 & 0 \end{bmatrix} + (1 - \mathbf{i} + 2\mathbf{j} - \mathbf{k}) \begin{bmatrix} \mathbf{i} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{i} - 6\mathbf{k} & 1 - \mathbf{i} + 2\mathbf{j} - \mathbf{k} \end{bmatrix}. \end{aligned}$$

By Theorem 1.1, *a desired* $f \in \mathbb{H}[z]$ *exists if and only if the equation*

$$\begin{bmatrix} \mathbf{i} & 0 \\ 1 & \mathbf{j} \end{bmatrix} X - X \begin{bmatrix} \mathbf{i} & 1 \\ 0 & \mathbf{k} \end{bmatrix} = \begin{bmatrix} 6\mathbf{k} & -1 + \mathbf{i} - 2\mathbf{j} + \mathbf{k} \\ \mathbf{j} & 0 \end{bmatrix} \tag{3.13}$$

has a solution. Passing to the associated complex equation (1.10) *we get*

$$\begin{bmatrix} \mathbf{i} & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -\mathbf{i} & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} Y - Y \begin{bmatrix} \mathbf{i} & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} \\ 0 & 0 & -\mathbf{i} & 1 \\ 0 & \mathbf{i} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 + \mathbf{i} & 6\mathbf{i} & -2 + \mathbf{i} \\ 0 & 0 & 1 & 0 \\ 6\mathbf{i} & 2 + \mathbf{i} & 0 & -1 - \mathbf{i} \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.14}$$

Using the tensor-product form of the latter equation we verify that it has solutions. Using the same tensor-product equation, we can find all solutions to the equation (3.14). *For example, the matrix*

$$Y = \begin{bmatrix} -2\mathbf{i} & -1 + \mathbf{i} & 3 & 0 \\ -2 & -2 & 2\mathbf{i} & 1 - \mathbf{i} \\ -3 & -2 & 2\mathbf{i} & 1 - \mathbf{i} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a solution to (3.14), *and we may use formula* (1.11) *to get*

$$X = \begin{bmatrix} -2\mathbf{i} & \mathbf{i} \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ \mathbf{i} & \frac{1-\mathbf{i}}{2} \end{bmatrix} \mathbf{j} = \begin{bmatrix} -2\mathbf{i} + 3\mathbf{j} & \mathbf{i} + \mathbf{j} \\ \mathbf{k} - 1 & -1 + \frac{\mathbf{j}-\mathbf{k}}{2} \end{bmatrix},$$

the corresponding solution to the equation (3.13). *Formula* (1.7) *then produces*

$$\begin{aligned} f_X(z) &= f_{\boldsymbol{\ell}}(z) + (z - \mathbf{i})(z - \mathbf{j}) \left(\mathbf{k} - 1 + \left(-1 + \frac{\mathbf{j} - \mathbf{k}}{2} \right) (z - \mathbf{i}) \right) \\ &= f_{\mathbf{r}}(z) + \left(\mathbf{i} + \mathbf{j} + (z - \mathbf{i}) \left(-1 + \frac{\mathbf{j} - \mathbf{k}}{2} \right) \right) (z - \mathbf{k})(z - \mathbf{i}), \end{aligned} \tag{3.15}$$

a low-degree solution to the problem (**P**) *(which is also confirmed by the second representation in* (3.15)*). Since* $\alpha_i \sim \beta_j$ *and* $\alpha_2 \neq \overline{\alpha}_1$, $\overline{\beta}_2 \neq \beta_2$, *it follows from* (3.12) *that* $\boldsymbol{\mathcal{M}}_{G,H}$ *is spanned by the polynomials*

$$\mathcal{X}_{[\alpha_1]} \boldsymbol{\rho}_{\alpha_1} \mu_1 = (z^2 + 1)(z - \mathbf{i})\mu_1 \quad \textit{and} \quad \boldsymbol{\rho}_{\alpha_1} \boldsymbol{\rho}_{\alpha_2} \mu_2 \boldsymbol{\rho}_{\beta_1} = (z - \mathbf{i})(z - \mathbf{j})\mu_2 (z - \mathbf{i})$$

with μ_1, μ_2 *subject to intertwining relations* $\mathbf{i}\mu_1 = \mu_1 \mathbf{i}$ *and* $\mathbf{j}\mu_2 = \mu_2 \mathbf{k}$, *or equivalently, with* μ_1, μ_2 *of the form*

$$\mu_1 = x_1 + x_2 \mathbf{i} \quad \textit{and} \quad \mu_2 = x_3(1 - \mathbf{i}) + x_4(\mathbf{j} + \mathbf{k}) \qquad (x_i \in \mathbb{R}). \tag{3.16}$$

According to (3.8)*, all low-degree solutions f to the problem* $\mathbf{P}$ *with the present choice of $G, H, f_{\boldsymbol{\ell}}, f_{\mathbf{r}}$ are produced by the formula*

$$f(z) = f_X(z) + (z^2+1)(z-\mathbf{i})\mu_1 + (z-\mathbf{i})(z-\mathbf{j})\mu_2(z-\mathbf{i}), \tag{3.17}$$

where f_X is given by (3.15) *and μ_1, μ_2 are any quaternions of the form* (3.16)*. It is readily seen from* (3.15) *that the leading coefficient f_3 of the cubic polynomial f in* (3.17) *equals $f_3 = -1 + \mathbf{j}/2 - \mathbf{k}/2 + \mu_1 + \mu_2$. Since $f_3 \neq 0$ for any μ_1, μ_2 of the form* (3.16)*, it follows that all low-degree solutions to the problem are of degree three. In general, finding minimal degree solutions to the problem (*$\mathbf{P}$*) (even finding the value of the minimally possible degree) is an open question.*

3.3. The sum of two ideals

If the polynomials $GG^\sharp$ and $HH^\sharp$ are coprime, the sum $\boldsymbol{\mathcal{S}}_{G,H} = \langle G\rangle_{\mathbf{r}} + \langle H\rangle_{\boldsymbol{\ell}}$ of ideals generated by G and H equals the whole $\mathbb{H}[z]$, since $\mathbb{H}[z] = \langle GG^\sharp\rangle + \langle HH^\sharp\rangle \subseteq \langle G\rangle_{\mathbf{r}} + \langle H\rangle_{\boldsymbol{\ell}} = \boldsymbol{\mathcal{S}}_{G,H}$. In general, $\boldsymbol{\mathcal{S}}_{G,H}$ can be contained in $\mathbb{H}[z]$ properly, and in this case, Theorem 1.1 gives us the condition verifying the containment of a given polynomial in $\boldsymbol{\mathcal{S}}_{G,H}$.

Remark 3.5. Let G and H be factored as in (1.3) and let the matrices $\mathcal{J}_{\boldsymbol{\alpha}}$, $\mathcal{J}_{\boldsymbol{\beta}}$ be as in Theorem 1.1. Given $g \in \mathbb{H}[z]$, let $\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g)$ be defined as in (2.10). Then g belongs to $\boldsymbol{\mathcal{S}}_{G,H}$ if and only if the equation

$$\mathcal{J}_{\boldsymbol{\alpha}} X - X\mathcal{J}_{\boldsymbol{\beta}}^\top = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g) E_m^\top \tag{3.18}$$

has a solution. Indeed, $g \in \boldsymbol{\mathcal{S}}_{G,H}$ if and only if the problem ($\mathbf{P}$) with $f_{\boldsymbol{\ell}} = g$ and $f_{\mathbf{r}} = 0$ has a solution. It remains to note that for this choice of $f_{\boldsymbol{\ell}}$ and $f_{\mathbf{r}}$, equations (1.6) and (3.18) are identical.

3.4. The product-ideal interpolation scheme

In analogy to ideal interpolation schemes introduced in [6], we may say that the problem considered in Theorem 3.2 is an example of a *product-ideal interpolation scheme*, as the solution set $\boldsymbol{\mathcal{B}}_{G,H}$ of its homogeneous counter-part is the the product of ideals $\langle G\rangle_{\mathbf{r}}$ and $\langle G\rangle_{\boldsymbol{\ell}}$ (see (3.2)). A more general example of a problem of this type is given in (3.20) and covered in Theorem 3.7. The next lemma characterizes $\boldsymbol{\mathcal{B}}_{G,H}$ (without any coprimeness assumptions) as the solution set of certain homogeneous interpolation problems.

Lemma 3.6. *Let G and H be factored as in* (1.3) *and let $\boldsymbol{\mathcal{Q}}_{G,H}$, $\boldsymbol{\mathcal{B}}_{G,H}$ be defined as in* (3.3)*. The following are equivalent:*

(a) $f \in \boldsymbol{\mathcal{B}}_{G,H}$ (*i.e., $f = GpH$ for some $p \in \mathbb{H}[z]$*);

(b) $f \in \langle G\rangle_{\mathbf{r}}$ *and* $[\alpha_1, \dots \alpha_k; f; \beta_1, \dots, \beta_j] = 0$ $(1 \le j \le m)$;

(c) $f \in \langle H\rangle_{\boldsymbol{\ell}}$ *and* $[\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_m] = 0$ $(1 \le i \le k)$;

(d) $f \in \boldsymbol{\mathcal{Q}}_{G,H}$ *and* $[\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_j] = 0$ $(1 \le i \le k;\ 1 \le j \le m)$.

(3.19)

Proof. If $f = GpH$, then $f \in \langle G\rangle_{\mathbf{r}} \cap \langle H\rangle_{\boldsymbol{\ell}} = \boldsymbol{\mathcal{Q}}_{G,H}$, and by (2.9),

$$[\alpha_1, \dots, \alpha_i; f; \beta_1] = (L_{\alpha_i} \cdots L_{\alpha_1} f)^{\boldsymbol{e_r}}(\beta_j) = (\boldsymbol{\rho}_{\alpha_{i+1}} \cdots \boldsymbol{\rho}_{\alpha_k}\, pH)^{\boldsymbol{e_r}}(\beta_1) = 0.$$

Furthermore, by the first formula in (2.8), we have for all $j \geq 2$,

$$\begin{aligned}[\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_j] &= (L_{\alpha_i} \cdots L_{\alpha_1} R_{\beta_{j-1}} \cdots R_{\beta_1} f)^{\boldsymbol{e_r}}(\beta_j) \\ &= (\boldsymbol{\rho}_{\alpha_{i+1}} \cdots \boldsymbol{\rho}_{\alpha_k}\, p\, \boldsymbol{\rho}_{\beta_m} \cdots \boldsymbol{\rho}_{\beta_j})^{\boldsymbol{e_r}}(\beta_j) = 0,\end{aligned}$$

from which the implication (a)⇒(d) follows.

To verify (b)⇒(a), assume that $f = Gh$ for some $h \in \mathbb{H}[z]$. Then $h = L_{\alpha_k} \dots L_{\alpha_1} f$, and according to (2.8), equalities in (3.19) can be written in terms of h as $[h; \beta_1, \dots, \beta_j] = 0$ $(1 \leq j \leq m)$. By Remark 2.1(2), the latter equalities imply that $h = pH$ for some $p \in \mathbb{H}[z]$, so that $f = Gh = GpH \in \boldsymbol{\mathcal{B}}_{G,H}$. Implication (c)⇒(a) is verified similarly, while implications (d)⇒(b), (c) are trivial. □

Here is the non-homogeneous version of the interpolation problem (3.3):

(**P′**): *Given collections of "points"* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ *and* $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$*, and given target values* c_i*,* d_j *and* γ_{ij}*, find a polynomial* $f \in \mathbb{H}[z]$ *such that*

$$[\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}} = c_i, \quad [f; \beta_1, \dots, \beta_j]_{\mathbf{r}} = d_j, \quad [\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_j] = \gamma_{ij} \tag{3.20}$$

for all $i = 1, \dots, k$ *and* $j = 1, \dots, m$.

With the rightmost conditions dropped in (3.20), the problem amounts (by Remark 2.1) to the problem (**P**) with

$$f_{\boldsymbol{\ell}} = c_1 + \sum_{i=2}^{k} \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_{i-1}} c_i \quad \text{and} \quad f_{\mathbf{r}} = d_1 + \sum_{j=2}^{m-1} d_j \boldsymbol{\rho}_{\beta_{j-1}} \cdots \boldsymbol{\rho}_{\beta_1}.$$

Although (3.20) contains $m + k + mk$ conditions, it is clear from Lemma 3.6 that only $k + m$ of them are independent. For the problem (**P′**) to have a solution, its data must satisfy certain consistency condition which (as is expected from the preceding analysis) can be written in terms of a certain Sylvester equality.

Theorem 3.7. *Let* G *and* H *be defined as in* (1.3) *and let* $\mathcal{J}_{\boldsymbol{\alpha}}$ *and* $\mathcal{J}_{\boldsymbol{\beta}}$ *be constructed via formula* (1.4)*. The problem* (**P′**) *has a solution if and only if the matrices*

$$\Gamma = [\gamma_{ij}]_{i=1,\dots,k}^{j=1,\dots,m}, \quad C = \begin{bmatrix} c_1 & \cdots & c_k \end{bmatrix}^{\top} \quad \text{and} \quad D = \begin{bmatrix} d_1 & \cdots & d_m \end{bmatrix}$$

satisfy the Sylvester equality

$$\mathcal{J}_{\boldsymbol{\alpha}} \Gamma - \Gamma \mathcal{J}_{\boldsymbol{\beta}}^{\top} = C E_m^{\top} - E_k D.$$

If this is the case, then the solution set to the problem (**P′**) *equals* $\widetilde{f} + \boldsymbol{\mathcal{B}}_{G,H}$*, where*

$$\widetilde{f} = c_1 + \sum_{i=2}^{k} \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_{i-1}} c_i + G \cdot \left(\gamma_{k1} + \sum_{j=2}^{m-1} \gamma_{kj} \boldsymbol{\rho}_{\beta_{j-1}} \cdots \boldsymbol{\rho}_{\beta_1} \right) \tag{3.21}$$

is a unique solution to the problem of degree less than $k + m$.

Proof. The first statement and formula (3.21) for the low-degree solution follows from Theorem 1.1. The rest follows since any solution f to the problem $(\mathbf{P}')$ can be written in the form (3.20) where $\widetilde{f}$ is a low-degree solution to the problem, and since the difference of any two solutions belongs to $\boldsymbol{\mathcal{B}}_{G,H}$, by Lemma 3.6. □

3.5. Further questions

With just minor modifications, the problem $(\mathbf{P})$ can be stated in the setting of the ring $\mathbb{H}[[z]]$ of formal power series over quaternions as follows: *given the polynomials* G, H, $f_{\boldsymbol{\ell}}$, $f_{\mathbf{r}} \in \mathbb{H}[z]$, *find* $f \in \mathbb{H}[z]$ *satisfying conditions* (1.1) *for some* $p, \widetilde{p} \in \mathbb{H}[[z]]$. The solvability criterion is the same as in the polynomial case, and the solution set is parametrized by the formula (3.8), where by $\boldsymbol{\mathcal{B}}_{G,H}$ we mean $G \cdot \mathbb{H}[[z]] \cdot H$. The results still are pretty much the same if one replaces $\mathbb{H}[[z]]$ by the Hardy space H^2 of square summable power series over quaternions [4, 2]. However, in this more specific context, two extra questions (which we leave open) arise. The first question is to find solutions with the minimally possible H^2-norm. For one-sided problems, the answer is known [4, 8] (see [1] for the two-sided complex matrix-valued case). Moreover, as was shown in [10, 8], conditions (1.1) (with $p, \widetilde{p} \in \mathrm{H}^2$) can be replaced equivalently by

$$f = h_{\boldsymbol{\ell}} + \Theta_{\boldsymbol{\ell}} p \quad \text{and} \quad f = h_{\mathbf{r}} + \widetilde{p}\Theta_{\mathbf{r}}$$

where the sums are orthogonal (in the metric of H^2) and where $\Theta_{\boldsymbol{\ell}}$ and $\Theta_{\mathbf{r}}$ are quaternionic finite Blaschke products having the same left (right) zero structure as G and H, respectively. The second question is to find an (orthogonal) analog of decompositions (3.7), (3.10) for the vector space $\boldsymbol{\mathcal{Q}}_{\Theta_{\boldsymbol{\ell}},\Theta_{\mathbf{r}}}$.

It is also of interest to consider the problem $(\mathbf{P})$ in the algebra of multipliers of H^2, that is, in the set of quaternionic power series F such that the left (or right) multiplication operator $M_{F,\boldsymbol{\ell}} : h \mapsto Fh$ (or, respectively, $M_{F,\mathbf{r}} : h \mapsto hF$) is bounded on H^2. The multiplier norm of F is defined to be $\|F\|_{\mathrm{mult}} := \|M_{F,\boldsymbol{\ell}}\| = \|M_{F,\mathbf{r}}\|$. The first question here is to find a solution to the problem $(\mathbf{P})$ with the minimally possible multiplier norm. It is expected (by convexity arguments) that such a solution $F_{\min}$ is unique and equals to a scaled finite Blaschke product (the one-sided version can be found in [3, 4]). The next question is to find the explicit formula for $\mu = \|F_{\min}\|_{\mathrm{mult}}$ in terms of the interpolation data set. The answer for this question will provide, in particular, necessary and sufficient conditions for the existence of a *contractive* multiplier F (with $\|F\|_{\mathrm{mult}} \le 1$) solving the problem $(\mathbf{P})$. The latter class is of particular interest: as was shown in [3] (in slightly different terms),

$$\|F\|_{\mathrm{mult}} = \sup_{|\alpha|<1} |F^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha)| = \sup_{|\alpha|<1} |F^{\boldsymbol{e}_{\mathbf{r}}}(\alpha)|,$$

and hence, contractive multipliers of H^2 are characterized by the property that via evaluation functionals $\alpha \mapsto F^{\boldsymbol{e}_{\boldsymbol{\ell}}}(\alpha)$ and $\alpha \mapsto F^{\boldsymbol{e}_{\mathbf{r}}}(\alpha)$, they map the unit quaternionic ball $\mathbb{B}$ into itself. Parametrization of the set of contractive multipliers of H^2 that are solutions to the problem $(\mathbf{P})$ is the next problem we leave open.

A separate question is to characterize the class of interpolation problems that are equivalent to a problem $(\mathbf{P}')$ and therefore, can be solved using the Newton interpolation scheme of Theorem 3.7. In the commutative case, such problems appear only in the matrix-valued context; we refer to [5] for a thorough account of the subject.

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Fischer Decomposition in Generalized Fractional Clifford Analysis

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This paper is dedicated to Professor Daniel Alpay, on the occasion of his 60th birthday.

Abstract. In this paper we present the basic tools of a fractional function theory in higher dimensions by means of a fractional correspondence to the Weyl relations via Gelfond–Leontiev operators of generalized differentiation. A Fischer decomposition is established. Furthermore, we give an algorithm for the construction of monogenic homogeneous polynomials of arbitrary degree.

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1. Introduction

What is nowadays called (classic) Clifford analysis, or hypercomplex function theories consists in the establishment of a function theory for null-solutions of the Dirac operator. Among other things it represents a higher-dimensional generalization of complex analysis, where these null-solutions, also called monogenic functions, replace the notion of a holomorphic function. While such functions can very well describe problems of a particle with internal $SU(2)$-symmetries, higher-order symmetries are beyond the reach of this theory. Although many modifications were suggested over the years (such as Yang–Mills theory), they could not address the principal problem, id est, the need of a n-fold factorization of the d'Alembert operator. One way to approach this problem is using generalized fractional calculus.

In the last decades the interest in generalized fractional calculus increased substantially. This fact is, in part, due to some of its interesting special cases and its applications to different topics of analysis. The basic operators of this calculus, the so-called generalized (multiple) operators of fractional integration are defined by

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means of convolutional-type single integrals involving the Meijer functions $G^{m,0}_{m,m}$ as kernels. The classical Riemann–Liouville fractional integrals are a particular case of these general fractional integrals. In a more general case, the H-functions of Fox appear as kernel functions of those convolutional-type single integrals. These same generalized fractional integral operators can be also considered as compositions of an arbitrary number of commuting Erdélyi–Kober fractional integrals written as repeated integrals without any use of special functions. Since such compositions arise quite often in several problems and branches of different areas of applied mathematics, this relationship proves to be the key to applications of the fractional calculus. From the long list of applications we have: differential and integral equations, operational calculi, convolutional calculi, integral transforms, univalent functions and special functions, among many more.

Nevertheless, in the used setting one often disregards the geometry of the domain. To overcome this difficulty, the authors proposed in [12, 19] to treat fractional calculi by means of Clifford analysis. Clifford analysis is a generalization of classical complex analysis in the plane to the case of an arbitrary dimension $d \in \mathbb{Z}$ (in the case of negative dimensions, one is dealing with the so-called super Clifford analysis). At the heart of the theory lies the Dirac operator D on $\mathbb{R}^d$, a conformally invariant first-order differential operator which plays the same role in classical Clifford analysis as the Cauchy–Riemann operator ∂_z does in complex analysis. Over the last decades F. Sommen and his collaborators developed a method for establishing a higher-dimensional function theory based on the so-called Weyl relations [3, 5, 6]. In more restrictive settings, this is called Howe dual pair technique (see [17]). Its focal point is the construction of an operator algebra (classically $\mathfrak{osp}(1|2)$) and to establish a Fischer decomposition. But this approach runs immediately into problems in the setting of Dirac operators with fractional derivatives. The main reason is that in general they do not allow a construction of a Howe dual pair. Hereby, the principal problem is not the invariance under a fractional spin group, but the construction of a Super-Lie-algebra $\mathfrak{osp}(1|2)$ for general fractional Dirac operators. This requires the application of a new approach and new methods, in particular the establishment of fractional Sommen–Weyl relations.

The traditional Fischer decomposition in harmonic analysis yields an orthogonal decomposition of the space of homogeneous polynomials of fixed degree of homogeneity in terms of spaces of harmonic homogeneous polynomials. In classical continuous Clifford analysis one obtains a refinement yielding an orthogonal decomposition with respect to the so-called Fischer inner product of homogeneous polynomials in terms of spaces of monogenic polynomials, i.e., null solutions of the Dirac operator (see [3]). Generalizations of the Fischer decomposition in other frameworks can be found, for example, in [1, 5, 6, 9, 12, 16, 17, 19] and are mainly based on the establishment of raising and lowering operators acting on a ground state.

The aim of this paper is to present the building blocks of a function theory for fractional Dirac operators based on Gelfond–Leontiev operators of generalized differentiation. These operators were studied in the 1980s and 1990s and allow particular realizations in form of the classical Caputo and Riemann–Liouville fractional derivatives. In this paper we will show that Dirac operators based on these fractional derivatives still allow for a Fischer decomposition, and while not providing an $\mathfrak{osp}(1|2)$ algebra realization, they still permit the explicit construction of the basis of fractional homogeneous monogenic polynomials acting on a ground state. The proposed algorithm will construct these fractional homogeneous monogenic polynomials as linear combinations of fractional homogeneous polynomials. To show the application of this algorithm, MATLAB programs for the three-dimensional case are provided.

The structure of the paper reads as follows: in the Preliminaries we recall some basic facts about Clifford analysis and generalized fractional calculus. In Section 3, we introduce the corresponding Sommen–Weyl relations for the generalized fractional setting, the fractional correspondence to the Fischer decomposition, and its extension to a fractional Almansi decomposition. Based on the calculation of the dimension of the space of fractional homogeneous monogenic polynomials we conclude the paper with the construction of an algorithm for the explicit expression of the basis for the space of fractional homogeneous monogenic polynomials.

2. Preliminaries

2.1. Analysis in higher dimensions

It is well known that the treatment of the two-dimensional vector spaces $\mathbb{R}^2$ in terms of complex numbers has the advantage of providing a multiplication operator. Appropriate higher-dimensional associative analogues of the complex numbers are the real Clifford algebras. For details about Clifford algebras and basic concepts of the associated function theory we refer the interested reader to [2, 3, 11].

Let $\{e_1, \ldots, e_d\}$ be the standard basis of the Euclidean vector space $\mathbb{R}^d$. The associated Clifford algebra $\mathbb{R}_{0,d}$ is the free algebra generated by $\mathbb{R}^d$ modulo $x^2 = -||x||^2$. The defining relation induces the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \quad i, j = 1, \ldots, d,$$

where $\delta_{i,j}$ denoted the Kronecker symbol. In particular, as we have $e_i^2 = -1$, the standard basis vectors operate as imaginary units.

A vector space basis for $\mathbb{R}_{0,d}$ is given by the set

$$\{e_\emptyset = 1, e_A = e_{l_1} e_{l_2} \ldots e_{l_r} : A = \{l_1, l_2, \ldots, l_r\} \subseteq M = \{1, \ldots, d\},\ 1 \leq l_1 < \ldots < l_r \leq d\}.$$

Each $a \in \mathbb{R}_{0,d}$ can be written in the form $a = \sum_A a_A\, e_A$, with $a_A \in \mathbb{R}$. Now, we introduce the complexified Clifford algebra $\mathbb{C}_d$ as the tensor product

$$\mathbb{C} \otimes \mathbb{R}_{0,d} = \left\{ w = \sum_A w_A e_A,\ w_A \in \mathbb{C},\ A \subseteq M \right\},$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j = e_j i$ for all $j = 1, \ldots, d$.

The conjugation in the Clifford algebra $\mathbb{C}_d$ is defined as the automorphism

$$w \mapsto \overline{w} = \sum_A \overline{w}_A\, \overline{e}_A,$$

where $\overline{w}_A$ denotes the usual complex conjugation and $\overline{e}_A = \overline{e}_{l_r}\, \overline{e}_{l_{r-1}} \ldots \overline{e}_{l_1}$, where $\overline{e}_\emptyset = 1$ and $\overline{e}_j = -e_j$ for $j = 1, \ldots, d$. For a vector $w = \sum_{j=1}^d w_j e_j$ we have $w\overline{w} = \|w\|^2 := \sum_{j=1}^d \|w_j\|^2$. Hence, each non-zero vector $w = \sum_{j=1}^d w_j e_j$ has an unique multiplicative inverse given by $w^{-1} = \frac{\overline{w}}{\|w\|^2}$.

A $\mathbb{C}_d$-valued function f over a non-empty domain $\Omega \subset \mathbb{R}^d$ is written as $f = \sum_A f_A e_A$, with components $f_A : \Omega \to \mathbb{C}$. Properties such as continuity are understood component-wisely. For example, $f = \sum_A f_A e_A$ is continuous if and only if all components f_A are continuous. Next, we recall the Euclidean Dirac operator $D = \sum_{j=1}^d e_j\, \partial_{x_j}$, which factorizes the d-dimensional Euclidean Laplacian, i.e., $D^2 = -\Delta = -\sum_{j=1}^d \partial_{x_j}^2$. A $\mathbb{C}_d$-valued function f is said to be *left-monogenic* if it satisfies $Df = 0$ on Ω (resp. *right-monogenic* if it satisfies $fD = 0$ on Ω).

From now on, we define $x^\alpha = \sum_{i=1}^d x_i^\alpha e_i \in \mathbb{C}^d$, where each component x_i^α is a fractional power which should be understood as

$$x_i^\alpha = \begin{cases} \exp(\alpha \ln |x_i^\alpha|); & x_i^\alpha > 0 \\ 0; & x_i^\alpha = 0 \\ \exp(\alpha \ln |x_i^\alpha| + \mathbf{i}\alpha\pi); & x_i^\alpha < 0 \end{cases}, \tag{2.1}$$

with $0 < \alpha < 1$, and $\mathbf{i}$ represents the imaginary unit. The authors remark that in the present manuscript they restrict themselves to the case of $\alpha \in]0, 1[$. Indeed, for values of α outside this range one can always reduce it to the previous case via $\alpha = [\alpha] + \tilde{\alpha}$, where $[\alpha]$ denotes its integer part and $\tilde{\alpha} \in]0, 1[$.

2.2. Generalized fractional derivatives

In this section we recall some basic facts about generalized fractional calculus (for more details we refer [15]). We start with the following definition of generalized differentiation and integration operators.

Definition 2.1. [10, 15] Let the function

$$\varphi(\lambda) = \sum_{k=0}^{\infty} \varphi_k\, \lambda^k$$

be an entire function with order $\rho > 0$ and $\sigma \neq 0$ such that $\lim_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{\frac{1}{\rho}}$. Then, the operation

$$f(z) = \sum_{k=0}^{\infty} a_k \, z^k \quad \xrightarrow{\qquad D_\varphi \qquad} \quad D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k} \, z^{k-1} \tag{2.2}$$

is said to be the Gelfond–Leontiev (G–L) operator of generalized differentiation with respect to the function φ and the corresponding G–L operator, or integration, is

$$I_\varphi f(z) = \sum_{k=0}^{\infty} a_k \, \frac{\varphi_{k+1}}{\varphi_k} \, z^{k+1}. \tag{2.3}$$

From the theory of entire functions, the conditions required for φ should be given in terms of $\limsup_{k\to\infty}$. However, we assume that the limit exits and, therefore, $\lim_{k\to\infty} \sqrt[k]{\left|\frac{\varphi_{k-1}}{\varphi_k}\right|} = 1$. By the Cauchy–Hadamard formula, both series, (2.2) and (2.3), have the same radius of convergence $R > 0$.

Example. Consider a function $f(z) = \sum_{k=0}^{\infty} a_k \, z^k$ analytic in the disk $\Delta_R = \{z : |z| < R\}$. Then, particular cases of series (2.2) and (2.3) are, respectively, the Riemann–Liouville (R–L) fractional derivative and integral of order $\delta > 0$ of f. They have the form

$$D^\delta f(z) = \sum_{k=0}^{\infty} a_k \, \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} \, z^{k-\delta}, \quad R^\delta f(z) = \sum_{k=0}^{\infty} a_k \, \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} \, z^{k+\delta},$$

and the usual differentiation $D = \frac{d}{dx}$, $\delta = 1$ is of the same form with a multiplier $\frac{\Gamma(k+1)}{\Gamma(k)}$. Moreover, due to the interchange formula between the Riemann–Liouville fractional derivative and the Caputo fractional derivative (see Expression (2.4.1) in [13]), it is possible also to obtain a particular case of series (2.2) and (2.3) in a form of a Caputo fractional derivative and integral of order $\delta > 0$ of f.

Example. Let $\varphi(\lambda)$ be a Mittag–Leffler function of the form

$$\varphi(\lambda) = E_{\frac{1}{\rho},\mu}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \quad \rho > 0, \quad \mu \in \mathbb{C},$$

with $\mathrm{Re}(\mu) > 0$. Then $\varphi_k(\lambda) = \frac{1}{\Gamma\left(\mu+\frac{k}{\rho}\right)}$ and operators (2.2), (2.3) turn into the so-called Dzrbashjan–Gelfond–Leontiev (D–G–L) operators of differentiation and

integration:

$$D_{\rho,\mu}f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k-1}{\rho}\right)} z^{k-1}, \tag{2.4}$$

$$I_{\rho,\mu}f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k+1}{\rho}\right)} z^{k+1}, \tag{2.5}$$

studied in [7,8,15].

In [15,18] the author studied the connections between the D–G–L operators (2.4), (2.5), and the so-called Erdélyi–Kober (E–K) fractional integrals and derivatives. In [15], the author presented transmutation operators relating the Riemann–Liouville (R–L) fractional integrals $R^{\frac{1}{\rho}}$ with the D–G–L generalized integrations $L_{\rho,1}$, and $L_{\rho,\mu}$, which where given in terms of E–K operators.

The above statements lead us to consider the fractional Dirac operator $D^\alpha = \sum_{j=1}^{d} e_i D_i^\alpha$, where D_i^α represents the G-L generalized derivative (2.2) with respect to the coordinate x_i^α. Analogous to the Euclidean case a $\mathbb{C}_d$-valued function u is called *fractional left-monogenic* if it satisfies $D^\alpha u = 0$ on Ω (resp. *fractional right-monogenic* if it satisfies $uD^\alpha = 0$ on Ω). As can be seen from the above exposition the most common fractional derivatives arise as special cases in our studies. We start with the discussion of one of the most important tools in Clifford analysis, the *Fischer decomposition*.

3. Sommen–Weyl relations and Fractional Fischer decomposition

The aim of this section is to provide the basic tools for a function theory for the fractional Dirac operator defined via generalized Gelfond–Leontiev differentiation operators.

3.1. Fractional Sommen–Weyl relations

The standard approach to the establishment of a function theory in higher dimensions is the construction of the analogues to the Euler and Gamma operators and the establishment of the corresponding Sommen–Weyl relations.

In order to achieve our goal, we want to study the commutator and the anti-commutator between x^α (see (2.1)) and $D^\alpha = \sum_{i=1}^{d} D_i^\alpha e_i$, where D_i^α is the G–L differentiation operator (see Definition 2.1) with respect to x_i^α. To this effect we start with the following fractional relations:

$$\begin{aligned}\left[\, D_i^\alpha, x_j^\alpha \,\right] (x_r^\alpha)^l &= \left(D_i^\alpha\, x_j^\alpha - x_j^\alpha\, D_i^\alpha\right) (x_r^\alpha)^l \\ &= \begin{cases} 0, & \text{if } i \neq j, \\ \varphi(l,0)\, (x_r^\alpha)^l, & \text{if } i = j \ \wedge\ i \neq r, \\ \varphi_D(l+1, l-1)\, (x_r^\alpha)^l, & \text{if } i = j = r, \end{cases}\end{aligned} \tag{3.1}$$

with $l \in \mathbb{N}$, $i = 1, \ldots, d$, $0 < \alpha < 1$, $\varphi(l,k) = \frac{\varphi_l}{\varphi_k}$, and $\varphi_D(l+1, l-1) = \varphi(l+1, l) - \varphi(l, l-1)$.

Example. For the case of Mittag–Leffler functions (see Example 2.2) we have $\varphi(a,b) = \frac{\Gamma(a\alpha+1)}{\Gamma(b\alpha+1)}$.

From (3.1) we get the following Sommen–Weyl relations for x, D^α and $(\underline{x}^\alpha)^{\underline{l}} = \prod_{i=1}^d (x_i^\alpha)^{l_i}$, with $\underline{l} = (l_1, \ldots, l_d)$, and $l = |\underline{l}| = l_1 + \ldots + l_d$:

$$\begin{aligned} \{D^\alpha, x\}\ (\underline{x}^\alpha)^{\underline{l}} &= -\left(\sum_{r=1}^d \varphi_D(l_r+1, l_r-1) + 2\mathbb{E}^\alpha\right)\ (\underline{x}^\alpha)^{\underline{l}} \\ [D^\alpha, x]\ (\underline{x}^\alpha)^{\underline{l}} &= -\left(\sum_{r=1}^d \varphi_D(l_r+1, l_r-1) + 2\Gamma^\alpha\right)\ (\underline{x}^\alpha)^{\underline{l}}, \end{aligned} \tag{3.2}$$

where $\mathbb{E}^\alpha$, Γ^α are, respectively, the fractional Euler and Gamma operators

$$\mathbb{E}^\alpha = \sum_{r=1}^d x_r^\alpha\, D_r^\alpha, \quad \Gamma^\alpha = -\sum_{r<s} e_r\, e_s\, (x_r^\alpha\, D_s^\alpha - D_r^\alpha\, x_s^\alpha)\,. \tag{3.3}$$

This can be easily checked by

$$\begin{aligned} \{D^\alpha, x\}\ (\underline{x}^\alpha)^{\underline{l}} &= -\sum_{r=1}^d (D_r^\alpha\, x_r^\alpha - x_r^\alpha\, D_r^\alpha)\, (\underline{x}^\alpha)^{\underline{l}} - 2\sum_{r=1}^d x_r^\alpha\, D_r^\alpha\ (\underline{x}^\alpha)^{\underline{l}} \\ &= -\left(\sum_{r=1}^d \varphi_D(l_r+1, l_r-1) + 2\mathbb{E}^\alpha\right)\ (\underline{x}^\alpha)^{\underline{l}}, \end{aligned}$$

and

$$\begin{aligned} [D^\alpha, x]\ (\underline{x}^\alpha)^{\underline{l}} &= -2\sum_{r<s} e_r\, e_s\ (D_r^\alpha x_s^\alpha - x_r^\alpha D_s^\alpha)\ (\underline{x}^\alpha)^{\underline{l}} - \sum_{r=1}^d (D_r^\alpha x_r^\alpha - x_r^\alpha D_r^\alpha)\ (\underline{x}^\alpha)^{\underline{l}} \\ &= -\left(\sum_{r=1}^d \varphi_D(l_r+1, l_r-1) + 2\Gamma^\alpha\right)\ (\underline{x}^\alpha)^{\underline{l}}. \end{aligned}$$

From (3.3) we derive, via straightforward calculations, the remaining Sommen–Weyl relations

$$[x, \mathbb{E}^\alpha]\ (\underline{x}^\alpha)^{\underline{l}} = -\sum_{i=1}^d e_i\, \varphi_D\,(l_i+1, l_i-1)\ x_i^\alpha\ (\underline{x}^\alpha)^{\underline{l}}, \tag{3.4}$$

$$[D^\alpha, \mathbb{E}^\alpha]\ (\underline{x}^\alpha)^{\underline{l}} = -\sum_{i=1}^d e_i\, \varphi(l_i, l_i-1)\, \varphi_D\,(l_i, l_i-2)\ (x_i^\alpha)^{-1}\ (\underline{x}^\alpha)^{\underline{l}}. \tag{3.5}$$

Now, the standard procedure would be to construct a finite-dimensional Lie-superalgebra generated by x and D^α which, in the classical case leads to an algebra

isomorphic to $\mathfrak{osp}(1|2)$. To achieve this, we would need to have the following relations:

$$[x, \mathbb{E}^\alpha]\ (\underline{x}^\alpha)^{\underline{l}} = -x\ (\underline{x}^\alpha)^{\underline{l}}, \quad [D^\alpha, \mathbb{E}^\alpha]\ (\underline{x}^\alpha)^{\underline{l}} = D^\alpha\ (\underline{x}^\alpha)^{\underline{l}}.$$

However, relations (3.4) and (3.5) show that this is only possible if

$$\varphi_D\,(l_i + 1, l_i - 1) = 1, \quad \varphi(l_i, l_i - 1)\,\varphi_D\,(l_i, l_i - 2) = 1.$$

This means that one cannot follow that path in the general case. Despite this setback, we can observe that the polynomial space generated by $(\underline{x}^\alpha)^{\underline{l}}$ allows a decomposition into homogeneous spaces Π_l, where $\Pi_0 = \text{span}\{[1]\}$, and $\Pi_l = \text{span}\left\{x^{\underline{\beta}}[1]\right\}$ with $\underline{\beta} = (\beta_1, \ldots, \beta_d)$ and $|\underline{\beta}| = \beta_1 + \ldots + \beta_d$, and where $[1]$ is the ground-state (usually, given by 1).

3.2. Fractional Fischer decomposition

The observation in the end of the previous subsection opens a path for defining a Fischer inner product for fractional homogeneous polynomials. A fractional Fischer inner product of two fractional homogeneous polynomials P and Q is given by:

$$\langle P(x), Q(x)\rangle = \text{Sc}\left[\overline{P\,(D^\alpha)}\,Q(x)\right]\Big|_{x=0}, \tag{3.6}$$

where $P\,(D^\alpha)$ is a differential operator obtained by replacing in the polynomial P each variable x_j^α by its corresponding fractional derivative D_j^α. From (3.6) we immediately get that for any polynomial P_{l-1} of homogeneity $l-1$ and any polynomial Q_l of homogeneity l it holds

$$\langle x\,P_{l-1},\, Q_l\rangle = \langle P_{l-1}, D^\alpha Q_l\rangle\,. \tag{3.7}$$

This fact allows us to prove the following result:

Theorem 3.1. *For each $l \in \mathbb{N}_0$ we have $\Pi_l = \mathcal{M}_l + x\,\Pi_{l-1}$, where Π_l denotes the space of fractional homogeneous polynomials of degree l and $\mathcal{M}_l$ denotes the space os fractional monogenic homogeneous polynomials of degree l. Moreover, the subspaces $\mathcal{M}_k$ and $x\,\Pi_{l-1}$ are orthogonal with respect to the Fischer inner product* (3.6).

Proof. Since $\Pi_l = x\,\Pi_{l-1} + (x\,\Pi_{l-1})^\perp$, it suffices to prove that $(x\,\Pi_{l-1})^\perp = \mathcal{M}_l$. Assume that $P_l \in \Pi_l$ is in $(x\,\Pi_{l-1})^\perp$. Then, we have $\langle x\,P_{l-1},\, P_l\rangle = 0$, for all $P_{l-1} \in \Pi_{l-1}$. From (3.7) we get $\langle P_{l-1},\, D^\alpha P_l\rangle = 0$, for all $P_{l-1} \in \Pi_{l-1}$. Hence, we obtain that $D^\alpha P_l = 0$, that is $P_l \in \mathcal{M}_l$. This means that $(x\,\Pi_{l-1})^\perp \subset \mathcal{M}_l$. Conversely, take $P_l \in \mathcal{M}_l$. Then, for every $P_{l-1} \in \Pi_{l-1}$ we have that:

$$\langle x\,P_{l-1},\, P_l\rangle = \langle P_{l-1},\, D^\alpha P_l\rangle = \langle P_{l-1},\, 0\rangle = 0,$$

from which it follows that $\mathcal{M}_l \subset (x\,\Pi_{l-1})^\perp$. Therefore $\mathcal{M}_l = (x\,\Pi_{l-1})^\perp$. □

As a consequence, we obtain the fractional Fischer decomposition with respect to the fractional Dirac operator D^α.

Theorem 3.2. *Let P_l be a fractional homogeneous polynomial of degree l. Then:*

$$P_l = M_l + x\, M_{l-1} + (x)^2\ M_{l-2} + \ldots + (x)^l\ M_0, \tag{3.8}$$

where each M_j denotes the fractional monogenic polynomial of degree j. More specifically,

$$M_0 \in \Pi_0, \quad \textit{and} \quad M_l \in \{u \in \Pi_l :\ D^\alpha u = 0\}\,.$$

The spaces represented in (3.8) are orthogonal to each other with respect to the Fischer inner product (3.6). Moreover, the above decomposition can be represented in form of an infinite triangle:

$$\begin{array}{ccccccccc}
\Pi_0 & & & \Pi_1 & & & \Pi_2 & & \Pi_3 & \\
\mathcal{M}_0 & \xleftarrow{D^\alpha} & & x\,\mathcal{M}_0 & \xleftarrow{D^\alpha} & & |x|^2\,\mathcal{M}_0 & \xleftarrow{D^\alpha} & (x)^3\,\mathcal{M}_0 & \ldots \\
 & & & \oplus & & & \oplus & & \oplus & \\
 & & & \mathcal{M}_1 & \xleftarrow{D^\alpha} & & x\,\mathcal{M}_1 & \xleftarrow{D^\alpha} & |x|^2\,\mathcal{M}_1 & \ldots \\
 & & & & & & \oplus & & \oplus & \\
 & & & & & & \mathcal{M}_2 & \xleftarrow{D^\alpha} & x\,\mathcal{M}_2 & \ldots \\
 & & & & & & & & \oplus & \\
 & & & & & & & & \mathcal{M}_3 & \ldots
\end{array}$$

While in the classic case all the summands in the same row are isomorphic to $\mathrm{Pin}(d)$-modules, and each row is an irreducible module for the Howe dual pair $\mathrm{Pin}(d) \times \mathfrak{osp}(1|2)$ (see [4] for more details), the same cannot be said at the moment for the fractional case. The reason for this fact was already stated in the previous subsection where we showed that for a general φ we do not have a finite-dimensional superalgebra generated by x and D^α isomorphic to $\mathfrak{osp}(1|2)$.

Nevertheless, we have that all the summands in the same row are indeed modules. The authors' conjecture is that these modules are invariant under a certain "fractional" Spin group which does not coincide with the classical one. However, such a study is beyond the scope of the present paper.

The Dirac operator shifts all spaces in the same row to the left, the multiplication by x shifts them to the right, and both of these actions establish isomorphisms between the modules.

From Theorem 3.2 we can derive the following direct extension to the fractional case of the Almansi decomposition:

Theorem 3.3. *For any fractional polyharmonic polynomial P_l of degree $l \in \mathbb{N}_0$ in a starlike domain Ω in $\mathbb{R}^d$ with respect to 0, i.e.,*

$$(D^\alpha)^2 P_l = 0, \quad \textit{in } \Omega,$$

there exist uniquely fractional harmonic functions $P_0, P_1, \ldots, P_{l-1}$ such that

$$P_l = P_0 + |x|^2\, P_1 + \ldots + |x|^{2(l-1)}\, P_{l-1}, \quad \textit{in } \Omega.$$

3.3. Explicit formulae

The aim of this subsection is to give an explicit algorithm for the construction of the projection $\pi_{\mathcal{M}}(P_l)$ of a given fractional homogeneous polynomial P_l into the space of fractional homogeneous monogenic polynomials $\mathcal{M}_l$. In order to reach our goal, we start by looking at the dimension of the space of fractional homogeneous monogenic polynomials of degree l. From the Fischer decomposition (3.8) we get:

$$\dim(\mathcal{M}_l) = \dim(\Pi_l) - \dim(\Pi_{l-1}),$$

with the dimension of the space of fractional homogeneous polynomials of degree l given by:

$$\dim(\Pi_l) = \frac{(l+d-1)!}{l!\,(d-1)!}.$$

This leads to the following theorem:

Theorem 3.4. *The space of fractional homogeneous monogenic polynomials of degree l has dimension*

$$\dim(\mathcal{M}_l) = \frac{(l+d-1)! - l(l+d-2)!}{l!\,(d-1)!} = \frac{(l+d-2)!}{l!\,(d-2)!}.$$

In the classical setting (see [3, 12, 19]) one usually considers the following scheme for the monogenic projection

$$r = a_0 P_l + a_1 x^\alpha D^\alpha P_l + a_2 (x^\alpha)^2 (D^\alpha)^2 P_l + \ldots + a_l (x^\alpha)^l (D^\alpha)^l P_l,$$

with $a_j \in \mathbb{R}, j = 0, \ldots, l$, and $a_0 = 1$. Let us take a closer look at the case $d = 3$ and $l = 3$,

$$r = a_0\, P_3 + a_1\, x D^\alpha P_3 + a_2\, (x)^2\, (D^\alpha)^2\, P_3 + a_3\, (x)^3\, (D^\alpha)^3\, P_3,$$

where $x = e_1 x_1^\alpha + e_2 x_2^\alpha + e_3 x_3^\alpha$ and $P_3 = (x_1^\alpha)^2\, (x_2^\alpha)^1\, (x_3^\alpha)^0$. In this case, we wish to obtain the real coefficients a_0, a_1, a_2, a_3 such that $D^\alpha P_3 = 0$. Taking into account that

$$\begin{aligned}
&D^\alpha\ (\underline{x}^\alpha)^3 = e_1\, \varphi(2,1)\, x_1^\alpha\, x_2^\alpha + e_2\, \varphi(1,0)\ (x_1^\alpha)^2\,,\\
&(D^\alpha)^2\ (\underline{x}^\alpha)^3 = -\varphi(2,0)\, x_2^\alpha,\\
&(D^\alpha)^3\ (\underline{x}^\alpha)^3 = -e_2\, \varphi(2,0)\, \varphi(1,0),\\
&D^\alpha x D^\alpha\ (\underline{x}^\alpha)^3 = -e_1\, \varphi(2,1)\, x_1^\alpha\, x_2^\alpha\, (2\varphi(2,1) + 2\varphi(1,0))\\
&\qquad\qquad - e_2 \left[\varphi(1,0)\ (x_1^\alpha)^2\, (\varphi(3,2) + 2\varphi(1,0) + \varphi(2,1)) + \varphi(2,0)\, (x_2^\alpha)^2\right]\\
&\qquad\qquad + e_3 \varphi(2,0) x_2^\alpha\, x_3^\alpha,\\
&D^\alpha\, (x)^2\, (D^\alpha)^2\ (\underline{x}^\alpha)^3 = e_1\, \varphi(2,0)\, \varphi(2,1)\, x_1^\alpha x_2^\alpha + e_3\, \varphi(2,0)\, \varphi(2,1)\, x_2^\alpha\, x_3^\alpha\\
&\qquad\qquad + e_2\, \varphi(2,0) \left[\varphi(1,0)\ (x_1^\alpha)^2 + \varphi(3,2)\ (x_2^\alpha)^2 + \varphi(1,0)\ (x_3^\alpha)^2\right],
\end{aligned}$$

$$D^{\alpha}\,(x)^3\,(D^{\alpha})^3\;(\underline{x}^{\alpha})^3 = e_2\,\varphi(2,0)\,\varphi(1,0)\,\Big[-(x_1^{\alpha})^2\,(\varphi(3,2)+2\varphi(1,0)) - (x_2^{\alpha})^2\,(\varphi(3,2)+2\varphi(1,0)) - (x_3^{\alpha})^2\,(\varphi(3,2)+2\varphi(1,0))\Big],$$

we obtain the equation

$$\begin{aligned}
0 &= D^{\alpha} r \\
&= a_0\,D^{\alpha}\;(\underline{x}^{\alpha})^3 + a_1\,D^{\alpha} x D^{\alpha}\;(\underline{x}^{\alpha})^3 + a_2 D^{\alpha}\,(x)^2\,(D^{\alpha})^2\;(\underline{x}^{\alpha})^3 \\
&\quad + a_3\,D^{\alpha}\,(x)^3\,(D^{\alpha})^3\;(\underline{x}^{\alpha})^3 \\
&= (a_0 + a_1\,D^{\alpha} x)\,\Big[e_1\,\varphi(2,1) x_1^{\alpha}\,x_2^{\alpha} + e_2\,\varphi(1,0)\;(x_1^{\alpha})^2\Big] \\
&\quad + a_2\,D^{\alpha}\,(x)^2\,[-\varphi(2,0) x_2^{\alpha}] + a_3\,D^{\alpha}\,(x)^3\,[-e_2\,\varphi(2,0)\,\varphi(1,0)] \\
&= e_1\,\varphi(2,1)\,x_1^{\alpha}\,x_2^{\alpha}\,[1 - a_1\,(2\varphi(2,1)+2\varphi(1,0)) + a_2\,\varphi(2,0)] \\
&\quad + e_2\,\varphi(1,0)\,\Big\{(x_1^{\alpha})^2\,[1 - a_1\,(\varphi(3,2)+2\varphi(1,0)+\varphi(2,1)) + a_2\varphi(2,0) \\
&\qquad -a_3\varphi(2,0)\,(\varphi(3,2)+2\varphi(1,0))] \\
&\qquad +\varphi(2,1)\,x_2^{\alpha}\,[-a_1 + a_2\,\varphi(3,2) - a_3\,(\varphi(3,2)+2\varphi(1,0))] \\
&\qquad +\varphi(2,0)\;(x_3^{\alpha})^2\,[a_2 - a_3\,(\varphi(3,2)+2\varphi(1,0))]\Big\} \\
&\quad + e_3\varphi(2,0)\,x_2^{\alpha}\,x_3^{\alpha}\,(a_1 + a_2\,\varphi(2,1))\,,
\end{aligned}$$

which leads to the following system of linear equations:

$$\begin{cases}
1 - a_1\,(2\varphi(2,1)+2\varphi(1,0)) + a_2\varphi(2,0) = 0, \\
1 - a_1\,(\varphi(3,2)+2\varphi(1,0)+\varphi(2,1)) + a_2\varphi(2,0) a_3\varphi(2,0)\,(\varphi(3,2)+2\varphi(1,0)) = 0, \\
-a_1 + a_2\varphi(3,2) - a_3\,(\varphi(3,2)+2\varphi(1,0)) = 0, \\
a_2 - a_3\,(\varphi(3,2)+2\varphi(1,0)) = 0, \\
a_1 + a_2\varphi(2,1) = 0,
\end{cases}$$

which has no solution if we assume $a_0,\, a_1,\, a_2,\, a_3 \in \mathbb{R}$. This fact is, of course, due to the lack of the $\mathfrak{osp}(1|2)$ property. Using the Fischer decomposition and the explicit knowledge of the dimensions of the spaces (see Theorem 3.4) we can use a more direct approach to determine the fractional homogeneous monogenic polynomial. In fact, any homogeneous polynomial of degree $l = |\underline{l}|$ can be written as

$$P_l(\underline{x}^{\alpha}) = \sum_{\underline{l}\in\mathbb{N}_0^d:\,|\underline{l}|=l} (\underline{x}^{\alpha})^{\underline{l}}\,a_{\underline{l}}, \quad a_{\underline{l}} \in \mathbb{R},$$

with $l = |\underline{l}| = l_1 + \ldots + l_d$ denoting the degree of the polynomial. We now check under which conditions we have $D^{\alpha} P_l = 0$, i.e.,

$$\begin{aligned}
0 &= D^{\alpha}\left(\sum_{\underline{l}\in\mathbb{N}_0^d:\,|\underline{l}|=l} (\underline{x}^{\alpha})^{\underline{l}}\,a_{\underline{l}}\right) \\
&= \sum_{\underline{l}\in\mathbb{N}_0^d:\,|\underline{l}|=l} D^{\alpha}\,(\underline{x}^{\alpha})^{\underline{l}}\,a_{\underline{l}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\underline{l}\in\mathbb{N}_0^d:\,|\underline{l}|=l} \left(\sum_{j=1}^{d} e_j\, D_j^{\alpha}\right) \left(\prod_{i=1}^{d} (x_i^{\alpha})^{l_i}\right) a_{\underline{l}} \\
&= \sum_{j=1}^{d} e_j \left(\sum_{\underline{l}\in\mathbb{N}_0^d:\,|\underline{l}|=l} \varphi(l_j, l_j-1)\, \left(x_j^{\alpha}\right)^{-1} (\underline{x}^{\alpha})^{\underline{l}}\, a_{\underline{l}}\right) \\
&= \big[e_1\,\varphi(d,d-1)\,a_{(d,0,\ldots,0)} + e_2\,\varphi(1,0)\,a_{(d-1,1,0,\ldots,0)} + \ldots \\
&\quad + e_d\,\varphi(1,0)\,a_{(d-1,0,\ldots,1)}\big]\,(x_1^{\alpha})^{d-1} \\
&\quad + \ldots \\
&\quad + \big[e_1\,\varphi(2,1)\,a_{(2,1,\ldots,0)} + e_2\,\varphi(2,1)\,a_{(1,2,1,\ldots,0)} + \ldots \\
&\quad + e_d\,\varphi(1,0)\,a_{(1,\ldots,1,0\ldots,0)}\big]\, x_1^{\alpha}\, x_2^{\alpha} \cdot \ldots \cdot x_{d-1}^{\alpha}. \qquad (3.9)
\end{aligned}
$$

The last equality leads to the following theorem:

Theorem 3.5. *Equation* (3.9) *is equivalent to the linear system*

$$M\,A = 0, \qquad (3.10)$$

where $A = [a_{(l_1,\ldots,l_d)}]_{\dim(\Pi_l)\times 1}$, $0 = [0]_{\dim(\Pi_{l-1})\times 1}$ *are vectors, and* M *is the matrix*

$$M = \left[M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)}\right]_{\dim(\Pi_{l-1})\times(\dim(\Pi_l))},$$

with entries given by:

$$M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)} = \begin{cases} e_i\,\varphi(l_i,k_i), & \text{if } k_i = l_i - 1 \wedge k_j = l_j\ \forall i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

Let us now indicate a possible ordering for the rows of system (3.10). In order to do that, let us consider the ordered set

$$L = \{\underline{L}^i = (l_1^i, \ldots, l_d^i) :\ |\underline{L}^i| = l = l_1^i + \ldots + l_d^i,\ i = 1, 2 \ldots, \dim(\Pi_l)\},$$

where the relation order is given by:

$$
\begin{aligned}
\underline{L}^i > \underline{L}^{i+1} &\Longleftrightarrow \left(l_1^i, \ldots, l_d^i\right) > \left(l_1^{i+1}, \ldots, l_d^{i+1}\right) \\
&\Longleftrightarrow l_i^i l_2^i \ldots l_d^i > l_1^{i+1} l_2^{i+1} \ldots l_d^{i+1},
\end{aligned}
$$

with

$$l_1^k l_2^k \ldots l_d^k := l_1^k \times 10^{d-1} + l_2^k \times 10^{d-2} + \ldots + l_d^k \times 10^0.$$

Applying this ordering we get the following corollary:

Corollary 3.6. *The matrix* M *has the structure*

$$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix},$$

where the sub-matrix $M_1 = [m_{ij}^1]_{\dim(\Pi_{l-1})\times\dim(\Pi_{l-1})}$ *is an upper triangular matrix with entries given by:*

$$M_1 = \begin{pmatrix} e_1\varphi(l,l-1) & e_2\varphi(1,0) & e_3\varphi(1,0) & e_4\varphi(1,0) & e_5\varphi(1,0) \\ 0 & e_1\varphi(l-1,l-2) & 0 & e_2\varphi(2,1) & e_3\varphi(1,0) \\ 0 & 0 & e_1\varphi(l-1,l-2) & 0 & e_2\varphi(1,0) \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & e_d\varphi(1,0) & 0 & 0 & \cdots & 0 \\ \cdots & e_{d-2}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \cdots & e_{d-3}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & 0 & 0 & 0 & \cdots & e_1\varphi(1,0) \end{pmatrix},$$

and the sub-matrix $M_2 = [m_{ij}^2]_{\dim(\Pi_{l-1})\times\dim(\Pi_l)}$ *has its entries given by:*

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots \\ e_2\varphi(l,l-1) & e_3\varphi(1,0) & e_4\varphi(1,0) & \cdots \\ 0 & e_2\varphi(l-1,l-2) & e_3\varphi(2,1) & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ e_d\varphi(1,0) & 0 & \cdots & 0 \\ e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & e_2\varphi(1,0) & \cdots & e_d\varphi(l,l-1) \end{pmatrix}.$$

For the resolution of system (3.10) we implement the following algorithm to obtain the coefficients.

Since M_1 is upper triangular matrix we can treat $a_{\dim(\Pi_{l-1})+1}$, $\ldots, a_{\dim(\Pi_l)}$ as free parameters and obtain the following formula for the coefficients. Let the entry (n,n) of M_1 correspond to the index $[(k_1,k_2,\ldots,k_d),(k_1+1,k_2,\ldots,k_d)]$, then

$$a_n = e_1\ (\varphi(k_1+1,k_1))^{-1}\ \left[\sum_{j=2}^{d} e_j\,\varphi(k_j+1,k_j)\,a_{(k_1,\ldots,k_j+1,\ldots,k_d)}\right], \qquad (3.11)$$

where

$$a_n \leftrightarrow M_{(n,n)} \leftrightarrow M_{(k_1,l\ldots,k_d),(k_1+1,k_2,\ldots,k_d)}.$$

Example. To illustrate the structure of M and A consider the case $d = 3$, $l = 3$, $\alpha = \frac{1}{2}$, and consider the Mittag–Leffler function, i.e., $\varphi(a,b) = \frac{\Gamma(a\alpha+1)}{\Gamma(b\alpha+1)}$ (see Example 2.2). Taking into account Corollary 3.6, the vector A and the matrixes M_1, M_2 take the form

$$\begin{aligned} A^T &= \begin{pmatrix} a_{(3,0,0)} & a_{(2,1,0)} & a_{(2,0,1)} & a_{(1,2,0)} & a_{(1,1,1)} & a_{(1,0,2)} & a_{(0,3,0)} & \cdots \\ \cdots & a_{(0,2,1)} & a_{(0,1,2)} & a_{(0,0,3)} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix}, \end{aligned}$$

$$M_1 = \begin{pmatrix} e_1\,\varphi(3,2) & e_2\,\varphi(1,0) & e_3\,\varphi(1,0) & 0 & 0 & 0 \\ 0 & e_1\,\varphi(2,1) & 0 & e_2\,\varphi(2,1) & e_3\,\varphi(1,0) & 0 \\ 0 & 0 & e_1\,\varphi(2,1) & 0 & e_2\,\varphi(1,0) & e_3\,\varphi(2,1) \\ 0 & 0 & 0 & e_1\,\varphi(1,0) & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1\,\varphi(1,0) & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1\,\varphi(1,0) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_2\,\varphi(3,2) & e_3\,\varphi(1,0) & 0 & 0 \\ 0 & e_2\,\varphi(2,1) & e_3\,\varphi(2,1) & 0 \\ 0 & 0 & e_2\,\varphi(1,0) & e_3\,\varphi(3,2) \end{pmatrix}.$$

Now we can see that the columns of the matrix M_2 are associated, respectively, to the last four elements of the matrix A. Hence, if we fix a_7, a_8, a_9, a_{10} we can obtain, via formula (3.11), the remaining elements of the matrix A:

$$\begin{aligned} a_1 &= a_{(3,0,0)} = e_1\ (\varphi(3,2))^{-1}\left[e_2\,\varphi(1,0)\,a_{(2,1,0)} + e_3\,\varphi(1,0)\,a_{(2,0,1)}\right], \\ a_2 &= a_{(2,1,0)} = e_1\ (\varphi(2,1))^{-1}\left[e_2\,\varphi(2,1)\,a_{(1,2,0)} + e_3\,\varphi(1,0)\,a_{(1,1,1)}\right], \\ a_3 &= a_{(2,0,1)} = e_1\ (\varphi(2,1))^{-1}\left[e_2\,\varphi(1,0)\,a_{(1,1,1)} + e_3\,\varphi(2,1)\,a_{(1,0,2)}\right], \\ a_4 &= a_{(1,2,0)} = e_1\ (\varphi(1,0))^{-1}\left[e_2\,\varphi(3,2)\,a_{(0,3,0)} + e_3\,\varphi(1,0)\,a_{(0,2,1)}\right], \\ a_5 &= a_{(1,1,1)} = e_1\ (\varphi(1,0))^{-1}\left[e_2\,\varphi(2,1)\,a_{(0,2,1)} + e_3\,\varphi(2,1)\,a_{(0,1,2)}\right], \\ a_6 &= a_{(1,0,2)} = e_1\ (\varphi(1,0))^{-1}\left[e_2\,\varphi(1,0)\,a_{(0,1,2)} + e_3\,\varphi(3,2)\,a_{(0,0,3)}\right], \end{aligned}$$

and, therefore, we solve system (3.10). Furthermore, we can use the previous conclusions to obtain the four polynomials which are the basis for the space of fractional homogeneous monogenic polynomials $\mathcal{M}_3$

$$V_1^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_3\,(x_1^\alpha)^3 - \frac{3}{2}\,(x_1^\alpha)^2\,(x_2^\alpha) + e_3\,\frac{3}{2}\,x_1^\alpha\,(x_2^\alpha)^2 + (x_2^\alpha)^3\,,$$
$$V_2^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = e_2\,\frac{2}{3}\,(x_1^\alpha)^3 - (x_1^\alpha)^2\,x_3^\alpha - e_2\,(x_1^\alpha)\,(x_2^\alpha)^2 + e_3\,\frac{4}{\pi}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha + (x_2^\alpha)^2\,x_3^\alpha,$$
$$V_3^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_3\,\frac{2}{3}\,(x_1^\alpha)^3 - (x_1^\alpha)^2\,x_2^\alpha - e_2\,\frac{4}{\pi}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha + e_3\,x_1^\alpha\,(x_3^\alpha)^2 + x_2^\alpha\,(x_3^\alpha)^2\,,$$
$$V_4^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = e_2\,(x_1^\alpha)^3 - \frac{3}{2}\,(x_1^\alpha)^2\,x_3^\alpha - e_2\,\frac{3}{2}\,x_1^\alpha\,(x_3^\alpha)^2 + (x_3^\alpha)^3\,.$$

For convenience of the reader we also give the basic monogenic polynomials for $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively,

$$V_1^{1,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_3\,x_1^\alpha + x_2^\alpha,$$
$$V_2^{1,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_2\,x_1^\alpha + x_3^\alpha,$$

$$V_1^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -(x_1^\alpha)^2 + e_3\,\frac{4}{\pi}\,x_1^\alpha\,x_2^\alpha + (x_2^\alpha)^2\,,$$
$$V_2^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_2\,x_1^\alpha\,x_2^\alpha + e_3\,x_1^\alpha\,x_3^\alpha + x_2^\alpha\,x_3^\alpha,$$
$$V_3^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -(x_1^\alpha)^2 - e_2\,\frac{4}{\pi}\,x_1^\alpha\,x_3^\alpha + (x_3^\alpha)^2\,.$$

Remark 3.7. The above algorithm can be easily implemented. For the convenience of the reader a MATLAB program for the D–G–L operators (see Example 2.2) in the three-dimensional case, is available at:

http://sweet.ua.pt/pceres/Webpage/Main_files/Frac_Code.zip

The main program is coef_frac(d,l,a). This program calculates the coefficients of the monogenic homogeneous polynomials that form the basis of the space of fractional homogeneous monogenic polynomials $\mathcal{M}_l$, i.e., it solves the system (3.10). The input data of this program consists of the dimension of the space (at this moment is fixed to d=3), the degree of homogeneity l, and the value of α a. For the cases presented in the previous example the program should be called, respectively, in the following forms:

coef_frac(3,3,0.5) coef_frac(3,1,0.5) coef_frac(3,2,0.5).

The output of the program is given as cell of 4×4 matrices representing quaternions, where the coefficients for each polynomial are given by each column ordered according to Multi-indices given by the function MultiindexIndexgen. We opted for the matrix representation of quaternions, but the program can be adapted to any Clifford algebra by using the appropriate matrix representation in terms of

a $2^n \times 2^n$ real or complex matrix (the latter for the case of complexified Clifford algebras).

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Free Products of C^*-Probability Spaces from p-Adic Hecke Algebras

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Dedicated to Prof. Daniel Alpay for his 60th birthday

Abstract. In this paper, by establishing suitable free-probabilistic models on the Hecke algebras $\mathcal{H}(G_p)$, we study (i) free probability on $\mathcal{H}(G_p)$, (ii) Hilbert-space representations of $\mathcal{H}(G_p)$, (iii) the C^*-algebras $\mathbb{H}(G_p)$ induced by $\mathcal{H}(G_p)$, and (iv) free probability on the free product C^*-probability space $\mathbb{H}_{\mathcal{P}}$ of $\mathbb{H}(G_p)$'s over primes p, where G_p are the general linear groups $GL_2(\mathbb{Q}_p)$ over the p-adic number fields $\mathbb{Q}_p$, for primes p. In particular, we are mainly interested in the operator-algebraic properties the free product C^*-probability space $\mathbb{H}_{\mathcal{P}}$, and operator-theoretic informations of the elements of $\mathbb{H}_{\mathcal{P}}$ in terms of their free-distributional data. As an application of the main results, an embedded free-probabilistic sub-structure $\mathfrak{S}$ generated by projections of $\mathbb{H}_{\mathcal{P}}$ is considered in detail.

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1. Introduction

In this paper, we establish free-probabilistic models (operator-theoretic structures equipped with noncommutative probabilistic tools) for *Hecke algebras* $\mathcal{H}(G_p)$ of the *general linear groups* $G_p = GL_2(\mathbb{Q}_p)$ *over p-adic number fields* $\mathbb{Q}_p$ (implying number-theoretic concepts and data), and corresponding *representations* (as algebraic tools). Although representations of Hecke algebras have been studied in various other texts (e.g., see [5], [6], [19], [21] and [23]), we concentrate on representations obtained from our free-probabilistic structures on Hecke algebras (e.g., [9], [10] and [11]). Based on our representation, one can study not only

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operator-algebraic structures of Hecke algebras, but also operator-theoretic properties of elements of the algebras, by investigating free-distributional data (and hence, spectral-theoretic information) of them.

Relations between *number theory* and *functional analysis* have been studied for both mathematical and scientific interests (e.g., [3], [4], [6], [8], [24] and [17]). Here, we are interested in Hecke algebras (from a number-theoretic point of view), and operator theory on them (from a free-probabilistic informational point of view), via certain Hilbert-space representations as in [9], [10], [11] and [12].

1.1. Motivation and Background

Connections between *primes* and *operators* have been studied in various earlier approaches. For instance, in [7] and [8], we studied how primes act "on" certain *von Neumann algebras* generated by *p -adic* and *Adelic measure spaces*. Independently, we studied primes as *linear functionals* acting on *arithmetic functions*. In such a case, one can understand arithmetic functions as *Krein-space operators* (for fixed primes), under *indefinite-inner-product-space* representations (See [13]). These studies are all motivated by number-theoretic consequences (e.g., [3], [6], [14], [15], [16] and [24]) applied in free probability techniques (e.g., [20], [22] and [26]).

The *p-adic number fields* $\mathbb{Q}_p$, and the *Adele ring* $\mathbb{A}_{\mathbb{Q}}$ play key roles in modern *number theory*; *analytic number theory*, *L-function theory*, and *algebraic geometry* (e.g., [14], [15] and [16]). Also, analysis on such Adelic structures gives ways for understanding vector analysis under non-Archimedean metric (e.g., [7], [8], [12], [13] and [24]).

In [12], the first-named author and Gillespie considered free-probabilistic models on embedded sub-structures of a *Hecke algebra* $\mathcal{H}(G_p)$, for a prime p. They characterized the "inner" free structures of $\mathcal{H}(G_p)$, determined by "normal" compact-open subgroups of G_p.

In [11], by extending the "inner" free-probabilistic structures of [12] to those fully on $\mathcal{H}(G_p)$, the first-named author established universal free-probabilistic models on Hecke algebras, and corresponding Hilbert-space representations. And he studied operator-theoretic properties of operators induced by generating elements of $\mathcal{H}(G_p)$. In particular, *self-adjointness*, *projection-property*, *normality*, *isometric property*, and *unitarity* of such operators were characterized.

In [9] and [10], from the free-probabilistic settings of [11], we studied certain embedded free-probabilistic *subgroups* of $\mathcal{H}(G_p)$, and corresponding *dynamical systems of* $\mathcal{H}(G_p)$. The main results of [9] and [10] illustrate how the number-theoretic information on $\mathcal{H}(G_p)$ affects operator-theoretic properties on $\mathcal{H}(G_p)$ with help of free probability.

1.2. Overview

In the first half of our paper, we prepare the ground for what will be needed in the proofs of our main theorems. The list of important notions we need in our analysis are as follows: our Hecke algebras $\mathcal{H}(G_p)$, normal Hecke subalgebras $\mathcal{H}_{Y_p}$, and their

noncommutative probability spaces (over $\mathbb{Q}_p$), and associated free-probabilistic processes of random variables (see Theorem 3.4, and the details in Section 4). We outline the concrete Hilbert spaces which carry our representations in Section 5, realized as free L^2-constructions (referring to our free random processes). Our main theorems are stated and proven in Sections 6, 7 and 8. We believe that they are of relevance to the crossroads of number theory, representation theory, and free-probabilistic operator theory. As is documented in the literature, these areas are relevant to statistical quantum physics, and to the associated phase-transition questions (see below).

In Section 2, we briefly introduce backgrounds, and basic concepts used in our proceeding works.

In Sections 3 and 4, we establish our free-probabilistic models on Hecke algebras $\mathcal{H}(G_p)$, for primes p, and consider free-distributional data of generating elements of $\mathcal{H}(G_p)$, implying the number-theoretic data.

Depending on our free probability on $\mathcal{H}(G_p)$, the Hilbert-space representations $(\mathfrak{H}_p, \alpha^p)$ are constructed, and the corresponding C^*-algebras $\mathbb{H}(G_p)$ of $\mathcal{H}(G_p)$ are obtained in Section 5, for primes p. One can get suitable C^*-probabilistic settings on $\mathbb{H}(G_p)$, implying the free-distributional data of $\mathcal{H}(G_p)$ (obtained in Sections 3 and 4), under the operator-algebraic framework in Section 6.

In Section 7, based on the main results of Sections 5 and 6, we construct-and-study the *free product C^*-probability space* $\mathbb{H}_{\mathcal{P}}$ generated by the family $\{\mathbb{H}(G_p), \tau_p\}_p$ of our C^*-probability spaces over primes p. Free-distributional data of generating operators of $\mathbb{H}_{\mathcal{P}}$ are characterized under *freeness*.

In Section 8, we concentrate on a certain embedded free-probabilistic substructure $\mathfrak{S}$ of the free product C^*-probability space $\mathbb{H}_{\mathcal{P}}$ of Section 7. We study not only free-distributional data on $\mathfrak{S}$, but also the estimations of such data on $\mathfrak{S}$.

Our main results demonstrate the close connections between number theory and operator algebra theory via free probability theory and representation theory.

2. Preliminaries

In this section, we review concepts and backgrounds of our proceeding works.

2.1. Free Probability

Readers can check Voiculescu's original analytic free probability theory in [22] (and cited papers therein), and Speicher's combinatorial free probability theory in [26] and cited papers therein. *Free probability* is a noncommutative operator-algebraic version of (classical) *probability* and *statistics* (and hence, naturally covering commutative operator-algebraic cases). The classical *independence* on probability is replaced by so-called the *freeness* on operators; *measures* for *measure spaces* are replaced by *linear functionals* on operator algebras, called *free probability spaces*,

or *noncommutative probability spaces*. It is not only important in pure mathematics (e.g., [2], [18] and [20]), but also applicable in related fields (e.g., [1], [9], [10] and [25])

Throughout this paper, we follow Speicher's combinatorial approach. Without detailed introduction, we use *free moments* and *free cumulants* of *free random variables* for studying *free distributions* (e.g., [22]), and consider *free product* of free probability spaces (e.g., [22] and [26]).

2.2. Hecke Algebras over $GL_2(\mathbb{Q}_p)$

Let p be a fixed *prime*, and let $\mathbb{Q}_p$ be the *p-adic number field* for p. This set $\mathbb{Q}_p$ is by definition the completion of the *rational numbers* $\mathbb{Q}$ with respect to the *p-adic norm*,

$$|q|_p = \left|p^k \frac{a}{b}\right| = \left(\frac{1}{p}\right)^k,$$

for $q = p^k \frac{a}{b} \in \mathbb{Q}$, for some k, a, $b \in \mathbb{Z}$. Remark here that the norm $|.|_p$ satisfies that

$$|q_1 + q_2|_p \leq \max\{|q_1|_p, |q_2|_p\},$$

and hence, it is *non-Archimedean*. The *topology for* $\mathbb{Q}_p$ is induced by the non-Archimedean metric d_p induced by the p-adic norm $|.|_p$,

$$d_p(q_1, q_2) = |q_1 - q_2|_p,$$

for all q_1, $q_2 \in \mathbb{Q}$.

Under topology, $\mathbb{Q}_p$ is *locally compact*, and *totally disconnected* as a *Banach space*, and it contains a maximal compact-open subring,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

We call $\mathbb{Z}_p$ the *unit disk of* $\mathbb{Q}_p$, and all elements of $\mathbb{Z}_p$ are said to be the *p-adic integers* in $\mathbb{Q}_p$.

As a topological space, $\mathbb{Q}_p$ has *neighborhood bases of zero* consisting of the *subsets*

$$p^k\mathbb{Z}_p = \{p^k x : x \in \mathbb{Z}_p\}, \text{ for } k \in \mathbb{Z}.$$

In fact, set-theoretically, one has

$$\mathbb{Q}_p = \underset{k\in\mathbb{Z}}{\cup} p^k\mathbb{Z}_p$$

(e.g., see [24]).

The *general linear group* $GL_2(\mathbb{Q}_p)$ consists of all *invertible* (2×2)-matrices over $\mathbb{Q}_p$, i.e.,

$$GL_2(\mathbb{Q}_p) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(\mathbb{Q}_p) \middle| \begin{array}{c} a, b, c, d \in \mathbb{Q}_p, \\ ad - bc \neq 0 \end{array} \right\},$$

where $M_2(\mathbb{Q}_p)$ means the set of all (2×2)-matrices over $\mathbb{Q}_p$.

Notation. Denote $GL_2(\mathbb{Q}_p)$ simply by G_p, if there is no confusion, for all primes p.

The group G_p is also *locally profinite* coming from the (product) topology of (those of) $\mathbb{Q}_p$, i.e., it has a neighborhood bases of the identity consisting of the *compact-open subgroups*,

$$K_k = u_p + (p^k)GL_2(\mathbb{Z}_p), \text{ for all } k \in \mathbb{N},$$

where $GL_2(\mathbb{Z}_p)$ means the subset of G_p, whose elements are invertible matrices in $GL_2(\mathbb{Q}_p)$, having their entries in $\mathbb{Z}_p$, and where

$$u_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ the identity matrix of } M_2(\mathbb{Q}_p).$$

Then the subgroup

$$K_0 = GL_2(\mathbb{Z}_p)$$

forms the maximal compact-open subgroup of G_p.

Define now the *Hecke algebra* $\mathcal{H}(G_p)$ of G_p by

$$\mathcal{H}(G_p) = \left\{ f : G_p \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ has compact support,} \\ \text{and} \\ \text{it is } \mu_p\text{-smooth} \end{array} \right\}. \tag{2.2.1}$$

The μ_p-smoothness means that $\mathcal{H}(G_p)$ is a smooth representation of G_p under right translation. In other words, for any element $f \in \mathcal{H}(G_p)$, there is a compact open subgroup K of G_p such that

$$\mu_p(y)f(g) = f(gy) = f(g)$$

for all $g \in G_p$.

We make $\mathcal{H}(G_p)$ into an associative algebra by taking $f_1, f_2 \in \mathcal{H}(G_p)$ and defining *convolution*,

$$(f_1 * f_2)(g) = \int_{G_p} f_1(x) f_2(x^{-1}g)\, d\mu_p(x) \tag{2.2.2}$$

where μ_p denotes a *left multiplication-invariant Haar measure* on the locally compact group G_p, satisfying

$$\mu_p\left(GL_2(\mathbb{Z}_p)\right) = 1.$$

If $f_j = \chi_{K_j}$, for some compact-open subgroups K_1 and K_2 of G_p, for $j = 1, 2$, then the convolution $(*)$ is well-defined on G_p by

$$(f_1 * f_2)(g) = \mu_p\left(K_1 \cap gK_2\right), \tag{2.2.3}$$

by (2.2.2). Note that, for any $g \in G_p$, and any μ_p-measurable subset S of G_p,

$$\mu_p(gS) = \mu_p(S) = \mu_p(Sg), \tag{2.2.4}$$

by (2.2.1). Note also that $\mathcal{H}(G_p)$ is an algebra "without identity," but has a *system of local units*. In this vein for any compact open subgroup of G_p, let

$$e_K = \frac{1}{\mu_p(K)}\chi_K \tag{2.2.5}$$

where χ_K are the *characteristic function of the compact-open subgroup* K of G_p. Then one has the following multiplication rules;

$$e_K * e_K = e_K, \tag{2.2.6}$$

and

$$e_K * f = f \Longleftrightarrow f(kg) = f(g),$$

for all $k \in K$ and $g \in G_p$, by (2.2.3) and (2.2.4).

3. Canonical Free-Probabilistic Models on $\mathcal{H}(G_p)$

In this section, we review free-probabilistic structures of $\mathcal{H}(G_p)$ introduced in [12], [9] and [10].

3.1. The Hecke Algebra $\mathcal{H}(G_p)$

Let $\mathcal{M}(G_p)$ be an algebra consisting of all μ_p-measurable functions on G_p, equipped with the usual functional-addition, and convolution (2.2.2). The formal definition (2.2.1) of Hecke algebras $\mathcal{H}(G_p)$ can be re-defined algebraically by

$$\mathcal{H}(G_p) = \mathbb{C}_* \left[\left\{ f = \sum_{j=1}^{N} t_j \chi_{x_j K} \;\middle|\; \begin{array}{c} N \in \mathbb{N}, \text{ and } t_j \in \mathbb{C}, \text{ and} \\ K \text{ is a compact-open subgroup of } G_p, \\ \text{depending on } f \\ \text{where } x_j \in G_p, \text{ for all } j = 1, \ldots, N \end{array} \right\} \right], \tag{3.1.1}$$

for all primes p, where $\mathbb{C}_*[X]$ mean the subalgebras generated by subsets X of $\mathcal{M}(G_p)$ under the usual functional addition, and convolution, and where χ_Y mean the usual *characteristic functions* of μ_p-measurable subsets Y of G_p, where μ_p is the *Haar measure* in (2.2.1) on G_p, satisfying (2.2.2), (2.2.3) and (2.2.4). The set

$$X_p = \left\{ f = \sum_{j=1}^{N} t_j \chi_{x_j K} \;\middle|\; \begin{array}{c} N \in \mathbb{N}, \text{ and } t_j \in \mathbb{C}, \text{ and} \\ K \text{ is a compact subgroup of } G_p, \\ \text{depending on } f \\ \text{where } x_j \in G_p, \text{ for all } j = 1, \ldots, N \end{array} \right\} \tag{3.1.1}'$$

generating the Hecke algebra $\mathcal{H}(G_p)$, is said to be the *generating set of* $\mathcal{H}(G_p)$, and call elements of X_p are said to be *generating elements of* $\mathcal{H}(G_p)$, i.e.,

$$\mathcal{H}(G_p) = \mathbb{C}_*[X_p], \tag{3.1.2}$$

where X_p is the generating set (3.1.1)′.

By (3.1.1) and (3.1.2), one may write

$$\mathcal{H}(G_p) = \left\{ \sum_{j=1}^{N} t_j \chi_{x_j K_j} \;\middle|\; \begin{array}{c} N \in \mathbb{N}, \text{ and } t_j \in \mathbb{C}, \text{ and} \\ K_j \text{ are compact subgroups of } G_p, \\ \text{where } x_j \in G_p, \text{ for all } j = 1, \ldots, N \end{array} \right\}, \tag{3.1.3}$$

set-theoretically.

Take now generating elements $\chi_{x_1K_1}, \chi_{x_2K_2}$ in the Hecke algebra $\mathcal{H}(G_p)$ of (3.1.3), for $x_j \in G_p$, and compact-open subgroups K_j of G_p, for $j = 1, 2$. Then they satisfy that

$$\begin{aligned}(\chi_{x_1K_1} * \chi_{x_2K_2})(g) &= \int_{G_p} \chi_{x_1K_1}(x)\, \chi_{x_2K_2}(x^{-1}g)\, d\mu_p(x) \\ &= \int_{G_p} \chi_{x_1K_1}(x)\, \chi_{x_2K_2g^{-1}}(x^{-1})\, d\mu_p(x) \\ &= \int_{G_p} \chi_{x_1K_1}(x)\, \chi_{gK_2x_2^{-1}}(x)\, d\mu_p(x)\end{aligned}$$

since $x^{-1} \in x_2K_2g^{-1}$ if and only if $x \in gK_2x_2^{-1}$

$$\begin{aligned}&= \int_{G_p} \chi_{x_1K_1 \cap gK_2x_2^{-1}}(x)\, d\mu_p(x) \\ &= \mu_p\left(x_1K_1 \cap gK_2x_2^{-1}\right), \end{aligned} \tag{3.1.4}$$

for all $g \in G_p$. This formula (3.1.4) provides how to do convolution on $\mathcal{H}(G_p)$, by (3.1.3).

Without loss of generality, one may understand

$$\chi_K(g) = \frac{\mu_p(K \cap gK)}{\mu_p(K)} \quad \text{and} \quad e_K(g) = \frac{\mu_p(K \cap gK)}{\mu_p(K)^2}, \tag{3.1.5}$$

for all $g \in G_p$ and all compact-open subgroups K of G_p.

Similar to (3.1.5), for any $x \in G_p$,

$$\chi_{xK}(g) = \frac{\mu_p(xK \cap gK)}{\mu_p(xK)} = \frac{\mu_p(xK \cap gK)}{\mu_p(K)}, \tag{3.1.6}$$

and

$$e_{xK}(g) = \frac{\mu_p(xK \cap gK)}{\mu_p(xK)^2} = \frac{\mu_p(xK \cap gK)}{\mu_p(K)^2}.$$

We now focus on generating elements χ_{xK} and e_{xK} in X_p, where K are "normal" compact-open subgroups of G_p. It is reasonable to restrict our interests to normal subgroups by the locally-profiniteness of G_p.

Recall that a subgroup K is *normal* in an arbitrary group Γ, if $gK = Kg$, for all $g \in \Gamma$. As usual, we denote the normal subgroup-inclusion by $K \lhd \Gamma$. Define a subset Y_p of the generating set X_p of $\mathcal{H}(G_p)$ by

$$Y_p \overset{\text{def}}{=} \left\{ \sum_{j=1}^{N} t_j \chi_{x_jK} \in X_p \,|K \lhd G_p \right\}. \tag{3.1.7}$$

Then one can have a subalgebra $\mathcal{H}_{Y_p}$ of $\mathcal{H}(G_p)$;

$$\mathcal{H}_{Y_p} \overset{\text{def}}{=} \mathbb{C}_*[Y_p] \text{ of } \mathcal{H}(G_p). \tag{3.1.8}$$

where Y_p is in the sense of (3.1.7) in X_p.

Proposition 3.1 (See [12].). *Let $\chi_{x_jK_j}$, $e_{x_jK_j} \in \mathcal{H}_{Y_p}$, where $x_j \in G_p$, and $K_j \lhd G_p$ compact-open, for $j = 1, 2$, where $\mathcal{H}_{Y_p}$ is the subalgebra (3.1.8) of $\mathcal{H}(G_p)$. Then*

$$\chi_{x_1K_1} * \chi_{x_2K_2} = \mu_p(K_1 \cap K_2)\, \chi_{x_1x_2K_1K_2}, \quad \text{and} \tag{3.1.9}$$

$$e_{x_1K_1} * e_{x_2K_2} = \frac{\mu_p(K_1 \cap K_2)\mu_p(K_1K_2)}{\mu_p(K_1)\mu_p(K_2)}\, e_{x_1x_2K_1K_2}, \tag{3.1.10}$$

where $K_1K_2 \lhd G_p$ is the product group of K_1 and K_2.

The formulas (3.1.9) and (3.1.10) provide the fundamental tools to establish our free-probabilistic structures on $\mathcal{H}_{Y_p}$ (and hence, those on $\mathcal{H}(G_p)$).

Definition 3.2. Let Y_p be the subset (3.1.7) of the generating set X_p, and let $\mathcal{H}_{Y_p} = \mathbb{C}_*[Y_p]$ be the subalgebra (3.1.8) of the Hecke algebra $\mathcal{H}(G_p)$, for a fixed prime p. Then we call Y_p and $\mathcal{H}_{Y_p}$ the normal sub-generating set of X_p, and the normal Hecke subalgebra of $\mathcal{H}(G_p)$, respectively.

3.2. On the Normal Hecke Subalgebra $\mathcal{H}_{Y_p}$ of $\mathcal{H}(G_p)$

In this section, we study the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ of the Hecke algebra $\mathcal{H}(G_p)$, in the sense of (3.1.8), where Y_p is the normal sub-generating set (3.1.7) of the generating set X_p of $\mathcal{H}(G_p)$, for a fixed prime p.

Notation. For convenience, denote $\prod_{j=1}^{N} x_j \in G_p$ and $\bigtimes_{j=1}^{N} K_j \in \sigma(G_p)$ simply by

$$x_{1,\dots,N} \text{ and } K_{1,\dots,N}, \text{ respectively,}$$

for all $N \in \mathbb{N}$. Also, we will write

$$K^o_{1,\dots,N} = K_{1,\dots,(N-1)} \cap K_N,$$

for all $N \in \mathbb{N} - \{1\}$. We will use these notations in the text below.

We obtain the following generalized results of (3.1.9) and (3.1.10).

Proposition 3.3. *Let $\chi_{x_jK_j}$, $e_{x_jK_j}$ be generating elements of the normal Hecke subalgebra $\mathcal{H}_{Y_p}$, for $j \in \mathbb{N}$. Then*

$$\overset{N}{\underset{j=1}{\circledast}} \chi_{x_jK_j} = \left(\prod_{l=2}^{N} \mu_p(K^o_{1,\dots,l}) \right) \chi_{x_{1,\dots,N}K_{1,\dots,N}}, \tag{3.2.1}$$

and

$$\overset{N}{\underset{j=1}{\circledast}} e_{x_jK_j} = \left(\frac{\prod_{l=2}^{N} \mu_p(K^o_{1,\dots,l})}{\prod_{l=1}^{N} \mu_p(K_l)} \right) e_{x_{1,\dots,N}K_{1,\dots,N}}, \tag{3.2.2}$$

for all $N \in \mathbb{N} - \{1\}$.

Proof. The formulas (3.2.1) and (3.2.2) are obtained by induction on (3.1.9), respectively (3.1.10). See [12], for more details. Here, the normality condition of $K_1, \dots, K_N$ is critical. □

Notation. Denote the convolution $f * \ldots * f$ of n copies of f simply by $f^{(n)}$, for all $n \in \mathbb{N}$, for all $f \in \mathcal{H}(G_p)$.

3.3. Free Probability on $\mathcal{H}_{Y_p}$

Let $\mathcal{H}(G_p)$ be the Hecke algebra generated by the general linear group $G_p = GL_2(\mathbb{Q}_p)$ over the p-adic number field $\mathbb{Q}_p$, for a fixed prime p. In this section, we restrict our interest to the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ of $\mathcal{H}(G_p)$.

Let u_p be the *group-identity of* G_p, i.e.,

$$u_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G_p = GL_2(\mathbb{Q}_p).$$

For the fixed u_p define now a linear functional φ_p on $\mathcal{H}_{Y_p}$ by

$$\varphi_p(f) \stackrel{\text{def}}{=} f(u_p), \text{ for all } f \in \mathcal{H}_{Y_p}. \tag{3.3.1}$$

Clearly, the morphism φ_p of (3.3.1) is a well-defined linear functional on $\mathcal{H}_{Y_p}$, and hence, the pair $(\mathcal{H}_{Y_p}, \varphi_p)$ forms a noncommutative probability space in the sense of [22] and [26].

Definition 3.4. The noncommutative probability space $(\mathcal{H}_{Y_p}, \varphi_p)$ is said to be the normal Hecke probability space (for a prime p), where φ_p is in the sense of (3.3.1).

On the normal Hecke probability space $\big(\mathcal{H}_{Y_p}, \varphi_p\big)$, one obtains the following fundamental free-moment computations.

Proposition 3.5 (See [12]). *Let $\chi_{x_jK_j}$, $e_{x_jK_j}$ be generating free random variables in the normal Hecke probability space $(\mathcal{H}_{Y_p}, \varphi_p)$, for all $j \in \mathbb{N}$. Then*

$$\varphi_p\left(\overset{N}{\underset{j=1}{\circledast}} \chi_{x_jK_j}\right) = \frac{\mu_p(K^o_{1,2}) \ldots \mu_p(K^o_{1,\ldots,N})\mu_p(x_{1,\ldots,N}K_{1,\ldots,N} \cap K_{1,\ldots,N})}{\mu_p(K_{1,\ldots,N})}, \tag{3.3.2}$$

$$\varphi_p\left(\overset{N}{\underset{j=1}{\circledast}} e_{x_jK_j}\right) = \frac{\mu_p(K^o_{1,2}) \ldots \mu_p(K^o_{1,\ldots,N})\mu_p(x_{1,\ldots,N}K_{1,\ldots,N} \cap K_{1,\ldots,N})}{\mu_p(K_1)\mu_p(K_2)\ldots\mu_p(K_N)\mu_p(K_{1,\ldots,N})}, \tag{3.3.3}$$

for all $N \in \mathbb{N} - \{1\}$. If $N = 1$, then one has

$$\varphi_p(\chi_{xK}) = \frac{\mu_p(xK \cap K)}{\mu_p(K)} \quad \text{and} \quad \varphi_p(e_{xK}) = \frac{\mu_p(xK \cap K)}{\mu_p(K)^2}. \tag{3.3.4}$$

Indeed, if $N > 1$ in $\mathbb{N}$, then one can get that

$$\varphi_p\left(\overset{N}{\underset{j=1}{\circledast}} \chi_{x_jK_j}\right) = \varphi_p\left(\left(\prod_{j=2}^{N} \mu_p(K^o_{1,\ldots,j})\right)\chi_{x_{1,\ldots,N}K_{1,\ldots,N}}\right)$$

$$\text{by (3.2.1)} \quad = \left(\prod_{j=2}^{N} \mu_p(K^o_{1,\ldots,j})\right)\left(\chi_{x_{1,\ldots,N}K_{1,\ldots,N}}(u_p)\right)$$

$$\text{by (3.3.1)} \quad = \left(\prod_{j=2}^{N} \mu_p(K^o_{1,\dots,j}) \right) \left(\frac{\mu_p(x_{1,\dots,N}K_{1,\dots,N} \cap u_pK_{1,\dots,N})}{\mu_p(K_{1,\dots,N})} \right)$$

$$\text{by (3.1.9)} \quad = \frac{\left(\prod_{j=2}^{N} \mu_p(K^o_{1,\dots,j}) \right) (\mu_p(x_{1,\dots,N}K_{1,\dots,N} \cap K_{1,\dots,N}))}{\mu_p(K_{1,\dots,N})}.$$

Thus, formula (3.3.2) holds. Similarly, formula (3.3.3) holds, because

$$e_{xK} = \frac{1}{\mu_p(K)} \chi_{xK}, \tag{3.3.5}$$

for all $x \in G_p$, and compact-open normal subgroups K of G_p, by (3.1.8).

Let $\chi_{x_1K_1}, \dots, \chi_{x_NK_N} \in (\mathcal{H}_{Y_p}, \varphi_p)$, for $N \in \mathbb{N}$. Then

$$k^p_N(\chi_{x_1K_1}, \dots, \chi_{x_NK_N}) = \sum_{\pi \in NC(N)} \left(\prod_{V \in \pi} \varphi_p\Big(\underset{j \in V}{*} \chi_{x_{i_j}K_{i_j}} \Big) \right) \mu(\pi, 1_N), \tag{3.3.6}$$

by the *Möbius inversion of* [22],

$$k^p_N(\chi_{x_1K_1}, \dots, \chi_{x_NK_N}) = \sum_{\pi \in NC(N)} \left(\prod_{V=(i_1,\dots,i_{|V|}) \in \pi} (\mu_p(V)) \right) \mu(\pi, 1_N), \tag{3.3.7}$$

by (3.3.4), where

$$\mu_p(V) = \begin{cases} \dfrac{\left(\prod_{l=2}^{|V|} \mu_p(K^o_{i_1,\dots,i_l}) \right) \mu_p(x_{i_1,\dots,i_{|V|}} K_{i_1,\dots,i_{|V|}} \cap K_{i_1,\dots,i_{|V|}})}{\mu_p(K_{i_1,\dots,i_{|V|}})} & \text{if } |V| > 1, \\ \dfrac{\mu_p\big(x_{i_1}K_{i_1} \cap K_{i_1}\big)}{\mu_p\big(K_{i_1}\big)} & \text{if } |V| = 1, \end{cases}$$

are the *block-depending free moments* for all $V \in \pi$, for all $\pi \in NC(N)$, where $k^p_n(\dots)$ means *free cumulant* in terms of φ_p, in the sense of [22].

Theorem 3.6. *Let $f_j = \chi_{K_j}$, $h_j = e_{K_j}$ be free random variables in the normal Hecke free probability space $(\mathcal{H}_{Y_p}, \varphi_p)$, for $j = 1, 2$. Then*

$$f_1 \text{ and } f_2 \text{ are free in } (\mathcal{H}_{Y_p}, \varphi_p) \Leftrightarrow \mu_p(K^o_{1,2}) = \mu_p(K_1)\mu_p(K_2). \tag{3.3.8}$$

$$h_1 \text{ and } h_2 \text{ are free in } (\mathcal{H}_{Y_p}, \varphi_p) \Leftrightarrow \mu_p(K^o_{1,2}) = \mu_p(K_1)\mu_p(K_2). \tag{3.3.9}$$

Proof. The freeness characterization (3.3.8) automatically implies the freeness characterization (3.3.9). Indeed, by [22] and [26], f_1 and f_2 are free, if and only if the subsets $\{f_1\}$ and $\{f_2\}$ are free, if and only if the subalgebras $\mathfrak{A}_j = \mathbb{C}_*[\{f_j\}]$ are free in $(\mathcal{H}_{Y_p}, \varphi_p)$, for $j = 1, 2$. Since $h_j \in \mathfrak{A}_j$, for $j = 1, 2$, they are free in $(\mathcal{H}_{Y_p}, \varphi_p)$, whenever f_1 and f_2 are free in $(\mathcal{H}_{Y_p}, \varphi_p)$.

Also, the proof of (3.3.8) is done, by computing "mixed" free cumulants of f_1 and f_2 based on the joint free-cumulant computation (3.3.7), and the fact that

$$\sum_{\pi \in NC(N)} \mu(\pi, 1_N) = 0,$$

for all $N \in \mathbb{N} - \{1\}$. See [12], for more details. □

The above freeness condition (3.3.8) (and (3.3.9)) characterizes the free structures in $\mathcal{H}_{Y_p}$.

4. Free Probability on $\mathcal{H}(G_p)$

In this section, we extend the free probability on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ (of Section 3.3) to that on the Hecke algebra $\mathcal{H}(G_p)$. The main tools would be certain normalization process of compact-open subgroups of G_p (e.g., see [11]).

Let G be an arbitrary group, and let K be a subgroup (not necessarily a normal subgroup) of G. Let K_G be the maximal subgroup of K, which is a normal subgroup of G, i.e., it is the biggest normal subgroup of G inside K. More precisely, K_G is the maximal subgroup of G, satisfying

$$K_G \lhd G, \text{ and } K_G \leq K \leq G. \tag{4.10}$$

We remark that K_G is not necessarily a normal subgroup of K, but it is normal in G.

Such a normal subgroup K_G exists for any given subgroup K in G, because a subgroup K of G contains its *normal core* $\mathrm{Core}_G(K)$ *of* K *in* G, defined by

$$\mathrm{Core}_G(K) \stackrel{\text{def}}{=} \bigcap_{g \in G} \left(g^{-1} K g\right). \tag{4.11}$$

It is the maximal normal subgroup of G contained in K (e.g., see [3]). So, the normal subgroup K_G in the sense of (4.1) is identified with the normal core $\mathrm{Core}_G(K)$ of (4.2) in G, i.e.,

$$K_G = \mathrm{Core}_G(K). \tag{4.12}$$

Notation. Let G be an arbitrary group, and let K be a subgroup of G. Then the normal core $\mathrm{Core}_G(K)$ of K in G is simply denoted by K_G throughout this paper, as in (4.3).

Define now a morphism E_p on the Hecke algebra $\mathcal{H}(G_p)$ by a linear transformation satisfying

$$E_p\left(\chi_{xK}\right) = \begin{cases} \chi_{xK_{G_p}} = \chi_{K_{G_p}x} & \text{if } xK = Kx, \\ 0_{\mathcal{H}(G_p)} & \text{otherwise,} \end{cases} \tag{4.13}$$

and

$$E_p\left(\chi_{x_1K_1} * \chi_{x_2K_2}\right) = \begin{cases} \mu_p(K^o_{1,2})\chi_{x_{1,2}K_{1,2:G_p}} & \text{if } K_ix_j = x_jK_i, \\ & \text{for all } i,j = 1,2, \\ 0_{\mathcal{H}(G_p)} & \text{otherwise,} \end{cases}$$

for all x, x_1, $x_2 \in G_p$, and compact-open subgroups K, K_1, K_2 of G_p, where $0_{\mathcal{H}(G_p)}$ means the *zero-element* of the Hecke algebra $\mathcal{H}(G_p)$, and

$$K_{G_p} = \mathrm{Core}_{G_p}(K),$$

are the normal cores (4.2) of K in G_p, as in (4.3), satisfying (4.1) in G_p.

By the proofs of (3.1.9), (3.1.10) and (3.2.1), if

$$K_ix_j = x_jK_i \Longleftrightarrow K_i = x_jK_ix_j^{-1},$$

for all $i,j = 1,2$, then

$$\chi_{x_1K_1} * \chi_{x_2K_2} = \mu_p(K^o_{1,2})\chi_{x_{1,2}K_{1,2}}, \tag{4.4}'$$

for such specific x_1, $x_2 \in G_p$, and compact-open subgroups K_1, K_2 of G_p.

So, the consideration (4.4)′ makes the second condition of (4.4) be reasonable; but, check the difference between (4.4) and (4.4)′.

Also, the second condition of (4.4) implies

$$E_p\left(\overset{N}{\underset{j=1}{\circledast}} \chi_{x_jK_j}\right) = \begin{cases} \left(\prod_{j=2}^{N} \mu_p(K^o_{1,\dots,j})\right)\chi_{x_{1,\dots,N}K_{1,\dots,N:G_p}} & \text{or} \\ 0_{\mathcal{H}(G_p)}, \end{cases} \tag{4.14}$$

inductively, for all $N \in \mathbb{N} - \{1\}$. In particular, if

$$x_iK_j = K_jx_i, \quad \forall i,j = 1,\dots,N,$$

then

$$E_p\left(\overset{N}{\underset{j=1}{\circledast}} \chi_{x_jK_j}\right) = \left(\prod_{j=2}^{N} \mu_p(K^o_{1,\dots,j})\right)\chi_{x_{1,\dots,N}K_{1,\dots,N:G_p}}, \tag{4.5}'$$

in $\mathcal{H}(G_p)$, for $N \in \mathbb{N} - \{1\}$.

Notation 4.1. In the rest of this paper, for convenience, we write the conditional relation (4.5) simply as (4.5)′, i.e., even though we simply write (4.5)′, it means (4.5) under the conditions (4.4) of E_p.

By definition, it is clear that, for any compact-open subgroup K of G_p,

$$E_p\left(\chi_K\right) = \chi_{K_{G_p}},$$

by (4.4) and (4.5). Notice that the linear morphism E_p mapped $\mathcal{H}(G_p)$ onto $\mathcal{H}_{Y_p}$, by (4.4). Note also that, even though the morphism E_p is not a conditional expectation in the sense of [22], it is a well-defined surjective linear transformation, satisfying that

$$E_p(\chi_{xK}) = \chi_{xK} = \chi_{Kx}, \text{ for all } \chi_{xK} \in \mathcal{H}_{Y_p},$$

and hence,

$$E_p^2(f) = E_p(f), \text{ for all } f \in \mathcal{H}_{Y_p} \text{ in } \mathcal{H}(G_p),$$

i.e., E_p is an *idempotent* linear transformation preserving (or fixing) the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ of the Hecke algebra $\mathcal{H}(G_p)$.

Moreover, one can get that

$$\begin{aligned} E_p\left(e_{xK}\right) &= E_p\left(\frac{1}{\mu_p(K)}\chi_{xK}\right) \\ &= \frac{1}{\mu_p(K)}\chi_{xK_{G_p}} = e_{xK_{G_p}}, \end{aligned} \tag{4.15}$$

by (4.5), for all $x \in G_p$ such that $xK = Kx$, for all compact-open subgroups K of G_p.

Thus, we obtain the following result.

Lemma 4.1. *Under the same hypothesis as above, we have that*

$$\begin{aligned} &E_p\left(e_{x_1K_1} * \ldots * e_{x_NK_N}\right) \\ &= \frac{\mu_p(K^o_{1,2})\ldots\mu_p(K^o_{1,\ldots,N})\mu_p(K_{1,\ldots,N:G_p})}{\mu_p(K_1)\ldots\mu_p(K_N)\mu_p(K_{1,\ldots,N})} e_{x_{1,\ldots,N}K_{1,\ldots,N:G_p}}, \end{aligned} \tag{4.16}$$

under Notation 4.1.

Proof. Observe that

$$\begin{aligned} &E_p\left(e_{x_1K_1} * \ldots * e_{x_NK_N}\right) \\ &= E_p\left(\frac{\mu_p(K^o_{1,2})\ldots\mu_p(K^o_{1,\ldots,N})\mu_p(K_{1,\ldots,N})}{\mu_p(K_1)\mu_p(K_2)\ldots\mu_p(K_N)} e_{x_{1,\ldots,N}K_{1,\ldots,N}}\right) \\ &= \left(\frac{\mu_p(K^o_{1,2})\ldots\mu_p(K^o_{1,\ldots,N})\mu_p(K_{1,\ldots,N})}{\mu_p(K_1)\mu_p(K_2)\ldots\mu_p(K_N)}\right)\left(\frac{1}{\mu_p(x_{1,\ldots,N}K_{1,\ldots,N})}\chi_{x_{1,\ldots,N}K_{1,\ldots,N:G_p}}\right) \\ &= \frac{\mu_p(K^o_{1,2})\ldots\mu_p(K^o_{1,\ldots,N})}{\mu_p(K_1)\ldots\mu_p(K_N)\mu_p(K_{1,\ldots,N})}\chi_{x_{1,\ldots,N}K_{1,\ldots,N:G_p}} \\ &= \frac{\mu_p(K^o_{1,2})\ldots\mu_p(K^o_{1,\ldots,N})\mu_p(K_{1,\ldots,N:G_p})}{\mu_p(K_1)\ldots\mu_p(K_N)\mu_p(K_{1,\ldots,N})} e_{x_{1,\ldots,N}K_{1,\ldots,N:G_p}} \end{aligned}$$

by (4.6), under Notation 4.1. □

With the help of the above lemma, one obtains the following proposition.

Proposition 4.2. *Let $f_j = \chi_{x_jK_j}$, $h_j = e_{x_jK_j}$ be generating elements of the Hecke algebra $\mathcal{H}(G_p)$, for $j = 1, \ldots, N$, for $N \in \mathbb{N} - \{1\}$, and let E_p be the linear transformation* (4.4) *on $\mathcal{H}(G_p)$. Then*

$$E_p\left(\overset{N}{\underset{j=1}{\circledast}} f_j\right) = \prod_{l=2}^{N} \mu_p(K^o_{1,\ldots,j})\chi_{x_{1,\ldots,N}K_{1,\ldots,N:G_p}}, \tag{4.17}$$

and

$$E_p\left(\mathop{\ast}_{j=1}^{N} h_j\right) = \frac{\mu_p(K_{1,2}^o)\dots\mu_p(K_{1,\dots,N}^o)\mu_p(K_{1,\dots,N:G_p})}{\mu_p(K_1)\dots\mu_p(K_N)} e_{x_{1,\dots,N}K_{1,\dots,N:G_p}}, \quad (4.18)$$

under Notation 4.1, *where* $K_{1,\dots,N:G_p}$ *is the normal core of* $K_{1,\dots,N}$.

Proof. The proofs of (4.8) and (4.9) are done by (4.5), (4.6) and (4.7). See [11] for more details. □

As we discussed above, the linear transformation E_p maps $\mathcal{H}(G_p)$ onto the normal Hecke subalgebra $\mathcal{H}_{Y_p}$, by (4.4) and (4.5).

Definition 4.3. We call the morphism E_p of (4.2) the normal-coring on $\mathcal{H}(G_p)$.

Define now a linear functional ψ_p on the Hecke algebra $\mathcal{H}(G_p)$ by

$$\psi_p \overset{\text{def}}{=} \varphi_p \circ E_p \text{ on } \mathcal{H}(G_p), \quad (4.19)$$

where φ_p is the linear functional (3.3.1) on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ of $\mathcal{H}(G_p)$, and E_p is in the sense of (4.4).

We call this linear functional ψ_p the *normal-core linear functional on* $\mathcal{H}(G_p)$. So, the pair $(\mathcal{H}(G_p), \psi_p)$ forms a noncommutative probability space.

Definition 4.4. The noncommutative probability space $(\mathcal{H}(G_p), \psi_p)$ of the Hecke algebra $\mathcal{H}(G_p)$, and the normal-core linear functional ψ_p of (4.10) is said to be the normal-core Hecke probability space (for a prime p).

Generally, we obtain the following joint free-moment computations.

Theorem 4.5. *Let* $(\mathcal{H}(G_p), \psi_p)$ *be the normal-core Hecke probability space, and let* $f_j = \chi_{x_jK_j}$ *and* $h_j = e_{x_jK_j}$ *be generating free random variables in* $(\mathcal{H}(G_p), \psi_p)$, *for* $j \in \mathbb{N}$. *Then*

$$\psi_p\left(\mathop{\ast}_{j=1}^{N} f_j\right) = \frac{\left(\prod_{l=2}^{N} \mu_p(K_{1,\dots,l}^o)\right) \mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G} \cap K_{1,\dots,N:G_p}\right)}{\mu_p\left(K_{1,\dots,N:G_p}\right)}, \quad (4.20)$$

and

$$\psi_p\left(\mathop{\ast}_{j=1}^{N} h_j\right) = \frac{\left(\prod_{l=2}^{N} \mu_p(K_{1,\dots,l}^o)\right) \mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G} \cap K_{1,\dots,N:G_p}\right)}{\left(\prod_{l=1}^{N} \mu_p(K_l)\right)\left(\mu_p(K_{1,\dots,N:G_p})\right)},$$

for all $N \in \mathbb{N} - \{1\}$, *under Notation 4.1, where* $K_{1,\dots,N:G_p}$ *is in the sense of* (4.2).

If $N = 1$, *then*

$$\psi_p(f_j) = \chi_{x_j(K_j)_{G_p}}, \text{ and } \psi_p(h_j) = e_{x_j(K_j)_{G_p}}, \quad (4.21)$$

under Notation 4.1, *for all* $j \in \mathbb{N}$.

Proof. Observe that, under the hypothesis, if $N > 1$ in $\mathbb{N}$, then

$$\psi_p\left(\overset{N}{\underset{j=1}{\ast}} f_j\right) = \psi_p\left(\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\chi_{x_{1,\dots,N}K_{1,\dots,N}}\right),$$

and by (4.4)′, under Notation 4.1,

$$\begin{aligned}
\psi_p\left(\overset{N}{\underset{j=1}{\ast}} f_j\right) &= \mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\psi_p\left(\chi_{x_{1,\dots,N}K_{1,\dots,N}}\right)\\
&= \mu_p(K^o_{1,2})\dots\mu_p(K^o_{1,\dots,N})\varphi_p\left(E_p(\chi_{x_{1,\dots,N}K_{1,\dots,N}})\right)\\
&= \mu_p(K^o_{1,2})\dots\mu_p(K^o_{1,\dots,N})\varphi_p\left(\chi_{x_{1,\dots,N}K_{1,\dots,N:G_p}}\right)\\
&= \mu_p(K^o_{1,2})\dots\mu_p(K^o_{1,\dots,N})\left(\frac{\mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G}\cap K_{1,\dots,N:G_p}\right)}{\mu_p\left(K_{1,\dots,N:G_p}\right)}\right)\\
&= \frac{\mu_p(K^o_{1,2})\dots\mu_p(K^o_{1,\dots,N})\mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G}\cap K_{1,\dots,N:G_p}\right)}{\mu_p\left(K_{1,\dots,N:G_p}\right)}.
\end{aligned}$$

So, the first formula of (4.11) holds, for all $N \in \mathbb{N}$.

Also, consider that

$$\psi_p\left(\overset{N}{\underset{j=1}{\ast}} h_j\right) = \frac{1}{\mu_p(K_1)\mu_p(K_2)\dots\mu_p(K_N)}\psi_p\left(\overset{N}{\underset{j=1}{\ast}} \chi_{x_jK_j}\right),$$

and by (4.4)′, under Notation 4.1,

$$\begin{aligned}
&\psi_p\left(\overset{N}{\underset{j=1}{\ast}} h_j\right)\\
&= \frac{1}{\mu_p(K_1)\dots\mu_p(K_N)}\\
&\quad\cdot\frac{\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G}\cap K_{1,\dots,N:G_p}\right)}{\mu_p(K_{1,\dots,N:G_p})}\\
&\overset{\text{by (4.11)}}{=} \frac{\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\mu_p\left(x_{1,\dots,N}K_{1,\dots,N:G}\cap K_{1,\dots,N:G_p}\right)}{\mu_p(K_1)\mu_p(K_2)\dots\mu_p(K_N)\mu_p(K_{1,\dots,N:G_p})}.
\end{aligned}$$

Therefore, the second formula of (4.11) holds true.

The formulas in (4.12) are trivial by (4.4), under Notation 4.1. □

Note the difference between (3.3.2) with (3.3.3), and (4.11). One can realize that: if all compact-open subgroups $K_1, \dots, K_N$ are normal in G_p, then the formulas of (4.11) are identified with the formulas (3.3.2) and (3.3.3), because

$$K_{1,\dots,N:G_p} = K_{1,\dots,N}$$

whenever $x_iK_j = K_jx_i$, for all $i, j \in \{1, \dots, N\}$, for all $N \in \mathbb{N} - \{1\}$.

So, by (4.11), we have that

$$\psi_p\left(\overset{N}{\underset{j=1}{\circledast}}\chi_{K_j}\right) = \frac{\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\mu_p(K_{1,\dots,N:G_p})}{\mu_p(K_{1,\dots,N:G_p})}$$

$$= \mu_p(K^o_{1,2,})\mu_p(K^o_{1,2,3,})\dots\mu_p(K^o_{1,\dots,N}), \tag{4.22}$$

$$\psi_p\left(\overset{N}{\underset{j=1}{\circledast}}e_{K_j}\right) = \frac{\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})\mu_p(K_{1,\dots,N:G_p})}{\mu_p(K_1)\mu_p(K_2)\dots\mu_p(K_N)\mu_p(K_{1,\dots,N:G_p})}$$

$$= \frac{\mu_p(K^o_{1,2})\mu_p(K^o_{1,2,3})\dots\mu_p(K^o_{1,\dots,N})}{\mu_p(K_1)\mu_p(K_2)\dots\mu_p(K_N)}, \tag{4.23}$$

for all $N \in \mathbb{N} - \{1\}$.

Now, let K_1 and K_2 be compact-open subgroups of G_p, and let χ_{K_j} be corresponding free random variables in the normal-core Hecke probability space $(\mathcal{H}(G_p), \psi_p)$. Suppose $k_N(\dots)$ is the free cumulant for the normal-core linear functional ψ_p of (4.10). Then, for any $(i_1, \dots, i_N) \in \{1,2\}^N$, for all $N \in \mathbb{N} - \{1\}$, we obtain the following free cumulant computation;

$$k_N\left(\chi_{K_{i_1}}, \dots, \chi_{K_{i_N}}\right) = \sum_{\pi \in NC((i_1,\dots,i_N))} \left(\prod_{V\in\pi}\mu_p(V)\right)\mu\left(\pi, 1_N\right), \tag{4.24}$$

with

$$\mu_p(V) = \mu_p(K^o_{i_{j_1},i_{j_2}})\mu_p\left(K^o_{i_{j_1},i_{j_2},i_{j_3}}\right)\dots\mu_p\left(K^o_{i_{j_1},\dots,i_{j_k}}\right),$$

by (4.11), under Notation 4.1, whenever $V = (j_1, \dots, j_k) \in \pi$, for all $\pi \in NC(N)$, for all $N \in \mathbb{N}$, where $\mu_p(V)$ are the V-block-depending free moments.

By the above joint free-cumulant formula (4.15), one obtains the following freeness condition on the normal-core Hecke probability space $(\mathcal{H}(G_p), \psi_p)$.

Theorem 4.6. *Let $f_j = \chi_{K_j}$ and $h_j = e_{K_j}$ be free random variables in the normalized Hecke probability space $(\mathcal{H}(G_p), \psi_p)$, for $j = 1, 2$. Then:*

$$f_1 \text{ and } f_2 \text{ are free in } (\mathcal{H}(G_p), \psi_p) \Leftrightarrow \mu_p(K^o_{1,2}) = \mu_p(K_1)\mu_p(K_2). \tag{4.25}$$

$$h_1 \text{ and } h_2 \text{ are free in } (\mathcal{H}(G_p), \mu_p) \Leftrightarrow \mu_p(K^o_{1,2}) = \mu_p(K_1)\mu_p(K_2). \tag{4.26}$$

Proof. ($\Leftarrow$) Assume first that two compact subgroups K_1 and K_2 of G_p satisfy

$$\mu_p(K_1 \cap K_2) = \mu_p(K_1)\mu_p(K_2).$$

Then, for all "mixed" N-tuples $(i_1, \dots, i_N) \in \{1,2\}^N$, for all $N \in \mathbb{N} - \{1\}$, the formulas (4.15) satisfy

$$\left(\mu_p(K_1)^{m(1)}\mu_p(K_2)^{m(2)}\right)\left(\sum_{\pi\in NC((i_1,\dots,i_N))}\mu(\pi, 1_N)\right) = 0, \tag{4.27}$$

where

$$m(1) = \text{the number of 1's in } (i_1, \dots, i_N),$$

and

$$m(2) = \text{the number of 2's in } (i_1, \ldots, i_N).$$

Since

$$\sum_{\pi \in NC((i_1,\ldots,i_N))} \mu(\pi, 1_N) = 0, \text{ for all } N \in \mathbb{N},$$

the above mixed free cumulants (4.18) vanish, for all mixed tuples (e.g., see [22]). Equivalently, χ_{K_1} and χ_{K_2} are free in $(\mathcal{H}(G_p), \psi_p)$.

($\Rightarrow$) Suppose now that χ_{K_1} and χ_{K_2} are free in $(\mathcal{H}(G_p), \psi_p)$, and assume that

$$\mu_p(K^o_{1,2}) \neq \mu_p(K_1)\mu_p(K_2).$$

Then it contradicts our freeness of χ_{K_1} and χ_{K_2}, because there exists at least one mixed n-tuple $(i_1, \ldots, i_n) \in \{1, 2\}^n$, for some $n \in \mathbb{N}$, such that

$$k_n\left(\chi_{K_{i_1}}, \ldots, \chi_{K_{i_n}}\right) \neq 0.$$

Indeed, if $n = 2$, then

$$k_2\left(\chi_{K_1}, \chi_{K_2}\right) = \mu_p(K^o_{1,2}) - \mu_p(K_1)\mu_p(K_2) \neq 0.$$

Therefore, f_1 and f_2 are free in $(\mathcal{H}(G_p), \psi_p)$ if and only if

$$\mu_p(K_1K_2) = \mu_p(K_1)\mu_p(K_2).$$

So, the statement (4.16) holds true.

Two free random variables f_1 and f_2 are free in $(\mathcal{H}(G_p), \psi_p)$, if and only if two subsets $\{f_1\}$ and $\{f_2\}$ are free in $(\mathcal{H}(G_p), \psi_p)$, if and only if corresponding subalgebras $\mathfrak{A}_j = \mathbb{C}_*[\{f_j\}]$ generated by $\{f_j\}$ are free in $(\mathcal{H}(G_p), \psi_p)$, for $j = 1, 2$. By the very definition, the generating elements h_j are contained in $\mathfrak{A}_j$, since

$$h_j = e_{K_j} = \frac{1}{\mu_p(K)}\chi_{K_j} = \frac{1}{\mu_p(K)}f_j \in \mathfrak{A}_j,$$

for all $j = 1, 2$. It shows that h_1 and h_2 are free in $(\mathcal{H}(G_p), \psi_p)$, if and only if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are free in $(\mathcal{H}(G_p), \psi_p)$, if and only if

$$\mu_p(K_1K_2) = \mu_p(K_1)\mu_p(K_2),$$

by (4.16).

Therefore, the freeness characterization (4.17) also holds. □

5. Representations on normal-core Hecke Probability Spaces

Let p be a fixed prime, and let $(\mathcal{H}(G_p), \psi_p)$ be the normal-core Hecke probability space in the sense of Section 4.

Define a form on the Hecke algebra $\mathcal{H}(G_p)$,

$$[,]_p : \mathcal{H}(G_p) \times \mathcal{H}(G_p) \to \mathbb{C}$$

by

$$[f_1, f_2]_p \overset{\text{def}}{=} \psi_p(f_1 * f_2^*), \text{ for all } f_1, f_2 \in \mathcal{H}(G_p), \tag{5.1}$$

where

$$f^*(x) \stackrel{\text{def}}{=} \overline{f(x)} \text{ in } \mathbb{C}, \text{ for all } x \in G_p,$$

where $\overline{z}$ mean the *conjugates of* z, for all $z \in \mathbb{C}$. We call the unary operation

$$f \in \mathcal{H}(G_p) \longmapsto f^* \in \mathcal{H}(G_p), \tag{5.2}$$

the *adjoint*. And the element f^* is said to be the *adjoint of* f. Since the adjoint (5.2) is well defined on $\mathcal{H}(G_p)$, one may understand our Hecke algebra $\mathcal{H}(G_p)$ as a $*$-*algebra* over $\mathbb{C}$.

By (5.1) of the form $[,]_p$, we have

$$[t_1 f_1 + t_2 f_2, f_3]_p = t_1[f_1, f_3]_p + t_2[f_2, f_3]_p, \tag{5.3}$$

and

$$[f_1, t_2 f_2 + t_3 f_3]_p = \overline{t_2}[f_1, f_2]_p + \overline{t_3}[f_1, f_3]_p,$$

for all $f_1, f_2, f_3 \in \mathcal{H}(G_p)$, and $t_1, t_2, t_3 \in \mathbb{C}$.

By (5.3), the form $[,]_p$ of (5.1) is a *sesqui-linear form on* $\mathcal{H}(G_p)$.

Consider now that, for any fixed generating element χ_{xK} of $\mathcal{H}(G_p)$, for $x \in G_p$, and a compact-open subgroup K of G_p, we have

$$[t\chi_{xK}, t\chi_{xK}]_p = \psi_p\left(t\chi_{xK} * \overline{t}\chi_{xK}\right) = |t|^2 \psi_p\left(\chi_{xK} * \chi_{xK}\right)$$

by the sesqui-linearity of $[,]_p$, where $|t|$ means the *modulus* $\sqrt{t\overline{t}}$ *of* t, hence

$$[t\chi_{xK}, t\chi_{xK}]_p = |t|^2 \psi_p\left(\mu_p(K)\chi_{x^2K}\right)$$

under Notation 4.1, by (3.2.1), thus

$$\begin{aligned}[t\chi_{xK}, t\chi_{xK}]_p &= \left(\mu_p(K)\,|t|^2\right)\left(\frac{\mu_p\left(x^2K_{G_p} \cap K_{G_p}\right)}{\mu_p(K_{G_p})}\right), \\ &= |t|^2\left(\frac{\mu_p(K)\mu_p\left(x^2K_{G_p} \cap K_{G_p}\right)}{\mu_p(K_{G_p})}\right),\end{aligned}$$

by (4.11), so

$$[t\chi_{xK}, t\chi_{xK}]_p = |t|^2\left(\frac{\mu_p(K)\mu_p(x^2K \cap K_{G_p})}{\mu_p(K_{G_p})}\right) \geq 0, \tag{5.4}$$

for all $x \in G_p$, and compact-open subgroups K of G_p, for all $t \in \mathbb{C}$.

By (5.4), one has

$$[f, f]_p \geq 0, \text{ for all } f \in \mathcal{H}(G_p). \tag{5.5}$$

Proposition 5.1. *The sesqui-linear form* $[,]_p$ *on the Hecke algebra* $\mathcal{H}(G_p)$ *forms a pseudo-inner product on* $\mathcal{H}(G_p)$*, i.e.,*

$$(\mathcal{H}(G_p), [,]_p) \text{ is a pseudo-inner-product space over } \mathbb{C}. \tag{5.6}$$

Proof. The form $[,]_p$ of (5.1) satisfies (5.3) and (5.5), on $\mathcal{H}(G_p)$. Observe further that, by the sesqui-linearity of $[,]_p$,

$$\begin{aligned}
[t_1\chi_{x_1K_1}, t_2\chi_{x_2K_2}]_p &= t_1\overline{t_2}\psi_p\left(\chi_{x_1K_1} * \chi_{x_2K_2}\right)\\
&= t_1\overline{t_2}\psi_p\left(\mu_p(K^o_{1,2})\chi_{x_{1,2}K_{1,2}}\right)\\
&= t_1\overline{t_2}\mu_p(K^o_{1,2})\varphi_p\left(\chi_{x_{1,2}K_{1,2:G_p}}\right)\\
&= t_1\overline{t_2}\mu_p(K^o_{1,2})\left(\frac{\mu_p\left(x_{1,2}K_{1,2,:G_p}\cap K_{1,2:G_p}\right)}{\mu_p(K_{1,2:G_p})}\right)\\
&= \left(t_1\overline{t_2}\right)\left(\frac{\mu_p(K^o_{1,2})\mu_p\left(x_{1,2}K_{1,2:G_p}\cap K_{1,2:G_p}\right)}{\mu_p\left(K_{1,2:G_p}\right)}\right), \qquad (5.7)
\end{aligned}$$

under Notation 4.1. Similarly, we also have that

$$[t_2\chi_{x_2K_2}, t_1\chi_{x_1K_1}]_p = \left(t_2\overline{t_1}\right)\left(\frac{\mu_p(K^o_{2,1})\mu_p\left(x_{2,1}K_{2,1:G_p}\cap K_{2,1:G_p}\right)}{\mu_p(K_{2,1:G_p})}\right), \qquad (5.8)$$

under Notation 4.1. Note that

$$\mu_p(K^o_{1,2}) = \mu_p(K^o_{2,1}), \text{ and } \mu_p(K_{1,2:G_p}) = \mu_p(K_{2,1:G_p}).$$

Therefore, formula (5.7) can be re-written by

$$[t_2\chi_{x_2K_2}, t_1\chi_{x_1K_1}]_p = \left(t_2\overline{t_1}\right)\left(\frac{\mu_p(K^o_{1,2})\mu_p\left(x_{1,2}K_{1,2:G_p}\cap K_{1,2:G_p}\right)}{\mu_p\left(K_{1,2:G_p}\right)}\right). \qquad (5.9)$$

So, by (5.6) and (5.8), we obtain that

$$[t_1\chi_{x_1K_1}, t_2\chi_{x_2K_2}]_p = \overline{[t_2\chi_{x_2K_2}, t_1\chi_{x_1K_1}]_p}, \qquad (5.10)$$

for all $t_1, t_2 \in \mathbb{C}$, $x_1, x_2 \in G_p$, and compact-open subgroups K_1 and K_2.

Therefore, by (5.9),

$$[f_1, f_2]_p = \overline{[f_2, f_1]_p}, \text{ for all } f_1, f_2 \in \mathcal{H}(G_p). \qquad (5.11)$$

Therefore, by (5.3), (5.5) and (5.10), the form $[,]_p$ is a pseudo-inner product on $\mathcal{H}(G_p)$. □

Suppose K is a proper normal compact-open subgroup of G_p and assume that

$$xK \neq Kx, \text{ for some } x \in G_p.$$

Then

$$[\chi_{xK}, \chi_{xK}]_p = 0,$$

under Notation 4.1, i.e., there do exist nonzero elements f of $\mathcal{H}(G_p)$, such that

$$[f, f]_p = 0.$$

So, the pseudo-inner product space $(\mathcal{H}(G_p), [,]_p)$ of (5.6) cannot be an inner product space.

Notation. When we understand our Hecke algebra $\mathcal{H}(G_p)$ as a pseudo-inner product space (5.6), we denote it by $\mathcal{H}_p$, i.e.,

$$\mathcal{H}_p = (\mathcal{H}(G_p), [,]_p)\,.$$

On the pseudo-inner product space $\mathcal{H}_p$ of (5.6), define an *equivalence relation* $\mathcal{R}_p$ by

$$f_1 \mathcal{R}_p f_2 \overset{\text{def}}{\Longleftrightarrow} [f_1, f_1]_p = [f_2, f_2]_p. \tag{5.12}$$

Definition 5.2. Let $\mathcal{H}_p$ be the pseudo-inner product space (5.6), and let $\mathcal{R}_p$ be the equivalence relation (5.11) on $\mathcal{H}_p$. Define the quotient space $\mathfrak{H}_p^a$ by

$$\mathfrak{H}_p^a = \mathcal{H}_p/\mathcal{R}_p, \tag{5.13}$$

equipped with the corresponding (quotient) pseudo-inner product, also denoted by $[,]_p$, on it, i.e.,

$$\left[[f_1]_{\mathcal{R}_p}, [f_2]_{\mathcal{R}_p}\right]_p = [f_1, f_2]_p\,,$$

for all equivalence classes

$$[f_j]_{\mathcal{R}_p} = \{g \in \mathcal{H}_p : g \mathcal{R}_p f_j\},$$

for all $j = 1, 2$. Then the pair

$$\mathfrak{H}_p^a \overset{\text{denote}}{=} (\mathfrak{H}_p^a, [,]_p) = (\mathcal{H}_p/\mathcal{R}_p, [,]_p)$$

is called the (normal-core) Hecke inner product space.

Indeed, our Hecke inner product space $\mathfrak{H}_p^a$ of (5.15) forms a well-determined inner product space by $\mathcal{R}_p$ of (5.11), i.e., it not only satisfies (5.3), (5.5) and (5.10) (under quotient), but also satisfies

$$[f, f]_p = 0 \Longleftrightarrow f = 0_{\mathfrak{H}_p^a} = \left[0_{\mathcal{H}_p}\right]_{\mathcal{R}_p}, \tag{5.14}$$

where 0_X mean the *zero vectors* of inner product spaces X.

Proposition 5.3. *The Hecke inner product space $\mathfrak{H}_p^a$ is an inner product space over $\mathbb{C}$.*

Proof. The proof is done by (5.3), (5.5), (5.10) (under quotient), and (5.12). □

Notation. For convenience, we denote equivalence classes

$$[f]_{\mathcal{R}_p} = \{h \in \mathcal{H}_p : h \mathcal{R}_p f\} \text{ of } \mathfrak{H}_p^a,$$

simply by f, if there is no confusion.

For the given inner product space $\mathfrak{H}_p^a$, one can define the corresponding *norm* $\|.\|_p$ *on* $\mathfrak{H}_p^a$ by

$$\|f\|_p \overset{\text{def}}{=} \sqrt{[f, f]_p}, \text{ for all } f \in \mathfrak{H}_p^a, \tag{5.15}$$

and the corresponding *metric* d_p *on* $\mathfrak{H}_p^a$ by

$$d_p(f_1, f_2) = \|f_1 - f_2\|_p\,, \text{ for all } f_1, f_2 \in \mathfrak{H}_p^a. \tag{5.16}$$

Definition 5.4. Construct the d_p-metric topology closure $\mathfrak{H}_p$ of $\mathfrak{H}_p^a$, where d_p is the metric (5.14) induced by the norm $\|.\|_p$ of (5.13). We call it the (normal-core) Hecke Hilbert space (for a prime p).

Then, by the very construction of the Hecke Hilbert space $\mathfrak{H}_p$ from the normal-core Hecke probability space $(\mathcal{H}(G_p), \psi_p)$, the Hecke algebra $\mathcal{H}(G_p)$ acts on $\mathfrak{H}_p$ via an $*$-*algebra-action* α^p,

$$\alpha^p(f)(h) = f * h, \text{ for all } h \in \mathfrak{H}_p, \tag{5.17}$$

for all $f \in \mathcal{H}(G_p)$.

More precisely, the definition (5.15) means that

$$\alpha^p(f)(h) = \alpha^p(f)\left([h]_{\mathcal{R}_p}\right) = [f * h]_{\mathcal{R}_p} = f * h, \tag{5.18}$$

in $\mathfrak{H}_p$.

Notation. For convenience, we denote $\alpha^p(f)$ by α_f^p, for all $f \in \mathcal{H}(G_p)$.

The above morphism α^p of (5.15) or (5.16) is indeed a well-defined $*$-algebra-action of $\mathcal{H}(G_p)$ acting on $\mathfrak{H}_p$, since

$$\begin{aligned}\alpha_{f_1*f_2}^p(h) &= f_1 * f_2 * h = f_1 * (f_2 * h)\\ &= f_1 * \left(\alpha_{f_2}^p(h)\right) = \alpha_{f_1}^p\left(\alpha_{f_2}^p(h)\right)\\ &= \alpha_{f_1}^p \alpha_{f_2}^p(h),\end{aligned}$$

for all $h \in \mathfrak{H}_p$, for all $f_1, f_2 \in \mathcal{H}(G_p)$, i.e.,

$$\alpha_{f_1*f_2}^p = \alpha_{f_1}^p \alpha_{f_2}^p \text{ on } \mathfrak{H}_p, \tag{5.19}$$

for all $f_1, f_2 \in \mathcal{H}(G_p)$. Also, α^p satisfies that

$$\begin{aligned}\left[\alpha_f^p(h_1), h_2\right]_p &= [f * h_1, h_2]_p\\ &= \psi_p\left((f * h_1) * h_2^*\right) = \psi_p\left(h_1 * f * h_2^*\right)\\ &= \psi_p\left(h_1 * (h_2^* * f)\right) = \psi_p\left(h_1 * (f^* * h_2)^*\right)\\ &= [h_1, f^* * h_2]_p = \left[h_1, \alpha_{f^*}^p(h_2)\right]_p,\end{aligned}$$

for all $h_1, h_2 \in \mathfrak{H}_p$, for all $f \in \mathcal{H}(G_p)$, i.e., one can get that:

$$\left(\alpha_f^p\right)^* = \alpha_{f^*}^p \text{ on } \mathfrak{H}_p, \text{ for all } f \in \mathcal{H}(G_p). \tag{5.20}$$

Theorem 5.5. *The pair $(\mathfrak{H}_p, \alpha^p)$ of the Hecke Hilbert space $\mathfrak{H}_p$ and the $*$-algebra-action α^p of* (5.15) *forms a well-determined Hilbert-space representation of the Hecke algebra $\mathcal{H}(G_p)$, induced by the normal-core Hecke probability space $(\mathcal{H}(G_p), \psi_p)$.*

Proof. By (5.17) and (5.18), the morphism α^p of (5.15) is a $*$-algebra-action of $\mathcal{H}(G_p)$ acting on $\mathfrak{H}_p$. Equivalently, the pair $(\mathfrak{H}_p, \alpha^p)$ forms a Hilbert-space representation of $\mathcal{H}(G_p)$. □

We call the $*$-algebra-action α^p of (5.15), the (*normal-core*) *Hecke*(*-algebra*) *action of* $\mathcal{H}(G_p)$ *acting on* $\mathfrak{H}_p$.

Definition 5.6. The Hilbert-space representation $(\mathfrak{H}_p, \alpha^p)$ of the Hecke Hilbert space $\mathfrak{H}_p$, and the Hecke action α^p of the Hecke algebra $\mathcal{H}(G_p)$ acting on $\mathfrak{H}_p$ is called the (normal-core) Hecke representation (of the normal-core Hecke probability space $(\mathcal{H}(G_p), \psi_p)$).

6. C^*-probability spaces induced by normal-core Hecke probability spaces

Let $(\mathcal{H}(G_p), \psi_p)$ be the normal-core Hecke probability space for a prime p, and $(\mathfrak{H}_p, \alpha^p)$, the Hecke representation of $\mathcal{H}(G_p)$, equipped with the Hecke Hilbert space $\mathfrak{H}_p$, and the $*$-algebra-action α^p of (5.15).

We now can understand each element f of the Hecke algebra $\mathcal{H}(G_p)$ as a *Hilbert-space operator* α_f^p on $\mathfrak{H}_p$ (under quotient by $\mathcal{R}_p$). Moreover, it is acting as a *convolution operator on* $\mathfrak{H}_p$, satisfying (5.17) and (5.18).

Let $B(\mathfrak{H}_p)$ be the operator algebra consisting of all bounded linear operators on $\mathfrak{H}_p$, equipped with its *operator-norm* $\|.\|$,

$$\|T\| = \sup\left\{ \|Th\|_p \,\middle|\, h \in \mathfrak{H}_p, \|h\|_p = 1 \right\}, \tag{6.1}$$

for all $T \in B(\mathfrak{H}_p)$, where $\|.\|_p$ is the Hilbert-space norm (5.13) on $\mathfrak{H}_p$. Then, under the $\|.\|$-norm topology, the operator algebra $B(\mathfrak{H}_p)$ forms a C^*-*algebra*.

Definition 6.1. Let $\mathcal{H}(G_p)$ be the Hecke algebra, and $(\mathfrak{H}_p, \alpha^p)$, the Hecke representation of $\mathcal{H}(G_p)$, and let $B(\mathfrak{H}_p)$ be the operator algebra consisting of all (bounded linear) operators on $\mathfrak{H}_p$. Define the C^*-subalgebra $\mathbb{H}(G_p)$ of $B(\mathfrak{H}_p)$ by

$$\mathbb{H}(G_p) = C^*\left(\mathcal{H}(G_p)\right) = \overline{\mathbb{C}\left[\alpha^p\left(\mathcal{H}(G_p)\right)\right]}^{\|.\|}, \tag{6.2}$$

where $\overline{Y}^{\|.\|}$ mean the $\|.\|$-closures of subsets Y of $B(\mathfrak{H}_p)$, where $\|.\|$ is the operator norm (6.1). This C^*-subalgebra $\mathbb{H}(G_p)$ of $B(\mathfrak{H}_p)$ is called the p-Hecke C^*-algebra (generated by $(\mathcal{H}(G_p), \psi_p)$ in $B(\mathfrak{H}_p)$).

Let $\mathbb{H}(G_p)$ be the p-Hecke C^*-algebra (6.2) in $B(\mathfrak{H}_p)$, and let $T \in \mathbb{H}(G_p)$. Then

$$T = \sum_{x \in G_p,\, K \leq G_p:\ \text{compact-open}} t_{xK} \alpha^p_{\chi_{xK}}, \text{ with } t_{xK} \in \mathbb{C}, \tag{6.3}$$

where $\sum$ in the expression (6.3) means finite, or infinite (limit of finite) sum(s) under the C^*-topology for $\mathbb{H}(G_p)$,i.e.,

$$T(h) = \sum_{x \in G_p,\, K \leq G_p:\ \text{compact-open}} t_{xK}\left(\chi_{xK} * h\right) \text{ in } \mathfrak{H}_p,$$

for all $h \in \mathfrak{H}_p$, whenever T is in the sense of (6.3) in $\mathbb{H}(G_p)$.

Notation. For convenience, we will denote the generating operators $\alpha^p_{\chi_{xK}}$ of the p-Hecke C^*-algebra $\mathbb{H}(G_p)$, simply by α^p_{xK}, for all $x \in G_p$, and K, compact-open subgroups of G_p.

The operator-theoretic properties of generating operators α^p_{xK} of p-Hecke C^*-algebra $\mathbb{H}(G_p)$ was studied in [11], [9] and [10].

Let Z_p be a (non-normal) compact-open subgroup G_p,

$$Z_p = GL_2(\mathbb{Z}_p) \text{ in } GL_2(\mathbb{Q}_p) = G_p, \tag{6.4}$$

which is a basis element of G_p, satisfying

$$\mu_p\left(Z_p\right) = 1.$$

For a fixed compact-open subgroup Z_p of (6.4), define a linear functional τ_p on $\mathbb{H}(G_p)$ by

$$\tau_p\left(\alpha^p_f\right) \stackrel{\text{def}}{=} \left[\alpha^p_f\left(\chi_{Z_p}\right), \chi_{Z_p}\right]_p, \tag{6.5}$$

for all $f \in \mathcal{H}(G_p)$, where $[,]_p$ is the inner product (5.1) (under $\mathcal{R}_p$) on the Hecke Hilbert space $\mathfrak{H}_p$.

Note that, for any compact-open subgroup K of G_p, one can define

$$e_K = \frac{1}{\mu_p(K)}\chi_K \in \mathcal{H}(G_p),$$

and

$$e_K * e_K = e_K$$

(see (2.2.5) and (2.2.6)). Remark that, for a compact-open subgroup Z_p of (6.4) in G_p,

$$e_{Z_p} = \frac{1}{\mu_p\left(Z_p\right)}\chi_{Z_p} = \frac{1}{1}\chi_{Z_p} = \chi_{Z_p},$$

and hence,

$$e_{Z_p} * e_{Z_p} = \chi_{Z_p} * \chi_{Z_p} = \chi_{Z_p} = e_{Z_p},$$

by (2.2.6), i.e.,

$$\chi^{(k)}_{Z_p} = e^{(k)}_{Z_p} = e_{Z_p} = \chi_{Z_p}, \text{ for all } k \in \mathbb{N}, \tag{6.6}$$

where $f^{(n)}$ means the convolution of n-copies of f.

By the definition (6.5) of τ_p, one can get that

$$\begin{aligned}
\tau_p\left(\alpha^p_{xK}\right) &= \left[\alpha^p_{xK}\left(\chi_{Z_p}\right), \chi_{Z_p}\right]_p \\
&= \left[\chi_{xK} * \chi_{Z_p}, \chi_{Z_p}\right]_p = \psi_p\left(\chi_{xK} * \chi_{Z_p} * \chi_{Z_p}\right) \\
&= \psi_p\left(\chi_{xK} * \chi_{Z_p}\right) \text{ by (6.6)} \\
&= \psi_p\left(\mu_p(xK \cap Z_p)\chi_{xKZ_p}\right) \text{ under Notation 4.1} \\
&= \mu_p\left(xK \cap Z_p\right)\left(\frac{\mu_p\left(xKZ_p \cap KZ_{p:G_p}\right)}{\mu_p\left(KZ_{p:G_p}\right)}\right),
\end{aligned} \tag{6.7}$$

under Notation 4.1, for all $x \in G_p$, and compact-open subgroups K of G_p, where $KZ_{p:G_p}$ is the normal core of the product group KZ_p.

The above computation (6.7) is non-vanishing only if

$$xK = Kx, \text{ and } xZ_p = Z_p x,$$

for $x \in G_p$, and K, a compact-open subgroup of G_p.

Thus, we will restrict our interests to the cases where x are from the *centralizer* $C_{G_p}(Z_p)$ *of* Z_p *in* G_p, where

$$C_{G_p}(Z_p) = \{x \in G_p : xZ_p = Z_p x\}. \tag{6.8}$$

Indeed, if $x \notin C_{G_p}(Z_p)$ in G_p, then

$$\tau_p\left(\alpha^p_{xK}\right) = 0 \text{ for all compact-open subgroups } K \text{ of } G_p$$

under Notation 4.1.

Thus it suffices to restrict our interests to the cases where the generating operators α^p_{xK} of $\mathbb{H}(G_p)$ satisfy $x \in C_{G_p}(Z_p)$, to obtain the non-vanishing free-distributional data. More precisely, one obtains the following proposition, refining (6.7).

Proposition 6.2. *Let $x \in C_{G_p}(K) \cap C_{G_p}(Z_p)$, where $C_{G_p}(K)$ and $C_{G_p}(Z_p)$ are the centralizer of K, respectively, that of Z_p, in G_p. Then*

$$\tau_p\left(\left(\alpha^p_{xK}\right)^n\right) = \frac{\mu_p(xK)^{n-1}\mu_p\left(x^nK \cap Z_p\right)\mu_p\left(x^nKZ_p \cap KZ_{p:G_p}\right)}{\mu_p\left(x^nKZ_{p:G_p}\right)} \neq 0, \tag{6.9}$$

for all $n \in \mathbb{N}$, where τ_p is the linear functional (6.5) *on the p-Hecke C^*-algebra $\mathbb{H}(G_p)$, and $KZ_{p:G_p}$ is the normal core of KZ_p in G_p.*

Meanwhile, if $x \notin C_{G_p}(Z_p) \cap C_{G_p}(K)$, then

$$\tau_p\left(\left(\alpha^p_{xK}\right)^n\right) = 0, \text{ for all } n \in \mathbb{N}. \tag{6.10}$$

Proof. First, assume that

$$x \in C_{G_p}(K) \cap C_{G_p}(Z_p), \text{ in } G_p. \tag{6.11}$$

Then, we obtain that

$$\tau_p\left(\alpha^p_{xK}\right) = \frac{\mu_p\left(xK \cap Z_p\right)\mu_p\left(xKZ_p \cap KZ_{p:G_p}\right)}{\mu_p\left(KZ_{p:G_p}\right)} \neq 0, \tag{6.12}$$

by (6.7). Now, suppose $n > 1$ in $\mathbb{N}$. Then

$$\tau_p\left(\left(\alpha^p_{xK}\right)^n\right) = \tau_p\left(\alpha^p(\chi^{(n)}_{xK})\right)$$

where $f^{(n)} = f * f * \ldots * f$ (n-times) in $\mathcal{H}(G_p)$, for all $f \in \mathcal{H}(G_p)$, thus

$$\tau_p\left(\left(\alpha^p_{xK}\right)^n\right) = \tau_p\left(\alpha^p\left(\mu_p(xK)^{n-1}\chi_{x^nK}\right)\right)$$

since $x \in C_{G_p}(K)$, by the assumption (6.11),

$$\begin{aligned}\tau_p\left((\alpha^p_{xK})^n\right) &= \mu_p(xK)^{n-1}\,\tau_p\left(\alpha^p_{x^nK}\right)\\ &= \mu_p(xK)^{n-1}\left(\frac{\mu_p\left(x^nK\cap Z_p\right)\mu_p\left(x^nKZ_p\cap KZ_{p:G_p}\right)}{\mu_p\left(KZ_{p:G_p}\right)}\right) \text{ by (6.7)}\\ &= \frac{\mu_p(xK)^{n-1}\,\mu_p\left(x^nK\cap Z_p\right)\mu_p\left(x^nKZ_p\cap KZ_{p:G_p}\right)}{\mu_p\left(KZ_{p:G_p}\right)} \neq 0, \qquad (6.13)\end{aligned}$$

by Assumption (6.11), since $x \in C_{G_p}(Z_p)$, too.

Therefore, formula (6.9) holds true, by (6.12) and (6.13).

Assume now that Condition (6.11) does not hold, i.e., suppose

$$x \notin C_{G_p}(K)\cap C_{G_p}(Z_p) \text{ in } G_p,$$

equivalently, either

$$x \in G_p - C_{G_p}(K), \text{ or } x \in G_p - C_{G_p}(Z_p).$$

Then, by (4.5), (4.5)′, (4.10), (5.1), and (6.5), we obtain that

$$\tau_p\left((\alpha^p_{xK})^n\right) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, formula (6.10) holds. □

By the above free-distributional data (6.9) and (6.10), we obtain the following generalized result.

Theorem 6.3. *Let $x_1, \ldots, x_N \in G_p$, and $K_1, \ldots, K_N$, compact-open subgroups of G_p, for $N \in \mathbb{N} - \{1\}$, and let $\alpha^p_{x_jK_j}$ be generating operators of our p-Hecke C^*-algebra $\mathbb{H}(G_p)$, for $j = 1, \ldots, N$. Then*

$$\begin{aligned}&\tau_p\left(\prod_{j=1}^N \Pi\alpha^p_{x_jK_j}\right)\\ &= \frac{\left(\prod_{j=2}^N \mu_p\left(K^o_{1,\ldots,j}\right)\right)\mu_p\left(x_{1,\ldots,N}K_{1,\ldots,N}\cap Z_p\right)\mu_p\left(x_{1,\ldots,N}K_{1,\ldots,N}Z_p\cap K_{1,\ldots,N}Z_{p:G_p}\right)}{\mu_p\left(K_{1,\ldots,N}Z_{p:G_p}\right)}.\end{aligned} \qquad (6.14)$$

The above joint free-moment formula (6.14) is non-vanishing, "only if"

$$x_j \in \left(\bigcap_{j=1}^N C_{G_p}(K_j)\right)\cap C_{G_p}(Z_p) \text{ in } G_p,$$

for all $j = 1, \ldots, N$.

Proof. Let $\alpha^p_{x_jK_j}$ be generating operators of $\mathbb{H}(G_p)$, for $j = 1, \ldots, N$, for $N \in \mathbb{N} - \{1\}$. Assume first that

$$x_j \in \left(\bigcap_{j=1}^{N} C_{G_p}(K_j) \right) \cap C_{G_p}(Z_p) \text{ in } G_p,$$

for all $j = 1, \ldots, N$. Then, for any $j = 1, \ldots, N$, one has

$$x_jK_i = K_ix_j, \text{ and } x_jZ_p = Z_px_j, \tag{6.15}$$

for all $i = 1, \ldots, N$. So,

$$\begin{aligned}
&\tau_p\left(\prod_{j=1}^{N}\alpha^p_{x_jK_j}\right) = \tau_p\left(\alpha^p\left(\overset{N}{\underset{j=1}{*}}\chi_{x_jK_j}\right)\right)\\
&= \tau_p\left(\alpha^p\left(\left(\prod_{j=2}^{N}\mu_p\left(K^o_{1,\ldots,j}\right)\right)\chi_{x_{1,\ldots,N}K_{1,\ldots,N}}\right)\right)\\
&= \left(\prod_{j=2}^{N}\mu_p\left(K^o_{1,\ldots,j}\right)\right)\tau_p\left(\alpha^p_{x_{1,\ldots,N}K_{1,\ldots,N}}\right)\\
&= \left(\prod_{j=2}^{N}\mu_p\left(K^o_{1,\ldots,j}\right)\right)\\
&\quad\cdot\left(\frac{\mu_p\left(x_{1,\ldots,N}K_{1,\ldots,N}\cap Z_p\right)\mu_p\left(x_{1,\ldots,N}K_{1,\ldots,N}Z_p\cap K_{1,\ldots,N}Z_{p:G_p}\right)}{\mu_p\left(K_{1,\ldots,N}Z_{p:G_p}\right)}\right) \text{ by (6.7)}\\
&\neq 0,
\end{aligned}$$

by Assumption (6.15). Therefore, the joint free-moment formula (6.14) holds, and it provides non-vanishing value under (6.15).

Indeed, if

$$x \notin \left(\bigcap_{j=1}^{N} C_{G_p}(K_j) \right) \cap C_{G_p}(Z_p) \text{ in } G_p,$$

and hence, if there exists at least one $j_0 \in \{j_1, \ldots, j_N\}$, such that

$$x \in G_p - C_{G_p}(K_{j_0}), \text{ or } x \in C_{G_p}\left(Z_p\right),$$

in G_p, then

$$\tau_p\left(\prod_{j=1}^{N}\alpha^p_{x_jK_j}\right) = 0,$$

by (4.5), (4.5)′, (4.10), (5.1) and (6.5). □

By our main results (6.9), (6.10) and (6.14), we obtain the following result.

Corollary 6.4. *Let $\alpha_K^p, \alpha_{K_j}^p$ be generating operators of the p-Hecke C^*-algebra $\mathbb{H}(G_p)$, for $j = 1, \ldots, N$, for $N \in \mathbb{N} - \{1\}$. Then*

$$\tau_p\left((\alpha_K^p)^n\right) = \mu_p(K)^{n-1}\,\mu_p(K \cap Z_p), \tag{6.16}$$

for all $n \in \mathbb{N}$, and

$$\tau_p\left(\prod_{j=1}^{N} \alpha_{K_j}^p\right) = \left(\prod_{j=2}^{N} \mu_p\left(K_{1,\ldots,j}^o\right)\right)\left(\mu_p\left(K_{1,\ldots,N} \cap Z_p\right)\right). \tag{6.17}$$

Proof. Note that $K = u_p K$, and $K_j = u_p K_j$, for all $j = 1, \ldots, N$, where u_p is the group-identity of G_p, satisfying

$$u_p \in C_{G_p}(K), \text{ and } u_p \in C_{G_p}(K_j), \text{ in } G_p, \tag{6.18}$$

for all $j = 1, \ldots, N$. Observe first that

$$\tau_p\left((\alpha_K^p)^n\right) = \frac{\mu_p(K)^{n-1}\,\mu_p(K \cap Z_p)\,\mu_p\left(KZ_p \cap KZ_{p:G_p}\right)}{\mu_p\left(KZ_{p:G_p}\right)}$$

by (6.9) and (6.10), under (6.18),

$$\tau_p\left((\alpha_K^p)^n\right) = \frac{\mu_p(K)^{n-1}\,\mu_p(K \cap Z_p)\,\mu_p\left(KZ_{p:G_p}\right)}{\mu_p\left(KZ_{p:G_p}\right)},$$

since $KZ_p \cap KZ_{p:G_p} = KZ_{p:G_p}$ in G_p,

$$\tau_p\left((\alpha_K^p)^n\right) = \mu_p(K)^{n-1}\,\mu_p(K \cap Z_p),$$

for all $n \in \mathbb{N}$. Therefore, the free-moment formula (6.16) holds.

Consider now that,

$$\tau_p\left(\prod_{j=1}^{N} \alpha_{K_j}^p\right)$$
$$= \frac{\left(\prod_{j=2}^{N} \mu_p\left(K_{1,\ldots,j}^o\right)\right)\mu_p\left(K_{1,\ldots,N} \cap Z_p\right)\mu_p\left(K_{1,\ldots,N}Z_p \cap K_{1,\ldots,N}Z_{p:G_p}\right)}{\mu_p\left(K_{1,\ldots,N}Z_{p:G_p}\right)}$$

by (6.14), under (6.18)

$$\tau_p\left(\prod_{j=1}^{N} \alpha_{K_j}^p\right) = \left(\prod_{j=2}^{N} \mu_p\left(K_{1,\ldots,j}^o\right)\right)\mu_p\left(K_{1,\ldots,N} \cap Z_p\right)\frac{\mu_p\left(K_{1,\ldots,N}Z_{p:G_p}\right)}{\mu_p\left(K_{1,\ldots,N}Z_{p:G_p}\right)}$$

since $K_{1,\ldots,N}Z_p \cap K_{1,\ldots,N}Z_{p:G_p} = K_{1,\ldots,N}Z_{p:G_p}$ in G_p

$$\tau_p\left(\prod_{j=1}^{N} \alpha_{K_j}^p\right) = \left(\prod_{j=2}^{N} \mu_p\left(K_{1,\ldots,j}^o\right)\right)\mu_p\left(K_{1,\ldots,N} \cap Z_p\right).$$

Therefore, the joint free-moment formula (6.17) holds true. □

Now, we focus on the joint free-moment formula (6.14), for $N \in \mathbb{N} - \{1\}$. It shows that there exists $r_{1,\dots,N} \in \mathbb{R}$, such that

$$0 < r_{1,\dots,N} \leq 1, \tag{6.19}$$

and

$$\mu_p\left(x_{1,\dots,N}K_{1,\dots,N} \cap Z_p\right) = r_{1,\dots,N},$$

because

$$\mu_p\left(x_{1,\dots,N}K_{1,\dots,N} \cap Z_p\right) \leq \mu_p\left(Z_p\right) = 1.$$

Thus, formula (6.14) becomes

$$\begin{aligned} &\tau_p\left(\prod_{j=1}^{N} \alpha^p_{x_jK_j}\right) \\ &= \frac{r_{1,\dots,N}\left(\prod_{j=2}^{N} \mu_p\left(K^o_{1,\dots,j}\right)\right)\left(\mu_p\left(x_{1,\dots,N}K_{1,\dots,N}Z_p \cap K_{1,\dots,N}Z_{p:G_p}\right)\right)}{\mu_p\left(K_{1,\dots,N}Z_{p:G_p}\right)}, \end{aligned} \tag{6.20}$$

where $r_{1,\dots,N}$ is in the sense of (6.19).

As we have seen above, for a p-Hecke C^*-algebra $\mathbb{H}(G_p)$, one can define a linear functional τ_p of (6.5), determined by the Hecke representation $(\mathfrak{H}_p, \alpha^p)$ of the Hecke probability space $(\mathcal{H}(G_p), \psi_p)$ for a fixed prime p, for all $p \in \mathcal{P}$, where $\mathcal{P}$ is the subset consisting of all primes in $\mathbb{N}$. It means that one can get C^*-probability spaces

$$\left\{(\mathbb{H}(G_p), \tau_p) : p \in \mathcal{P}\right\}. \tag{6.21}$$

Definition 6.5. Let $p \in \mathcal{P}$ and let $(\mathbb{H}(G_p), \tau_p)$ be the corresponding C^*-probability space in (6.21), where $\mathbb{H}(G_p)$ is the p-Hecke C^*-algebra of $\mathcal{H}(G_p)$, and τ_p is in the sense of (6.5). Then we call $(\mathbb{H}(G_p), \tau_p)$, the p-Hecke C^*-probability space.

7. Free product C^*-probability spaces induced by $\left\{(\mathbb{H}(G_p), \tau_p)\right\}_{p\in\mathcal{P}}$

Let $(\mathbb{H}(G_p), \tau_p)$ be the p-Hecke C^*-probability spaces, for $p \in \mathcal{P}$, and

$$\left\{(\mathbb{H}(G_p), \tau_p)\right\}_{p\in\mathcal{P}},$$

the system (6.21) of such C^*-probability spaces over $\mathcal{P}$. Recall that, for any fixed prime $p \in \mathcal{P}$, if $\alpha^p_{x_jK_j} \in (\mathbb{H}(G_p), \tau_p)$ are generating operators, for $j = 1, \dots, N$, for $N \in \mathbb{N} - \{1\}$, then

$$\tau_p\left(\left(\alpha^p_{x_jK_j}\right)^n\right) = \frac{\left(\mu_p\left(K_j\right)^{n-1}\right)\mu_p\left(x^nK_j \cap Z_p\right)\mu_p\left(x^n_jK_jZ_p \cap K_jZ_{p:G_p}\right)}{\mu_p\left(K_jZ_{p:G_p}\right)} \neq 0, \tag{7.0.1}$$

for all $n \in \mathbb{N}$, only if

$$x_j \in C_{G_p}\left(K_j\right) \cap C_{G_p}\left(Z_p\right), \text{ for all } j = 1, \dots, N,$$

and

$$\tau_p\left(\prod_{j=1}^{N}\alpha^p_{x_jK_j}\right) \tag{7.0.2}$$
$$= \frac{\left(\prod_{j=2}^{N}\mu_p\left(K^o_{1,\ldots,j}\right)\right)\left(\mu_p\left(x_{1,\ldots,N}K_{1,\ldots,N}\cap Z_p\right)\right)\left(\mu_p\left(x_{1,\ldots,N}K_{1,\ldots,N}Z_p\cap K_{1,\ldots,N}Z_{p:G_p}\right)\right)}{\mu_p\left(K_{1,\ldots,N}Z_{p:G_p}\right)}$$

is non-vanishing, only if

$$x_j \in \left(\bigcap_{j=1}^{N} C_{G_p}\left(K_j\right)\right)\cap C_{G_p}\left(Z_p\right),$$

for all $j = 1, \ldots, N$, for all $N \in \mathbb{N}-\{1\}$, by (6.9), (6.10) and (6.14), where $C_{G_p}(K)$ are the centralizers of compact-open subgroups K of G_p.

7.1. Free Product C^*-Probability Spaces

Let (A_k, φ_k) be arbitrary C^*-probability spaces, consisting of C^*-algebras A_k, and linear functionals φ_k on A_k, for $k \in \Delta$, where Δ is an arbitrary countable (finite or infinite) index set. The *free product C^*-algebra* A,

$$A = \underset{l\in\Delta}{\star} A_l$$

is the C^*-algebra generated by the *noncommutative reduced words* in $\bigcup_{l=1}^{N} A_l$, having a new linear functional

$$\varphi = \underset{l\in\Delta}{\star}\varphi_l.$$

The C^*-algebra A is understood as a *Banach space*,

$$\mathbb{C}\oplus\left(\bigoplus_{n=1}^{\infty}\left(\bigoplus_{(i_1,\ldots,i_n)\in \mathrm{alt}(\Delta^n)}\left(\bigotimes_{k=1}^{n}A^o_{i_k}\right)\right)\right) \tag{7.1.1}$$

with

$$A^o_{i_k} = A_{i_k}\ominus\mathbb{C}, \text{ for all } k = 1, \ldots, n,$$

where

$$\mathrm{alt}\left(\Delta^n\right) = \left\{(i_1,\ldots,i_n)\;\middle|\;\begin{array}{c}(i_1,\ldots,i_n)\in\Delta^n\\ i_1\neq i_2, i_2\neq i_3,\\ \ldots, i_{n-1}\neq i_n\end{array}\right\},$$

for all $n \in \mathbb{N}$, and where the *direct product* $\oplus$, and the *tensor product* $\otimes$ are topological on Banach spaces.

In particular, if an element $a \in A$ is a *free* "*reduced*" *word*,

$$\prod_{l=1}^{n} a_{i_l} \text{ in } A,$$

then one can understand a as an equivalent Banach-space vector,

$$\bigotimes_{l=1}^{n} a_{i_l} \text{ in the Banach space } A \text{ of (7.1.1)},$$

contained in a direct summand, $\bigotimes_{k=1}^{n} A_{i_k}^{o}$ of (7.1.1), for

$$(i_1, \ldots, i_n) \in \text{alt}\,(\Delta^n)\,, \text{ for } n \in \mathbb{N}.$$

Here, one may / can understand $\bigotimes_{l=1}^{n} a_{i_l}$ as an operator in the tensor product C^*-algebra,

$$\mathbb{C} \oplus \left(\bigotimes_{k=1}^{n}{}_{\mathbb{C}} A_{i_k}^{o} \right) = \bigotimes_{k=1}^{n}{}_{\mathbb{C}} \left(\mathbb{C} \oplus A_{i_k}^{o} \right) = \bigotimes_{k=1}^{n}{}_{\mathbb{C}} A_{i_k} \text{ in } A,$$

where $\otimes_{\mathbb{C}}$ means the *tensor product of C^*-algebras*, i.e., the given free reduced word a is regarded as a Banach-space vector in A of (7.1.1), or as an operator under tensor product.

We denote this relation by

$$a \overset{\text{equi}}{=} \bigotimes_{l=1}^{n} a_{i_l} \text{ in } A, \tag{7.1.2}$$

where "$\overset{\text{equi}}{=}$" means "being equivalent to".

We remark that, if a is a free reduced word in A, and if it is regarded as an operator under the equivalence (7.1.2), then

$$a^k \overset{\text{equi}}{=} \left(\bigotimes_{l=1}^{n} a_{i_l} \right)^k = \bigotimes_{l=1}^{n} a_{i_l}^k \overset{\text{equi}}{=} \prod_{l=1}^{n} a_{i_l}^k \text{ in } A, \tag{7.1.3}$$

for all $k \in \mathbb{N}$.

Notation and Remark 7.1.1 (in short, NR 7.1.1 below). Let $a = \prod_{l=1}^{n} a_{i_l}$ be a free reduced word in A, as above, for

$$(i_1, \ldots, i_n) \in \text{alt}\,(\Delta^n)\,, \text{ for } n \in \mathbb{N}.$$

The power a^k in (7.1.3) means the k-th power of a as a new element in A, equivalent to the operator in

$$\bigotimes_{l=1}^{n}{}_{\mathbb{C}} A_{i_l} \subset A \text{ in the sense of (7.1.1)}.$$

To avoid confusion, we use the notation $a^{(k)}$, as a construction of new free "*non-reduced*" word,

$$a^{(k)} = \underbrace{a \cdot a \cdots a}_{k\text{-times}} \text{ in } A,$$

induced by a in A.

For example, let $a = a_{i_1} a_{i_2} a_{i_1}$ be a free reduced word, where

$$(i_1, i_2, i_1) \in \text{alt}\,(\Delta^3)\,,$$

as a Banach-space vector,

$$a_{i_1} \otimes a_{i_2} \otimes a_{i_1} \text{ in } A, \text{ by (7.1.3).}$$

Then

$$a^3 \overset{\text{equi}}{=} (a_{i_1} \otimes a_{i_2} \otimes a_{i_3})^3 \overset{\text{equi}}{=} a_{i_1}^3 a_{i_2}^3 a_{i_1}^3,$$

in A, but

$$\begin{aligned} a^{(3)} &= (a_{i_1} a_{i_2} a_{i_1})^{(3)} \\ &= a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_2} a_{i_1} \text{ (non-reduced)} \\ &= a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}, \text{ (reduced),} \end{aligned}$$

i.e.,

$$a^{(3)} = a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1},$$

as a free reduced word in A. At this moment, we emphasize the difference between a^k and $a^{(k)}$ in A.

In a similar manner, we can have

$$a^* \overset{\text{equi}}{=} \left(\bigotimes_{l=1}^n a_{i_l} \right)^* = \bigotimes_{l=1}^n a_{i_l}^* \overset{\text{equi}}{=} \prod_{l=1}^n a_{i_l}^* \text{ in } A,$$

and

$$a^{(*)} = (a_{i_1} a_{i_2} \dots a_{i_n})^{(*)} = a_{i_n}^* a_{i_{n-1}}^* \dots a_{i_2}^* a_{i_1}^* \text{ in } A,$$

where $(*)$ means the adjoint.

So, we remark also the difference between the adjoints a^* and $a^{(*)}$ of a free reduced word a in A.

The free product linear functional $\varphi = \underset{i \in \Delta}{\star} \varphi_i$ on A satisfies that, whenever a is a reduced free word in A satisfying (7.1.2), then

$$\varphi(a^k) = \varphi \left(\prod_{l=1}^n a_{i_l}^k \right) = \prod_{l=1}^n \left(\varphi_{i_l}(a_{i_l}^k) \right), \tag{7.1.4}$$

by (7.1.3), for all $k \in \mathbb{N}$. By abusing (7.1.3), one can / may write

$$\varphi(a^k) \overset{\text{equi}}{=} \varphi \left(\bigotimes_{l=1}^n a_{i_l}^k \right) = \prod_{l=1}^n \varphi(a_{i_l}^k),$$

whenever $a = \prod_{l=1}^n a_{i_l}$ is a free reduced word in A, for all $k \in \mathbb{N}$.

Now, let

$$b = \sum_{l=1}^n b_{i_l} \in (A, \varphi),$$

where b_{i_l} are free reduced words in A, for all $l = 1, \dots, n$.

We say that a is a *free sum in* A, if all summands $b_{i_1}, \ldots, b_{i_n}$ of b are contained in "mutually-distinct" direct summands of a Banach space A of (7.1.1), as free reduced words. Then, similar to the above observation, one can write

$$b \overset{\text{equi}}{=} \bigoplus_{l=1}^{n} b_{i_l} \text{ in the Banach space } A \text{ of (7.1.1)}, \tag{7.1.5}$$

satisfying

$$\varphi\left(b^k\right) \overset{\text{equi}}{=} \varphi\left(\left(\bigoplus_{l=1}^{n} b_{i_l}\right)^k\right) = \varphi\left(\bigoplus_{l=1}^{n} b_{i_l}^k\right)$$
$$\overset{\text{equi}}{=} \varphi\left(\sum_{l=1}^{n} b_{i_l}^k\right) = \sum_{l=1}^{n} \varphi\left(b_{i_l}^k\right),$$

for all $k \in \mathbb{N}$.

Here, we remark that each summand $\varphi\left(b_{i_l}^k\right)$ of (7.1.5) satisfies (7.1.4).

Notation and Remark 7.1.2 (in short, **NR 7.1.2** below). Similar to the free-reduced-word case, if b is a free sum in the sense of (6.1.5), then one can consider

$$b^{(k)} = \left(\sum_{l=1}^{n} b_{i_l}\right)^{(k)}$$
$$= \sum_{(l_1,\ldots,l_k)\in\{1,\ldots,n\}^k} \left(b_{i_{l_1}} b_{i_{l_2}} \ldots b_{i_{l_k}}\right),$$

where the summands are free "non-reduced" words in A.

7.2. The adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$

Let $\{(\mathbb{H}(G_p), \tau_p)\}_{p\in\mathcal{P}}$ be the system (7.0.1) of our p-Hecke C^*-probability spaces over $\mathcal{P}$. Construct the free product C^*-probability space $(\mathbb{H}_{\mathcal{P}}, \tau)$ by

$$(\mathbb{H}_{\mathcal{P}}, \tau) \overset{\text{def}}{=} \underset{p\in\mathcal{P}}{\star} (\mathbb{H}(G_p), \tau_p) \tag{7.2.1}$$

$$= \left(\underset{p\in\mathcal{P}}{\star} \mathbb{H}(G_p), \underset{p\in\mathcal{P}}{\star} \tau_p\right), \tag{7.2.2}$$

as in Section 7.1.

Definition 7.1. Let $(\mathbb{H}_{\mathcal{P}}, \tau)$ be the free product C^*-probability space (7.2.1) induced by the system (7.0.1). We call it the Adelic Hecke C^*-probability space (over primes). For convenience, we denote $(\mathbb{H}_{\mathcal{P}}, \tau)$ simply by $\mathbb{H}_{\mathcal{P}}$. So, the term "$\mathbb{H}_{\mathcal{P}}$" will mean both the free product C^*-algebra, and the free product C^*-probability space, case-by-case, below.

Let $N \in \mathbb{N}$, and let

$$T_{p_l} = \alpha^{p_l}_{x_l K_l} \in (\mathbb{H}(G_{p_l}), \tau_{p_l}) \tag{7.2.3}$$

be generating operators of $\mathbb{H}(G_{p_l})$, with

$$x_l \in C_{G_{p_l}}(K_l) \cap C_{G_{p_l}}(Z_p) \text{ in } G_{p_l},$$

and K_l, compact-open subgroups of G_{p_l}, for all $l = 1, \ldots, N$.

Construct a free random variable T in the Adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$,

$$T = \prod_{l=1}^{N} T_{p_l}^{n_l} \in \mathbb{H}_{\mathcal{P}}, \tag{7.2.4}$$

where T_{p_l} are in the sense of (7.2.2), for $n_l \in \mathbb{N}$, for all $l = 1, \ldots, N$.

Let us consider the free-distributional data of a free random variable T of (7.2.3) in $\mathbb{H}_{\mathcal{P}}$. Recall the notations in NR 7.1.1.

Theorem 7.2. *Let T be in the sense of* (7.2.3) *in the Adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$, where $N \in \mathbb{N} - \{1\}$. Assume that*

$$(p_1, \ldots, p_N) \in \text{alt}\left(\mathcal{P}^N\right)$$

is an alternating N-tuple. Then

$$\tau\left(T^n\right) = \prod_{l=1}^{N} \left(\frac{\mu_{p_l}(K_l)^{nn_l - 1} \mu_{p_l}\left(K_l \cap Z_p\right) \mu_{p_l}\left(x_l^{nn_l} K_l Z_p \cap K_l Z_{p:G_p}\right)}{\mu_{p_l}\left(K_l Z_{p:G_p}\right)} \right), \tag{7.2.5}$$

for all $n \in \mathbb{N}$. Furthermore, if $p_1 \neq p_N$ in $\mathcal{P}$, then

$$\tau\left(T^{(n)}\right) = \prod_{l=1}^{N} \left(\frac{\mu_p\left(K_l\right)^{n_l - 1} \mu_p\left(K_l \cap Z_p\right) \mu_p\left(x_l^{n_l} K_l Z_p \cap K_l Z_{p:G_p}\right)}{\mu_p\left(K_l Z_{p:G_p}\right)} \right)^n, \tag{7.2.6}$$

for all $n \in \mathbb{N}$.

Proof. Let T be a free random variable (7.2.3) in the Adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$, and suppose the N-tuple $(p_1, \ldots, p_N)$ is alternating in $\mathcal{P}$. Then the operator T forms a free reduced word in $\mathbb{H}_{\mathcal{P}}$, by (7.2.1). Thus, one can get that

$$T^n \overset{\text{equi}}{=} \left(\bigotimes_{l=1}^{N} T_{p_l}^{n_l} \right)^n = \bigotimes_{l=1}^{N} T_{p_l}^{n_l n} \overset{\text{equi}}{=} \prod_{l=1}^{N} T_{p_l}^{n_l n}, \tag{7.2.7}$$

for all $n \in \mathbb{N}$, by (7.1.3). So, for any $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} \tau\left(T^n\right) &= \tau\left(\prod_{l=1}^{N} T_{p_l}^{n_l n} \right) = \prod_{l=1}^{N} \tau_{p_l}\left(T_{p_l}^{n_l n}\right) \\ &= \prod_{l=1}^{N} \left(\frac{\mu_{p_l}(K_l)^{n_l n - 1} \mu_{p_l}\left(K_l \cap Z_p\right) \mu_{p_l}\left(x_l^{n_l n} K_l Z_p \cap K_l Z_{p:G_p}\right)}{\mu_{p_l}\left(K_l Z_{p:G_p}\right)} \right), \end{aligned}$$

by (7.2.6), (7.0.1) and (7.2.2). Therefore, the free-moment formula (7.2.4) holds.

Now, assume that $p_1 \neq p_N$ in $\mathcal{P}$. Then, for any $n \in \mathbb{N}$, the nN-tuple,

$$W_n = \left(\underbrace{(p_1, \ldots, p_N), (p_1, \ldots, p_N), \ \ldots, \ (p_1, \ldots, p_N)}_{n\text{-times}} \right)$$

is an alternating nN-tuple in $\mathcal{P}$, i.e.,

$$W_n \in \text{alt}\left(\mathcal{P}^{nN}\right), \text{ for all } n \in \mathbb{N}. \tag{7.2.8}$$

Thus, the free random variables $T^{(n)}$ are again free reduced words,

$$T^{(n)} = \underbrace{\left(T_{p_1}^{n_1} \cdots T_{p_N}^{n_N}\right)\left(T_{p_1}^{n_1} \cdots T_{p_N}^{n_N}\right) \cdots \left(T_{p_1}^{n_1} \cdots T_{p_N}^{n_N}\right)}_{n\text{-times}}$$

in $\mathbb{H}_{\mathcal{P}}$, for all $n \in \mathbb{N}$, by (7.2.7). So, one obtains that

$$\begin{aligned}
\tau\left(T^{(n)}\right) &= \tau\left((T_{p_1}^{n_1} \ldots T_{p_N}^{n_N}) \ldots (T_{p_1}^{n_1} \ldots T_{p_N}^{n_N})\right) \\
&= \left(\prod_{l=1}^{N} \tau_{p_l}\left(T_{p_l}^{n_l}\right)\right) \cdots \left(\prod_{l=1}^{N} \tau_{p_l}\left(T_{p_l}^{n_l}\right)\right) \\
&= \left(\prod_{l=1}^{N} \tau_{p_l}\left(T_{p_l}^{n_l}\right)\right)^n \\
&= \left(\prod_{l=1}^{N} \left(\frac{\mu_p\left(K_l\right)^{n_l - 1} \mu_p\left(K_l \cap Z_p\right) \mu_p\left(x_l^{n_l} K_l Z_p \cap K_l Z_{p:G_p}\right)}{\mu_p\left(K_l Z_{p:G_p}\right)} \right)\right)^n \\
&= \prod_{l=1}^{N} \left(\frac{\mu_p\left(K_l\right)^{n_l - 1} \mu_p\left(K_l \cap Z_p\right) \mu_p\left(x_l^{n_l} K_l Z_p \cap K_l Z_{p:G_p}\right)}{\mu_p\left(K_l Z_{p:G_p}\right)} \right)^n,
\end{aligned}$$

by (7.0.1) and (7.2.2), for all $n \in \mathbb{N}$. Therefore, the free-moment formula (7.2.5) holds true. □

The above two free-distributional data (7.2.4) and (7.2.5) characterize the free distributions of free reduced words generated by the generating operators in our adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$.

Also, the above two formulas (7.2.4) and (7.2.5) illustrate the differences between two free distributions,

$$\left(\tau(T^n)\right)_{n=1}^{\infty}, \text{ and } \left(\tau(T^{(n)})\right)_{n=1}^{\infty},$$

where T^n and $T^{(n)}$ are in the sense of NR 7.1.1, for all $n \in \mathbb{N}$. Of course, if $N = 1$ in the above theorem, then

$$\tau\left(T^n\right) = \tau\left(T^{(n)}\right), \text{ for all } n \in \mathbb{N}, \tag{7.2.5}'$$

since

$$T^{(n)} = T^n \text{ in } (\mathbb{H}(G_{p_1}), \tau_{p_1}), \text{ in } \mathbb{H}_{\mathcal{P}},$$

as length-1 free reduced words, for all $n \in \mathbb{N}$.

Theorem 7.3. *Let $S_k = \prod_{l_k=1}^{N_k} T_{p_{l_k}}^{n_{l_k}}$ be free random variables (7.2.3) of the adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$, where the $T_{p_{l_k}} \in \mathbb{H}(G_{p_{l_k}})$ are in the sense of (7.2.2), for all $k = 1, \ldots, N$, for $N \in \mathbb{N} - \{1\}$. Also, let*

$$T = \sum_{k=1}^{N} S_k \in \mathbb{H}_{\mathcal{P}}.$$

Suppose

$$W_k = (p_{1_k}, p_{2_k}, \ldots, p_{N_k}) \in \text{alt}\left(\mathcal{P}^{N_k}\right), \tag{7.2.9}$$

for all $k = 1, \ldots, N$, and

$$W_{k_1} \neq W_{k_2} \text{ in } \mathcal{P}^{\infty}, \text{ whenever } k_1 \neq k_2 \text{ in } \{1, \ldots, N\}.$$

Then

$$\tau\left(T^n\right) = \sum_{k=1}^{N} \left(\prod_{l_k=1}^{N_k} \left(\frac{\mu_p\left(K_{l_k}\right)^{n-1} \mu_p\left(x_{l_k}^n K_{l_k} \cap Z_p\right) \mu_p\left(x_{l_k}^n K_{l_k} Z_p \subset K_{l_k} Z_{p:G_p}\right)}{\mu_p\left(K_{l_k} Z_{p:G_p}\right)} \right) \right), \tag{7.2.10}$$

for all $n \in \mathbb{N}$.

Proof. Let S_k be in the sense of (7.2.3) in $\mathbb{H}_{\mathcal{P}}$, for all $k = 1, \ldots, N$. Then, by the first condition of the assumption (7.2.8), every operator S_k is a free reduced word in $\mathbb{H}_{\mathcal{P}}$, for all $k = 1, \ldots, N$. Moreover, by the second condition of the assumption (7.2.8), the operator $T = \sum_{k=1}^{N} S_k$ forms a free sum in $\mathbb{H}_{\mathcal{P}}$, satisfying

$$T^n \overset{\text{equi}}{=} \left(\bigoplus_{k=1}^{N} S_k \right)^n = \bigoplus_{k=1}^{N} S_k^n \overset{\text{equi}}{=} \sum_{k=1}^{N} S_k^n \text{ in } \mathbb{H}_{\mathcal{P}},$$

for all $n \in \mathbb{N}$ (see NR 7.1.2). So, one obtains that

$$\tau\left(T^n\right) = \sum_{k=1}^{N} \tau\left(S_k^n\right), \text{ for all } n \in \mathbb{N},$$

where

$$\tau\left(S_k^n\right) = \prod_{l_k=1}^{N_k} \left(\frac{\mu_p\left(K_{l_k}\right)^{n-1} \mu_p\left(x_{l_k}^n K_{l_k} \cap Z_p\right) \mu_p\left(x_{l_k}^n K_{l_k} Z_p \cap K_{l_k} Z_{p:G_p}\right)}{\mu_p\left(K_{l_k} Z_{p:G_p}\right)} \right),$$

for all $n \in \mathbb{N}$, by (7.2.4).

Therefore, the free-moment formula (7.2.9) holds. □

The above free-distributional data (7.2.9) characterizes the free distributions of free sums of free reduced words in $\mathbb{H}_{\mathcal{P}}$. Remark that, in (7.2.9), the free summands S_k of T are generated by their free-factors in the sense of (7.2.2).

8. On a Certain C^*-Subalgebra $\mathfrak{S}$ of $\mathbb{H}_{\mathcal{P}}$

In this section, we consider a certain C^*-subalgebra $\mathfrak{S}$ of the free product C^*-algebra $\mathbb{H}_{\mathcal{P}}$, the adelic Hecke C^*-algebra, having interesting free-probabilistic sub-structure.

For any $p \in \mathcal{P}$, define a subset $\mathcal{K}_p$ of the power set of G_p by

$$\mathcal{K}_p = \{K \lhd G_p : K \text{ is compact-open in } G_p\}, \tag{8.1}$$

i.e., the set $\mathcal{K}_p$ is the collection of all compact-open normal subgroups of the general linear group $G_p = GL_2(\mathbb{Q}_p)$, for all $p \in \mathcal{P}$.

Now, define the C^*-subalgebras $\mathfrak{S}_p$ of the p-Hecke C^*-algebras $\mathbb{H}(G_p)$ by the C^*-algebras generated by $\mathcal{K}_p$ of (8.1), i.e.,

$$\mathfrak{S}_p \overset{\text{def}}{=} C^* \left(\{\alpha_K \in \mathbb{H}(G_p) : K \in \mathcal{K}_p\}\right) \text{ in } \mathbb{H}(G_p), \tag{8.2}$$

for all $p \in \mathcal{P}$. Then the C^*-algebras $\mathfrak{S}_p$ of (8.2) are well-defined C^*-subalgebras of $\mathbb{H}(G_p)$, inducing the free-probabilistic sub-structures,

$$\mathfrak{S}_p \overset{\text{denote}}{=} (\mathfrak{S}_p, \tau_p), \text{ with } \tau_p = \tau_p \mid_{\mathfrak{S}_p}, \tag{8.3}$$

in the p-Hecke C^*-probability spaces $(\mathbb{H}(G_p), \tau_p)$, where the τ_p are in the sense of (6.5), for all $p \in \mathcal{P}$.

Then, from the system

$$\{\mathfrak{S}_p = (\mathfrak{S}_p, \tau_p) : p \in \mathcal{P}\},$$

consisting of all C^*-probability spaces $\mathfrak{S}_p$ of (8.3), one can obtain the free-product C^*-probability space

$$\mathfrak{S} = (\mathfrak{S}, \tau) \overset{\text{def}}{=} \underset{p\in\mathcal{P}}{\star} frakS_p = \left(\underset{p\in\mathcal{P}}{\star} \mathfrak{S}_p, \underset{p\in\mathcal{P}}{\star} \tau_p\right). \tag{8.4}$$

Then this new C^*-probability space $\mathfrak{S}$ of (8.4) is a well-determined C^*-probabilistic sub-structure of our adelic Hecke C^*-probability space

$$\mathbb{H}_{\mathcal{P}} = (\mathbb{H}_{\mathcal{P}}, \tau)$$

of (7.2.1).

Definition 8.1. Let $\mathfrak{S}$ be the free-product C^*-probability space (8.4) of the system $\{\mathfrak{S}_p\}_{p\in\mathcal{P}}$ of C^*-probability spaces $\mathfrak{S}_p$'s of (8.3). Then it is called the (adelic-normal-compact-open) sub-Hecke C^*-probability space of the adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$.

Then, by the main results of Section 7, we obtain the following free-distributional data on the sub-Hecke C^*-probability space $\mathfrak{S}$ of $\mathbb{H}_{\mathcal{P}}$.

Corollary 8.2. *Let α_K^p be a generating operator of the sub-Hecke C^*-probability space $\mathfrak{S}$ of $\mathbb{H}_{\mathcal{P}}$, for $K \in \mathcal{K}_p$, for $p \in \mathcal{P}$. Then*

$$\tau\left((\alpha_K^p)^n\right) = \mu_p(K)^{n-1}\, \mu_p(K \cap Z_p) \ = \tau\left((\alpha_K^p)^{(n)}\right),$$

for all $n \in \mathbb{N}$.

Proof. Recall that, by (7.2.4), (7.2.5) and (7.2.5)′

$$\begin{aligned}\tau\left((\alpha_K^p)^n\right) &= \tau_p\left((\alpha_K^p)^n\right)\\ &= \left(\frac{\mu_p(K)^{n-1}\mu_p\left(K\cap Z_p\right)\mu_p\left(KZ_p\cap KZ_{p:G_p}\right)}{\mu_p\left(KZ_{p:G_p}\right)}\right)\\ &= \tau_p\left(\left(\alpha_K^p\right)^{(n)}\right) = \tau\left(\left(\alpha_K^p\right)^{(n)}\right),\end{aligned}$$

for all $K \in \mathcal{K}_p$, for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}\tau\left((\alpha_K^p)^n\right) &= \frac{\mu_p(K)^{n-1}\mu_p\left(K\cap Z_p\right)\mu_p\left(KZ_p\cap KZ_p\right)}{\mu_p\left(KZ_p\right)}\\ &= \frac{\mu_p(K)^{n-1}\mu_p\left(K\cap Z_p\right)\mu_p\left(KZ_p\right)}{\mu_p\left(KZ_p\right)}\\ &= \mu_p(K)^{n-1}\mu_p\left(K\cap Z_p\right) = \tau\left(\left(\alpha_K^p\right)^{(n)}\right),\end{aligned}$$

for all $K \in \mathcal{K}_p$, and $n \in \mathbb{N}$, for all $p \in \mathcal{P}$. □

More general to (8.5), one obtains the following result.

Corollary 8.3. *Let* $(p_1, \ldots, p_N) \in \text{alt}\left(\mathcal{P}^N\right)$*, for* $N \in \mathbb{N} - \{1\}$*, and let*

$$T = \prod_{l=1}^{N}\left(\alpha_{K_l}^{p_l}\right)^{n_l} \in \mathfrak{S}, \tag{8.5}$$

where $K_l \in \mathcal{K}_{p_l}$*, and* $n_l \in \mathbb{N}$*, for all* $l = 1, \ldots, N$*.. Then*

$$\tau(T^n) = \prod_{l=1}^{N}\left(\mu_{p_l}\left(K_l\right)^{nn_l-1}\mu_{p_l}\left(K_l\cap Z_p\right)\right), \tag{8.6}$$

for all $n \in \mathbb{N}$.

Assume further that $p_N \neq p_1$ *in* $\mathcal{P}$*. Then*

$$tau(T^{(n)}) = \prod_{l=1}^{N}\left(\mu_{p_l}\left(K_l\right)^{nn_l-1}\mu_{p_l}\left(K_l\cap Z_p\right)\right)^n, \tag{8.7}$$

for all $n \in \mathbb{N}$.

Proof. By the assumption that the N-tuple $(p_1, \ldots, p_N)$ is an alternating sequence in $\mathcal{P}^N$, for $N \in \mathbb{N} - \{1\}$, the operator T of (8.6) is a free reduced word in the sub-Hecke C^*-probability space $\mathfrak{S}$ of the adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$, by

(8.4). So, it satisfies that

$$\begin{aligned}\tau\left(T^n\right) &\overset{\text{equi}}{=} \tau\left(\prod_{l=1}^{N}\left(\alpha_{K_l}^{p_l}\right)^{nn_l}\right)\\ &= \prod_{l=1}^{N}\tau_{p_l}\left(\left(\alpha_{K_l}^{p_l}\right)^{nn_l}\right)\\ &= \prod_{l=1}^{N}\left(\mu_{p_l}\left(K_l\right)^{nn_l-1}\mu_{p_l}\left(K_l\cap Z_p\right)\right) \text{ by (7.2.4)}\end{aligned}$$

by (8.5), for all $n\in\mathbb{N}$. Therefore, the formula (8.7) holds.

Assume now that $(p_1,\ldots,p_N)$ is an alternating N-tuple, and suppose $p_N\neq p_1$ in $\mathcal{P}$. Then the given operator T is a free reduced word, and the free (non-reduced) word $T^{(n)}$ becomes a free reduced word in the sub-Hecke C^*-probability space $\mathfrak{S}$ of $\mathbb{H}_{\mathcal{P}}$, for all $n\in\mathbb{N}$. Therefore, by (7.2.5), one obtains that

$$\tau(T^{(n)}) = (\tau(T))^n, \text{ for all } n\in\mathbb{N}.$$

So, by the formula (8.7), we obtain

$$\tau\left(T^{(n)}\right) = \left(\prod_{l=1}^{N}\left(\mu_{p_l}\left(K_l\right)^{nn_l-1}\mu_{p_l}\left(K_l\cap Z_p\right)\right)\right)^n$$

for all $n\in\mathbb{N}$. Thus, the formula (8.8) holds true. □

The free-moment formulas (8.5), (8.7) and (8.8) characterize the free distributions of free reduced words in the sub-Hecke C^*-probability space $\mathfrak{S}$ of our adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$.

For $K\in\mathcal{K}_p$, for any $p\in\mathcal{P}$, one can take the operator

$$\beta_K^p \overset{\text{def}}{=} \frac{1}{\mu_p(K)}\alpha_K^p \in \mathfrak{S}_p \text{ in } \mathfrak{S}. \tag{8.8}$$

Observe that if β_K^p is in the sense of (8.9) in $\mathfrak{S}$, then

$$\begin{aligned}\beta_K^p &= \frac{1}{\mu_p(K)}\alpha_K^p = \frac{1}{\mu_p(K)}\alpha_{\chi_K}^p\\ &= \alpha^p\left(\frac{1}{\mu_p(K)}\chi_K\right) = \alpha_{e_K}^p,\end{aligned} \tag{8.9}$$

in the free block $\mathbb{H}(G_p)$ of $\mathbb{H}_{\mathcal{P}}$, where e_K are in the sense of (2.2.5) in $(\mathcal{H}(G_p),\psi_p)$, for all $K\in\mathcal{K}_p$, for all $p\in\mathcal{P}$.

By (8.10), the operators β_K^p of (8.9) are well determined in $\mathfrak{S}$, for all $K\in\mathcal{K}_p$, for all $p\in\mathcal{P}$.

Corollary 8.4. *Let β_K^p be an operator* (8.9) *in the free block $\mathfrak{S}_p$ of the sub-Hecke C^*-probability space $\mathfrak{S}$ of $\mathbb{H}_{\mathcal{P}}$, for $K \in \mathcal{K}_p$, for $p \in \mathcal{P}$. Then*

$$\tau\left((\beta_K^p)^n\right) = \frac{\mu_p\left(K \cap Z_p\right)}{\mu_p(K)} = \tau\left((\beta_K^p)^{(n)}\right), \tag{8.10}$$

for all $n \in \mathbb{N}$.

Proof. Observe that

$$\tau\left((\beta_K^p)^n\right) = \tau_p\left((\beta_K^p)^n\right) = \left(\frac{1}{\mu_p(K)^n}\right)\tau_p\left((\alpha_K^p)^n\right)$$

by (8.10)

$$\begin{aligned}\tau\left((\beta_K^p)^n\right) &= \frac{\mu_p\left(K \cap Z_p\right)}{\mu_p(K)}, \\ &= \left(\frac{1}{\mu_p(K)^n}\right)\left(\mu_p(K)^{n-1}\mu_p\left(K \cap Z_p\right)\right)\end{aligned}$$

for all $n \in \mathbb{N}$. Similarly, one obtains that

$$\tau\left((\beta_K^p)^{(n)}\right) = \frac{\mu_p\left(K \cap Z_p\right)}{\mu_p(K)}, \text{ for all } n \in \mathbb{N}.$$

Therefore, we obtain the free-distributional data (8.11) of β_K^p, for all $K \in \mathcal{K}_p$, for all $p \in \mathcal{P}$, in $\mathfrak{S}$. □

Also, we obtain the following free-probabilistic information.

Corollary 8.5. *Let $\beta_{K_l}^{p_l}$ be free random variables* (8.9) *in the free blocks $\mathfrak{S}_{p_l}$ of the sub-Hecke C^*-probability space $\mathfrak{S}$ of $\mathbb{H}_{\mathcal{P}}$, for $l = 1, \ldots, N$, for $N \in \mathbb{N} - \{1\}$. Let*

$$T = \prod_{l=1}^{N}\left(\beta_{K_l}^{p_l}\right)^{n_l} \in \mathfrak{S}, \tag{8.11}$$

where $n_1, \ldots, n_N \in \mathbb{N}$. If $(p_1, \ldots, p_N) \in \text{alt}\left(\mathcal{P}^N\right)$, and T is in the sense of (8.12)*, then*

$$\tau\left(T^n\right) = \prod_{l=1}^{N}\left(\frac{\mu_{p_l}\left(K_l \cap Z_p\right)}{\mu_{p_l}\left(K_l\right)}\right), \text{ for all } n \in \mathbb{N}. \tag{8.12}$$

Suppose $(p_1, \ldots, p_N) \in \text{alt}\left(\mathcal{P}^N\right)$, and assume further $p_N \neq p_1$ in $\mathcal{P}$. If T is in the sense of (8.12)*, then*

$$\tau(T^{(n)}) = \prod_{l=1}^{N}\left(\frac{\mu_{p_l}\left(K_l \cap Z_p\right)}{\mu_{p_l}\left(K_l\right)}\right)^n, \text{ for all } n \in \mathbb{N}. \tag{8.13}$$

Proof. Let T be in the sense of (8.12) in $\mathfrak{S}$. Then, by the assumption that the N-tuple $(p_1, \ldots, p_N)$ is alternating, the operator T is a free reduced word in $\mathfrak{S}$. So, one has

$$\tau(T^n) \overset{\text{equi}}{=} \tau\left(\prod_{l=1}^{N}\left(\left(\beta_{K_l}^{p_l}\right)^{nn_l}\right)\right)$$
$$= \prod_{l=1}^{N}\tau_{p_l}\left(\left(\beta_{K_l}^{p_l}\right)^{nn_l}\right) = \prod_{l=1}^{N}\left(\frac{\mu_{p_l}\left(K\cap Z_p\right)}{\mu_{p_l}(K)}\right),$$

for all $n \in \mathbb{N}$, by (7.2.4) and (8.11). Therefore, the free-distributional data (8.13) holds.

Now, for the fixed operator T of (8.12), if $(p_1, \ldots, p_N)$ is an alternating N-tuple, satisfying $p_N \neq p_1$ in $\mathcal{P}$, then the operators $T^{(n)}$ are free reduced words in $\mathfrak{S}$, for all $n \in \mathbb{N}$. So, by (7.2.5), we have that

$$\tau\left(T^{(n)}\right) = \left(\tau(T)\right)^n, \text{ for all } n \in \mathbb{N}. \tag{8.14}$$

Thus, one obtains the free distribution (8.14), by (8.13) and (8.15). □

Now, we concentrate on the free-distributional data (8.11) of the operator $\beta_K^p \in \mathfrak{S}$, for $K \in \mathcal{K}_p$, for $p \in \mathcal{P}$.

In the rest of this section, let us fix $K \in \mathcal{K}_p$ and $p \in \mathcal{P}$, and the corresponding free random variable β_K^p in the sub-Hecke C^*-probability space $\mathfrak{S}$ of the adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$. One can easily verify that

$$\mu_p\left(K\cap Z_p\right) \leq \mu_p\left(Z_p\right) = 1. \tag{8.15}$$

Now, let $r_K^p \in \mathbb{R}$ in $\mathbb{C}$, such that

$$0 < r_K^p \overset{\text{denote}}{=} \mu_p\left(K\cap Z_p\right) \leq 1. \tag{8.16}$$

The quantity r_K^p of (8.17) is well determined in the interval $(0, 1]$ of $\mathbb{R}$, by (8.16). Then, by (8.11),

$$\tau\left(\left(\beta_K^p\right)^n\right) = \frac{r_K^p}{\mu_p(K)}, \text{ for all } n \in \mathbb{N}, \tag{8.17}$$

where r_K^p is in the sense of (8.17) in $(0, 1]$.

Note that

$$r_K^p = \mu_p\left(K\cap Z_p\right) \leq \mu_p\left(K\right). \tag{8.18}$$

It shows that the quantity in (8.18) is contained in $(0, 1]$ in $\mathbb{R}$, by (8.19).

So, by (8.17) and (8.19), one obtains the following estimations for the free-distributional data of the operator β_K^p in $\mathfrak{S}$, for all $K \in \mathcal{K}_p$, and $p \in \mathcal{P}$.

Theorem 8.6. *Let $\beta_K^p = \frac{1}{\mu_p(K)}\alpha_K^p$ be free random variables* (8.9) *in the sub-Hecke C^*-probability space $\mathfrak{S}$ of the adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$, for all $K \in \mathcal{K}_p$, for all $p \in \mathcal{P}$. Then*

$$0 < \tau\left(\left(\beta_K^p\right)^n\right) = \tau\left(\left(\beta_K^p\right)^{(n)}\right) \leq 1 \text{ in } \mathbb{R}, \tag{8.19}$$

for all $n \in \mathbb{N}$.

Proof. Let β_K^p be in the sense of (8.9) in $\mathfrak{S}$, for $K \in \mathcal{K}_p$, for $p \in \mathcal{P}$. Then,

$$0 < \tau\left(\left(\beta_K^p\right)^n\right) = \frac{\mu_p\left(K \cap Z_p\right)}{\mu_p(K)} = \tau\left(\left(\beta_K^p\right)^{(n)}\right) \leq 1,$$

in $\mathbb{R}$, for all $n \in \mathbb{N}$, by (8.11), (8.17), (8.18) and (8.19). □

More general to (8.20), we obtain the following estimation.

Theorem 8.7. *Let* $K_l \in \mathcal{K}_{p_l}$, *for* $p_l \in \mathcal{P}$, *where* $l = 1, \ldots, N$, *for* $N \in \mathbb{N} - \{1\}$. *Suppose the* N*-tuple* $(p_1, \ldots, p_N)$ *is alternating in* $\mathcal{P}^N$, *and let*

$$T = \prod_{l=1}^{N} \left(\beta_{K_l}^{p_l}\right)^{n_l} \in \mathfrak{S}, \text{ with } \beta_{K_l}^{p_l} = \frac{1}{\mu_{p_l}(K_l)} \alpha_{K_l}^{p_l} \in \mathfrak{S}_{p_l}, \tag{8.20}$$

where $n_l \in \mathbb{N}$, *for all* $l = 1, \ldots, N$. *If* T *is in the sense of* (8.21) *in* $\mathfrak{S}$, *then we obtain*

$$0 < \tau\left(T^n\right) \leq 1, \text{ for all } n \in \mathbb{N}, \tag{8.21}$$

and

$$0 < \tau\left(T^{(n)}\right) \leq 1, \text{ for all } n \in \mathbb{N}.$$

Proof. Let T be a free random variable (8.21) in $\mathfrak{S}$. Then, by the assumption that the N-tuple $(p_1, \ldots, p_N)$ is an alternating N-tuple, the operator T is a free reduced word in $\mathfrak{S}$, satisfying

$$\tau(T^n) = \prod_{l=1}^{N} \left(\frac{\mu_{p_l}\left(K_l \cap Z_p\right)}{\mu_{p_l}\left(K_l\right)}\right),$$

and

$$\tau(T^{(n)}) = \prod_{l=1}^{N} \left(\frac{\mu_{p_l}\left(K_l \cap Z_p\right)}{\mu_{p_l}(K_l)}\right)^n,$$

for all $n \in \mathbb{N}$, by (8.13), respectively, by (8.14).

The factors of $\tau\left(T^n\right)$ satisfy

$$0 < \frac{\mu_{p_l}\left(K_l \cap Z_p\right)}{\mu_{p_l}\left(K_l\right)} = \tau\left(\beta_{K_l}^{p_l}\right) \leq 1, \text{ in } \mathbb{R},$$

y (8.20), for all $n \in \mathbb{N}$, and hence, the factors of $\tau\left(T^{(n)}\right)$ satisfy

$$0 < \left(\frac{\mu_{p_l}\left(K_l \cap Z_p\right)}{\mu_{p_l}\left(K_l\right)}\right)^n = \left(\tau\left(\beta_{K_l}^{p_l}\right)\right)^n \leq 1, \text{ in } \mathbb{R},$$

too, by (8.20), for all $n \in \mathbb{N}$.

Since all factors of $\tau\left(T^n\right)$ and $\tau\left(T^{(n)}\right)$ are contained in the interval $(0, 1]$ in $\mathbb{R}$,

$$\tau(T^n), \tau(T^{(n)}) \in (0, 1], \text{ in } \mathbb{R},$$

for all $n \in \mathbb{N}$.

Therefore, the estimations in (8.22) hold. □

Based on the above estimations (8.20) and (8.22), we obtain the following interesting result in terms of the freeness condition on the Hecke probability spaces $(\mathcal{H}(G_p), \psi_p)$ for primes $p \in \mathcal{P}$. First, recall that, for an arbitrarily fixed prime $p \in \mathcal{P}$, two free random variables χ_{K_1} and χ_{K_2} are free in the Hecke probability space $(\mathcal{H}(G_p), \psi_p)$, if and only if

$$\mu_p(K_1 \cap K_2) = \mu_p(K_1)\mu_p(K_2), \tag{8.22}$$

by (4.16). So, the two free random variables e_{K_1} and e_{K_2} are free in $(\mathcal{H}(G_p), \psi_p)$, if and only if the condition (8.23) holds, by (4.17).

Theorem 8.8. *Let $K_l \in \mathcal{K}_p$, for $p_l \in \mathcal{P}$, and let $\beta_{K_l}^{p_l} = \frac{1}{\mu_{p_l}(K_l)}\alpha_{K_l}^{p_l} \in \mathfrak{S}_{p_l}$ be generating operators* (8.9) *in the sub-Hecke probability space $\mathfrak{S}$ of the adelic Hecke C^*-probability space $\mathbb{H}_{\mathcal{P}}$, for $l = 1, \ldots, N$, for $N \in \mathbb{N} - \{1\}$. Assume that the μ_{p_l}-measurable functions χ_{K_l} and $\chi_{Z_{p_l}}$ are free in the Hecke probability space $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, for all $l = 1, \ldots, N$, where $Z_{p_l} = GL_2(\mathbb{Z}_{p_l})$ in G_{p_l}. Then*

$$\tau\left(\left(\beta_{K_l}^{p_l}\right)^n\right) = 1 = \tau\left(\left(\beta_{K_l}^{p_l}\right)^{(n)}\right), \text{ for all } n \in \mathbb{N}, \tag{8.23}$$

for all $l = 1, \ldots, N$. More precisely, if $(p_1, \ldots, p_N)$ is an alternating N-tuple in $\mathcal{P}^N$, and if

$$T = \prod_{l=1}^{N} \left(\beta_{K_l}^{p_l}\right)^{n_l} \in \mathfrak{S}, \text{ for } n_1, \ldots, n_N \in \mathbb{N},$$

then

$$\tau\left(T^n\right) = 1 = \tau\left(T^{(n)}\right), \text{ for all } n \in \mathbb{N}. \tag{8.24}$$

Proof. Note first that

$$\begin{aligned}\beta_K^p &= \frac{1}{\mu_p(K)}\alpha_K^p = \frac{1}{\mu_p(K)}\alpha_{\chi_K}^p \\ &= \alpha^p\left(\frac{1}{\mu_p(K)}\chi_K\right) = \alpha^p\left(e_K\right),\end{aligned}$$

for all $K \in \mathcal{K}_p$, for all $p \in \mathcal{P}$, in the free block $\mathfrak{S}_p$ of $\mathfrak{S}$.

Let $K_1, \ldots, K_N$ be given as above, and suppose χ_{K_l} and $\chi_{Z_{p_l}}$ are free in $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, for all $l = 1, \ldots, N$. Then, by (8.23), one has that

$$\mu_{p_l}\left(K_l \cap Z_{p_l}\right) = \mu_{p_l}\left(K_l\right)\mu_{p_l}\left(Z_{p_l}\right) = \mu_{p_l}\left(K_l\right),$$

for all $l = 1, \ldots, N$, because $\mu_{p_l}\left(Z_{p_l}\right) = 1$, for all $p_l \in \mathcal{P}$, in G_{p_l}, i.e., χ_{K_l} and $\chi_{Z_{p_l}}$ are free in $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, if and only if

$$\mu_{p_l}\left(K_l \cap Z_{p_l}\right) = \mu_{p_l}\left(K_l\right), \tag{8.25}$$

if and only if e_{K_l} and $e_{Z_{p_l}}$ are free in $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, for all $l = 1, \ldots, N$.

Under hypothesis, by (8.11), one has

$$\tau\left(\left(\beta_{K_l}^{p_l}\right)^n\right) = \tau\left(\left(\beta_{K_l}^{p_l}\right)^{(n)}\right) \tag{8.26}$$
$$= \frac{\mu_{p_l}\left(K_l \cap Z_p\right)}{\mu_{p_l}\left(K_l\right)} = \frac{\mu_{p_l}\left(K_l\right)}{\mu_{p_l}\left(K_l\right)} = 1,$$

by (8.26), for all $n \in \mathbb{N}$. Therefore, the formula (8.24) holds true.

If the N-tuple $(p_1, \ldots, p_N)$ is an alternating finite sequence, then the operator $T = \prod^N l = 1 \left(\beta_{K_l}^{p_l}\right)^{n_l}$ is a free reduced word in $\mathfrak{S}$, satisfying

$$\tau\left(T^n\right) = \prod_{l=1}^{N}\left(\frac{\mu_{p_l}\left(K_l \cap Z_{p_l}\right)}{\mu_{p_l}\left(K_l\right)}\right),$$

and

$$\tau(T^{(n)}) = \prod_{l=1}^{N}\left(\frac{\mu_{p_l}\left(K_l \cap Z_{p_l}\right)}{\mu_{p_l}\left(K_l\right)}\right)^n,$$

for all $n \in \mathbb{N}$, by (8.13) and (8.14), respectively. So, if χ_{K_l} and $\chi_{Z_{p_l}}$ are free in $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, for all $l = 1, \ldots, N$, then

$$\tau(T^n) = 1^N = 1, \text{ and } \tau(T^{(n)}) = 1^{nN} = 1, \tag{8.27}$$

for all $n \in \mathbb{N}$, by (8.27).

Therefore, by (8.28), the formula (8.25) holds true. □

The following corollary is directly obtained from (8.20), (8.22), (8.24) and (8.25).

Corollary 8.9. *Let $\beta_{K_l}^{p_l}$ be the operators* (8.9) *in $\mathfrak{S}$, for $l = 1, \ldots, N$, for $N \in \mathbb{N} - \{1\}$, and let us fix n in $\mathbb{N}$, arbitrarily. Let*

$$T = \prod_{l=1}^{N}\left(\beta_{K_l}^{p_l}\right)^{n_l} \in \mathfrak{S}, \text{ for } n_1, \ldots, n_N \in \mathbb{N}.$$

If two μ_{p_l}-measurable functions χ_{K_l} and $\chi_{Z_{p_l}}$ are free in the Hecke probability space $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, where $Z_{p_l} = GL_2\left(\mathbb{Z}_{p_l}\right)$ in G_{p_l}, for all $l = 1, \ldots, N$, then

$$\tau\left(\left(\beta_{K_l}^{p_l}\right)^n\right) = \tau\left(\left(\beta_{K_l}^{p_l}\right)^{(n)}\right) \text{ is maximal.} \tag{8.28}$$

$$\tau\left(T^n\right), \text{ and } \tau\left(T^{(n)}\right) \text{ are maximal.} \tag{8.29}$$

Proof. Under hypothesis, we have

$$0 < \tau\left(\left(\beta_{K_l}^{p_l}\right)^n\right) = \tau\left(\left(\beta_{K_l}^{p_l}\right)^{(n)}\right) \leq 1,$$

and

$$0 < \tau(T^n) \leq 1, \text{ and } 0 < \tau(T^{(n)}) \leq 1,$$

for any fixed $n \in \mathbb{N}$, by (8.20) and (8.22), respectively.

It is shown in (8.24) and (8.25) that

$$\tau\left(\left(\beta_{K_l}^{p_l}\right)^n\right) = 1 = \tau\left(\left(\beta_{K_l}^{p_l}\right)^{(n)}\right),$$

and

$$\tau(T^n) = 1 = \tau(T^{(n)}),$$

if χ_{K_l} and $\chi_{Z_{p_l}}$ are free in $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, for all $l = 1, \ldots, N$.

Therefore, the freeness of χ_{K_l} and $\chi_{Z_{p_l}}$ in $(\mathcal{H}(G_{p_l}), \psi_{p_l})$, for all $l = 1, \ldots, N$, implies the maximality of linear functional values of such operators. □

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An Introduction to Fractional Powers of Quaternionic Operators and New Fractional Diffusion Processes

Fabrizio Colombo and Jonathan Gantner

This paper is dedicated to our friend and colleague Daniel Alpay on the occasion of his 60th birthday

Abstract. The spectral theory based on the notion of S-spectrum, for quaternionic operators and in particular for vector operators, has been introduced about ten years ago and it is nowadays well developed. It turned out that this theory allows to define new fractional diffusion processes. In this note we announce new results on the quaternionic version of the H^∞ functional calculus and we use them to compute the fractional powers of vector operators. Such fractional powers of generalized gradient operators define a new Fourier's law for the propagation of the heat. This paper is devoted to researchers in different research fields such as: quaternionic analysis, operator theory and fractional diffusion problems.

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1. Introduction

In recent years there has been an important activity focused on the use of so-called fractional diffusion operators to replace the standard Laplace operator, but also other elliptic operators with variable coefficients, with the aim of taking into account long range interactions. The new operators do not act by pointwise differentiation, but by a global integration with respect to a singular kernel. One of the most studied nonlocal operators is the fractional Laplacian, which can be defined in different ways, for example, by the Fourier transform

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{\mathbb{R}^n} u(x)e^{-2\pi i x\cdot\xi}\,dx, \qquad (\mathcal{F}^{-1}u)(x) = \int_{\mathbb{R}^n} \hat{u}(\xi)e^{2\pi i x\cdot\xi}\,d\xi.$$

For $u \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions which is defined as

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \ : \ \forall \beta, \gamma \in \mathbf{N}_0^n \ \sup_{x\in\mathbb{R}^n} |x^\beta \partial_x^\gamma u(x)| < \infty\},$$

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we define

$$(-\Delta)^{\alpha}u(x) = \mathcal{F}^{-1}((2\pi|\xi|^{2\alpha}\hat{u}(\xi)), \quad \text{for } \alpha \in (0,1).$$

A different approach considers the semigroup generated by $-\Delta$. For $u \in \mathcal{S}(\mathbb{R}^n)$, we consider the solution of the Cauchy problem

$$\partial_t U(t,x) = \Delta U(t,x), \quad t > 0; \quad U(0,x) = u(x).$$

We then have

$$(-\Delta)^{\alpha}u(x) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} t^{-\alpha-1}(U(t,x) - u(x))dt,$$

and this definition can be generalized to more general elliptic operators that are the generators of semigroups. Another approach to define the fractional Laplace operator, which can be found in [13], is via an extension problem. An explicit integral representation of the fractional Laplacian is

$$(-\Delta)^{\alpha}u(x) = c(n,\alpha)P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\alpha}}dy,$$

where the integral is defined in the sense of the principal value. The constant is

$$c(n,\alpha) = 4^{\alpha}\alpha\Gamma(\alpha + n/2)/(\pi^{n/2}\Gamma(1-\alpha)).$$

In the literature there are several nonlinear models that involve the fractional Laplacian and even the fractional powers of more general elliptic operators. With no claim of completeness, for a more extended introduction to fractional diffusion problems we refer to the books [3, 41] and the references therein, and for more recent results see also [14, 15, 31, 36]. Balakrishnan, see [11], obtains a construction for fractional powers of an operator A, in which it is not required that A generates a semigroup. He assumes that the linear operator A is closed with domain and range in a Banach space X. He proved that if any $\lambda > 0$ belongs to the resolvent set of A and there exists a positive constant M such that

$$\|\lambda(\lambda - A)^{-1}\| < M, \quad \lambda > 0,$$

i.e., if $-A$ is a sectorial operator in today's terminology, then the fractional powers of $-A$ can be defined by the integral

$$(-A)^{\alpha}x = \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} \lambda^{\alpha-1}(\lambda - A)^{-1}(-A)x\, d\lambda, \quad x \in \mathcal{D}(A),$$

for $\alpha \in (0,1)$. This formula can also be obtained as a consequence of the H^{∞}-functional calculus introduced by A. McIntosh in [37]. Some of the original papers on fractional powers of operators are [11, 32, 34] and also [35, 39, 40].

In this paper we offer a different point of view, on defining fractional diffusion problems, that requires the quaternionic version of the S-functional calculus that applies, in particular, to vector operators.

Precisely we observe that the heat equation is based on Fourier's law

$$\mathbf{q}(t,x) = -\mathcal{K}\nabla u(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^3,$$

where u is the temperature, $\mathbf{q}$ is the heat flow and $\mathcal{K}$ is the thermal diffusivity, and on the principle of conservation of energy

$$\partial_t u(t,x) + \operatorname{div} \mathbf{q}(t,x) = 0.$$

Their combination yields the heat equation

$$\partial_t u(t,x) - \mathcal{K} \operatorname{div}(\nabla u(t,x)) = 0.$$

Setting $\mathcal{K} = 1$, for sake of simplicity, and having at hand the fractional Laplacian we obtain the fractional evolution equation

$$\partial_t u(t,x) + (-\Delta)^{\alpha} u(t,x) = 0.$$

We denote by $\mathbb{H} = \{s = s_0 + e_1 s_1 + e_2 s_2 + e_3 s_3 \ : \ s_i \in \mathbb{R}\}$ the algebra of quaternions, where e_1, e_2 and e_3 are the basis elements whose properties are stated in the next section. Our new approach to fractional diffusion processes consists in identifying the space $\mathbb{R}^3$ with the set of purely imaginary quaternions and considering a Fourier law with variable coefficients

$$\mathbf{q}(x, \partial_x) := e_1 \, \alpha_1(x) \partial_{x_1} + e_2 \, \alpha_2(x) \partial_{x_2} + e_3 \, \alpha_3(x) \partial_{x_3}$$

where the imaginary units of the quaternions are used as unit vectors of the axis, α_1, α_2 and α_3 are suitable real-valued functions that depend on the material, and $\mathbf{q}(x, \partial_x)$ can be seen as a quaternionic linear operator of which we define the fractional powers as

$$\partial_t u(t,x) - \operatorname{div}(\operatorname{Vec}(\mathbf{q}(x, \partial_x)^{\alpha}) u(t,x)) = 0,$$

where, roughly speaking, the operator $\operatorname{Vec}(\mathbf{q}(x, \partial_x)^{\alpha})$ is the vector part of the quaternionic operator $\mathbf{q}(x, \partial_x)^{\alpha}$. We will be more precise in the sequel. This approach has several advantages:

(I) It modifies the Fourier law but keeps the law of Conservation of Energy.
(II) It is applicable to a large class of operators that includes the gradient but also operators with variable coefficients such as the operator $\mathbf{q}(x, \partial_x)$.
(III) The fractional powers of the operator $\mathbf{q}(x, \partial_x)$ is more realistic for nonhomogeneous materials.
(IV) The fact that we keep the evolution equation in divergence form allows an immediate definition of the weak solution of the fractional evolution problem.

Our strategy to define fractional powers of vector operators is based on the quaternionic version of the Riesz–Dunford functional calculus. Just for the sake of simplicity consider a complex linear bounded operator B on a Banach space. The spectrum of B is defined as

$$\sigma(B) = \{\lambda \in \mathbb{C} \ : \ \lambda \mathcal{I} - B \ \text{ is not invertible}\}$$

and the resolvent set as

$$\rho(B) = \mathbb{C} \setminus \sigma(B).$$

The resolvent operator is defined as

$$R_B(\lambda) := (\lambda \mathcal{I} - B)^{-1}$$

for $\lambda \in \rho(B)$ and it is a holomorphic operator-valued function. The holomorphic functional calculus (due to Riesz and Dunford) is defined as

$$f(B) = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathcal{I} - B)^{-1} f(\lambda)\, d\lambda,$$

where Γ is a rectifiable Jordan curve that surrounds the spectrum $\sigma(B)$ of B. It is then natural to define the fractional powers of B as

$$B^\alpha = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathcal{I} - B)^{-1} \lambda^\alpha\, d\lambda, \quad \alpha \in (0,1).$$

Quaternionic linear operators are of the form

$$T = T_0 + e_1 T_1 + e_2 T_2 + e_3 T_3,$$

where T_ℓ, $\ell = 0,1,2,3$ are real operators, and this class contains many vector operators if we identify $\mathbb{R}^3$ with the set of all purely imaginary quaternions. The quaternionic version (or vector version) of the Riesz–Dunford functional calculus requires the notion of S-spectrum. For a bounded quaternionic linear operator T we define the S-spectrum $\sigma_S(T)$ as

$$\sigma_S(T) = \{s \in \mathbb{H} \quad : \quad T^2 - 2s_0 T + |s|^2 \mathcal{I} \text{ is not invertible}\},$$

where $|s|^2 = s_0^2 + s_1^2 + s_2^2 + s_3^2$, and the S-resolvent set is

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

Due to the noncommutativity of the quaternions there are two resolvent operators associated with a quaternionic linear operator and hence there are two formulations of the quaternionic functional calculus. The left S-resolvent operator is defined as

$$S_L^{-1}(s,T) := -(T^2 - 2s_0 T + |s|^2 \mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \quad s \in \rho_S(T), \tag{1.1}$$

while the right S-resolvent operator is defined as

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})(T^2 - 2s_0 T + |s|^2 \mathcal{I})^{-1}, \quad s \in \rho_S(T). \tag{1.2}$$

Let $U \subset \mathbb{H}$ be a suitable domain that contains the S-spectrum of T. The quaternionic functional calculus for left slice hyperholomorphic functions $f : U \to \mathbb{H}$ is defined as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s,T)\; ds_I\; f(s), \tag{1.3}$$

where $ds_I = -ds I$. For right slice hyperholomorphic functions, we define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} f(s)\; ds_I\; S_R^{-1}(s,T). \tag{1.4}$$

These definitions are well posed since the integrals depend neither on the open set U nor on the complex plane $\mathbb{C}_I$. (For the definition of slice hyperholomorphic functions see next section.) From here we can define the fractional powers of a vector operator.

In the case we chose $f(s) = s^\alpha$ we can define

$$T^\alpha = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} s^\alpha \, ds_I \, S_R^{-1}(s,T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s,T) \, ds_I \, s^\alpha,$$

since the function $s \mapsto s^\alpha$ is both left and right slice-hyperholomorphic. The case of unbounded operators is more complicated and will be the topic of this note.

The plan of the paper is as follows: Section 2 contains the preliminary results on the theory of slice hyperholomorphic functions and the S-functional calculus. Section 3 deals with the definition of the fractional powers of quaternionic operators via the H^∞-functional calculus. Section 4 presents a new approach to fractional evolution via quaternionic operators. Finally, Section 5 contains some concluding remarks.

2. Slice hyperholomorphicity and the S-functional calculus

The real algebra of quaternions consists of the real vector space

$$\mathbb{H} := \left\{ p = \sum_{i=0}^{3} \xi_i e_i : \xi_i \in \mathbb{R} \right\},$$

which is endowed with a noncommutative product. This product is $\mathbb{R}$-linear and hence defined by its action on the basis elements $e_0, \ldots, e_3$ of the quaternions: e_0 is the neutral element of the product, which we will threrefore also denote by 1 in the following, and $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i, j \in \{1,2,3\}$ with $i \neq j$. We shall hence identify $\mathbb{R} \cong \text{span}\{e_0\}$ and we shall identify the euclidean space $\mathbb{R}^3$ with the set of all purely imaginary quaternions, i.e. quaternions of the form $\sum_{i=1}^3 \xi_i e_i$.

The real part of a quaternion $p = \xi_0 + \sum_{i=1}^3 \xi_i e_i$ is defined as $\text{Re}(p) := \xi_0$, its imaginary part as $\underline{p} := \sum_{i=1}^3 \xi_i e_i$ and its conjugate as $\overline{p} := \text{Re}(p) - \underline{p}$. Moreover, its modulus is given by $|p|^2 = p\overline{p} = \sum_{i=0}^3 \xi_i^2$ and hence for $p \neq 0$ we have $p^{-1} = \overline{p}/|p|^2$.

Each element I in the unit sphere $\mathbb{S}$ of $\mathbb{R}^3$, i.e., of the set

$$\mathbb{S} := \{p \in \mathbb{H} : \text{Re}(p) = 0, |p| = 1\},$$

satisfies $I^2 = -1$ and is therefore called an imaginary unit. The real subspace $\mathbb{C}_I := \{p_0 + Ip_1 : p_0, p_1 \in \mathbb{R}\}$ of $\mathbb{H}$ is an isomorphic copy of the field of complex numbers. If $I, J \in \mathbb{S}$ with $I \perp J$, set $K = IJ = -JI$. Then 1, I, J and K form an orthonormal basis of $\mathbb{H}$ as a real vector space and 1 and J form a basis of $\mathbb{H}$ as a left or right vector space over the complex plane $\mathbb{C}_I$, that is,

$$\mathbb{H} = \mathbb{C}_I + \mathbb{C}_I J \quad \text{and} \quad \mathbb{H} = \mathbb{C}_I + J\mathbb{C}_I.$$

Any quaternion p belongs to a complex plane: if we set

$$I_p := \begin{cases} \frac{1}{|\underline{p}|}\underline{p}, & \text{if } \underline{p} \neq 0, \\ \text{any } I \in \mathbb{S}, & \text{if } \underline{p} = 0, \end{cases}$$

then $p = p_0 + I_p p_1$ with $p_0 = \mathrm{Re}(p)$ and $p_1 = |\underline{p}|$. The set

$$[p] := \{p_0 + Ip_1 : I \in \mathbb{S}\}$$

is a 2-sphere that reduces to a single point if p is real. A quaternion s belongs to $[p]$ if and only if there exist $h \in \mathbb{H} \setminus \{0\}$ such that $s = h^{-1}ph$.

2.1. Slice hyperholomorphic functions

The theory of complex linear operators is based on the theory of holomorphic functions. In a similar way, the theory of quaternionic linear operators is based on the theory of slice hyperholomorphic functions. We shall recall its most important results, which will be essential later on. For a detailed introduction we refer to the book [23]. We start by introducing the natural domains of definitions of slice hyperholomorphic functions.

Definition 2.1. A set $U \subset \mathbb{H}$ is called axially symmetric if $[p] \subset U$ for any $p \in U$.

Definition 2.2. Let $U \subset \mathbb{H}$ be open and axially symmetric. A function $f : U \to \mathbb{H}$ is called left slice hyperholomorphic, if it is of the form

$$f(p) = \alpha(p_0, p_1) + I_p \beta(p_0, p_1) \quad \forall p \in U, \tag{2.1}$$

where α and β are functions that take values in $\mathbb{H}$, satisfy the compatibility condition

$$\alpha(p_0, p_1) = \alpha(p_0, -p_1) \quad \text{and} \quad \beta(p_0, p_1) = -\beta(p_0, -p_1) \tag{2.2}$$

and the Cauchy–Riemann-differential equations

$$\frac{\partial}{\partial p_0}\alpha(p_0, p_1) = \frac{\partial}{\partial p_1}\beta(p_0, p_1) \quad \text{and} \quad \frac{\partial}{\partial p_1}\alpha(p_0, p_1) = -\frac{\partial}{\partial p_0}\beta(p_0, p_1). \tag{2.3}$$

A function $f : U \to \mathbb{H}$ is called right slice hyperholomorphic, if it is of the form

$$f(p) = \alpha(p_0, p_1) + \beta(p_0, p_1) I_p \quad \forall p \in U, \tag{2.4}$$

with functions α and β satisfying (2.2) and (2.3). Finally, a left slice hyperholomorphic function $f = \alpha + I\beta$ is called intrinsic, if α and β are real-valued.

We denote the set of all left slice hyperholomorphic functions on U by $\mathcal{SH}_L(U)$, the set of all right slice hyperholomorphic functions on U by $\mathcal{SH}_R(U)$ and the set of all intrinsic functions by $\mathcal{N}(U)$.

Remark 2.3. Observe that any nonreal quaternion can be represented by two different imaginary units, namely, $p = p_0 + I_p p_1 = p_0 + (-I_p)(-p_1)$. If p is real, then we can even choose I_p arbitrarily in $\mathbb{S}$. The compatibility condition (2.2) assures that the choice of the imaginary unit in this representation does not matter.

The term slice hyperholomorphic refers to the fact that the restriction of a slice hyperholomorphic function to a complex plane $\mathbb{C}_I$ is holomorphic in the classical sense with values in complex vector space $\mathbb{H} = \mathbb{C}_I + \mathbb{C}_I J$ resp. $\mathbb{H} = \mathbb{C}_I + J\mathbb{C}_I$ over $\mathbb{C}_I$ with $I, J \in \mathbb{S}$ and $J \perp I$.

Lemma 2.4. *Let $U \subset \mathbb{H}$ be an axially symmetric open set. If $f \in \mathcal{SH}_L(U)$, then for any $I \in \mathbb{S}$ the restriction $f_I := f|_{U\cap\mathbb{C}_I}$ is left holomorphic, i.e.,*

$$\frac{1}{2}\left(\frac{\partial}{\partial p_0}f(p) + I\frac{\partial}{\partial p_1}f(p)\right) = 0, \quad \forall p = p_0 + Ip_1 \in U \cap \mathbb{C}_I. \tag{2.5}$$

If $f \in \mathcal{SH}_R(U)$, then for any $I \in \mathbb{S}$ the restriction $f_I := f|_{U\cap\mathbb{C}_I}$ is right holomorphic, i.e.,

$$\frac{1}{2}\left(\frac{\partial}{\partial p_0}f(p) + \frac{\partial}{\partial p_1}f(p)I\right) = 0, \quad \forall p = p_0 + Ip_1 \in U \cap \mathbb{C}_I. \tag{2.6}$$

Moreover, all values of f are determined by its values on a single complex plane. Indeed, for left slice hyperholomorphic $f = \alpha + I\beta$, one has $\alpha(p_0, p_1) = \frac{1}{2}(f(p) + f(\overline{p}))$ and $\beta(p_0, p_1) = \frac{1}{2}I_p(f(\overline{p}) - f(p))$. We thus obtain the following important representation formula.

Theorem 2.5 (Representation Formula). *Let $U \subset \mathbb{H}$ be an axially symmetric open set and let $I \in \mathbb{S}$. For any $p = p_0 + I_p p_1 \in U$ set $p_I := p_0 + Ip_1$. If $f \in \mathcal{SH}_L(U)$, then*

$$f(p) = \frac{1}{2}(1 - I_pI)f(p_I) + \frac{1}{2}(1 + I_pI)f(\overline{p_I}) \quad \textit{for all } p \in U.$$

If $f \in \mathcal{SH}_R(U)$, then

$$f(p) = f(p_I)(1 - II_p)\frac{1}{2} + f(\overline{p_I})(1 + II_p)\frac{1}{2} \quad \textit{for all } p \in U.$$

We want to stress the role of intrinsic functions. An intrinsic function $f = \alpha + I\beta$ is both left and right slice hyperholomorphic because β commutes with the imaginary unit I. The converse is not true: the constant function $p \mapsto b \in \mathbb{H} \setminus \mathbb{R}$ is left and right slice hyperholomorphic, but it is not intrinsic. The importance of this class of functions is due to the fact that the multiplication and the composition with intrinsic functions preserve slice hyperholomorphicity. This is not true for arbitrary slice hyperholomorphic functions.

Corollary 2.6. *If $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{N}(U)$, then $fg \in \mathcal{SH}_R(U)$.*

If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_L(g(U))$, then $f \circ g \in \mathcal{SH}_L(U)$. If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_R(g(U))$, then $f \circ g \in \mathcal{SH}_R(U)$.

Finally, an analogue of Cauchy's integral theorem and an integral formula of Cauchy type hold true for slice hyperholomorphic functions.

Definition 2.7. We define the left slice hyperholomorphic Cauchy kernel as

$$S_L^{-1}(s, p) = -(p^2 - 2\mathrm{Re}(s)p + |s|^2)^{-1}(p - \overline{s}) \quad \text{for } p \notin [s]$$

and the right slice hyperholomorphic Cauchy kernel as

$$S_R^{-1}(s, p) = -(p - \overline{s})(p^2 - 2\mathrm{Re}(s)p + |s|^2)^{-1} \quad \text{for } p \notin [s].$$

Corollary 2.8. *The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,p)$ is left slice hyperholomorphic in the variable p and right slice hyperholomorphic in the variable s on its domain of definition. Moreover, we have $S_R^{-1}(s,p) = -S_L^{-1}(p,s)$.*

Remark 2.9. If p and s belong to the same complex plane, they commute and the slice hyperholomorphic Cauchy kernels reduce to the classical one:

$$\frac{1}{s-p} = S_L^{-1}(s,p) = S_R^{-1}(s,p).$$

Theorem 2.10 (Cauchy's integral theorem). *Let $U \subset \mathbb{H}$ be an axially symmetric open set, let $I \in \mathbb{S}$ and let D_I be an open subset of $O \cap \mathbb{C}_I$ with $\overline{D_I} \subset O \cap \mathbb{C}_I$ such that its boundary consists of a finite number of continuously differentiable Jordan curves. For any $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{SH}_L(U)$,*

$$\int_{\partial D_I} f(s)\, ds_I\, g(s) = 0,$$

where $ds_I = -I\, ds$.

Definition 2.11. An axially symmetric open set $U \subset \mathbb{H}$ is called a slice Cauchy domain if $U \cap \mathbb{C}_I$ is a Cauchy domain for any $I \in \mathbb{S}$, that is,

1. $U \cap \mathbb{C}_I$ is open;
2. $U \cap \mathbb{C}_I$ has a finite number of components (i.e., maximal connected subsets), the closures of any two of which are disjoint;
3. the boundary of $U \cap \mathbb{C}_I$ consists of a finite positive number of closed piecewise continuously differentiable Jordan curves.

Remark 2.12. A slice Cauchy domain is either bounded or has exactly one unbounded component. If it is unbounded, then the unbounded component contains a neighborhood of infinity.

Theorem 2.13 (Cauchy's integral formula). *Let $U \subset \mathbb{H}$ be a slice Cauchy domain. Let $I \in \mathbb{S}$ and set $ds_I = -I\, ds$. If f is left slice hyperholomorphic on an open set that contains $\overline{U}$, then*

$$f(p) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s,p)\, ds_I\, f(s) \quad \text{for all } p \in U.$$

If f is right slice hyperholomorphic on an open set that contains $\overline{U}$, then

$$f(p) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,p) \quad \text{for all } p \in U.$$

Remark 2.14. Observe that only values of f on one complex plane $\mathbb{C}_I$ enter the integral formula since we integrate along a path in $\mathbb{C}_I$. This is coherent with the fact that all values of f are determined by its values on $\mathbb{C}_I$, cf. Theorem 2.5.

Remark 2.15. The results presented in this section can be extended to functions with values in a two-sided quaternionic Banach space. As in the complex setting, problems concerning vector-valued functions can be reduced to scalar problems by applying elements of the dual space. For the details see [2].

2.2. Quaternionic linear operators and the S-functional calculus

Throughout the paper V shall denote a two-sided quaternionic Banach space, i.e., a Banach space with scalars in the quaternions on which the multiplications with scalars from both sides is defined. Any vector $v \in V$ is of the form $v = \sum_{i=0}^{3} v_i e_i$, where all v_i belong to the set of vectors $V_{\mathbb{R}}$ that satisfy $av = va$ for all $a \in \mathbb{H}$, cf. [38]. The set $V_{\mathbb{R}}$ is a real Banach space and we write $V = V_{\mathbb{R}} \otimes \mathbb{H}$.

For more rigorous definitions and the proofs of the following results, we refer again to [23].

Definition 2.16. A mapping $T : \mathcal{D}(T) \subset V \to V$ is called right linear if $T(u+va) = T(v) + T(v)a$ for all $u, v \in \mathcal{D}(T)$ and all $a \in \mathbb{H}$.

The set $\mathcal{B}(V)$ of bounded right linear operators on V is again a two-sided quaternionic Banach space, if we set $(aT)v = a(Tv)$ and $(Ta)v = T(av)$. We have $\mathcal{B}(V) = \mathcal{B}(V_{\mathbb{R}}) \otimes \mathbb{H}$, such that $T = \sum_{i=0}^{3} T_i e_i$ with $T_i \in \mathcal{B}(V_{\mathbb{R}})$. We hence call any operator acting on $V_{\mathbb{R}}$ a scalar operator.

In the quaternionic setting the notion of right linearity replaces the notion of linearity – one can also define the notion of left linearity, but both concepts lead to equivalent theories.

When mathematicians tried to generalize concepts and techniques of classical operator theory to the quaternionic setting, several problems occurred. In particular defining a proper notion of spectrum for quaternionic linear operators turned out to be surprisingly difficult. In the complex setting the spectrum of an operator is a generalization of the concept of eigenvalues: if the operator of the eigenvalue equation $(\lambda\mathcal{I} - T)v = 0$ does not have a bounded inverse, then λ belongs to the spectrum of T.

In the quaternionic setting however, it was not even clear what notion of eigenvalue to use. Due to the non-commutativity, one can formulate to different eigenvalue problems, namely

$$Tv = vs \quad \text{and} \quad Tv = sv.$$

For a right linear operator, the notion of right eigenvalue seemed to be more plausible. It was also the one that proved useful both in applications such as quaternionic quantum mechanics [1] and in the mathematical theory, for instance for showing the spectral theorem for matrices with quaternionic entries [26]. However, several problematic phenomena accompanied the notion of right eigenvalues. If $T(v) = vs$, then

$$T(va) = T(v)a = vsa = (va)(a^{-1}sa) \neq (va)s,$$

and hence the associated operator is not right linear as s and a do not commute in general. Moreover, if s is a right eigenvalue of T, then this equation shows that any quaternion of the form $a^{-1}sa$, i.e., any element of the sphere $[s]$, is also a right eigenvalue. Hence, the set of right eigenvalues is axially symmetric and in general not discrete. The notion of left eigenvalues, on the other hand, is associated to a right linear operator, but did not yet prove useful for any application.

The introduction of the notion of S-spectrum about ten years ago resolved these problems.

We denote the set of all bounded quaternionic right-linear operators on V by $\mathcal{B}(V)$ and the set of all closed and quaternionic right-linear operators on V by $\mathcal{K}(V)$. For $T \in \mathcal{K}(V)$, we set

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} \quad \text{for } s \in \mathbb{H}.$$

Definition 2.17. Let $T \in \mathcal{K}(V)$. We define the S-resolvent set of T as

$$\rho_S(T) := \{s \in \mathbb{H} : \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V)\}$$

and the S-spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

If $s \in \rho_S(T)$, then $\mathcal{Q}_s(T)^{-1}$ is called the pseudo-resolvent of T at s. Furthermore, we define the extended S-spectrum $\sigma_{SX}(T)$ as

$$\sigma_{SX}(T) := \begin{cases} \sigma_S(T) & \text{if } T \text{ is bounded,} \\ \sigma_S(T) \cup \{\infty\} & \text{if } T \text{ is unbounded.} \end{cases}$$

The S-spectrum has properties that are similar to those of the spectrum of a complex linear operator. Moreover, it generalizes the notion of right eigenvalues in a similar way as the spectrum of an operator generalizes the notion of eigenvalues in the complex setting.

Theorem 2.18. *Let $T \in \mathcal{K}(V)$.*

1. *The S-spectrum $\sigma_S(T)$ is a closed subset of $\mathbb{H}$ and the extended S-spectrum $\sigma_{SX}(T)$ is a closed and hence compact subset of $\mathbb{H}_\infty := \mathbb{H} \cup \{\infty\}$.*
2. *If T is bounded, then $\sigma_S(T)$ is nonempty and bounded by the norm of T, i.e., $\sigma_S(T) \subset \overline{B_{\|T\|}(0)}$.*
3. *The S-spectrum $\sigma_S(T)$ of T is axially symmetric. It contains the set of right eigenvalues and if V is finite-dimensional it even agrees with it.*

Definition 2.19. Let $T \in \mathcal{K}(V)$. For $s \in \rho_S(T)$, the left S-resolvent operator is defined as

$$S_L^{-1}(s,T) := \mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1} \tag{2.7}$$

and the right S-resolvent operator is defined as

$$S_R^{-1}(s,T) := -(T - \mathcal{I}\overline{s})\mathcal{Q}_s(T)^{-1}. \tag{2.8}$$

Remark 2.20. One clearly obtains the right S-resolvent operator by formally replacing the variable p in the right slice hyperholomorphic Cauchy kernel by the operator T. The same procedure yields

$$S_L^{-1}(s,T)v = -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I})v \quad \text{for } v \in \mathcal{D}(T) \tag{2.9}$$

for the left S-resolvent operator. This operator is only defined on the domain $\mathcal{D}(T)$ of T and not on the entire space V. However, $\mathcal{Q}_s(T)^{-1}Tv = T\mathcal{Q}_s(T)^{-1}v$ for $v \in \mathcal{D}(T)$, and applying this relation in (2.9) yields (2.7), which is a bounded

operator defined on all of V. In particular, if T is bounded, then $S_L^{-1}(s,T)$ can directly be defined by (2.9).

If one considers left-linear operators, then one must modify the definition of the right S-resolvent operator for the same reasons.

Remark 2.21. The S-resolvent operators reduce to the classical resolvent if T and s commute, that is,

$$S_L^{-1}(s,T) = S_R^{-1}(s,T) = (s\mathcal{I} - T)^{-1}.$$

This is, in particular, the case if s is real.

The proof of the following crucial lemma can be found, for instance, in [23], for the case of bounded operators. For the case of unbounded operators, several additional technical difficulties have to be overcome. The respective proof can be found in [17].

Lemma 2.22. *Let $T \in \mathcal{K}(V)$. The map $s \mapsto S_L^{-1}(s,T)$ is a right slice-hyperholomorphic function on $\rho_S(T)$ with values in the two-sided quaternionic Banach space $\mathcal{B}(V)$. The map $s \mapsto S_R^{-1}(s,T)$ is a left slice-hyperholmorphic function on $\rho_S(T)$ with values in the two-sided quaternionic Banach space $\mathcal{B}(V)$.*

The S-resolvent equation is the analogue of the classical resolvent equation in the quaternionic setting. It has first been proved for the case that T is a bounded operator in [5] and was generalized to the case that T is unbounded in [17]. We want to stress that in contrast to the complex setting, it involves both S-resolvent operators and that a generalization of the resolvent equation that includes just one of them does not exist.

Theorem 2.23 (S-resolvent equation). *Let $T \in \mathcal{K}(V)$. For $s, p \in \rho_S(T)$ with $s \notin [p]$, one has*

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T) = &\big[[S_R^{-1}(s,T) - S_L^{-1}(p,T)]p \\ &- \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(p,T)]\big](p^2 - 2s_0p + |s|^2)^{-1}. \end{aligned} \tag{2.10}$$

We also recall [5, Lemma 3.23], a technical lemma that we shall need later on.

Lemma 2.24. *Let $B \in \mathcal{B}(V)$, let U be a bounded slice Cauchy domain and let $f \in \mathcal{N}(\overline{U})$. For any $p \in U$ and any $I \in \mathbb{S}$, we have*

$$Bf(p) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} f(s)\, ds_I\, (\overline{s}B - Bp)\left(p^2 - 2s_0p + |s|^2\right)^{-1}.$$

The idea of the famous Riesz–Dunford functional calculus is simple: if f is holomorphic on a suitable set O that contains the spectrum $\sigma(A)$ of A, then we write f in terms of the Cauchy formula as

$$f(z) = \frac{1}{2\pi} \int_{\partial O} \frac{f(\xi)}{\xi - z}\, d\xi$$

and formally replace z by the operator A in this formula. Hence, we replace the Cauchy kernel $1/(\xi - z)$ by the resolvent $R_\xi(A) = (\xi\mathcal{I} - A)^{-1}$ and define

$$f(A) := \frac{1}{2\pi} \int_{\partial O} f(\xi) R_\xi(A)\, d\xi. \tag{2.11}$$

Following this idea, we can replace the slice hyperholomorphic Cauchy kernels in the Cauchy formula by the S-resolvent operators. We obtain in this way the natural generalization of the Riesz–Dunford functional calculus to quaternionic linear operators, see (1.3) and (1.4).

This definition can be extended to closed operators which can be done via a transformation to the bounded case as it is usually done in the complex setting [23]. One then obtains an integral representation of Cauchy type for this operator. One can, however, use these integrals as a definition, which we do here for brevity and which is sometimes useful for technical reasons [27].

We say that a function is left (or right) slice hyperholomorphic at infinity, if f is left (or right) slice hyperholomorphic, if there exists $r > 0$ such that the complement of the ball $B_r(0)$ of radius $r > 0$ centered at 0 is contained in the domain $\mathrm{dom}(f)$ of f and if $f(\infty) = \lim_{p\to\infty} f(p)$ exists.

Definition 2.25 (S-functional calculus for closed operators). Let $T \in \mathcal{K}(V)$ with $\rho_S(T) \neq 0$, choose $I \in \mathbb{S}$ and set $ds_I = -I\, ds$. For $f \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$, we choose an unbounded slice Cauchy domain $U \subset \mathrm{dom}(f)$ with $\sigma_S(T) \subset U$ and define

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f(s). \tag{2.12}$$

For $f \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$, we choose again a bounded slice Cauchy domain $U \subset \mathrm{dom}(f)$ with $\sigma_S(T) \subset U$ and define

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,T). \tag{2.13}$$

These integrals are independent of the choices of the slice domain U and the imaginary unit $I \in \mathbb{S}$.

Recently in the paper [28] it has been proved that for intrinsic functions the S-functional calculus can be defined on one-sided quaternionic linear spaces, and the theory of spectral operators has been introduced in this setting.

3. Fractional powers of quaternionic operators via the H^∞-functional calculus

As pointed out in the introduction, there exist several approaches to fractional powers of operators. The most general one introduces them via the H^∞-functional calculus. This functional calculus was introduced by A. McIntosh for complex linear operators in [37]. We shall briefly recall the main ideas behind it.

A closed linear operator A on a reflexive complex Banach space X is said to be sectorial of angle $\omega \in (0, \pi)$ if the spectrum $\sigma(A)$ is contained in the closure of the sector

$$S_\omega := \{z \in \mathbb{C} \ : \ |\arg(z)| < \omega, z \neq 0\}$$

of angle ω around the positive real axis and if for any $\varphi > \omega$ there exists $C_\varphi > 0$ such that the resolvent $R_\lambda(A) = (\lambda \mathcal{I} - A)^{-1}$ satisfies the estimate

$$\|\lambda R_\lambda(A)\| \leq C_\varphi \qquad \forall \lambda \in \mathbb{C} \setminus S_\varphi. \tag{3.1}$$

If A is, in addition, injective with dense range, then it is possible to proceed as follows in order to construct operators $f(A)$ that might even be unbounded. For any function g that is holomorphic on S_φ for some $\varphi \in (\omega, \pi)$ and satisfies

$$|g(z)| \leq \frac{c|z|^\alpha}{1 + |z|^{2\alpha}} \quad \forall z \in S_\varphi, \tag{3.2}$$

with two positive constants α and c, we define analogue to the Riesz–Dunford functional calculus given by (2.11) the operator

$$g(A) = \frac{1}{2\pi i} \int_{\partial S_\varphi} g(\lambda) R_\lambda(A) \, d\lambda. \tag{3.3}$$

The estimates (3.1) and (3.2) guarantee that this integral is convergent and independent of the choice of φ.

Let now f be holomorphic on S_φ and not necessarily bounded, but instead satisfy

$$|f(z)| \leq c \left(|z|^k + |z|^{-k}\right) \tag{3.4}$$

for some $c > 0$ and some $k \in \mathbb{N}$. We can then consider the function $\psi(z) = \left(\frac{z}{1+z^2}\right)^{k+1}$ and set

$$\psi(A) = \left(A \left(\mathcal{I} + A^2\right)^{-1}\right)^{k+1} \tag{3.5}$$

using the rational functional calculus. The function ψf does then satisfy (3.2) such that we can define $(\psi f)(T)$ via (3.3). Moreover, our assumptions on A imply that $\psi(T)$ is injective and hence we can define

$$f(A) = (\psi(A))^{-1} (f\psi)(A). \tag{3.6}$$

The definition above is well posed since the operator $f(A)$ does not depend on the suitable rational function ψ that we choose to regularise f.

Now observe that z^α satisfies (3.4) for any α with $\mathrm{Re}(\alpha) > 0$ such that we can define A^α using this method. For a systematic development of this calculus in the classical setting we refer the reader to the book [33].

3.1. The H^∞-functional calculus for quaternionic operators

The H^∞-functional calculus was generalized to quaternionic right-linear operators following the strategy of McIntosh in [10]. We shall give a short explanation of how to define the H^∞-functional calculus for quaternionic right-linear operators following the strategy of [33], which applies to arbitrary and not necessarily injective sectorial operators. We just want to illustrate the strategy and point out the differences with respect to the classical case and refer to the forthcoming paper [16] for the technical details.

In order to define sectorial operators we introduce the sector Σ_φ for $\varphi \in (0, \pi]$ as

$$\Sigma_\varphi := \{s \in \mathbb{H} : \arg(s) < \varphi\}.$$

Definition 3.1. Let $\omega \in [0, \pi)$. An operator $T \in \mathcal{K}(V)$ is called sectorial of angle ω if

1. we have $\sigma_S(T) \subset \overline{\Sigma_\omega}$ and
2. for every $\varphi \in (\omega, \pi)$ there exists a constant $C > 0$ such that for $s \notin \Sigma_\varphi$

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{C}{|s|} \quad \text{and} \quad \left\|S_R^{-1}(s,T)\right\| \leq \frac{C}{|s|}. \tag{3.7}$$

We denote the infimum of all these constants by C_φ resp. by $C_{\varphi,T}$ if we also want to stress its dependence on T.

We denote the set of all operators in $\mathcal{K}(V)$ that are sectorial of angle ω by $\mathrm{Sect}(\omega)$. Furthermore, if T is a sectorial operator, we call $\omega_T = \min\{\omega : T \in \mathrm{Sect}(\omega)\}$ the spectral angle of T.

Finally, a family of operators $(T_\ell)_{\ell\in\Lambda}$ is called uniformly sectorial of angle ω if $T_\ell \in \mathrm{Sect}(\omega)$ for all $\ell \in \Lambda$ and $\sup_{\ell\in\Lambda} C_{\varphi,T_\ell} < \infty$ for all $\varphi \in (\omega, \pi)$.

Definition 3.2. We say that a slice hyperholomorphic function f has polynomial limit $c \in \mathbb{H}$ in Σ_φ at 0 if there exists $\alpha > 0$ such that $f(p) - c = O\left(|p|^\alpha\right)$ as $p \to 0$ in Σ_φ and that it has polynomial limit ∞ in Σ_φ at 0 if $1/|f(p)| \in O(|p|^\alpha)$ for some $\alpha > 0$ as $p \to 0$ in Σ_φ.

Similarly, we say that f has polynomial limit $c \in \mathbb{H}_\infty$ at ∞ in Σ_φ if $p \mapsto f(p^{-1})$ has polynomial limit c at 0. If a function has polynomial limit 0 at 0 or at ∞, we say that it decays regularly at 0 resp. ∞.

Definition 3.3. Let $\varphi \in (0, \pi]$. We define $\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ (and $\mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ resp. $\mathcal{N}^\infty_0(\Sigma_\varphi)$) as the set of all bounded functions in $\mathcal{SH}_L(\Sigma_\varphi)$ (or $\mathcal{SH}_R(\Sigma_\varphi)$ resp. $\mathcal{N}(\Sigma_\varphi)$) that decay regularly at 0 and ∞.

These spaces are closed under the usual operations supported by slice hyperholomorphic functions.

Lemma 3.4. *Let $\varphi \in (0, \pi]$.*

1. *If $f, g \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ and $a \in \mathbb{H}$, then $fa + g \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$. If in addition even $f \in \mathcal{N}^\infty_0(\Sigma_\varphi)$, then also $fg \in \mathcal{N}^\infty_0(\Sigma_\varphi)$.*

2. *If* $f, g \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ *and* $a \in \mathbb{H}$*, then* $af + g \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$*. If in addition even* $g \in \mathcal{N}^\infty_0(\Sigma_\varphi)$*, then also* $fg \in \mathcal{N}^\infty_0(\Sigma_\varphi)$*.*
3. *The space* $\mathcal{N}^\infty_0(\Sigma_\varphi)$ *is a real algebra.*

Similar to the complex case we can now define the operator $f(T)$ for regularly decaying f via a Cauchy integral.

Definition 3.5 (S-functional calculus for sectorial operators). Let $T \in \mathrm{Sect}(\omega)$. For any $f \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ with $\omega < \varphi < \pi$, we choose φ' with $\omega < \varphi' < \varphi$ and $I \in \mathbb{S}$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I\, f(s). \tag{3.8}$$

Similarly, for $f \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ with $\omega < \varphi < \pi$, we choose φ' with $\omega < \varphi' < \varphi$ and $I \in \mathbb{S}$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s, T). \tag{3.9}$$

Remark 3.6. Since T is sectorial of angle ω, the estimates in (3.7) assure the convergence of the above integrals. Moreover, if f is intrinsic, then (3.8) and (3.9) yield the same operator, which can be shown as in [27].

In order to illustrate two important techniques in the quaternionic setting and to point out some differences to the complex setting, we give the proofs of the well-definedness and the product rule of this functional calculus.

Lemma 3.7. *Let* $T \in \mathrm{Sect}(\omega)$*. If* $f \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ *or* $f \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ *with* $\omega < \varphi < \pi$*, then* $f(T)$ *is well defined, i.e., independent of the choice of* φ *and* $I \in \mathbb{S}$ *in* (3.8) *resp.* (3.9)*.*

Proof. We only consider the case that $f \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ since the other one follows by analogous arguments. The independence of φ is an immediate consequence of Cauchy's integral theorem. In order to show the independence of the imaginary unit, we consider two different units I and J and choose two different angles $\omega < \varphi' < \varphi'' < \varphi$. By (2.13) and Cauchy's integral theorem, we have with the notation $U(r, R) = (\Sigma_{\varphi'} \cap B_R(0)) \setminus B_r(0)$ that for any $s \in \partial(\Sigma_{\varphi'})$

$$\begin{aligned} f(s) &= \lim_{r \to 0} \lim_{R \to +\infty} \frac{1}{2\pi} \int_{\partial(U(r,R) \cap \mathbb{C}_J)} S_L^{-1}(p, s)\, ds_I\, f(p) \\ &= \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi''} \cap \mathbb{C}_J)} S_L^{-1}(p, s)\, dp_I\, f(p), \end{aligned}$$

where the fact that f is regularly decaying at 0 and at infinity assures that we can take these limits. Hence we find

$$\begin{aligned} f(T) &= \frac{1}{2\pi}\int_{\partial(\Sigma_{\varphi'}\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s) \\ &= \frac{1}{(2\pi)^2}\int_{\partial(\Sigma_{\varphi'}\cap\mathbb{C}_I)}\int_{\partial(\Sigma_{\varphi''}\cap\mathbb{C}_J)} S_L^{-1}(s,T)\,ds_I\,S_L^{-1}(p,s)\,dp_J\,f(p) \\ &= \frac{1}{2\pi}\int_{\partial(\Sigma_{\varphi''}\cap\mathbb{C}_J)}\left(\frac{1}{2\pi}\int_{\partial(\Sigma_{\varphi'}\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_L^{-1}(p,s)\right)dp_J\,f(p), \quad (3.10) \end{aligned}$$

where we used Fubini's theorem to exchange the order of integration. Recall that $S_L^{-1}(s,T)$ is right slice hyperholomorphic in s on $\rho_S(T)$ and that $S_L^{-1}(s,p) = -S_R^{-1}(p,s)$. Hence, with $U(r,R) = \mathbb{H}\setminus(\Sigma_{\varphi'}\cup B_r(0))\cap B_R(0)$, we have again because of (2.13) and Cauchy's integral theorem that

$$\begin{aligned} &\int_{\partial(\Sigma_{\varphi'}\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_L^{-1}(p,s) \\ &= -\frac{1}{2\pi}\int_{\partial(\Sigma_{\varphi'}\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_R^{-1}(s,p) \\ &= \frac{1}{2\pi}\int_{\partial(\mathbb{H}\setminus\Sigma_{\varphi'}\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_R^{-1}(s,p) \\ &= \lim_{R\to+\infty}\lim_{r\to 0}\frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_R^{-1}(s,p) = S_L^{-1}(p,T) \end{aligned}$$

such that finally

$$f(T) = \frac{1}{2\pi}\int_{\partial(\Sigma_{\varphi''}\cap\mathbb{C}_J)} S_L^{-1}(p,T)\,dp_I\,f(p). \qquad \square$$

Lemma 3.8. *Let $T\in\mathrm{Sect}(\omega)$ and $\omega<\varphi<\pi$. The following statements hold true.*

1. *If $f\in\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ or $f\in\mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$, then the operator $f(T)$ is bounded.*
2. *If $f,g\in\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ and $a\in\mathbb{H}$, then $(fa+g)(T)=f(T)a+g(T)$. If $f,g\in\mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ and $a\in\mathbb{H}$, then $(af+g)(T)=af(T)+g(T)$.*
3. *If $f\in\mathcal{N}^\infty_0(\Sigma_\varphi)$ and $g\in\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$, then $(fg)(T)=f(T)g(T)$. If $f\in\mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ and $g\in\mathcal{N}^\infty_0(\Sigma_\varphi)$, then also $(fg)(T)=f(T)g(T)$.*

Proof. The boundedness of the operator follows from the convergence of the integrals due to (3.7) and the regular decay of f at 0 and infinity. The linearity properties follow from the linearity of the integral.

Let us now show the product rule. We only consider the case $f\in\mathcal{N}^\infty_0(\Sigma_\omega)$ and $g\in\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ because the other one follows again by analogous arguments. We choose $\omega<\varphi_p<\varphi_s<\varphi$, where the subscripts s and p refer to the corresponding variable of integration along $\Gamma_p := \partial(\Sigma_{\varphi_p}\cap\mathbb{C}_I)$ resp. $\Gamma_s := \partial(\Sigma_{\varphi_s}\cap\mathbb{C}_I)$ in the following calculation. Since f is an intrinsic function, we can write $f(T)$ in two

ways, namely, as (3.8) using the left S-resolvent operator or as (3.9) using the right S-resolvent operator. Using the latter option, we find

$$f(T)g(T) = \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_I\,S_R^{-1}(s,T)\int_{\Gamma_p} S_L^{-1}(p,T)\,dp_I\,g(p).$$

The S-resolvent equation (2.10) allows us to split this product into

$$\begin{aligned}
&f(T)g(T)\\
&= \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_I S_R^{-1}(s,T)\int_{\Gamma_p} p\left(p^2-2s_0p+|s|^2\right)^{-1}\,dp_I\,g(p)\\
&\quad - \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_I\,\overline{s}S_R^{-1}(s,T)\int_{\Gamma_p}\left(p^2-2s_0p+|s|^2\right)^{-1}\,dp_I\,g(p)\\
&\quad + \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_I\int_{\Gamma_p}\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right) \qquad (3.11)\\
&\qquad\qquad \times\left(p^2-2s_0p+|s|^2\right)^{-1}\,dp_I\,g(p).
\end{aligned}$$

Now observe that since we chose $\varphi_s > \varphi_p$ the functions $p\mapsto p\left(p^2-2s_0p+|s|^2\right)^{-1}$ and $p\mapsto\left(p^2-2s_0p+|s|^2\right)^{-1}$ are intrinsic on Σ_{φ_p} such that the first two integrals vanish because of Theorem 2.10. In the remaining integral we want to apply Fubini's theorem in order to exchange the order of integration. Since s and p belong to the same complex plane $\mathbb{C}_I$, we have $(p^2-2s_0p+|s|^2)^{-1} = (p-s)(p-\overline{s})$ and hence

$$\begin{aligned}
\frac{1}{|p^2-2s_0p+|s|^2|} &= \frac{1}{|p-s||p-\overline{s}|}\\
&\le \frac{1}{|s|^2}\frac{1}{\left|\frac{|p|}{|s|}-e^{I(\theta_s-\theta_p)}\right|\left|\frac{|p|}{|s|}-e^{I(-\theta_s-\theta_p)}\right|} \le \frac{1}{|s|^2}K_1,
\end{aligned}$$

where $\theta_s\in\{\pm\varphi_s\}$ and $\theta_p\in\{\pm\varphi_p\}$ are such that $s=|s|e^{I\theta_s}$ and $p=|p|e^{I\theta_p}$ and the last two fractions are bounded by a constant $K_1>0$ because $|p|/|s|$ is real but $0<\varphi_p<\varphi_s<\pi$ such that $\pm\theta_s-\theta_p\notin\{0,\pm2\pi\}$ such that $e^{I(\pm\theta_s-\theta_p)}$ has positive distance to the real line. Similarly, we can also estimate

$$\begin{aligned}
\frac{1}{|p^2-2s_0p+|s|^2|} &= \frac{1}{|p-s||p-\overline{s}|}\\
&\le \frac{1}{|p||s|}\frac{1}{\left|e^{I(\theta_p-\theta_s)}-\frac{|s|}{|p|}\right|\left|\frac{|p|}{|s|}-e^{I(-\theta_s-\theta_p)}\right|} \le \frac{1}{|p||s|}K_2.
\end{aligned}$$

Because of (3.7) we can therefore estimate the integrand in (3.11) by

$$\begin{aligned}
&\int_{\Gamma_s}\int_{\Gamma_p}|f(s)|\,d|s|\left(|\overline{s}|\frac{C_{\varphi_p,T}}{|p|}+\frac{C_{\varphi_p,T}|p|}{|p|}\right)\frac{1}{|p^2-2s_0p+|s|^2|}\,d|p|\,|g(p)|\\
&\le 2C_{\varphi_p,T}(K_1+K_2)\int_{\Gamma_s}\frac{|f(s)|}{|s|}\,d|s|\int_{\Gamma_p}\frac{|g(p)|}{|p|}\,d|p| < +\infty,
\end{aligned}$$

because f and g decay regularly at 0 and infinity. Thus we can actually apply Fubini's theorem and exchanging the order of integration in (3.11) yields

$$f(T)g(T) = \frac{1}{(2\pi)^2}\int_{\Gamma_p}\left(\int_{\Gamma_s} f(s)\,ds_I \right.$$
$$\left.\times \left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)\left(p^2 - 2s_0p + |s|^2\right)^{-1}\right) dp_I\, g(p).$$

With the notation $U(r,R) = (\Sigma_{\varphi_s}\setminus B_r(0))\cap B_R(0)$ and applying Lemma 2.24 with $B = S_L^{-1}(p,T)$, we find that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\Gamma_s} f(s)\,ds_I\,\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)\left(p^2 - 2s_0p + |s|^2\right)^{-1}\\
&= \lim_{R\to\infty}\lim_{r\to 0}\frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_I)} f(s)\,ds_I\,\left(\overline{s}S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right)\left(p^2 - 2s_0p + |s|^2\right)^{-1}\\
&= S_L^{-1}(p,T)f(p).
\end{aligned}$$

Since f is intrinsic, $f(p)$ commutes with dp_I such that we finally obtain

$$f(T)g(T) = \frac{1}{(2\pi)^2}\int_{\Gamma_p} S_L^{-1}(p,T)\,dp_I\,f(p)g(p) = (fg)(T). \qquad \square$$

The class of functions that are admissible to this functional calculus can be extended. Following [33], we define the analogue of the extended Riesz class.

Lemma 3.9. *For $0 < \varphi < \pi$, we define*

$$\mathcal{E}_L(\Sigma_\varphi) = \left\{f(p) = \tilde{f}(p) + (1+p)^{-1}a + b : \tilde{f}\in\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi), a,b\in\mathbb{H}\right\}$$

and similarly

$$\mathcal{E}_R(\Sigma_\varphi) = \left\{f(p) = \tilde{f}(p) + a(1+p)^{-1} + b : \tilde{f}\in\mathcal{SH}^\infty_{R,0}(\Sigma_\varphi), a,b\in\mathbb{H}\right\}.$$

Finally, we define $\mathcal{E}_{int}(\Sigma_\varphi)$ as the set of all intrinsic functions in $\mathcal{E}_L(\Sigma_\varphi)$, i.e.,

$$\mathcal{E}_{int}(\Sigma_\varphi) = \left\{f(p) = \tilde{f}(p) + (1+p)^{-1}a + b : \tilde{f}\in\mathcal{N}^\infty_0(\Sigma_\varphi), a,b\in\mathbb{R}\right\}.$$

Simple calculations as in [33, Lemma 2.2.3] show the following two corollaries.

Corollary 3.10. *Let $0 < \varphi < \pi$.*

1. *The set $\mathcal{E}_L(\Sigma_\varphi)$ is a quaternionic right vector space and closed under multiplication with functions in $\mathcal{E}_{int}(\Sigma_\varphi)$ from the left.*
2. *The set $\mathcal{E}_R(\Sigma_\varphi)$ is a quaternionic left vector space and closed under multiplication with functions in $\mathcal{E}_{int}(\Sigma_\varphi)$ from the right.*
3. *The set $\mathcal{E}_{int}(\Sigma_\varphi)$ is a real algebra.*

Corollary 3.11. *Let $0 < \varphi < \pi$. A function $f\in\mathcal{SH}_L(\Sigma_\varphi)$ (or $f\in\mathcal{SH}_R(\Sigma_\varphi)$ or $f\in\mathcal{N}(\Sigma_\varphi)$) belongs to $\mathcal{E}_L(\Sigma_\varphi)$ (resp. $\mathcal{E}_R(\Sigma_\varphi)$ or $\mathcal{E}_{int}(\Sigma_\varphi)$) if and only if it is bounded and has finite polynomial limits at 0 and infinity.*

Definition 3.12. Let $T \in \mathrm{Sect}(\omega)$ and $\varphi \in (\omega, \pi)$. We define for $f \in \mathcal{E}_L(\Sigma_\varphi)$ with $f(s) = \tilde{f}(s) + (1+s)^{-1}a + b$ the bounded operator

$$f(T) := \tilde{f}(T) + (1+T)^{-1}a + \mathcal{I}b$$

and for $f \in \mathcal{E}_R(\Sigma_\varphi)$ with $f(s) = \tilde{f}(s) + a(1+s)^{-1} + b$ the bounded operator

$$f(T) := \tilde{f}(T) + a(1+T)^{-1} + b\mathcal{I},$$

where $\tilde{f}(T)$ is intended as in Definition 3.5.

Corollary 3.13. *The S-functional calculus for closed operators and the S-functional calculus for sectorial operators are compatible.*

Definition 3.12 is compatible with the algebraic structures of the underlying function spaces.

Lemma 3.14. *If $T \in \mathrm{Sect}(\omega)$, then the following statements hold true.*

1. *If $f, g \in \mathcal{E}_L[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(fa+g)(T) = f(T)a + g(T)$. If $f, g \in \mathcal{E}_R[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(af+g)(T) = af(T) + g(T)$.*
2. *If $f \in \mathcal{E}_{int}[\Sigma_\omega]$ and $g \in \mathcal{E}_L[\Sigma_\omega]$, then $(fg)(T) = f(T)g(T)$. If $f \in \mathcal{E}_R[\Sigma_\omega]$ and $g \in \mathcal{E}_{int}[\Sigma_\omega]$, then also $(fg)(T) = f(T)g(T)$.*

The H^∞-functional calculus for complex linear sectorial operators in [33] applies to meromorphic functions that are regularizable. We point out that introducing a notion of order for zeros and poles of slice hyperholomorphic functions (and hence a theory of generalized meromorphy) is a nontrivial task in our setting. However, for our purposes the following simple definition shall be sufficient.

Definition 3.15. Assume $s \in \mathbb{H}$ and define $\mathcal{Q}_s(p) = p^2 - 2s_0 p + |s|^2$. Let f be a left slice hyperholomorphic function on an axially symmetric neighborhood $[B_r(s)]\backslash\{s\}$ of s with $[B_r(s)] = \{p \in \mathbb{H} : \mathrm{dist}([s], p) < r\}$ and assume that f does not have a left slice hyperholomorphic continuation to all of $[B_r(s)]$. We say that f has a pole at the sphere $[s]$ if there exists $n \in \mathbb{N}$ such that $p \mapsto \mathcal{Q}_s(p)^n f(p)$ has a left slice hyperholomorphic continuation to $[B_r(s)]$. In the case $s \in \mathbb{R}$, se say that f has a pole at s if there exists $n \in \mathbb{N}$ such that $p \mapsto (p-s)^n f(p)$ has a left slice hyperholomorphic continuation to $[B_r(s)]$ for $s \in \mathbb{R}$.

Definition 3.16. Let $U \subset \mathbb{H}$ be axially symmetric. A function f belongs to $\mathcal{M}_L(U)$ if there exist isolated spheres $[p_n] \subset U$ with $n \in \Theta$, where Θ is any subset of $\mathbb{N}$, such that $f|_{\tilde{U}} \in \mathcal{SH}_L(\tilde{U})$ with $\tilde{U} = U \setminus \bigcup_{n\in\Theta}[p_n]$ and such that each sphere $[p_n]$ is a pole of f. By $\mathcal{M}_{\mathrm{int}}(U)$ we denote the set of all functions in $\mathcal{M}_L(U)$ that are intrinsic.

Definition 3.17. Let $T \in \mathrm{Sect}(\omega)$. A left slice hyperholomorphic function f is said to be regularizable if $f \in \mathcal{M}_L(\Sigma_\varphi)$ for some $\omega < \varphi < \pi$ and there exists $e \in \mathcal{E}_{int}(\Sigma_\varphi)$ such that $e(T)$ defined in the sense of Definition 3.12 is injective and $ef \in \mathcal{E}_L(\Sigma_\varphi)$. In this case we call e a regulariser for f.

We denote the set of all such functions by $\mathcal{M}_L[\Sigma_\omega]_T$. Furthermore, we denote the subset of intrinsic functions in $\mathcal{M}_L[\Sigma_\omega]_T$ by $\mathcal{M}_{\mathrm{int}}[\Sigma_\omega]_T$.

These function spaces do again have the usual algebraic structure. Moreover, if T is not injective, then any function in $\mathcal{M}_L[\Sigma_\omega]_T$ needs to have a proper limit behavior at 0.

Corollary 3.18. *Let $T \in \mathrm{Sect}(\omega)$.*

1. *If $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$ and $a \in \mathbb{H}$, then $fa + g \in \mathcal{M}_L[\Sigma_\omega]_T$. If furthermore $f \in \mathcal{M}_{\mathrm{int}}[\Sigma_\omega]_T$, then also $fg \in \mathcal{M}_L[\Sigma_\omega]_T$.*
2. *The space $\mathcal{M}_{\mathrm{int}}[\Sigma_\omega]_T$ is a real algebra.*
3. *If T is not injective, then $f \in \mathcal{M}_L[\Sigma_\omega]_T$ has finite polynomial limit $c \in \mathbb{H}$ at 0.*

Remark 3.19. If T is injective, it might happen that f does not have a finite polynomial limit at 0. The function $p \mapsto p\left(1+p^2\right)^{-1}$ and its powers can then be used as regularizers to compensate a singularity at 0. This yields exactly the procedure presented in [10], where the H^∞-functional calculus was first introduced for quaternionic linear operators following the original approach of McIntosh.

Definition 3.20 (H^∞-functional calculus). Let $T \in \mathrm{Sect}(\omega)$. For regularizable $f \in \mathcal{M}_L[\Sigma_\omega]_T$, we define

$$f(T) := e(T)^{-1}(ef)(T),$$

where $e(T)^{-1}$ is the closed inverse of $e(T)$ and $(ef)(T)$ is intended in the sense of Definition 3.12.

Remark 3.21. The operator $f(T)$ is independent of the regulariser e and hence well defined. Observe, however, that, since we are considering right-linear operators, a similar definition for right slice hyperholomorphic functions is not possible: since right slice hyperholomorphic functions maintain slice hyperholomorphicity under multiplication with intrinsic functions from the right, we would have to define a regulariser of a functions f as a function e such that $e(T)$ is injective and $fe \in \mathcal{E}_R(\Sigma_\varphi)$. The operator $f(T)$ would then be defined as $(fe)(T)e(T)^{-1}$. It can hence not be independent of the choice of e, because this operator is only defined on $\mathrm{ran}(e(T))$. If we consider left-linear operators, we of course end up with vice versa situation. This is a common phenomenon in quaternionic operator theory, see for instance [4].

With this definition we obtain a functional calculus whose properties correspond to those of the complex analogue.

Lemma 3.22. *Let $T \in \mathrm{Sect}(\omega)$.*

1. *If $A \in \mathcal{B}(V)$ commutes with T, then A commutes with $f(T)$ for any $f \in \mathcal{M}_{\mathrm{int}}[\Sigma_\omega]_T$. Moreover, if $f(T) \in \mathcal{B}(V)$ for some $f \in \mathcal{M}_{\mathrm{int}}[\Sigma_\omega]_T$, then $f(T)$ commutes with T.*
2. *If $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$, then*

$$f(T) + g(T) \subset (f+g)(T).$$

If furthermore $f \in \mathcal{M}_{\mathrm{int}}[\Sigma_\omega]_T$, then

$$f(T)g(T) \subset (fg)(T)$$

with $\mathcal{D}(f(T)g(T)) = \mathcal{D}((fg)(T)) \cap \mathcal{D}(g(T))$. *In particular, the above inclusion turns into an equality if* $g(T) \in \mathcal{B}(V)$.

3. *Let* $f \in \mathcal{M}_{\text{int}}[\Sigma_\omega]_T$ *and* $g \in \mathcal{M}_{\text{int}}[\Sigma_\omega]$ *such that* $fg \equiv 1$. *Then* $g \in \mathcal{M}_{\text{int}}[\Sigma_\omega]_T$ *if and only if* $f(T)$ *is injective. In this case* $f(T) = g(T)^{-1}$.
4. *This functional calculus is compatible with intrinsic rational functions. More precisely, if* r *is an intrinsic rational function (that is,* $r(p) = P(p)Q(p)^{-1}$ *with intrinsic polynomials* P *and* Q*) whose poles lie in* $\rho_S(T)$, *then* $r \in \mathcal{M}_{\text{int}}[\Sigma_\omega]_T$ *and the operator* $r(T)$ *is given by* $r(T) = P[T]Q[T]^{-1}$, *where* $P[T]$ *and* $Q[T]$ *are defined in the usual way, i.e.,* $\psi[T] = \sum_{k=0}^{n} T^k a_k$ *with* $\mathcal{D}(\psi[T]) = \mathcal{D}(T^n)$ *for any polynomial* $\psi(p) = \sum_{k=0}^{n} p^k a_k$.

Finally, the composition rule and the spectral mapping theorem also hold true for the quaternionic H^∞-functional calculus. We, however, want to point out that severe technical difficulties appear due to the noncommutativity of the quaternionic multiplication, when trying to adapt the corresponding proofs of the classical results to the quaternionic setting, cf. [16].

Theorem 3.23. *Let* $T \in \text{Sect}(\omega)$ *and* $g \in \mathcal{M}_{\text{int}}[\Sigma_\omega]_T$ *such that* $g(T) \in \text{Sect}(\omega')$. *Furthermore assume that for any* $\varphi' \in (\omega', \pi)$ *there exists* $\varphi \in (\omega, \pi)$ *such that* $g \in \mathcal{M}_{\text{int}}(\Sigma_\varphi)$ *and* $g(\Sigma_\varphi) \subset \overline{\Sigma_{\varphi'}}$. *In this case, for any* $f \in \mathcal{M}_L[\Sigma_{\omega'}]$, *we have* $f \circ g \in \mathcal{M}_{\text{int}}[\Sigma_\omega]_T$ *and*

$$(f \circ g)(T) = f(g(T)).$$

Corollary 3.24. *Let* $T \in \text{Sect}(\omega)$ *be injective and let* $f \in \mathcal{M}_L[\omega]$. *Then* $f \in \mathcal{M}_L[\omega]_T$ *if and only if* $p \mapsto f(p^{-1}) \in \mathcal{M}_L[\omega]_{T^{-1}}$ *and in this case*

$$f(T) = f(p^{-1})(T^{-1}).$$

Theorem 3.25 (Spectral Mapping Theorem). *Let* $T \in \text{Sect}(\omega)$ *and* $f \in \mathcal{M}_{\text{int}}[\Sigma_\omega]_T$ *have polynomial limits at* $\{0, \infty\} \cap \sigma_{SX}(T)$. *Then*

$$f(\sigma_{SX}(T)) = \sigma_{SX}(f(T)).$$

Remark 3.26. We want to stress that the spectral mapping theorem is only true for intrinsic functions, which is again a common phenomenon in the quaternionic setting. This is in any case plausible, since only this class of functions preserves the axially symmetric structure of the spectrum. Indeed, if $f = \alpha + I\beta$ is an intrinsic function and $p_0 + I_p p_1 = p = f(s)$, then the relations $p_0 = \alpha(s_0, s_1)$ and $p_1 = |\beta(s_0, s_1)|$ and $I_p = \text{sgn}(\beta(s_0, s_1))I_s$ hold true since in this case α and β take real values. Hence,

$$f([s|)] = \{\alpha(s_0, s_1) + I\beta(s_0, s_1) : I \in \mathbb{S}\} = \{p_0 + Ip_1 : I \in \mathbb{S}\} = [p] = [f(s)]$$

such that the axially symmetric structure of the S-spectrum is actually preserved under f. For general left slice hyperholomorphic functions this is not true.

3.2. Fractional powers via the H^∞-functional calculus

We can now apply the H^∞-functional calculus introduced above in order to define fractional powers of sectorial operators. Again we followed the strategy used in [33] to obtain our results. We start with recalling the definition of fractional powers of quaternionic scalars.

Definition 3.27. The (slice hyperholomorphic) logarithm on $\mathbb{H}$ is defined as

$$\log s := \ln|s| + I_s \arg(s) \quad \text{for } s \in \mathbb{H} \setminus (-\infty, 0],$$

where $\arg(s) = \arccos(s_0/|s|)$ is the unique angle $\varphi \in [0, \pi]$ such that $s = |s|e^{I_s \varphi}$.

If $s = s_0 \in [0, \infty)$, then $\arccos(s_0/|s|) = 0$ and in turn $\log s = \ln s$ such that $\log s$ is independent of I_s and hence well defined on the positive real axis. One has

$$e^{\log s} = s \quad \text{for } s \in \mathbb{H}$$

and

$$\log e^s = s \quad \text{for } s \in \mathbb{H} \quad \text{with } |\underline{s}| < \pi.$$

The quaternionic logarithm is intrinsic on $\mathbb{H} \setminus (-\infty, 0]$. For any $I \in \mathbb{S}$ its restriction to the complex plane $\mathbb{C}_I$ is the principal branch of the complex logarithm on $\mathbb{C}_I$.

Definition 3.28. Let $\alpha \in \mathbb{R}$. We define the fractional power of exponent α of a quaternion $s \in \mathbb{H} \setminus (-\infty, 0)$ as

$$s^\alpha := e^{\alpha \log s} = e^{\alpha(\ln|s| + I_s \arccos(s_0/|s|))}.$$

The function $s \mapsto s^\alpha$ is also intrinsic on $\mathbb{H} \setminus (-\infty, 0]$. Note, however, that we only define fractional powers with real exponents. The functions $e^{\alpha \log s}$ or $e^{\log(s)\alpha}$ with $\alpha \in \mathbb{H} \setminus \mathbb{R}$ are not slice hyperholomorphic, because the composition of two slice hyperholomorphic functions is in general only slice hyperholomorphic if the inner function is intrinsic.

3.2.1. Fractional powers with positive real part. For arbitrary sectorial operators one can define fractional powers with nonnegative real exponent: let $T \in \mathrm{Sect}(\omega)$ and let $\alpha \in (0, +\infty)$. The function $s \mapsto s^\alpha$ does then belong to $\mathcal{M}_{\mathrm{int}}[\Sigma_\omega]_T$ such that we can choose $n \in \mathbb{N}$ with $n > \alpha$ and define according to the H^∞-functional calculus

$$T^\alpha := s^\alpha(T) = (\mathcal{I} + T)^n \left(s^\alpha (1+s)^{-n}\right)(T), \tag{3.12}$$

where $\left(s^\alpha(1+s)^{-n}\right)(T)$ is defined via a slice hyperholomorphic Cauchy integral as in (3.8) or (3.9).

Definition 3.29. Let $T \in \mathrm{Sect}(\omega)$ and $\alpha > 0$. We call the operator defined in (3.12) the fractional power with exponent α of T.

From the properties of the H^∞-functional calculus, one immediately deduces the following lemma.

Lemma 3.30. *Let $T \in \mathrm{Sect}(\omega)$ and let $\alpha \in (0, +\infty)$.*

1. *If T is injective, then $\left(T^{-1}\right)^{\alpha} = (T^{\alpha})^{-1}$. Thus $0 \in \rho_S(T)$ if and only if $0 \in \rho_S(T^{\alpha})$.*
2. *Any bounded operator that commutes with T commutes also with T^{α}.*
3. *The spectral mapping theorem holds, namely,*

$$\sigma_S(T^{\alpha}) = \{s^{\alpha} : s \in \sigma_S(T)\}.$$

Also the usual computational rules for fractional powers of an operator are true with this approach.

Proposition 3.31 (First Law of Exponents). *Let $T \in \mathrm{Sect}(\omega)$. For all $\alpha, \beta > 0$ the identity $T^{\alpha+\beta} = T^{\alpha}T^{\beta}$ holds. In particular $\mathcal{D}(T^{\gamma}) \subset \mathcal{D}(T^{\alpha})$ for $0 < \alpha < \gamma$.*

Proposition 3.32 (Scaling Property). *Let $T \in \mathrm{Sect}(\omega)$ and let $\Lambda = [\delta_1, \delta_2] \subset (0, \pi/\omega)$ be a compact interval. Then the family $(T^{\alpha})_{\alpha\in\Lambda}$ is uniformly sectorial of angle $\delta_2\omega$. In particular, for every $\alpha \in (0, \pi/\omega_T)$, the operator T^{α} is sectorial with angle $\omega_{T^{\alpha}} = \alpha\omega_T$.*

Proposition 3.32 and the composition rule Theorem 3.23 imply the following two results. Moreover, we also have analyticity in the exponent.

Proposition 3.33. *Let $T \in \mathrm{Sect}(\omega)$ for some $\omega \in (0, \pi)$ and let $\alpha \in (0, \pi\omega)$ and $\varphi \in (\omega, \pi/\alpha)$. If $f \in \mathcal{SH}^{\infty}_{L,0}(\Sigma_{\alpha\varphi})$ (or $f \in \mathcal{M}_L[\Sigma_{\alpha\omega}]_{T^{\alpha}}$), then the function $p \mapsto f(p^{\alpha})$ belongs to $\mathcal{SH}^{\infty}_{L,0}(\Sigma_{\varphi})$ (resp. $\mathcal{M}_L[\Sigma_{\omega}]_T$) and*

$$f(T^{\alpha}) = (f(p^{\alpha}))(T).$$

Corollary 3.34 (Second Law of Exponents). *Let $T \in \mathrm{Sect}(\omega)$ with $\omega \in (0, \pi)$ and let $\alpha \in (0, \pi/\omega)$. For all $\beta > 0$, we have*

$$(T^{\alpha})^{\beta} = T^{\alpha\beta}.$$

Theorem 3.35. *Let $T \in \mathrm{Sect}(\omega)$ and $\gamma > 0$. For any $v \in \mathcal{D}(T^{\gamma})$, the mapping $\alpha \mapsto T^{\alpha}v$ defined on $(0, \gamma)$ is analytic in α and the radius of convergence of its power series expansion at any point $\alpha_0 \in (0, \gamma)$ is given by $\min\{\gamma - \alpha_0, \alpha_0\}$.*

Finally, also the famous Balakrishnan Representation introduced in [11] as one of the first approaches to fractional powers of sectorial operators holds true in this setting. This result shall be essential for our approach to fractional evolution.

Theorem 3.36 (Balakrishnan Representation). *Let $T \in \mathrm{Sect}(\omega)$. For $0 < \alpha < 1$, we have*

$$T^{\alpha}v = \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^{\alpha-1}(t+T)^{-1}Tv\,dt \quad \forall v \in \mathcal{D}(A). \tag{3.13}$$

More generally, for $0 < \alpha < n \leq m$, we have

$$T^{\alpha}v = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{+\infty} t^{\alpha-1}[T(t+T)^{-1}]^m v\,dt \quad \forall v \in \mathcal{D}(T^n). \tag{3.14}$$

3.2.2. Fractional powers with negative real part. For $\alpha < 0$, the fractional power p^α has polynomial limit infinity in any sector Σ_φ with $\varphi > \pi$. By (3.18) it is, therefore, not in the class of admissible functions $\mathcal{M}_L[\Sigma_\omega]_T$ if T is not injective. If, on the other hand, T is injective, then it some power of $p\left(1+p^2\right)^{-1}$regularises p^α such that $p^\alpha \in \mathcal{M}_L[\Sigma_\omega]_T$. Consequently, we can define T^α for injective sectorial operators.

Definition 3.37. Let $T \in \mathrm{Sect}(\omega)$ be injective. For $\alpha \in \mathbb{R}$ we call the operator $T^\alpha := (p^\alpha)(T)$ the fractional power of T with exponent α.

Again we obtain properties and representations analogue to the complex setting, which we once more state without a proof for the sake of neatness.

Proposition 3.38. *Let $T \in \mathrm{Sect}(\omega)$ be injective and let $\alpha, \beta \in \mathbb{R}$.*

1. *The operator T^α is injective and $(T^\alpha)^{-1} = T^{-\alpha} = \left(T^{-1}\right)^\alpha$.*
2. *We have $T^\alpha T^\beta \subset T^{\alpha+\beta}$ with $\mathcal{D}\left(T^\alpha T^\beta\right) = \mathcal{D}\left(T^\beta\right) \cap \mathcal{D}\left(T^{\alpha+\beta}\right)$.*
3. *If $\overline{\mathcal{D}(T)} = V = \overline{\mathrm{ran}(T)}$, then $T^{\alpha+\beta} = \overline{T^\alpha T^\beta}$.*
4. *If $0 < \alpha < 1$, then*
$$T^{-\alpha} v = \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^{-\alpha} (t\mathcal{I} + T)^{-1} v \, dt, \quad v \in \mathrm{ran}(A).$$
5. *If $\alpha \in \mathbb{R}$ with $|\alpha|\pi/\omega$, then $T^\alpha \in \mathrm{Sect}(|\alpha|\omega)$ and*
$$(T^\alpha)^\beta = \left(T^{\alpha\beta}\right).$$
6. *If $0 < \alpha_1$, $0 < \alpha_2$, then $\mathcal{D}(T^{\alpha_2}) \cap \mathrm{ran}(T^{\alpha_1})$ for each $\alpha \in (-\alpha_1, \alpha_2)$, the mapping $\alpha \mapsto T^\alpha v$ is analytic on $(-\alpha_1, \alpha_2)$ for any $v \in \mathcal{D}(T^{-\alpha_2}) \cap \mathrm{ran}(T^{\alpha_1})$.*

Proposition 3.39 (Komatsu Representation). *Let $T \in \mathrm{Sect}(\omega)$ be injective. For $v \in \mathcal{D}(A) \cap \mathrm{ran}(A)$ and $\alpha \in (-1, 1)$, one has*

$$\begin{aligned} T^\alpha v &= \frac{\sin(\alpha\pi)}{\pi} \left[\frac{1}{\alpha} v - \frac{1}{1+\alpha} T^{-1} v \right. \\ &\qquad \left. + \int_0^1 t^{\alpha+1} (t\mathcal{I} + T)^{-1} T^{-1} v \, dt + \int_1^{+\infty} t^{\alpha-1} (t\mathcal{I} + T)^{-1} T v \, dt \right] \\ &= \frac{\sin(\alpha\pi)}{\pi} \left[\frac{1}{\alpha} v + \int_0^1 t^{-\alpha} (\mathcal{I} + tT)^{-1} T v \, dt - \int_0^1 t^\alpha \left(\mathcal{I} + tT^{-1}\right)^{-1} T^{-1} v \, dt \right]. \end{aligned}$$

3.2.3. Other approaches to and results on fractional powers of quaternionic operators. Although defining fractional powers via the H^∞-functional calculus is the most general approach to this problem, other approaches have already been studied in [17]. We shall briefly mention several chosen results.

Fractional powers with negative exponents of sectorial operators that have a bounded inverse can be defined directly via a Cauchy integral. Indeed, in this case a ball $B_\varepsilon(0)$ centered at 0 is contained in the S-resolvent set of T and for $\alpha > 0$ one can set

$$T^{-\alpha} := \frac{1}{2\pi} \int_\Gamma s^{-\alpha} \, ds_I \, S_R^{-1}(s, T), \tag{3.15}$$

where Γ is a piecewise smooth path that goes from $\infty e^{I\theta}$ to $\infty e^{-I\theta}$ for some $I \in \mathbb{S}$ and some $\theta \in (\omega, \pi)$ in $(\mathbb{H} \setminus \Sigma_\omega) \cup B_\varepsilon(0)$ and that avoids the negative real axis.

In this way one obtains a strongly continuous semi-group of bounded and injective quaternionic linear operators. We want to mention three integral representations for these operators. The first two are completely analogous to the corresponding complex results, the third one is however different from any representation one meets in the complex setting because of the occurrence of the quadratic polynomial in T that comes from the pseudo-resolvent of T.

Proposition 3.40. *Let $n \in \mathbb{N}$. For $\alpha \in (0, n+1)$ with $\alpha \notin \mathbb{N}$, the operator $T^{-\alpha}$ defined in* (3.15) *has the representation*

$$T^{-\alpha} = -\frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^{-\alpha} S_R^{-1}(-t, T)\, dt. \tag{3.16}$$

Corollary 3.41. *For $\alpha \in (0, n+1)$ with $\alpha(0,1)$, the operator $T^{-\alpha}$ defined in* (3.15) *has the representation*

$$T^{-\alpha} = (-1)^{n+1} \frac{\sin(\alpha\pi)}{\pi} \frac{n!}{(n-\alpha)\cdots(1-\alpha)} \int_0^{+\infty} t^{n-\alpha} S_R^{-(n+1)}(-t, T)\, dt. \tag{3.17}$$

Proposition 3.42. *Assume that $\sigma_S(T) \subset \{s \in \mathbb{H} : \mathrm{Re}(s) > 0\}$ and the spectral angle of T is lower or equal to $\pi/2$. For $\alpha \in (0,1)$, one has*

$$T^{-\alpha} = \frac{1}{\pi} \int_0^{\infty} t^{-\alpha} \left(\cos\left(\frac{\alpha\pi}{2}\right) T + \sin\left(\frac{\alpha\pi}{2}\right) t\mathcal{I} \right) (T^2 + t^2)^{-1}\, dt.$$

We want to point out that with these definitions it is possible to establish a theory of interpolation spaces for strongly continuous quaternionic semigroups analogue to the one for complex operator semigroups.

Finally, also the approach chosen by Kato in [34] can be generalized to the quaternionic setting.

Theorem 3.43. *Let $T \in \mathrm{Sect}(\omega)$, let $\alpha \in (0,1)$ and let $\varphi_0 > \max(\alpha\pi, \omega)$. There exists a densely defined closed operator B_α such that*

$$S_R^{-1}(p, B_\alpha) = \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^\alpha (p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha})^{-1} S_R^{-1}(-t, T)\, dt$$

for $p \in \Sigma_{\varphi_0}$.

Since the above integral corresponds to a Cauchy Integral of the form (3.15) for the function $S_R^{-1}(p, s^\alpha)$, the following definition is plausibile

Definition 3.44. Let $T \in \mathrm{Sect}(\omega)$. For $\alpha \in (0,1)$ we define $T^\alpha := B_\alpha$.

Indeed, this definition is consistent with the approaches mentioned previously such that Kato's famous formula for the S-resolvent of fractional powers of operators has an analogue in the quaternionic setting.

4. A new approach to fractional evolution via quaternionic operators

Our new approach uses the following idea: instead of replacing the negative Laplacian in the heat equation by its fractional power, we wanted to replace the gradient in the Fourier law. We saw two main advantages in this approach: on the one hand, this approach would allow a physical interpretation as a modification of the law for heat flow, but it would preserve the law of conservation of energy. On the other hand, this procedure would be applicable to a large class of vector operators, in particular, for instance, to the generalized gradient with variable coefficients

$$T := \begin{pmatrix} \alpha_1(x)\partial_{x_1} \\ \alpha_2(x)\partial_{x_2} \\ \alpha_3(x)\partial_{x_3} \end{pmatrix},$$

which acts as

$$Tv = \begin{pmatrix} \alpha_1(x)\partial_{x_1} v(x) \\ \alpha_2(x)\partial_{x_2} v(x) \\ \alpha_3(x)\partial_{x_3} v(x) \end{pmatrix},$$

where v in this section is a $\mathbb{R}$-valued function. In order to define ∇^α, we would "quaternionify" the problem and consider v as an element of a space of quaternion-valued functions and identify $\mathbb{R}^3$ with the space of purely imaginary quaternions. The gradient ∇ would then correspond to the quaternionic nabla operator (in the Clifford analysis setting called Dirac operator)

$$\nabla = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3,$$

where e_i, $i = 1, 2, 3$ belong to the generating basis of $\mathbb{H}$ and which acts as

$$\nabla v(x) = e_1 \partial_{x_1} v(x) + e_2 \partial_{x_2} v(x) + e_3 \partial_{x_3} v(x).$$

However, we found that this procedure is not directly applicable: in important function spaces such as in the L^2-setting, which we shall keep in mind in the following, the quaternionic operator ∇ is self-adjoint and hence has real S-spectrum. Moreover, we shall see in Remark 4.2 that $\sigma_S(\nabla) = \sigma_S(-\nabla) = -\sigma_S(\nabla)$. Hence, the operator ∇ has spectral values on $(-\infty, 0)$ and is, therefore, not sectorial such that the we cannot use the techniques introduced above in order to define ∇^α. Even worse, since we cannot choose different branches of the logarithm in the slice hyperholomorphic setting, we cannot hope to define ∇^α at all – not even with a different slice hyperholomorphic functional calculus – because s^α is not defined on $\sigma_S(\nabla)$. Nevertheless, we present below a method to generalize the heat equation towards a fractional evolution equation based on these considerations. Again, we will focus on an explanatory perspective and refer to the forthcoming paper [16] for the technical details.

Computing the S-spectrum of the gradient directly is quite complicated. However, the components of the gradient commute and hence one can simplify the computations considerably: for an operator $T = \sum_{i=0}^3 T_i e_i \in \mathcal{B}(V)$ with commuting components on some quaternionic Banach space V , one can set $\overline{T} := \sum_{i=0}^3 T_\ell \overline{e_i} =$

$T_0 - \sum_{i=1}^3 T_i e_i$ and then finds $2T_0 = T + \overline{T}$ and $T\overline{T} = \overline{T}T = \sum_{i=0}^3 T_i^2$, which are both scalar operators and hence commute with any scalar in $\mathbb{H}$. For $s \in \mathbb{H}$, we therefore have

$$\begin{aligned}
&(\overline{s}\mathcal{I} - T)\left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)\\
&= |s|^2 s\mathcal{I} - Ts^2 - 2|s|^2 T_0 + 2TT_0 s + \overline{s}T\overline{T} - T^2\overline{T}\\
&= |s|^2 s\mathcal{I} - Ts^2 - |s|^2 T - |s|^2\overline{T} + T^2 s + T\overline{T}s + \overline{s}T\overline{T} - T^2\overline{T}\\
&= |s|^2\left(s\mathcal{I} - \overline{T}\right) - 2s_0 T\left(s\mathcal{I} - \overline{T}\right) + T^2\left(s\mathcal{I} - \overline{T}\right)\\
&= \left(T^2 - 2s_0 T + |s|^2\mathcal{I}\right)\left(s\mathcal{I} - \overline{T}\right)
\end{aligned}$$

where we used the identities $2s_0 = s + \overline{s}$ and $|s|^2 = s\overline{s}$. With the notation $\mathcal{Q}_s(T) = T^2 - 2s_0 T + |s|^2\mathcal{I}$ and $\mathcal{Q}_{c,s}(T) = s^2\mathcal{I} - 2sT_0 + T\overline{T}$, this reads

$$(\overline{s} - T)\,\mathcal{Q}_{c,s}(T) = \mathcal{Q}_s(T)\left(s\mathcal{I} - \overline{T}\right).$$

If $\mathcal{Q}_s(T)$ and $\mathcal{Q}_{c,s}(T)$ are both invertible, we, therefore, have

$$S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}(\overline{s}\mathcal{I} - T) = \left(s\mathcal{I} - \overline{T}\right)\mathcal{Q}_{c,s}(T)^{-1}.$$

Similarly, one also shows

$$S_R^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}).$$

Indeed, one can show that in case T has commuting components, $\mathcal{Q}_s(T)$ has a bounded inverse if and only if $\mathcal{Q}_{c,s}(T)$ has one [21]. Hence,

$$\rho_S(T) = \left\{s \in \mathbb{H} : \mathcal{Q}_{c,s}(T)^{-1} = \left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)^{-1} \in \mathcal{B}(V)\right\}.$$

Let us now turn back to the gradient and let us consider it as a quaternionic operator on the quaternionic right Hilbert space $L^2(\mathbb{R}^3, \mathbb{H})$ of quaternion-valued functions on $\mathbb{R}^3$ endowed with the scalar product $(f,g) := \int_{\mathbb{R}^3} \overline{f(x)}g(x)\,dx$. We have

$$\nabla^2 = \sum_{i,j=1}^3 \partial_{x_i}\partial_{x_j} e_i e_j = \sum_{i=1}^3 -\partial_{x_i}^2 + \sum_{\substack{1\le i,j\le 3\\ i\neq j}} \partial_{x_i}\partial_{x_j} e_i e_j = \sum_i -\partial_{x_i}^2 = -\Delta,$$

where the last identity follows from $e_i e_j = -e_j e_i$ for $1 \le i, j \le 3$ with $i \neq j$ and Δ denotes the Laplace operator. Since $\nabla_0 = 0$, we have $\overline{\nabla} = -\nabla$ and hence

$$\mathcal{Q}_{c,s}(\nabla) = s^2\mathcal{I} + \Delta = s^2\mathcal{I} - (-\Delta).$$

If we choose $J_s \in \mathbb{S}$ with $J \perp I_s$, then we can write any quaternion-valued function f in the function space we consider as $f = f_1 + f_2 J$ with components f_1, f_2 that take values in $\mathbb{C}_{I_s}$. The component space of all $\mathbb{C}_{I_s}$-valued functions, however, is the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}_{I_s})$ of all square-integrable functions over the field of complex numbers $\mathbb{C}_{I_s}$. This space is invariant under $\mathcal{Q}_{c,s}(\nabla)$ and hence $\mathcal{Q}_{c,s}(\nabla)$ is invertible on all of $L^2(\mathbb{R}^3, \mathbb{H})$ if and only if it is invertible on $L^2(\mathbb{R}^3, \mathbb{C}_{I_s})$. The following theorem holds:

Theorem 4.1. *A quaternion $s \in \mathbb{H}$ belongs to the S-spectrum of ∇ as an operator on $L^2(\mathbb{R}^3, \mathbb{H})$ if and only if s^2 belongs to the spectrum of the negative Laplace operator $-\Delta$ as a complex linear operator on $L^2(\mathbb{R}^3, \mathbb{C}_{I_s})$.*

Remark 4.2. Observe that this implies that $\sigma_S(\nabla)$ is symmetric under reflection across 0 as $s^2 = (-s)^2$ and hence $s \in \sigma_S(\nabla)$ if and only if $-s \in \sigma_S(\nabla)$.

Remark 4.3. We point out that the above relation between the S-spectrum of ∇ and the spectrum of the Laplacian is not restricted to the space $L^2(\mathbb{R}^3, \mathbb{H})$ but holds on any function space whose Banach space structure is compatible with the splitting into the complex component space.

Since the spectrum of the negative Laplacian on $L^2(\mathbb{R}^2, \mathbb{C}_{I_s})$ is $[0, +\infty)$ as one can easily see via the Fourier transform, we find

$$\sigma_S(\nabla) = \mathbb{R}.$$

We choose now $I \in \mathbb{S}$ and set for $\alpha \in (0, 1)$ and sufficiently regular v

$$f_\alpha(\nabla)v := \frac{1}{2\pi} \int_{-I\mathbb{R}} S_L^{-1}(s, T)\, ds_I\, s^{\alpha-1} \nabla v.$$

Intuitively, this corresponds to Balakrishnan's formula for the ∇^α, in which we only take spectral values on the positive real axis into account, i.e., points where ∇^α is actually defined. We find

$$\begin{aligned} f_\alpha(\nabla)v &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-It\mathcal{I} + \nabla) \mathcal{Q}_{c,-It}(\nabla)^{-1} (-I)^2 (-tI)^{\alpha-1} \nabla v\, dt \\ &= -\frac{1}{2\pi} \int_0^{+\infty} (-It\mathcal{I} + \nabla) \mathcal{Q}_{c,-It}(\nabla)^{-1} t^{\alpha-1} e^{-I(\alpha-1)\frac{\pi}{2}} \nabla v\, dt \\ &\quad - \frac{1}{2\pi} \int_0^{+\infty} (It\mathcal{I} + \nabla) \mathcal{Q}_{c,It}(\nabla)^{-1} t^{\alpha-1} e^{I(\alpha-1)\frac{\pi}{2}} \nabla v\, dt. \end{aligned} \tag{4.1}$$

We observe now that

$$\mathcal{Q}_{c,It}(\nabla)^{-1} = (-t^2 + \Delta)^{-1} = \mathcal{Q}_{c,-It}(\nabla)^{-1}$$

and hence

$$\begin{aligned} f_\alpha(\nabla)v &= -\frac{1}{2\pi} \int_0^{+\infty} t^\alpha \mathcal{Q}_{c,It}(\nabla)^{-1} I \left(e^{I(\alpha-1)\frac{\pi}{2}} - e^{-I(\alpha-1)\frac{\pi}{2}} \right) \nabla v\, dt \\ &\quad - \frac{1}{2\pi} \int_0^{+\infty} \nabla \mathcal{Q}_{c,It}(\nabla)^{-1} t^{\alpha-1} \left(e^{I(\alpha-1)\frac{\pi}{2}} + e^{-I(\alpha-1)\frac{\pi}{2}} \right) \nabla v\, dt \\ &= \frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi} \int_0^{+\infty} t^\alpha \mathcal{Q}_{c,It}(\nabla)^{-1} \nabla v\, dt \\ &\quad - \frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi} \int_0^{+\infty} \nabla \mathcal{Q}_{c,It}(\nabla)^{-1} t^{\alpha-1} \nabla v\, dt. \end{aligned} \tag{4.2}$$

For the first integral, we obtain

$$\begin{aligned}
&\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^{\alpha}\mathcal{Q}_{c,it}(\nabla)^{-1}\nabla v\,dt\\
&=\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^{\alpha}(-t^2+\Delta)^{-1}\nabla v\,dt\\
&=\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} \tau^{\frac{\alpha-1}{2}}(-\tau+\Delta)^{-1}\nabla v\,d\tau\\
&=\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v.
\end{aligned}\tag{4.3}$$

The last identity follows from the integral representation for the fractional power A^{β} with $\mathrm{Re}(\beta)\in(0,1)$ of a complex linear sectorial operator A given in [33, Corollary 3.1.4], namely,

$$A^{\beta}v=\frac{\sin(\pi\beta)}{\pi}\int_0^{+\infty}\tau^{\beta}\left(\tau+A^{-1}\right)^{-1}v\,d\tau,\quad v\in\mathcal{D}(A),$$

and the facts that with $-\Delta$ also its closed inverse $(-\Delta)^{-1}$ is a sectorial operator and that $\left((-\Delta)^{-1}\right)^{\alpha}=(-\Delta)^{-\alpha}$. For the second, we obtain in a similar way that

$$\begin{aligned}
&-\frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty}\nabla\mathcal{Q}_{c,it}(\nabla)^{-1}t^{\alpha-1}\nabla v\,dt\\
&=\nabla\frac{\sin\left((\alpha-2)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty}(-t^2+\Delta)^{-1}t^{\alpha-1}\nabla v\,dt\\
&=\nabla\frac{\sin\left((\alpha-2)\frac{\pi}{2}\right)}{2\pi}\int_0^{+\infty}(-\tau+\Delta)^{-1}\tau^{\frac{\alpha-2}{2}}\nabla v\,d\tau\\
&=\frac{1}{2}\nabla(-\Delta)^{\frac{\alpha}{2}-1}\nabla v.
\end{aligned}\tag{4.4}$$

Altogether, we obtain

$$f_{\alpha}(\nabla)v=\left[\frac{1}{2}(-\Delta)^{\frac{1}{2}}+\frac{1}{2}\nabla\right](-\Delta)^{\frac{\alpha}{2}-1}\nabla v.\tag{4.5}$$

Now observe that the vector part of this operator is given by (4.5), i.e.,

$$\mathrm{Vec}\,f_{\alpha}(\nabla)v=\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v.$$

Applying the divergence to this operator, we find

$$\mathrm{div}\,(\mathrm{Vec}\,f_{\alpha}(\nabla)v)=-\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}(-\Delta)v=-\frac{1}{2}(-\Delta)^{\frac{\alpha+1}{2}}.$$

We can thus reformulate the fractional heat equation with $\alpha\in(1/2,1)$ as

$$\partial_t v-2\,\mathrm{div}\,(\mathrm{Vec}\,f_{\beta}(\nabla)v)=0,\quad \beta=2\alpha-1.$$

4.1. An example with nonconstant coefficients

As pointed out before, the advantage of the above procedure is that is does not only apply to the gradient and allows to reproduce the fractional Laplacian, but instead to a large class of vector operators, in particular, general gradients with nonconstant coefficients. As a first example, we consider the operator

$$T := \xi_1 \frac{\partial}{\partial \xi_1} e_1 + \xi_2 \frac{\partial}{\partial \xi_2} e_2 + \xi_3 \frac{\partial}{\partial \xi_3} e_3$$

on the space $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu)$ of $\mathbb{H}$-valued functions on $\mathbb{R}^3_+ = \{(\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3 : \xi_\ell > 0\}$ that are square integrable with respect to $d\mu(\xi) = \frac{1}{\xi_1, \xi_2, \xi_3} d\lambda(\xi)$. In order to determine $\mathcal{Q}_s(T)^{-1}$ we observe that the operator given by the change of variables $J : f \mapsto f \circ \iota$ with $\iota(x) = (e^{x_1}, e^{x_2}, e^{x_3})^T$ is an isometric isomorphism between $L^2(\mathbb{R}^3, \mathbb{H}, d\lambda(x))$ and $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu(\xi))$. Moreover, $T = J^{-1} \nabla J$ such that

$$Q_s(T) = (s^2 \mathcal{I} + T\overline{T}) = J^{-1}(s^2 \mathcal{I} + \Delta) J$$

and, in turn,

$$Q_s(T)^{-1} := (s^2 \mathcal{I} - T\overline{T})^{-1} = J^{-1}(s^2 \mathcal{I} + \Delta)^{-1} J.$$

We, therefore, have for sufficiently regular v with calculations analogue to those in (4.1) and (4.2) that

$$\begin{aligned} f_\alpha(T)v = &\frac{\sin((\alpha-1)\pi)}{\pi} \int_0^{+\infty} t^\alpha (-t^2 + T\overline{T})^{-1} T \, dt \\ &+ \frac{\cos((\alpha-1)\pi)}{\pi} \int_0^{+\infty} t^{\alpha-1} T (-t^2 + T\overline{T})^{-1} T v \, dt. \end{aligned}$$

Clearly, the vector part of this operator is again given by the first integral such that

$$\begin{aligned} \operatorname{Vec} f_\alpha(T)v &= \frac{\sin((\alpha-1)\pi)}{\pi} \int_0^{+\infty} t^\alpha (-t^2 + T\overline{T})^{-1} T v \, dt \\ &= \frac{\sin((\alpha-1)\pi)}{\pi} \int_0^{+\infty} t^\alpha J^{-1} (-t^2 + \Delta)^{-1} J T v \, dt \\ &= J^{-1} \frac{\sin((\alpha-1)\pi)}{\pi} \int_0^{+\infty} t^\alpha (-t^2 + \Delta)^{-1} \, dt \, J T v \\ &= \frac{1}{2} J^{-1} (-\Delta)^{\frac{\alpha-1}{2}} J T v, \end{aligned}$$

where the last equation follows from computations as in (4.5). Choosing $\beta = 2\alpha + 1$ we thus find for sufficiently regular v that

$$\begin{aligned} &\operatorname{Vec} f_\beta(T)v(\xi) \\ &= \frac{1}{2} J^{-1} (-\Delta)^\alpha J T v(\xi_1, \xi_2, \xi_3) \end{aligned}$$

$$= \frac{1}{2} J^{-1} (-\Delta)^{\alpha} \begin{pmatrix} e^{x_1} v_{\xi_1}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_2} v_{\xi_2}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_3} v_{\xi_3}(e^{x_1}, e^{x_2}, e^{x_3}) \end{pmatrix}$$

$$= \frac{1}{2} J^{-1} (-\Delta)^{\alpha} \begin{pmatrix} e^{x_1} v_{\xi_1}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_2} v_{\xi_2}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_3} v_{\xi_3}(e^{x_1}, e^{x_2}, e^{x_3}) \end{pmatrix}$$

$$= \frac{1}{2} J^{-1} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} -|y|^{2\alpha} e^{iz\cdot y} e^{-ix\cdot y} \begin{pmatrix} e^{x_1} v_{\xi_1}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_2} v_{\xi_2}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_3} v_{\xi_3}(e^{x_1}, e^{x_2}, e^{x_3}) \end{pmatrix} dx\, dy$$

$$= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} -|y|^{2\alpha} e^{i \sum_{k=1}^{3} \xi_k y_k} e^{-ix\cdot y} \begin{pmatrix} e^{x_1} v_{\xi_1}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_2} v_{\xi_2}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_3} v_{\xi_3}(e^{x_1}, e^{x_2}, e^{x_3}) \end{pmatrix} dx\, dy.$$

5. Concluding remarks

The development of the theory of slice hyperholomorphic functions, see the books [9, 23, 25, 29], was the crucial step towards the discovery of the S-spectrum, which was inspired by their Cauchy formula. The Cauchy kernels suggested the notion of S-resolvent operators, and the formulations of the S-functional calculus as a consequence, see the original papers [5, 18, 19, 20, 27].

The notion of S-spectrum is the natural notion of spectrum for quaternionic linear operators, which contain as a particular case many vector operators.

The quaternionic formulation of quantum mechanics, see [1, 12], has stimulated the research on the slice hyperholomorphic functional calculus and the spectral theorem based on the S-spectrum, see [6, 7]. Nowadays, there is an application of this theory to fractional evolution problems, see [10, 17]. This calculus also allows for the definition of fractional Maxwell equations.

Using Clifford algebra-valued slice hyperholomorphic functions, one can define the S-functional calculus for n-tuples of not necessarily commuting operators, see [24].

An important direction of research is also Schur analysis in the slice hyperholomorphic setting, where the S-resolvent operators strongly enter into the definition of the realization of Schur functions in this setting, see the book [9].

In the case of quaternionic or vector operators are generators of groups or semigroups, see [8, 22, 30], it is possible to define the function of the generators [4] via the Laplace–Stieltjes transform analogue to the Philips functional calculus.

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Finite-dimensional Self-adjoint Extensions of a Symmetric Operator with Finite Defect and their Compressions

Aad Dijksma and Heinz Langer

Dedicated to our colleague, coauthor and dear friend Daniel Alpay on the occasion of his 60th birthday

Abstract. Let S be a symmetric operator with finite and equal defect numbers d in the Hilbert space $\mathfrak{H}$, and with a boundary triplet $(\mathbb{C}^d, \Gamma_1, \Gamma_2)$. Following the method of E.A. Coddington, we describe all self-adjoint extensions $\widetilde{A}$ of S in a Hilbert space $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}_1$ where $\dim \mathfrak{H}_1 < \infty$. The parameters in this description are matrices $\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}$ and $\mathcal{E}$, where $\mathcal{A}$ and $\mathcal{B}$ determine the compression A_0 of $\widetilde{A}$ to $\mathfrak{H}$. According to a result of W. Stenger, this compression A_0 is self-adjoint. Being a canonical self-adjoint extension of S, A_0 can be chosen as the fixed extension in M.G. Krein's formula for the description of all generalized resolvents of S. Among other results, we describe those parameters which in Krein's formula correspond to self-adjoint extensions of S having A_0 as their compression to $\mathfrak{H}$.

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1. Introduction

1.1. Throughout this paper S is a closed densely defined symmetric operator in some Hilbert space $\mathfrak{H}$ with finite and equal defect numbers $d \in \mathbb{N} = \{1, 2, \dots\}$. We study the self-adjoint extensions $\widetilde{A}$ of S in a larger Hilbert space $\widetilde{\mathfrak{H}}$ where the extending space is of finite dimension $m \in \mathbb{N}$:

$$\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}_1, \quad \dim \mathfrak{H}_1 = m. \tag{1.1}$$

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Denote the orthogonal projection in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ by P. Then, since $\dim \mathfrak{H}_1 < \infty$, by Stenger's lemma (see [19] and Subsection 1.2), the compression $C_{\mathfrak{H}}(\widetilde{A})$ of $\widetilde{A}$ to $\mathfrak{H}$:

$$C_{\mathfrak{H}}(\widetilde{A}) := P\widetilde{A}\big|_{\mathfrak{H}}, \quad \text{that is, } C_{\mathfrak{H}}(\widetilde{A})x = P\widetilde{A}x, \ x \in \operatorname{dom} C_{\mathfrak{H}}(\widetilde{A}) = (\operatorname{dom} \widetilde{A}) \cap \mathfrak{H},$$

is a self-adjoint operator in $\mathfrak{H}$, in fact, it is a canonical self-adjoint extension of S. Among other results, we are interested in a description of all self-adjoint extensions of S in $\widetilde{\mathfrak{H}}$ which have the same compression to $\mathfrak{H}$.

A basic result in the extension theory of symmetric operators is Krein's resolvent formula (see (1.2) below). It describes the set of all generalized resolvents of S, that is, the set of all operator functions $P(\widetilde{A}-z)^{-1}\big|_{\mathfrak{H}}$ in $\mathfrak{H}$, if $\widetilde{A}$ runs through all minimal self-adjoint extensions of S with the extending space being of any (not necessarily finite) dimension, by means of a parameter set which is, roughly, the set of proper or non-proper Nevanlinna $d \times d$ matrix functions; for Krein's formula and related notions see Subsection 1.3 and the Appendix.

The main goal of this paper is a description of the parameters $\mathcal{T}(z)$ that correspond in Krein's formula to the generalized resolvents for which the associated extensions $\widetilde{A}$ of S in the space $\widetilde{\mathfrak{H}}$ from (1.1) have the given canonical self-adjoint extension A_0 of S as their compression: $C_{\mathfrak{H}}(\widetilde{A}) = A_0$.

Our starting point is a closed densely defined symmetric operator S with defect index $(d;d)$ and a boundary triplet $(\mathbb{C}^d, \Gamma_1, \Gamma_2)$. In Theorem 2.3 we describe the self-adjoint extensions $\widetilde{A}$ of S in $\widetilde{\mathfrak{H}}$ from (1.1) by means of $d \times d$ matrices $\mathcal{A}, \mathcal{B}$, $m \times d$ matrices $\mathcal{U}, \mathcal{V}$, and a symmetric $m \times m$ matrix $\mathcal{E}$ as parameters. Here $\mathcal{A}$ and $\mathcal{B}$ determine the compression of $\widetilde{A}$ to $\mathfrak{H}$. This theorem is a special case of the extension theorem [8, Theorem 7.5] by Coddington and the first author, which was formulated even for subspaces (linear relations) in a Banach space setting. It also coincides with Theorem 2.1 of [10], giving a geometric characterization of the extensions. In the present version and proof of this result we make a consequent use of boundary triplets. Theorem 2.1 in [10] was used to derive eigenfunction expansions for ordinary differential operators with boundary conditions which depend polynomially on the eigenvalue parameter. For similar results in extension theory and ordinary differential operators we refer to [5, 6, 7, 15].

In Section 3 we give the block operator matrix representation of the resolvent $(\widetilde{A}-z)^{-1}$ of such an extension $\widetilde{A}$ with respect to the decomposition (1.1) by means of the parameters $\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}$, and $\mathcal{E}$. This representation involves a mapping

$$u_z : \mathfrak{H}_1 = \widetilde{\mathfrak{H}} \ominus \mathfrak{H} \quad \longrightarrow \quad \ker(S^* - z)$$

and a Nevanlinna matrix function $\mathcal{P}(z)$.

Instead of generalized resolvents, also A.V. Straus extensions ([20]) can be used to describe the minimal self-adjoint extensions. In Section 4 we describe these Straus extensions for S through the parameters from Section 2.

In Section 5, upon a suitable reduction, u_z and $\mathcal{P}(z)$ yield a γ-field and a Q-function for the closed densely defined symmetric operator $S_0 := \widetilde{A} \cap \mathfrak{H}^2$ [1] and

[1] We freely consider operators as linear relations and use the corresponding notation.

the compression $A_0 = C_{\mathfrak{H}}(\widetilde{A})$ as canonical self-adjoint extension of S_0. As the main results, in Theorem 5.4 and Theorem 5.5 we derive a characterization of those parameters $\mathcal{T}(z)$ in Krein's formula (1.2) that correspond to the self-adjoint extensions $\widetilde{A}$ of S_0 with $C_{\mathfrak{H}}(\widetilde{A}) = A_0$. Roughly speaking these $\mathcal{T}(z)$ are Schur complements of a z-linear matrix pencil.

In the Appendix for the convenience of the reader we prove some statements related to Krein's formula.

1.2. In this note an essential role is played by Stenger's lemma which states that the compression of a self-adjoint operator in a Hilbert space to a subspace of finite co-dimension is self-adjoint in this subspace. Recently this result was generalized by M.A. Nudelman [18] who proved that the compression of a densely defined maximal dissipative operator in a Hilbert space to a subspace of finite co-dimension is densely defined and maximal dissipative in this subspace. Alternative proofs, converses of these statements and generalizations can be found in [2] and [11]. The results of Stenger and Nudelman are two answers to the question about which properties a compression of an operator T to a subspace of finite co-dimension may or may not inherit from T. More results in this direction can be found in [3, 4]. In the last two papers and also in [11] the results of Stenger and Nudelman are considered in an indefinite setting. Also the results of the present paper can be generalized, for example, to the case where the exit space $\mathfrak{H}_1$ is replaced by a finite-dimensional Pontryagin space as in [10].

1.3. In this subsection we collect some notions and results. A *Nevanlinna* (*matrix*) *function* is a (square matrix) function $\mathcal{P}$, defined and analytic on $\mathbb{C}^+ \cup \mathbb{C}^-$, such that $\mathcal{P}(z^*)^* = \mathcal{P}(z),\ z \in \mathbb{C}^+ \cup \mathbb{C}^-$, and

$$\frac{\mathcal{P}(z) - \mathcal{P}(z)^*}{z - z^*} \ \geq\ 0, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^-.$$

If S is a densely defined symmetric operator in $\mathfrak{H}$ with equal defect numbers, a self-adjoint operator extension A of S is called *canonical* if it acts in the same Hilbert space $\mathfrak{H}$. If A acts in a larger space $\widetilde{\mathfrak{H}}$ it is called an *extension with exit* and $\widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ is the *exit space*. Let $\widetilde{A}$ be a self-adjoint operator extension of S, possibly with exit. It is called *minimal* if

$$\overline{\operatorname{span}}\{(\widetilde{A} - z)^{-1}x :\ x \in \mathfrak{H}, z \in \rho(\widetilde{A})\} = \widetilde{\mathfrak{H}}.$$

The operator function $z \mapsto P(\widetilde{A} - z)^{-1}\big|_{\mathfrak{H}},\ z \in \rho(\widetilde{A})$, whose values are bounded linear operators on $\mathfrak{H}$, is called a *generalized resolvent of* S. Krein's resolvent formula describes the set of all these generalized resolvents of S. Here we recall this description for the case of a densely defined symmetric operator S with finite defect index $(d; d)$. Choose any (canonical) self-adjoint extension A_0 of S in $\mathfrak{H}$, any point $z_0 \neq z_0^*$, and any bijective mapping $\gamma_{z_0} : \mathbb{C}^d \to \ker(S^* - z_0)$. Define a so-called γ-*field* $\gamma_z : \mathbb{C}^d \to \ker(S^* - z)$, *corresponding to* A_0, by

$$\gamma_z := (A_0 - z_0)(A_0 - z)^{-1}\gamma_{z_0}, \quad z \in \rho(A_0),$$

and a *corresponding Q-function* $\mathcal{Q}$, which is a $d\times d$ matrix function, by the relation

$$\frac{\mathcal{Q}(z)-\mathcal{Q}(w)^*}{z-w^*}=\gamma_w^*\gamma_z,\quad z,w\in\rho(A_0).$$

Then the set of all generalized resolvents of S is given by *Krein's formula based on the canonical extension A_0 of S*:

$$P(\widetilde{A}-z)^{-1}\big|_{\mathfrak{H}}=(A_0-z)^{-1}-\gamma_z(\mathcal{Q}(z)+\mathcal{T}(z))^{-1}\gamma_{z^*}^*,\quad z\in\rho(A_0)\cap\rho(\widetilde{A}).\tag{1.2}$$

Here the parameter function $\mathcal{T}$ runs through the set of all families of $d\times d$ relations (or subspaces) $\mathcal{T}$ of the form

$$\mathcal{T}(z)=\Big\{\big\{P_0\mathbf{x},P_0\mathcal{T}_0(z)P_0\mathbf{x}\big\}+\big\{0,(I-P_0)\mathbf{x}\big\}:\mathbf{x}\in\mathbb{C}^d\Big\},\quad z\in\mathbb{C}\setminus\mathbb{R},\tag{1.3}$$

where P_0 is a projection in $\mathbb{C}^d$ onto a d_0-dimensional subspace with $0\le d_0\le d$, and, after identifying $\operatorname{ran}P_0$ with $\mathbb{C}^{d_0}$, $\mathcal{T}_0(z)$ is a Nevanlinna $d_0\times d_0$ matrix function. Then the inverse of the matrix or relation on the right-hand side of (1.2) is

$$\big(\mathcal{Q}(z)+\mathcal{T}(z)\big)^{-1}=P_0\big(P_0\mathcal{Q}(z)P_0+\mathcal{T}_0(z)\big)^{-1}P_0,\quad z\in\mathbb{C}\setminus\mathbb{R},$$

see [16, Theorem 5.1] (or [17, formula (1.8)]). From the proof of (1.2) in [16] it follows that

$$P_0=I\iff\widetilde{A}\cap A_0=S,\tag{1.4}$$

or, equivalently, $\mathcal{T}(z)$ in (1.2) is a Nevanlinna $d\times d$ matrix function (not a relation). For the convenience of the reader we give a proof of (1.4) in Lemma 6.1 of the Appendix.

Two self-adjoint extensions $\widetilde{A}$, A_0 of S with the property $\widetilde{A}\cap A_0=S$ as in (1.4) are called *disjoint*. The exit space $\widetilde{\mathfrak{H}}\ominus\mathfrak{H}$ is of finite dimension m if and only if the matrix function $\mathcal{T}_0(z)$ in (1.3) is rational with the total number of poles equal to m.

Q-functions, also known as Weyl or Titchmarsh–Weyl functions, play an important role in the spectral theory of differential operators. For a recent paper in this area we refer to [1].

2. A parametrization of the extensions in $\widetilde{\mathfrak{H}}$

Let $S\subset S^*$ be a closed densely defined symmetric operator in a Hilbert space $\big(\mathfrak{H},(\,\cdot\,,\,\cdot\,)\big)$ with defect index $(d;d)$. We suppose that for S a *boundary triplet* $(\mathbb{C}^d,\Gamma_1,\Gamma_2)$ is given. Recall that this means that Γ_1 and Γ_2 are linear operators from $\operatorname{dom}S^*$ to $(\mathbb{C}^d,\langle\,\cdot\,,\,\cdot\,\rangle)$ such that

$$(S^*f,g)-(f,S^*g)=\langle\Gamma_2f,\Gamma_1g\rangle-\langle\Gamma_1f,\Gamma_2g\rangle,\quad f,g\in\operatorname{dom}S^*,\tag{2.1}$$

and

$$\text{the mapping } x\mapsto\begin{pmatrix}\Gamma_1x\\ \Gamma_2x\end{pmatrix}\text{ from }\operatorname{dom}S^*\text{ to }\mathbb{C}^{2d}\text{ is surjective.}\tag{2.2}$$

Let $\widetilde{A}$ be a self-adjoint extension of S in some Hilbert space

$$\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}_1, \tag{2.3}$$

where $\mathfrak{H}_1$ is finite-dimensional: $\dim \mathfrak{H}_1 = m$. We identify $\mathfrak{H}_1$ with $\mathbb{C}^m$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and consider, with respect to a chosen basis, the elements of $\mathfrak{H}_1$ as vectors and the operators in $\mathfrak{H}_1$ as matrices. The elements of $\widetilde{\mathfrak{H}}$ are denoted by $\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix}$, $x \in \mathfrak{H}$, $\mathbf{x} \in \mathbb{C}^m$, and P, P_1 are the orthogonal projections in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ and $\mathbb{C}^m$, respectively. According to Stenger's lemma, with $\widetilde{A}$ also the compression $A_0 := C_{\mathfrak{H}}(\widetilde{A})$ is a self-adjoint extension of S. Hence there exist $d \times d$-matrices $\mathcal{A}$, $\mathcal{B}$ with the properties

$$\operatorname{rank}(\mathcal{A} \;\; \mathcal{B}) = m, \quad \mathcal{A}\mathcal{B}^* = \mathcal{B}\mathcal{A}^*, \tag{2.4}$$

and

$$x \in \operatorname{dom} A_0 \quad \Longleftrightarrow \quad x \in \operatorname{dom} S^*, \;\; \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x = 0$$

(see V. Derkach [9], V. and M. Gorbachuk [13, Theorem 3.1.6]). The matrices $\mathcal{A}$ and $\mathcal{B}$ are unique modulo multiplication from the left by an invertible $d \times d$ matrix. Since $\operatorname{dom} A_0$ consists of all $x \in \mathfrak{H}$ such that $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \operatorname{dom} \widetilde{A}$, the first claim of the next lemma follows.

Lemma 2.1. *There exist $m \times m$-matrices $\mathcal{A}, \mathcal{B}$ with the properties* (2.4) *and such that*

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \in \operatorname{dom} \widetilde{A} \quad \Longleftrightarrow \quad x \in \operatorname{dom} S^*, \;\; \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x = 0. \tag{2.5}$$

$\mathcal{A}$ and $\mathcal{B}$ can be chosen such that $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix}$ is J-unitary with $J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

To prove the second claim, consider $\mathcal{A}$ and $\mathcal{B}$ with the properties (2.4) and (2.5), and replace them by $\mathcal{F}\mathcal{A}$ and $\mathcal{F}\mathcal{B}$, respectively, where $\mathcal{F} := (\mathcal{A}\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)^{-1/2}$. It can be checked straightforwardly that the new matrix has the asserted property.

Denote

$$\mathcal{L} := \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : \; x \in \operatorname{dom} A_0 \right\} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : \; x \in \operatorname{dom} S^*, \;\; \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x = 0 \right\}. \tag{2.6}$$

Then

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{L} \;\; \Longrightarrow \;\; \widetilde{A} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} S^* x \\ \mathbf{w}(x) \end{pmatrix}$$

with some linear mapping $\mathbf{w}$ from $\operatorname{dom} A_0$ to $\mathfrak{H}_1$.

Lemma 2.2. *There exist $m \times d$-matrices $\mathcal{U}$, $\mathcal{V}$ such that*

$$\mathbf{w}(x) = \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x, \quad x \in \operatorname{dom} A_0. \tag{2.7}$$

Proof. Since $\widetilde{A} \supset S$ we have

$$x \in \operatorname{dom} S \implies \mathbf{w}(x) = 0,$$

and hence

$$x \in \operatorname{dom} A_0,\ \Gamma_1 x = \Gamma_2 x = 0 \implies \mathbf{w}(x) = 0. \tag{2.8}$$

Consider $\begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}$ as a linear operator from $\operatorname{dom} A_0$, equipped with the graph norm, to $\mathbb{C}^{2d}$. The relation (2.8) implies

$$\begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix} x = 0 \implies \mathbf{w}(x) = 0, \quad \text{that is,} \quad \ker \begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix} \subset \ker \mathbf{w}. \tag{2.9}$$

Further, the ranges of the operators $\mathbf{w}^*$ and $\begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}^*$ are finite-dimensional and therefore closed. Hence (2.9) implies $\operatorname{ran} \begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}^* \supset \operatorname{ran} \mathbf{w}^*$. Now from a result of R.G. Douglas and P.R. Halmos ([12, Theorem 1], see also [14, Problem 59]), it follows that there exist $d \times m$-matrices $\mathcal{U}'$, $\mathcal{V}'$ such that $\mathbf{w}^* = \begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}^* \begin{pmatrix}\mathcal{U}'\\ \mathcal{V}'\end{pmatrix}$, and (2.7) follows with $\mathcal{U} = \mathcal{U}'^*$, $\mathcal{V} = \mathcal{V}'^*$. □

The next theorem is the basic result for the rest of the paper. It describes all self-adjoint extensions of S in the space $\widetilde{\mathfrak{H}}$, which is an m-dimensional extension of $\mathfrak{H}$; for the history of this theorem see the Introduction.

Theorem 2.3. *Let S be a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$ with defect index $(d;d)$ and with a boundary triplet $(\mathbb{C}^d, \Gamma_1, \Gamma_2)$. If $\widetilde{A}$ is a self-adjoint operator extension of S in the larger Hilbert space $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}_1$, then there exist $d \times d$ matrices $\mathcal{A}$ and $\mathcal{B}$ satisfying* (2.4)*, $m \times d$ matrices $\mathcal{U}$ and $\mathcal{V}$, and a symmetric $m \times m$ matrix $\mathcal{E}$, such that*

$$\begin{pmatrix}x\\ \mathbf{x}\end{pmatrix} \in \operatorname{dom} \widetilde{A} \iff \begin{cases} x \in \operatorname{dom} S^*,\ \mathbf{x} \in \mathfrak{H}_1, \\ \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x + (\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*)\mathbf{x} = 0, \end{cases} \tag{2.10}$$

and then

$$\widetilde{A}\begin{pmatrix}x\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}S^* x\\ \mathbf{y}\end{pmatrix} \quad \text{with} \quad \mathbf{y} = \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + \left(\mathcal{E} + \tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*)\right)\mathbf{x}. \tag{2.11}$$

Conversely, given $d \times d$ matrices $\mathcal{A}$ and $\mathcal{B}$ satisfying (2.4)*, $m \times d$ matrices $\mathcal{U}$ and $\mathcal{V}$ and a symmetric $m \times m$ matrix $\mathcal{E}$, then the relations* (2.10) *and* (2.11) *define a self-adjoint operator extension $\widetilde{A}$ of S in $\widetilde{\mathfrak{H}}$.*

Proof. Assume that $\widetilde{A}$ is a self-adjoint operator extension of S in $\widetilde{\mathfrak{H}}$. By Lemma 2.1, there exist $d \times d$ matrices $\mathcal{A}$, $\mathcal{B}$ with the properties (2.4), (2.5):

$$\begin{pmatrix}x\\ 0\end{pmatrix} \in \operatorname{dom} \widetilde{A} \iff x \in \operatorname{dom} S^*,\ \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x = 0$$

and such that $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix}$ is J-unitary. Further, according to Lemma 2.2, there exist $m \times d$ matrices $\mathcal{U}$, $\mathcal{V}$ such that

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \in \operatorname{dom} \widetilde{A} \Longrightarrow \widetilde{A} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} S^* x \\ \mathbf{w}(x) \end{pmatrix} = \begin{pmatrix} S^* x \\ \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x \end{pmatrix}. \tag{2.12}$$

Since $\widetilde{A}$ is densely defined and P_1 is finite-dimensional, the relation $P_1 \operatorname{dom} \widetilde{A} = \mathfrak{H}_1$ follows. With the standard basis $\{\mathbf{e}_j\}_{j=1}^m$ of $\mathfrak{H}_1$ we choose elements $u(\mathbf{e}_j) \in \mathfrak{H}$ such that

$$\begin{pmatrix} u(\mathbf{e}_j) \\ \mathbf{e}_j \end{pmatrix} \in \operatorname{dom} \widetilde{A}, \quad j = 1, \dots, m,$$

and extend u to a linear map from $\mathfrak{H}_1$ to $\mathfrak{H}$. Define $\mathbf{v} : \mathfrak{H}_1 \to \mathfrak{H}_1$ by

$$\mathbf{v}(\mathbf{x}) = P_1 \widetilde{A} \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix}.$$

Thus, for $\mathbf{x} \in \mathfrak{H}_1$,

$$\begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} \in \operatorname{dom} \widetilde{A} \quad \text{and} \quad \widetilde{A} \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} S^* u(\mathbf{x}) \\ \mathbf{v}(\mathbf{x}) \end{pmatrix}. \tag{2.13}$$

We determine $u(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ in terms of the matrices $\mathcal{A}$, $\mathcal{B}$ from the boundary triplet and the matrices $\mathcal{U}$, $\mathcal{V}$. Set

$$\mathcal{L}_1 := \left\{ \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} : \mathbf{x} \in \mathfrak{H}_1 \right\} \subset \operatorname{dom} \widetilde{A}. \tag{2.14}$$

Since $\dim \mathcal{L}_1 = m$ we have

$$\operatorname{dom} \widetilde{A} = \mathcal{L} \dotplus \mathcal{L}_1, \text{ direct sum in } \widetilde{\mathfrak{H}}, \tag{2.15}$$

with $\mathcal{L}$ defined in (2.6). Hence for all $x \in \operatorname{dom} A_0$ and all $\mathbf{x} \in \mathbb{C}^m$ it holds

$$\begin{aligned} 0 = \Bigg(\widetilde{A} \begin{pmatrix} x \\ 0 \end{pmatrix} \; &+ \widetilde{A} \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} \Bigg) \\ &- \left(\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix}, \widetilde{A} \begin{pmatrix} x \\ 0 \end{pmatrix} + \widetilde{A} \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} \right) \\ = \left(\begin{pmatrix} S^* x \\ \mathbf{w}(x) \end{pmatrix} + \begin{pmatrix} S^* u(\mathbf{x}) \\ \mathbf{v}(\mathbf{x}) \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} \right) \\ - \left(\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} S^* x \\ \mathbf{w}(x) \end{pmatrix} + \begin{pmatrix} S^* u(\mathbf{x}) \\ \mathbf{v}(\mathbf{x}) \end{pmatrix} \right) \end{aligned}$$

$$
\begin{aligned}
=(S^*x,x) \quad &+\big(S^*u(\mathbf{x}),x\big)+\big(S^*x,u(\mathbf{x})\big)+\big(S^*u(\mathbf{x}),u(\mathbf{x})\big)\\
&+\big\langle \mathbf{w}(x),\mathbf{x}\big\rangle+\big\langle \mathbf{v}(\mathbf{x}),\mathbf{x}\big\rangle-(x,S^*x)-\big(u(\mathbf{x}),S^*x\big)\\
&-\big(x,S^*u(\mathbf{x})\big)-\big(u(\mathbf{x}),S^*u(\mathbf{x})\big)-\big\langle \mathbf{x},\mathbf{w}(x)\big\rangle-\big\langle \mathbf{x},\mathbf{v}(\mathbf{x})\big\rangle\\
=\big\langle \Gamma_2x,\Gamma_1x\big\rangle &-\big\langle \Gamma_1x,\Gamma_2x\big\rangle+\big\langle \Gamma_2u(\mathbf{x}),\Gamma_1x\big\rangle-\big\langle \Gamma_1u(\mathbf{x}),\Gamma_2x\big\rangle\\
&+\big\langle \Gamma_2x,\Gamma_1u(\mathbf{x})\big\rangle-\big\langle \Gamma_1x,\Gamma_2u(\mathbf{x})\big\rangle\\
&+\big\langle \Gamma_2u(\mathbf{x}),\Gamma_1u(\mathbf{x})\big\rangle-\big\langle \Gamma_1u(\mathbf{x}),\Gamma_2u(\mathbf{x})\big\rangle\\
&+\big\langle \mathbf{w}(x),\mathbf{x}\big\rangle+\big\langle \mathbf{v}(\mathbf{x}),\mathbf{x}\big\rangle-\big\langle \mathbf{x},\mathbf{w}(x)\big\rangle-\big\langle \mathbf{x},\mathbf{v}(\mathbf{x})\big\rangle.
\end{aligned}
$$

The difference of the first two terms vanishes because A_0 is self-adjoint. Setting $x=0$ the relation

$$\big\langle \Gamma_1u(\mathbf{x}),\Gamma_2u(\mathbf{x})\big\rangle-\big\langle \Gamma_2u(\mathbf{x}),\Gamma_1u(\mathbf{x})\big\rangle=\big\langle \mathbf{v}(\mathbf{x}),\mathbf{x}\big\rangle-\big\langle \mathbf{x},\mathbf{v}(\mathbf{x})\big\rangle \tag{2.16}$$

follows. The remaining terms contain x and $\mathbf{x}$, and with a replacement of x by ix it follows that

$$0=\big\langle \Gamma_2x,\Gamma_1u(\mathbf{x})\big\rangle-\big\langle \Gamma_1x,\Gamma_2u(\mathbf{x})\big\rangle+\big\langle \mathbf{w}(x),\mathbf{x}\big\rangle.$$

Finally, (2.7) implies

$$0=\big\langle \Gamma_2x,\Gamma_1u(\mathbf{x})\big\rangle-\big\langle \Gamma_1x,\Gamma_2u(\mathbf{x})\big\rangle+\big\langle \Gamma_1x,\mathcal{U}^*\mathbf{x}\big)\big\rangle+\big\langle \Gamma_2x,\mathcal{V}^*\mathbf{x}\big\rangle.$$

This relation can be written as

$$0=\left(\begin{pmatrix}0&-I\\I&0\end{pmatrix}\begin{pmatrix}\Gamma_1x\\\Gamma_2x\end{pmatrix},\begin{pmatrix}-\Gamma_1u(\mathbf{x})-\mathcal{V}^*\mathbf{x}\\-\Gamma_2u(\mathbf{x})+\mathcal{U}^*\mathbf{x}\end{pmatrix}\right). \tag{2.17}$$

With the J-unitary matrix $\begin{pmatrix}\mathcal{A}&\mathcal{B}\\-\mathcal{B}&\mathcal{A}\end{pmatrix}$ the relation (2.17) yields

$$
\begin{aligned}
0&=\left(\begin{pmatrix}0&-I\\I&0\end{pmatrix}\begin{pmatrix}\mathcal{A}&\mathcal{B}\\-\mathcal{B}&\mathcal{A}\end{pmatrix}\begin{pmatrix}\Gamma_1x\\\Gamma_2x\end{pmatrix},\begin{pmatrix}\mathcal{A}&\mathcal{B}\\-\mathcal{B}&\mathcal{A}\end{pmatrix}\begin{pmatrix}-\Gamma_1u(\mathbf{x})-\mathcal{V}^*\mathbf{x}\\-\Gamma_2u(\mathbf{x})+\mathcal{U}^*\mathbf{x}\end{pmatrix}\right)\\
&=\left(\begin{pmatrix}0&-I\\I&0\end{pmatrix}\begin{pmatrix}0\\-\mathcal{B}\Gamma_1x+\mathcal{A}\Gamma_2x\end{pmatrix},\begin{pmatrix}\mathcal{A}&\mathcal{B}\\-\mathcal{B}&\mathcal{A}\end{pmatrix}\begin{pmatrix}-\Gamma_1u(\mathbf{x})-\mathcal{V}^*\mathbf{x}\\-\Gamma_2u(\mathbf{x})+\mathcal{U}^*\mathbf{x}\end{pmatrix}\right).
\end{aligned}
$$

If x runs through the set $\operatorname{dom}A_0$ the element to the left of the comma runs through a hypermaximal neutral subspace which coincides with its J-orthogonal complement. Therefore also the element to the right of the comma belongs to this subspace, hence

$$\mathcal{A}(-\Gamma_1u(\mathbf{x})-\mathcal{V}^*\mathbf{x})+\mathcal{B}(-\Gamma_2u(\mathbf{x})+\mathcal{U}^*\mathbf{x})=0,$$

or

$$\mathcal{A}\Gamma_1u(\mathbf{x})+\mathcal{B}\Gamma_2u(\mathbf{x})=-\mathcal{A}\mathcal{V}^*\mathbf{x}+\mathcal{B}\mathcal{U}^*\mathbf{x}.$$

The element $u(\mathbf{x})$ is determined by this relation only modulo an element of $\operatorname{dom}A_0$. Since $\dim\big(\operatorname{dom}A_0/\operatorname{dom}S\big)=d$ we can choose it such that, additionally, the relation

$$-\mathcal{B}\Gamma_1u(\mathbf{x})+\mathcal{A}\Gamma_2u(\mathbf{x})=\mathcal{B}\mathcal{V}^*\mathbf{x}+\mathcal{A}\mathcal{U}^*\mathbf{x}$$

holds, or, in matrix form,

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \Gamma_1 u(\mathbf{x}) \\ \Gamma_2 u(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix} \begin{pmatrix} -\mathcal{V}^*\mathbf{x} \\ \mathcal{U}^*\mathbf{x} \end{pmatrix}.$$

Hence the mapping u can be chosen such that

$$\Gamma_1 u(\mathbf{x}) = -\mathcal{V}^*\mathbf{x}, \quad \Gamma_2 u(\mathbf{x}) = \mathcal{U}^*\mathbf{x}, \tag{2.18}$$

and then $u(\mathbf{x})$ is uniquely determined modulo $\operatorname{dom} S$. Now (2.16) implies

$$\begin{aligned} \big\langle \mathbf{v}(\mathbf{x}), \mathbf{x} \big\rangle - \big\langle \mathbf{x}, \mathbf{v}(\mathbf{x}) \big\rangle &= \big\langle \Gamma_1 u(\mathbf{x}), \Gamma_2 u(\mathbf{x}) \big\rangle - \big\langle \Gamma_2 u(\mathbf{x}), \Gamma_1 u(\mathbf{x}) \big\rangle \\ &= -\big\langle \mathcal{V}^*\mathbf{x}, \mathcal{U}^*\mathbf{x} \big\rangle + \big\langle \mathcal{U}^*\mathbf{x}, \mathcal{V}^*\mathbf{x} \big\rangle, \end{aligned}$$

and we obtain

$$\operatorname{Im} \mathbf{v} = \frac{\mathcal{V}\mathcal{U}^* - \mathcal{U}\mathcal{V}^*}{2\,\mathrm{i}}.$$

With $\operatorname{Re} \mathbf{v} =: \mathcal{E}$, a symmetric $m \times m$-matrix which can be arbitrary, the extension $\widetilde{A}$ is determined by $\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}$, and $\mathcal{E}$. Indeed, with the direct sum decomposition (2.15):

$$\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} x - u(\mathbf{x}) \\ 0 \end{pmatrix} + \begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix}, \quad x \in \operatorname{dom} S^*,\ \mathbf{x} \in \mathcal{H}_1,$$

and because of (2.14) , $\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} \in \operatorname{dom} \widetilde{A}$ if and only if $x - u(\mathbf{x}) \in \operatorname{dom} A_0$, which, by (2.5) and (2.18) is equivalent to

$$0 = \mathcal{A}\Gamma_1(x - u(\mathbf{x})) + \mathcal{B}\Gamma_2(x - u(\mathbf{x})) = \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x + (\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*)\mathbf{x}.$$

This proves (2.10). It follows then that, on account of (2.12) and (2.13),

$$\widetilde{A} \begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} S^*(x - u(\mathbf{x})) \\ \mathbf{w}(x - u(\mathbf{x})) \end{pmatrix} + \begin{pmatrix} S^* u(\mathbf{x}) \\ \mathbf{v}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} S^* x \\ \mathbf{y} \end{pmatrix},$$

where, by (2.7), (2.18) and the expressions for the real and imaginary parts of $\mathbf{v}$,

$$\begin{aligned} \mathbf{y} &= \mathbf{w}\big(x - u(\mathbf{x})\big) + \mathbf{v}(\mathbf{x}) \\ &= \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x - \big(\mathcal{U}\Gamma_1 u(\mathbf{x}) + \mathcal{V}\Gamma_2 u(\mathbf{x})\big) + \mathbf{v}(\mathbf{x}) \\ &= \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + \big(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*\big)\mathbf{x} + \big(\mathcal{E} - \tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*)\big)\,\mathbf{x} \\ &= \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + \big(\mathcal{E} + \tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*)\big)\,\mathbf{x}. \end{aligned}$$

This proves the first part of the theorem.

Now we prove the converse. Given $d \times d$ matrices $\mathcal{A}, \mathcal{B}$ with the properties (2.4), $m \times d$ matrices $\mathcal{U}, \mathcal{V}$ and a symmetric $m \times m$ matrix $\mathcal{E}$. We show that a self-adjoint extension $\widetilde{A}$ of S is defined as in Theorem 2.3, that is by the relation

$$\widetilde{A} \begin{pmatrix} x + u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} A_0 x + S^* u(\mathbf{x}) \\ \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + (\mathcal{E} - \mathrm{i}\Delta)\mathbf{x} \end{pmatrix}, \quad x \in \operatorname{dom} A_0,\ \mathbf{x} \in \mathbb{C}^m,$$

with $\Delta = \frac{\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*}{2\mathrm{i}} (= \Delta^*)$. First we indicate a proof of the symmetry. With $x \in \operatorname{dom} A_0$ and $\mathbf{x} \in \mathbb{C}^m$ consider

$$\begin{aligned}
&\left(\begin{pmatrix} A_0 x + S^* u(\mathbf{x}) \\ \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + (\mathcal{E} - \mathrm{i}\Delta)\mathbf{x} \end{pmatrix}, \begin{pmatrix} x + u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} \right) \\
&\qquad - \left(\begin{pmatrix} x + u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} A_0 x + S^* u(\mathbf{x}) \\ \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + (\mathcal{E} - \mathrm{i}\Delta)\mathbf{x} \end{pmatrix} \right) \\
&= \big(A_0 x + S^* u(\mathbf{x}), x + u(\mathbf{x})\big) + \big\langle \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + (\mathcal{E} - \mathrm{i}\Delta)\mathbf{x}, \mathbf{x} \big\rangle \\
&\quad - \big(x + u(\mathbf{x}), A_0 x + S^* u(\mathbf{x})\big) - \big\langle \mathbf{x}, \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + (\mathcal{E} - \mathrm{i}\Delta)\mathbf{x} \big\rangle \\
&= \big(A_0 x, u(\mathbf{x})\big) + \big(S^* u(\mathbf{x}), x\big) + \big(S^* u(\mathbf{x}), u(\mathbf{x})\big) - \big(u(\mathbf{x}), A_0 x\big) - \big(x, S^* u(\mathbf{x})\big) \\
&\quad - \big(u(\mathbf{x}), S^* u(\mathbf{x})\big) + \big\langle \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x - \mathrm{i}\Delta\mathbf{x}, \mathbf{x} \big\rangle - \big\langle \mathbf{x}, \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x - \mathrm{i}\Delta\mathbf{x} \big\rangle.
\end{aligned}$$

Now using that $A_0 x = S^* x$, (2.1) and (2.18) it easily follows that this expression vanishes, and hence $\widetilde{A}$ is symmetric.

To show that $\widetilde{A}$ is self-adjoint we consider the symmetric operator $\widetilde{S}$ in $\widetilde{\mathfrak{H}}$ which is induced by S, that is $\operatorname{dom} \widetilde{S} = \operatorname{dom} S$, considered as a subspace of $\widetilde{\mathfrak{H}}$, and $\widetilde{S}x = Sx$, $x \in \operatorname{dom} \widetilde{S}$. Then $\widetilde{S}$ is a non-densely defined symmetric operator with defect index $(d+m; d+m)$, $\widetilde{A}$ is a canonical symmetric extension of $\widetilde{S}$ in $\widetilde{\mathfrak{H}}$, and we have

$$\widetilde{S} \subset \widetilde{A} \subset \widetilde{A}^* \subset \widetilde{S}^*.$$

Since $\dim \widetilde{S}^* / \widetilde{S} = 2(m+d)$ and $\dim \widetilde{A} / \widetilde{S} = \dim \widetilde{S}^* / \widetilde{A}^* = m + d$, it follows that $\widetilde{A} = \widetilde{A}^*$. □

In the following remarks and corollaries S is a closed densely defined symmetric operator in $\mathfrak{H}$ with defect index $(d; d)$ and a boundary triplet $(\mathbb{C}^d, \Gamma_1, \Gamma_2)$, as in Theorem 2.3.

Remark 2.4. In Theorem 2.3 the set of all self-adjoint extensions $\widetilde{A}$ of S in $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}^m$ is described by $d \times d$ matrices $\mathcal{A}$, $\mathcal{B}$ with the property (2.4), arbitrary $m \times d$ matrices $\mathcal{U}$, $\mathcal{V}$, and an arbitrary symmetric $m \times m$ matrix $\mathcal{E}$ as parameters. Here the pair of matrices $\mathcal{A}$, $\mathcal{B}$ determines the compression A_0 of $\widetilde{A}$ to $\mathfrak{H}$ through

$$\operatorname{dom} A_0 = \{x \in \operatorname{dom} S^* : \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x = 0\}, \quad A_0 x := S^* x.$$

Instead of the pair of matrices $\mathcal{A}$, $\mathcal{B}$, this compression A_0, which is an arbitrary canonical self-adjoint extension of S, can be used as a parameter. If it is fixed, all self-adjoint extensions with this compression are described through the parameters $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{E}$.

Remark 2.5. Let $\widetilde{A}$ be a self-adjoint subspace extension of S with multi-valued part $\widetilde{A}(0) = \{\widetilde{x} : \{0, \widetilde{x}\} \in \widetilde{A}\}$. Then, since S is densely defined in $\mathfrak{H}$, $\widetilde{A}(0) \subset \mathfrak{H}_1$. If $\mathfrak{H}_1^0$ is the orthogonal complement of $\widetilde{A}(0)$ in $\mathfrak{H}_1$, then Theorem 2.3 can be applied to the operator part $\widetilde{A}_{\mathrm{op}}$ of $\widetilde{A}$ in $\widetilde{\mathfrak{H}}^0 = \mathfrak{H} \oplus \mathfrak{H}_1^0$. Thus, after this reduction, Theorem 2.3 yields also a description of all self-adjoint subspace extensions of S in $\widetilde{\mathfrak{H}}$.

Since $\mathfrak{H}_1$ is finite-dimensional, the operator extension $\widetilde{A}$ is minimal if and only if it does not have an eigenvalue with an eigenvector from $\mathfrak{H}_1$. According to Theorem 2.3 the latter means, that there does not exist a real λ such that the system

$$(\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*)\mathbf{x} = 0,$$
$$\left(\mathcal{E} + \tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*)\right)\mathbf{x} - \lambda\mathbf{x} = 0$$

has a non-trivial solution $\mathbf{x}$. Theorem 2.3 therefore implies the following result.

Corollary 2.6. *The self-adjoint operator extension $\widetilde{A}$ of S in Theorem* 2.3 *is minimal if and only if for all $\lambda \in \mathbb{R}$ the $(d+m) \times m$ matrix*

$$\begin{pmatrix} \mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^* \\ \mathcal{E} + \frac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*) - \lambda \end{pmatrix}$$

has rank m. Since $\mathrm{i}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*)$ is a symmetric matrix, this implies that $\widetilde{A}$ is minimal if the $m \times m$ matrix $\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*$ has full rank m. In a similar way, if $\operatorname{rank}\left(\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*\right) = m$ then $\widetilde{A}$ is minimal.

Since, according to Theorem 2.3, for $\mathbf{x} \in \ker \mathcal{U}^* \cap \ker \mathcal{V}^*$ it holds $\widetilde{A}\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{E}\mathbf{x} \end{pmatrix}$, we also have:

Corollary 2.7. *If $\mathcal{L}_0 = (\operatorname{ran}\mathcal{U})^\perp \cap (\operatorname{ran}\mathcal{V})^\perp \left(= \ker \mathcal{U}^* \cap \ker \mathcal{V}^*\right) \neq \{0\}$ and $\mathcal{E}\mathcal{L}_0 \subset \mathcal{L}_0$, then $\mathcal{L}_0$ is a non-trivial invariant subspace of $\widetilde{A}$ and hence $\widetilde{A}$ is not minimal.*

If the compression $C_{\mathfrak{H}}(\widetilde{A})$ is fixed, that is the matrices $\mathcal{A}, \mathcal{B}$ are fixed, in the following theorem we characterize those matrix triplets $\mathcal{U}, \mathcal{V}, \mathcal{E}$ which determine the same extension $\widetilde{A}$.

Theorem 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be $d \times d$ matrices which satisfy* (2.4). *For $j = 1, 2$, denote by $\widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}_j, \mathcal{V}_j, \mathcal{E}_j)$ the self-adjoint extension of S determined by the parameters $\mathcal{A}, \mathcal{B}$, the $m \times d$ matrices $\mathcal{U}_j, \mathcal{V}_j$ and the symmetric $m \times m$ matrix $\mathcal{E}_j$ as described by Theorem* 2.3. *Then $\widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}_1, \mathcal{V}_1, \mathcal{E}_1) = \widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}_2, \mathcal{V}_2, \mathcal{E}_2)$ if and only if there exists an $m \times d$ matrix $\mathcal{K}$ such that, with $\mathcal{M} := \mathcal{K}(\mathcal{A}\mathcal{V}_1^* - \mathcal{B}\mathcal{U}_1^*)$,*

$$\mathcal{U}_2 = \mathcal{U}_1 + \mathcal{K}\mathcal{A}, \quad \mathcal{V}_2 = \mathcal{V}_1 + \mathcal{K}\mathcal{B}, \quad \mathcal{E}_2 = \mathcal{E}_1 + \tfrac{1}{2}(\mathcal{M} + \mathcal{M}^*). \tag{2.19}$$

Proof. We use the abbreviation $\widetilde{A}_j := \widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}_j, \mathcal{V}_j, \mathcal{E}_j)$, $j = 1, 2$. Assume $\widetilde{A}_1 = \widetilde{A}_2$. For $\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix}$ in the domain of these operators we have

$$\mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x + (\mathcal{A}\mathcal{V}_j^* - \mathcal{B}\mathcal{U}_j^*)\mathbf{x} = 0, \ j = 1, 2. \tag{2.20}$$

Since for every $\mathbf{x} \in \mathfrak{H}_1$ there is an $x \in \operatorname{dom} S^*$ such that $\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix}$ satisfies (2.20), we conclude that for all $\mathbf{x} \in \mathbb{C}^m$

$$\mathcal{A}(\mathcal{V}_2 - \mathcal{V}_1)^*\mathbf{x} - \mathcal{B}(\mathcal{U}_2 - \mathcal{U}_1)^*\mathbf{x} = 0.$$

Now (2.4) implies that for every $\mathbf{x} \in \mathbb{C}^m$ there is a unique $\mathbf{y} \in \mathbb{C}^d$ such that

$$\begin{pmatrix} (\mathcal{V}_2 - \mathcal{V}_1)^* \\ (\mathcal{U}_2 - \mathcal{U}_1)^* \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathcal{B}^* \\ \mathcal{A}^* \end{pmatrix} \mathbf{y};$$

we set $y = \mathcal{K}\mathbf{x}$. This readily implies the first two equalities in (2.19). If we apply $\widetilde{A}_1$ and $\widetilde{A}_2$ to $\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} \in \operatorname{dom} \widetilde{A}_1 = \operatorname{dom} \widetilde{A}_2$, we obtain

$$\begin{aligned} \mathcal{U}_1\Gamma_1 x + \mathcal{V}_1\Gamma_2 x + \big(\mathcal{E}_1 + \tfrac{1}{2}(\mathcal{U}_1\mathcal{V}_1^* - \mathcal{V}_1\mathcal{U}_1^*)\big)\mathbf{x} \\ = \mathcal{U}_2\Gamma_1 x + \mathcal{V}_2\Gamma_2 x + \big(\mathcal{E}_2 + \tfrac{1}{2}(\mathcal{U}_2\mathcal{V}_2^* - \mathcal{V}_2\mathcal{U}_2^*)\big)\mathbf{x}. \end{aligned}$$

From this, (2.20) and (2.4) it follows that

$$\begin{aligned} (\mathcal{E}_2 - \mathcal{E}_1)\mathbf{x} &= -\mathcal{K}(\mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x) + \tfrac{1}{2}(\mathcal{U}_1\mathcal{V}_1^* - \mathcal{V}_1\mathcal{U}_1^*)\mathbf{x} - \tfrac{1}{2}(\mathcal{U}_2\mathcal{V}_2^* - \mathcal{V}_2\mathcal{U}_2^*)\mathbf{x} \\ &= \big(\mathcal{K}(\mathcal{A}\mathcal{V}_1^* - \mathcal{B}\mathcal{U}_1^*) - \tfrac{1}{2}\mathcal{K}(\mathcal{A}\mathcal{V}_1^* - \mathcal{B}\mathcal{U}_1^*) - \tfrac{1}{2}(\mathcal{U}_1\mathcal{B}^* - \mathcal{V}_1\mathcal{A}^*)\mathcal{K}^*\big)\,\mathbf{x} \\ &= \tfrac{1}{2}(\mathcal{M} + \mathcal{M}^*)\mathbf{x}. \end{aligned}$$

This proves the last equality in the theorem.

Conversely, (2.19) yields, by straightforward calculation, $\widetilde{A}_1 = \widetilde{A}_2$. □

If the parameters $\mathcal{A}, \mathcal{B}, \mathcal{U}$ and $\mathcal{V}$ are fixed, then

$$\text{the map } \mathcal{E} \mapsto \widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}, \mathcal{E}) \text{ is injective.} \tag{2.21}$$

Indeed,

$$\widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}, \mathcal{E}_1) = \widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}, \mathcal{E}_2) \quad \Longrightarrow \quad \mathcal{E}_2 = \mathcal{E}_1,$$

because then in (2.19) with $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$ and $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$ the matrix $\mathcal{K}$ is the zero matrix. In this sense the matrix $\mathcal{E}$ is a "completely free" parameter. Minimality is not stable under the map (2.21) as the following example shows.

Example 2.9. Take $m = d = 2$ and

$$\mathcal{A} = \mathcal{B} = \mathcal{V} = \mathcal{I}_2,\ \mathcal{U} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathcal{U}^*,\ \mathcal{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ \mathcal{E}_2 = 0.$$

Then, by Corollary 2.6, the operator $\widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}, \mathcal{E}_1)$ is minimal, because

$$\operatorname{rank} \begin{pmatrix} \mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^* \\ \mathcal{E}_1 + \frac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*) - \lambda \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \\ 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = 2 \text{ for all } \lambda \in \mathbb{R}.$$

The operator $\widetilde{A}(\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}, \mathcal{E}_2)$ is not minimal, because

$$\operatorname{rank} \begin{pmatrix} \mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^* \\ \mathcal{E}_2 + \frac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*) - \lambda \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \\ -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 1 \text{ for } \lambda = 0.$$

In this case $\widetilde{A}(\mathcal{A},\mathcal{B},\mathcal{U},\mathcal{V},\mathcal{E}_2)$ has an eigenvector in $\mathfrak{H}_1$:

$$\widetilde{A}(\mathcal{A},\mathcal{B},\mathcal{U},\mathcal{V},\mathcal{E}_2)\begin{pmatrix}0\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}0\\ \mathbf{0}\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}.$$

3. A block operator matrix representation of the resolvent

In this section S and $\widetilde{A}$, the boundary triplet $(\mathbb{C}^d,\Gamma_1,\Gamma_2)$, the compression A_0, and also the matrices $\mathcal{A},\mathcal{B},\mathcal{U},\mathcal{V}$, and $\mathcal{E}$ are as in Section 2. We establish a block operator matrix representation of the resolvent $\widetilde{R}(z)=(\widetilde{A}-z)^{-1}$ with respect to the decomposition (2.3) of $\widetilde{\mathfrak{H}}$.

To this end, with the mapping $u:\mathbb{C}^m\longrightarrow \operatorname{dom} S^*\subset\mathfrak{H}$ as given by (2.18):

$$\Gamma_1 u(\mathbf{x})=-\mathcal{V}^*\mathbf{x},\quad \Gamma_2 u(\mathbf{x})=\mathcal{U}^*\mathbf{x},$$

we define for $z\in\rho(A_0)$ the mapping u_z by

$$u_z(\mathbf{x}):=\big(I-R_0(z)(S^*-z)\big)u(\mathbf{x}),\quad \mathbf{x}\in\mathbb{C}^m,\ z\in\rho(A_0),\tag{3.1}$$

and also an $m\times m$ matrix function $\mathcal{P}$ by

$$\mathcal{P}(z)\mathbf{x}:=-\mathcal{U}\Gamma_1 u_z(\mathbf{x})-\mathcal{V}\Gamma_2 u_z(\mathbf{x})-\tfrac{1}{2}(\mathcal{U}\mathcal{V}^*-\mathcal{V}\mathcal{U}^*)\mathbf{x},\quad \mathbf{x}\in\mathbb{C}^m,\ z\in\rho(A_0).\tag{3.2}$$

These notions are also considered in [10, Section 4]. Item (d) of the next proposition coincides with [10, Lemma 4.1 (b)].

Proposition 3.1. (a) *The operator u_z maps $\mathbb{C}^m$ into* $\ker(S^*-z)$ *and*

$$u_z=(A_0-w)(A_0-z)^{-1}u_w\ \big(=\big(I+(z-w)R_0(z)\big)u_w\big),\quad z,w\in\rho(A_0).$$

(b) $u_z^*=-\mathcal{V}\Gamma_2 R_0(z^*)-\mathcal{U}\Gamma_1 R_0(z^*),\quad z\in\rho(A_0).$

(c) $\dfrac{\mathcal{P}(z)-\mathcal{P}(w)^*}{z-w^*}=u_w^*u_z,\quad z,w\in\rho(A_0),$
and hence $\mathcal{P}$ is a Nevanlinna $m\times m$ matrix function.

(d) $\rho(A_0)\cap\sigma_p(\widetilde{A})=\{\lambda\in\rho(A_0)\,:\,\det\big(\mathcal{P}(\lambda)-\mathcal{E}+\lambda\big)=0\}.$

Proof. To prove (a) we first observe that

$$(S^*-z)u_z=\big((S^*-z)-(A_0-z)R_0(z)(S^*-z)\big)u=0.$$

Further,

$$\begin{aligned}(A_0-w)R_0(z)u_w-u_z&=(A_0-w)R_0(z)\big(I-R_0(w)(S^*-w)\big)u-\big(I-R_0(z)(S^*-z)\big)u\\&=(A_0-w)R_0(z)u-R_0(z)(-w)u-\big(I-R_0(z)(-z)\big)u\\&=0,\end{aligned}$$

and claim (a) follows. If $x \in \mathfrak{H}$ and $\mathbf{x} \in \mathbb{C}^m$ then with (2.18) and (2.1) it follows that

$$\begin{aligned}
-\big\langle (\mathcal{V}\Gamma_2 R_0(z) + \mathcal{U}\Gamma_1 R_0(z))x, \mathbf{x}\big\rangle &= -\big\langle \Gamma_2 R_0(z)x, \mathcal{V}^*\mathbf{x}\big\rangle - \big\langle \Gamma_1 R_0(z)x, \mathcal{U}^*\mathbf{x}\big\rangle \\
&= \big(\Gamma_2 R_0(z)x, \Gamma_1 u(\mathbf{x})\big) - \big(\Gamma_1 R_0(z)x, \Gamma_2 u(\mathbf{x})\big) \\
&= \big(S^* R_0(z)x, u(\mathbf{x})\big) - \big(R_0(z)x, S^* u(\mathbf{x})\big) \\
&= \big((S^* + z - z)R_0(z)x, u(\mathbf{x})\big) - \big(R_0(z)x, S^* u(\mathbf{x})\big) \\
&= \big(x, u(\mathbf{x})\big) - \big(x, R_0(z^*)(S^* - z^*)u(\mathbf{x})\big) \\
&= (x, u_z(\mathbf{x})),
\end{aligned}$$

and this proves (b).

To prove (c) we use (2.18), (2.1), and (3.1) to obtain

$$\begin{aligned}
\big\langle \mathcal{P}(z)\mathbf{x}, \mathbf{x}\big\rangle &= \big\langle -\mathcal{U}\Gamma_1 u_z(\mathbf{x}) - \mathcal{V}\Gamma_2 u_z(\mathbf{x}) - \tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*)\mathbf{x}, \mathbf{x}\big\rangle \\
&= \big\langle -\mathcal{U}\Gamma_1\big(u_z(\mathbf{x}) - \tfrac{1}{2}u(\mathbf{x})\big) - \mathcal{V}\Gamma_2\big(u_z(\mathbf{x}) - \tfrac{1}{2}u(\mathbf{x})\big), \mathbf{x}\big\rangle \\
&= \big(S^*\big(u_z(\mathbf{x}) - \tfrac{1}{2}u(\mathbf{x})\big), u(\mathbf{x})\big) - \big(u_z(\mathbf{x}) - \tfrac{1}{2}u(\mathbf{x}), S^* u(\mathbf{x})\big) \\
&= \tfrac{1}{2}\big(S^* u(\mathbf{x}), u(\mathbf{x})\big) - \big((S^* - z + z)R_0(z)(S^* - z)u(\mathbf{x}), u(\mathbf{x})\big) \\
&\quad - \tfrac{1}{2}\big(u(\mathbf{x}), S^* u(\mathbf{x})\big) + \big(R_0(z)(S^* - z)u(\mathbf{x}), S^* u(\mathbf{x})\big) \\
&= \tfrac{1}{2}\big(S^* u(\mathbf{x}), u(\mathbf{x})\big) - \tfrac{1}{2}\big(u(\mathbf{x}), S^* u(\mathbf{x})\big) - \big((S^* - z)u(\mathbf{x}), u(\mathbf{x})\big) \\
&\quad + \big(R_0(z)(S^* - z)u(\mathbf{x}), (S^* - z^*)u(\mathbf{x})\big) \\
&= -\tfrac{1}{2}\big((S^* u(\mathbf{x}), u(\mathbf{x})) + (u(\mathbf{x}), S^* u(\mathbf{x}))\big) + z\big(u(\mathbf{x}), u(\mathbf{x})\big) \\
&\quad + \big(R_0(z)(S^* - z)u(\mathbf{x}), (S^* - z^*)u(\mathbf{x})\big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\big\langle (\mathcal{P}(z) - \mathcal{P}(w)^*)\mathbf{x}, \mathbf{x}\big\rangle \\
&\quad = (z - w^*)\big(u(\mathbf{x}), u(\mathbf{x})\big) + \big(R_0(z)(S^* - z)u(\mathbf{x}), (S^* - z^*)u(\mathbf{x})\big) \\
&\qquad - \big((S^* - w^*)u(\mathbf{x}), R_0(w)(S^* - w)u(\mathbf{x})\big) \\
&\quad = (z - w^*)\big(u(\mathbf{x}), u(\mathbf{x})\big) + \big(R_0(z)(S^* - z)u(\mathbf{x}), (S^* - w)u(\mathbf{x})\big) \\
&\qquad - (z - w^*)\big(R_0(z)(S^* - z)u(\mathbf{x}), u(\mathbf{x})\big) \\
&\qquad - \big(R_0(w)^*(S^* - z)u(\mathbf{x}), (S^* - w)u(\mathbf{x})\big) \\
&\qquad - (z - w^*)\big(u(\mathbf{x}), R_0(w)(S^* - w)u(\mathbf{x})\big) \\
&\quad = (z - w^*)\big(u(\mathbf{x}), u(\mathbf{x})\big) - (z - w^*)\big(R_0(z)(S^* - z)u(\mathbf{x}), u(\mathbf{x})\big) \\
&\qquad - (z - w^*)\big(u(\mathbf{x}), R_0(w)(S^* - w)u(\mathbf{x})\big) \\
&\qquad + (z - w^*)\big(R_0(z)(S^* - z)u(\mathbf{x}), R_0(w)(S^* - w)u(\mathbf{x})\big) \\
&\quad = (z - w^*)\big(u(\mathbf{x}) - R_0(z)(S^* - z)u(\mathbf{x}), u(\mathbf{x}) - R_0(z)(S^* - z)u(\mathbf{x})\big) \\
&\quad = (z - w^*)\big(u_z(\mathbf{x}), u_w(\mathbf{x})\big).
\end{aligned}$$

This proves item (c). We now prove item (d). Assume $\lambda \in \rho(A_0) \cap \sigma_p(\widetilde{A})$. Then there is a non-zero element $\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} \in \operatorname{dom} \widetilde{A}$ such that

$$S^*x = \lambda x, \quad \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x + \big(\mathcal{E} + \tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*)\big)\,\mathbf{x} = \lambda \mathbf{x}. \tag{3.3}$$

From

$$0 = \mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x + (\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*)\mathbf{x} = \mathcal{A}\Gamma_1\big(x - u(\mathbf{x})\big) + \mathcal{B}\Gamma_2\big(x - u(\mathbf{x})\big)$$

it follows that $x - u(\mathbf{x}) \in \operatorname{dom} A_0$. We claim that $\mathbf{x} \neq 0$: Assume it is zero. Then $x \in \operatorname{dom} A_0$. From $A_0 x = S^* x = \lambda x$ and $\lambda \in \rho(A_0)$ it follows that $x = 0$. Hence $0 \neq \begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} = 0$ and this contradiction implies the claim. From

$$(A_0 - \lambda)(x - u(\mathbf{x})) = (S^* - \lambda)\big(x - u(\mathbf{x})\big) = (S^* - \lambda)u(\mathbf{x})$$

it follows that $x = u_\lambda(\mathbf{x})$. Substituting this in the second equality in (3.3) we obtain the equality

$$(\mathcal{P}(\lambda) - \mathcal{E} + \lambda)\mathbf{x} = 0_{m\times 1}$$

which implies, since $\mathbf{x} \neq 0$, that $\det(\mathcal{P}(\lambda) - \mathcal{E} + \lambda) = 0$.

Conversely, assume $\lambda \in \rho(A_0)$ and $\det(\mathcal{P}(\lambda) - \mathcal{E} + \lambda) = 0$. Then there is a non-zero vector $\mathbf{x} \in \mathbb{C}^m$ such that $(\mathcal{P}(\lambda) - \mathcal{E} + \lambda)\mathbf{x} = 0$ and after some straightforward checking we find that $\begin{pmatrix} u_\lambda(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} \in \operatorname{dom} \widetilde{A}$ and that it is an eigenelement of $\widetilde{A}$ with eigenvalue λ. Hence $\lambda \in \rho(A_0) \cap \sigma_p(\widetilde{A})$. □

Remark 3.2. If, in the notation of Theorem 2.8, the $m \times m$ matrix function $\mathcal{P}_j(z)$ is associated with the parameters $\mathcal{A}, \mathcal{B}, \mathcal{U}_j, \mathcal{V}_j$, and $\mathcal{E}_j$, $j = 1, 2$, in the same way as $\mathcal{P}(z)$ in (3.2) is associated with the parameters $\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}$, and $\mathcal{E}$, then

$$\mathcal{P}_2(z) = \mathcal{P}_1(z) + \tfrac{1}{2}(\mathcal{M} + \mathcal{M}^*) \text{ with } \mathcal{M} = \mathcal{K}(\mathcal{A}\mathcal{V}_1^* - \mathcal{B}\mathcal{U}_1^*). \tag{3.4}$$

Indeed, let $u_j : \mathbb{C}^m \to \operatorname{dom} S^* \subset \mathfrak{H}$ be defined such that

$$\Gamma_1 u_j(\mathbf{x}) = -\mathcal{V}_j^* \mathbf{x}, \quad \Gamma_2 u_j(\mathbf{x}) = \mathcal{U}_j^* \mathbf{x}, \quad \mathbf{x} \in \mathbb{C}^m,\ j = 1, 2.$$

Then, by (2.4) and (2.19),

$$\mathcal{A}\Gamma_1(u_2(\mathbf{x}) - u_1(\mathbf{x})) + \mathcal{B}\Gamma_2(u_2(\mathbf{x}) - u_1(\mathbf{x})) = 0$$

and hence $u_2(\mathbf{x}) - u_1(\mathbf{x}) \in \operatorname{dom} A_0$ for all $\mathbf{x} \in \mathbb{C}^m$. This readily implies that $u_{2z} = u_{1z}$ for all $z \in \rho(A_0)$. Straightforward calculations now yield the equality (3.4).

Theorem 3.3. *The resolvent operator $\widetilde{R}(z) = (\widetilde{A} - \lambda)^{-1}$ of the self-adjoint extension $\widetilde{A}$ on $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}^m$ described in Theorem* 2.3 *admits the block operator matrix*

representation

$$\widetilde{R}(z) = \begin{pmatrix} R_0(z) - u_z\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} u_{z^*}^* & -u_z\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} \\ -\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} u_{z^*}^* & -\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} \end{pmatrix} \tag{3.5}$$
$$= \begin{pmatrix} R_0(z) & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} u_z \\ I \end{pmatrix} \big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} \begin{pmatrix} u_{z^*}^* & I \end{pmatrix}, \quad z \in \rho(\widetilde{A}) \cap \rho(A_0).$$

Proof. For $z \in \rho(\widetilde{A}) \cap \rho(A_0)$, $f, g \in \mathfrak{H}$ and $\mathbf{f}, \mathbf{g} \in \mathbb{C}^m$ assume

$$\widetilde{R}(z) \begin{pmatrix} f \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}. \tag{3.6}$$

The block matrix representation is obtained by expressing g and $\mathbf{g}$ in terms of f and $\mathbf{f}$. The right-hand side of (3.6) belongs to $\operatorname{dom} \widetilde{A}$ and

$$(\widetilde{A} - z) \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} f \\ \mathbf{f} \end{pmatrix},$$

that is,

$$\begin{cases} (S^* - z)g = f, \\ \mathcal{A}\Gamma_1\big(g - u(\mathbf{g})\big) + \mathcal{B}\Gamma_2\big(g - u(\mathbf{g})\big) = 0_{d\times 1}, \\ \mathcal{U}\Gamma_1 g + \mathcal{V}\Gamma_2 g + \big(\mathcal{E} + \frac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*) - z\big)\mathbf{g} = \mathbf{f}. \end{cases} \tag{3.7}$$

By the second equality in (3.7), $g - u(\mathbf{g}) \in \operatorname{dom} A_0$, and then from the first equality, written as

$$(A_0 - z)\big(g - u(\mathbf{g})\big) + (S^* - z)u(\mathbf{g}) = f,$$

we obtain that

$$g = R_0(z)f + u_z(\mathbf{g}). \tag{3.8}$$

The third relation in (3.7), (3.8) and item (b) of Proposition 3.1 yield

$$\begin{aligned} \mathbf{f} &= \mathcal{U}\Gamma_1 R_0(z)f + \mathcal{V}\Gamma_2 R_0(z)f + \mathcal{U}\Gamma_1 u_z(\mathbf{g}) + \mathcal{V}\Gamma_2 u_z(\mathbf{g}) \\ &\quad + \big(\mathcal{E} + \tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*) - z\big)\mathbf{g} \\ &= -u_{z^*}^* f - \big(\mathcal{P}(z) - \mathcal{E} + z\big)\mathbf{g}, \end{aligned}$$

and hence, by item (d) of Proposition 3.1,

$$\mathbf{g} = -\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} u_{z^*}^* f - \big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1}\mathbf{f}.$$

This equality leads to the bottom row of the block matrix for $\widetilde{R}(z)$ and to the top row by substituting it in formula (3.8) for g. □

Remark 3.4. Note that the expression in the left upper corner in the representation (3.5) is the generalized resolvent of $\widetilde{A}$:

$$P(\widetilde{A} - z)^{-1}\big|_{\mathfrak{H}} = (A_0 - z)^{-1} - u_z\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} u_{z^*}^*. \tag{3.9}$$

In Krein's formula (1.2) the matrix function in the second term on the right-hand side is of size $d \times d$, whereas in (3.9) the corresponding term has size $m \times m$. This term and u_z in (3.9) are reduced to obtain formula (5.6); see Proposition 5.3.

Remark 3.5. If for $z \in \mathbb{C} \setminus \mathbb{R}$ the mappings u_z and the $m \times m$ matrices $\mathcal{P}(z)$ have the properties (a) and (c) of Proposition 3.1 and $\mathcal{E}$ is a symmetric $m \times m$ matrix, then $\widetilde{R}(z)$ as defined by the block operator matrix (3.5) in Theorem 3.3 is the resolvent of a self-adjoint operator extension of S in $\widetilde{\mathfrak{H}}$ with $C_{\mathfrak{H}}(\widetilde{A}) = A_0$.

Indeed, the operator function $\widetilde{R}(z)$ satisfies

$$\widetilde{R}(z)^* = \widetilde{R}(z^*), \quad \widetilde{R}(z) - \widetilde{R}(w) = (z-w)\widetilde{R}(z)\widetilde{R}(w)$$

and

$$\widetilde{R}(z)\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} = 0 \quad \Longrightarrow \quad x = 0,\ \mathbf{x} = 0.$$

Hence

$$\widetilde{A} := \left\{ \left\{ \widetilde{R}(z)\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} + z\widetilde{R}(z)\begin{pmatrix} x \\ \mathbf{x} \end{pmatrix} \right\} : x \in \mathfrak{H},\ \mathbf{x} \in \mathfrak{H}_1 \right\}$$

has the asserted properties.

In the following proposition we give equivalent formulations for the disjointness of $\widetilde{A}$ and its compression A_0, also in terms of the parameters.

Proposition 3.6. *The following statements are equivalent:*

(i) $\widetilde{A} \cap A_0 = S$, *that is, the extensions* $\widetilde{A}$ *and* A_0 *of* S *are disjoint.*
(ii) $u_z : \mathbb{C}^m \to \ker(S^* - z)$ *is surjective for one (and then for all)* $z \in \mathbb{C} \setminus \mathbb{R}$.
(iii) $\operatorname{rank}\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{U} & \mathcal{V} \end{pmatrix} = 2d$.
(iv) $\operatorname{rank}(\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*) = d$.

Proof. (ii)⇒(i) Assume u_z is surjective for some $z \in \mathbb{C}\setminus\mathbb{R}$. To show that $\widetilde{A} \cap A_0 = S$ it suffices to show that $\operatorname{dom}(\widetilde{A} \cap A_0) \subset \operatorname{dom} S$. Let $x \in \operatorname{dom}(\widetilde{A} \cap A_0)$. Then $x \in \operatorname{dom} A_0$ and for all $\mathbf{x} \in \mathbb{C}^m$

$$\begin{aligned} 0 &= -\big\langle \mathcal{U}\Gamma_1 x + \mathcal{V}\Gamma_2 x, \mathbf{x} \big\rangle \\ &= \big(S^*x, u(\mathbf{x})\big) - \big(x, S^*u(\mathbf{x})\big) \\ &= \big(S^*x, u_z(\mathbf{x})\big) - \big(x, S^*u_z(\mathbf{x})\big) \\ &= \big(S^*x, u_z(\mathbf{x})\big) - \big(x, zu_z(\mathbf{x})\big) \\ &= \big((A_0 - z^*)x, u_z(\mathbf{x})\big). \end{aligned}$$

The surjectivity of u_z implies that

$$(A_0 - z^*)x \in \ker(S^* - z)^\perp = \operatorname{ran}(S - z^*).$$

Hence there exist a $y \in \operatorname{dom} S$ such that $(A_0 - z^*)x = (S - z^*)y = (A_0 - z^*)y$. It follows that $x - y$ is an eigenvector of A_0 with eigenvalue z^*. Since A_0 is self-adjoint and z non-real, we conclude $x = y \in \operatorname{dom} S$.

(i)⇒(ii) Now we assume $\widetilde{A} \cap A_0 = S$. Consider a point $z \in \mathbb{C} \setminus \mathbb{R}$. We show that u_z is surjective. Consider $x \in \ker(S^* - z) \cap (\operatorname{ran} u_z)^\perp$. To prove that u_z is surjective, it is enough to show that $x = 0$. Set $y = R_0(z^*)x$. Then $y \in \operatorname{dom} A_0$ and the above chain of equalities applied to y shows that $\mathcal{U}\Gamma_1 y + \mathcal{V}\Gamma_2 y = 0$, that

is $y \in \operatorname{dom}(\widetilde{A} \cap A_0) = \operatorname{dom} S$. It follows that $x = (S - z^*)y \in \ker(S^* - z) = \operatorname{ran}(S - z^*)^\perp$, hence $x = 0$.

The equivalence of (iii) with (i) and (ii) follows from (2.2), $\ker \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \operatorname{dom} S$ and the fact that $x \in \operatorname{dom}(\widetilde{A} \cap A_0)$ if and only if $x \in \operatorname{dom} S^*$ and

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{U} & \mathcal{V} \end{pmatrix} \begin{pmatrix} \Gamma_1 x \\ \Gamma_2 x \end{pmatrix} = 0.$$

The equivalence (iii)$\Longleftrightarrow$(iv) follows from (2.4) and from multiplying the matrix in item (iii) from the right by the invertible matrix

$$\begin{pmatrix} \mathcal{A}^* & -\mathcal{B}^* \\ \mathcal{B}^* & \mathcal{A}^* \end{pmatrix}.$$

For then we obtain that

$$\operatorname{rank} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{U} & \mathcal{V} \end{pmatrix} = \operatorname{rank}(\mathcal{A}\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) + \operatorname{rank}(\mathcal{V}\mathcal{A}^* - \mathcal{U}\mathcal{B}^*) = d + \operatorname{rank}(\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*).$$

The left-hand side equals $2d$ if and only if $\operatorname{rank}(\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*) = d$. □

4. Straus extension

Recall that the *Straus extension* corresponding to the self-adjoint extension $\widetilde{A}$, acting in the Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$, of the symmetric operator S in $\mathfrak{H}$ is the operator function $T(z)$, $z \in \rho(\widetilde{A})$, in $\mathfrak{H}$, such that

$$\operatorname{dom} T(z) \subset \operatorname{dom} S^*, \quad P(\widetilde{A} - z)^{-1}\big|_{\mathfrak{H}} = (T(z) - z)^{-1}, \quad z \in \rho(\widetilde{A}).$$

In this section we describe the Straus extensions of the operator $\widetilde{A}$ in Theorem 2.3.

Lemma 4.1. *There exists a unique holomorphic $d \times m$ matrix function $\mathcal{L}$ on $\rho(A_0)$ such that for $z \in \rho(A_0)$ and $\mathbf{x} \in \mathfrak{H}_1$*

$$\begin{aligned} \Gamma_1 R_0(z)(S^* - z)u(\mathbf{x}) &= \mathcal{B}^*\mathcal{L}(z)\mathbf{x}, \\ \Gamma_2 R_0(z)(S^* - z)u(\mathbf{x}) &= -\mathcal{A}^*\mathcal{L}(z)\mathbf{x}, \end{aligned} \tag{4.1}$$

and then

$$\begin{aligned} \Gamma_1 u_z(\mathbf{x}) &= -(\mathcal{V}^* + \mathcal{B}^*\mathcal{L}(z))\mathbf{x}, \\ \Gamma_2 u_z(\mathbf{x}) &= (\mathcal{U}^* + \mathcal{A}^*\mathcal{L}(z))\mathbf{x}, \\ u_z^*(S^* - z^*)g &= -(\mathcal{U}^* + \mathcal{A}^*\mathcal{L}(z))^*\Gamma_1 g - (\mathcal{V}^* + \mathcal{B}^*\mathcal{L}(z))^*\Gamma_2 g, \ g \in \operatorname{dom} S^*, \\ \mathcal{P}(z) &= (\mathcal{U}\mathcal{B}^* - \mathcal{V}\mathcal{A}^*)\mathcal{L}(z) + \frac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*). \end{aligned}$$

Proof. Let $\mathbf{x} \in \mathfrak{H}_1$ and $z \in \rho(A_0)$. Since $R_0(z)(S^* - z)u(\mathbf{x}) \in \operatorname{dom} A_0$,

$$\mathcal{A}\Gamma_1 R_0(z)(S^* - z)u(\mathbf{x}) + \mathcal{B}\Gamma_2 R_0(z)(S^* - z)u(\mathbf{x}) = 0_{d\times 1}.$$

From $\mathcal{A}\mathcal{B}^* - \mathcal{B}\mathcal{A}^* = 0$ and rank $\begin{pmatrix}\mathcal{A} & \mathcal{B}\end{pmatrix} = d$ it follows that there is a unique $d \times m$ matrix function $\mathcal{L}(z)$ such that

$$\begin{pmatrix}\Gamma_1 R_0(z)(S^* - z)u(\mathbf{x})\\ \Gamma_2 R_0(z)(S^* - z)u(\mathbf{x})\end{pmatrix} = \begin{pmatrix}\mathcal{B}^*\\ -\mathcal{A}^*\end{pmatrix}\mathcal{L}(z)\mathbf{x}.$$

In fact, $\mathcal{L}(z)$ is given by

$$\mathcal{L}(z)\mathbf{x} = (\mathcal{A}\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)^{-1}\big(\mathcal{B}\Gamma_1 R_0(z)(S^* - z)u(\mathbf{x}) - \mathcal{A}\Gamma_2 R_0(z)(S^* - z)u(\mathbf{x})\big). \quad (4.2)$$

and this formula shows that $\mathcal{L}$ is holomorphic on $\rho(A_0)$. This proves the first part of the lemma. Substituting (4.1) in the defining formulas for u_z and $\mathcal{P}(z)$ we obtain the first, the second and the fourth relation in the second part of the lemma. The third relation follows from the first two and is obtained as follows:

$$\begin{aligned}\big\langle u_z^*(S^* - z)g, \mathbf{x}\big\rangle &= \big((S^* - z)g, u_z(\mathbf{x})\big)\\ &= \big(S^*g, u_z(\mathbf{x})\big) - \big(g, S^*u_z(\mathbf{x})\big)\\ &= \big\langle \Gamma_2 g, \Gamma_1 u_z(\mathbf{x})\big\rangle - \big\langle \Gamma_1 g, \Gamma_2 u_z(\mathbf{x})\big\rangle\\ &= -\big\langle \big(\mathcal{V}^* + \mathcal{B}^*\mathcal{L}(z)\big)^*\Gamma_2 g, \mathbf{x}\big\rangle - \big\langle \big(\mathcal{U}^* + \mathcal{A}^*\mathcal{L}(z)\big)^*\Gamma_1 g, \mathbf{x}\big\rangle.\\ &= -\big\langle \big(\mathcal{U}^* + \mathcal{A}^*\mathcal{L}(z)\big)^*\Gamma_1 g + \big(\mathcal{V}^* + \mathcal{B}^*\mathcal{L}(z)\big)^*\Gamma_2 g, \mathbf{x}\big\rangle. \qquad \square\end{aligned}$$

Theorem 4.2. *Let $\widetilde{A}$ in $\widetilde{\mathfrak{H}}$ be a self-adjoint extension of S described in Theorem* 2.3. *Then the Straus extension $T(z)$ of S, $z \in \rho(A_0) \cap \rho(\widetilde{A})$, determined by $\widetilde{A}$ is the restriction of S^* to the domain*

$$\operatorname{dom} T(z) = \Big\{g \in \operatorname{dom} S^* \,:\, \mathcal{A}_s(z)\Gamma_1 g + \mathcal{B}_s(z)\Gamma_2\, g = 0_{d\times 1}\Big\},$$

where, with the $d \times m$ matrix functions $\mathcal{U}_s(z) = (\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*)(\mathcal{P}(z) - \mathcal{E} + z)^{-1}$ and $\mathcal{L}(z)$ as in Lemma 4.1,

$$\begin{aligned}\mathcal{A}_s(z) &= \mathcal{A} + \mathcal{U}_s(z)(\mathcal{U} + \mathcal{L}(z^*)^*\mathcal{A}),\\ \mathcal{B}_s(z) &= \mathcal{B} + \mathcal{U}_s(z)(\mathcal{V} + \mathcal{L}(z^*)^*\mathcal{B}),\end{aligned}$$

and the kernel $\dfrac{\mathcal{A}_s(z)\mathcal{B}_s(w)^* - \mathcal{B}_s(z)\mathcal{A}_s(w)^*}{z - w^*}$ *is non-negative, in fact,*

$$\frac{\mathcal{A}_s(z)\mathcal{B}_s(w)^* - \mathcal{B}_s(z)\mathcal{A}_s(w)^*}{z - w^*} = \mathcal{U}_s(z)\mathcal{U}_s(w)^*.$$

Remark 4.3. Since $\lim_{y\to\infty} R_0(iy)(S^* - iy) = I$, we derive from (4.2) and (3.2)

$$\begin{aligned}\lim_{y\to\infty} \mathcal{L}(iy) &= -(\mathcal{A}\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)(\mathcal{A}\mathcal{U}^* + \mathcal{B}\mathcal{V}^*),\\ \lim_{y\to\infty} \mathcal{P}(iy) &= -\tfrac{1}{2}(\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*),\\ \lim_{y\to\infty} \mathcal{U}_s(iy) &= 0_{d\times m},\end{aligned}$$

and hence $\lim_{y\to\infty} \mathcal{A}_s(iy) = \mathcal{A}$ and $\lim_{y\to\infty} \mathcal{B}_s(iy) = \mathcal{B}$. In this limit sense

$$T(i\infty) := \lim_{y\to\infty} T(iy) = A_0 = \mathcal{C}_{\widetilde{\mathfrak{H}}}(\widetilde{A}).$$

For a similar result, see [11, Theorem 3.3].

Proof of Theorem 4.2. From $g = (T(z)-z)^{-1}f = P\widetilde{R}(z)f$, $f\in\mathfrak{H}$, and Theorem 3.3 we obtain

$$T(z)g = f + zg, \quad g = R_0(z)f - u_z(\mathcal{P}(z) - \mathcal{E} + z)^{-1}u_{z^*}^* f.$$

Applying $(S^* - z)$ to both sides of the last equality we find $(S^* - z)g = f$, hence $T(z)g = S^*g$ and

$$g + u_z(\mathcal{P}(z) - \mathcal{E} + z)^{-1}u_{z^*}^*(S^* - z)g = R_0(z)f \in \operatorname{dom} A_0.$$

If we apply $\mathcal{A}\Gamma_1 + \mathcal{B}\Gamma_2$ to $R_0(z)f$ we get 0. Using the formulas for $\Gamma_1 u_z$, $\Gamma_2 u_z$, and $u_{z^*}^*(S^* - z)g$ as in Lemma 4.1 and the formula for $\mathcal{U}_s(z)$ we find

$$\begin{aligned}
0 &= \mathcal{A}\Gamma_1 g + \mathcal{B}\Gamma_2 g - \big(-\mathcal{A}\mathcal{V}^* + \mathcal{B}\mathcal{U}^*\big)\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1} \\
&\qquad \times \Big(\big(\mathcal{U}^* + \mathcal{A}^*\mathcal{L}(z^*)\big)^*\Gamma_1 g + \big(\mathcal{V}^* + \mathcal{B}^*\mathcal{L}(z^*)\big)^*\Gamma_2 g\Big) \\
&= \mathcal{A}\Gamma_1 g + \mathcal{B}\Gamma_2 g + \mathcal{U}_s(z)\Big(\big(\mathcal{U} + \mathcal{L}(z^*)^*\mathcal{A}\big)\Gamma_1 g + \big(\mathcal{V} + \mathcal{L}(z^*)^*\mathcal{B}\big)\Gamma_2 g\Big) \\
&= \mathcal{A}_s(z)\Gamma_1 g + \mathcal{B}_s(z)\Gamma_2 g.
\end{aligned}$$

This proves the inclusion

$$T(z) \subseteq \Big\{\{g, S^*g\} \,:\, g \in \operatorname{dom} S^*, \mathcal{A}_s(z)\Gamma_1 g + \mathcal{B}_s(z)\Gamma_2\, g = 0_{d\times 1}\Big\}.$$

The reverse inclusion can be shown by tracing the above arguments and calculations backwards. The kernel formula follows from direct calculations, the formula for $\mathcal{P}(z)$ in Lemma 4.1 and $\mathcal{P}(z) = \mathcal{P}(z^*)^*$. □

5. Krein's resolvent formula

Let $\widetilde{A}$ in $\widetilde{\mathfrak{H}}$ be a self-adjoint operator extension of S as described by Theorem 2.3 with parameters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{E}$ and let A_0 be its compression to $\mathfrak{H}$. With $\mathcal{U}$ and $\mathcal{V}$ there is defined a linear map $u : \mathfrak{H}_1 \to \operatorname{dom} S^* \subset \mathfrak{H}$ as in (2.18):

$$\Gamma_1 u(\mathbf{x}) = -\mathcal{V}^*\mathbf{x}, \quad \Gamma_2 u(\mathbf{x}) = \mathcal{U}^*\mathbf{x}, \quad \mathbf{x} \in \mathfrak{H}_1. \tag{5.1}$$

Then u is unique modulo elements of $\operatorname{dom} S$ and it has the property that

$$\begin{pmatrix} u(\mathbf{x}) \\ \mathbf{x} \end{pmatrix} \in \operatorname{dom}\widetilde{A}, \quad \mathbf{x} \in \mathfrak{H}_1.$$

Define the non-negative integer d_0 by

$$d_0 := \operatorname{rank}(\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*). \tag{5.2}$$

Then, since $\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*$ is a $d \times m$ matrix,

$$d_0 \le \min(d, m).$$

Since, by (5.1),

$$(\mathcal{A}\Gamma_1 + \mathcal{B}\Gamma_2)u(\mathbf{x}) = -(\mathcal{A}\mathcal{V}^* - \mathcal{B}\mathcal{U}^*)\mathbf{x}$$

and $x \in \operatorname{dom} A_0$ if and only if $\mathcal{A}\Gamma_1 x + \mathcal{B}\Gamma_2 x = 0$, d_0 can be characterized as

$$d_0 = \dim\big((\operatorname{dom} A_0 + \{u(\mathbf{x}) \colon \mathbf{x} \in \mathbb{C}^m\})/\operatorname{dom} A_0\big) = \dim\big(\{u(\mathbf{x}) \colon \mathbf{x} \in \mathbb{C}^m\}/\operatorname{dom} A_0\big).$$

We define the operator S_0 in $\mathfrak{H}$ by

$$S_0 := \widetilde{A} \cap \mathfrak{H}^2.$$

Then S_0 is a symmetric extension of S with defect index $(d_0; d_0)$ and

$$\begin{aligned} S_0 &= \Big\{\{h, S^*h\} \,:\, h \in \operatorname{dom} S^*,\ \mathcal{A}\Gamma_1 h + \mathcal{B}\Gamma_2 h = 0,\ \mathcal{U}\Gamma_1 h + \mathcal{V}\Gamma_2 h = 0\Big\} \\ &= \Big\{\{h, S^*h\} \,:\, h \in \operatorname{dom} A_0,\ \mathcal{U}\Gamma_1 h + \mathcal{V}\Gamma_2 h = 0\Big\} \\ &= A_0 \cap \widetilde{A}. \end{aligned}$$

That is, S_0 is the maximal symmetric extension of S in $\mathfrak{H}$ which is contained in $\widetilde{A}$, and it is at the same time the common part of $\widetilde{A}$ and A_0.

The following lemma splits $\mathbb{C}^m$ into two subspaces. One subspace is mapped by u_z into $\operatorname{dom} A_0$, and the range of u_z on the other subspace has a zero intersection with $\operatorname{dom} A_0$.

Lemma 5.1. *There exist matrices $\mathcal{N}_1$ and $\mathcal{N}_2$ of sizes $m \times d_0$ and $m \times (m - d_0)$, respectively, such that*

(1) *the $m \times m$ matrix $\mathcal{N} := \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \end{pmatrix}$ is invertible,*
(2) *if $\widehat{u}(\mathbf{x}) := u(\mathcal{N}_1\mathbf{x})$, $\mathbf{x} \in \mathbb{C}^{d_0}$, then $\operatorname{dom} A_0 \cap \operatorname{ran} \widehat{u} = \{0\}$,*
(3) *$u(\mathcal{N}_2\mathbf{x}) \in \operatorname{dom} A_0$ for all $\mathbf{x} \in \mathbb{C}^{m-d_0}$,*
(4) *the map*

$$\widehat{u}_z(\mathbf{x}) := u_z(\mathcal{N}_1\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}^{d_0}, \tag{5.3}$$

is a bijection from $\mathbb{C}^{d_0}$ onto $\ker(S_0^* - z)$,
(5) *$u_z(\mathcal{N}_2\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{C}^{m-d_0}$, and*
(6) *for $z, w \in \rho(A_0)$ and $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$, $\mathbf{x}_1 \in \mathbb{C}^{d_0}$, $\mathbf{x}_2 \in \mathbb{C}^{m-d_0}$ we have*

$$\big\langle \mathcal{N}^*\big(\mathcal{P}(z) - \mathcal{P}(w)^*\big)\mathcal{N}\mathbf{x}, \mathbf{x}\big\rangle = (z - w^*)\big(\widehat{u}_z(\mathbf{x}_1), \widehat{u}_w(\mathbf{x}_1)\big).$$

Proof. Properties (1)–(3) of the matrices $\mathcal{N}_1$ and $\mathcal{N}_2$ and their indicated sizes follow from the definition of d_0. Items (4) and (5) can be proved using $u_z(\mathbf{x}) \in \ker(S^* - z)$, $\mathbf{x} \in \mathbb{C}^m$, and the following equivalences for $\mathbf{x} \in \mathbb{C}^m$:

$$\exists z \in \rho(A_0) : u_z(\mathbf{x}) = 0 \iff u(\mathbf{x}) \in \operatorname{dom} A_0 \iff u_z(\mathbf{x}) = 0\ \forall z \in \rho(A_0).$$

Item (6) follows from Proposition 3.1 and items (4) and (5). □

Using the notation of Lemma 5.1 we define the $d_0 \times d_0$ matrix function $\widehat{\mathcal{P}}$ by

$$\widehat{\mathcal{P}}(z) := \mathcal{N}_1^*\mathcal{P}(z)\mathcal{N}_1, \quad z \in \rho(A_0). \tag{5.4}$$

Proposition 5.2. *Let $\widetilde{A}$ in $\widetilde{\mathfrak{H}}$ be a self-adjoint operator extension of S as described by Theorem 2.3, and let $A_0 := C_{\mathfrak{H}}(\widetilde{A})$ be its compression to $\mathfrak{H}$. Then the matrix function $\widehat{\mathcal{P}}(z)$ is the Q-function for S_0 and A_0 with γ-field $\widehat{u}_z$:*

$$\frac{\widehat{\mathcal{P}}(z)-\widehat{\mathcal{P}}(w)^*}{z-w^*}=\widehat{u}_w^*\widehat{u}_z,\quad \widehat{u}_w=(A_0-z)(A_0-w)^{-1}\widehat{u}_z,\quad z,w\in\rho(A_0). \tag{5.5}$$

Proof. The defining relations (5.5) for $\widehat{\mathcal{P}}(z)$ to be the Q-function and for $\widehat{u}_z$ to be the γ-field associated with S_0 and A_0 follow directly from Lemma 5.1 and Proposition 3.1. □

Krein's resolvent formula for the symmetric operator S_0 with defect index $(d_0;d_0)$ and based on the canonical self-adjoint extension $A_0=C_{\mathfrak{H}}(\widetilde{A})$ reads now as follows:

$$P(\widetilde{A}-z)^{-1}\big|_{\mathfrak{H}}=(A_0-z)^{-1}-\widehat{u}_z\big(\widehat{\mathcal{P}}(z)+\widehat{\mathcal{T}}(z)\big)^{-1}\widehat{u}_{z^*}^*. \tag{5.6}$$

Here $\widehat{u}_z$ and $\widehat{\mathcal{P}}(z)$ are given by (5.3) and (5.4), and if $m=d_0$ they specialize to $\widehat{u}_z=u_z$ and $\widehat{\mathcal{P}}(z)=\mathcal{P}(z)$ as in (3.1) and (3.2). The parameter $\widehat{\mathcal{T}}(z)$ is a Nevanlinna $d_0\times d_0$ matrix function, which will be characterized in Proposition 5.3.

To formulate this proposition, with the symmetric $m\times m$ matrix $\mathcal{E}$ from Theorem 2.3 and

$$\Delta=\frac{\mathcal{U}\mathcal{V}^*-\mathcal{V}\mathcal{U}^*}{2\mathrm{i}},$$

we introduce the block Nevanlinna $m\times m$ matrix function

$$\begin{pmatrix}\mathcal{F}(z) & \mathcal{H}(z)\\ \mathcal{H}(z^*)^* & \mathcal{G}(z)\end{pmatrix}:=\begin{pmatrix}\mathcal{N}_1^*\\ \mathcal{N}_2^*\end{pmatrix}\big(z\mathcal{I}_m-\mathcal{E}\big)\begin{pmatrix}\mathcal{N}_1 & \mathcal{N}_2\end{pmatrix}+\begin{pmatrix}0 & -\mathrm{i}\mathcal{N}_1^*\Delta\mathcal{N}_2\\ \mathrm{i}\mathcal{N}_2^*\Delta\mathcal{N}_1 & 0\end{pmatrix}, \tag{5.7}$$

where $\mathcal{F}(z)$ and $\mathcal{G}(z)$ are the $d_0\times d_0$ and $(m-d_0)\times(m-d_0)$ matrix functions given by

$$\mathcal{F}(z)\;=\;\mathcal{N}_1^*(z\mathcal{I}_m-\mathcal{E})\mathcal{N}_1,\quad \mathcal{G}(z)\;=\;\mathcal{N}_2^*(z\mathcal{I}_m-\mathcal{E})\mathcal{N}_2,$$

and $\mathcal{H}(z)=\mathcal{N}_1^*(z\mathcal{I}_m-\mathcal{E})\mathcal{N}_2-\mathrm{i}\mathcal{N}_1^*\Delta\mathcal{N}_2$.

Its *first Schur complement* (see [21])

$$\widehat{\mathcal{T}}(z):=\mathcal{F}(z)-\mathcal{H}(z)\mathcal{G}(z)^{-1}\mathcal{H}(z^*)^*, \tag{5.8}$$

which is the inverse of the left upper component of $\begin{pmatrix}\mathcal{F}(z) & \mathcal{H}(z)\\ \mathcal{H}(z^*)^* & \mathcal{G}(z)\end{pmatrix}^{-1}$, is a Nevanlinna $d_0\times d_0$ matrix function.

Proposition 5.3. *Let S be a densely defined symmetric operator in $\mathfrak{H}$ with defect index $(d;d)$, let $\widetilde{A}$ in $\widetilde{\mathfrak{H}}=\mathfrak{H}\oplus\mathbb{C}^m$ be a minimal self-adjoint extension of S with parameters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{E}$ as in Theorem 2.3, and let $A_0=C_{\mathfrak{H}}(\widetilde{A})$ be the compression of $\widetilde{A}$ to $\mathfrak{H}$. Consider the symmetric operator $S_0=A_0\cap\widetilde{A}$ with defect index $(d_0;d_0)$ where $d_0\,(\le m)$ is given by (5.2). In Krein's resolvent formula (5.6) for S_0, based on A_0, the $d_0\times d_0$ matrix function $\widehat{\mathcal{T}}(z)$ corresponding to the generalized resolvent $P(\widetilde{A}-z)^{-1}\big|_{\mathfrak{H}}$ is as follows:*

(i) *If $m = d_0$, then*
$$\widehat{\mathcal{T}}(z) = z\mathcal{I}_m - \mathcal{E}.$$

(ii) *If $m > d_0$, then $\widehat{\mathcal{T}}(z)$ is the first Schur complement* (5.8) *of the z-linear matrix pencil in* (5.7).

A more explicit formula for $\widehat{\mathcal{T}}(z)$ in case (ii) is as follows:

$$\widehat{\mathcal{T}}(z) = \mathcal{N}_1^*\Big(z\mathcal{I}_m - \mathcal{E} - \big((z\mathcal{I}_m - \mathcal{E})\mathcal{N}_2 - \mathrm{i}\Delta\mathcal{N}_2\big)\big(\mathcal{N}_2^*(z\mathcal{I}_m - \mathcal{E})\mathcal{N}_2\big)^{-1}\big(\mathcal{N}_2^*(z\mathcal{I}_m - \mathcal{E}) + \mathrm{i}\mathcal{N}_2^*\Delta\big)\Big)\mathcal{N}_1.$$

Clearly, if $m = d_0$ then $\widehat{T}(z)$ has a pole of multiplicity m at ∞, if $m > d_0$ it has a pole of multiplicity d_0 at ∞ and also real poles of total multiplicity $m - d_0$ at the eigenvalues of the matrix pencil $\mathcal{N}_2^*(z\mathcal{I}_m - \mathcal{E})\mathcal{N}_2$ in $\mathbb{C}^{m-d_0}$.

Proof of Proposition 5.3. We first note that by the last relations in Lemma 4.1 and Lemma 5.1 (5) for $\mathbf{x} \in \mathbb{C}^m$ it holds

$$\begin{aligned}\mathcal{P}(z)\mathcal{N}_2\mathbf{x} &= \mathcal{U}\big(\mathcal{V}^* + \mathcal{B}^*L(z)\big)\mathcal{N}_2\mathbf{x} - \mathcal{V}\big(\mathcal{U}^* + \mathcal{A}^*L(z)\big)\mathcal{N}_2\mathbf{x} + \tfrac{1}{2}\big(\mathcal{V}\mathcal{U}^* - \mathcal{U}\mathcal{V}^*\big)\mathcal{N}_2\mathbf{x}\\ &= \mathcal{U}\Gamma_1 u_z(\mathcal{N}_2\mathbf{x}) - \mathcal{V}\Gamma_2 u_z(\mathcal{N}_2\mathbf{x}) + \tfrac{1}{2}\big(\mathcal{V}\mathcal{U}^* - \mathcal{U}\mathcal{V}^*\big)\mathcal{N}_2\mathbf{x}\\ &= -\mathrm{i}\Delta\mathcal{N}_2\mathbf{x}\end{aligned}$$

and

$$\mathcal{N}_2^*\mathcal{P}(z)\mathcal{N}_2 = \big(\mathcal{N}_2^*\mathcal{P}(z^*)\mathcal{N}_2\big)^* = -\mathcal{N}_2^*\mathcal{P}(z)\mathcal{N}_2 = 0.$$

Hence

$$\mathcal{N}^*\mathcal{P}(z)\mathcal{N} = \begin{pmatrix}\widehat{\mathcal{P}}(z) & \mathcal{N}_1^*\mathcal{P}(z)\mathcal{N}_2\\ \mathcal{N}_2^*\mathcal{P}(z)\mathcal{N}_1 & \mathcal{N}_2^*\mathcal{P}(z)\mathcal{N}_2\end{pmatrix} = \begin{pmatrix}\widehat{\mathcal{P}}(z) & -\mathrm{i}\mathcal{N}_1^*\Delta\mathcal{N}_2\\ \mathrm{i}\mathcal{N}_2^*\Delta\mathcal{N}_1 & 0\end{pmatrix},$$

where only the left upper component depends on z. To obtain a formula for $\widehat{\mathcal{T}}(z)$ (and, at the same time, to give a direct proof of (5.6)) we use (3.9) and rewrite the last summand on the right-hand side:

$$\begin{aligned}u_z\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1}u_{z^*}^* &= u_z\mathcal{N}\mathcal{N}^{-1}\big(\mathcal{P}(z) - \mathcal{E} + z\big)^{-1}\mathcal{N}^{-*}\mathcal{N}^*u_{z^*}^*\\ &= \begin{pmatrix}\widehat{u}_z & 0\end{pmatrix}\big(\mathcal{N}^*\mathcal{P}(z)\mathcal{N} - \mathcal{N}^*\mathcal{E}\mathcal{N} + z\mathcal{N}^*\mathcal{N}\big)^{-1}\begin{pmatrix}\widehat{u}_{z^*}^*\\ 0\end{pmatrix}\\ &= \begin{pmatrix}\widehat{u}_z & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathcal{P}}(z) + \mathcal{F}(z) & \mathcal{H}(z)\\ \mathcal{H}(z^*)^* & \mathcal{G}(z)\end{pmatrix}^{-1}\begin{pmatrix}\widehat{u}_{z^*}^*\\ 0\end{pmatrix}\\ &= \widehat{u}_z(\widehat{\mathcal{P}}(z) + \widehat{\mathcal{T}}(z))^{-1}\widehat{u}_{z^*}^*\end{aligned}$$

with $\widehat{\mathcal{T}}(z)$ from (5.8). □

In the following theorem, if $m \geq d$, the generalized resolvents of the minimal self-adjoint extensions $\widetilde{A}$ with an m-dimensional exit space and $C_{\mathfrak{H}}(\widetilde{A}) = A_0$ and which are disjoint from A_0 are described by Krein's formula (1.2):

$$P(\widetilde{A} - z)^{-1}\big|_{\mathfrak{H}} = (A_0 - z)^{-1} - \gamma_z\big(\mathcal{Q}(z) + \mathcal{T}(z)\big)^{-1}\gamma_{z^*}^*. \tag{5.9}$$

If $m < d$, then the set of self-adjoint extensions with the properties (5.10) below is empty because of Proposition 3.6(ii); hence the assumption $m \geq d$ is no restriction.

Theorem 5.4. *Let S be a closed densely defined symmetric operator in $\mathfrak{H}$ with defect index $(d;d)$, let A_0 be a canonical self-adjoint extension of S, and suppose that $m \geq d$. Then Krein's formula* (5.9) *for S, based on A_0, establishes a bijective correspondence between*

the set of all minimal self-adjoint extensions $\widetilde{A}$ of S in $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}^m$ which satisfy

$$C_{\mathfrak{H}}(\widetilde{A}) = A_0 \quad \text{and} \quad \widetilde{A} \cap A_0 = S \tag{5.10}$$

and

the set of all Nevanlinna $d \times d$ matrix functions $\mathcal{T}(z)$ of the form

$$\mathcal{T}(z) = \mathcal{X}_{11}(z) - \mathcal{X}_{12}(z)\mathcal{X}_{22}(z)^{-1}\mathcal{X}_{21}(z); \tag{5.11}$$

here $\mathcal{X}_{ij}(z)$ are the entries of the block matrix of an arbitrary z-linear $m \times m$ matrix pencil

$$\mathcal{X}(z) = \mathcal{X}_0 + z\mathcal{X}_1 =: \begin{pmatrix} \mathcal{X}_{11}(z) & \mathcal{X}_{12}(z) \\ \mathcal{X}_{21}(z) & \mathcal{X}_{22}(z) \end{pmatrix} \text{ where } \mathcal{X}_0 = \mathcal{X}_0^*,\ \mathcal{X}_1 > 0, \tag{5.12}$$

with respect to the decomposition $\mathbb{C}^m = \mathbb{C}^d \oplus \mathbb{C}^{m-d}$. If $m = d$, (5.11) *reduces to $\mathcal{T}(z) = \mathcal{X}_0 + z\mathcal{X}_1$.*

If $\widetilde{A} \cap A_0 = S$ holds and $m = d$, then, by Corollary 2.6 and Proposition 3.6, $\widetilde{A}$ is already minimal.

Proof of Theorem 5.4. Suppose that a self-adjoint extension $\widetilde{A}$ with the properties (5.10) is given. Then $d_0 = d$ and hence, in the notation of Proposition 5.2, $\widehat{u}_z$, like γ_z, is a bijection from $\mathbb{C}^d$ onto $\ker(S^* - z)$. For a fixed point $z_1 \neq z_1^*$, we choose an invertible $d \times d$ matrix $\mathcal{M}$ such that $\gamma_{z_1} = \widehat{u}_{z_1}\mathcal{M}$. If we apply $(I + (z - z_1)R_0(z))$ to both sides of this equality we get that $\gamma_z = \widehat{u}_z\mathcal{M}$. It follows that

$$\frac{\mathcal{Q}(z) - \mathcal{Q}(w)^*}{z - w^*} = \gamma_w^*\gamma_z = \mathcal{M}^*\widehat{u}_w^*\widehat{u}_z\mathcal{M} = \mathcal{M}^*\frac{\widehat{\mathcal{P}}(z) - \widehat{\mathcal{P}}(w)^*}{z - w^*}\mathcal{M}$$

and hence $\mathcal{Q}(z) = \mathcal{M}^*(\widehat{\mathcal{P}}(z) + \mathcal{R})\mathcal{M}$ for some symmetric $d \times d$ matrix $\mathcal{R} = \mathcal{R}^*$. From the two formulas (5.6) and (1.2) for the generalized resolvent and using (1.4) we obtain that

$$\begin{aligned} \gamma_z\big(\mathcal{Q}(z) + \mathcal{T}(z)\big)^{-1}\gamma_{z^*}^* &= \widehat{u}_z\big(\widehat{\mathcal{P}}(z) + \widehat{\mathcal{T}}(z)\big)^{-1}\widehat{u}_{z^*}^* \\ &= \gamma_z\big(\mathcal{Q}(z) + \mathcal{M}^*(\widehat{\mathcal{T}}(z) - \mathcal{R})\mathcal{M}\big)^{-1}\gamma_{z^*}^*, \end{aligned}$$

and, consequently,

$$\mathcal{T}(z) = \mathcal{M}^*(\widehat{\mathcal{T}}(z) - \mathcal{R})\mathcal{M}.$$

By Proposition 5.3,

$$\widehat{\mathcal{T}}(z) = \mathcal{F}(z) - \mathcal{H}(z)\mathcal{G}(z)^{-1}\mathcal{H}(z^*)^*,$$

where

$$\begin{pmatrix} \mathcal{F}(z) & \mathcal{H}(z) \\ \mathcal{H}(z^*)^* & \mathcal{G}(z) \end{pmatrix} = \mathcal{N}^*(z + \mathcal{E}_1)\mathcal{N}$$

with

$$\mathcal{E}_1 = -\mathcal{E} + \mathcal{N}^{-*} \begin{pmatrix} 0 & -\mathrm{i}\mathcal{N}_1^* \Delta \mathcal{N}_2 \\ \mathrm{i}\mathcal{N}_2^* \Delta \mathcal{N}_1 & 0 \end{pmatrix} \mathcal{N}^{-1} = \mathcal{E}_1^*, \ \Delta = \frac{\mathcal{U}\mathcal{V}^* - \mathcal{V}\mathcal{U}^*}{2\mathrm{i}}.$$

It follows that $\mathcal{T}(z)$ is the $d \times d$ matrix function

$$\mathcal{T}(z) = \mathcal{X}_{11}(z) - \mathcal{X}_{12}(z)\mathcal{X}_{22}(z)^{-1}\mathcal{X}_{12}(z^*)^*,$$

in which the $\mathcal{X}_{ij}$'s are given through the relation

$$\begin{aligned} \begin{pmatrix} \mathcal{X}_{11}(z) & \mathcal{X}_{12}(z) \\ \mathcal{X}_{12}(z^*)^* & \mathcal{X}_{22}(z) \end{pmatrix} &= \begin{pmatrix} \mathcal{M}^* & 0 \\ 0 & I \end{pmatrix} \left(\begin{pmatrix} \mathcal{F}(z) & \mathcal{H}(z) \\ \mathcal{H}(z^*)^* & \mathcal{G}(z) \end{pmatrix} - \begin{pmatrix} \mathcal{R} & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \mathcal{M} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M}^* & 0 \\ 0 & I \end{pmatrix} \mathcal{N}^* \left(z + \mathcal{E}_1 - \mathcal{N}^{-*} \begin{pmatrix} \mathcal{R} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{N}^{-1} \right) \mathcal{N} \begin{pmatrix} \mathcal{M} & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

and the right-hand side is of the form $\mathcal{X}_0 + z\mathcal{X}_1$ with $\mathcal{X}_0 = \mathcal{X}_0^*$ and $\mathcal{X}_1 = \mathcal{X}_1^* > 0$, as claimed in the theorem.

Now suppose that, conversely, a function $\mathcal{T}(z)$ as in (5.11) is given. Clearly, it is a Nevanlinna $d \times d$ matrix function. We have to show that the generalized resolvent of S, given by formula (5.9), defines an extension $\widetilde{A}$ of S such that (5.10) holds. To this end we factor $\mathcal{X}_1$ into $\mathcal{X}_1 = \mathcal{N}^*\mathcal{N}$ and define

$$u_z := \begin{pmatrix} \gamma_z & 0 \end{pmatrix} \mathcal{N}^{-1}, \quad \mathcal{P}(z) := \mathcal{N}^{-*} \begin{pmatrix} \mathcal{Q}(z) & 0 \\ 0 & 0 \end{pmatrix} \mathcal{N}^{-1}, \quad \mathcal{E} := -\mathcal{N}^{-*}\mathcal{X}_0\mathcal{N}^{-1}.$$

Then u_z and $\mathcal{P}(z)$ have the properties as in items (a) and (c) of Proposition 3.1. Moreover, $u_z \colon \mathbb{C}^m \to \ker(S^* - z)$ is surjective,

$$\mathcal{P}(z) - \mathcal{E} + z = \mathcal{N}^{-*} \left(\begin{pmatrix} \mathcal{Q}(z) & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{X}_0 + z\mathcal{X}_1 \right) \mathcal{N}^{-1}$$

and

$$\begin{aligned} u_z(\mathcal{P}(z) - \mathcal{E} + z)^{-1} u_{z^*}^* &= \begin{pmatrix} \gamma_z & 0 \end{pmatrix} \left(\begin{pmatrix} \mathcal{Q}(z) & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{X}_0 + z\mathcal{X}_1 \right)^{-1} \begin{pmatrix} \gamma_{z^*}^* \\ 0 \end{pmatrix} \\ &= \gamma_z \big(\mathcal{Q}(z) + \mathcal{T}(z) \big)^{-1} \gamma_{z^*}^*. \end{aligned}$$

Now define $\widetilde{R}(z)$ by the block matrix (3.5) as in Theorem 3.3 and apply Remark 3.5. Then $C_{\mathfrak{H}}(\widetilde{A}) = A_0$. The equality $\widetilde{A} \cap A_0 = S$ follows from Proposition 3.6. It remains to apply Remark 3.4. □

In the following theorem we drop the assumptions that $\widetilde{A}$ and A_0 are disjoint and $m \geq d$.

Theorem 5.5. *Let S be a closed densely defined symmetric operator in $\mathfrak{H}$ with defect index $(d; d)$ and let A_0 be a canonical self-adjoint extension of S. Then Krein's formula* (5.9) *for S and based on A_0 establishes a bijective correspondence between*

all minimal self-adjoint extensions $\widetilde{A}$ *of* S *in* $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}^m$ *with* $C_{\mathfrak{H}}(\widetilde{A}) = A_0$ *and*

all subspaces $\mathcal{T}(z)$ *in* $\mathbb{C}^d \times \mathbb{C}^d$ *of the form*

$$\mathcal{T}(z) = \{\{P_0\mathbf{x}, P_0\mathcal{T}_0(z)P_0\mathbf{x}\} + \{0, (I - P_0)\mathbf{x}\} \,:\, \mathbf{x} \in \mathbb{C}^d\}, \tag{5.13}$$

in which P_0 *is a projection in* $\mathbb{C}^d$ *with* $d_1 := \dim \operatorname{ran} P_0 \le m$ *and, after identifying* $\operatorname{ran} P_0$ *with* $\mathbb{C}^{d_1}$, $\mathcal{T}_0(z)$ *is the* $d_1 \times d_1$ *first Schur complement of a* z*-linear* $m \times m$ *matrix pencil* $\mathcal{X}_0 + z\mathcal{X}_1$ *with* $\mathcal{X}_0 = \mathcal{X}_0^*$ *and* $\mathcal{X}_1 > 0$.

Proof. Consider a minimal self-adjoint extension $\widetilde{A}$ of S in $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}^m$ with $C_{\mathfrak{H}}(\widetilde{A}) = A_0$, and let $\mathcal{T}(z)$ be the corresponding parameter in Krein's formula (5.9). Then $\mathcal{T}(z)$ has the form (5.13) and Krein's formula for S and based on A_0 can written as

$$P(\widetilde{A} - z)^{-1}|_{\mathfrak{H}} = R_0(z) - (\gamma_z P_0)(P_0\mathcal{Q}(z)P_0 + \mathcal{T}_0(z))^{-1}(\gamma_{z^*}P_0)^*. \tag{5.14}$$

By Lemma 6.2 in the Appendix, $\gamma_z P_0$ and $P_0\mathcal{Q}(z)P_0$ are a γ-field and a Q-function for a symmetric operator S_1 with $S \subset S_1 \subset A_0$ and defect index $(d_1; d_1)$ with $d_1 = \dim \operatorname{ran} P_0$. Since $\mathcal{T}_0(z)$ is a matrix function (not a relation), $\widetilde{A} \cap A_0 = S_1$. Hence $d_1 \le m$. Now apply Theorem 5.4 with S, d and $\mathcal{T}(z)$ replaced by S_1, d_1 and $\mathcal{T}_0(z)$. Conversely, let P_0 be a projection in $\mathbb{C}^d$ with $d_1 = \dim \operatorname{ran} P_0 \le m$ and let $\mathcal{T}_0(z)$ be a Nevanlinna $d_1 \times d_1$ matrix function which is a first Schur complement as described in the theorem. Then, according to Theorem 5.4, the operator $\widetilde{A}$ in (5.14) has the asserted properties. □

As a special case we obtain:

Corollary 5.6. *Suppose that the symmetric operator* S *in Theorem* 5.5 *has defect index* $(1; 1)$ *and* $m = 1$*, and choose a canonical self-adjoint extension* A_0 *of* S*. In the description of the generalized resolvents of* S *by Krein's formula* (1.2)*, based on* A_0*, a self-adjoint extension* $\hat{A}$ *in* $\mathfrak{H} \oplus \mathbb{C}$ *has the compression* A_0 *if and only if* $\mathcal{T}(z) = \xi_0 + z\xi_1$ *with a real number* ξ_0 *and a positive number* ξ_1.

6. Appendix

Here we first prove a lemma which is equivalent to (1.4).

Lemma 6.1. *The Nevanlinna parameter* $\mathcal{T}(z)$ *in Krein's resolvent formula* (1.2) *corresponding to the self-adjoint extension* $\widetilde{A}$ *of* S *and based on* A_0 *is a matrix function* (*not a linear relation*) *if and only if* $\widetilde{A} \cap A_0 = S$.

Proof. Let $\mathcal{T}(z)$ be given by the linear relation (1.3). For $z \in \mathbb{C} \setminus \mathbb{R}$ the following six statements are equivalent:

. $\mathcal{T}(z)$ is a matrix function.
- P_0 is the identity in $\mathbb{C}^d$.
- $\gamma_z(\mathcal{Q}(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^*$ is a bijection from $\ker(S^* - z^*)$ onto $\ker(S^* - z)$.

- $\ker\left(\gamma_z(\mathcal{Q}(z)+\mathcal{T}(z))^{-1}\gamma_{z^*}^*\right) \subseteq \operatorname{ran}(S-z)$.
- $\ker\left(P(\widetilde{A}-z)^{-1}|_{\mathfrak{H}} - (A_0-z)^{-1}\right) \subseteq \operatorname{ran}(S-z)$.
- $\widetilde{A} \cap A_0 \subseteq S$.

Clearly, the inclusions in the last three items are in fact equalities. The equivalence of the first five statements is evident. We prove the implication

$$\widetilde{A} \cap A_0 \subseteq S \Rightarrow \ker\left(P(\widetilde{A}-z)^{-1}|_{\mathfrak{H}} - (A_0-z)^{-1}\right) \subseteq \operatorname{ran}(S-z).$$

Suppose x belongs to the kernel. Then there are elements $y \in \operatorname{dom} A_0$ and $u \in \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ such that

$$\{y, zy+x\} \in A_0, \quad \{y+u, z(y+u)+x\} \in \widetilde{A}.$$

Since A_0 is self-adjoint we have $(zy+x, y) - (y, zy+x) = 0$ and therefore, since also $\widetilde{A}$ is self-adjoint,

$$0 = (zy+y, y) - (y, zy+x) + (zu, u) - (u, zu) = (z-z^*)(u,u).$$

It follows that $u = 0$, that is,

$$\{y, zy+x\} \in \widetilde{A} \cap A_0 \subseteq S.$$

Hence $x = (S-z)y \in \operatorname{ran}(S-z)$.

We now prove the converse implication. Let $\{u, v\} \in \widetilde{A} \cap A_0$. Then

$$\left(P(\widetilde{A}-z)^{-1}|_{\mathfrak{H}} - (A_0-z)^{-1}\right)(v-zu) = u - u = 0,$$

and hence $v - zu \in \operatorname{dom}(S-z)$ Thus, for some $x \in \operatorname{dom} S$,

$$(A_0-z)u = v - zu = (S-z)x = (A_0-z)x.$$

It follows that $u - x$ is an eigenvector of A_0 with eigenvalue z. Since A_0 is self-adjoint and z is non-real, $u = x \in \operatorname{dom} S$. It follows that $\{u, v\} = \{u, A_0 u\} = \{u, Su\} \in S$. □

The following lemma is used in the proof of Theorem 5.5.

Lemma 6.2. *Let S be a closed densely defined symmetric operator in a Hilbert space $\mathfrak{H}$ with defect index $(d;d)$, let A be a canonical self-adjoint extension of S and let $z_1 \in \mathbb{C} \setminus \mathbb{R}$. If $\mathfrak{M}$ is a linear subset of $\ker(S^*-z_1)$ and $d_1 = \dim \mathfrak{M}$, then there is a symmetric operator S_1 in $\mathfrak{H}$ with defect index $(d_1;d_1)$ such that $S \subset S_1 \subset A$ and $\mathfrak{M} = \ker(S_1^*-z_1)$.*

Proof. Set $R(z) = (A-z)^{-1}$. Then the mapping $U := I + (z_1^*-z_1)R(z_1^*)$ is a bijection from $\ker(S^*-z_1) \to \ker(S^*-z_1^*)$. Let $u_+ : \mathbb{C}^{d_1} \to \mathfrak{M}$ and $v_+ : \mathbb{C}^{d-d_1} \to \ker(S^*-z_1) \ominus \mathfrak{M}$ be bijective mappings. Then $u_- := Uu_+$ is a bijection from $\mathbb{C}^{d_1} \to U\mathfrak{M} \subset \ker(S^*-z_1^*)$ and $v_- := Uv_+$ is a bijection from $\mathbb{C}^{d-d_1} \to U(\ker(S^*-z_1^*) \ominus \mathfrak{M}) = \ker(S^*-z_1^*) \ominus U\mathfrak{M}$. Define S_1 as the direct sum

$$S_1 = S \dot{+} \{v_+(x) - v_-(x), z_1 v_+(x) - z_1^* v_-(x) \,:\, x \in \mathbb{C}^{d-d_1}\}. \tag{6.1}$$

Then $S \subset S_1$. Since for all $x \in \mathbb{C}^{d-d_1}$

$$\{v_+(x) - v_-(x), z_1 v_+(x) - z_1^* v_-(x)\} \\ = (z_1 - z_1^*)\{R(z_1^*)v_+(x), (I + z_1^* R(z_1^*))v_+(x)\} \in A,$$

we also have that $S_1 \subset A$. This implies that S_1 is symmetric and that it has equal defect numbers. Since the dimension of the set on the right-hand side of (6.1) has dimension $d - d_1$ and $\dim A/S = d$, the defect index of S_1 equals $(d_1; d_1)$.

Let $u \in \ker(S_1^* - z_1)$. Then $\{u, z_1 u\} \in S_1^*$. By (6.1), $\{u, z_1 u\} \in S^*$ that is $u \in \ker(S^* - z_1)$. It also follows that for all $x \in \mathbb{C}^{d-d_1}$

$$0 = (z_1 u, v_+(x) - v_-(x)) - (u, z_1 v_+(x) - z_1^* v_-(x)) = (z_1 - z_1^*)(u, v_+(x)),$$

that is, $(u, v_+(x)) = 0$. Hence $u \in \mathfrak{M}$. This shows that $\ker(S_1^* - z_1) \subset \mathfrak{M}$. By retracing these calculations backwards we find that $\mathfrak{M} \subset \ker(S_1^* - z_1)$ and hence equality prevails. □

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On the Structure of Hausdorff Moment Sequences of Complex Matrices

Bernd Fritzsche, Bernd Kirstein and Conrad Mädler

Dedicated to Daniel Alpay on the occasion of his 60th birthday

Abstract. The paper treats several aspects of the truncated matricial $[\alpha, \beta]$-Hausdorff type moment problems. It is shown that each $[\alpha, \beta]$-Hausdorff moment sequence has a particular intrinsic structure. More precisely, each element of this sequence varies within a closed bounded matricial interval. The case that the corresponding moment coincides with one of the endpoints of the interval plays a particular important role. This leads to distinguished molecular solutions of the truncated matricial $[\alpha, \beta]$-Hausdorff moment problem, which satisfy some extremality properties. The proofs are mainly of algebraic character. The use of the parallel sum of matrices is an essential tool in the proofs.

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Keywords. Truncated matricial Hausdorff moment problem, canonical molecular solutions, matricial intervals associated with matricial Hausdorff moment sequences, parallel sum of matrices.

1. Introduction

The starting point of this paper was a question connected to matricial versions of the truncated power moment problem on a compact interval $[\alpha, \beta]$ of the real axis. In joint work with A.E. Choque Rivero and Yu.M. Dyukarev (see [8, 9]), the first and second authors could extend the characterizations of solvability of this moment problem, which were given in the scalar case by M.G. Krein [26] (see also Krein/Nudelman [27, Ch. III]) to the matrix case. In the case that $q \in \mathbb{N}$, $n \in \mathbb{N}$, and $(s_j)_{j=0}^{2n-1}$ is a sequence of complex $q \times q$ matrices for which the moment problem in question is solvable, in their joint paper [13] with Yu.M. Dyukarev the authors constructed a concrete molecular solution (i.e., a discrete non-negative Hermitian $q \times q$ measure concentrated on finitely many points of the interval $[\alpha, \beta]$) of the

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moment problem. The motivation for this paper was to find an explicit molecular solution for the case of a given sequence $(s_j)_{j=0}^{2n}$ of prescribed moments. A closer look at our method used in [13] shows that the realization of our aim can be reached by a thorough study of the structure of finite $[\alpha, \beta]$-Hausdorff non-negative definite sequences of complex $q \times q$ matrices (see Definition 4.2). The key information comes from Theorem 4.9, which says that if $m \in \mathbb{N}$ and if $(s_j)_{j=0}^{m}$ is an $[\alpha, \beta]$-Hausdorff non-negative definite sequence, then we can always find a complex $q \times q$ matrix s_{m+1} such that the sequence $(s_j)_{j=0}^{m+1}$ is $[\alpha, \beta]$-Hausdorff non-negative definite. We are even able to describe all complex $q \times q$ matrices s_{m+1} which can be chosen to realize this aim. More precisely, the set of all these matrices s_{m+1} turns out to be a closed matricial interval of complex $q \times q$ matrices. The left and right endpoint as well as the midpoint of this interval play (as clearly expected) a distinguished role (see Section 10). The main part of this paper is concerned with the investigation of the structure of $[\alpha, \beta]$-Hausdorff non-negative definite sequences of complex $q \times q$ matrices and the study of the above-mentioned extension problem for such sequences. These results lead us to interesting new insights concerning a whole family of particular molecular solutions of the matrix version of the truncated $[\alpha, \beta]$-Hausdorff moment problem. In particular, we guess that the choice of the endpoints of the interval exactly leads to those extremal molecular solutions which were studied by M.G. Krein [26] (see also Krein/Nudelman [27, Ch. III, §5]). M.G. Krein found them via the lower and upper principal representation of the given moment sequence (see Section 12).

A more careful view shows that the situation is in some sense similar as in the case of non-negative definite sequences from $\mathbb{C}^{q \times q}$ (see [16]) or $p \times q$ Schur sequences (see [17]). If $n \in \mathbb{N}$ and if $(C_j)_{j=0}^{n}$ is a sequence of one of the just mentioned types, then, for each $m \in \{1, 2, \ldots, n\}$, the matrix C_m belongs to a matrix ball the parameters of which depend on the sequence $(C_j)_{j=0}^{m-1}$. Having in mind the geometry of a matrix ball, we see that there are two types of distinguished points, namely, on the one hand, the center of the matrix ball and, on the other hand, its boundary points. The $q \times q$ non-negative definite sequences or $p \times q$ Schur sequences which are starting from some index consist only of the centers of the matrix balls in question occupy an extremal position in the set of all sequences of the considered types. Similar things can be said about those sequences which contain an element of the boundary of the relevant matrix ball. A similar situation will be met for $[\alpha, \beta]$-Hausdorff non-negative definite sequences. This will be discussed in detail in Section 10.

The study of moment spaces was initiated by C. Carathéodory [4, 5] in the context of the trigonometric moment problem. The approach of Carathéodory was based on the theory of convexity. O. Toeplitz [34] observed that the results of Carathéodory can be reformulated in terms of non-negative Hermitian Toeplitz matrices. This view was then generally accepted and is also the basis for the approach in the matrix case in [16, 17].

In the study of the moment space connected with the $[\alpha, \beta]$-Hausdorff moment problem, the theory of convexity played an important role from the very beginning. These developments were strongly influenced by M.G. Krein's landmark paper [26], which essentially determined the further direction of research reflected in the monographs Karlin/Shapley [23], Karlin/Studden [24], and Krein/Nudelman [27]. It should be mentioned that Skibinsky [31, 32] considered probability measures on $[0, 1]$ and observed that the $(n + 1)$-th moment of such measures can vary within a closed bounded interval of the real axis. The work of Skibinsky was also an important source of inspiration for the theory of canonical moments, which was worked out by Dette/Studden [10].

2. On the matrix version of the truncated $[\alpha, \beta]$-Hausdorff moment problem

In order to formulate the moment problem we are going to study, we first give some notation. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. For every choice of $\kappa, \tau \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\kappa,\tau}$ be the set of all integers ℓ for which $\kappa \leq \ell \leq \tau$ holds. Throughout this paper, let p and q be positive integers. If $\mathcal{X}$ is a non-empty set, then $\mathcal{X}^{p\times q}$ stands for the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^p$ is short for $\mathcal{X}^{p\times 1}$. The null matrix which belongs to $\mathbb{C}^{p\times q}$ is denoted by $0_{p\times q}$, whereas I_p is the identity matrix belonging to $\mathbb{C}^{p\times p}$. We will write $\mathbb{C}_{\mathrm{H}}^{q\times q}$, $\mathbb{C}_{\geq}^{q\times q}$, and $\mathbb{C}_{>}^{q\times q}$ for the set of all Hermitian complex $q \times q$ matrices, the set of all non-negative Hermitian complex $q \times q$ matrices, and the set of all positive Hermitian complex $q \times q$ matrices, respectively. If A and B are complex $q \times q$ matrices, then $A \geq B$ or $B \leq A$ (resp., $A > B$ or $B < A$) means that A and B are Hermitian and $A - B$ is non-negative Hermitian (resp., positive Hermitian).

Let $(\Omega, \mathfrak{A})$ be a measurable space. Each countably additive mapping whose domain is $\mathfrak{A}$ and whose values belong to $\mathbb{C}_{\geq}^{q\times q}$ is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$. For the integration theory with respect to non-negative Hermitian measures, we refer to Kats [25] and Rosenberg [30]. We denote by $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ the set of all $\mathfrak{A}$-measurable complex-valued functions f on Ω which are integrable with respect to a non-negative Hermitian measure μ on $(\Omega, \mathfrak{A})$.

Let $\mathfrak{B}_{\mathbb{R}}$ (resp., $\mathfrak{B}_{\mathbb{C}}$) be the σ-algebra of all Borel subsets of $\mathbb{R}$ (resp., $\mathbb{C}$). For all $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, let $\mathfrak{B}_\Omega$ be the σ-algebra of all Borel subsets of Ω and let $\mathcal{M}_{\geq}^q(\Omega)$ be the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{B}_\Omega)$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{M}_{\geq,\kappa}^q(\Omega)$ be the set of all $\sigma \in \mathcal{M}_{\geq}^q(\Omega)$ such that for each $j \in \mathbb{Z}_{0,\kappa}$ the function $f_j\colon \Omega \to \mathbb{C}$ defined by $f_j(x) := x^j$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$. If $\sigma \in \mathcal{M}_{\geq,\kappa}^q(\Omega)$, then let

$$s_j^{[\sigma]} := \int_\Omega x^j \sigma(\mathrm{d}x) \tag{2.1}$$

for all $j \in \mathbb{Z}_{0,\kappa}$. Obviously, $\mathcal{M}^q_{\geq}(\Omega) \supseteq \mathcal{M}^q_{\geq,k}(\Omega) \supseteq \mathcal{M}^q_{\geq,k+1}(\Omega) \supseteq \mathcal{M}^q_{\geq,\infty}(\Omega)$ holds true for every choice of $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$ and $k \in \mathbb{N}_0$. Furthermore, if Ω is a non-empty bounded Borel subset of $\mathbb{R}$, then one can easily see that $\mathcal{M}^q_{\geq}(\Omega) = \mathcal{M}^q_{\geq,\infty}(\Omega)$. In particular, for all $\alpha \in \mathbb{R}$ and $\beta \in (\alpha, \infty)$, we have

$$\mathcal{M}^q_{\geq}([\alpha,\beta]) = \mathcal{M}^q_{\geq,\infty}([\alpha,\beta]). \tag{2.2}$$

Let $\mathcal{M}^{q,\mathrm{mol}}_{\geq}(\Omega)$ be the set of all $\sigma \in \mathcal{M}^q_{\geq}(\Omega)$ for which there exists a finite subset B of Ω satisfying $\sigma(\Omega \setminus B) = 0_{q\times q}$. The elements of $\mathcal{M}^{q,\mathrm{mol}}_{\geq}(\Omega)$ are called *molecular*. Obviously, $\mathcal{M}^{q,\mathrm{mol}}_{\geq}(\Omega)$ is the set of all $\sigma \in \mathcal{M}^q_{\geq}(\Omega)$ for which there exist an $m \in \mathbb{N}$ and sequences $(\xi_\ell)_{\ell=0}^m$ and $(M_\ell)_{\ell=0}^m$ from Ω and $\mathbb{C}^{q\times q}_{\geq}$ such that $\sigma = \sum_{\ell=1}^m \delta_{\xi_\ell} M_\ell$, where δ_{ξ_ℓ} is the Dirac measure on $\mathfrak{B}_\Omega$ with unit mass at ξ_ℓ, $\ell \in \mathbb{Z}_{1,m}$. In particular, we have $\mathcal{M}^{q,\mathrm{mol}}_{\geq}(\Omega) \subseteq \mathcal{M}^q_{\geq,\infty}(\Omega)$.

The following matricial moment problem lies in the background of our considerations:

$\mathsf{M}[\Omega; (s_j)_{j=0}^\kappa, =]$**:** Let $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{q\times q}$. Parametrize the set $\mathcal{M}^q_{\geq}[\Omega; (s_j)_{j=0}^\kappa, =]$ of all $\sigma \in \mathcal{M}^q_{\geq,\kappa}(\Omega)$ such that $s_j^{[\sigma]} = s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$.

In this paper, we often use the procedure of reflecting measures on the real axis. For this reason, we introduce some terminology. Let $R\colon \mathbb{R} \to \mathbb{R}$ be defined by $x \mapsto -x$. Then R is a continuous involution of $\mathbb{R}$ and consequently $\mathfrak{B}_{\mathbb{R}}$-$\mathfrak{B}_{\mathbb{R}}$-measurable. In view of $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, the set $\Omega^\vee := R^{-1}(\Omega)$ belongs to $\mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$ and the mapping $R_\Omega := \mathrm{Rstr}_\Omega R$ is $\mathfrak{B}_\Omega$-$\mathfrak{B}_{\Omega^\vee}$-measurable, whereas the mapping $R_{\Omega^\vee} := \mathrm{Rstr}_{\Omega^\vee} R$ is $\mathfrak{B}_{\Omega^\vee}$-$\mathfrak{B}_\Omega$-measurable. Moreover $R_\Omega \circ R_{\Omega^\vee} = \mathrm{Id}_\Omega$ and $R_{\Omega^\vee} \circ R_\Omega = \mathrm{Id}_{\Omega^\vee}$. For each $\sigma \in \mathcal{M}^q_{\geq}(\Omega)$, we denote by $\sigma^\vee$ the image measure of σ with respect to R_Ω, i.e., for $B \in \mathfrak{B}_{\Omega^\vee}$, we have

$$\sigma^\vee(B) := \sigma\big(R_\Omega^{-1}(B)\big). \tag{2.3}$$

By construction then $\sigma^\vee \in \mathcal{M}^q_{\geq}(\Omega^\vee)$. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and if $\sigma \in \mathcal{M}^q_{\geq,\kappa}(\Omega)$, then it is easily checked that $\sigma^\vee \in \mathcal{M}^q_{\geq,\kappa}(\Omega^\vee)$ and that $s_j^{[\sigma^\vee]} = (-1)^j s_j^{[\sigma]}$ for all $j \in \mathbb{Z}_{0,\kappa}$.

Using the preceding considerations we get the following result:

Remark 2.1. Let $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{q\times q}$. Let $\Omega^\vee := \{x \in \mathbb{R}\colon -x \in \Omega\}$ and let $r_j := (-1)^j s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$. Then $\Omega^\vee \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$ and, using the notation introduced in (2.3), we get $\mathcal{M}^q_{\geq}[\Omega^\vee; (r_j)_{j=0}^\kappa, =] = \{\sigma^\vee\colon \sigma \in \mathcal{M}^q_{\geq}[\Omega; (s_j)_{j=0}^\kappa, =]\}$.

The discussions of this paper are mostly concentrated on the case that the set Ω is a bounded and closed interval of the real axis. Such moment problems are called to be of Hausdorff-type. The following special case of Remark 2.1 is of particular interest for us.

Remark 2.2. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{q\times q}$. Let $r_j := (-1)^j s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$. Then $\mathcal{M}^q_{\geq}[[-\beta,-\alpha]; (r_j)_{j=0}^\kappa, =] = \{\sigma^\vee\colon \sigma \in \mathcal{M}^q_{\geq}[[\alpha,\beta]; (s_j)_{j=0}^\kappa, =]\}$.

The solvability of the truncated matricial $[\alpha, \beta]$-Hausdorff-type moment problem can be characterized as follows:

Theorem 2.3 ([8, Theorem 1.2]). *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n-1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{2n-1},=] \neq \emptyset$ if and only if the block Hankel matrices*

$$[-\alpha s_{j+k} + s_{j+k+1}]_{j,k=0}^{n-1} \qquad \text{and} \qquad [\beta s_{j+k} - s_{j+k+1}]_{j,k=0}^{n-1} \tag{2.4}$$

are both non-negative Hermitian.

Theorem 2.4 ([9, Theorem 1.3]). *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{2n},=] \neq \emptyset$ if and only if the block Hankel matrices*

$$[s_{j+k}]_{j,k=0}^{n} \qquad \text{and} \qquad [-\alpha\beta s_{j+k} + (\alpha+\beta)s_{j+k+1} - s_{j+k+2}]_{j,k=0}^{n-1} \tag{2.5}$$

are both non-negative Hermitian.

In the scalar case $q = 1$, Theorems 2.3 and 2.4 are due to M.G. Krein [26] (see also [27, Ch. III]). The approach of M.G. Krein is essentially based on the use of the theory of convexity along the lines applied by C. Carathéodory [4,5] in the context of the trigonometric moment problem. The proofs of Theorems 2.3 and 2.4 given in [8,9], respectively, are rather complicated and not constructive. They are based on Potapov's method of fundamental matrix inequalities and an extensive explicit solving procedure of the corresponding system of these inequalities in the non-degenerate case. Indeed, if the block Hankel matrices in (2.4) and (2.5) are both positive Hermitian, then [8, Corollary 6.13] and [9, Corollary 6.15] show that Problem $\mathsf{M}[[\alpha,\beta];(s_j)_{j=0}^{2n-1},=]$ and Problem $\mathsf{M}[[\alpha,\beta];(s_j)_{j=0}^{2n},=]$ have solutions. In the proofs of [8, Theorem 1.2] and [9, Theorem 1.3], a perturbation argument for the construction of a sequence of corresponding approximating non-degenerate problems is used and then a matricial version of the Helly–Prohorov Theorem (see [18, Satz 9] or [12, Lemma 2.2.1]) is applied.

3. On Hankel non-negative definite sequences

In this section, we summarize essential properties on two classes of sequences of complex $q \times q$ matrices which are a main tool for the investigations of this paper. This material is mainly taken from [14, 19].

Definition 3.1. Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Let $H_n := [s_{j+k}]_{j,k=0}^{n}$. Then $(s_j)_{j=0}^{2n}$ is called *Hankel non-negative definite* (resp., *Hankel positive definite*) if $H_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ (resp., $\mathbb{C}_{>}^{(n+1)q\times(n+1)q}$). We denote by $\mathcal{H}_{q,2n}^{\geq}$ (resp., $\mathcal{H}_{q,2n}^{>}$) the set of all Hankel non-negative definite (resp., Hankel positive definite) sequences $(s_j)_{j=0}^{2n}$ from $\mathbb{C}^{q\times q}$.

Definition 3.2. (a) Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n}$ is called *Hankel non-negative definite extendable* (resp., *Hankel positive definite extendable*) if there exist matrices s_{2n+1} and s_{2n+2} from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2n+2} \in \mathcal{H}_{q,2n+2}^{\geq}$ (resp., $(s_j)_{j=0}^{2n+2} \in \mathcal{H}_{q,2n+2}^{>}$).

(b) Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n+1}$ is called *Hankel non-negative definite extendable* (resp., *Hankel positive definite extendable*) if there exists a matrix s_{2n+2} from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2n+2} \in \mathcal{H}_{q,2n+2}^{\geq}$ (resp., $(s_j)_{j=0}^{2n+2} \in \mathcal{H}_{q,2n+2}^{>}$).

If $m \in \mathbb{N}_0$, then the notation $\mathcal{H}_{q,m}^{\geq,\mathrm{e}}$ (resp., $\mathcal{H}_{q,m}^{>,\mathrm{e}}$) stands for the set of all sequences $(s_j)_{j=0}^{m}$ from $\mathbb{C}^{q\times q}$ which are Hankel non-negative definite extendable (resp., Hankel positive definite extendable).

The importance of the class $\mathcal{H}_{q,m}^{\geq,\mathrm{e}}$ in the context of moment problems is caused by the following observation:

Theorem 3.3 (see [14, Theorem 4.17]). *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{M}_{\geq}^{q}[\mathbb{R};(s_j)_{j=0}^{m},=] \neq \emptyset$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{H}_{q,m}^{\geq,\mathrm{e}}$.*

Now we are going to indicate some essential features of the structure of Hankel non-negative definite sequences. First we introduce some matrices which occupy a key role in the sequel. For each matrix $A \in \mathbb{C}^{p\times q}$, we denote by $A^{\dagger}$ its Moore–Penrose inverse. This means that $A^{\dagger}$ is the unique matrix X from $\mathbb{C}^{q\times p}$ which satisfies the four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$. For every choice of $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n \in \mathbb{C}^{p\times q}$, let $\operatorname{col}(A_j)_{j=1}^{n} := [A_1^*, A_2^*, \ldots, A_n^*]^*$ and let $\operatorname{row}(A_j)_{j=1}^{n} := [A_1, A_2, \ldots, A_n]$.

Notation 3.4. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$.

(a) Let $H_n := [s_{j+k}]_{j,k=0}^{n}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $K_n := [s_{j+k+1}]_{j,k=0}^{n}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, and let $G_n := [s_{j+k+2}]_{j,k=0}^{n}$ for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$.

(b) Let $y_{\ell,m} := \operatorname{col}(s_j)_{j=\ell}^{m}$ and $z_{\ell,m} := \operatorname{row}(s_j)_{j=\ell}^{m}$ for all $\ell, m \in \mathbb{N}_0$ with $\ell \leq m \leq \kappa$.

(c) Let $\Theta_0 := 0_{p\times q}$ and, for all $n \in \mathbb{N}$ with $2n-1 \leq \kappa$, let $\Theta_n := z_{n,2n-1} H_{n-1}^{\dagger} y_{n,2n-1}$. Furthermore, for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $L_n := s_{2n} - \Theta_n$.

(d) Let $M_0 := 0_{p\times q}$, and, for all $n \in \mathbb{N}$ with $2n \leq \kappa$, let $M_n := z_{n,2n-1} H_{n-1}^{\dagger} y_{n+1,2n}$.

If we build the matrices introduced in Notation 3.4 from an other given sequence, e.g., $(t_j)_{j=0}^{\kappa}$, then this is indicated by a superscript $\langle t\rangle$, i.e., $H_n^{\langle t\rangle} := [t_{j+k}]_{j,k=0}^{n}$, etc.

Remark 3.5. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Then the block Hankel matrix H_n admits the block partition

$$H_n = \begin{bmatrix} H_{n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{bmatrix}. \tag{3.1}$$

Let $\mathcal{R}(A) := \{Ax\colon x \in \mathbb{C}^q\}$ and $\mathcal{N}(A) := \{x \in \mathbb{C}^q\colon Ax = 0_{p\times 1}\}$ be the column space and the null space of a matrix $A \in \mathbb{C}^{p\times q}$.

Remark 3.6. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. In view of (3.1) and Lemma A.12, then:

(a) $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ if and only if $(s_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{\geq}$, $L_n \in \mathbb{C}_{\geq}^{q\times q}$, $\mathcal{R}(y_{n,2n-1}) \subseteq H_{n-1}$, and $s_{2n+1} \in \mathbb{C}_{\mathrm{H}}^{q\times q}$.

(b) $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$ if and only if $(s_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{>}$, $L_n \in \mathbb{C}_{>}^{q\times q}$, and $s_{2n+1} \in \mathbb{C}_{\mathrm{H}}^{q\times q}$.

A sequence $(s_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$ is called *Hankel non-negative definite* (resp., *Hankel positive definite*) if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ (resp., $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$) for all $n \in \mathbb{N}_0$. We denote by $\mathcal{H}_{q,\infty}^{\geq}$ (resp., $\mathcal{H}_{q,\infty}^{>}$) the set of all Hankel non-negative definite (resp., Hankel positive definite) sequences $(s_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$.

Remark 3.7. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. Then:

(a) If $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\geq}$, then $s_j \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,2\kappa}$ and $s_{2k} \in \mathbb{C}_{\geq}^{q\times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

(b) If $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{>}$, then $s_j \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,2\kappa}$ and $s_{2k} \in \mathbb{C}_{>}^{q\times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

Lemma 3.8. *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n-1}$ be a sequence from $\mathbb{C}_{\mathrm{H}}^{q\times q}$. Then:*

(a) *If $(s_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{\geq}$, then $\Theta_n \in \mathbb{C}_{\geq}^{q\times q}$.*

(b) *If $(s_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{>}$ and $n \geq 2$, then $\Theta_n \in \mathbb{C}_{>}^{q\times q}$.*

Proof. (a) This follows immediately from Remarks A.10 and A.2.

(b) Now suppose that $(s_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{>}$ and $n \geq 2$. Then H_{n-1} is positive Hermitian. In particular, H_{n-1}^{-1} is positive Hermitian. From Remark 3.7(b) we obtain $s_{2n-2} \in \mathbb{C}_{>}^{q\times q}$. In particular, $\det s_{2n-2} \neq 0$. Because of $n \geq 2$, the matrix s_{2n-2} is a block in $y_{n,2n-1}$. Thus, $\operatorname{rank} y_{n,2n-1} = q$. Hence, $\Theta_n \in \mathbb{C}_{>}^{q\times q}$ follows from Remark A.2. □

Remark 3.9. Let $n \in \mathbb{N}_0$. Then $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}} \subseteq \mathcal{H}_{q,2n}^{\geq}$ and, in the case $n \in \mathbb{N}$, furthermore $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}} \neq \mathcal{H}_{q,2n}^{\geq}$.

For all $n \in \mathbb{N}_0$ let

$$J_{q,n} := \operatorname{diag}\big((-1)^j I_q\big)_{j=0}^{n}.$$

Remark 3.10. Let $n \in \mathbb{N}_0$. Then $J_{q,n}^* = J_{q,n}$ and $J_{q,n}^2 = I_{(n+1)q}$. In particular, $J_{q,n}$ is unitary.

Lemma 3.11. *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Let the sequence $(r_j)_{j=0}^{2n}$ be given by $r_j := (-1)^j s_j$. Then:*

(a) *Let $H_n^{\langle s\rangle} := [s_{j+k}]_{j,k=0}^{n}$ and $H_n^{\langle r\rangle} := [r_{j+k}]_{j,k=0}^{n}$. Then $H_n^{\langle r\rangle} = J_{q,n}^* H_n^{\langle s\rangle} J_{q,n}$.*
(b) *$(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ if and only if $(r_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$.*
(c) *$(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$ if and only if $(r_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$.*

Proof. Part (a) follows by direct computation. Parts (b) and (c) follow from (a). □

Lemma 3.12. *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Let the sequence $(r_j)_{j=0}^{m}$ be given by $r_j := (-1)^j s_j$. Then $(s_j)_{j=0}^{m} \in \mathcal{H}_{q,m}^{\geq,\mathrm{e}}$ if and only if $(r_j)_{j=0}^{m} \in \mathcal{H}_{q,m}^{\geq,\mathrm{e}}$.*

Proof. Use Lemma 3.11(b). □

4. On $[\alpha,\beta]$-Hausdorff non-negative definite sequences from $\mathbb{C}^{q\times q}$

Against to the background of Theorems 2.3 and 2.4 now we are going to introduce one of the central notions of this paper. Before doing this, we introduce some notation.

Notation 4.1. Let $\alpha,\beta \in \mathbb{C}$, let $\kappa \in \mathbb{N}\cup\{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Then let the sequences $\big((_\alpha s)_j\big)_{j=0}^{\kappa-1}$ and $\big((s_\beta)_j\big)_{j=0}^{\kappa-1}$ be given by

$$(_\alpha s)_j := -\alpha s_j + s_{j+1} \quad \text{and} \quad (s_\beta)_j := \beta s_j - s_{j+1} \quad \text{for each } j \in \mathbb{Z}_{0,\kappa-1}.$$

Furthermore, if $\kappa \geq 2$, then let $\big((_\alpha s_\beta)_j\big)_{j=0}^{\kappa-2}$ be defined by

$$(_\alpha s_\beta)_j := -\alpha\beta s_j + (\alpha+\beta)s_{j+1} - s_{j+2} \quad \text{for all } j \in \mathbb{Z}_{0,\kappa-2}.$$

When using the sequences from Notation 4.1, we write

$$(_\alpha H)_n := [(_\alpha s)_{j+k}]_{j,k=0}^{n}, \quad (H_\beta)_n := [(s_\beta)_{j+k}]_{j,k=0}^{n}, \quad (_\alpha H_\beta)_n := [(_\alpha s_\beta)_{j+k}]_{j,k=0}^{n}, \quad \text{etc.}$$

Definition 4.2. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, and let $n \in \mathbb{N}_0$.

(a) Let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n+1}$ is called $[\alpha,\beta]$-*Hausdorff non-negative definite* (resp., $[\alpha,\beta]$-*Hausdorff positive definite*) if both sequences $\big((_\alpha s)_j\big)_{j=0}^{2n}$ and $\big((s_\beta)_j\big)_{j=0}^{2n}$ are Hankel non-negative definite (resp., Hankel positive definite).
(b) Let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n}$ is called $[\alpha,\beta]$-*Hausdorff non-negative definite* (resp., $[\alpha,\beta]$-*Hausdorff positive definite*) if $(s_j)_{j=0}^{2n}$ and, in the case $n \geq 1$, moreover $\big((_\alpha s_\beta)_j\big)_{j=0}^{2(n-1)}$ is Hankel non-negative definite (resp., Hankel positive definite).

If $m \in \mathbb{N}_0$, then the notation $\mathcal{F}_{q,m,\alpha,\beta}^{\geq}$ (resp., $\mathcal{F}_{q,m,\alpha,\beta}^{>}$) stands for the set of all $[\alpha,\beta]$-Hausdorff non-negative definite (resp., $[\alpha,\beta]$-Hausdorff positive definite) sequences $(s_j)_{j=0}^{m}$ from $\mathbb{C}^{q\times q}$.

In view of Definition 4.2, now Theorems 2.3 and 2.4 can be summarized and reformulated as follows:

Theorem 4.3. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^m,=] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$.*

Definition 4.4. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^m$ is called $[\alpha,\beta]$-*Hausdorff non-negative definite extendable* (resp., $[\alpha,\beta]$-*Hausdorff positive definite extendable*) if there exists a matrix $s_{m+1} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{m+1}$ is $[\alpha,\beta]$-Hausdorff non-negative definite (resp., $[\alpha,\beta]$-Hausdorff positive definite). We denote by $\mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{e}}$ (resp., $\mathcal{F}_{q,m,\alpha,\beta}^{>,\mathrm{e}}$) the set of all $[\alpha,\beta]$-Hausdorff non-negative definite extendable (resp., $[\alpha,\beta]$-Hausdorff positive definite extendable) sequences $(s_j)_{j=0}^m$ from $\mathbb{C}^{q\times q}$.

Using Theorem 4.3, we derive now several algebraic results on the matrix sequences introduced in Definitions 4.2 and 4.4, respectively.

Proposition 4.5. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then $(s_j)_{j=0}^{\ell} \in \mathcal{F}_{q,\ell,\alpha,\beta}^{\geq}$ for all $\ell \in \mathbb{Z}_{0,m}$.*

Proof. In view of Theorem 4.3, we have $\mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^m,=] \neq \emptyset$. Let $\ell \in \mathbb{Z}_{0,m}$. Then $\mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^m,=] \subseteq \mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^{\ell},=]$. Hence $\mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^{\ell},=] \neq \emptyset$. Thus, again applying Theorem 4.3, we get $(s_j)_{j=0}^{\ell} \in \mathcal{F}_{q,\ell,\alpha,\beta}^{\geq}$. □

Note that an alternate purely algebraic proof will be given in Section 7. Proposition 4.5 leads us to the following notions:

Definition 4.6. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, and let $(s_j)_{j=0}^{\infty}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{\infty}$ is called $[\alpha,\beta]$-*Hausdorff non-negative definite* (resp., $[\alpha,\beta]$-*Hausdorff positive definite*) if for all $m \in \mathbb{N}_0$ the sequence $(s_j)_{j=0}^m$ is $[\alpha,\beta]$-Hausdorff non-negative definite (resp., $[\alpha,\beta]$-Hausdorff positive definite). The notation $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ (resp., $\mathcal{F}_{q,\infty,\alpha,\beta}^{>}$) stands for set of all $[\alpha,\beta]$-Hausdorff non-negative definite (resp., $[\alpha,\beta]$-Hausdorff positive definite) sequences $(s_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$.

Lemma 4.7. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, and let $\sigma \in \mathcal{M}_{\geq}^q([\alpha,\beta])$. Then $\sigma \in \mathcal{M}_{\geq,\infty}^q([\alpha,\beta])$ and the sequence $(s_j^{[\sigma]})_{j=0}^{\infty}$ given via (2.1) belongs to $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$.*

Proof. By the choice of σ we have $\sigma \in \bigcap_{m=0}^{\infty} \mathcal{M}_{\geq,m}^q([\alpha,\beta])$. Thus, Theorem 4.3 implies $(s_j^{[\sigma]})_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$ for each $m \in \mathbb{N}_0$. Hence $(s_j^{[\sigma]})_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$. □

Proposition 4.8. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then there is a sequence $(s_j)_{j=m+1}^{\infty}$ from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$.*

Proof. In view of Theorem 4.3 we have $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{m},=] \neq \emptyset$. Let $\sigma \in \mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{m},=]$. In view of Lemma 4.7, we have then $\sigma \in \mathcal{M}_{\geq,\infty}^{q}([\alpha,\beta])$, $(s_j^{[\sigma]})_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$, and $(s_j^{[\sigma]})_{j=0}^{m} = (s_j)_{j=0}^{m}$. □

Theorem 4.9. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, and let $m \in \mathbb{N}_0$. Then $\mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{e}} = \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$.*

Proof. Combine Propositions 4.8 and 4.5. □

Comparing Theorem 4.9 with Remark 3.9, we see that both statements are completely different. Against the background of Theorems 4.3 and 3.3 we can now immediately see the reason for this phenomenon, namely, in view of (2.2), we have $\mathcal{M}_{\geq}^{q}([\alpha,\beta]) = \mathcal{M}_{\geq,\infty}^{q}([\alpha,\beta])$, whereas, on the other hand, it can be easily checked that, for each $k \in \mathbb{N}$, the proper inclusion $\mathcal{M}_{\geq,k+1}^{q}(\mathbb{R}) \subset \mathcal{M}_{\geq,k}^{q}(\mathbb{R})$ is satisfied.

In view of Lemma 4.7, we will consider the following problem:

P$[[\alpha,\beta];(s_j)_{j=0}^{m},=]$**:** Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Describe the set

$$\mathcal{P}\big[[\alpha,\beta];(s_j)_{j=0}^{m}\big] := \left\{ \int_{[\alpha,\beta]} x^{m+1}\sigma(\mathrm{d}x) \colon \sigma \in \mathcal{M}_{\geq}^{q}\big[[\alpha,\beta];(s_j)_{j=0}^{m},=\big] \right\}. \tag{4.1}$$

Proposition 4.10. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{P}[[\alpha,\beta];(s_j)_{j=0}^{m}] \neq \emptyset$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. In this case, $\mathcal{P}[[\alpha,\beta];(s_j)_{j=0}^{m}] = \{s_{m+1} \in \mathbb{C}^{q\times q} \colon (s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^{\geq}\}$.*

Proof. Use Theorem 4.3 and Lemma 4.7. □

Proposition 4.10 leads us to the problem of describing the set $\{s_{m+1} \in \mathbb{C}^{q\times q} \colon (s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^{\geq}\}$.

Remark 4.11. Let $\alpha,\beta \in \mathbb{C}$, let $\kappa \in \mathbb{N}\cup\{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$. Then

$$(\beta-\alpha)s_j = ({}_{\alpha}s)_j + (s_\beta)_j, \quad \text{and} \quad (\beta-\alpha)s_{j+1} = \beta({}_{\alpha}s)_j + \alpha(s_\beta)_j,$$

for all $j \in \mathbb{Z}_{0,\kappa-1}$. Furthermore, if $\kappa \geq 2$, then

$$({}_{\alpha}s_\beta)_j = -\alpha(s_\beta)_j + (s_\beta)_{j+1} = \Big({}_{\alpha}(s_\beta)\Big)_j, \quad ({}_{\alpha}s_\beta)_j = \beta({}_{\alpha}s)_j - ({}_{\alpha}s)_{j+1} = \Big(({}_{\alpha}s)_\beta\Big)_j,$$

and $(\beta-\alpha)s_{j+2} = \beta^2({}_{\alpha}s)_j + \alpha^2(s_\beta)_j - (\beta-\alpha)({}_{\alpha}s_\beta)_j$ for all $j \in \mathbb{Z}_{0,\kappa-2}$.

Remark 4.12. Let $\alpha,\beta \in \mathbb{C}$, let $\kappa \in \mathbb{N}\cup\{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$. Then $({}_{\alpha}H)_n = -\alpha H_n + K_n$ and $(H_\beta)_n = \beta H_n - K_n$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Furthermore, if $\kappa \geq 2$, then $({}_{\alpha}H_\beta)_n = -\alpha\beta H_n + (\alpha+\beta)K_n - G_n$ for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$.

Using Remark 4.12, we obtain:

Remark 4.13. Let $\alpha, \beta \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$. Then $(\beta-\alpha)H_n = ({}_\alpha H)_n + (H_\beta)_n$ and $(\beta-\alpha)K_n = \beta({}_\alpha H)_n + \alpha(H_\beta)_n$ for all $n \in \mathbb{N}_0$ with $2n+1 \le \kappa$. Furthermore, if $\kappa \ge 2$, then

$$({}_\alpha H_\beta)_n = -\alpha(H_\beta)_n + (K_\beta)_n = \Big({}_\alpha(H_\beta)\Big)_n,$$
$$({}_\alpha H_\beta)_n = \beta({}_\alpha H)_n - ({}_\alpha K)_n = \Big(({}_\alpha H)_\beta\Big)_n,$$

and $(\beta-\alpha)G_n = \beta^2({}_\alpha H)_n + \alpha^2(H_\beta)_n - (\beta-\alpha)({}_\alpha H_\beta)_n$ for all $n \in \mathbb{N}_0$ with $2n+2 \le \kappa$.

Lemma 4.14. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. If $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\ge}_{q,2n+1,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{>}_{q,2n+1,\alpha,\beta}$), then $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\ge}_{q,2n}$ (resp., $(s_j)_{j=0}^{2n} \in \mathcal{H}^{>}_{q,2n}$).*

Proof. By virtue of Remark 4.13, we have $H_n = (\beta-\alpha)^{-1}[({}_\alpha H)_n + (H_\beta)_n]$. In view of Definition 4.2(a) and Definition 3.1, the assertions follow. □

Lemma 4.15. *Let $\alpha \in \mathbb{R}$ and $\beta \in (\alpha,\infty)$. Then $\mathcal{F}^{\ge}_{q,\infty,\alpha,\beta} \subseteq \mathcal{H}^{\ge}_{q,\infty}$ and $\mathcal{F}^{>}_{q,\infty,\alpha,\beta} \subseteq \mathcal{H}^{>}_{q,\infty}$.*

Proof. This follows in view of Definition 4.6 from Lemma 4.14. □

Lemma 4.16. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}^{\ge}_{q,m,\alpha,\beta}$. Then $s_j \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,m}$ and $s_{2k} \in \mathbb{C}_{\ge}^{q\times q}$ for all $k \in \mathbb{N}_0$ with $2k \le m$.*

Proof. In the case $m = 2n$ with some $n \in \mathbb{N}_0$, the assertion follows from Definition 4.2 and Remark 3.7(a).

Now suppose that $m = 2n+1$ for some $n \in \mathbb{N}_0$. By virtue of Lemma 4.14, then $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\ge}_{q,2n}$. According to Remark 3.7(a), we obtain $s_j \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,2n}$ and $s_{2k} \in \mathbb{C}_{\ge}^{q\times q}$ for all $k \in \mathbb{Z}_{0,n}$. In view of Definition 4.2(a), furthermore $(({}_\alpha s)_j)_{j=0}^{2n} \in \mathcal{H}^{\ge}_{q,2n}$. Because of Remark 3.7(a), in particular ${}_\alpha s)_{2n} \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Since the matrix s_{2n+1} admits the representation $s_{2n+1} = ({}_\alpha s)_{2n} + \alpha s_{2n}$, we see that $s_{2n+1} \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ holds true as well. □

Lemma 4.17. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$, and let the sequence $(r_j)_{j=0}^{\kappa}$ be given by $r_j := (-1)^j s_j$. Then $H_n^{\langle r\rangle} = J_{p,n}H_nJ_{q,n}$ for all $n \in \mathbb{N}_0$ with $2n \le \kappa$ and $K_n^{\langle r\rangle} = -J_{p,n}K_nJ_{q,n}$ for all $n \in \mathbb{N}_0$ with $2n+1 \le \kappa$ and $G_n^{\langle r\rangle} = J_{p,n}G_nJ_{q,n}$ for all $n \in \mathbb{N}_0$ with $2n+2 \le \kappa$.*

Proof. Apply Lemma 3.11(a). □

Lemma 4.18. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$, and let the sequence $(r_j)_{j=0}^{\kappa}$ be given by $r_j := (-1)^j s_j$. Then:*

(a) *$\Theta_n^{\langle r\rangle} = \Theta_n$ for all $n \in \mathbb{N}_0$ with $2n-1 \le \kappa$.*
(b) *$M_n^{\langle r\rangle} = -M_n$ and $L_n^{\langle r\rangle} = L_n$ for all $n \in \mathbb{N}_0$ with $2n \le \kappa$.*

Proof. In view of Notation 3.4, we have $\Theta_0^{\langle r\rangle} = 0_{p\times q} = \Theta_0$ and $M_0^{\langle r\rangle} = 0_{p\times q} = -M_0$.

Suppose that $n \in \mathbb{N}$ with $2n-1 \leq \kappa$. We have $z_{n,2n-1}^{\langle r\rangle} = (-1)^n z_{n,2n-1} J_{q,n-1}$ and $y_{n,2n-1}^{\langle r\rangle} = (-1)^n J_{p,n-1} y_{n,2n-1}$. Lemma 4.17 and Remarks 3.10 and A.9 yield $(H_{n-1}^{\langle r\rangle})^\dagger = (J_{p,n-1} H_{n-1} J_{q,n-1})^\dagger = J_{q,n-1}(H_{n-1})^\dagger J_{p,n-1}$. In view of Remark 3.10, we obtain $\Theta_n^{\langle r\rangle} = (-1)^{2n} z_{n,2n-1} H_{n-1}^\dagger y_{n,2n-1} = \Theta_n$.

Now suppose that $n \in \mathbb{N}$ with $2n \leq \kappa$. We have then furthermore $y_{n+1,2n}^{\langle r\rangle} = (-1)^{n+1} J_{p,n-1} y_{n+1,2n}$. Using Remark 3.10, we obtain $M_n^{\langle r\rangle} = (-1)^{2n+1} z_{n,2n-1} H_{n-1}^\dagger y_{n+1,2n} = -M_n$. The rest follows from (a). □

Lemma 4.19. *Let $\alpha, \beta \in \mathbb{C}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{p\times q}$. Let the sequence $(r_j)_{j=0}^m$ be given by $r_j := (-1)^j s_j$.*

(a) *Let $m = 2n+1$ with some $n \in \mathbb{N}_0$. Then*

$$-(-\beta)H_n^{\langle r\rangle} + K_n^{\langle r\rangle} = J_{p,n}^*(H_\beta)_n J_{q,n} \quad \text{and} \quad (-\alpha)H_n^{\langle r\rangle} - K_n^{\langle r\rangle} = J_{p,n}^*({}_\alpha H)_n J_{q,n}.$$

(b) *Let $m = 2n$ with some $n \in \mathbb{N}$. Then*

$$-(-\beta)(-\alpha)H_{n-1}^{\langle r\rangle} + [(-\beta)+(-\alpha)]K_{n-1}^{\langle r\rangle} - G_{n-1}^{\langle r\rangle} = J_{p,n-1}^*({}_\alpha H_\beta)_{n-1} J_{q,n-1}.$$

Proof. (a) According to Lemma 4.17 and Remark 3.10, we have $H_n^{\langle r\rangle} = J_{p,n}^* H_n J_{q,n}$ and $K_n^{\langle r\rangle} = -J_{p,n}^* K_n J_{q,n}$. Hence, $-(-\beta)H_n^{\langle r\rangle} + K_n^{\langle r\rangle} = \beta J_{p,n}^* H_n J_{q,n} - J_{p,n}^* K_n J_{q,n} = J_{p,n}^*(H_\beta)_n J_{q,n}$ and, analogously, $(-\alpha)H_n^{\langle r\rangle} - K_n^{\langle r\rangle} = J_{p,n}^*({}_\alpha H)_n J_{q,n}$.

(b) According to Lemma 4.17 and Remark 3.10, we have $H_{n-1}^{\langle r\rangle} = J_{p,n-1}^* H_{n-1} J_{q,n-1}$, $K_{n-1}^{\langle r\rangle} = -J_{p,n-1}^* K_{n-1} J_{q,n-1}$, and $G_{n-1}^{\langle r\rangle} = J_{p,n-1}^* G_{n-1} J_{q,n-1}$. Hence,

$$\begin{aligned}
&-(-\beta)(-\alpha)H_{n-1}^{\langle r\rangle} + [(-\beta)+(-\alpha)]K_{n-1}^{\langle r\rangle} - G_{n-1}^{\langle r\rangle} \\
&= -\beta\alpha J_{p,n-1}^* H_{n-1} J_{q,n-1} + (\beta+\alpha) J_{p,n-1}^* K_{n-1} J_{q,n-1} - J_{p,n-1}^* G_{n-1} J_{q,n-1} \\
&= J_{p,n-1}^*({}_\alpha H_\beta)_{n-1} J_{q,n-1}.
\end{aligned}$$

□

The following technical result is useful for our further purposes.

Lemma 4.20. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Let the sequence $(r_j)_{j=0}^m$ be given by $r_j := (-1)^j s_j$. Then:*

(a) $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{F}_{q,m,-\beta,-\alpha}^{\geq}$.
(b) $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{>}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{F}_{q,m,-\beta,-\alpha}^{>}$.
(c) $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{e}}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{F}_{q,m,-\beta,-\alpha}^{\geq,\mathrm{e}}$.
(d) $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{>,\mathrm{e}}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{F}_{q,m,-\beta,-\alpha}^{>,\mathrm{e}}$.

Proof. (a) The case $m = 0$ is trivial. Assume that $m = 2n+1$ with some $n \in \mathbb{N}_0$. If $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq}$, then $\{({}_\alpha H)_n, (H_\beta)_n\} \subseteq \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$. Hence, $\{-(-\beta)H_n^{\langle r\rangle} + K_n^{\langle r\rangle}, (-\alpha)H_n^{\langle r\rangle} - K_n^{\langle r\rangle}\} \subseteq \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ follows from

Lemma 4.19(a). Thus, $(r_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,-\beta,-\alpha}$. If $(r_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,-\beta,-\alpha}$, then, in view of $(s_j)_{j=0}^{2n+1} = ((-1)^j r_j)_{j=0}^{2n+1}$, we get from the already proved implication that $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$. Assume that $m = 2n$ with some $n \in \mathbb{N}$. If $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$, then the matrices H_n and ${}_{\alpha}H_{\beta})_{n-1}$ are non-negative Hermitian. From $H_n^{\langle r\rangle} = J_{q,n}^* H_n J_{q,n}$ and Lemma 4.19(b) we see that the matrices $H_n^{\langle r\rangle}$ and $-(-\beta)(-\alpha)H_{n-1}^{\langle r\rangle} + [(-\beta) + (-\alpha)]K_{n-1}^{\langle r\rangle} - G_{n-1}^{\langle r\rangle}$ are non-negative Hermitian. Hence, $(r_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,-\beta,-\alpha}$. If $(r_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,-\beta,-\alpha}$, then, by $(s_j)_{j=0}^{2n} = ((-1)^j r_j)_{j=0}^{2n}$, we get from the already proved implication that $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$.

(b) This can be proved analogously to (a) using $\det J_{q,n} \neq 0$.

(c)–(d) The assertions of (c) and (d) follow immediately from (a) and (b), respectively. □

5. On the canonical CD-measure associated with a sequence belonging to $\mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$

In our joint paper [13] with Yu.M. Dyukarev, we constructed explicitly a molecular solution of Problem $\mathsf{M}[[\alpha,\beta]; (s_j)_{j=0}^{2n-1}, =]$. This solution comes from a solution of Problem $\mathsf{M}[\mathbb{R}; (s_j)_{j=0}^{2n-1}, =]$ obtained in [14]. Now we are looking for an appropriate modification of our method, which allows us to handle Problem $\mathsf{M}[[\alpha,\beta]; (s_j)_{j=0}^{2n}, =]$ in a similar way .

Definition 5.1 ([14, Definition 2.2]). Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n}$. Then $(s_j)_{j=0}^{2n}$ is called *Hankel completely degenerate* if $L_n = 0_{q\times q}$, where L_n is given in Notation 3.4(c). We denote by $\mathcal{H}^{\geq,\mathrm{cd}}_{q,2n}$ the set of all sequences $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n}$ which are Hankel completely degenerate.

Definition 5.2 ([14, Definition 2.35]). Let $(s_j)_{j=0}^{\infty} \in \mathcal{H}^{\geq}_{q,\infty}$.

(a) Let $n \in \mathbb{N}_0$. Then $(s_j)_{j=0}^{\infty}$ is called *Hankel completely degenerate of order* n if $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq,\mathrm{cd}}_{q,2n}$.

(b) The sequence $(s_j)_{j=0}^{\infty}$ is called *Hankel completely degenerate* if there exists an $n \in \mathbb{N}_0$ such that $(s_j)_{j=0}^{\infty}$ is Hankel completely degenerate of order n

We denote by $\mathcal{H}^{\geq,\mathrm{cd}}_{q,\infty}$ the set of all sequences $(s_j)_{j=0}^{\infty} \in \mathcal{H}^{\geq}_{q,\infty}$ which are Hankel completely degenerate.

Theorem 5.3. *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n-1} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1}$. Then:*

(a) *There is a unique sequence $(s_k)_{k=2n}^{\infty}$ from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{\infty}$ is a Hankel completely degenerate Hankel non-negative definite sequence of order n.*

(b) *The set $\mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{\infty}, =]$ contains exactly one element σ_n.*

(c) *The measure σ_n is molecular.*

Proof. From [14, Proposition 2.38] we get (a). In view of (a) the assertions of (b) and (c) follow from parts (a) and (b) of Proposition 4.9 in [14], respectively. □

If $n \in \mathbb{N}$ and if $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}$, then the measure σ_n given via Theorem 5.3(b) is called the *Hankel completely degenerate non-negative Hermitian measure* (short *CD-measure*) associated with $(s_j)_{j=0}^{2n-1}$. (For additional information, we refer to [14, Chs. 4 and 5].) In [13], we found a molecular solution for Problem $\mathsf{M}[[\alpha,\beta]; (s_j)_{j=0}^{2n-1}, =]$ in the case of a sequence $(s_j)_{j=0}^{2n-1} \in \mathcal{F}_{q,2n-1,\alpha,\beta}^{\geq}$. This will be explained now in more detail.

Theorem 5.4. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n-1} \in \mathcal{F}_{q,2n-1,\alpha,\beta}^{\geq}$. Then $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}$. Moreover, the CD-measure σ_n associated with $(s_j)_{j=0}^{2n-1}$ fulfills $\sigma_n(\mathbb{R} \setminus [\alpha,\beta]) = 0_{q\times q}$ and the measure $\underline{\sigma}_{2n-1} := \mathrm{Rstr}_{\mathfrak{B}_{[\alpha,\beta]}} \sigma_n$ is molecular and satisfies $\underline{\sigma}_{2n-1} \in \mathcal{M}_{\geq}^{q}[[\alpha,\beta]; (s_j)_{j=0}^{2n-1}, =]$.*

Proof. Because of [13, Lemma 2.21], the sequence $(s_j)_{j=0}^{2n-1}$ belongs to $\mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}$. In view of $(s_j)_{j=0}^{2n-1} \in \mathcal{F}_{q,2n-1,\alpha,\beta}^{\geq}$, we see from formulas (1.2) and (1.3) in [13] that in [13, Theorem 1.1] we can particularly choose $\Omega = [\alpha,\beta]$. Then from [13, Proof of Theorem 1.1, p. 921] we see that $\sigma_n(\mathbb{R} \setminus [\alpha,\beta]) = 0_{q\times q}$. From Theorem 5.3(c) we see that σ_n is molecular. In view of $\sigma_n(\mathbb{R} \setminus [\alpha,\beta]) = 0_{q\times q}$ and Theorem 5.3(b), we obtain the rest. □

Proposition 5.5. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n} \in \mathcal{F}_{q,2n,\alpha,\beta}^{\geq,\mathrm{e}}$. Further, let $s_{2n+1} \in \mathbb{C}^{q\times q}$ be such that $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq}$. Then $\underline{\sigma}_{2n+1}$ given in Theorem 5.4 belongs to $\mathcal{M}_{\geq}^{q}[[\alpha,\beta]; (s_j)_{j=0}^{2n}, =] \cap \mathcal{M}_{\geq}^{q,\mathrm{mol}}([\alpha,\beta])$.*

Proof. Use Theorem 5.4. □

Corollary 5.6. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n} \in \mathcal{F}_{q,2n,\alpha,\beta}^{\geq,\mathrm{e}}$. Then $\mathcal{M}_{\geq}^{q}[[\alpha,\beta]; (s_j)_{j=0}^{2n}, =] \cap \mathcal{M}_{\geq}^{q,\mathrm{mol}}([\alpha,\beta]) \neq \emptyset$.*

Proof. In view of Definition 4.4, we have $\{s_{2n+1} \in \mathbb{C}^{q\times q} \colon (s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq}\} \neq \emptyset$. Now the assertion follows from Proposition 5.5. □

In the situation of Proposition 5.5, we see that each matrix $s_{2n+1} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq}$ generates a molecular solution of Problem $\mathsf{M}[[\alpha,\beta]; (s_j)_{j=0}^{2n}, =]$. In order to get more information about this family of molecular measures, we are led to the problem of describing the set $\{s_{2n+1} \in \mathbb{C}^{q\times q} \colon (s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq}\}$. This problem is of purely algebraic nature as well as our method to handle it. We will carefully study the intrinsic structure of finite $[\alpha,\beta]$-Hausdorff non-negative definite sequences. More precisely, if $m \in \mathbb{N}$ and if $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$, then we will show in Section 10 that, for each $k \in \mathbb{Z}_{1,m}$, the matrix s_k belongs to some matrix interval $[\mathfrak{a}_{k-1}, \mathfrak{b}_{k-1}]$, where $\mathfrak{a}_{k-1}$ and $\mathfrak{b}_{k-1}$ will be explicitly computed in terms of the sequence $(s_j)_{j=0}^{k-1}$. Particular attention is drawn to the case that s_k coincides with one of the interval ends $\mathfrak{a}_{k-1}$ or $\mathfrak{b}_{k-1}$.

6. On Hankel non-negative definite extendable sequences

The class of $[\alpha, \beta]$-Hausdorff non-negative definite sequences stands in the center of this paper. If we look back at Definition 4.2, where we have introduced it, then we see immediately that such a sequence is determined by the interplay of two Hankel non-negative definite sequences. Hence, our further considerations will be mainly based on the theory of Hankel non-negative definite sequences as well as the theory of Hankel non-negative definite extendable sequences. For this reason, in this section we sketch some features of this theory, which turn out to be important for our further considerations.

Lemma 6.1 ([22, Lemma 3.1(a)–(d)]). *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$. Then:*

(a) *$s_j \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $s_{2k} \in \mathbb{C}_{\geq}^{q\times q}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$.*

(b) *$\bigcup_{j=2k}^{\kappa} \mathcal{R}(s_j) \subseteq \mathcal{R}(s_{2k})$ and $\mathcal{N}(s_{2k}) \subseteq \bigcap_{j=2k}^{\kappa} \mathcal{N}(s_j)$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$.*

Lemma 6.2 (cf. [14, Lemma 2.7]). *Let $n \in \mathbb{N}$. Then*

$$\mathcal{H}_{q,2n}^{\geq,\mathrm{e}} = \Big\{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \colon \mathcal{R}(y_{n+1,2n}) \subseteq \mathcal{R}(H_{n-1})\Big\}.$$

Remark 6.3. For each $n \in \mathbb{N}_0$ we see from [14, Proposition 2.24] that $\mathcal{H}_{q,2n}^{>,\mathrm{e}} = \mathcal{H}_{q,2n}^{>}$.

Proposition 6.4 (cf. [14, Proposition 2.13]). *Let $n \in \mathbb{N}$. Then*

$$\mathcal{H}_{q,2n}^{\geq,\mathrm{e}} = \Big\{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \colon \mathcal{N}(L_{n-1}) \subseteq \mathcal{N}(L_n)\Big\}.$$

Proposition 6.4 indicates an essential difference between the set $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ and the set of finite non-negative definite sequences from $\mathbb{C}^{q\times q}$. Proposition 6.4 tells us that there is an inclusion of the null spaces of the consecutive associated Schur complements introduced in Notation 3.4(c), whereas [16, Lemma 6] shows that the corresponding consecutive Schur complements associated with a (Toeplitz-)non-negative definite sequence from $\mathbb{C}^{q\times q}$ are even decreasing with respect to the Löwener semiordering in the set $\mathbb{C}_{\mathrm{H}}^{q\times q}$. Clearly, this implies the inclusion for the corresponding null spaces.

Lemma 6.5 (cf. [14, Lemma 2.16]). *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$, $s_{2n+1}^* = s_{2n+1}$, and $\mathcal{R}(y_{n+1,2n+1}) \subseteq \mathcal{R}(H_n)$.*

Remark 6.6. Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then from [14, Proposition 2.24] we see that $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{>,\mathrm{e}}$ if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$ and $s_{2n+1}^* = s_{2n+1}$.

Proposition 6.7 (cf. [14, Proposition 2.22(a)]). *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$. Then*

$$\Big\{s_{2n+2} \in \mathbb{C}^{q\times q} \colon (s_j)_{j=0}^{2n+2} \in \mathcal{H}_{q,2n+2}^{\geq}\Big\} = \big\{X \in \mathbb{C}_{\mathrm{H}}^{q\times q} \colon X \geq \Theta_{n+1}\big\}.$$

Proposition 6.8. *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{>,\mathrm{e}}$. Then*

$$\big\{s_{2n+2} \in \mathbb{C}^{q\times q}\colon (s_j)_{j=0}^{2n+2} \in \mathcal{H}_{q,2n+2}^{>}\big\} = \big\{X \in \mathbb{C}_{\mathrm{H}}^{q\times q}\colon X > \Theta_{n+1}\big\}.$$

Proof. This follows from Remark 6.6 and [14, Proposition 2.24]. □

We denote by $\mathcal{D}_{p\times q,\kappa}$ the set of all sequences $(s_j)_{j=0}^{\kappa}$ from $\mathbb{C}^{p\times q}$ which fulfill $\bigcup_{j=0}^{\kappa} \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)$ and $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^{\kappa} \mathcal{N}(s_j)$.

Proposition 6.9 ([22, Proposition 4.24]). *Let $m \in \mathbb{N}_0$. Then $\mathcal{H}_{q,m}^{\geq,\mathrm{e}} \subseteq \mathcal{D}_{q\times q,m}$.*

7. Algebraic approach to Proposition 4.5

A main theme of our following considerations is to gain more insights into the intrinsic structure of $[\alpha,\beta]$-Hausdorff non-negative definite sequences. Such sequences are characterized by a particular interplay between some Hankel non-negative definite sequences. More precisely, we have to organize the interplay between several non-negative Hermitian block Hankel matrices. The difficulty of the situation results from the fact that the block Hankel matrix of step n is not a principle submatrix of the block Hankel matrix of stage $n+1$. It will turn out (see Lemma 7.2) that the parallel sum of matrices (see Appendix B) will be the essential tool to overcome this difficulty.

The construction of a purely algebraic approach to Proposition 4.5 stands in the center of this section. Our proof of Proposition 4.5 above is based on Theorem 4.3, which gives a characterization of the solvability of the truncated $[\alpha,\beta]$-Hausdorff moment problem. Since Proposition 4.5 is a statement of purely algebraic character, for our further purposes it is advantageous to have an independent algebraic proof as well.

The key instrument of our purely algebraic approach to Theorem 4.9 is to find convenient expressions for the relevant block Hankel matrices in terms of parallel sums of matrices and applying Lemma B.2. For every choice of complex $p\times q$ matrices A and B, let $A \mathbin{\overline{+}} B$ be the *parallel sum of A and B*, i.e., let

$$A \mathbin{\overline{+}} B := A(A+B)^{\dagger}B.$$

Remark 7.1. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then the block Hankel matrix H_n admits the block representations $H_n = \mathrm{row}(y_{k,k+n})_{k=0}^{n}$ and $H_n = \mathrm{col}(z_{j,j+n})_{j=0}^{n}$.

For all $n \in \mathbb{N}$, let $\Delta_{q,n} := \left[\begin{smallmatrix} I_{nq} \\ 0_{q\times nq} \end{smallmatrix}\right]$ and $\nabla_{q,n} := \left[\begin{smallmatrix} 0_{q\times nq} \\ I_{nq} \end{smallmatrix}\right]$.

In [33, proof of Lemma 2.5], the following result (formulated for the square case $p=q$) was already proved. Now we give an alternative proof.

Lemma 7.2. *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{p\times q}$ such that*

$$\mathcal{R}(y_{n+1,2n+1}) \subseteq \mathcal{R}(H_n) \quad \text{and} \quad \mathcal{N}(H_n) \subseteq \mathcal{N}(z_{n+1,2n+1}). \tag{7.1}$$

For every choice of complex numbers α and β with $\alpha \neq \beta$, then

$$({}_{\alpha}H_{\beta})_{n-1} = (\beta-\alpha)\Delta_{p,n}^{*}\big[({}_{\alpha}H)_n \mathbin{\overline{+}} (H_{\beta})_n\big]\Delta_{q,n}.$$

Proof. Using Remark 4.13, we see that

$$({}_\alpha H)_n \mathbin{\overline{+}} (H_\beta)_n = ({}_\alpha H)_n \big[({}_\alpha H)_n + (H_\beta)_n\big]^\dagger (H_\beta)_n = \frac{1}{\beta-\alpha} ({}_\alpha H)_n H_n^\dagger (H_\beta)_n. \tag{7.2}$$

Setting $Y := \operatorname{row}(y_{k,k+n})_{k=1}^n$ and $Z := \operatorname{col}(z_{j,j+n})_{j=1}^n$, from Remark 7.1 we see that H_n and K_n can be represented via

$$H_n = [y_{0,n}, Y], \quad H_n = \begin{bmatrix} z_{0,n} \\ Z \end{bmatrix}, \quad K_n = [Y, y_{n+1,2n+1}], \quad \text{and} \quad K_n = \begin{bmatrix} Z \\ z_{n+1,2n+1} \end{bmatrix}.$$

Thus, (7.1) implies $\mathcal{R}(K_n) \subseteq \mathcal{R}(H_n)$ and $\mathcal{N}(H_n) \subseteq \mathcal{N}(K_n)$. Consequently, Remarks A.5 and A.6 provide us $H_n H_n^\dagger K_n = K_n$ and $K_n H_n^\dagger H_n = K_n$. Hence, in view of Remark 4.12, we get then

$$\begin{aligned} ({}_\alpha H)_n H_n^\dagger (H_\beta)_n &= (-\alpha H_n + K_n) H_n^\dagger (\beta H_n - K_n) \\ &= -\alpha\beta H_n H_n^\dagger H_n + \alpha H_n H_n^\dagger K_n + \beta K_n H_n^\dagger H_n - K_n H_n^\dagger K_n \\ &= -\alpha\beta H_n + (\alpha+\beta) K_n - K_n H_n^\dagger K_n. \end{aligned} \tag{7.3}$$

Obviously, $\Delta_{p,n}^* K_n = Z = \nabla_{p,n}^* H_n$ and $K_n \Delta_{q,n} = Y = H_n \nabla_{q,n}$ and, consequently,

$$\Delta_{p,n}^* K_n H_n^\dagger K_n \Delta_{q,n} = \nabla_{p,n}^* H_n H_n^\dagger H_n \nabla_{q,n} = \nabla_{p,n}^* H_n \nabla_{q,n} = G_{n-1}. \tag{7.4}$$

Furthermore, Remark 3.5 yields $\Delta_{p,n}^* H_n \Delta_{q,n} = H_{n-1}$ and $\Delta_{p,n}^* K_n \Delta_{q,n} = K_{n-1}$. Using the last two equations, (7.4), (7.3), and (7.2), we get then

$$\begin{aligned} ({}_\alpha H_\beta)_{n-1} &= -\alpha\beta H_{n-1} + (\alpha+\beta) K_{n-1} - G_{n-1} \\ &= \Delta_{p,n}^* \big[-\alpha\beta H_n + (\alpha+\beta) K_n - K_n H_n^\dagger K_n\big] \Delta_{q,n} \\ &= \Delta_{p,n}^* ({}_\alpha H)_n H_n^\dagger (H_\beta)_n \Delta_{q,n} = (\beta-\alpha) \Delta_{p,n}^* \big[({}_\alpha H)_n \mathbin{\overline{+}} (H_\beta)_n\big] \Delta_{q,n}. \quad \square \end{aligned}$$

Remark 7.3. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{p\times q}$. For each $n \in \mathbb{N}$ with $2n \le \kappa$, then the block Hankel matrix H_n admits the block representations

$$H_n = \begin{bmatrix} H_{n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{bmatrix}, \qquad H_n = \begin{bmatrix} y_{0,n} & K_{n-1} \\ s_n & z_{n+1,2n} \end{bmatrix},$$

$$H_n = \begin{bmatrix} z_{0,n-1} & s_n \\ K_{n-1} & y_{n+1,2n} \end{bmatrix}, \qquad \text{and} \qquad H_n = \begin{bmatrix} s_0 & z_{1,n} \\ y_{1,n} & G_{n-1} \end{bmatrix}.$$

Lemma 7.4. *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+2}$ be a sequence from $\mathbb{C}^{p\times q}$. For every choice of α and β in $\mathbb{C}$, then*

$$(\beta-\alpha)({}_\alpha H)_n = (\nabla_{p,n+1} - \overline{\alpha}\Delta_{p,n+1})^* H_{n+1} (\nabla_{q,n+1} - \alpha\Delta_{q,n+1}) + ({}_\alpha H_\beta)_n \tag{7.5}$$

and

$$(\beta-\alpha)(H_\beta)_n = (\overline{\beta}\Delta_{p,n+1} - \nabla_{p,n+1})^* H_{n+1} (\beta\Delta_{q,n+1} - \nabla_{q,n+1}) + ({}_\alpha H_\beta)_n. \tag{7.6}$$

Proof. Using Remark 7.3, we conclude

$$\Delta_{p,n+1}^{*} H_{n+1} \Delta_{q,n+1} = H_n, \quad \Delta_{p,n+1}^{*} H_{n+1} \nabla_{q,n+1} = K_n,$$
$$\nabla_{p,n+1}^{*} H_{n+1} \Delta_{q,n+1} = K_n, \quad \text{and} \quad \nabla_{p,n+1}^{*} H_{n+1} \nabla_{q,n+1} = G_n.$$

For each $\alpha \in \mathbb{C}$ and each $\beta \in \mathbb{C}$, this implies (7.5) and (7.6). □

Lemma 7.5. *Let $\alpha \in \mathbb{R}$, $\beta \in (\alpha, \infty)$, and $n \in \mathbb{N}_0$. Then $\mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta} \subseteq \mathcal{H}^{\geq,\mathrm{e}}_{q,2n+1}$.*

Proof. This can be seen from [13, Lemma 2.21]. □

Remark 7.6. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, and let $n \in \mathbb{N}_0$. Then, we see from Lemma 7.5 and Proposition 6.9 that $\mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta} \subseteq \mathcal{D}_{q\times q,2n+1}$.

Proposition 7.7. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *If $(s_j)_{j=0}^{m} \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$, then $(s_j)_{j=0}^{\ell} \in \mathcal{F}^{\geq}_{q,\ell,\alpha,\beta}$ for each $\ell \in \mathbb{Z}_{0,m}$.*
(b) *If $(s_j)_{j=0}^{m} \in \mathcal{F}^{>}_{q,m,\alpha,\beta}$, then $(s_j)_{j=0}^{\ell} \in \mathcal{F}^{>}_{q,\ell,\alpha,\beta}$ for each $\ell \in \mathbb{Z}_{0,m}$.*

Proof. The case $m = 0$ is trivial. Assume that $m \geq 1$ and $(s_j)_{j=0}^{m} \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{m} \in \mathcal{F}^{>}_{q,m,\alpha,\beta}$). It is sufficient to prove that $(s_j)_{j=0}^{m-1}$ belongs to $\mathcal{F}^{\geq}_{q,m-1,\alpha,\beta}$ (resp., $\mathcal{F}^{>}_{q,m-1,\alpha,\beta}$).

First we suppose that there is an integer $n \in \mathbb{N}_0$ such that $m = 2n + 1$. Then Lemma 4.14 yields that the matrix H_n is non-negative Hermitian (resp., positive Hermitian). If $n = 0$, then $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n} \in \mathcal{F}^{>}_{q,2n,\alpha,\beta}$) immediately follows from Definition 4.2. Now we assume that $n \geq 1$. In view of Lemma 4.16, we have $s_j^{*} = s_j$ for all $j \in \mathbb{Z}_{0,2n+1}$. The combination of Lemmata 7.5 and 6.5 provides us with $\mathcal{R}(y_{n+1,2n+1}) \subseteq \mathcal{R}(H_n)$ and $\mathcal{N}(H_n) = \mathcal{N}(H_n^{*}) = \mathcal{R}(H_n)^{\perp} \subseteq \mathcal{R}(y_{n+1,2n+1})^{\perp} = \mathcal{N}(y_{n+1,2n+1}^{*}) = \mathcal{N}(z_{n+1,2n+1})$. From Lemma 7.2 we get then

$$({}_{\alpha}H_{\beta})_{n-1} = (\beta - \alpha)\Delta_{q,n}^{*}\big[({}_{\alpha}H)_n \mathbin{\dot{-}} (H_{\beta})_n\big]\Delta_{q,n}. \tag{7.7}$$

Because of $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{>}_{q,2n+1,\alpha,\beta}$), Definition 4.2(a) shows that the matrices $({}_{\alpha}H)_n$ and $(H_{\beta})_n$ are both non-negative Hermitian (resp., positive Hermitian). Thus, Lemma B.2 (resp., Remark B.1) implies that the matrix $({}_{\alpha}H)_n \mathbin{\dot{-}} (H_{\beta})_n$ is non-negative Hermitian (resp., positive Hermitian) as well. Because of $\beta - \alpha > 0$ and $\operatorname{rank}\Delta_{q,n} = nq$, then (7.7) und Remark A.2 yield that the matrix $({}_{\alpha}H_{\beta})_{n-1}$ is non-negative Hermitian (resp., positive Hermitian). Consequently, in view of Definition 4.2(b), we see that $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$ (resp., $\mathcal{F}^{>}_{q,2n,\alpha,\beta}$).

It remains to consider the case that $m = 2n$ holds true with some $n \in \mathbb{N}$. By virtue of $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n} \in \mathcal{F}^{>}_{q,2n,\alpha,\beta}$) and Definition 4.2(b), the matrix $({}_{\alpha}H_{\beta})_{n-1}$ is non-negative Hermitian (resp., positive Hermitian). Lemma 7.4 and $\beta > \alpha$ yield $({}_{\alpha}H)_{n-1} \geq (\beta - \alpha)^{-1}({}_{\alpha}H_{\beta})_{n-1}$ and $(H_{\beta})_{n-1} \geq (\beta - \alpha)^{-1}({}_{\alpha}H_{\beta})_{n-1}$. Hence, the matrices $({}_{\alpha}H)_{n-1}$ and $(H_{\beta})_{n-1}$ are

both non-negative Hermitian (resp., positive Hermitian). Thus, Definition 4.2(a) implies $(s_j)_{j=0}^{2n-1} \in \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n-1} \in \mathcal{F}^{>}_{q,2n-1,\alpha,\beta}$). □

Corollary 7.8. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *If $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$, then $L_n \in \mathbb{C}^{q\times q}_{\geq}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, furthermore $\{({}_\alpha L)_n, (L_\beta)_n\} \subseteq \mathbb{C}^{q\times q}_{\geq}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, and $({}_\alpha L_\beta)_n \in \mathbb{C}^{q\times q}_{\geq}$ for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$.*

(b) *If $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{>}_{q,\kappa,\alpha,\beta}$, then $L_n \in \mathbb{C}^{q\times q}_{>}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, furthermore $\{({}_\alpha L)_n, (L_\beta)_n\} \subseteq \mathbb{C}^{q\times q}_{>}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, and $({}_\alpha L_\beta)_n \in \mathbb{C}^{q\times q}_{>}$ for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$.*

Proof. Use Proposition 7.7, Definitions 4.2 and 4.6, and Remark 3.6. □

Corollary 7.9. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, and let $m \in \mathbb{N}_0$. Then $\mathcal{F}^{\geq,\mathrm{e}}_{q,m,\alpha,\beta} \subseteq \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$ and $\mathcal{F}^{>,\mathrm{e}}_{q,m,\alpha,\beta} \subseteq \mathcal{F}^{>}_{q,m,\alpha,\beta}$.*

Proof. Use Proposition 7.7. □

Proposition 7.10. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, and let $m \in \mathbb{N}_0$. Then $\mathcal{F}^{\geq}_{q,m,\alpha,\beta} \subseteq \mathcal{H}^{\geq,\mathrm{e}}_{q,m}$ and $\mathcal{F}^{>}_{q,m,\alpha,\beta} \subseteq \mathcal{H}^{>,\mathrm{e}}_{q,m}$.*

Proof. We first show $\mathcal{F}^{\geq}_{q,m,\alpha,\beta} \subseteq \mathcal{H}^{\geq,\mathrm{e}}_{q,m}$. Because of $\mathcal{H}^{\geq,\mathrm{e}}_{q,0} = \mathcal{H}^{\geq}_{q,0}$, the case $m = 0$ is trivial. If $m = 2n+1$ for some $n \in \mathbb{N}_0$, then Lemma 7.5 yields the assertion.

Now let $m = 2n$ with some $n \in \mathbb{N}$. We consider an arbitrary sequence $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$. Then Definition 4.2(b) and Remark 7.3 provide us $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n}$ and $\{({}_\alpha H_\beta)_{n-1}, G_{n-1}\} \subseteq \mathbb{C}^{nq\times nq}_{\geq}$. Thus, Remark A.1 yields $\mathcal{R}(G_{n-1}+({}_\alpha H_\beta)_{n-1}) = \mathcal{R}(G_{n-1})+\mathcal{R}(({}_\alpha H_\beta)_{n-1})$. According to Proposition 7.7(a), the sequence $(s_j)_{j=0}^{2n-1}$ belongs to $\mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$. Consequently, Definition 4.2(a) yields $\{({}_\alpha H)_{n-1}, (H_\beta)_{n-1}\} \subseteq \mathbb{C}^{nq\times nq}_{\geq}$. Hence, using Remark A.1 again, we conclude $\mathcal{R}(({}_\alpha H)_{n-1} + (H_\beta)_{n-1}) = \mathcal{R}(({}_\alpha H)_{n-1}) + \mathcal{R}((H_\beta)_{n-1})$. Since Remark 4.13 shows that $H_{n-1} = \frac{1}{\beta-\alpha}[({}_\alpha H)_{n-1}+(H_\beta)_{n-1}]$ and $G_{n-1}+({}_\alpha H_\beta)_{n-1} = \frac{1}{\beta-\alpha}[\beta^2({}_\alpha H)_{n-1}+\alpha^2(H_\beta)_{n-1}]$, we get then

$$\begin{aligned}\mathcal{R}(G_{n-1}) &\subseteq \mathcal{R}(G_{n-1}) + \mathcal{R}\big(({}_\alpha H_\beta)_{n-1}\big)\\ &= \mathcal{R}\big(\beta^2({}_\alpha H)_{n-1} + \alpha^2(H_\beta)_{n-1}\big) \subseteq \mathcal{R}\big(\beta^2({}_\alpha H)_{n-1}\big) + \mathcal{R}\big(\alpha^2(H_\beta)_{n-1}\big)\\ &\subseteq \mathcal{R}\big(({}_\alpha H)_{n-1}\big) + \mathcal{R}\big((H_\beta)_{n-1}\big) = \mathcal{R}(H_{n-1}).\end{aligned}$$

Taking into account that $y_{n+1,2n}$ is the last $nq \times q$ block column of G_{n-1}, we infer $\mathcal{R}(y_{n+1,2n}) \subseteq \mathcal{R}(H_{n-1})$. Thus, Lemma 6.2 yields $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n}$.

Now we show $\mathcal{F}^{>}_{q,m,\alpha,\beta} \subseteq \mathcal{H}^{>,\mathrm{e}}_{q,m}$. If $m = 2n$ with some $n \in \mathbb{N}_0$, this follows from Remark 6.3 and Definition 4.2(b). If $m = 2n+1$ with some $n \in \mathbb{N}_0$, this follows from Remark 6.6 in view of Lemmata 4.14 and 4.16. □

8. On Stieltjes non-negative definite sequences

In this section, we discuss two dual classes of finite sequences of complex $q \times q$ matrices which turn out to be intimately connected with the class $\mathcal{F}^{\geq}_{q,m,\alpha,\beta}$. The first and second types of this classes will be associated with the left endpoint α and the right endpoint β of the interval $[\alpha, \beta]$, respectively. These classes were already studied by the authors in earlier work (see [14, 20, 21]). First we consider the class of sequences connected with the left endpoint α of $[\alpha, \beta]$.

Definition 8.1. Let $\alpha \in \mathbb{R}$ and let $n \in \mathbb{N}_0$.

(a) Let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n+1}$ is called $[\alpha, \infty)$-*Stieltjes non-negative definite* (resp., $[\alpha, \infty)$-*Stieltjes positive definite*) if both sequences $(s_j)_{j=0}^{2n}$ and $((_\alpha s)_j)_{j=0}^{2n}$ are Hankel non-negative definite (resp., Hankel positive definite).

(b) Let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n}$ is called $[\alpha, \infty)$-*Stieltjes non-negative definite* (resp., $[\alpha, \infty)$-*Stieltjes positive definite*) if $(s_j)_{j=0}^{2n}$ and, in the case $n \geq 1$, also $((_\alpha s)_j)_{j=0}^{2(n-1)}$ is Hankel non-negative definite (resp., Hankel positive definite).

If $m \in \mathbb{N}_0$, then $\mathcal{K}^{\geq}_{q,m,\alpha}$ (resp., $\mathcal{K}^{>}_{q,m,\alpha}$) stands for the set of all $[\alpha, \infty)$-Stieltjes non-negative definite (resp., $[\alpha, \infty)$-Stieltjes positive definite) sequences $(s_j)_{j=0}^{m}$ from $\mathbb{C}^{q\times q}$.

Definition 8.2. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{m}$ is called $[\alpha, \infty)$-*Stieltjes non-negative definite extendable* (resp., $[\alpha, \infty)$-*Stieltjes positive definite extendable*) if there exists a matrix $s_{m+1} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{m+1} \in \mathcal{K}^{\geq}_{q,m+1,\alpha}$ (resp., $(s_j)_{j=0}^{m+1} \in \mathcal{K}^{>}_{q,m+1,\alpha}$). We denote by $\mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$ (resp., $\mathcal{K}^{>,\mathrm{e}}_{q,m,\alpha}$) the set of all $[\alpha, \infty)$-Stieltjes non-negative definite extendable (resp., $[\alpha, \infty)$-Stieltjes positive definite extendable) sequences $(s_j)_{j=0}^{m}$ from $\mathbb{C}^{q\times q}$.

The importance of the class $\mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$ in the context of moment problems is caused by the following observation:

Theorem 8.3 ([13, Theorem 1.3]). *Let $\alpha \in \mathbb{R}$, $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{M}^{q}_{\geq}[[\alpha, \infty); (s_j)_{j=0}^{m}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$.*

Lemma 8.4 ([20, Lemma 2.9(a)–(b)]). *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq}_{q,m,\alpha}$. Then $s_j \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ for all $j \in \mathbb{Z}_{0,m}$ and $s_{2k} \in \mathbb{C}^{q\times q}_{\geq}$ for all $k \in \mathbb{N}_0$ with $2k \leq m$. Furthermore, $(_\alpha s)_j \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ for all $j \in \mathbb{Z}_{0,m-1}$ and $(_\alpha s)_{2k} \in \mathbb{C}^{q\times q}_{\geq}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \leq m$.*

Remark 8.5. Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$. Then $\mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha} \subseteq \mathcal{K}^{\geq}_{q,m,\alpha}$ and, in the case $m \in \mathbb{N}$, furthermore $\mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha} \neq \mathcal{K}^{\geq}_{q,m,\alpha}$.

Remark 8.6. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq}_{q,m,\alpha}$. Then $(s_j)_{j=0}^{\ell} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,\ell,\alpha}$ for all $\ell \in \mathbb{Z}_{0,m-1}$.

Remark 8.7. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$ (resp., $\mathcal{K}_{q,m,\alpha}^{>}$). Then $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq,\mathrm{e}}$ (resp., $\mathcal{K}_{q,\ell,\alpha}^{>}$) for all $\ell \in \mathbb{Z}_{0,m}$.

Definition 8.8. Let $\alpha \in \mathbb{R}$. A sequence $(s_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$ is called $[\alpha,\infty)$-*Stieltjes non-negative definite* (resp., $[\alpha,\infty)$-*Stieltjes positive definite*) if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ (resp., $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{>}$) for all $m \in \mathbb{N}_0$. We denote by $\mathcal{K}_{q,\infty,\alpha}^{\geq}$ (resp., $\mathcal{K}_{q,\infty,\alpha}^{>}$) the set of all $[\alpha,\infty)$-Stieltjes non-negative definite (resp., $[\alpha,\infty)$-Stieltjes positive definite) sequences $(s_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$. Furthermore, let $\mathcal{K}_{q,\infty,\alpha}^{\geq,\mathrm{e}} := \mathcal{K}_{q,\infty,\alpha}^{\geq}$ and let $\mathcal{K}_{q,\infty,\alpha}^{>,\mathrm{e}} := \mathcal{K}_{q,\infty,\alpha}^{>}$.

Lemma 8.9. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$. Then $L_n \in \mathbb{C}_{\geq}^{q\times q}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, and $({}_{\alpha}L)_n \in \mathbb{C}_{\geq}^{q\times q}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.*

Proof. Combine Definition 8.1, Remarks 8.6 and 8.5, and Remark 3.6(a). □

Proposition 8.10 ([20, Proposition 2.19]). *Let $\alpha \in \mathbb{R}$. Then:*

(a) *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}}$, then*

$$\mathcal{R}(L_0) \supseteq \mathcal{R}(({}_{\alpha}L)_0) \supseteq \mathcal{R}(L_1) \supseteq \mathcal{R}(({}_{\alpha}L)_1) \supseteq \cdots \supseteq \mathcal{R}(L_n) \supseteq \mathcal{R}(({}_{\alpha}L)_n)$$

and

$$\mathcal{N}(L_0) \subseteq \mathcal{N}(({}_{\alpha}L)_0) \subseteq \mathcal{N}(L_1) \subseteq \mathcal{N}(({}_{\alpha}L)_1) \subseteq \cdots \subseteq \mathcal{N}(L_n) \subseteq \mathcal{N}(({}_{\alpha}L)_n).$$

(b) *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq,\mathrm{e}}$, then*

$$\mathcal{R}(L_0) \supseteq \mathcal{R}(({}_{\alpha}L)_0) \supseteq \mathcal{R}(L_1) \supseteq \mathcal{R}(({}_{\alpha}L)_1) \supseteq \cdots \supseteq \mathcal{R}\big(({}_{\alpha}L)_{n-1}\big) \supseteq \mathcal{R}(L_n)$$

and

$$\mathcal{N}(L_0) \subseteq \mathcal{N}(({}_{\alpha}L)_0) \subseteq \mathcal{N}(L_1) \subseteq \mathcal{N}(({}_{\alpha}L)_1) \subseteq \cdots \subseteq \mathcal{N}\big(({}_{\alpha}L)_{n-1}\big) \subseteq \mathcal{N}(L_n).$$

Now we consider the constructions connected with the right endpoint β of the interval $[\alpha,\beta]$.

Definition 8.11. Let $\beta \in \mathbb{R}$ and let $n \in \mathbb{N}_0$.

(a) Let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n+1}$ is called $(-\infty,\beta]$-*Stieltjes non-negative definite* (resp., $(-\infty,\beta]$-*Stieltjes positive definite*) if both sequences $(s_j)_{j=0}^{2n}$ and $((s_\beta)_j)_{j=0}^{2n}$ are Hankel non-negative definite (resp., Hankel positive definite).

(b) Let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n}$ is called $(-\infty,\beta]$-*Stieltjes non-negative definite* (resp., $(-\infty,\beta]$-*Stieltjes positive definite*) if $(s_j)_{j=0}^{2n}$ and, in the case $n \geq 1$, also $((s_\beta)_j)_{j=0}^{2(n-1)}$ is Hankel non-negative definite (resp., Hankel positive definite).

If $m \in \mathbb{N}_0$, then $\mathcal{L}_{q,m,\beta}^{\geq}$ (resp., $\mathcal{L}_{q,m,\beta}^{>}$) stands for the set of all $(-\infty,\beta]$-Stieltjes non-negative definite (resp., $(-\infty,\beta]$-Stieltjes positive definite) sequences $(s_j)_{j=0}^m$ from $\mathbb{C}^{q\times q}$.

Definition 8.12. Let $\beta \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^m$ is called $(-\infty,\beta]$-*Stieltjes non-negative definite extendable* (resp., $(-\infty,\beta]$-*Stieltjes positive definite extendable*) if there exists a matrix $s_{m+1} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{m+1} \in \mathcal{L}^{\geq}_{q,m+1,\beta}$ (resp., $(s_j)_{j=0}^{m+1} \in \mathcal{L}^{>}_{q,m+1,\beta}$). We denote by $\mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$ (resp., $\mathcal{L}^{>,\mathrm{e}}_{q,m,\beta}$) the set of all $(-\infty,\beta]$-Stieltjes non-negative definite extendable (resp., $(-\infty,\beta]$-Stieltjes positive definite extendable) sequences $(s_j)_{j=0}^m$ from $\mathbb{C}^{q\times q}$.

Definition 8.13. A sequence $(s_j)_{j=0}^\infty$ from $\mathbb{C}^{q\times q}$ is called $(-\infty,\beta]$-*Stieltjes non-negative definite* (resp., $(-\infty,\beta]$-*Stieltjes positive definite*) if $(s_j)_{j=0}^m \in \mathcal{L}^{\geq}_{q,m,\beta}$ (resp., $(s_j)_{j=0}^m \in \mathcal{L}^{>}_{q,m,\beta}$) for all $m \in \mathbb{N}_0$. We denote by $\mathcal{L}^{\geq}_{q,\infty,\beta}$ (resp., $\mathcal{L}^{>}_{q,\infty,\beta}$) the set of all $(-\infty,\beta]$-Stieltjes non-negative definite (resp., $(-\infty,\beta]$-Stieltjes positive definite) sequences $(s_j)_{j=0}^\infty$ from $\mathbb{C}^{q\times q}$. Furthermore, let $\mathcal{L}^{\geq,\mathrm{e}}_{q,\infty,\beta} := \mathcal{L}^{\geq}_{q,\infty,\beta}$ and let $\mathcal{L}^{>,\mathrm{e}}_{q,\infty,\beta} := \mathcal{L}^{>}_{q,\infty,\beta}$.

Lemma 8.14. *Let $\beta \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Let the sequence $(r_j)_{j=0}^m$ be given by $r_j := (-1)^j s_j$. Then:*

(a) $(s_j)_{j=0}^m \in \mathcal{L}^{\geq}_{q,m,\beta}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{K}^{\geq}_{q,m,-\beta}$.
(b) $(s_j)_{j=0}^m \in \mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{K}^{\geq,\mathrm{e}}_{q,m,-\beta}$.
(c) $(s_j)_{j=0}^m \in \mathcal{L}^{>}_{q,m,\beta}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{K}^{>}_{q,m,-\beta}$.
(d) $(s_j)_{j=0}^m \in \mathcal{L}^{>,\mathrm{e}}_{q,m,\beta}$ *if and only if* $(r_j)_{j=0}^m \in \mathcal{K}^{>,\mathrm{e}}_{q,m,-\beta}$.

Proof. This can be verified analogously to the proof of Lemma 4.20. □

The importance of the class $\mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$ in the context of moment problems is caused by the following observation:

Theorem 8.15. *Let $\beta \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{M}^q_{\geq}[(-\infty,\beta];(s_j)_{j=0}^m,=] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$.*

Proof. Combine Lemma 8.14(b), Theorem 8.3, and Remark 2.1. □

Remark 8.16. Let $\beta \in \mathbb{R}$ and let $m \in \mathbb{N}_0$. Then $\mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta} \subseteq \mathcal{L}^{\geq}_{q,m,\beta}$ and, in the case $m \in \mathbb{N}$, furthermore $\mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta} \neq \mathcal{L}^{\geq}_{q,m,\beta}$.

Remark 8.17. Let $\beta \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^m \in \mathcal{L}^{\geq}_{q,m,\beta}$. Then $(s_j)_{j=0}^\ell \in \mathcal{L}^{\geq,\mathrm{e}}_{q,\ell,\beta}$ for all $\ell \in \mathbb{Z}_{0,m-1}$.

Remark 8.18. Let $\beta \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$ (resp., $\mathcal{L}^{>}_{q,m,\beta}$). Then $(s_j)_{j=0}^\ell \in \mathcal{L}^{\geq,\mathrm{e}}_{q,\ell,\beta}$ (resp., $\mathcal{L}^{>}_{q,\ell,\beta}$) for all $\ell \in \mathbb{Z}_{0,m}$.

Remark 8.19. Let $\beta \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{L}^{\geq}_{q,\kappa,\beta}$. In view of Definition 8.11, Remarks 8.17 and 8.16, and Remark 3.6(a), then $L_n \in \mathbb{C}^{q\times q}_{\geq}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, and $(L_\beta)_n \in \mathbb{C}^{q\times q}_{\geq}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.

Remark 8.20. For each $n \in \mathbb{N}$, it is readily checked that

$$\mathcal{K}^{\geq,\mathrm{e}}_{q,2n-1,\alpha} = \Big\{(s_j)_{j=0}^{2n-1} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1} \colon \big({}_{(\alpha}s)_j\big)_{j=0}^{2(n-1)} \in \mathcal{H}^{\geq}_{q,2(n-1)}\Big\},$$
$$\mathcal{L}^{\geq,\mathrm{e}}_{q,2n-1,\beta} = \Big\{(s_j)_{j=0}^{2n-1} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1} \colon \big((s_\beta)_j\big)_{j=0}^{2(n-1)} \in \mathcal{H}^{\geq}_{q,2(n-1)}\Big\},$$
$$\mathcal{K}^{\geq,\mathrm{e}}_{q,2n,\alpha} = \Big\{(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n} \colon \big({}_{(\alpha}s)_j\big)_{j=0}^{2n-1} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1}\Big\},$$
$$\mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta} = \Big\{(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n} \colon \big((s_\beta)_j\big)_{j=0}^{2n-1} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1}\Big\}.$$

Proposition 8.21. *Let $\alpha, \beta \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then:*

(a) $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n-1,\alpha} = \{(s_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq}_{q,2n-1,\alpha} \colon \mathcal{N}(L_{n-1}) \subseteq \mathcal{N}({}_{(\alpha}L)_{n-1})\}$.
(b) $\mathcal{L}^{\geq,\mathrm{e}}_{q,2n-1,\beta} = \{(s_j)_{j=0}^{2n-1} \in \mathcal{L}^{\geq}_{q,2n-1,\beta} \colon \mathcal{N}(L_{n-1}) \subseteq \mathcal{N}((L_\beta)_{n-1})\}$.
(c) $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n,\alpha} = \{(s_j)_{j=0}^{2n} \in \mathcal{K}^{\geq}_{q,2n,\alpha} \colon \mathcal{N}({}_{(\alpha}L)_{n-1}) \subseteq \mathcal{N}(L_n)\}$.
(d) $\mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta} = \{(s_j)_{j=0}^{2n} \in \mathcal{L}^{\geq}_{q,2n,\beta} \colon \mathcal{N}((L_\beta)_{n-1}) \subseteq \mathcal{N}(L_n)\}$.

Proof. Parts (a) and (c) are proved in [13, Lemmata 4.15 and 4.16].

(b) Let $r_j := (-1)^j s_j$ for all $j \in \mathbb{Z}_{0,2n-1}$. First assume $(s_j)_{j=0}^{2n-1} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n-1,\beta}$. Lemma 8.14(b) provides us with $(r_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n-1,-\beta}$. Thus, using

$$({}_{-\beta}r)_j = -(-\beta)(-1)^j s_j + (-1)^{j+1} s_{j+1} = (-1)^j(\beta s_j - s_{j+1}) = (-1)^j (s_\beta)_j \quad (8.1)$$

for each $j \in \mathbb{Z}_{0,2n-1}$, part (a), Lemma 8.14(a), and Lemma 4.18(b) we get

$$(s_j)_{j=0}^{2n-1} \in \mathcal{L}^{\geq}_{q,2n-1,\beta} \quad \text{and} \quad \mathcal{N}(L_{n-1}) \subseteq \mathcal{N}\big((L_\beta)_{n-1}\big). \quad (8.2)$$

Conversely, suppose (8.2). Then Lemma 8.14(a) yields $(r_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq}_{q,2n-1,-\beta}$. From (8.1), Lemma 4.18(b), and Lemma 8.14(b) we conclude $(s_j)_{j=0}^{2n-1} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n-1,\beta}$.

(d) Analogously, as part (b) can be proved by virtue of part (a), part (d) can be checked using part (c). □

If $\mathcal{M}$ is a non-empty subset of $\mathbb{C}^q$, then let $\mathcal{M}^\perp$ be the set of all $x \in \mathbb{C}^q$ which fulfill $\langle x, y\rangle_{\mathrm{E}} = 0$ for each $y \in \mathcal{M}$, where $\langle \cdot, \cdot \rangle_{\mathrm{E}}$ is the Euclidean inner product in $\mathbb{C}^q$.

Proposition 8.22. *Let $\alpha, \beta \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then:*

(a) $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n-1,\alpha} = \{(s_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq}_{q,2n-1,\alpha} \colon \mathcal{R}({}_{(\alpha}L)_{n-1}) \subseteq \mathcal{R}(L_{n-1})\}$.
(b) $\mathcal{L}^{\geq,\mathrm{e}}_{q,2n-1,\beta} = \{(s_j)_{j=0}^{2n-1} \in \mathcal{L}^{\geq}_{q,2n-1,\beta} \colon \mathcal{R}((L_\beta)_{n-1}) \subseteq \mathcal{R}(L_{n-1})\}$.
(c) $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n,\alpha} = \{(s_j)_{j=0}^{2n} \in \mathcal{K}^{\geq}_{q,2n,\alpha} \colon \mathcal{R}(L_n) \subseteq \mathcal{R}({}_{(\alpha}L)_{n-1})\}$.
(d) $\mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta} = \{(s_j)_{j=0}^{2n} \in \mathcal{L}^{\geq}_{q,2n,\beta} \colon \mathcal{R}(L_n) \subseteq \mathcal{R}((L_\beta)_{n-1})\}$.

Proof. In view of the Definition of the set $\mathcal{K}^{\geq}_{q,2n-1,\alpha}$ and Lemma 8.9, we see that, for each $(s_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq}_{q,2n-1,\alpha}$, the matrices L_{n-1} and ${}_{(\alpha}L)_{n-1}$ are both non-negative Hermitian. Thus, $\mathcal{N}(L_{n-1}) = \mathcal{N}(L_{n-1}^*) = \mathcal{R}(L_{n-1})^\perp$ and $\mathcal{N}({}_{(\alpha}L)_{n-1}) = \mathcal{N}({}_{(\alpha}L)_{n-1}^*) = \mathcal{R}({}_{(\alpha}L)_{n-1})^\perp$. Consequently, the application of Proposition 8.21 completes the proof of (a). Parts (b), (c), and (d) can be checked analogously. □

Now we are going to study several aspects of the interplay between those four Hankel non-negative definite sequences which determine the sections of an $[\alpha,\beta]$-Hausdorff non-negative definite sequence. In particular, we derive formulas connecting the matrices which were introduced in parts (c) and (d) of Notation 3.4 for each of the sequences $(s_j)_{j=0}^{\kappa}$, $({}_{(\alpha}s)_j)_{j=0}^{\kappa-1}$, $((s_\beta)_j)_{j=0}^{\kappa-1}$, and $({}_{(\alpha}s_{\beta)}{}_j)_{j=0}^{\kappa-2}$. These formulas play an important role in the proof of Theorem 10.14, which is of central importance for our further considerations.

Lemma 8.23. *Let $\alpha,\beta\in\mathbb{R}$ and let $n\in\mathbb{N}$. Then:*

(a) *If $(s_j)_{j=0}^{2n}\in\mathcal{K}_{q,2n,\alpha}^{\geq}$, then*

$$L_n = ({}_\alpha s)_{2n-1} - ({}_\alpha M)_{n-1} - ({}_\alpha L)_{n-1}L_{n-1}^{\dagger}(s_{2n-1}-M_{n-1}).$$

(b) *If $(s_j)_{j=0}^{2n}\in\mathcal{L}_{q,2n,\beta}^{\geq}$, then*

$$L_n = -\big[(s_\beta)_{2n-1} - (M_\beta)_{n-1}\big] + (L_\beta)_{n-1}L_{n-1}^{\dagger}(s_{2n-1}-M_{n-1}).$$

Proof. A proof of part (a) is given in [20, Proposition 6.4]. To prove part (b), we consider an arbitrary sequence $(s_j)_{j=0}^{2n}\in\mathcal{L}_{q,2n,\beta}^{\geq}$. Setting $r_j := (-1)^j s_j$ for all $j\in\mathbb{Z}_{0,2n}$, we see from Lemma 8.14(a) that $(r_j)_{j=0}^{2n}$ belongs to $\mathcal{K}_{q,2n,-\beta}^{\geq}$. Because of part (a), we have then

$$L_n^{\langle r\rangle} = -(-\beta)r_{2n-1} + r_{2n} - ({}_{-\beta}M^{\langle r\rangle})_{n-1} - ({}_{-\beta}L^{\langle r\rangle})_{n-1}(L_{n-1}^{\langle r\rangle})^{\dagger}(r_{2n-1}-M_{n-1}^{\langle r\rangle}). \tag{8.3}$$

Obviously, $r_{2n-1}=-s_{2n-1}$ and $r_{2n}=s_{2n}$. Lemma 4.18 yields $L_\ell^{\langle r\rangle}=L_\ell$ for $\ell\in\{n-1,n\}$ and $M_{n-1}^{\langle r\rangle}=-M_{n-1}$. Taking into account that (8.1) holds true for all $j\in\mathbb{Z}_{0,2n-1}$, from Lemma 4.18 we also get $({}_{-\beta}M^{\langle r\rangle})_{n-1}=-(M_\beta)_{n-1}$ and $({}_{-\beta}L^{\langle r\rangle})_{n-1}=(L_\beta)_{n-1}$. Thus, in view of (8.3), the proof is complete. □

Lemma 8.24. *Let $\alpha,\beta\in\mathbb{R}$ and let $n\in\mathbb{N}$. Then:*

(a) *If $(s_j)_{j=0}^{2n-1}\in\mathcal{K}_{q,2n-1,\alpha}^{\geq,\mathrm{e}}$, then*

$$\Theta_n = \alpha s_{2n-1} + ({}_\alpha M)_{n-1} + ({}_\alpha L)_{n-1}L_{n-1}^{\dagger}(s_{2n-1}-M_{n-1}). \tag{8.4}$$

(b) *If $(s_j)_{j=0}^{2n-1}\in\mathcal{L}_{q,2n-1,\beta}^{\geq,\mathrm{e}}$, then*

$$\Theta_n = \beta s_{2n-1} - \Big[(M_\beta)_{n-1} + (L_\beta)_{n-1}L_{n-1}^{\dagger}(s_{2n-1}-M_{n-1})\Big].$$

Proof. First we suppose that $(s_j)_{j=0}^{2n-1}\in\mathcal{K}_{q,2n-1,\alpha}^{\geq,\mathrm{e}}$. Then there is an $s_{2n}\in\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2n}\in\mathcal{K}_{q,2n,\alpha}^{\geq}$. Using Lemma 8.23(a), the equation $\Theta_n=s_{2n}-L_n$, and the equation $({}_\alpha s)_{2n-1}=-\alpha s_{2n-1}+s_{2n}$, we get (8.4). Part (a) is proved. In view of Lemma 8.23(b), part (b) can be checked analogously. □

Lemma 8.25. *Let $\alpha,\beta\in\mathbb{R}$ and let $n\in\mathbb{N}$. Then:*

(a) *If $(s_j)_{j=0}^{2n+1}\in\mathcal{K}_{q,2n+1,\alpha}^{\geq}$, then*

$$({}_\alpha L)_n = s_{2n+1} - M_n - L_n\Big(\alpha I_q + \big[({}_\alpha L)_{n-1}\big]^{\dagger}\big[({}_\alpha s)_{2n-1} - ({}_\alpha M)_{n-1}\big]\Big).$$

(b) *If* $(s_j)_{j=0}^{2n+1} \in \mathcal{L}^{\geq}_{q,2n+1,\beta}$, *then*

$$(L_\beta)_n = -(s_{2n+1} - M_n) + L_n\Big(\beta I_q + \big[(L_\beta)_{n-1}\big]^\dagger \big[(s_\beta)_{2n-1} - (M_\beta)_{n-1}\big]\Big). \quad (8.5)$$

Proof. Part (a) is proved in [20, Proposition 6.5]. In order to check part (b), we consider an arbitrary sequence $(s_j)_{j=0}^{2n+1} \in \mathcal{L}^{\geq}_{q,2n+1,\beta}$. Let $r_j := (-1)^j s_j$ for each $j \in \mathbb{Z}_{0,2n+1}$. According to Lemma 8.14(a), the sequence $(r_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}^{\geq}_{q,2n+1,-\beta}$. Thus, part (a) provides us with

$$\begin{aligned} &({}_{-\beta}L^{\langle r\rangle})_n \\ &= r_{2n+1} - M_n^{\langle r\rangle} - L_n^{\langle r\rangle}\big((-\beta)I_q + ({}_{-\beta}L^{\langle r\rangle})_{n-1}\big[({}_{-\beta}r)_{2n-1} - ({}_{-\beta}M^{\langle r\rangle})_{n-1}\big]\big). \end{aligned} \quad (8.6)$$

Obviously, $r_{2n+1} = -s_{2n+1}$ and, by (8.1), furthermore $({}_{-\beta}r)_{2n-1} = -(s_\beta)_{2n-1}$. Lemma 4.18 yields $M_n^{\langle r\rangle} = -M_n$ and $L_n^{\langle r\rangle} = L_n$. Since (8.1) holds true for all $j \in \mathbb{Z}_{0,2n}$, Lemma 4.18 also shows that $({}_{-\beta}L^{\langle r\rangle})_\ell = (L_\beta)_\ell$ for $\ell \in \{n-1, n\}$ and that $({}_{-\beta}M^{\langle r\rangle})_{n-1} = -(M_\beta)_{n-1}$. Thus, (8.5) follows from (8.6). □

Lemma 8.26. *Let* $\alpha, \beta \in \mathbb{R}$ *and let* $n \in \mathbb{N}$. *Then:*

(a) *If* $(s_j)_{j=0}^{2n} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n,\alpha}$, *then*

$$({}_\alpha\Theta)_n = -\alpha s_{2n} + M_n + L_n\Big(\alpha I_q + \big[({}_\alpha L)_{n-1}\big]^\dagger \big[({}_\alpha s)_{2n-1} - ({}_\alpha M)_{n-1}\big]\Big). \quad (8.7)$$

(b) *If* $(s_j)_{j=0}^{2n} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta}$, *then*

$$(\Theta_\beta)_n = \beta s_{2n} - \Big[M_n + L_n\Big(\beta I_q + \big[(L_\beta)_{n-1}\big]^\dagger \big[(s_\beta)_{2n-1} - (M_\beta)_{n-1}\big]\Big)\Big]. \quad (8.8)$$

Proof. Choose a matrix $s_{2n+1} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}^{\geq}_{q,2n+1,\alpha}$ (resp., $\mathcal{L}^{\geq}_{q,2n+1,\beta}$) and apply Lemma 8.25. □

Lemma 8.27. *Let* $\alpha, \beta \in \mathbb{R}$ *and let* $\kappa \in \mathbb{N} \cup \{\infty\}$. *Then:*

(a) *If* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq}_{q,\kappa,\alpha}$, *then* $(({}_\alpha s)_j)_{j=0}^{m} \in \mathcal{K}^{\geq}_{q,m,\alpha}$ *for all* $m \in \mathbb{Z}_{0,\kappa-1}$.

(b) *If* $(s_j)_{j=0}^{\kappa} \in \mathcal{L}^{\geq}_{q,\kappa,\beta}$, *then* $((s_\beta)_j)_{j=0}^{m} \in \mathcal{L}^{\geq}_{q,m,\beta}$ *for all* $m \in \mathbb{Z}_{0,\kappa-1}$.

Proof. (a) In view of Remarks 8.6 and 8.5, part (a) is proved in [21, Proposition 5.1].

(b) Use Lemma 8.14 and part (a). □

The following result gives us a first impression of the interplay between $\mathcal{F}^{\geq}_{q,m,\alpha,\beta}$ and the classes studied in this section.

Proposition 8.28. *Let* $\alpha \in \mathbb{R}$, *let* $\beta \in (\alpha, \infty)$, *and let* $n \in \mathbb{N}_0$. *Then* $\mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta} = \mathcal{K}^{\geq}_{q,2n+1,\alpha} \cap \mathcal{L}^{\geq}_{q,2n+1,\beta}$.

Proof. Let $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$. Then Definition 4.2(a) shows that $\{((_\alpha s)_j)_{j=0}^{2n}, ((s_\beta)_j)_{j=0}^{2n}\} \subseteq \mathcal{H}^{\geq}_{q,2n}$. Moreover, from Lemma 4.14 we know that $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n}$. Hence, $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}^{\geq}_{q,2n+1,\alpha} \cap \mathcal{L}^{\geq}_{q,2n+1,\beta}$.

Conversely, now assume that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}^{\geq}_{q,2n+1,\alpha} \cap \mathcal{L}^{\geq}_{q,2n+1,\beta}$. Then the definitions of the sets $\mathcal{K}^{\geq}_{q,2n+1,\alpha}$ and $\mathcal{L}^{\geq}_{q,2n+1,\beta}$ in combination with Definition 4.2(a) yield immediately $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$. □

9. On the inclusion $\mathcal{F}^{\geq}_{q,m,\alpha,\beta} \subseteq \mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$

In this section we investigate interrelations between the set $\mathcal{F}^{\geq}_{q,m,\alpha,\beta}$ on the one hand and the intersection $\mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$ on the other hand.

Proposition 9.1. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. Then:*

(a) $\{((_\alpha s)_j)_{j=0}^{\kappa-1}, ((s_\beta)_j)_{j-0}^{\kappa-1}\} \subseteq \mathcal{F}^{\geq}_{q,\kappa-1,\alpha,\beta}$.

(b) *If $\kappa \geq 2$, then $((_\alpha s_\beta)_j)_{j=0}^{\kappa-2} \in \mathcal{F}^{\geq}_{q,\kappa-2,\alpha,\beta}$.*

Proof. (a) First we consider the case that $\kappa = 2n+1$ with some $n \in \mathbb{N}_0$. Then Definition 4.2(a) yields $\{(_\alpha H)_n, (H_\beta)_n\} \subseteq \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$. If $n = 0$, then (a) is proved. Assume $n \geq 1$. Then Proposition 7.7(a) provides us with $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$ and, consequently, $(_\alpha H_\beta)_{n-1} \in \mathbb{C}_{\geq}^{nq\times nq}$. From Remark 4.13 we have $(_\alpha H_\beta)_{n-1} = (_\alpha(H_\beta))_{n-1} = ((_\alpha H)_\beta)_{n-1}$. Thus, $((s_\beta)_j)_{j=0}^{2n} \in \mathcal{K}^{\geq}_{q,2n,\alpha}$ and $((_\alpha s)_j)_{j=0}^{2n} \in \mathcal{L}^{\geq}_{q,2n,\beta}$. Applying Lemma 8.27(a) to $((s_\beta)_j)_{j=0}^{2n}$, from Remark 4.11 we get $((_\alpha s_\beta)_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq}_{q,2n-1,\alpha}$. In particular, $-\alpha(_\alpha H_\beta)_{n-1} + (_\alpha K_\beta)_{n-1} \in \mathbb{C}_{\geq}^{nq\times nq}$. Taking into account that Remark 4.11 shows that $-\alpha(_\alpha s_\beta)_j + (_\alpha s_\beta)_{j+1} = -\alpha\beta(_\alpha s)_j + (\alpha+\beta)(_\alpha s)_{j+1} - (_\alpha s)_{j+2}$ holds true for all $j \in \mathbb{Z}_{0,2n-2}$, then $-\alpha\beta(_\alpha H)_{n-1} + (\alpha+\beta)(_\alpha K)_{n-1} - (_\alpha G)_{n-1} \in \mathbb{C}_{\geq}^{nq\times nq}$ and $((_\alpha s)_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$ follow. Applying Lemma 8.27(b) to the sequence $((_\alpha s)_j)_{j=0}^{2n}$, we obtain from Remark 4.11 that $((_\alpha s_\beta)_j)_{j=0}^{2n-1}$ belongs to $\mathcal{L}^{\geq}_{q,2n-1,\beta}$. In particular, $\beta(_\alpha H_\beta)_{n-1} - (_\alpha K_\beta)_{n-1} \in \mathbb{C}_{\geq}^{nq\times nq}$. Since Remark 4.11 yields $\beta(_\alpha s_\beta)_j - (_\alpha s_\beta)_{j+1} = -\alpha\beta(s_\beta)_j + (\alpha+\beta)(s_\beta)_{j+1} - (s_\beta)_{j+2}$ for all $j \in \mathbb{Z}_{0,2n-2}$, we conclude then $-\alpha\beta(H_\beta)_{n-1} + (\alpha+\beta)(K_\beta)_{n-1} - (G_\beta)_{n-1} \in \mathbb{C}_{\geq}^{nq\times nq}$ and $((s_\beta)_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$ and, because of Remark 4.11, furthermore $((_\alpha s_\beta)_j)_{j=0}^{2n-1} \in \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$.

Now we consider the case that $\kappa = 2n$ with some $n \in \mathbb{N}$. Then the matrices H_n and $(_\alpha H_\beta)_{n-1}$ are both non-negative Hermitian. Proposition 7.7(a) provides us with $(s_j)_{j=0}^{2n-1} \in \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$. Hence, $\{(_\alpha H)_{n-1}(H_\beta)_{n-1}\} \in \mathbb{C}_{\geq}^{nq\times nq}$. Thus, $(s_j)_{j=0}^{2n} \in \mathcal{K}^{\geq}_{q,2n,\alpha} \cap \mathcal{L}^{\geq}_{q,2n,\beta}$. Applying Lemma 8.27(a) to $(s_j)_{j=0}^{2n}$, we infer $((_\alpha s)_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq}_{q,2n-1,\alpha}$. Thus, $-\alpha(_\alpha H)_{n-1} + (_\alpha K)_{n-1} \in \mathbb{C}_{\geq}^{nq\times nq}$. Since Remark 4.13 shows

that $\beta({}_{\alpha}H)_{n-1} - ({}_{\alpha}K)_{n-1} = ({}_{\alpha}H_{\beta})_{n-1}$ is valid, from Definition 4.2(a) we get $(({}_{\alpha}s)_j)_{j=0}^{2n-1} \in \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$. Lemma 8.27(b) provides us with $(s_\beta)_{j=0}^{2n-1} \in \mathcal{L}^{\geq}_{q,2n-1,\beta}$. Consequently, the matrix $\beta(H_\beta)_{n-1} - (K_\beta)_{n-1}$ is non-negative Hermitian. Since Remark 4.13 shows that $-\alpha(H_\beta)_{n-1} + (K_\beta)_{n-1} = ({}_{\alpha}H_{\beta})_{n-1}$ holds true, then Definition 4.2(a) shows that $((s_\beta)_j)_{j=0}^{2n-1} \in \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$.

(b) In view of Remark 4.11, this follows by applying part (a) to one of the sequences $(({}_{\alpha}s)_j)_{j=0}^{\kappa-1}$ or $((s_\beta)_j)_{j=0}^{\kappa-1}$ which, according to part (a), belong to $\mathcal{F}^{\geq}_{q,\kappa-1,\alpha,\beta}$. □

Obviously, $\mathcal{F}^{\geq}_{q,0,\alpha,\beta} = \mathcal{K}^{\geq,\mathrm{e}}_{q,0,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,0,\beta}$ and $\mathcal{F}^{\geq}_{q,0,\alpha,\beta} = \mathcal{K}^{\geq}_{q,0,\alpha} \cap \mathcal{L}^{\geq}_{q,0,\beta}$ for all $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$.

Proposition 9.2. *Let $\alpha \in \mathbb{R}$ and let $\beta \in (\alpha, \infty)$. Then $\mathcal{F}^{\geq}_{q,m,\alpha,\beta} \subseteq \mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$ for all $m \in \mathbb{N}_0$.*

Proof. The case $m = 0$ is trivial. Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^{m} \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$.

We consider now the case that $m = 2n + 1$ with some $n \in \mathbb{N}_0$. From Lemma 7.5 we know that $(s_j)_{j=0}^{2n+1} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n+1}$. Furthermore, Definition 4.2(a) yields $\{(({}_{\alpha}s)_j)_{j=0}^{2n}, ((s_\beta)_j)_{j=0}^{2n}\} \subseteq \mathcal{H}^{\geq}_{q,2n}$. Thus, Remark 8.20 shows that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,2n+1,\beta}$.

It remains to consider the case that $m = 2n$ with some $n \in \mathbb{N}$. Since $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$, from Definition 4.2(b) we have $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n}$. Proposition 9.1(a) yields $\{(({}_{\alpha}s)_j)_{j=0}^{2n-1}, ((s_\beta)_j)_{j=0}^{2n-1}\} \subseteq \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$. Consequently, Lemma 7.5 provides us with $\{(({}_{\alpha}s)_j)_{j=0}^{2n-1}, ((s_\beta)_j)_{j=0}^{2n-1}\} \subseteq \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1}$. Hence, Remark 8.20 yields that $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta}$. □

Alternative proof of Proposition 9.2. Let $(s_j)_{j=0}^{m} \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$. In view of Theorem 4.3, then $\mathcal{M}^{q}_{\geq}[[\alpha,\beta];(s_j)_{j=0}^{m},=] \neq \emptyset$. Defining $\sigma_{[\alpha]}\colon \mathfrak{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ by $\sigma_{[\alpha]}(B) := \sigma(B \cap [\alpha,\beta])$ and $\sigma^{[\beta]}\colon \mathfrak{B}_{(-\infty,\beta]} \to \mathbb{C}^{q\times q}$ by $\sigma^{[\beta]}(C) := \sigma(C \cap [\alpha,\beta])$, we immediately see $\sigma_{[\alpha]} \in \mathcal{M}^{q}_{\geq}[[\alpha,\infty);(s_j)_{j=0}^{m},=]$ and $\sigma^{[\beta]} \in \mathcal{M}^{q}_{\geq}[(-\infty,\beta];(s_j)_{j=0}^{m},=]$, respectively. Thus, Theorems 8.3 and 8.15 yield $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$ and $(s_j)_{j=0}^{m} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,m,\beta}$. □

Proposition 9.3. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, and let $n \in \mathbb{N}_0$. Then $\mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta} = \mathcal{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,2n+1,\beta}$.*

Proof. In view of Definition 4.2(a), Remark 8.20 provides us with $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,2n+1,\beta} \subseteq \mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$. Thus, the application of Proposition 9.2 completes the proof. □

In view of Proposition 9.2, the following example shows that $\mathcal{F}^{\geq}_{1,2,\alpha,\beta}$ is a proper subset of $\mathcal{K}^{\geq,\mathrm{e}}_{1,2,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{1,2,\beta}$ if α and β are real numbers such that $\alpha < \beta$.

Example 9.4. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, and let

$$s_0 := 4, \quad s_1 := 2(\alpha+\beta), \quad \text{and} \quad s_2 := (\alpha+\beta)^2 + 3(\beta-\alpha)^2.$$

Then $H_1 = \left[\begin{smallmatrix} 2 & 0 \\ \alpha+\beta & \sqrt{3}(\beta-\alpha) \end{smallmatrix}\right] \left[\begin{smallmatrix} 2 & 0 \\ \alpha+\beta & \sqrt{3}(\beta-\alpha) \end{smallmatrix}\right]^*$ is positive Hermitian and $L_1 = 3(\beta-\alpha)$, $({}_\alpha H)_0 = ({}_\alpha L)_0 = 2(\beta-\alpha)$, and $(H_\beta)_0 = (L_\beta)_0 = 2(\beta-\alpha)$ are positive. Thus, parts (c) and (d) of Proposition 8.21 show that $(s_j)_{j=0}^2$ belongs to $\mathcal{K}_{1,2,\alpha}^{\geq,\mathrm{e}} \cap \mathcal{L}_{1,2,\beta}^{\geq,\mathrm{e}}$. However, taking into account that $({}_\alpha H_\beta)_0 = -2(\beta-\alpha)^2$ is negative, we see that $(s_j)_{j=0}^2 \notin \mathcal{F}_{1,2,\alpha,\beta}^{\geq}$.

10. On the structure of $[\alpha, \beta]$-Hausdorff non-negative definite sequences

In this section, we discuss essential aspects of the structure of the elements of the set $\mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. In particular, we will see that each element of such a sequence varies within a closed matricial subinterval of $\mathbb{C}_{\mathrm{H}}^{q\times q}$ the endpoints of which are completely determined by the preceding elements of the sequence. First we introduce two types of matricial intervals:

Let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Then we set

$$[A,B] := \{X \in \mathbb{C}_{\mathrm{H}}^{q\times q} \colon A \leq X \leq B\} \quad \text{and} \quad (A,B) := \{X \in \mathbb{C}_{\mathrm{H}}^{q\times q} \colon A < X < B\}.$$

The matrix A (resp., B) is called the *matricial left* (resp., *right*) *endpoint* of $[A, B]$ or (A, B). The matrix $C := \frac{1}{2}(A+B)$ (resp., $D := B - A$) is called the *matricial midpoint* (resp., *length*) of $[A, B]$ or (A, B).

The following result indicates that closed and open matricial intervals are intimately related to closed and open matricial matrix balls.

Lemma 10.1. *Let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and let $D := B - A$. Then:*

(a) *$[A, B] \neq \emptyset$ if and only if $D \in \mathbb{C}_{\geq}^{q\times q}$. In this case,*

$$[A,B] = \Big\{A + \sqrt{D}K\sqrt{D} \colon K \in [0_{q\times q}, I_q]\Big\}.$$

(b) *$(A, B) \neq \emptyset$ if and only if $D \in \mathbb{C}_{>}^{q\times q}$. In this case,*

$$(A,B) = \Big\{A + \sqrt{D}K\sqrt{D} \colon K \in (0_{q\times q}, I_q)\Big\}.$$

Proof. (a) If $[A, B] \neq \emptyset$, then there exists a matrix $X \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $A \leq X \leq B$. In particular, $A \leq B$, i.e., $D \in \mathbb{C}_{\geq}^{q\times q}$. Conversely, assume $D \in \mathbb{C}_{\geq}^{q\times q}$. Obviously, $[0_{q\times q}, I_q] \neq \emptyset$. Let $K \in [0_{q\times q}, I_q]$. Then $X := A + \sqrt{D}K\sqrt{D}$ is Hermitian. By virtue of Remark A.2, furthermore $X - A = \sqrt{D}K\sqrt{D} \geq 0_{q\times q}$ and $B - X = D - \sqrt{D}K\sqrt{D} = \sqrt{D}(I_q - K)\sqrt{D} \geq 0_{q\times q}$. Hence, $X \in [A, B]$. In particular, $[A, B] \neq \emptyset$. Now let $X \in [A, B]$. In view of Remark A.8, then $K := \sqrt{D}^\dagger(X-A)\sqrt{D}^\dagger$ is Hermitian and we have

$$0_{q\times q} \leq X - A \leq B - A = D. \tag{10.1}$$

Hence, $\mathcal{R}(X-A) \subseteq \mathcal{R}(D)$ and $\mathcal{N}(D) \subseteq \mathcal{N}(X-A)$ by virtue of Remark A.11. According to Remark A.10, thus $\mathcal{R}(X-A) \subseteq \mathcal{R}(\sqrt{D})$ and $\mathcal{N}(\sqrt{D}) \subseteq \mathcal{N}(X-A)$. Using Remarks A.5 and A.6, we obtain then $\sqrt{D}K\sqrt{D} = X - A$. Hence, $X = A + \sqrt{D}K\sqrt{D}$. With Remark A.2, we conclude from (10.1) that $0_{q\times q} \leq K \leq \sqrt{D}^{\dagger} D \sqrt{D}^{\dagger}$. Using Remarks A.8 and A.3 and Proposition A.4, we have

$$\sqrt{D}^{\dagger} D \sqrt{D}^{\dagger} = \sqrt{D}^{\dagger}\sqrt{D}\sqrt{D}\sqrt{D}^{\dagger} = \sqrt{D}\sqrt{D}^{\dagger}\sqrt{D}\sqrt{D}^{\dagger} = \sqrt{D}\sqrt{D}^{\dagger} = P_{\mathcal{R}(\sqrt{D})} \leq I_q.$$

Thus, $0_{q\times q} \leq K \leq I_q$, i.e., $K \in [0_{q\times q}, I_q]$.

(b) Since every positive Hermitian matrix is invertible, this can be shown more easy in a similar manner. □

From Lemma 10.1 we conclude the following result:

Remark 10.2. Let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and let $C := \frac{1}{2}(A+B)$ and $D := B - A$. Then:

(a) If $[A,B] \neq \emptyset$, then $\{A, C, B\} \subseteq \{A + \eta D \colon \eta \in [0,1]\} \subseteq [A,B]$.

(b) If $(A,B) \neq \emptyset$, then $C \in \{A + \eta D \colon \eta \in (0,1)\} \subseteq (A,B)$.

Now we are going to introduce those sequences of matrices which will be needed to describe that sequences of matricial intervals which turn out to be associated with an $[\alpha,\beta]$-Hausdorff non-negative definite sequence:

Definition 10.3. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\geq}$. For all $k \in \mathbb{N}_0$ with $2k \leq \kappa$, let

$$\mathfrak{a}_{2k} := \alpha s_{2k} + ({}_{\alpha}\Theta)_k \quad \text{and} \quad \mathfrak{b}_{2k} := \beta s_{2k} - (\Theta_{\beta})_k.$$

For all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$, let

$$\mathfrak{a}_{2k+1} := \Theta_{k+1} \quad \text{and} \quad \mathfrak{b}_{2k+1} := -\alpha\beta s_{2k} + (\alpha+\beta)s_{2k+1} - ({}_{\alpha}\Theta_{\beta})_k.$$

Then the sequences $(\mathfrak{a}_j)_{j=0}^{\kappa}$ and $(\mathfrak{b}_j)_{j=0}^{\kappa}$ are called the *sequence of left matricial interval endpoints associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha,\beta]$ and the *sequence of right matricial interval endpoints associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha,\beta]$, respectively.

Remark 10.4. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\geq}$. Then $\mathfrak{a}_0 = \alpha s_0$ and $\mathfrak{b}_0 = \beta s_0$. If $\kappa \geq 1$, further $\mathfrak{a}_1 = s_1 s_0^{\dagger} s_1$ and $\mathfrak{b}_1 = -\alpha\beta s_0 + (\alpha+\beta)s_1$.

Remark 10.5. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\geq}$. Because of Definition 10.3, Notation 3.4, and Remarks 4.16 and A.7, then $\{\mathfrak{a}_j, \mathfrak{b}_j\} \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$.

Definition 10.6. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\geq}$. Then the sequences $(A_j)_{j=0}^{\kappa}$ and $(B_j)_{j=1}^{\kappa}$ given with $\mathfrak{a}_{-1} := 0_{q\times q}$ by

$$A_j := s_j - \mathfrak{a}_{j-1} \quad \text{and} \quad B_j := \mathfrak{b}_{j-1} - s_j$$

are called the *sequence of lower Schur complements associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha,\beta]$ and the *sequence of upper Schur complements associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha,\beta]$, respectively.

Taking into account that Proposition 10.15(a) will show $\mathfrak{b}_{j-1} - \mathfrak{a}_{j-1} \in \mathbb{C}^{q\times q}_{\geq}$, the matrices A_j and B_j introduced in Definition 10.6 determine the position of the matrix s_j in the interval $[\mathfrak{a}_{j-1}, \mathfrak{b}_{j-1}]$.

Remark 10.7. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. Then $A_0 = s_0$. If $\kappa \geq 1$, then, furthermore, $A_1 = ({}_\alpha s)_0$ and $B_1 = (s_\beta)_0$. If $\kappa \geq 2$, then, moreover, $B_2 = ({}_\alpha s_\beta)_0$.

Remark 10.8. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. Then $A_{2n} = L_n$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $A_{2n+1} = ({}_\alpha L)_n$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Furthermore, $B_{2n+1} = (L_\beta)_n$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and $B_{2n+2} = ({}_\alpha L_\beta)_n$ for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$.

From Remark 10.8 and Notation 3.4(c) we see now that the matrices introduced in Definition 10.6 are indeed Schur complements in block Hankel matrices which are responsible for the property of belonging to the set $\mathcal{F}^{\geq}_{q,m,\alpha,\beta}$. For this reason, we had chosen the terminology introduced in Definition 10.6.

Lemma 10.9. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$.*

(a) *If $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$, then $A_j \in \mathbb{C}^{q\times q}_{\geq}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $B_j \in \mathbb{C}^{q\times q}_{\geq}$ for all $j \in \mathbb{Z}_{1,\kappa}$.*

(b) *If $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{>}_{q,\kappa,\alpha,\beta}$, then $A_j \in \mathbb{C}^{q\times q}_{>}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $B_j \in \mathbb{C}^{q\times q}_{>}$ for all $j \in \mathbb{Z}_{1,\kappa}$.*

Proof. Use Remark 10.8 and Corollary 7.8. □

In the sequel, we need detailed information on the interplay between the ranges of consecutive elements of the sequences of lower and upper Schur complements associated with an $[\alpha, \beta]$-Hausdorff non-negative definite sequence. The next result gives us a first impression on this theme.

Lemma 10.10. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. Then $\mathcal{R}(A_j) + \mathcal{R}(B_j) \subseteq \mathcal{R}(A_{j-1}) \cap \mathcal{R}(B_{j-1})$ for each $j \in \mathbb{Z}_{2,\kappa}$.*

Proof. Suppose $\kappa \geq 2$. We consider an arbitrary $n \in \mathbb{N}$ with $2n \leq \kappa$. In view of Proposition 7.7(a), we have $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$. Thus, Proposition 9.2 yields $(s_j)_{j=0}^{2n} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta}$. Hence, using Proposition 8.22, we get

$$\mathcal{R}(L_n) \subseteq \mathcal{R}\big(({}_\alpha L)_{n-1}\big) \cap \mathcal{R}\big((L_\beta)_{n-1}\big). \tag{10.2}$$

Proposition 9.1(a) provides us with $\{(({}_\alpha s)_j)_{j=0}^{2n-1}, ((s_\beta)_j)_{j=0}^{2n-1}\} \subseteq \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$. Hence, from Proposition 9.2 we conclude $((s_\beta)_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n-1,\alpha}$ and $(({}_\alpha s)_j)_{j=0}^{2n-1} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n-1,\beta}$. Using Remark 4.11 and Proposition 8.22, we infer then

$$\mathcal{R}\big(({}_\alpha L_\beta)_{n-1}\big) \subseteq \mathcal{R}\big(({}_\alpha L)_{n-1}\big) \cap \mathcal{R}\big((L_\beta)_{n-1}\big). \tag{10.3}$$

From (10.2), (10.3), and Remark 10.8 we get $\mathcal{R}(A_{2n}) + \mathcal{R}(B_{2n}) \subseteq \mathcal{R}(A_{2n-1}) \cap \mathcal{R}(B_{2n-1})$.

Now we consider an arbitrary $n \in \mathbb{N}$ with $2n+1 \leq \kappa$. Then Proposition 7.7(a) shows that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$. According to Proposition 9.2, this implies $(s_j)_{j=0}^{2n+1} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha} \cap \mathcal{L}^{\geq,\mathrm{e}}_{q,2n+1,\beta}$. Thus, from Proposition 8.22 we obtain

$$\mathcal{R}((_\alpha L)_n) \cup \mathcal{R}\big((L_\beta)_n\big) \subseteq \mathcal{R}(L_n). \tag{10.4}$$

Proposition 9.1(a) yields $\{((_\alpha s)_j)_{j=0}^{2n}, ((s_\beta)_j)_{j=0}^{2n}\} \subseteq \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$. Consequently, Proposition 9.2 provides us with $((_\alpha s)_j)_{j=0}^{2n} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n-1,\beta}$ and $((s_\beta)_j)_{j=0}^{2n} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n,\alpha}$. Taking into account Remark 4.11, then Proposition 8.22 shows that $\mathcal{R}((_\alpha L)_n) \cup \mathcal{R}((L_\beta)_n) \subseteq \mathcal{R}((_\alpha L_\beta)_{n-1})$. Therefore, in view of (10.4) and Remark 10.8, we get then the inclusion $\mathcal{R}(A_{2n+1}) + \mathcal{R}(B_{2n+1}) \subseteq \mathcal{R}(A_{2n}) \cap \mathcal{R}(B_{2n})$. □

Corollary 10.19 will show, that in the situation of Lemma 10.10 we even have the equality $\mathcal{R}(A_j) + \mathcal{R}(B_j) = \mathcal{R}(A_{j-1}) \cap \mathcal{R}(B_{j-1})$ for all $j \in \mathbb{Z}_{2,\kappa}$.

Definition 10.11. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. Then the sequences $(\mathfrak{c}_j)_{j=0}^{\kappa}$ and $(\mathfrak{d}_j)_{j=0}^{\kappa}$ given by

$$\mathfrak{c}_j := \frac{1}{2}(\mathfrak{a}_j + \mathfrak{b}_j) \quad \text{and} \quad \mathfrak{d}_j := \mathfrak{b}_j - \mathfrak{a}_j$$

are called the *sequence of matricial interval midpoints associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha, \beta]$ and the *sequence of matricial interval lengths associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha, \beta]$, respectively.

Remark 10.12. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. If $m \in \mathbb{Z}_{0,\kappa}$, then it is easily checked that $(\mathfrak{d}_j)_{j=0}^{m}$ is the sequence of matricial interval lengths associated with $(s_j)_{j=0}^{m}$ and $[\alpha, \beta]$. Obviously, $\mathfrak{d}_0 = (\beta - \alpha)s_0$. If $\kappa \geq 1$, then $\mathfrak{d}_1 = -\alpha\beta s_0 + (\alpha + \beta)s_1 - s_1 s_0^{\dagger} s_1$ and Definitions 10.11 and 10.6 also show that $\mathfrak{d}_j = A_{j+1} + B_{j+1}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.

Remark 10.13. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. Because of Definition 10.11 and Remark 10.5, then $\{\mathfrak{c}_j, \mathfrak{d}_j\} \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$.

The next result is of fundamental importance for our further considerations. Again we make essential use of the parallel sum of matrices:

Theorem 10.14. *Let* $\alpha \in \mathbb{R}$, *let* $\beta \in (\alpha, \infty)$, *let* $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, *and let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. *Then* $\mathfrak{d}_0 = (\beta - \alpha)A_0$. *If* $\kappa \geq 1$, *then* $\mathfrak{d}_k = (\beta - \alpha)(A_k \mathbin{\overline{+}} B_k)$ *and* $\mathfrak{d}_k = (\beta - \alpha)(B_k \mathbin{\overline{+}} A_k)$ *for all* $k \in \mathbb{Z}_{1,\kappa}$.

Proof. In view of Notation 3.4(c) and Definition 10.6, we have $s_0 = L_0 = A_0$ and, taking into account Remark 10.4, consequently, $\mathfrak{d}_0 = \mathfrak{b}_0 - \mathfrak{a}_0 = (\beta - \alpha)A_0$. Assume $\kappa \geq 1$. According to Lemma 10.9(a), we know that $\{A_k, B_k\} \subseteq \mathbb{C}_{\geq}^{q\times q}$ for each

$k \in \mathbb{Z}_{1,\kappa}$. Thus, Lemmata B.2 and B.3 provide us with $A_k \mathbin{\overline{+}} B_k = B_k \mathbin{\overline{+}} A_k$ for each $k \in \mathbb{Z}_{1,\kappa}$. From Remark 10.12 we see that

$$(\beta - \alpha)(A_k \mathbin{\overline{+}} B_k) = (\beta - \alpha) A_k (A_k + B_k)^\dagger B_k = A_k \left(\frac{1}{\beta - \alpha} \mathfrak{d}_{k-1} \right)^\dagger B_k \tag{10.5}$$

is valid for each $k \in \mathbb{Z}_{1,\kappa}$. Using Proposition 7.7(a) and Remark 7.6, we conclude that $(s_j)_{j=0}^1$ belongs to $\mathcal{D}_{q\times q,1}$. Hence, Remarks A.5 and A.6 yield $s_0 s_0^\dagger s_1 = s_1$ and $s_1 s_0^\dagger s_0 = s_1$. Taking into account (10.5) and Remarks 10.8 and 10.12, we obtain then

$$\begin{aligned}(\beta - \alpha)(A_1 \mathbin{\overline{+}} B_1) &= A_1 \left(\frac{1}{\beta - \alpha} \mathfrak{d}_0 \right)^\dagger B_1 = [({}_\alpha L)_0] s_0^\dagger [(L_\beta)_0] = [({}_\alpha s)_0] s_0^\dagger [(s_\beta)_0] \\ &= (-\alpha s_0 + s_1) s_0^\dagger (\beta s_0 - s_1) = -\alpha\beta s_0 s_0^\dagger s_0 + \alpha s_0 s_0^\dagger s_1 + \beta s_1 s_0^\dagger s_0 - s_1 s_0^\dagger s_1 = \mathfrak{d}_1.\end{aligned}$$

In the cases $\kappa = 0$ and $\kappa = 1$, the proof is complete. Now suppose $\kappa \geq 2$. We assume that $m \in \mathbb{Z}_{2,\kappa}$ and that

$$\mathfrak{d}_j = (\beta - \alpha)(A_j \mathbin{\overline{+}} B_j) \quad \text{for each } j \in \mathbb{Z}_{0,m-1}. \tag{10.6}$$

Since the matrices A_{m-1} and B_{m-1} are both non-negative Hermitian, from Lemma B.2 and Proposition B.5 we get

$$(A_{m-1} \mathbin{\overline{+}} B_{m-1})^\dagger = P_{\mathcal{R}_{m-1}} (A_{m-1}^\dagger + B_{m-1}^\dagger) P_{\mathcal{R}_{m-1}} \tag{10.7}$$

where $\mathcal{R}_{m-1} := \mathcal{R}(A_{m-1}) \cap \mathcal{R}(B_{m-1})$. By virtue of Lemma 10.10, we infer $\mathcal{R}(A_m) + \mathcal{R}(B_m) \subseteq \mathcal{R}_{m-1}$ and, consequently, $P_{\mathcal{R}_{m-1}} A_m = A_m$ and $P_{\mathcal{R}_{m-1}} B_m = B_m$. Since the matrices $P_{\mathcal{R}_{m-1}}$ and A_m are Hermitian, then $A_m P_{\mathcal{R}_{m-1}} = A_m$ also holds true. Using (10.5), (10.6), and (10.7), it follows

$$\begin{aligned}(\beta - \alpha)(A_m \mathbin{\overline{+}} B_m) &= A_m \left(\frac{1}{\beta - \alpha} \mathfrak{d}_{m-1} \right)^\dagger B_m = A_m (A_{m-1} \mathbin{\overline{+}} B_{m-1})^\dagger B_m \\ &= A_m P_{\mathcal{R}_{m-1}} (A_{m-1}^\dagger + B_{m-1}^\dagger) P_{\mathcal{R}_{m-1}} B_m \\ &= A_m (A_{m-1}^\dagger + B_{m-1}^\dagger) B_m = A_m A_{m-1}^\dagger B_m + A_m B_{m-1}^\dagger B_m.\end{aligned} \tag{10.8}$$

First we consider now the case that $m = 2n$ with some positive integer n. Then from (10.8) and Remark 10.8 we obtain

$$(\beta - \alpha)(A_{2n} \mathbin{\overline{+}} B_{2n}) = L_n \big[({}_\alpha L)_{n-1} \big]^\dagger ({}_\alpha L_\beta)_{n-1} + L_n \big[(L_\beta)_{n-1} \big]^\dagger ({}_\alpha L_\beta)_{n-1}. \tag{10.9}$$

According to Proposition 7.7(a), the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{F}_{q,2n,\alpha,\beta}^{\geq}$. Thus, Proposition 9.2 yields $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq,\mathrm{e}} \cap \mathcal{L}_{q,2n,\beta}^{\geq,\mathrm{e}}$. Hence, Lemma 8.26 shows that (8.7) and (8.8) are true. This implies

$$\begin{aligned}\mathfrak{d}_{2n} = \mathfrak{b}_{2n} - \mathfrak{a}_{2n} &= \beta s_{2n} - (\Theta_\beta)_n - [\alpha s_{2n} + ({}_\alpha \Theta)_n] \\ &= \beta L_n + L_n \big[(L_\beta)_{n-1} \big]^\dagger \big[(s_\beta)_{2n-1} - (M_\beta)_{n-1} \big] \\ &\quad - \alpha L_n - L_n \big[({}_\alpha L)_{n-1} \big]^\dagger \big[({}_\alpha s)_{2n-1} - ({}_\alpha M)_{n-1} \big].\end{aligned} \tag{10.10}$$

Since $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{K}_{q,2n,\alpha}^{\geq,\mathrm{e}} \cap \mathcal{L}_{q,2n,\beta}^{\geq,\mathrm{e}}$, we get from Proposition 8.21 that $\mathcal{N}({}_{\alpha}L)_{n-1}) \cup \mathcal{N}((L_{\beta})_{n-1}) \subseteq \mathcal{N}(L_n)$. Consequently, Remark A.6 shows that

$$L_n\big[({}_{\alpha}L)_{n-1}\big]^{\dagger}({}_{\alpha}L)_{n-1} = L_n \quad \text{and} \quad L_n\big[(L_{\beta})_{n-1}\big]^{\dagger}(L_{\beta})_{n-1} = L_n. \tag{10.11}$$

Remark 4.11 and Notation 3.4 yield

$$({}_{\alpha}L_{\beta})_0 = ({}_{\alpha}s_{\beta})_0 = \beta({}_{\alpha}s)_0 - ({}_{\alpha}s)_1 = \beta({}_{\alpha}L)_0 - ({}_{\alpha}s)_1 + ({}_{\alpha}M)_0 \tag{10.12}$$

and

$$({}_{\alpha}L_{\beta})_0 = ({}_{\alpha}s_{\beta})_0 = -\alpha(s_{\beta})_0 + (s_{\beta})_1 = -\alpha(L_{\beta})_0 + (s_{\beta})_1 - (M_{\beta})_0. \tag{10.13}$$

Using (10.11) for $n = 1$, from (10.12) and (10.13) we obtain then $L_1({}_{\alpha}L)_0^{\dagger}({}_{\alpha}L_{\beta})_0 = L_1({}_{\alpha}L)_0^{\dagger}[({}_{\alpha}M)_0 - ({}_{\alpha}s)_1] + \beta L_1$ and $L_1(L_{\beta})_0^{\dagger}({}_{\alpha}L_{\beta})_0 = L_1(L_{\beta})_0^{\dagger}[(s_{\beta})_1 - (M_{\beta})_0] - \alpha L_1$. This implies

$$\begin{aligned} &L_1({}_{\alpha}L)_0^{\dagger}({}_{\alpha}L_{\beta})_0 + L_1(L_{\beta})_0^{\dagger}({}_{\alpha}L_{\beta})_0 \\ &\quad = L_1({}_{\alpha}L)_0^{\dagger}[({}_{\alpha}M)_0 - ({}_{\alpha}s)_1] + \beta L_1 + L_1(L_{\beta})_0^{\dagger}\big[(s_{\beta})_1 - (M_{\beta})_0\big] - \alpha L_1. \end{aligned} \tag{10.14}$$

Comparing (10.9) and (10.10) for $n = 1$ with (10.14), we conclude then $\mathfrak{d}_2 = (\beta - \alpha)(A_2 \mathbin{\dot{\mp}} B_2)$.

Now we suppose that $n \geq 2$. From Proposition 7.7(a) we know that $(s_j)_{j=0}^{2n-1}$ belongs to $\mathcal{F}_{q,2n-1,\alpha,\beta}^{\geq}$. Hence, Proposition 9.1(a) implies $\{(({}_{\alpha}s)_j)_{j=0}^{2n-2}, ((s_{\beta})_j)_{j=0}^{2n-2}\} \subseteq \mathcal{F}_{q,2n-2,\alpha,\beta}^{\geq}$. Consequently, because of Proposition 9.2, we have $(({}_{\alpha}s)_j)_{j=0}^{2n-2} \in \mathcal{L}_{q,2n-2,\beta}^{\geq,\mathrm{e}}$ and $((s_{\beta})_j)_{j=0}^{2n-2} \in \mathcal{K}_{q,2n-2,\alpha}^{\geq,\mathrm{e}}$. In view Remark 4.11, the application of Lemma 8.26(b) to the sequence $(({}_{\alpha}s)_j)_{j=0}^{2n-2}$ and of Lemma 8.26(a) to the sequence $((s_{\beta})_j)_{j=0}^{2n-2}$ provides us with

$$\begin{aligned} &(({}_{\alpha}\Theta)_{\beta})_{n-1} \\ &= \beta({}_{\alpha}s)_{2n-2} \\ &\quad - \Big[({}_{\alpha}M)_{n-1} + ({}_{\alpha}L)_{n-1}\Big(\beta I_q + ((\alpha L)_{\beta})_{n-2}^{\dagger}[(({}_{\alpha}s)_{\beta})_{2n-3} - (({}_{\alpha}M)_{\beta})_{n-2}]\Big)\Big] \\ &= \beta({}_{\alpha}s)_{2n-2} - \Big[({}_{\alpha}M)_{n-1} + ({}_{\alpha}L)_{n-1}\Big(\beta I_q + ({}_{\alpha}L_{\beta})_{n-2}^{\dagger}\big[({}_{\alpha}s_{\beta})_{2n-3} - ({}_{\alpha}M_{\beta})_{n-2}\big]\Big)\Big] \end{aligned}$$

and, analogously,

$$\begin{aligned} &({}_{\alpha}(\Theta_{\beta}))_{n-1} \\ &= -\alpha(s_{\beta})_{2n-2} + (M_{\beta})_{n-1} + (L_{\beta})_{n-1}\Big(\alpha I_q + ({}_{\alpha}L_{\beta})_{n-2}^{\dagger}\big[({}_{\alpha}s_{\beta})_{2n-3} - ({}_{\alpha}M_{\beta})_{n-2}\big]\Big). \end{aligned}$$

By virtue of Remark 4.11, we get

$$\begin{aligned} ({}_{\alpha}L_{\beta})_{n-1} &= ((\alpha L)_{\beta})_{n-1} = \beta({}_{\alpha}s)_{2n-2} - ({}_{\alpha}s)_{2n-1} - (({}_{\alpha}\Theta)_{\beta})_{n-1} \\ &= ({}_{\alpha}M)_{n-1} - ({}_{\alpha}s)_{2n-1} \\ &\quad + ({}_{\alpha}L)_{n-1}\Big(\beta I_q + ({}_{\alpha}L_{\beta})_{n-2}^{\dagger}\big[({}_{\alpha}s_{\beta})_{2n-3} - ({}_{\alpha}M_{\beta})_{n-2}\big]\Big) \end{aligned}$$

and, analogously,

$$\begin{aligned}&({}_\alpha L_\beta)_{n-1}\\&= (s_\beta)_{2n-1} - (M_\beta)_{n-1} - (L_\beta)_{n-1}\Big(\alpha I_q + ({}_\alpha L_\beta)^\dagger_{n-2}\big[({}_\alpha s_\beta)_{2n-3} - ({}_\alpha M_\beta)_{n-2}\big]\Big).\end{aligned}$$

Using (10.11), we conclude then

$$\begin{aligned}L_n\big[({}_\alpha L)_{n-1}\big]^\dagger({}_\alpha L_\beta)_{n-1} &= L_n\big[({}_\alpha L)_{n-1}\big]^\dagger\big[({}_\alpha M)_{n-1} - ({}_\alpha s)_{2n-1}\big]\\&\quad + L_n\Big(\beta I_q + ({}_\alpha L_\beta)^\dagger_{n-2}\big[({}_\alpha s_\beta)_{2n-3} - ({}_\alpha M_\beta)_{n-2}\big]\Big)\end{aligned}$$

and

$$\begin{aligned}L_n\big[(L_\beta)_{n-1}\big]^\dagger({}_\alpha L_\beta)_{n-1} &= L_n\big[(L_\beta)_{n-1}\big]^\dagger\big[(s_\beta)_{2n-1} - (M_\beta)_{n-1}\big]\\&\quad - L_n\Big(\alpha I_q + ({}_\alpha L_\beta)^\dagger_{n-2}\big[({}_\alpha s_\beta)_{2n-3} - ({}_\alpha M_\beta)_{n-2}\big]\Big).\end{aligned}$$

This implies

$$\begin{aligned}&L_n\big[({}_\alpha L)_{n-1}\big]^\dagger({}_\alpha L_\beta)_{n-1} + L_n\big[(L_\beta)_{n-1}\big]^\dagger({}_\alpha L_\beta)_{n-1}\\&= L_n\big[({}_\alpha L)_{n-1}\big]^\dagger\big[({}_\alpha M)_{n-1} - ({}_\alpha s)_{2n-1}\big] + \beta L_n\\&\quad + L_n\big[(L_\beta)_{n-1}\big]^\dagger\big[(s_\beta)_{2n-1} - (M_\beta)_{n-1}\big] - \alpha L_n.\end{aligned}$$

Comparing the foregoing equation with (10.9) and (10.10), we see that the equation $\mathfrak{d}_{2n} = (\beta - \alpha)(A_{2n} \mp B_{2n})$ holds true.

It remains to consider the case that $m = 2n+1$ with some $n \in \mathbb{N}$. Because of Remark 10.8 and (10.8), we have

$$(\beta - \alpha)(A_{2n+1} \mp B_{2n+1}) = ({}_\alpha L)_n L_n^\dagger (L_\beta)_n + ({}_\alpha L)_n ({}_\alpha L_\beta)^\dagger_{n-1}(L_\beta)_n. \tag{10.15}$$

From Proposition 7.7(a) we know that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$. Proposition 9.1(a) shows that $(({}_\alpha s)_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$. Consequently, Proposition 9.2 yields $(({}_\alpha s)_j)_{j=0}^{2n} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta}$. Using Remark 4.11 and applying Lemma 8.26(b) to the sequence $(({}_\alpha s)_j)_{j=0}^{2n}$, we get

$$\begin{aligned}&\beta({}_\alpha s)_{2n} - ({}_\alpha \Theta_\beta)_n = \beta({}_\alpha s)_{2n} - (({}_\alpha \Theta)_\beta)_n\\&= ({}_\alpha M)_n + ({}_\alpha L)_n\Big(\beta I_q + ((\alpha L)_\beta)^\dagger_{n-1}[((\alpha s)_\beta)_{2n-1} - ((\alpha M)_\beta)_{n-1}]\Big)\\&= ({}_\alpha M)_n + ({}_\alpha L)_n\Big(\beta I_q + ({}_\alpha L_\beta)^\dagger_{n-1}\big[({}_\alpha s_\beta)_{2n-1} - ({}_\alpha M_\beta)_{n-1}\big]\Big).\end{aligned} \tag{10.16}$$

Since $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$, we see from Proposition 9.2 that $(s_j)_{j=0}^{2n+1} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha}$ is valid. Thus, Lemma 8.24(a) provides us with

$$\alpha s_{2n+1} - \Theta_{n+1} = -({}_\alpha M)_n - ({}_\alpha L)_n L_n^\dagger (s_{2n+1} - M_n). \tag{10.17}$$

In view of Definitions 10.11 and 10.3, (10.16), and (10.17), we conclude

$$\begin{aligned}\mathfrak{d}_{2n+1} &= \mathfrak{b}_{2n+1} - \mathfrak{a}_{2n+1} = -\alpha\beta s_{2n} + (\alpha+\beta)s_{2n+1} - ({}_{\alpha}\Theta_{\beta})_n - \Theta_{n+1} \\ &= \beta({}_{\alpha}s)_{2n} - ({}_{\alpha}\Theta_{\beta})_n + \alpha s_{2n+1} - \Theta_{n+1} \\ &= ({}_{\alpha}L)_n\Big(\beta I_q + ({}_{\alpha}L_{\beta})^{\dagger}_{n-1}\big[({}_{\alpha}s_{\beta})_{2n-1} - ({}_{\alpha}M_{\beta})_{n-1}\big]\Big) - ({}_{\alpha}L)_n L_n^{\dagger}(s_{2n+1} - M_n).\end{aligned} \tag{10.18}$$

According to Proposition 7.7(a), the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$. Consequently, Proposition 9.2 shows that $(s_j)_{j=0}^{2n} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta}$. Therefore, because of Lemma 8.26(b), equation (8.8) holds true. By virtue of Notation 3.4(c) and (8.8), we have then

$$\begin{aligned}(L_{\beta})_n &= (s_{\beta})_{2n} - (\Theta_{\beta})_n \\ &= -s_{2n+1} + M_n + L_n\Big(\beta I_q + \big[(L_{\beta})_{n-1}\big]^{\dagger}\big[(s_{\beta})_{2n-1} - (M_{\beta})_{n-1}\big]\Big).\end{aligned} \tag{10.19}$$

Since $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}^{\geq,\mathrm{e}}_{q,2n+1,\alpha}$, we see from Proposition 8.21(a) that $\mathcal{N}(L_n) \subseteq \mathcal{N}(({}_{\alpha}L)_n)$. Hence, Remark A.6 shows that ${}_{\alpha}L)_n L_n^{\dagger} L_n = ({}_{\alpha}L)_n$. Thus, because of (10.19), we obtain

$$\begin{aligned}&({}_{\alpha}L)_n L_n^{\dagger}(L_{\beta})_n \\ &= -({}_{\alpha}L)_n L_n^{\dagger}(s_{2n+1} - M_n) + ({}_{\alpha}L)_n\Big(\beta I_q + \big[(L_{\beta})_{n-1}\big]^{\dagger}\big[(s_{\beta})_{2n-1} - (M_{\beta})_{n-1}\big]\Big).\end{aligned} \tag{10.20}$$

Taking into account $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\geq}_{q,2n,\alpha,\beta}$ and Proposition 9.1(a), we get $((s_{\beta})_j)_{j=0}^{2n-1} \in \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta}$. By virtue of Proposition 9.2, this implies $((s_{\beta})_j)_{j=0}^{2n-1} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,2n-1,\alpha}$. Applying Lemma 8.24(a) to the sequence $((s_{\beta})_j)_{j=0}^{2n-1}$ and using Remark 4.11, we infer

$$\begin{aligned}(\Theta_{\beta})_n &= \alpha(s_{\beta})_{2n-1} + ({}_{\alpha}(M_{\beta}))_{n-1} + ({}_{\alpha}(L_{\beta}))_{n-1}\big[(L_{\beta})_{n-1}\big]^{\dagger}\big[(s_{\beta})_{2n-1} - (M_{\beta})_{n-1}\big] \\ &= \alpha(s_{\beta})_{2n-1} + ({}_{\alpha}M_{\beta})_{n-1} + ({}_{\alpha}L_{\beta})_{n-1}\big[(L_{\beta})_{n-1}\big]^{\dagger}\big[(s_{\beta})_{2n-1} - (M_{\beta})_{n-1}\big].\end{aligned}$$

In view of Notation 3.4(c) and Remark 4.11, then

$$\begin{aligned}(L_{\beta})_n &= (s_{\beta})_{2n} - (\Theta_{\beta})_n \\ &= ({}_{\alpha}s_{\beta})_{2n-1} - ({}_{\alpha}M_{\beta})_{n-1} - ({}_{\alpha}L_{\beta})_{n-1}\big[(L_{\beta})_{n-1}\big]^{\dagger}\big[(s_{\beta})_{2n-1} - (M_{\beta})_{n-1}\big]\end{aligned} \tag{10.21}$$

follows. Since $(({}_{\alpha}s)_j)_{j=0}^{2n} \in \mathcal{L}^{\geq,\mathrm{e}}_{q,2n,\beta}$ is true, Proposition 8.21(d) yields $\mathcal{N}((({}_{\alpha}L)_{\beta})_{n-1}) \subseteq \mathcal{N}(({}_{\alpha}L)_n)$. Since Remark 4.11 shows that $(({}_{\alpha}L)_{\beta})_{n-1} = ({}_{\alpha}L_{\beta})_{n-1}$, then Remark A.6 yields ${}_{\alpha}L)_n({}_{\alpha}L_{\beta})^{\dagger}_{n-1}({}_{\alpha}L_{\beta})_{n-1} = ({}_{\alpha}L)_n$. Consequently,

from (10.21) we obtain

$$\begin{aligned}
&({}_\alpha L)_n({}_\alpha L_\beta)^\dagger_{n-1}(L_\beta)_n \\
&= ({}_\alpha L)_n({}_\alpha L_\beta)^\dagger_{n-1}\big[({}_\alpha s_\beta)_{2n-1} - ({}_\alpha M_\beta)_{n-1}\big] \\
&\quad - ({}_\alpha L)_n\big[(L_\beta)_{n-1}\big]^\dagger\big[(s_\beta)_{2n-1} - (M_\beta)_{n-1}\big]. \qquad (10.22)
\end{aligned}$$

Combining (10.20) and (10.22), we conclude

$$\begin{aligned}
&({}_\alpha L)_n L_n^\dagger(L_\beta)_n + ({}_\alpha L)_n({}_\alpha L_\beta)^\dagger_{n-1}(L_\beta)_n \\
&= -({}_\alpha L)_n L_n^\dagger(s_{2n+1} - M_n) + \beta({}_\alpha L)_n + ({}_\alpha L)_n({}_\alpha L_\beta)^\dagger_{n-1}\big[({}_\alpha s_\beta)_{2n-1} - ({}_\alpha M_\beta)_{n-1}\big]. \qquad (10.23)
\end{aligned}$$

Comparing (10.23) with (10.18) and (10.15), we get that the equation $\mathfrak{d}_{2n+1} = (\beta-\alpha)(A_{2n+1} \mp B_{2n+1})$ holds true. □

The following result contains an important property of the interval lengths introduced in Definition 10.11:

Proposition 10.15. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *If $(s_j)_{j=0}^\kappa \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$, then $\mathfrak{d}_j \in \mathbb{C}^{q\times q}_{\geq}$ for each $j \in \mathbb{Z}_{0,\kappa}$.*

(b) *If $(s_j)_{j=0}^\kappa \in \mathcal{F}^{>}_{q,\kappa,\alpha,\beta}$, then $\mathfrak{d}_j \in \mathbb{C}^{q\times q}_{>}$ for each $j \in \mathbb{Z}_{0,\kappa}$.*

Proof. The case $j < \kappa$ is an easy consequence of Lemma 10.9 and Remark 10.12. In general, the assertion follows from Theorem 10.14, Lemmata 10.9 and B.2, and Remark B.1. □

Observe that Proposition 10.15 is the reason that $(\mathfrak{d}_j)_{j=0}^\kappa$ is said to be the sequence of matricial interval lengths associated with $(s_j)_{j=0}^\kappa$ and $[\alpha,\beta]$.

Proposition 10.16. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *If $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq}_{q,m+1,\alpha,\beta}$, then $s_{m+1} \in [\mathfrak{a}_m, \mathfrak{b}_m]$.*

(b) *If $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{>}_{q,m+1,\alpha,\beta}$, then $s_{m+1} \in (\mathfrak{a}_m, \mathfrak{b}_m)$.*

Proof. Suppose $(s_j)_{j=0}^m \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$. According to Remark 10.5, we have $\mathfrak{a}_m^* = \mathfrak{a}_m$ and $\mathfrak{b}_m^* = \mathfrak{b}_m$. Proposition 7.10 provides us with $(s_j)_{j=0}^m \in \mathcal{H}^{\geq,\mathrm{e}}_{q,m}$. From $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq}_{q,m+1,\alpha,\beta}$ and Lemma 4.16 we know that $s_{m+1}^* = s_{m+1}$. Since $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq}_{q,m+1,\alpha,\beta}$ and Lemma 10.9(a) show that $\{A_{m+1}, B_{m+1}\} \subseteq \mathbb{C}^{q\times q}_{\geq}$ holds true, in view of Definition 10.6, then (a) follows. If $(s_j)_{j=0}^m$ even belongs to $\mathcal{F}^{>}_{q,m,\alpha,\beta}$, Lemma 10.9(b) shows that A_{m+1} and B_{m+1} are positive Hermitian. Because of Definition 10.6, then (b) follows. □

Corollary 10.17. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *If* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$, *then*

$$\alpha s_{2k} \leq \mathfrak{a}_{2k} \leq \mathfrak{c}_{2k} \leq \mathfrak{b}_{2k} \leq \beta s_{2k} \tag{10.24}$$

for all $k \in \mathbb{N}$ *with* $2k \leq \kappa$, *and*

$$0_{q\times q} \leq \mathfrak{a}_{2k+1} \leq \mathfrak{c}_{2k+1} \leq \mathfrak{b}_{2k+1} \leq -\alpha\beta s_{2k} + (\alpha+\beta)s_{2k+1} \tag{10.25}$$

for all $k \in \mathbb{N}$ *with* $2k+1 \leq \kappa$.

(b) *If* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{>}_{q,\kappa,\alpha,\beta}$, *then*

$$\alpha s_{2k} < \mathfrak{a}_{2k} < \mathfrak{c}_{2k} < \mathfrak{b}_{2k} < \beta s_{2k} \tag{10.26}$$

for all $k \in \mathbb{Z}_{2,\infty}$ *with* $2k \leq \kappa$, *and*

$$0_{q\times q} < \mathfrak{a}_{2k+1} < \mathfrak{c}_{2k+1} < \mathfrak{b}_{2k+1} < -\alpha\beta s_{2k} + (\alpha+\beta)s_{2k+1} \tag{10.27}$$

for all $k \in \mathbb{Z}_{2,\infty}$ *with* $2k+1 \leq \kappa$.

Proof. Suppose $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{>}_{q,\kappa,\alpha,\beta}$). In view of Lemma 4.16 and Remarks 10.5 and 10.13, all the matrices involved are Hermitian. We consider a number $k \in \mathbb{N}$ (resp., $k \in \mathbb{Z}_{2,\infty}$).

First assume that $2k \leq \kappa$. Because of Proposition 10.15, then $\mathfrak{d}_{2k}$ is non-negative Hermitian (resp., positive Hermitian). Since $\mathfrak{c}_{2k} - \mathfrak{a}_{2k} = \frac{1}{2}\mathfrak{d}_{2k}$ and $\mathfrak{b}_{2k} - \mathfrak{c}_{2k} = \frac{1}{2}\mathfrak{d}_{2k}$, by virtue of Definition 10.11, the two inner inequalities in (10.24) (resp., (10.26)) follow. According to Proposition 7.7, we have $(s_j)_{j=0}^{2k-1} \in \mathcal{F}^{\geq}_{q,2k-1,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2k-1} \in \mathcal{F}^{>}_{q,2k-1,\alpha,\beta}$). In view of Definition 4.2(a), hence $\{(({}_{\alpha}s)_j)_{j=0}^{2k-2}, ((s_{\beta})_j)_{j=0}^{2k-2}\} \subseteq \mathcal{H}^{\geq}_{q,2k-2}$ (resp., $\{(({}_{\alpha}s)_j)_{j=0}^{2k-2}, ((s_{\beta})_j)_{j=0}^{2k-2}\} \subseteq \mathcal{H}^{>}_{q,2k-2}$). Since the matrices $({}_{\alpha}s)_{2k-1}$ and $(s_{\beta})_{2k-1}$ are, according to Lemma 4.16 and Notation 4.1, both Hermitian, we see from Lemma 3.8 that $({}_{\alpha}\Theta)_k$ and $(\Theta_{\beta})_k$ are both non-negative Hermitian (resp., positive Hermitian). Consequently, the two outer inequalities in (10.24) (resp., (10.26)) can be seen from Definition 10.3.

Now assume that $2k+1 \leq \kappa$. Because of Proposition 10.15, then $\mathfrak{d}_{2k+1}$ is non-negative Hermitian (resp., positive Hermitian). Since $\mathfrak{c}_{2k+1} - \mathfrak{a}_{2k+1} = \frac{1}{2}\mathfrak{d}_{2k+1}$ and $\mathfrak{b}_{2k+1} - \mathfrak{c}_{2k+1} = \frac{1}{2}\mathfrak{d}_{2k+1}$ by virtue of Definition 10.11, the two inner inequalities in (10.25) (resp., (10.27)) follow. According to Proposition 7.7, we have $(s_j)_{j=0}^{2k} \in \mathcal{F}^{\geq}_{q,2k,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2k} \in \mathcal{F}^{>}_{q,2k,\alpha,\beta}$). In view of Definition 4.2(b), hence $(s_j)_{j=0}^{2k} \in \mathcal{H}^{\geq}_{q,2k}$ and $(({}_{\alpha}s_{\beta})_j)_{j=0}^{2k-2} \in \mathcal{H}^{\geq}_{q,2k-2}$ (resp., $(s_j)_{j=0}^{2k} \in \mathcal{H}^{>}_{q,2k}$ and $(({}_{\alpha}s_{\beta})_j)_{j=0}^{2k-2} \in \mathcal{H}^{>}_{q,2k-2}$). Since the matrices s_{2k+1} and $({}_{\alpha}s_{\beta})_{2k-1}$ are, according to Lemma 4.16 and Notation 4.1, both Hermitian, we see from Lemma 3.8 that Θ_{k+1} and $({}_{\alpha}\Theta_{\beta})_k$ are both non-negative Hermitian (resp., positive Hermitian). Consequently, the two outer inequalities in (10.25) (resp., (10.27)) can be seen from Definition 10.3. □

Proposition 10.18. *Let* $\alpha \in \mathbb{R}$, *let* $\beta \in (\alpha, \infty)$, *let* $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, *and let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. *Then* $\mathcal{R}(\mathfrak{d}_0) = \mathcal{R}(A_0)$ *and* $\mathcal{N}(\mathfrak{d}_0) = \mathcal{N}(A_0)$. *If* $\kappa \geq 1$, *then*

$$\mathcal{R}(\mathfrak{d}_j) = \mathcal{R}(A_j) \cap \mathcal{R}(B_j) \quad \text{and} \quad \mathcal{N}(\mathfrak{d}_j) = \mathcal{N}(A_j) + \mathcal{N}(B_j) \tag{10.28}$$

hold true for all $j \in \mathbb{Z}_{1,\kappa}$ *and*

$$\mathcal{R}(\mathfrak{d}_j) = \mathcal{R}(A_{j+1}) + \mathcal{R}(B_{j+1}) \quad \text{and} \quad \mathcal{N}(\mathfrak{d}_j) = \mathcal{N}(A_{j+1}) \cap \mathcal{N}(B_{j+1}) \tag{10.29}$$

are fulfilled for all $j \in \mathbb{Z}_{0,\kappa-1}$.

Proof. Because of $\alpha < \beta$ and Theorem 10.14, we have $\mathcal{R}(\mathfrak{d}_0) = \mathcal{R}(A_0)$ and $\mathcal{N}(\mathfrak{d}_0) = \mathcal{N}(A_0)$. Assume now $\kappa \geq 1$. Then Lemma 10.9(a) provides us with $\{A_j, B_j\} \subseteq \mathbb{C}^{q\times q}_{\geq}$ for all $j \in \mathbb{Z}_{1,\kappa}$. Using Theorem 10.14, Lemmata B.2 and B.4, and $\alpha < \beta$, we get (10.28) for all $j \in \mathbb{Z}_{1,\kappa}$. By virtue of Remarks 10.12 and A.1, we also obtain (10.29) for each $j \in \mathbb{Z}_{0,\kappa-1}$. □

The following result improves Lemma 10.10.

Corollary 10.19. *Let* $\alpha \in \mathbb{R}$, *let* $\beta \in (\alpha, \infty)$, *let* $\kappa \in \mathbb{N} \cup \{\infty\}$, *and let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. *Then* $\mathcal{R}(A_j) + \mathcal{R}(B_j) = \mathcal{R}(A_{j-1}) \cap \mathcal{R}(B_{j-1})$ *and* $\mathcal{N}(A_{j-1}) + \mathcal{N}(B_{j-1}) = \mathcal{N}(A_j) \cap \mathcal{N}(B_j)$ *for all* $j \in \mathbb{Z}_{2,\kappa}$

Proof. This is a direct consequence of Proposition 10.18. □

Corollary 10.20. *Let* $\alpha \in \mathbb{R}$, *let* $\beta \in (\alpha, \infty)$, *let* $\kappa \in \mathbb{N} \cup \{\infty\}$, *and let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. *Then* $\mathcal{R}(\mathfrak{d}_j) \subseteq \mathcal{R}(\mathfrak{d}_{j-1})$, $\mathcal{N}(\mathfrak{d}_{j-1}) \subseteq \mathcal{N}(\mathfrak{d}_j)$, $\mathcal{R}(A_j) \subseteq \mathcal{R}(A_{j-1})$, *and* $\mathcal{N}(A_{j-1}) \subseteq \mathcal{N}(A_j)$ *for all* $j \in \mathbb{Z}_{1,\kappa}$. *Furthermore, if* $\kappa \geq 2$, *then* $\mathcal{R}(B_j) \subseteq \mathcal{R}(B_{j-1})$ *and* $\mathcal{N}(B_{j-1}) \subseteq \mathcal{N}(B_j)$ *for each* $j \in \mathbb{Z}_{2,\kappa}$.

Proof. Proposition 10.18 provides us with $\mathcal{R}(\mathfrak{d}_j) = \mathcal{R}(A_j) \cap \mathcal{R}(B_j) \subseteq \mathcal{R}(A_j) + \mathcal{R}(B_j) = \mathcal{R}(\mathfrak{d}_{j-1})$ for all $j \in \mathbb{Z}_{1,\kappa}$ and $\mathcal{R}(A_j) \subseteq \mathcal{R}(A_j) + \mathcal{R}(B_j) = \mathcal{R}(\mathfrak{d}_{j-1}) = \mathcal{R}(A_{j-1}) \cap \mathcal{R}(B_{j-1}) \subseteq \mathcal{R}(A_{j-1})$ as well as $\mathcal{R}(B_j) \subseteq \mathcal{R}(\mathfrak{d}_{j-1}) \subseteq \mathcal{R}(B_{j-1})$ for all $j \in \mathbb{Z}_{2,\kappa}$. Proposition 10.18, Remark 10.12, and Definition 10.6 also yield $\mathcal{R}(A_1) \subseteq \mathcal{R}(A_1) + \mathcal{R}(B_1) = \mathcal{R}(\mathfrak{d}_0) = \mathcal{R}(s_0) = \mathcal{R}(A_0)$. The asserted inclusions for the null spaces follow analogously. □

Corollary 10.21. *Let* $\alpha \in \mathbb{R}$, *let* $\beta \in (\alpha, \infty)$, *let* $\kappa \in \mathbb{N} \cup \{\infty\}$, *and let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. *For all* $j \in \mathbb{Z}_{1,\kappa}$, *then*

$$\mathfrak{d}_j = (\beta - \alpha) A_j \mathfrak{d}_{j-1}^{\dagger} B_j \quad \text{and} \quad \mathfrak{d}_j = (\beta - \alpha) B_j \mathfrak{d}_{j-1}^{\dagger} A_j.$$

Proof. In view of (B.1), this is a consequence of Theorem 10.14 and Remark 10.12. □

Now we are going to investigate the sequence of matricial interval lengths $(\mathfrak{d}_j)_{j=0}^{\kappa}$ associated with a sequence $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. Using Theorem 10.14 and the arithmetics of the parallel sum of matrices, we derive a formula connecting two consecutive interval lengths $\mathfrak{d}_{j+1}$ and $\mathfrak{d}_j$:

Proposition 10.22. *Let* $\alpha \in \mathbb{R}$, *let* $\beta \in (\alpha, \infty)$, *let* $\kappa \in \mathbb{N} \cup \{\infty\}$, *and let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$. *For all* $j \in \mathbb{Z}_{0,\kappa-1}$, *then*

$$\mathfrak{d}_{j+1} = \frac{\beta - \alpha}{4} \mathfrak{d}_j - (\beta - \alpha)(s_{j+1} - \mathfrak{c}_j) \mathfrak{d}_j^{\dagger} (s_{j+1} - \mathfrak{c}_j).$$

Proof. We consider an arbitrary $j \in \mathbb{Z}_{0,\kappa-1}$. According to Lemma 10.9(a), the matrices A_{j+1} and B_{j+1} are both non-negative Hermitian. In view of Lemmata B.2 and B.6, then

$$(A_{j+1}+B_{j+1})-4(A_{j+1}\mp B_{j+1})=(A_{j+1}-B_{j+1})(A_{j+1}+B_{j+1})^{\dagger}(A_{j+1}-B_{j+1}).$$

Using Remark 10.12 and Theorem 10.14, we get

$$(\beta-\alpha)\mathfrak{d}_j-4\mathfrak{d}_{j+1}=(\beta-\alpha)(A_{j+1}-B_{j+1})\mathfrak{d}_j^{\dagger}(A_{j+1}-B_{j+1}).$$

Since $2(s_{j+1}-\mathfrak{c}_j)=A_{j+1}-B_{j+1}$ holds true, we obtain

$$4\mathfrak{d}_{j+1}=(\beta-\alpha)\mathfrak{d}_j-4(\beta-\alpha)(s_{j+1}-\mathfrak{c}_j)\mathfrak{d}_j^{\dagger}(s_{j+1}-\mathfrak{c}_j). \qquad \square$$

Proposition 10.22 leads us now to an important monotonicity property for the sequence of interval lengths. Moreover, the extremal role of the midpoint of the interval in this context will be clear.

Proposition 10.23. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $\kappa \in \mathbb{N}\cup\{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\geq}_{q,\kappa,\alpha,\beta}$, and let $m \in \mathbb{Z}_{1,\kappa}$. Then $\mathfrak{d}_m \leq \frac{\beta-\alpha}{4}\mathfrak{d}_{m-1}$. Furthermore, $\mathfrak{d}_m = \frac{\beta-\alpha}{4}\mathfrak{d}_{m-1}$ if and only if $s_m = \mathfrak{c}_{m-1}$.*

Proof. The matrices $\mathfrak{d}_{m-1}$ and $\mathfrak{d}_m$ are both non-negative Hermitian according to Proposition 10.15. Let $D := \frac{\beta-\alpha}{4}\mathfrak{d}_{m-1}-\mathfrak{d}_m$. Because of Proposition 10.22, we have $D=(\beta-\alpha)(s_m-\mathfrak{c}_{m-1})\mathfrak{d}_{m-1}^{\dagger}(s_m-\mathfrak{c}_{m-1})$. Since the matrices s_m and $\mathfrak{c}_{m-1}$ are both Hermitian by virtue of Lemma 4.16 and Remark 10.13, hence

$$D=(\beta-\alpha)(s_m-\mathfrak{c}_{m-1})^{*}\mathfrak{d}_{m-1}^{\dagger}(s_m-\mathfrak{c}_{m-1}). \tag{10.30}$$

According to Remark A.10, the matrix $\mathfrak{d}_{m-1}^{\dagger}$ is non-negative Hermitian. In view of $\beta>\alpha$, then $D \in \mathbb{C}^{q\times q}_{\geq}$ follows.

Now assume $D=0_{q\times q}$. Because of $\beta>\alpha$ and $\mathfrak{d}_{m-1}^{\dagger}\in\mathbb{C}^{q\times q}_{\geq}$, we obtain from (10.30) then $\sqrt{\mathfrak{d}_{m-1}^{\dagger}}(s_m-\mathfrak{c}_{m-1})=0_{q\times q}$. In particular, $\mathfrak{d}_{m-1}\mathfrak{d}_{m-1}^{\dagger}(s_m-\mathfrak{c}_{m-1})=0_{q\times q}$. In view of Definitions 10.11 and 10.6, we get $s_m-\mathfrak{c}_{m-1}=\frac{1}{2}(A_m-B_m)$. Hence, $\mathcal{R}(s_m-\mathfrak{c}_{m-1})\subseteq\mathcal{R}(A_m)+\mathcal{R}(B_m)$. Since $\mathcal{R}(A_m)+\mathcal{R}(B_m)=\mathcal{R}(\mathfrak{d}_{m-1})$ by virtue of Proposition 10.18, we have $\mathcal{R}(s_m-\mathfrak{c}_{m-1})\subseteq\mathcal{R}(\mathfrak{d}_{m-1})$. Using Remark A.5, we obtain $s_m-\mathfrak{c}_{m-1}=\mathfrak{d}_{m-1}\mathfrak{d}_{m-1}^{\dagger}(s_m-\mathfrak{c}_{m-1})$ and hence $s_m-\mathfrak{c}_{m-1}=0_{q\times q}$.

Conversely, if $s_m=\mathfrak{c}_{m-1}$, then $D=0_{q\times q}$ follows from (10.30). $\square$

We turn our attention to an interesting subclass of $[\alpha,\beta]$-Hausdorff non-negative definite sequences:

Definition 10.24. Let $\alpha\in\mathbb{R}$, let $\beta\in(\alpha,\infty)$, let $m\in\mathbb{N}_0$ and let $(s_j)_{j=0}^{m}\in\mathcal{F}^{\geq}_{q,m,\alpha,\beta}$. Then $(s_j)_{j=0}^{m}$ is called $[\alpha,\beta]$-*Hausdorff completely degenerate* if $\mathfrak{d}_m=0_{q\times q}$, where $\mathfrak{d}_m$ is given in Definition 10.11. We denote by $\mathcal{F}^{\geq,\mathrm{cd}}_{q,m,\alpha,\beta}$ the set of all sequences $(s_j)_{j=0}^{m}\in\mathcal{F}^{\geq}_{q,m,\alpha,\beta}$ which are $[\alpha,\beta]$-Hausdorff completely degenerate.

Definition 10.25. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, and let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$.

(a) Let $m \in \mathbb{N}_0$. Then $(s_j)_{j=0}^{\infty}$ is called $[\alpha,\beta]$-*Hausdorff completely degenerate of order* m if $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{cd}}$.

(b) The sequence $(s_j)_{j=0}^{\infty}$ is called $[\alpha,\beta]$-*Hausdorff completely degenerate* if there exists an $m \in \mathbb{N}_0$ such that $(s_j)_{j=0}^{\infty}$ is Hankel completely degenerate of order m.

We denote by $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq,\mathrm{cd}}$ the set of all sequences $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ which are $[\alpha,\beta]$-Hausdorff completely degenerate.

Lemma 10.26. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then $(s_j)_{j=0}^{m}$ is $[\alpha,\beta]$-Hausdorff completely degenerate if and only if $\mathcal{R}(A_m) \cap \mathcal{R}(B_m) = \{0_{q\times 1}\}$.*

Proof. Use Proposition 10.18. □

Lemma 10.27. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ be $[\alpha,\beta]$-Hausdorff completely degenerate of order m. Then:*

(a) *If $m = 2n$ with some $n \in \mathbb{N}_0$, then $(s_j)_{j=0}^{\infty}$ is Hankel completely degenerate of order $n+1$ and the sequences $({}_{\alpha}s)_j)_{j=0}^{\infty}$, $((s_\beta)_j)_{j=0}^{\infty}$, and $({}_{\alpha}s_\beta)_j)_{j=0}^{\infty}$ are Hankel completely degenerate of order n.*

(b) *If $m = 2n+1$ with some $n \in \mathbb{N}_0$, then $({}_{\alpha}s_\beta)_j)_{j=0}^{\infty}$ is Hankel completely degenerate of order n and the sequences $(s_j)_{j=0}^{\infty}$, $({}_{\alpha}s)_j)_{j=0}^{\infty}$, and $((s_\beta)_j)_{j=0}^{\infty}$ are Hankel completely degenerate of order $n+1$.*

Proof. First observe that the sequences $(s_j)_{j=0}^{\infty}$, $({}_{\alpha}s)_j)_{j=0}^{\infty}$, $((s_\beta)_j)_{j=0}^{\infty}$, and $({}_{\alpha}s_\beta)_j)_{j=0}^{\infty}$ belong to $\mathcal{H}_{q,\infty}^{\geq}$ by virtue of Lemma 4.15 and Proposition 9.1. In view of Definitions 10.25 and 10.24, we have $\mathfrak{d}_m = 0_{q\times q}$. According to Proposition 10.18, we get $\mathcal{R}(A_{m+1}) + \mathcal{R}(B_{m+1}) = \{0_{q\times 1}\}$. In particular, $\mathcal{R}(A_{m+1}) = \{0_{q\times 1}\}$ and $\mathcal{R}(B_{m+1}) = \{0_{q\times 1}\}$. Using Corollary 10.20, this implies $\mathcal{R}(A_{m+2}) = \{0_{q\times 1}\}$ and $\mathcal{R}(B_{m+2}) = \{0_{q\times 1}\}$. Thus, each of the matrices A_{m+1}, B_{m+1}, A_{m+2}, and B_{m+2} is the zero matrix $0_{q\times q}$.

(a) In this situation each of the matrices A_{2n+1}, B_{2n+1}, A_{2n+2}, and B_{2n+2} is the zero matrix $0_{q\times q}$. In view of Remark 10.8, hence $({}_{\alpha}L)_n = 0_{q\times q}$, $(L_\beta)_n = 0_{q\times q}$, $L_{n+1} = 0_{q\times q}$, and $({}_{\alpha}L_\beta)_n = 0_{q\times q}$.

(b) In this situation each of the matrices A_{2n+2}, B_{2n+2}, A_{2n+3}, and B_{2n+3} is the zero matrix $0_{q\times q}$. In view of Remark 10.8, hence $L_{n+1} = 0_{q\times q}$, $({}_{\alpha}L_\beta)_n = 0_{q\times q}$, $({}_{\alpha}L)_{n+1} = 0_{q\times q}$, and $(L_\beta)_{n+1} = 0_{q\times q}$. □

Lemma 10.28. *Let $\alpha \in \mathbb{R}$ and let $\beta \in (\alpha, \infty)$. Then $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq,\mathrm{cd}} \subseteq \mathcal{H}_{q,\infty}^{\geq,\mathrm{cd}}$.*

Proof. This is a direct consequence of Lemma 10.27. □

Lemma 10.29. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$, and let $m \in \mathbb{N}_0$. If $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff completely degenerate of order m, then $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff completely degenerate of order ℓ for all $\ell \in \mathbb{Z}_{m,\infty}$.*

Proof. This can be seen from Corollary 10.20 □

Proposition 10.30. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ be $[\alpha,\beta]$-Hausdorff completely degenerate of order m. Then $s_j = \mathfrak{c}_{j-1} = \mathfrak{a}_{j-1} = \mathfrak{b}_{j-1}$ for all $j \in \mathbb{Z}_{m+1,\infty}$.*

Proof. Let $\ell \in \mathbb{Z}_{m,\infty}$. In view of Lemma 10.29 and Definitions 10.25 and 10.24, then $\mathfrak{d}_\ell = 0_{q\times q}$. Because of Definition 10.11, hence $\mathfrak{a}_\ell = \mathfrak{b}_\ell$ and thus $\mathfrak{c}_\ell = \mathfrak{a}_\ell$. In particular, $[\mathfrak{a}_\ell, \mathfrak{b}_\ell] = \{\mathfrak{c}_\ell\}$. According to Definition 4.6, furthermore $(s_j)_{j=0}^{\ell+1} \in \mathcal{F}_{q,\ell+1,\alpha,\beta}^{\geq}$. Proposition 10.16(a) yields then $s_{\ell+1} \in [\mathfrak{a}_\ell, \mathfrak{b}_\ell]$. Since $[\mathfrak{a}_\ell, \mathfrak{b}_\ell] = \{\mathfrak{c}_\ell\}$, this implies $s_{\ell+1} = \mathfrak{c}_\ell$. □

Proposition 10.31. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{cd}}$. Then there exists a unique sequence $(s_j)_{j=m+1}^{\infty}$ from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$.*

Proof. Combine Propositions 4.8 and 10.30. □

Proposition 10.31 leads us to the following notion:

Definition 10.32. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{cd}}$. Let $(s_j)_{j=m+1}^{\infty}$ be the unique sequence from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$. Then $(s_j)_{j=0}^{\infty}$ is called the $[\alpha,\beta]$-*Hausdorff completely degenerate sequence associated with* $(s_j)_{j=0}^{m}$.

The considerations of this section lead us to a further interesting subclass of $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$:

Definition 10.33. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, and let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$. If $m \in \mathbb{N}$ is such that $s_j = \mathfrak{c}_{j-1}$ for all $j \in \mathbb{Z}_{m,\infty}$, where $\mathfrak{c}_{j-1}$ is given by Definition 10.11, then $(s_j)_{j=0}^{\infty}$ is called $[\alpha,\beta]$-*Hausdorff central of order* m. If there exists an $\ell \in \mathbb{N}$ such that $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff central of order ℓ, then $(s_j)_{j=0}^{\infty}$ is simply called $[\alpha,\beta]$-*Hausdorff central*.

Remark 10.34. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$, and let $m \in \mathbb{N}$. If $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff central of order m, then $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff central of order ℓ for all $\ell \in \mathbb{Z}_{m,\infty}$.

Proposition 10.35. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$. Then $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff central of order m if and only if $\mathfrak{d}_\ell = \frac{\beta-\alpha}{4}\mathfrak{d}_{\ell-1}$ for all $\ell \in \mathbb{Z}_{m,\infty}$.*

Proof. This is a direct consequence of Proposition 10.23. □

A closer look at Propositions 10.23 and 10.35 shows that the role of the central sequences in the context of $[\alpha,\beta]$-Hausdorff non-negative definite sequences is comparable with the role of the central sequences in the context of non-negative definite sequences from $\mathbb{C}^{q\times q}$ or of the central sequences of $p \times q$ Schur sequences.

On the one hand, we see that starting from some index the interval lengths are maximal with respect to the Löwner semi-ordering in the set $\mathbb{C}_{\mathrm{H}}^{q\times q}$. On the other hand, [16, Lemma 6] and [17, Lemma 7] indicate that the semi-radii of the corresponding matrix balls are maximal with respect to the Löwner semi-ordering in the set $\mathbb{C}_{\mathrm{H}}^{q\times q}$ as well for central non-negative definite sequences from $\mathbb{C}^{q\times q}$ as for central $p\times q$ Schur sequences, respectively.

In view of Definition 10.33, we obtain from Proposition 10.30 the following result:

Remark 10.36. Let $\alpha\in\mathbb{R}$, let $\beta\in(\alpha,\infty)$, let $m\in\mathbb{N}_0$, and let $(s_j)_{j=0}^{\infty}\in\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ be $[\alpha,\beta]$-Hausdorff completely degenerate of order m. Then $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff central of order $m+1$.

Lemma 10.37. *Let $\alpha\in\mathbb{R}$, let $\beta\in(\alpha,\infty)$, let $m\in\mathbb{N}_0$, and let $(s_j)_{j=0}^{m}\in\mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{cd}}$. Then the $[\alpha,\beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{m}$ is $[\alpha,\beta]$-Hausdorff completely degenerate of order m and $[\alpha,\beta]$-Hausdorff central of order $m+1$.*

Proof. The $[\alpha,\beta]$-Hausdorff completely degenerate sequence $(s_j)_{j=0}^{\infty}$ associated to $(s_j)_{j=0}^{m}$ belongs to $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ by definition and fulfills $(s_j)_{j=0}^{m}\in\mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{cd}}$. In view of Definition 10.25(a), hence $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff completely degenerate of order m. According to Remark 10.36, then $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff central of order $m+1$. □

From Lemmata 10.37 and 10.27 we obtain the following result:

Remark 10.38. Let $\alpha\in\mathbb{R}$, let $\beta\in(\alpha,\infty)$, let $m\in\mathbb{N}_0$, and let $(s_j)_{j=0}^{m}\in\mathcal{F}_{q,m,\alpha,\beta}^{\geq,\mathrm{cd}}$. Then the $[\alpha,\beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{m}$ is Hankel completely degenerate of order $n+1$, where n is the unique number from $\mathbb{N}_0$ with $m=2n$ or $m=2n+1$.

Section 10 has some points of touch with the paper Dette/Studden [11], where the moment space of a matrix measure $\sigma\in\mathcal{M}_{\geq}^{q}([0,1])$ is studied and where it is assumed that the corresponding block Hankel matrices are non-singular. The strategy used in [11] is mainly based on using convexity techniques and constructing canonical representations for the points of the moment space as this was done in the classical scalar case by M.G. Krein [26] (see also Krein/Nudelman [27, Ch. 3]). An important tool used in [11] is given by a matricial generalization of the theory of classical canonical moments (see the monograph Dette/Studden [10]).

11. On the problem of $[\alpha,\beta]$-Hausdorff non-negative definite extension

Let $m\in\mathbb{N}_0$ and let $(s_j)_{j=0}^{m}\in\mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then against the background of Proposition 10.16 we will be able to describe the set $\{s_{m+1}\in\mathbb{C}^{q\times q}\colon (s_j)_{j=0}^{m+1}\in\mathcal{F}_{q,m+1,\alpha,\beta}^{\geq}\}$.

Lemma 11.1. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *If $(s_j)_{j=0}^m \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$, then $[\mathfrak{a}_m, \mathfrak{b}_m] \neq \emptyset$ and $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq}_{q,m+1,\alpha,\beta}$ for all $s_{m+1} \in [\mathfrak{a}_m, \mathfrak{b}_m]$.*

(b) *If $(s_j)_{j=0}^m \in \mathcal{F}^{>}_{q,m,\alpha,\beta}$, then $(\mathfrak{a}_m, \mathfrak{b}_m) \neq \emptyset$ and $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{>}_{q,m+1,\alpha,\beta}$ for all $s_{m+1} \in (\mathfrak{a}_m, \mathfrak{b}_m)$.*

Proof. Assume $(s_j)_{j=0}^m \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$ (resp., $(s_j)_{j=0}^m \in \mathcal{F}^{>}_{q,m,\alpha,\beta}$). From Proposition 10.15 we know that $\mathfrak{d}_m$ it non-negative Hermitian (resp., positive Hermitian). In view of Definition 10.11, Lemma 10.1 yields then $[\mathfrak{a}_m, \mathfrak{b}_m] \neq \emptyset$ (resp., $(\mathfrak{a}_m, \mathfrak{b}_m) \neq \emptyset$). Proposition 7.10 provides us with $(s_j)_{j=0}^m \in \mathcal{H}^{\geq,\mathrm{e}}_{q,m}$ (resp., $(s_j)_{j=0}^m \in \mathcal{H}^{>,\mathrm{e}}_{q,m}$). We consider an arbitrary $s_{m+1} \in [\mathfrak{a}_m, \mathfrak{b}_m]$ (resp., $s_{m+1} \in (\mathfrak{a}_m, \mathfrak{b}_m)$). Then $s_{m+1} \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and $\mathfrak{a}_m \leq s_{m+1} \leq \mathfrak{b}_m$ (resp., $\mathfrak{a}_m < s_{m+1} < \mathfrak{b}_m$).

First we discuss the case $m = 0$. In view of Remark 10.4 and Notation 4.1, then ${}_\alpha H)_0$ and $(H_\beta)_0$ are both non-negative Hermitian (resp., positive Hermitian). Thus, $(s_j)_{j=0}^1 \in \mathcal{F}^{\geq}_{q,1,\alpha,\beta}$ (resp., $(s_j)_{j=0}^1 \in \mathcal{F}^{>}_{q,1,\alpha,\beta}$) follows from Definition 4.2(a).

Now let $m = 1$. Then Remark 10.4 and Notation 4.1 yield $\Theta_1 \leq s_2$ and $({}_\alpha H_\beta)_0 \in \mathbb{C}^{q\times q}_{\geq}$ (resp., $\Theta_1 < s_2$ and $({}_\alpha H_\beta)_0 \in \mathbb{C}^{q\times q}_{>}$). Since $(s_j)_{j=0}^1$ belongs to $\mathcal{H}^{\geq,\mathrm{e}}_{q,1}$ (resp., $\mathcal{H}^{>,\mathrm{e}}_{q,1}$), then we see from Propositions 6.7 and 6.8 that $(s_j)_{j=0}^2$ belongs to $\mathcal{H}^{\geq}_{q,2}$ (resp., $\mathcal{H}^{>}_{q,2}$), i.e., the matrix H_1 is non-negative Hermitian (resp., positive Hermitian). Thus, Definition 4.2(b) shows that $(s_j)_{j=0}^2 \in \mathcal{F}^{\geq}_{q,2,\alpha,\beta}$ (resp., $(s_j)_{j=0}^2 \in \mathcal{F}^{>}_{q,2,\alpha,\beta}$).

Now we consider the case that $m = 2n$ is valid where n is some positive integer. According to $\mathfrak{a}_{2n} \leq s_{2n+1} \leq \mathfrak{b}_{2n}$ (resp., $\mathfrak{a}_{2n} < s_{2n+1} < \mathfrak{b}_{2n}$), Definition 10.3 and Notation 4.1, then

$$({}_\alpha\Theta)_n \leq ({}_\alpha s)_{2n}, \quad (\Theta_\beta)_n \leq (s_\beta)_{2n} \quad (\text{resp., } ({}_\alpha\Theta)_n < ({}_\alpha s)_{2n}, \quad (\Theta_\beta)_n < (s_\beta)_{2n}). \tag{11.1}$$

Because of Proposition 9.1(a) and Proposition 7.10, we have

$$\{(({}_\alpha s)_j)_{j=0}^{2n-1}, ((s_\beta)_j)_{j=0}^{2n-1}\} \subseteq \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta} \subseteq \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1}.$$

For $(s_j)_{j=0}^{2n} \in \mathcal{F}^{>}_{q,2n,\alpha,\beta}$, we obtain $(s_j)_{j=0}^{2n-1} \in \mathcal{F}^{>}_{q,2n-1,\alpha,\beta}$ by virtue of Proposition 7.7(b) and $\{({}_\alpha s)_{2n-1}, (s_\beta)_{2n-1}\} \subseteq \mathbb{C}^{q\times q}_{\mathrm{H}}$ according to Lemma 4.16. Because of Definition 4.2(a) and Remark 6.6, thus $\{(({}_\alpha s)_j)_{j=0}^{2n-1}, ((s_\beta)_j)_{j=0}^{2n-1}\} \subseteq \mathcal{H}^{>,\mathrm{e}}_{q,2n-1}$ if $(s_j)_{j=0}^{2n} \in \mathcal{F}^{>}_{q,2n,\alpha,\beta}$. Using (11.1) and Propositions 6.7 and 6.8, we conclude $\{(({}_\alpha s)_j)_{j=0}^{2n}, ((s_\beta)_j)_{j=0}^{2n}\} \subseteq \mathcal{H}^{\geq}_{q,2n}$ (resp., $\{(({}_\alpha s)_j)_{j=0}^{2n}, ((s_\beta)_j)_{j=0}^{2n}\} \subseteq \mathcal{H}^{>}_{q,2n}$). Hence, Definition 4.2(a) implies $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{>}_{q,2n+1,\alpha,\beta}$).

Finally, we consider now the case that there is a positive integer n such that $m = 2n + 1$. From $\mathfrak{a}_{2n+1} \leq s_{2n+2} \leq \mathfrak{b}_{2n+1}$ (resp., $\mathfrak{a}_{2n+1} < s_{2n+2} < \mathfrak{b}_{2n+1}$),

Definition 10.3 and Notation 4.1 we get then

$$\Theta_{n+1} \leq s_{2n+2}, \quad ({}_{\alpha}\Theta_{\beta})_n \leq ({}_{\alpha}s_{\beta})_{2n} \quad (\text{resp., } \Theta_{n+1} < s_{2n+2}, \quad ({}_{\alpha}\Theta_{\beta})_n < ({}_{\alpha}s_{\beta})_{2n}). \tag{11.2}$$

From $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\geq}_{q,2n+1,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{>}_{q,2n+1,\alpha,\beta}$) and Proposition 7.10 we get $(s_j)_{j=0}^{2n+1} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n+1}$ (resp., $(s_j)_{j=0}^{2n+1} \in \mathcal{H}^{>,\mathrm{e}}_{q,2n+1}$). In view of Proposition 9.1(b) and Proposition 7.10, we have $(({}_{\alpha}s_{\beta})_j)_{j=0}^{2n-1} \in \mathcal{F}^{\geq}_{q,2n-1,\alpha,\beta} \subseteq \mathcal{H}^{\geq,\mathrm{e}}_{q,2n-1}$. For $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{>}_{q,2n+1,\alpha,\beta}$, we obtain $(s_j)_{j=0}^{2n} \in \mathcal{F}^{>}_{q,2n,\alpha,\beta}$ by virtue of Proposition 7.7(b) and ${}_{\alpha}s_{\beta})_{2n-1} \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ according to Lemma 4.16 and Notation 4.1. Because of Definition 4.2(b) and Remark 6.6, thus $(({}_{\alpha}s_{\beta})_j)_{j=0}^{2n-1} \in \mathcal{H}^{>,\mathrm{e}}_{q,2n-1}$ if $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{>}_{q,2n+1,\alpha,\beta}$. Taking into account (11.2) and Propositions 6.7 and 6.8, we infer then $(s_j)_{j=0}^{2n+2} \in \mathcal{H}^{\geq}_{q,2n+2}$ and $(({}_{\alpha}s_{\beta})_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n}$ (resp., $(s_j)_{j=0}^{2n+2} \in \mathcal{H}^{>}_{q,2n+2}$ and $(({}_{\alpha}s_{\beta})_j)_{j=0}^{2n} \in \mathcal{H}^{>}_{q,2n}$). Because of Definition 4.2(b), we conclude $(s_j)_{j=0}^{2n+2} \in \mathcal{F}^{\geq}_{q,2n+2,\alpha,\beta}$ (resp., $(s_j)_{j=0}^{2n+2} \in \mathcal{F}^{>}_{q,2n+2,\alpha,\beta}$). □

Theorem 11.2. *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *If $(s_j)_{j=0}^{m} \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$, then $[\mathfrak{a}_m, \mathfrak{b}_m] \neq \emptyset$ and*

$$\left\{s_{m+1} \in \mathbb{C}^{q\times q} \colon (s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq}_{q,m+1,\alpha,\beta}\right\} = [\mathfrak{a}_m, \mathfrak{b}_m].$$

(b) *If $(s_j)_{j=0}^{m} \in \mathcal{F}^{>}_{q,m,\alpha,\beta}$, then $(\mathfrak{a}_m, \mathfrak{b}_m) \neq \emptyset$ and*

$$\left\{s_{m+1} \in \mathbb{C}^{q\times q} \colon (s_j)_{j=0}^{m+1} \in \mathcal{F}^{>}_{q,m+1,\alpha,\beta}\right\} = (\mathfrak{a}_m, \mathfrak{b}_m).$$

Proof. Combine Lemma 11.1 and Proposition 10.16. □

A closer look at the proof of Theorem 11.2 shows that we now immediately obtain purely algebraic proofs of some statements formulated in Section 4, which were proved there using results on moment problems. This concerns Theorem 4.3, Proposition 4.5, Proposition 4.8, and Theorem 4.9.

Theorem 11.3. *Let $\alpha \in \mathbb{R}$, $\beta \in (\alpha, \infty)$, and $m \in \mathbb{N}_0$. Then $\mathcal{F}^{>,\mathrm{e}}_{q,m,\alpha,\beta} = \mathcal{F}^{>}_{q,m,\alpha,\beta}$.*

Proof. Combine Corollary 7.9 and Theorem 11.2(b). □

Proposition 11.4. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{m} \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$, and let $s_{m+1} \in \{\mathfrak{a}_m, \mathfrak{b}_m\}$. Then $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq,\mathrm{cd}}_{q,m+1,\alpha,\beta}$.*

Proof. From Theorem 11.2(a) and Remark 10.2(a) we obtain $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq}_{q,m+1,\alpha,\beta}$. In view of Definition 10.6, we have furthermore $A_{m+1} = 0_{q\times q}$ or $B_{m+1} = 0_{q\times q}$. In particular, $\mathcal{R}(A_{m+1}) \cap \mathcal{R}(B_{m+1}) = \{0_{q\times 1}\}$. Thus, Lemma 10.26 yields $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\geq,\mathrm{cd}}_{q,m+1,\alpha,\beta}$. □

In view of Propositions 10.31, 11.4, and 10.30, the following notions seem to be natural:

Definition 11.5. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Let the sequence $(s_j)_{j=m+1}^{\infty}$ be recursively defined by $s_j := \mathfrak{a}_{j-1}$ (resp., $s_j := \mathfrak{b}_{j-1}$). Then $(s_j)_{j=0}^{\infty}$ is called the *lower* (resp., *upper*) $[\alpha, \beta]$-*Hausdorff completely degenerate sequence associated with* $(s_j)_{j=0}^m$.

Proposition 11.6. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then the lower (resp., upper) $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^m$ belongs to $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ and is $[\alpha, \beta]$-Hausdorff completely degenerate of order $m+1$ as well as $[\alpha, \beta]$-Hausdorff central of order $m+2$.*

Proof. Denote by $(s_j)_{j=0}^{\infty}$ the lower (resp., upper) $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^m$. In view of Definition 4.6 and Remark 10.2(a), we conclude $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ by successive application of Theorem 11.2(a). According to Definition 11.5 and Proposition 11.4, furthermore $(s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^{\geq,\mathrm{cd}}$. In view of Definition 10.25(a), thus $(s_j)_{j=0}^{\infty}$ is $[\alpha, \beta]$-Hausdorff completely degenerate of order $m+1$. By virtue of Remark 10.36, then $(s_j)_{j=0}^{\infty}$ is $[\alpha, \beta]$-Hausdorff central of order $m+2$. □

From Proposition 11.6 and Lemma 10.27 we obtain the following result:

Remark 11.7. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then the lower as well as the upper $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^m$ is Hankel completely degenerate of order $n+1$, where n is the unique number from $\mathbb{N}_0$ with $m = 2n-1$ or $m = 2n$.

Lemma 11.8. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n-1} \in \mathcal{F}_{q,2n-1,\alpha,\beta}^{\geq}$. Then the lower $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n-1}$ is Hankel completely degenerate of order n.*

Proof. Denote by $(s_j)_{j=0}^{\infty}$ the lower $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n-1}$. According to Proposition 11.6, then $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$. In particular, $(s_j)_{j=0}^{\infty} \in \mathcal{H}_{q,\infty}^{\geq}$ by virtue of Lemma 4.15. In view of Definition 11.5, we have furthermore $s_{2n} = \mathfrak{a}_{2n-1}$. Thus, $s_{2n} = \Theta_n$ according to Definition 10.3. From Notation 3.4(c) we see then $L_n = 0_{q \times q}$. □

Definition 11.9. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Let the sequence $(s_j)_{j=m+1}^{\infty}$ be recursively defined by $s_j := \mathfrak{c}_{j-1}$, where $\mathfrak{c}_{j-1}$ is given by Definition 10.11. Then $(s_j)_{j=0}^{\infty}$ is called the $[\alpha, \beta]$-*Hausdorff central sequence associated with* $(s_j)_{j=0}^m$.

Proposition 11.10. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then the $[\alpha, \beta]$-Hausdorff central sequence associated with $(s_j)_{j=0}^m$ is $[\alpha, \beta]$-Hausdorff non-negative definite and $[\alpha, \beta]$-Hausdorff central of order $m+1$.*

Proof. Denote by $(s_j)_{j=0}^{\infty}$ the $[\alpha,\beta]$-Hausdorff central sequence associated with $(s_j)_{j=0}^{m}$. In view of Definition 4.6 and Remark 10.2(a), we conclude $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$ by successive application of Theorem 11.2(a). By virtue of Definitions 11.9 and 10.33, then $(s_j)_{j=0}^{\infty}$ is $[\alpha,\beta]$-Hausdorff central of order $m+1$. □

Proposition 11.11. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{>}$. Then the $[\alpha,\beta]$-Hausdorff central sequence associated with $(s_j)_{j=0}^{m}$ is $[\alpha,\beta]$-Hausdorff positive definite.*

Proof. In view of Definition 4.6 and Remark 10.2(b), we conclude by successive application of Theorem 11.2(b), that the $[\alpha,\beta]$-Hausdorff central sequence associated with $(s_j)_{j=0}^{m}$ belongs to $\mathcal{F}_{q,\infty,\alpha,\beta}^{>}$. □

Proposition 11.12. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{>}$. Then there exists a sequence $(s_j)_{j=m+1}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{>}$.*

Proof. Use Proposition 11.11. □

12. Applications to the moment problem $\mathsf{M}[[\alpha,\beta];\,(s_j)_{j=0}^{m},\,=]$

In this section, we discuss first applications of the preceding investigations on the structure of matricial $[\alpha,\beta]$-Hausdorff non-negative definite sequences. The following statement concretizes Proposition 4.10.

Theorem 12.1. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then $\mathcal{P}[[\alpha,\beta];(s_j)_{j=0}^{m}]$ given by* (4.1) *admits the representation $\mathcal{P}[[\alpha,\beta];(s_j)_{j=0}^{m}] = [\mathfrak{a}_m,\mathfrak{b}_m]$ where $\mathfrak{a}_m$ and $\mathfrak{b}_m$ are given in Definition* 10.3.

Proof. Combine Proposition 4.10 and Theorem 11.2(a). □

Theorem 12.2. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, and let $(s_j)_{j=0}^{\infty}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{\infty},=] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\geq}$. In this case, the set $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{\infty},=]$ contains exactly one element.*

Proof. In view of a matricial version of the Helly-Prohorov Theorem (see, e.g., [18, Satz 9, Bemerkung 2]), it is readily checked that $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{\infty},=] \neq \emptyset$ if and only if $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{m},=] \neq \emptyset$ for all $m \in \mathbb{N}_0$. (The essential idea of this argumentation is originated in [1, proof of Theorem 2.1.1]). In view of Definition 4.6, then the asserted equivalence follows using Theorem 4.3. Since the interval $[\alpha,\beta]$ is bounded, it is a well-known fact that the set $\mathcal{M}_{\geq}^{q}[[\alpha,\beta];(s_j)_{j=0}^{\infty},=]$ contains at most one element. This completes the proof. □

Now we turn our attention to the special molecular solutions of truncated matricial $[\alpha,\beta]$-Hausdorff moment problems, which were constructed in Section 5 (see Theorem 5.4 and Proposition 5.5).

Proposition 12.3. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then:*

(a) *The lower (resp., upper) $[\alpha,\beta]$-Hausdorff completely degenerate sequence $(\underline{s}_j)_{j=0}^{\infty}$ (resp., $(\overline{s}_j)_{j=0}^{\infty}$) associated to $(s_j)_{j=0}^m$ belongs to $\mathcal{F}_{q,\infty,\alpha,\beta}^{\geq,\mathrm{cd}}$.*

(b) *The set $\mathcal{M}_{\geq}^q[[\alpha,\beta];(\underline{s}_j)_{j=0}^{\infty},=]$ contains exactly one element $\underline{\sigma}_m$ and the set $\mathcal{M}_{\geq}^q[[\alpha,\beta];(\overline{s}_j)_{j=0}^{\infty},=]$ contains exactly one element $\overline{\sigma}_m$.*

Proof. (a) Use Proposition 11.6.

(b) In view of (a), this follows from Theorem 12.2. □

Definition 12.4. Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then the non-negative Hermitian $q \times q$ measure $\underline{\sigma}_m$ (resp., $\overline{\sigma}_m$) is called the *lower* (resp., *upper*) *CD-measure associated with* $(s_j)_{j=0}^m$ *and* $[\alpha,\beta]$.

Proposition 12.5. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha,\infty)$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n-1} \in \mathcal{F}_{q,2n-1,\alpha,\beta}^{\geq}$. Then:*

(a) *Denote by $\underline{\sigma}_{2n-1}$ the lower CD-measure associated with $(s_j)_{j=0}^{2n-1}$ and $[\alpha,\beta]$. Then $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}$ and $\underline{\sigma}_{2n-1}$ is the restriction onto $\mathfrak{B}_{[\alpha,\beta]}$ of the CD-measure σ_n associated with $(s_j)_{j=0}^{2n-1}$.*

(b) *Denote by $\overline{\sigma}_{2n-1}$ the upper CD-measure associated with $(s_j)_{j=0}^{2n-1}$ and $[\alpha,\beta]$. Let $s_{2n} := \mathfrak{b}_{2n-1}$ and let $s_{2n+1} := \mathfrak{b}_{2n}$. Then $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ and $\overline{\sigma}_{2n-1}$ is the restriction onto $\mathfrak{B}_{[\alpha,\beta]}$ of the CD-measure σ_{n+1} associated with $(s_j)_{j=0}^{2n+1}$.*

Proof. (a) In view of Lemma 7.5, we have $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}$. Denote by $(s_j)_{j=0}^{\infty}$ the lower $[\alpha,\beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n-1}$. According to Lemma 11.8, then $(s_j)_{j=0}^{\infty}$ is Hankel completely degenerate of order n. By virtue of Theorem 5.3, thus $(s_j)_{j=2n}^{\infty}$ coincides with the unique sequence from Theorem 5.3(a) and therefore $\mathcal{M}_{\geq}^q[\mathbb{R};(s_j)_{j=0}^{\infty},=] = \{\sigma_n\}$. By definition, $\underline{\sigma}_{2n-1}$ belongs to $\mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^{\infty},=]$. Then, $\mu\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}_{\geq}^{q\times q}$ defined by $\mu(B) := \underline{\sigma}_{2n-1}(B \cap [\alpha,\beta])$ belongs to $\mathcal{M}_{\geq}^q[\mathbb{R};(s_j)_{j=0}^{\infty},=]$. Consequently, $\mu = \sigma_n$. Since the restriction of μ onto $\mathfrak{B}_{[\alpha,\beta]}$ is $\underline{\sigma}_{2n-1}$, the proof of part (a) is complete.

(b) Applying Proposition 11.4 twice, we see that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq,\mathrm{cd}}$. Denote by $(s_j)_{j=0}^{\infty}$ the upper $[\alpha,\beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n+1}$. In view of Definition 11.5, the sequence $(s_j)_{j=0}^{\infty}$ coincides with the upper $[\alpha,\beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n-1}$. Hence, $\{\overline{\sigma}_{2n-1}\} = \mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^{\infty},=]$ by definition. In view of $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq,\mathrm{cd}}$, Proposition 10.30 yields $s_j = \mathfrak{a}_{j-1}$ for all $j \in \mathbb{Z}_{2n+2,\infty}$. According to Definition 11.5, then $(s_j)_{j=0}^{\infty}$ is the lower $[\alpha,\beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n+1}$. By definition, then $\{\underline{\sigma}_{2n+1}\} = \mathcal{M}_{\geq}^q[[\alpha,\beta];(s_j)_{j=0}^{\infty},=]$. Hence, $\overline{\sigma}_{2n-1} = \underline{\sigma}_{2n+1}$. The application of part (a) to the sequence $(s_j)_{j=0}^{2n+1}$ completes the proof. □

Proposition 12.6. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n} \in \mathcal{F}_{q,2n,\alpha,\beta}^{\geq}$. Then:*

(a) *Denote by $\underline{\sigma}_{2n}$ the lower CD-measure associated with $(s_j)_{j=0}^{2n}$ and $[\alpha, \beta]$ and let $s_{2n+1} := \mathfrak{a}_{2n}$. Then $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ and $\underline{\sigma}_{2n}$ is the restriction onto $\mathfrak{B}_{[\alpha,\beta]}$ of the CD-measure σ_{n+1} associated with $(s_j)_{j=0}^{2n+1}$.*

(b) *Denote by $\overline{\sigma}_{2n}$ the upper CD-measure associated with $(s_j)_{j=0}^{2n}$ and $[\alpha, \beta]$ and let $s_{2n+1} := \mathfrak{b}_{2n}$. Then $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ and $\overline{\sigma}_{2n}$ is the restriction onto $\mathfrak{B}_{[\alpha,\beta]}$ of the CD-measure σ_{n+1} associated with $(s_j)_{j=0}^{2n+1}$.*

Proof. (a) From Proposition 11.4 we see that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq}$. In view of Definition 11.5, the lower $[\alpha, \beta]$-Hausdorff completely degenerate sequence $(s_j)_{j=0}^{\infty}$ associated with $(s_j)_{j=0}^{2n}$ coincides with the lower $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n+1}$. Hence, $\underline{\sigma}_{2n}$ is the lower CD-measure $\underline{\sigma}_{2n+1}$ associated with $(s_j)_{j=0}^{2n+1}$ and $[\alpha, \beta]$. The application of Proposition 12.5(a) to the sequence $(s_j)_{j=0}^{2n+1}$ completes the proof of (a).

(b) From Proposition 11.4 we see that $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq,\mathrm{cd}}$. Denote by $(s_j)_{j=0}^{\infty}$ the upper $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n+1}$. In view of Definition 11.5, then $(s_j)_{j=0}^{\infty}$ is the upper $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n}$. Hence, $\{\overline{\sigma}_{2n}\} = \mathcal{M}_{\geq}^{q}[[\alpha, \beta]; (s_j)_{j=0}^{\infty}, =]$ by definition. In view of $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\geq,\mathrm{cd}}$, Proposition 10.30 yields $s_j = \mathfrak{a}_{j-1}$ for all $j \in \mathbb{Z}_{2n+2,\infty}$. According to Definition 11.5, then $(s_j)_{j=0}^{\infty}$ is the lower $[\alpha, \beta]$-Hausdorff completely degenerate sequence associated with $(s_j)_{j=0}^{2n+1}$. By definition, then $\{\underline{\sigma}_{2n+1}\} = \mathcal{M}_{\geq}^{q}[[\alpha, \beta]; (s_j)_{j=0}^{\infty}, =]$. Hence, $\overline{\sigma}_{2n} = \underline{\sigma}_{2n+1}$. The application of Proposition 12.5(a) to the sequence $(s_j)_{j=0}^{2n+1}$ completes the proof of part (b). □

Corollary 12.7. *Let $\alpha \in \mathbb{R}$, let $\beta \in (\alpha, \infty)$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. Then $\{\underline{\sigma}_m, \overline{\sigma}_m\} \subseteq \mathcal{M}_{\geq}^{q}[[\alpha, \beta]; (s_j)_{j=0}^{m}, =] \cap \mathcal{M}_{\geq}^{q,\mathrm{mol}}([\alpha, \beta])$*

Proof. Combine Propositions 12.5 and 12.6 with Theorem 5.3(c). □

Now we want to add some comments concerning the case $q = 1$. Let $(s_j)_{j=0}^{m} \in \mathcal{F}_{1,m,\alpha,\beta}^{\geq}$. Then it can be expected that the lower (resp., upper) CD-measure $\underline{\sigma}_m$ (resp., $\overline{\sigma}_m$) associated with $(s_j)_{j=0}^{m}$ and $[\alpha, \beta]$ coincides with the lower (resp., upper) principal solution constructed by M.G. Krein in [26] in [27, Ch. III]. Furthermore, the other molecular solutions obtained in Proposition 5.5 should be canonical solutions in the terminology of M.G. Krein. The case of a sequence $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{>}$ deserves particular attention. In this case, the theory of orthogonal matrix polynomials can be effectively applied and promises to produce a whole collection of useful explicit formulas. This will be confirmed by the strategy used by M.G. Krein [26] (see also [27, Ch. III]). It should be mentioned that first steps in this direction are already contained in the papers [6, 7] by A.E. Choque Rivero, where the resolvent

matrices for the non-degenerate truncated $[\alpha, \beta]$-Hausdorff moment problem were expressed in terms of orthogonal matrix polynomials and moreover multiplicative decompositions of these resolvent matrices were derived. We want to handle this theme in separate work.

Appendix A. Some facts from matrix theory

Remark A.1. Let $n \in \mathbb{N}$ and let $(A_j)_{j=0}^n$ be a sequence of non-negative Hermitian complex $q \times q$ matrices. Then $\mathcal{R}(\sum_{j=1}^n A_j)) = \sum_{j=1}^n \mathcal{R}(A_j)$ and $\mathcal{N}(\sum_{j=1}^n A_j) = \bigcap_{j=1}^n \mathcal{N}(A_j)$.

Remark A.2. If $A \in \mathbb{C}_{\geq}^{q\times q}$ and if $X \in \mathbb{C}^{q\times p}$, then $X^*AX \in \mathbb{C}_{\geq}^{p\times p}$. If $A \in \mathbb{C}_{>}^{q\times q}$ and if $X \in \mathbb{C}^{q\times p}$ with $\operatorname{rank} X = p$, then $X^*AX \in \mathbb{C}_{>}^{p\times p}$.

Denote by $P_{\mathcal{U}}$ the matrix associated with the orthogonal projection in the Euclidean space $\mathbb{C}^q$ onto a linear subspace $\mathcal{U}$.

Remark A.3. Let $\mathcal{U}$ be a linear subspace of $\mathbb{C}^q$, then $P_{\mathcal{U}} \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and $0_{q\times q} \leq P_{\mathcal{U}} \leq I_q$.

For the convenience of the reader, we state some well-known and some special results on Moore–Penrose inverses of matrices (see, e.g., Rao/Mitra [29] or [12, Section 1]). If $A \in \mathbb{C}^{p\times q}$, then (by definition) the Moore–Penrose inverse $A^\dagger$ of A is the unique matrix $A^\dagger \in \mathbb{C}^{q\times p}$ which satisfies the four equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$$

Proposition A.4 (see, e.g., [12, Theorem 1.1.1]). *If $A \in \mathbb{C}^{p\times q}$, then a matrix $X \in \mathbb{C}^{q\times p}$ is the Moore–Penrose inverse of A if and only if $AX = P_{\mathcal{R}(A)}$ and $XA = P_{\mathcal{R}(X)}$.*

Remark A.5. Let $A \in \mathbb{C}^{p\times q}$ and $B \in \mathbb{C}^{p\times r}$. Then $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ if and only if $AA^\dagger B = B$.

Remark A.6. Let $A \in \mathbb{C}^{p\times q}$ and $B \in \mathbb{C}^{n\times q}$. Then $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ if and only if $BA^\dagger A = B$.

Remark A.7. $(A^*)^\dagger = (A^\dagger)^*$ for each $A \in \mathbb{C}^{p\times q}$.

Remark A.8. Let $A \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Then $A^\dagger \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and $A^\dagger A = AA^\dagger$.

Remark A.9. Let $A \in \mathbb{C}^{p\times q}$. If $U \in \mathbb{C}^{n\times p}$ fulfills $U^*U = I_p$ and if $V \in \mathbb{C}^{q\times r}$ is such that $VV^* = I_q$, then $(UAV)^\dagger = V^*A^\dagger U^*$.

Remark A.10. Let $A \in \mathbb{C}_{\geq}^{q\times q}$. Then $A^\dagger \in \mathbb{C}_{\geq}^{q\times q}$ and $\sqrt{A^\dagger} = \sqrt{A}^\dagger$. Furthermore, $\mathcal{R}(\sqrt{A}) = \mathcal{R}(A)$ and $\mathcal{N}(\sqrt{A}) = \mathcal{N}(A)$.

Remark A.11. Let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $0_{q\times q} \leq A \leq B$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.

Lemma A.12 ([2,15]). *Let* $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *be the block representation of a complex* $(p+q) \times (p+q)$ *matrix* M *with* $p \times p$ *block* A*. Then:*

(a) M *is non-negative Hermitian if and only if* A *and* $D - CA^{\dagger}B$ *are both non-negative Hermitian,* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$*, and* $C = B^{*}$*.*

(b) M *is positive Hermitian if and only if* A *and* $D - CA^{\dagger}B$ *are both positive Hermitian and* $C = B^{*}$*.*

Appendix B. Parallel sum of matrices

For every choice of complex $p \times q$ matrices A and B, the *parallel sum* $A \mathbin{\overline{+}} B$ of A and B is defined by

$$A \mathbin{\overline{+}} B := A(A+B)^{\dagger}B. \tag{B.1}$$

Furthermore, let $\mathcal{P}_{p\times q}$ be the set of all pairs $(A,B) \in \mathbb{C}^{p\times q} \times \mathbb{C}^{p\times q}$ such that $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$ and $\mathcal{N}(A+B) \subseteq \mathcal{N}(A)$ hold true.

Let A and B be non-singular matrices from $\mathbb{C}^{q\times q}$ such that $\det(A+B) \neq 0$ and $\det(A^{-1}+B^{-1}) \neq 0$. Then $(A,B) \in \mathcal{P}_{q\times q}$ and $A \mathbin{\overline{+}} B = (A^{-1}+B^{-1})^{-1}$.

Remark B.1. If $A, B \in \mathbb{C}_{>}^{q\times q}$, then $(A,B) \in \mathcal{P}_{q\times q}$ and $A \mathbin{\overline{+}} B \in \mathbb{C}_{>}^{q\times q}$.

Lemma B.2 ([3, Lemma 4]). *If* $A, B \in \mathbb{C}_{\geq}^{q\times q}$*, then* $(A,B) \in \mathcal{P}_{q\times q}$ *and* $A \mathbin{\overline{+}} B \in \mathbb{C}_{\geq}^{q\times q}$*.*

Lemma B.3 ([28, Theorem 2.2(a)]). *If* $(A,B) \in \mathcal{P}_{p\times q}$*, then* $(B,A) \in \mathcal{P}_{p\times q}$ *and* $A \mathbin{\overline{+}} B = B \mathbin{\overline{+}} A$*.*

Lemma B.4 ([28, Theorem 2.2(f)]). *If* $(A,B) \in \mathcal{P}_{p\times q}$*, then* $\mathcal{R}(A \mathbin{\overline{+}} B) = \mathcal{R}(A) \cap \mathcal{R}(B)$ *and* $\mathcal{N}(A \mathbin{\overline{+}} B) = \mathcal{N}(A) + \mathcal{N}(B)$*.*

Proposition B.5 ([28, Theorem 2.2(g)]). *If* $(A,B) \in \mathcal{P}_{q\times q}$*, then* $(A \mathbin{\overline{+}} B)^{\dagger} = P_{\mathcal{R}(A^{*})\cap\mathcal{R}(B^{*})}(A^{\dagger}+B^{\dagger})P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$*.*

Lemma B.6. *If* $(A,B) \in \mathcal{P}_{p\times q}$*, then* $(A+B)-4(A \mathbin{\overline{+}} B) = (A-B)(A+B)^{\dagger}(A-B)$*.*

Proof. Let $(A,B) \in \mathcal{P}_{p\times q}$ and let $C := A+B$. In view of Lemma B.3, we get

$$(A \pm B)C^{\dagger}(A \pm B) = AC^{\dagger}A \pm AC^{\dagger}B \pm BC^{\dagger}A + BC^{\dagger}B = AC^{\dagger}A + BC^{\dagger}B \pm 2(A \mathbin{\overline{+}} B).$$

Hence,

$$(A+B)-(A-B)(A+B)^{\dagger}(A-B) = (A+B)C^{\dagger}(A+B) - (A-B)C^{\dagger}(A-B) = 4(A \mathbin{\overline{+}} B).$$

□

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On Some Extremal Problems for Analytic Functions with Constraints on Real or Imaginary Parts

Juliette Leblond and Dmitry Ponomarev

Abstract. We study some approximation problems by functions in the Hardy space H^2 of the upper half-plane or by their real or imaginary parts, with constraint on their real or imaginary parts on the boundary. Situations where the criterion acts on subsets of the boundary or of horizontal lines inside the half-plane are considered. Existence and uniqueness results are established, together with novel solution formulas and techniques. As a by-product, we derive a regularized inversion scheme for Poisson and conjugate Poisson integral transforms.

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1. Introduction

Bounded extremal problems in Hardy classes of complex valued analytic functions have been widely considered in [4, 11, 12, 17] and the references therein and below. Being given a function on a subset K of the boundary $\partial\Omega$ of a set $\Omega \subset \mathbb{C}$, they consist in best approximating that function on K by a function in the Hardy space of Ω, subject to some norm constraint on the complementary set $\partial\Omega \setminus K$ of the boundary where no data are available. Whenever Ω is the unit disk and $K = \partial\Omega$ is equal to its full boundary (the unit circle), they are similar to classical extremal problems for analytic functions, see [19, Ch. 8], [21, Ch. IV].

Such constrained best approximation problems arise in the context of system theory, for harmonic identification purposes and recovery of transfer functions from partial boundary data, see [13], in Hardy classes of the unit disk (for discrete time

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systems) or of the right-half plane (for continuous time). They also happen to furnish regularized resolution schemes for overdetermined boundary value problems concerning Laplace or elliptic partial differential equations in domains of dimension 2, see [16, 23, 27, 29]. In these frameworks, data are provided by approximate pointwise values (measurements, corrupted by errors) of a function belonging to some Hardy class or of its real or imaginary part, partially available on a subset of the domain or of its boundary. From these data, the aim is to recover the function itself.

Further, properties of Hardy spaces allow to establish stability results of such inverse problems, as in [15], and robustness properties of the approximation schemes, as in [22].

Here, we work in the Hilbertian setting and we consider situations where the domain Ω is the upper half-plane.

One typical example of such a problem is the approximation of a given function on a segment by traces of analytic functions controlling its behavior on the complement of the segment. Posed in a half-plane setting, the formulation of this problem reads as Problem 0 below.

Let $\Pi_+ := \{x + iy \in \mathbb{C} : \quad x \in \mathbb{R},\ y > 0\}$ be the upper half-plane and

$$H^2_+ := H^2(\Pi_+) := \left\{ g \text{ analytic in } \Pi_+ : \quad \sup_{y>0} \int_{\mathbb{R}} |g(x+iy)|^2 \, dx < \infty \right\}$$

be the Hardy space of Π_+ [21, Ch. II] which is similar to $H^2(\mathbb{D})$ (Hardy space for the unit disk) but not equivalent to it.

We will also identify functions $g \in H^2_+$ with their (non-tangential) boundary values $\lim_{y\to 0^+} g(\cdot + iy) \in L^2(\mathbb{R})$.

Given real numbers $-\infty < a_k < b_k < a_{k+1} < b_{k+1} < \infty$, $k = 1, \ldots, N-1$, $N > 1$, let $K := \bigcup_{k=1}^{N} (a_k, b_k) \subset \mathbb{R}$ and $J := \mathbb{R} \backslash K$. For the sake of explicitness of constructive aspects, without loss of generality, we simply assume that $K = (a, b)$.

Problem 0. Given functions $f_0 \in L^2(K)$, $h_0 \in L^2(J)$, and a constant $M_0 > 0$, find g_0 such that

$$g_0 = \arg \min_{g \in H^2_+,\ \|g - h_0\|_{L^2(J)} \leq M_0} \|g - f_0\|_{L^2(K)} .$$

In this paper, we focus on two particular instances of bounded extremal problems involving approximation and constraints on real or imaginary parts, Problems 1 and 2, also mentioning some relevant problems.

Problem 1. Given a function $f_1 \in L^2(K)$, a real-valued function $h_1 \in L^2_{\mathbb{R}}(J)$, and a constant $M_1 \geq 0$, find g_1 such that

$$g_1 = \arg \min_{g \in H^2_+,\ \|\mathrm{Im} g - h_1\|_{L^2(J)} \leq M_1} \|g - f_1\|_{L^2(K)} .$$

In these two problems, both the criterion and the constraint act on the boundary values of H^2_+ functions. This is not the case of Problem 2 below, where the

constraint applies on the boundary values while the criterion applies to the function on some subset of the upper half-plane.

Let $S := \bigcup_{k=1}^{N} (a_k, b_k) \subset \mathbb{R}$, for $-\infty < a_k < b_k < a_{k+1} < b_{k+1} < \infty$, $k = 1, \ldots, N-1$, $N > 1$.

In what follows, we will write "supp" for the support of a function.

Problem 2. Given a real-valued functions $f_2, h_2 \in L^2_{\mathbb{R}}(S)$ and constants $M_2, y_0 > 0$, find $g_2 \in \mathcal{B}_2 \subset H^2_+$ such that

$$g_2 = \arg \min_{g \in \mathcal{B}_2} \|\mathrm{Re}\, g(\cdot + iy_0) - f_2\|_{L^2(S)},$$

where

$$\mathcal{B}_2 := \left\{ g \in H^2_+ : \mathrm{supp} \lim_{y \to 0^+} \mathrm{Re}\, g(\cdot + iy) \subset S, \right.$$
$$\left. \left\| \lim_{y \to 0^+} \mathrm{Re}\, g(\cdot + iy) - h_2 \right\|_{L^2(S)} \le M_2 \right\}$$
$$= \left\{ g \in H^2_+ : \quad \mathrm{supp}\, \mathrm{Re}\, g \subset S,\ \|\mathrm{Re}\, g - h_2\|_{L^2(S)} \le M_2 \right\}.$$

Problem 1 (precisely, its version on a unit disk) was introduced in [23] where it has been qualitatively analyzed, and its solution has been implicitly characterized in terms of Toeplitz and Hankel operators, see also [27]. An approximate procedure based on numerical inversion of finite Toeplitz/Hankel matrices has also been proposed. As it was noted there, this problem reduces to Problem 0 whose solution can be explicitly (up to solving some trancendental equation for the value of a Lagrange parameter) written in an integral form generalizing in some sense a well-known Carleman formula [1, 26, 31, 34] for the case when $h_0 \neq 0$. The obtained implicit characterization of the solution then reduces to an integral equation on J with a symmetric kernel which is smooth in the interior of J and bounded at the endpoints.

Problem 2 arises as a toy (two-dimensional) version of some physical inverse problems. More precisely, in Earth and planetary sciences, for study in paleomagnetism, an important inverse problem is to recover the magnetization contained in thin rock samples from partial measurements of their weak magnetic field, specifically of its normal component, taken on a planar set parallel to and at some distance of the sample, see [10, 30]. This is actually a three-dimensional issue, that involve both the Poisson kernel of the upper half-space and its conjugate, and their truncations to planar subsets (rectangles) of the upper half-space or its boundary.

Even though it has a real part constraint, Problem 2 is of a different type than Problems 0 and 1 as the approximation is performed on a subset of the analyticity domain rather than on a part of its boundary (see also Section 3.4). We show that solution of Problem 2 can be obtained from solution of an auxiliary integral equation with a simple (Poisson) kernel. Particular cases of the latter

have been studied over the years. Recently, asymptotic resolution schemes have been developed for the case when S is a single interval [36], depending on whether the ratio $|S|/y_0$ is large or small. Solutions are obtained as the eigenfunctions of the truncated Poisson operator and furnish a suitable basis for computation of solutions to Problem 2.

Problem 2 and its variation Problem 2″ (introduced in Section 3.4) can also be viewed as regularized inversion settings for Poisson and conjugate Poisson transforms [35, 38] of $L^2(\mathbb{R})$ functions supported on S.

In Section 2, we recall the properties of H^2_+ that we will need to establish existence and uniqueness of solutions to Problems 0, 1, 2 in Section 3. In Section 4, we obtain explicit constructive forms of the solutions. Some concluding comments are given in Section 5.

2. Properties of Hardy spaces

We list below the basic properties of H^2_+ that will be used for the study of Problems 0, 1, 2, see [2, Sec. 8.6], [19, Ch. 11], [21, Ch. II, III], [24, Ch. 8], [25, Ch. VI].

- Functions in H^2_+ possess $L^2(\mathbb{R})$ boundary values that form a closed subspace of $L^2(\mathbb{R})$ and thus also form a Hilbert space, with the inner product
$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(x)\,\overline{g(x)}dx. \tag{2.1}$$
- Functions in H^2_+ are uniquely determined by their boundary values on subsets of $\mathbb{R}$ of positive measure [21, Ch. II, Cor. 4.2], or [24, Ch. 8] (F. and M. Riesz theorem).
- $L^2(\mathbb{R}) = H^2_+ \oplus H^2_-$ (identified with their boundary values) with
$$H^2_- := H^2(\Pi_-) := \left\{ g \text{ analytic in } \Pi_- : \quad \sup_{y<0} \int_{\mathbb{R}} |g(x+iy)|^2\, dx < \infty \right\}$$
being the Hardy space of the lower half-plane
$$\Pi_- := \{x + iy \in \mathbb{C} : \quad x \in \mathbb{R},\ y < 0\}\ ,$$
which can be established using Poisson and Cauchy representation formula [21, Ch. II, Sec. 3], [24, Ch. 8], [25, Ch. VI] (or from Paley–Wiener theorem [37, Ch. 2]).

For $y > 0$, let us define the Hilbert transform, the Poisson and conjugate Poisson transforms $\mathcal{H}$, P_y, Q_y, respectively, as operators $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by, see [21, Ch. I, III]:
$$\mathcal{H}[f](x) := \frac{1}{\pi} \fint_{\mathbb{R}} \frac{f(t)}{x-t} dt,$$
$$P_y[f](x) := \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)\,dt}{(x-t)^2 + y^2}, \quad Q_y[f](x) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{(x-t)\,f(t)}{(x-t)^2 + y^2} dt.$$

The Hilbert transform $\mathcal{H}$ (or the harmonic conjugation operator) is bounded and isometric in $L^2(\mathbb{R})$ (see also Section 3). The projection operators $P_\pm$ from $L^2(\mathbb{R})$ onto $H^2_\pm$ can be written as

$$P_\pm [f](x) = \lim_{y\to 0^+} (P_{\pm y} + iQ_{\pm y})[f](x) = \pm\frac{1}{2} f(x) + \frac{i}{2}\mathcal{H}[f](x)$$
$$= \pm\frac{1}{2} f(x) + \frac{1}{2\pi i} \int\!\!\!\!\!\!- _{\mathbb{R}} \frac{f(t)\,dt}{t-x},$$

which can also be seen as Plemelj–Sokhotskii formulas for L^2 functions ([20, Ch. 1] and [32, Ch. 2]).

The following density and unstability properties of Hardy functions provide motivation for bounded extremal problems [28, Lem. 2.1, Prop. 2.2]. Let $U \subset \mathbb{R}$, such that $|U| > 0$ and $|\mathbb{R} \setminus U| > 0$.

- The restrictions to U of boundary values of H^2_+ functions form a dense subset of $L^2(U)$.
- Let $f \in L^2(U)$ and $g_n \in H^2_+$ such that $\|f - g_n\|_{L^2(U)} \to 0$; then, either f coincides with the restriction to U of some H^2_+ function, or $\|g_n\|_{L^2(\mathbb{R}\setminus U)} \to \infty$ as $n \to \infty$.

Note that similar properties for Hardy spaces $H^p(\mathbb{D})$ of the unit disk were also established and used in [4, 11, 12] in order to study bounded extremal problems similar to Problem 0.

Before passing to the next section, let us adopt a few conventions. We will denote the characteristic function of any subset $U \subset \mathbb{R}$ as χ_U. We will also use concatenation to construct a function from two complementary subsets of the line, namely, we write $f \vee h$ to mean a function which equals f on $U \subset \mathbb{R}$ and h on $V := \mathbb{R} \setminus U$. For simplicity, we will put $\chi_U u = u \vee 0$ on $\mathbb{R}$ even if u is defined on U only.

3. Analysis of the problems

We are going to show that the problems formulated above admit a unique solution which also saturates the constraint under additional assumptions, and mention some variations of these problems.

3.1. Preliminaries

We will need the following results established in [17, Sec. 2] for complex Hilbert spaces which remain also true for real Hilbert spaces as only real-linear arguments were used in the proofs.

Let $\mathcal{X}$, $\mathcal{K}$, $\mathcal{J}$ be Hilbert spaces and $A : \mathcal{X} \to \mathcal{K}$, $B : \mathcal{X} \to \mathcal{J}$ be bounded linear operators such that there exists a constant $\eta > 0$,

$$\|Ag\|^2_{\mathcal{K}} + \|Bg\|^2_{\mathcal{J}} \geq \eta \|g\|^2_{\mathcal{X}}, \quad \forall g \in \mathcal{X}, \tag{3.1}$$

Assume that A and B have dense ranges and, for $f \in \mathcal{K}$, $h \in \mathcal{J}$, $M > 0$, consider the nonempty closed convex set:

$$\mathcal{B} = \mathcal{B}(h, M) := \left\{ g \in \mathcal{X} : \quad \|Bg - h\|_{\mathcal{J}} \leq M \right\}.$$

We will write $A^\star$ and $B^\star$ for the adjoint operators to A and B, respectively.

Lemma 1. [17, Lem. 2.1] *Let $f \in \mathcal{K}$, $h \in \mathcal{J}$, and $M > 0$. There exists a solution $g_\star \in \mathcal{X}$ to the bounded extremal problem:*

$$g_\star = \arg \min_{g \in \mathcal{B}} \|Ag - f\|_{\mathcal{K}},$$

Moreover, if $f \notin A(\mathcal{B})$, the solution $g_\star$ is unique and saturates the constraint: $\|Bg_\star - h\|_{\mathcal{J}} = M$.

Lemma 2. [17, Thm. 2.1] *Let $f \in \mathcal{K} \setminus A(\mathcal{B})$, $h \in \mathcal{J}$, and $M > 0$. The solution to the bounded extremal problem in Lemma* 1 *satisfies the operator equation*

$$(A^\star A + \gamma B^\star B)\, g_\star = A^\star f + \gamma B^\star h, \tag{3.2}$$

where $\gamma > 0$ is the unique constant such that $\|Bg_\star - h\|_{\mathcal{J}} = M$.

Remark 3. Note that recovery situations where $f \in A(\mathcal{B})$ correspond to $A\, g_\star = f$, the preimage $g_\star \in \mathcal{B}$ of f by A being a solution that may not saturate the constraint (in this case, $\gamma = 0$ in equation (3.2)). Uniqueness of $g_\star$ in this case holds if, and only if, $A|_{\mathcal{B}}$ is injective.

We observe that we can view the complex-linear space H^2_+ as a real-linear Hilbert space with the inner product $\langle f, g \rangle_{L^2_{\mathbb{R}}} = \operatorname{Re} \langle f, g \rangle_{L^2}$, where $\langle f, g \rangle_{L^2}$ is as in (2.1) for f, $g \in H^2_+$. Indeed, because harmonic conjugation is realized by the Hilbert transform operator $\mathcal{H}$, the Hardy space H^2_+ can be represented as $(1 + i\mathcal{H})\, L^2(\mathbb{R})$. The latter complex-linear space is isomorphic to the real-linear space $(1 + i\mathcal{H})\, L^2_{\mathbb{R}}(\mathbb{R})$. To see this, take an arbitrary function $F = \operatorname{Re} F + i \operatorname{Im} F \in L^2(\mathbb{R})$, and using the operator identity $\mathcal{H}\mathcal{H} = -1$, we proceed as

$$\begin{aligned}(1 + i\mathcal{H})\, F &= (1 + i\mathcal{H}) \operatorname{Re} F + (i - \mathcal{H}) \operatorname{Im} F = (1 + i\mathcal{H}) \operatorname{Re} F - \mathcal{H}\,(1 + i\mathcal{H}) \operatorname{Im} F \\ &= (1 + i\mathcal{H})\, (\operatorname{Re} F - \mathcal{H}\, [\operatorname{Im} F]).\end{aligned}$$

Now, if we decompose $F = F_+ + F_-$ with $F_\pm \in H^2_\pm$, we note that $\operatorname{Re} F_+ = -\mathcal{H}\, [\operatorname{Im} F_+]$ and $\operatorname{Re} F_- = \mathcal{H}\, [\operatorname{Im} F_-]$, and therefore,

$$(1 + i\mathcal{H})\, F = 2\, (1 + i\mathcal{H}) \operatorname{Re} F_+.$$

We thus established that H^2_+ is isomorphic to the real-linear space $(1 + i\mathcal{H})\, L^2_{\mathbb{R}}(\mathbb{R})$ which is further isomorphic to the real vector-function space $\mathrm{H}^2_+ := (1,\, \mathcal{H})^T\, L^2_{\mathbb{R}}(\mathbb{R}) \subset L^2_{\mathbb{R}}(\mathbb{R}) \times L^2_{\mathbb{R}}(\mathbb{R})$ with the inner product

$$\begin{aligned}(w_1, w_2)_{L^2_{\mathbb{R}}} &= \langle u_1, u_2 \rangle_{L^2} + \langle \mathcal{H}u_1, \mathcal{H}u_2 \rangle_{L^2} = 2 \langle u_1, u_2 \rangle_{L^2}, \\ w_k &= (u_k, \mathcal{H}u_k)^T \in \mathrm{H}^2_+, \quad u_k \in L^2_{\mathbb{R}}(\mathbb{R}), \quad k = 1, 2,\end{aligned}$$

where the isometric property of the Hilbert transform was taken into account. On the other hand, by the same property, we have, for $f, g \in H^2_+$,

$$\langle f, g\rangle_{L^2_{\mathbb{R}}} = \operatorname{Re}\,\langle f, g\rangle_{L^2} = \langle \operatorname{Re} f,\, \operatorname{Re} g\rangle_{L^2} + \langle \operatorname{Im}\, f,\, \operatorname{Im}\, g\rangle_{L^2} = 2\,\langle \operatorname{Re} f,\, \operatorname{Re} g\rangle_{L^2}\,,$$

which, upon identification $u_1 \leftrightarrow \operatorname{Re} f$, $u_2 \leftrightarrow \operatorname{Re} g$, justifies the use of the real inner product.

3.2. Problems 0 and 1: solution existence and uniqueness

We establish existence and uniqueness properties for solutions to Problems 0 and 1, beginning with Problem 0.

Theorem 4. *Problem* 0 *admits a unique solution* g_0. *Moreover, if*

$$f_0 \notin \left\{ g \in H^2_+\,,\ \|g - h_0\|_{L^2(J)} \le M_0 \right\}\Big|_K\,,$$

then it saturates the constraint: $\|g_0 - h_0\|_{L^2(J)} = M_0$.

Proof. Here, we consider H^2_+ as a Hilbert space with the complex inner product (2.1) and define the bounded linear operators $A_0 : H^2_+ \to L^2(K)$, $B_0 : H^2_+ \to L^2(J)$ as restrictions to K and J, respectively, of H^2_+ functions. Clearly, both A_0 and B_0 are bounded linear operators, and because of the density property of restrictions of boundary values of H^2_+ functions (recall Section 2), we see that these operators have dense ranges and that the approximation set is not empty. Also, since K and J are complementary sets, we have, for all $g \in H^2_+$,

$$\|A_0 g\|^2_{L^2(K)} + \|B_0 g\|^2_{L^2(J)} = \|g\|^2_{L^2(\mathbb{R})}\,,$$

and hence (3.1) holds true with $\eta = 1$. Therefore, Lemma 1 implies existence of the solution which, by Lemma 2, is unique and saturates the constraint if $f_0 \notin \left\{ g \in H^2_+\,,\ \|g - h_0\|_{L^2(J)} \le M_0 \right\}\Big|_K$.

Finally, if $f_0 \in \left\{ g \in H^2_+\,,\ \|g - h_0\|_{L^2(J)} \le M_0 \right\}\Big|_K$, Remark 3 is to the effect that uniqueness still holds. Indeed, A_0 is injective on H^2_+, because H^2_+ functions cannot vanish on the set K of positive Lebesgue measure unless they identically vanish. □

For the above proof, observe that we could also proceed as in [37] using [28] and the orthogonal decomposition $L^2(\mathbb{R}) = H^2_+ \oplus H^2_-$.

We now turn to Problem 1.

Theorem 5. *Problem* 1 *admits a unique solution* g_1. *Moreover, if*

$$f_1 \notin \left\{ g \in H^2_+\,,\ \|Im\, g - h_1\|_{L^2(J)} \le M_1 \right\}\Big|_K\,,$$

then it saturates the constraint: $\|Im\, g_1 - h_1\|_{L^2(J)} = M_1$.

Proof. Considering the equivalent real Hilbertian structure for H^2_+ discussed in Section 3.1, we can prove Theorem 5 using Lemmas 1 and 2. Let the operator $A_1 = A_0$ be as in the proof of Theorem 4 and B_1: $H^2_+ \to L^2_{\mathbb{R}}(J)$ be the imaginary part of the restriction to J of H^2_+ functions, which, clearly, is also a bounded linear operator. As for Problem 0, A_1 has dense range property, so is the case for B_1 (in fact, B_1 is surjective: for any $h \in L^2_{\mathbb{R}}(J)$, we can construct the function $(-\mathcal{H}[\chi_J h] + i(\chi_J h)) \in H^2_+$). Further, the approximation set is not empty and property (3.1) holds for all $g \in H^2_+$ with $B = B_1$. Indeed, for $g \in H^2_+$, because $\mathrm{Im}\, g$ is the conjugate function of $\mathrm{Re}\, g$ and Hilbert transform is an isometry on $L^2(\mathbb{R})$, we get

$$\begin{aligned}\|A_1 g\|^2_{L^2(K)} + \|B_1 g\|^2_{L^2(J)} &= \|g\|^2_{L^2(K)} + \|\mathrm{Im}\, g\|^2_{L^2(J)} \\ &= \|\mathrm{Re}\, g\|^2_{L^2(K)} + \|\mathrm{Im}\, g\|^2_{L^2(\mathbb{R})} \geq \frac{1}{2}\|g\|^2_{L^2(\mathbb{R})},\end{aligned}$$

whence (3.1) holds true with $\eta = \frac{1}{2}$. If $f_1 \notin \left\{ g \in H^2_+,\ \|\mathrm{Im}\, g - h_1\|_{L^2(J)} \leq M_1 \right\}\Big|_K$, then Lemmas 1, 2 again directly lead to the results.

If $f_1 \in \left\{ g \in H^2_+,\ \|\mathrm{Im}\, g - h_1\|_{L^2(J)} \leq M_1 \right\}\Big|_K$, we argue as at the end of the proof of Theorem 4 using Remark 3. □

For proving Theorems 4, 5, other possibilities consist in using best approximation projections on closed convex subsets of a Hilbert space as in [23], or weak-compactness arguments as in [11].

3.3. Problem 2: solution existence and uniqueness

We now consider Problem 2.

Theorem 6. *Problem* 2 *admits a unique solution* g_2*, i.e., there exists a unique function* $g_2 \in \mathcal{B}_2 \subset H^2_+$ *such that*

$$g_2 = \arg \min_{g \in \mathcal{B}_2} \|Re\, g(\cdot + iy_0) - f_2\|_{L^2(S)}.$$

Moreover, if $f_2 \notin Re\, \mathcal{B}_2|_{S \times \{y_0\}}$*, then* g_2 *saturates the constraint:* $\|Re\, g_2 - h_2\|_{L^2(S)} = M_2$.

Proof. As in the proof of Theorem 5, we view H^2_+ as a real-linear Hilbert space with the inner product $\langle f, g \rangle_{L^2_{\mathbb{R}}} = \mathrm{Re}\, \langle f, g \rangle_{L^2}$ for $f, g \in H^2_+$. Consider the subspace

$$\mathcal{T} := \left\{ g \in H^2_+ : \quad \mathrm{supp}\, \mathrm{Re}\, g \subset S \right\} \subset H^2_+,$$

which is closed, and hence is a Hilbert space on its own account.

Define the bounded linear operators $A_2 : \mathcal{T} \to L^2_{\mathbb{R}}(S)$, $B_2 : \mathcal{T} \to L^2_{\mathbb{R}}(S)$ as

$$\begin{aligned} A_2 g(\cdot) &= \mathrm{Re}\, g(\cdot + iy_0) = P_{y_0}[\mathrm{Re}\, g](\cdot) \qquad \text{and} \\ B_2 g(\cdot) &= \lim_{y \to 0^+} \mathrm{Re}\, g(\cdot + iy) = \mathrm{Re}\, g(\cdot) \qquad \text{on } S. \end{aligned}$$

Note that B_2 is surjective, as discussed in the proof of Theorem 5. To verify that A_2 has a dense range, it is enough to show that it is the case for the same map

when its domain of definition is $L^2_{\mathbb{R}}(S)$. The latter is an operator still given by P_{y_0} which is now a self-adjoint operator (as a convolution operator with an even kernel) on $L^2_{\mathbb{R}}(S)$. Hence, to show the density of its range, it is enough to check its injectivity. The injectivity can be deduced from the non-vanishing property of harmonic functions on non-empty open subsets of the domain of harmonicity [6, Thms 1.27, 1.28] (note that $\operatorname{Re}\mathcal{B}_2$ consists of functions harmonic in Π_+). Combined with the density of $\operatorname{Re}\mathcal{T}|_S$ in $L^2_{\mathbb{R}}(S)$ (given $f \in L^2_{\mathbb{R}}(S)$, we construct $(f\chi_S + i\mathcal{H}[f\chi_S]) \in \mathcal{T}$), this leads to the dense range property of A_2. The fact that (3.1) is satisfied by A_2 and B_2 can easily be checked, for $g \in \mathcal{T}$,

$$\|\operatorname{Re} g(\cdot + iy_0)\|^2_{L^2(S)} + \|\operatorname{Re} g\|^2_{L^2(S)} \geq \|\operatorname{Re} g\|^2_{L^2(S)} = \|\operatorname{Re} g\|^2_{L^2(\mathbb{R})} = \frac{1}{2}\|g\|^2_{L^2(\mathbb{R})},$$

as in the proof of Theorem 5. Therefore, Lemmas 1 and 2 again furnish the proof whenever $f_2 \notin \operatorname{Re}\mathcal{B}_2|_{S\times\{y_0\}}$.

Finally, if $f_2 \in \operatorname{Re}\mathcal{B}_2|_{S\times\{y_0\}}$, thanks again to Remark 3, g_2 is unique for A_2 is injective (as discussed in the previous paragraph). □

Observe that in the present two-dimensional case, Theorems 4, 5 still hold whenever K is a more general set of finite positive Lebesgue measure [4, 11, 23]. This should also be the case for Theorem 6 if S is of finite positive Lebesgue measure, see [7].

3.4. More about Problems 1 and 2

Note that Problem 1 is equivalent to a similar one, Problem 1′, with the constraint acting on $\operatorname{Re} g$ (multiplying by $\pm i$ the function f_1 and the approximant g): given a function $f_1 \in L^2(K)$, a real-valued function $h_1 \in L^2_{\mathbb{R}}(J)$, and a constant $M_1 \geq 0$, find g_1' such that

$$g_1' = \arg \min_{g\in H^2_+,\ \|\operatorname{Re} g - h_1\|_{L^2(J)} \leq M_1} \|g - f_1\|_{L^2(K)}.$$

In Problem 1, situations when $M_1 = 0$ could be analyzed as they correspond to the constraint $\operatorname{Im} g = h_1$ a.e. in K, or $\operatorname{Re} g = h_1$ if we discuss its Problem 1′ version. In this case, solutions $g_1' \in H^2_+$ are such that on $\mathbb{R}$, $g_1' = (1 + i\mathcal{H})[\phi_1 \vee h_1]$ for some $\phi_1 \in L^2_{\mathbb{R}}(K)$, i.e., we look for ϕ_1 such that

$$\phi_1 = \arg \min_{\phi\in L^2_{\mathbb{R}}(K)} \|f_1 - \phi - i\mathcal{H}[\phi \vee h_1]\|_{L^2(K)}.$$

The above Problem 1′ is related to the following Problem 3 (see also [27, Prob. 2.2]). Given real-valued functions $f_3 \in L^2_{\mathbb{R}}(K)$, $h_3 \in L^2_{\mathbb{R}}(J)$, and a constant $M_3 \geq 0$, find g_3 such that

$$g_3 = \arg \min_{g\in H^2_+,\ \|\operatorname{Re} g - h_3\|_{L^2(J)} \leq M_3} \|\operatorname{Re} g - f_3\|_{L^2(K)}.$$

It is interesting to compare Problems 2 and 3, and to point out the links and differences between their criteria and constraints. Indeed, with $S = J$, $h_2 = h_3$, and $M_2 = M_3$, the approximation set $\mathcal{B}_2$ in Problem 2 is a strict subset of the one in Problem 3, because of the additional constraint on the support of the real

part of the approximant in Problem 2. A notable difference between them is that in Problem 2, the criterion acts on the subset $S \times \{y_0\}$ of Π_+ while in Problem 3 it acts on the part $K = \mathbb{R} \setminus J$ of the boundary $\mathbb{R}$.

Concerning Problem 2, note that $\mathcal{T} = (P_y + iQ_y)[\chi_S u]$ for $u \in L^2_{\mathbb{R}}(S)$ (whence $\mathcal{T}|_{\mathbb{R}} = (1 + i\mathcal{H})[\chi_S u]$) and $\mathcal{B}_2 = (P_y + iQ_y)[\chi_S u]$ for $u \in L^2_{\mathbb{R}}(S)$ such that $\|u - h_2\|_{L^2(S)} \leq M_2$. Then, Problem 2 rephrases as the one of finding u_2 solution to

$$u_2 = \arg \min_{u \in L^2_{\mathbb{R}}(S),\ \|u-h_2\|_{L^2(S)} \leq M_2} \|P_{y_0}[\chi_S u] - f_2\|_{L^2(S)},$$

where only the restriction to S of $P_{y_0}[\chi_S u]$ is involved in the above criterion. In particular, for $M_2 = 0$, this furnishes the unique solution $g_2 = (P_y + iQ_y)[\chi_S h_2]$ to Problem 2 (which necessarily saturates the constraint and is equal to $(1+i\mathcal{H})[\chi_S h_2]$ on $\mathbb{R}$). For $M_2 > 0$, we can see that $f_2 \in \mathrm{Re}\ \mathcal{B}_2|_{S\times\{y_0\}}$ if and only if $f_2 = P_{y_0}[\chi_S u]$ on S, for some $u \in L^2_{\mathbb{R}}(S)$ such that $\|u - h_2\|_{L^2(S)} \leq M_2$. This reformulation shows that Problem 2 is, in fact, nothing but a regularization scheme for inversion of Poisson transform [35] on $L^2(S)$.

This also furnishes another equivalent setting for establishing well-posedness of Problem 2, yet appropriate for Lemmas 1, 2. One can also make use of [14, Thm 5.2] in order to show that there exists $\gamma > 0$ such that

$$u_2 = \arg \min_{u \in L^2_{\mathbb{R}}(S)} \left(\|P_{y_0}[\chi_S u] - f_2\|^2_{L^2(S)} + \gamma \|u - h_2\|^2_{L^2(S)} \right).$$

The saturation of the constraint then implies by differentiation that

$$\langle P_{y_0}[\chi_S u_2] - f_2,\, P_{y_0}[\chi_S \varphi] \rangle_{L^2(S)} + \gamma \langle u_2 - h_2,\, \varphi \rangle_{L^2(S)} = 0,\ \forall \varphi \in L^2(S),$$

for γ such that $\|u_2 - h_2\|_{L^2(S)} = M_2$.

We also note that Problem 2 is equivalent to the following Problem 2′ (multiplying by $\pm i$ the functions f_2, h_2 and the approximant g): given real-valued functions f_2, $h_2 \in L^2_{\mathbb{R}}(S)$, find g_2' such that

$$g_2' = \arg \min_{g \in \mathcal{B}_2'} \|\mathrm{Im}\ g(\cdot + iy_0) - f_2\|_{L^2(S)},$$

where

$$\mathcal{B}_2' := \left\{ g \in H^2_+ :\ \ \mathrm{supp}\,\mathrm{Im}\ g \subset S,\ \|\mathrm{Im}\ g - h_2\|_{L^2(S)} \leq M_2 \right\}.$$

Another interesting variation of Problem 2 involves both real and imaginary parts. Namely, we introduce Problem 2″: given real-valued functions f_2, $h_2 \in L^2_{\mathbb{R}}(S)$, and the set $\mathcal{B}_2$ as in Problem 2, find g_2'' such that

$$g_2'' = \arg \min_{g \in \mathcal{B}_2} \|\mathrm{Im}\, g(\cdot + iy_0) - f_2\|_{L^2(S)}.$$

The only difference in the treatment of this problem with respect to Problem 2 is the replacement of A_2 by the bounded linear operator $A_2'' : \mathcal{T} \to L^2_{\mathbb{R}}(S)$ acting as

$$A_2'' g(\cdot) = \mathrm{Im}\, g(\cdot + iy_0) = Q_{y_0}[\mathrm{Re}\, g](\cdot)\ \text{ on } S.$$

In order to deduce existence and uniqueness of the solution, we only have to verify the range density property (the proof that (3.1) is satisfied is exactly the same as for Problem 2). Since $Q_{y_0\chi_S} = \mathcal{H}P_{y_0\chi_S}$, it follows directly from the facts that the range of $P_{y_0\chi_S}$ is dense in $L^2_{\mathbb{R}}(S)$ and that $\mathcal{H}$ is surjective.

4. Computation of solutions

4.1. Problem 0

A version of this problem for the unit disk has been considered in [4]. In [11], its solution has been obtained in an explicit integral form in terms of the so-called quenching function arising from the Goluzin–Krylov method (a Carleman formula) for the recovery of analytic functions from their partial boundary values [1, 26, 31, 34]. We obtain an analogous integral formula using reduction to a Riemann-Hilbert problem [20, 32], which further allows us to treat Problem 1.

In order to make use of Lemma 2 for the characterization of the solution, we compute the adjoint operators $A_0^\star$: $L^2(K) \to H^2_+$, $B_0^\star : L^2(J) \to H^2_+$. Letting $g \in H^2_+$, $\phi \in L^2(K)$, we have

$$\langle A_0 g, \phi\rangle_{L^2(K)} = \langle A_0 g, \chi_K \phi\rangle_{L^2} = \langle g, P_+\chi_K\phi\rangle_{L^2},$$

yielding $A_0^\star = P_+\chi_K$, and similarly, $B_0^\star = P_+\chi_J$.

Equation (3.2) then leads to the critical point equation that must be satisfied by the solution g_0 to Problem 0:

$$P_+\left[\left(1 + (\gamma_0 - 1)\,\chi_J\right) g_0\right] = P_+\left(f_0 \vee \gamma_0 h_0\right), \tag{4.1}$$

where the parameter $\gamma_0 > 0$ is chosen such that $\|g_0 - h_0\|_{L^2(K)} = M_0$.

Equation (4.1) can be rewritten as

$$\left(1 + (\gamma_0 - 1)\,\chi_J\right) g_0 = \left(f_0 \vee \gamma_0 h_0\right) + \psi$$

with some unknown function $\psi \in H^2_-$. Equivalently,

$$\gamma_0 g_0 = G\psi + L, \tag{4.2}$$

where $G \in L^\infty(\mathbb{R})$ and $L \in L^2(\mathbb{R})$ are defined by:

$$G(x) := \frac{\gamma_0}{1 + (\gamma_0 - 1)\,\chi_J(x)} = \begin{cases} 1, & x \in J, \\ \gamma_0, & x \in K, \end{cases}$$

$$L(x) := G(x)\left(f_0 \vee \gamma_0 h_0\right)(x) = \gamma_0\left(f_0 \vee h_0\right)(x).$$

Since $\gamma_0 g_0 \in H^2_+$ and $\psi \in H^2_-$, equation (4.2) is a Riemann–Hilbert problem.

The standard resolution procedure (see, for instance, [20, Ch. 2] or [32, Ch. 5]) requires factorizing the coefficient of the problem G as $G = G_+/G_-$ with zero-free functions G_+ and G_- analytic in the upper and lower half-planes, respectively. To construct such a factorization, we consider the decomposition $\log G = \log G_+ - \log G_-$ which can be achieved by Plemelj–Sokhotskii formulas applied to the Cauchy integral of $\log G$. Due to its vanishing at infinity and to

the positivity of G, this integral is well defined, except at the endpoints $x = a, b$ of K where it is merely bounded. Indeed,

$$\log G_{\pm}(x) = \lim_{y\to 0^{\pm}} \int_{\mathbb{R}} \frac{\log G(t)}{t-(x+iy)} dt = \pm\frac{1}{2}\log\gamma_0 \chi_K(x) + \frac{\log\gamma_0}{2\pi i} \fint_K \frac{dt}{t-x}$$
$$= \begin{cases} \dfrac{1}{2\pi i}\log\gamma_0 \log\left(\dfrac{x-a}{x-b}\right), & x\in J, \\ \pm\dfrac{1}{2}\log\gamma_0 + \dfrac{1}{2\pi i}\log\gamma_0 \log\left(\dfrac{a-x}{x-b}\right), & x\in K, \end{cases}$$

where the branches of logarithms in both expressions are real-valued on J and K, respectively. Therefore,

$$G_{\pm}(x) = e^{\log G_{\pm}(x)} = \begin{cases} \left(\dfrac{x-a}{x-b}\right)^{\frac{1}{2\pi i}\log\gamma_0}, & x\in J, \\ \pm\sqrt{\gamma_0}\left(\dfrac{a-x}{x-b}\right)^{\frac{1}{2\pi i}\log\gamma_0}, & x\in K, \end{cases} \tag{4.3}$$

are the boundary values on $\mathbb{R}$, from above and below, of the Cauchy integral

$$G(z) = e^{\log G(z)} = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log G(t)\, dt}{t-z}\right].$$

Equation (4.2) now becomes

$$\gamma_0 g_0/G_+ - \psi/G_- = L/G_+,$$

which is a jump problem since in the left-hand side we have the difference of H^2_+ and H^2_- functions and the right-hand side is in $L^2(\mathbb{R})$.

To solve this jump problem, we consider the Cauchy integral

$$Y(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{L(t)}{G_+(t)}\frac{dt}{t-z}$$

and, by Plemelj–Sokhotskii formulas,

$$Y_{\pm}(x) = \pm\frac{1}{2}\frac{L(x)}{G_+(x)} + \frac{1}{2\pi i}\fint_{\mathbb{R}} \frac{L(t)}{G_+(t)}\frac{dt}{t-x},$$

we have, for $x\in\mathbb{R}$, $x\neq a, b$,

$$L(x)/G_+(x) = Y_+(x) - Y_-(x).$$

This leads to

$$\gamma_0 g_0/G_+ - Y_+ = \psi/G_- - Y_-,$$

from which we deduce that both left- and right-hand sides are restrictions to $\mathbb{R}$ of one entire function. Since both g_0/G_+ and Y_+ decay at infinity in the upper half-plane, by Liouville theorem, we conclude that this entire function is identically zero. This entails that, for $x\in\mathbb{R}$,

$$g_0(x) = \frac{1}{\gamma_0} G_+(x) Y_+(x) = \frac{1}{2\gamma_0} L(x) + \frac{G_+(x)}{2\gamma_0\pi i}\fint_{\mathbb{R}} \frac{L(t)}{G_+(t)}\frac{dt}{t-x}, \tag{4.4}$$

and, for $z \in \Pi_+$,

$$g_0(z) = \frac{1}{\gamma_0} G(z) Y(z) = \frac{G(z)}{2\gamma_0 \pi i} \int_{\mathbb{R}} \frac{L(t)}{G_+(t)} \frac{dt}{t-z}. \tag{4.5}$$

4.2. Problem 1

According to [23, Prop. 5 (i)], the solution g_1 to Problem 1 also solves Problem 0 with $f_0 = f_1$, $h_0 = ih_1 + \operatorname{Re} g_1$ and $\gamma_0 = \gamma_1 > 0$ chosen such that $\|\operatorname{Im} g_1 - h_1\|_{L^2(J)} = M_1$.

Applying the above formulas (4.4)-(4.5), we arrive at

$$g_1(x) = \frac{1}{2}\operatorname{Re} g_1(x) + \frac{G_+(x)}{2\pi i} \fint_J \frac{\operatorname{Re} g_1(t)\, dt}{G_+(t)(t-x)} + F_0(x), \quad x \in J, \tag{4.6}$$

$$g_1(x) = \frac{1}{2} f_1(x) + \frac{G_+(x)}{2\pi i} \left[\fint_K \frac{f_1(t)\, dt}{G_+(t)(t-x)} + \int_J \frac{(\operatorname{Re} g_1(t) + ih_1(t))\, dt}{G_+(t)(t-x)} \right], \quad x \in K, \tag{4.7}$$

$$g_1(z) = \frac{G_+(z)}{2\pi i} \left[\int_K \frac{f_1(t)\, dt}{G_+(t)(t-z)} + \int_J \frac{(\operatorname{Re} g_1(t) + ih_1(t))\, dt}{G_+(t)(t-z)} \right], \quad z \in \Pi_+, \tag{4.8}$$

where

$$F_0(x) := \frac{i}{2} h_1(x) + \frac{G_+(x)}{2\pi} \fint_J \frac{h_1(t)\, dt}{G_+(t)(t-x)} + \frac{G_+(x)}{2\pi i} \int_K \frac{f_1(t)\, dt}{G_+(t)(t-x)}, \quad x \in J.$$

Observe that the solution expressions (4.7)-(4.8) will be complete once the involved values of $\operatorname{Re} g_1$ on J are known. The latter can, in principle, be found from the solution of the integral equation obtained from (4.6) by taking real parts and using the fact that $\overline{G}_+ = 1/G_+$

$$\operatorname{Re} g_1(x) = \frac{1}{2\pi i} \int_J \frac{\operatorname{Re} g_1(t)}{t-x} \left(\frac{G_+(x)}{G_+(t)} - \frac{G_+(t)}{G_+(x)} \right) dt + 2 \operatorname{Re} F_0(x), \quad x \in J.$$

Note that the integral equation for $\operatorname{Re} g_1$ has a regular symmetric kernel

$$\mathcal{K}(x,t) := \frac{1}{x-t} \left(\frac{G_+(x)}{G_+(t)} - \frac{G_+(t)}{G_+(x)} \right) = \frac{G_+(x) - G_+(t)}{x-t} \left(\frac{G_+(x) + G_+(t)}{G_+(x) G_+(t)} \right),$$

whose smoothness is seen from (4.3). Indeed, G_+ is non-vanishing and smooth (except at the endpoints where it still remains bounded), which eliminates the zero of the denominator at $t = x$ in the first factor. This makes the obtained integral equation amenable to effective numerical resolution.

4.3. Problem 2

Let us compute the adjoint operators $A_2^\star : L^2_{\mathbb{R}}(S) \to \mathcal{T}$, $B_2^\star : L^2_{\mathbb{R}}(S) \to \mathcal{T}$.

For $g \in \mathcal{T}$, $\phi \in L^2_{\mathbb{R}}(S)$, using that P_{y_0} is a self-adjoint operator (as a convolution integral operator with even kernel), we have:

$$\begin{aligned}\langle A_2 g, \phi\rangle_{L^2(S)} &= \mathrm{Re}\,\langle P_{y_0} g, \chi_S \phi\rangle_{L^2} = \mathrm{Re}\,\langle g,\, P_{y_0}\chi_S\phi\rangle_{L^2} = \langle \mathrm{Re}\, g,\, P_{y_0}\chi_S\phi\rangle_{L^2(S)} \\ &= \mathrm{Re}\,\langle g, \chi_S P_{y_0}\chi_S\phi\rangle_{L^2} = \langle g,\, P_+\chi_S P_{y_0}\chi_S\phi\rangle_{L^2_{\mathbb{R}}}, \\ \langle B_2 g, \phi\rangle_{L^2(S)} &= \mathrm{Re}\,\langle g, \chi_S\phi\rangle_{L^2} = \langle g,\, P_+\chi_S\phi\rangle_{L^2_{\mathbb{R}}},\end{aligned}$$

and hence $A_2^\star = P_+\chi_S P_{y_0}\chi_S$, $B_2^\star = P_+\chi_S$.

The characterization (3.2) of the solution g_2 now reads

$$P_+\chi_S P_{y_0}\chi_S P_{y_0}\mathrm{Re}\, g_2 + \gamma_2 P_+\chi_S \mathrm{Re}\, g_2 = P_+\left[\chi_S P_{y_0}\chi_S f_2 + \gamma_2\chi_S h_2\right],$$

where the parameter $\gamma_2 > 0$ is chosen such that $\|\mathrm{Re}\, g_2 - h_2\|_{L^2(S)} = M_2$.

This implies

$$\chi_S P_{y_0}\chi_S P_{y_0}\mathrm{Re}\, g_2 + \gamma_2\chi_S \mathrm{Re}\, g_2 = \chi_S P_{y_0}\chi_S f_2 + \gamma_2\chi_S h_2 + \psi,$$

with some function $\psi \in H^2_-$. However, since all the terms in the equation vanish outside of the set S and so must do ψ, but since $|\mathbb{R}\backslash S| > 0$, this vanishing is prohibited by analyticity unless $\psi \equiv 0$ which is what we have to conclude. We thus arrive at the double integral equation on S

$$\left[\left(P_{y_0}\chi_S\right)^2 + \gamma_2\right]\mathrm{Re}\, g_2 = P_{y_0}\chi_S f_2 + \gamma_2 h_2, \tag{4.9}$$

which can be solved, for example, in terms of eigenfunction expansion for the compact self-adjoint operator $f \mapsto \left[P_{y_0}\,\chi_S\, f\right]|_S$ on $L^2_{\mathbb{R}}(S)$ considered in [36, Part II]. Indeed, this operator is compact (due to regularity of the kernel and boundedness of S) and self-adjoint, hence the spectral theorem for linear compact self-adjoint operators [33, Thm 6.11.1] implies the existence of a set of mutually orthogonal basis functions $(\phi_n)_{n=0}^\infty$ in $L^2_{\mathbb{R}}(S)$ and a set of numbers $(\lambda_n)_{n=0}^\infty \subset \mathbb{R}$ such that

$$P_{y_0}\left[\chi_S\phi_n\right](x) = \frac{y_0}{\pi}\int_S \frac{\phi_n(t)}{(x-t)^2 + y_0^2}dt = \lambda_n\phi_n(x), \quad x \in S. \tag{4.10}$$

In [36, Part II], equation (4.10) was solved asymptotically when S is a finite interval.

Expanding both the solution $\mathrm{Re}\, g_2$ and the right-hand side of (4.9) over eigenfunctions ϕ_n and assuming, without loss of generality, that their $L^2(S)$ norms are normalized to one, we conclude

$$\mathrm{Re}\, g_2(x) = \sum_{n=0}^\infty \frac{1}{\lambda_n^2 + \gamma_2}\langle P_{y_0}\chi_S f_2 + \gamma_2 h_2,\, \phi_n\rangle_{L^2(S)}\,\phi_n(x). \tag{4.11}$$

This allows us to determine the parameter γ_2 from $\|\mathrm{Re}\, g_2 - h_2\|_{L^2(S)} = M_2$, that can now be rewritten as (upon expansion of h_2 over ϕ_n):

$$\sum_{n=0}^\infty\left[\left\langle \frac{1}{\lambda_n^2 + \gamma_2}\left(P_{y_0}\chi_S f_2 + \gamma_2 h_2\right) - h_2,\, \phi_n\right\rangle_{L^2(S)}\right]^2 = M_2^2. \tag{4.12}$$

Then, for $z = x + iy \in \Pi_+$, the solution to Problem 2 is given by

$$g_2(x+iy) = \frac{1}{2}\left(P_y + iQ_y\right)\left[\chi_S \mathrm{Re}\, g_2\right](x). \tag{4.13}$$

4.4. Problem $2''$

The resolution procedure can be repeated in the same way as for Problem 2 in Section 4.3; however, we now have $(A_2'')^\star = -P_+\chi_S Q_{y_0}\chi_S$. Therefore, equations (4.9)-(4.13) get replaced, respectively, by

$$\left[\left(Q_{y_0}\chi_S\right)^2 - \gamma_2''\right]\mathrm{Re}\, g_2'' = Q_{y_0}\chi_S f_2 - \gamma_2'' h_2,$$

$$Q_{y_0}\left[\chi_S\varphi_n\right](x) = \frac{1}{\pi}\int_S \frac{(x-t)\,\varphi_n(t)}{(x-t)^2 + y_0^2}dt = \lambda_n\varphi_n(x), \quad x \in S,$$

$$\mathrm{Re}\, g_2''(x) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n^2 - \gamma_2''}\left\langle Q_{y_0}\chi_S f_2 - \gamma_2'' h_2,\, \varphi_n\right\rangle_{L^2(S)}\varphi_n(x),$$

$$\sum_{n=0}^{\infty}\left[\left\langle \frac{1}{\lambda_n^2 - \gamma_2''}\left(P_{y_0}\chi_S f_2 - \gamma_2'' h_2\right) - h_2,\, \varphi_n\right\rangle_{L^2(S)}\right]^2 = M_2^2,$$

$$g_2''(x+iy) = \frac{1}{2}\left(P_y + iQ_y\right)\left[\chi_S \mathrm{Re}\, g_2''\right](x), \quad x+iy \in \Pi_+,$$

the logic remains precisely the same except that the existence of complete set of eigenfunctions φ_n and eigenvalues λ_n of $Q_{y_0}\,\chi_S$ is still guaranteed by the spectral theorem for compact normal operators (note that $f \mapsto \left[Q_{y_0}\,\chi_S\, f\right]|_S$ is a compact anti-self-adjoint operator on $L^2_{\mathbb{R}}(S)$) [33, Thm 6.11.1].

5. Conclusion

Observe that in the particular situations where the function to be approximated already belongs to the approximant class, the above bounded extremal problems 0 and 1 coincide, in the absence of constraint, with recovery or extrapolation issues as in the works [1, 26, 34], and with a constraint, to Loewner-type interpolation issues as in [3]. It would be interesting to study similar interpolation / extrapolation issues related to Problem 2, for which the behaviour of the error (criterion) with respect to the constraint must be further analyzed as well. This should go together with numerical computation of the solutions.

Generalizations of Problem 2 to situations involving also the conjugate Poisson kernel are under study. This is also the case of some extensions of Problem 2 in the three-dimensional case, see [8] and [36, Part III]. Note that three-dimensional versions of Problem 0 have been considered in Hardy classes of harmonic gradients in spherical domains [5], while Hardy classes of the upper half-space are studied and used in [10].

Finally, observe that Problem 2 still makes sense in Hardy classes of generalized analytic (or pseudo-holomorphic) functions, as does Problem 0, see [9]. This deserves further study.

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A Burns–Krantz-type Theorem for Pseudo-contractive Mappings

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Abstract. In this note we prove a rigidity result for pseudo-contractive mappings on the unit disk of the complex plane in the spirit of the Burns–Krantz theorem. As an auxiliary assertion, for pseudo-contractive mappings fixing the origin and 1, we establish an inequality connecting the first derivatives at these points.

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1. Preliminaries

Let Δ be the open unit disk in the complex plane $\mathbb{C}$. By $\mathrm{Hol}(\Delta, \mathbb{C})$ we denote the family of all holomorphic functions on Δ. For the special case when $F \in \mathrm{Hol}(\Delta, \Delta)$ is a self-mapping of Δ we will simply write $F \in \mathrm{Hol}(\Delta)$.

The famous rigidity theorem of D. M. Burns and S. G. Krantz [4], which can be considered as a boundary version of the second part of the classical Schwarz Lemma, asserts:

Theorem 1. *Let $F \in \mathrm{Hol}(\Delta)$ be such that*

$$F(z) = 1 + (z-1) + O\left((z-1)^4\right),$$

as $z \to 1$. Then $F(z) \equiv z$ on Δ.

It was also mentioned in [4] that the exponent 4 is sharp, and it follows from the proof of the theorem that $O\left((z-1)^4\right)$ can be replaced by $o\left((z-1)^3\right)$. There are many generalized versions of this result in different settings for one-dimensional, finite-dimensional and infinite-dimensional situations (see, for example, [1, 2, 3, 6, 7, 8, 12], and references therein).

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Similar results appeared earlier in the literature of conformal mappings (see, for instance, [14]) with the additional hypothesis that F be univalent (and often the function is assumed to be quite smooth – even analytic – in a neighborhood of 1). The theorem presented in [4] has no such hypothesis. The exponent 4 is sharp: simple geometric arguments show that the function

$$F(z) = z - \frac{1}{10}(z-1)^3$$

satisfies the conditions of the theorem with 4 replaced by 3.

One says that a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ has *angular limit* L at a point $\tau \in \partial\Delta$ and write $L := \angle \lim_{z\to\tau} f(z)$ if $f(z) \to L$ as $z \to \tau$ in each nontangential approach region

$$\Gamma(\tau, k) = \left\{ z \in \Delta : \frac{|z-\tau|}{1-|z|} < k \right\}, \quad k > 1.$$

If L is finite and the angular limit (finite or infinite)

$$M := \angle \lim_{z\to\tau} \frac{f(z) - L}{z - \tau}$$

exists, then M is said to be the *angular derivative* of f at τ. We denote it by $f'(\tau)$.

It is known (see [11], p. 79) that this angular derivative exists finitely if and only if the angular limit $\angle \lim_{z\to\tau} f'(z)$ exists finitely, hence $f'(\tau) = \angle \lim_{z\to\tau} f'(z)$.

The Burns–Krantz theorem was improved in 1995 by Thomas L. Kriete and Barbara D. MacCluer [9], who replaced F with its real part and considered the radial limit in $o((z-1)^3)$ instead of the unrestricted limit. Here is a more precise statement of their result.

Theorem 2. *Let $F \in \mathrm{Hol}(\Delta)$ with radial limit $F(1) = 1$ and angular derivative $F'(1) = 1$. If*

$$\liminf_{r\to 1^-} \frac{\mathrm{Re}(F(r) - r)}{(1-r)^3} = 0,$$

then $F(z) \equiv z$ on Δ.

In [13], Tauraso and Vlacci investigated how rigid is the set of holomorphic self-mappings of the unit disk Δ after imposing some conditions on the boundary Schwarzian derivative of F defined by

$$\mathcal{S}_F(z) := \frac{F'''(z)}{F'(z)} - \frac{3}{2}\left(\frac{F''(z)}{F'(z)}\right)^2, \quad z \in \partial\Delta.$$

It is known that the Schwarzian derivative carries global information about F: it vanishes identically if and only if F is a Möbius transformation.

Initially, the original rigidity result of Burns and Krantz was extended in [13] from the identity to the parabolic automorphisms.

Theorem 3. *Let $F \in \mathrm{Hol}(\Delta) \bigcap C^3_A(1)$. If*

$$F(1) = 1, \quad F'(1) = 1, \quad \mathrm{Re}\, F''(1) = 0 \quad \textit{and} \quad \mathrm{Re}\, \mathcal{S}_F(1) = 0,$$

then F is the parabolic automorphism of Δ defined by

$$\frac{1+F(z)}{1-F(z)} = \frac{1+z}{1-z} + ib,$$

where $b = \operatorname{Im} F''(1)$.

In the particular case $F''(1) = F'''(1) = 0$, this reduces to the result of Burns and Krantz, i.e., $F(z) \equiv z$ on Δ.

In [5] (2010), Contreras, Díaz-Madrigal and Pommerenke supplemented Theorem 3 as follows.

Theorem 4. (1) *A non-trivial (i.e., $F \neq I$) holomorphic map $F \in \operatorname{Hol}(\Delta)$ is a parabolic automorphism if and only if there exists $\zeta \in \partial\Delta$ such that $F \in C_A^3(\zeta)$ and*

$$F(\zeta) = \zeta, \quad F'(\zeta) = 1, \quad \operatorname{Re}(\zeta F''(\zeta)) = 0 \quad \text{and} \quad \mathcal{S}_F(\zeta) = 0.$$

(2) *$F \in \operatorname{Hol}(\Delta)$ is a hyperbolic automorphism if and only if there exist $\zeta \in \partial\Delta$ and $\alpha \in (0,1)$ such that $F \in C_A^3(\zeta)$ and*

$$F(\zeta) = \zeta, \quad F'(\zeta) = \alpha, \quad \operatorname{Re}(\zeta F''(\zeta)) = \alpha(\alpha - 1) \quad \text{and} \quad \mathcal{S}_F(\zeta) = 0.$$

The following boundary rigidity principles are given in [12]. In particular, some conditions on behavior of a holomorphic self-mapping F of Δ in a neighborhood of a boundary regular fixed point (not necessarily the Denjoy–Wolff point) under which F is a linear-fractional transformation have been established.

It is known that if a mapping $F \in \operatorname{Hol}(\Delta)$ with the boundary regular fixed point $\tau = 1$ and $F'(1) =: \alpha$ is linear fractional, then for all $k > 0$,

$$F\left(D(1,k)\right) = D\left(1, \frac{\alpha k}{1+\alpha k a}\right),$$

where $D(1,k) := \left\{ z \in \Delta : \frac{|1-z|^2}{1-|z|^2} < k \right\}$ and $a = \frac{1}{\alpha^2} \operatorname{Re}\left(F''(1) + \alpha(1-\alpha)\right)$. In other words, for all $z \in \Delta$,

$$\frac{|1-F(z)|^2}{1-|F(z)|^2} = \frac{\alpha\,|1-z|^2}{(1-|z|^2) + \alpha a\,|1-z|^2}\,. \tag{1}$$

Moreover, F is an automorphism of Δ (either hyperbolic, $\alpha \neq 1$, or parabolic, $\alpha = 1$) if and only if $a = 0$.

For a boundary point $\zeta \in \partial\Delta$, we denote by $\mathcal{C}_A^n(\zeta)$, $n \in \mathbb{N}$, the class of functions $f \in \operatorname{Hol}(\Delta, \mathbb{C})$, which admit representation

$$f(z) = \sum_{k=0}^{n} a_k (z-\zeta)^k + \gamma(z),$$

where $\gamma \in \operatorname{Hol}(\Delta, \mathbb{C})$ and $\angle \lim\limits_{z\to\zeta} \frac{\gamma(z)}{(z-\zeta)^n} = 0$.

It turns out that, under some smoothness conditions, equality (1) (and even some weaker condition) is also sufficient for $F \in \operatorname{Hol}(\Delta)$ to be linear fractional. Namely:

Theorem 5. *Let $F \in \mathrm{Hol}(\Delta) \cap C^3_A(1)$, $F(1) = 1$ and $F'(1) = \alpha$. Then F is a linear fractional transformation if and only if the following conditions hold.*

(i) $F(\Delta) \subseteq D\left(1, \dfrac{1}{a}\right)$, *i.e.,* $\dfrac{|1 - F(z)|^2}{1 - |F(z)|^2} \leq \dfrac{1}{a}, \quad z \in \Delta;$

(ii) *the Schwarzian derivative* $\mathcal{S}_F(1) = 0$.

So, if conditions (i) and (ii) are satisfied, then equality (1) *holds for all* $z \in \Delta$.

Corollary 6. *Let $F \in \mathrm{Hol}(\Delta) \cap C^3_A(1)$ with $F(1) = 1$ and $\alpha = F'(1) \in (0, 1]$. The following are equivalent.*

(i) $F''(1) = F'''(1) = 0$ *and*

$$F(\Delta) \subseteq D\left(1, \frac{\alpha}{1 - \alpha}\right),$$

or, equivalently,

$$F(D(1, k)) \subseteq D\left(1, \frac{k\alpha}{1 + k(1 - \alpha)}\right), \quad k > 0.$$

(ii) F *is an affine mapping of the form* $F(z) = \alpha z + 1 - \alpha$.

In this note we intend to generalize the Burns–Krantz Theorem in two directions. The first one is to establish some rigidity property for those functions the third derivative of which is not necessarily zero. In other words, we will assume that

$$\angle \lim_{z \to \tau} \frac{F(z) - z}{(z - 1)^2} = 0, \text{ but } \angle \lim_{z \to \tau} \frac{F(z) - z}{(z - 1)^3} = k.$$

It turns, out that number k is always nonnegative number and the value $F(0)$ lies always in the disk of radius k centred in k. Moreover, $F(0)$ lies on the circle-the boundary of this disk if and only if F has a special form which immediately becomes the identity mapping whenever $k = 0$.

The second direction we will be interested in is to extend the above-mentioned result for those functions which are not necessarily self mappings of Δ, but satisfy the so-called property to be pseudo-contractive on the open unit disk. Despite the latter class is much wider than the class of self mappings of Δ, it preserves many properties of its fixed points as well as the rigidity property in the spirit of the Burns–Krantz theorem.

To explain our approach we just make the following observation.

If F belongs to $\mathrm{Hol}(\Delta)$, then for each $t \in [0, 1)$ and each fixed $w \in \Delta$, function $G \in \mathrm{Hol}(\Delta)$ defined by

$$G(z) = tF(z) + (1 - t)w \tag{2}$$

has a unique fixed point $z := J_t(w)$. Note that for each $t \in [0, 1)$, the function J_t belongs to $\mathrm{Hol}(\Delta)$ as a function of $w \in \Delta$. However this is not always true if F does not map Δ into itself. This brings us to the following notion.

Definition 7. A function $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ is said to be *pseudo-contractive* if for each $t \in [0,1)$ and each fixed $w \in \Delta$, the equation

$$z = tF(z) + (1-t)w \tag{3}$$

has a unique solution $z := J_t(w)$ holomorphically depending on $w \in \Delta$.

As we have mentioned above, the class of pseudo-contractive functions is much wider then the class of self-mappings of Δ. For example, the function $F(z) = az$, $a < 1$, is pseudo-contractive, but for $a < -1$, it does not map Δ into itself. Also, it can be seen by solving the quadratic equation, that the holomorphic function F given by $F(z) = z - z^2 + 1$ is pseudo-contractive, but it is not a self-mapping of Δ.

The main result of this paper is the following generalization of the Burns–Krantz theorem.

Theorem 8. *Let $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ be pseudo-contractive on Δ with*

$$F(1) = F'(1) = 1, \ \angle \lim_{z\to 1} \frac{F(z)-z}{(z-1)^2} = 0, \ \text{and} \angle \lim_{z\to 1} \frac{F(z)-z}{(z-1)^3} = \mu.$$

Then μ is a real nonpositive number with

$$|F(0) + \mu| \le |\mu|.$$

Moreover, if $F(0) = -2\mu$, then

$$F(z) = z + 2\mu \frac{(z-1)^3}{1+z}$$

In particular, $\mu = 0$ if and only if $F(z) \equiv z$.

To prove our theorem we need some auxiliary assertions which might be of interest by themselves. Also on the way to do this we establish another rigidity theorem which seems to be new even for holomorphic self mappings.

Lemma 9. *A function $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ is pseudo-contractive if and only if there is a real number b such that F satisfies the inequality*

$$\mathrm{Re}\,(F(z)\overline{z}) \le |z|^2 + b\left(1 - |z|^2\right), \ z \in \Delta. \tag{4}$$

Proof. Let $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ be pseudo-contractive. Then for the family $\{J_t\}_{t\ge 0}$ defined by equation (3):

$$J_t(w) = tF(J_t(w)) + (1-t)w, \quad w \in \Delta, \tag{5}$$

we have by the continuity and the maximum modulus principle the following properties:

$$\lim_{t\to 0^+} J_t(w) = w, \quad w \in \Delta, \tag{6}$$

and

$$\begin{aligned}\lim_{t\to 0^+}\frac{1}{t}\left(J_t(w)-w\right)&=\lim_{t\to 0^+}\frac{1}{t}\left(tF\left(J_t(w)\right)+(1-t)w-w\right)\\&=\lim_{t\to 0^+}\left(F\left(J_t(w)\right)-w\right)=F(w)-w,\quad w\in\Delta.\end{aligned}\tag{7}$$

Since $\{J_t\}_{t\geq 0}\subset \mathrm{Hol}(\Delta)$ one can apply the Schwarz–Pick Lemma as follows:

$$|J_t(w)-J_t(0)|^2\leq |w|^2\left|1-J_t(w)\,\overline{J_t(0)}\right|^2,$$

or

$$\begin{aligned}&|J_t(w)|^2-2\,\mathrm{Re}\left(J_t(w)\,\overline{J_t(0)}\right)+|J_t(0)|^2\\&\leq |w|^2\left(1-2\,\mathrm{Re}\left(J_t(w)\,\overline{J_t(0)}\right)+|J_t(w)|^2\,|J_t(0)|^2\right).\end{aligned}$$

In turn, the latter inequality can be written as

$$|J_t(w)|^2-|w|^2\leq 2\,\mathrm{Re}\,J_t(w)\,\overline{J_t(0)}\left(1-|w|^2\right)+|J_t(0)|^2\left(|J_t(w)|^2|w|^2-1\right).\tag{8}$$

In addition, by using resolvent equation (5) and equalities (6) and (7) we have

$$\begin{aligned}&\lim_{t\to 0^+}\frac{1}{t}\left(|J_t(w)|^2-|w|^2\right)\\&=\lim_{t\to 0^+}\frac{1}{t}\left(t^2|F(J_t(w))|^2+2t(1-t)\,\mathrm{Re}\left(F(J_t(w))\,\overline{w}\right)+(1-t)^2|w|^2-|w|^2\right)\\&=2\,\mathrm{Re}\left(F(w)\,\overline{w}\right)+|w|^2\lim_{t\to 0^+}\frac{1}{t}\left((1-t)^2-1\right)=2\left[\mathrm{Re}\left(F(w)\,\overline{w}\right)-|w|^2\right].\end{aligned}\tag{9}$$

In particular, for $w=0$,

$$\lim_{t\to 0^+}\frac{1}{t}|J_t(0)|^2=0.$$

So we get by (8) and (9) that

$$\mathrm{Re}\left(F(w)\,\overline{w}\right)\leq |w|^2+\mathrm{Re}\left(F(0)\,\overline{w}\right)\left(1-|w|^2\right).$$

Setting $b=|F(0)|$ we have (4) as needed.

Conversely, let us assume that F satisfies condition (4). For $\delta\in(0,1)$ and fixed $w\in\Delta$ we define the family $\{G_t\}_{t\in[0,1-\delta]}\subset \mathrm{Hol}(\Delta,\mathbb{C})$ by

$$G_t(z)=z-tF(z)-(1-t)w.$$

Then by (4) we obtain

$$\begin{aligned}\mathrm{Re}(G_t(z)\,\overline{z})&=|z|^2-t\,\mathrm{Re}(F(z)\,\overline{z})-(1-t)\,\mathrm{Re}(w\overline{z})\\&\geq |z|^2-t\left(|z|^2+b\left(1-|z|^2\right)\right)-(1-t)\,|w|\,|z|.\end{aligned}$$

Let us take now $r\in(|w|,1)$ and consider z on the circle $|z|=r$. Then the latter inequality can be written as

$$\mathrm{Re}(G_t(z)\,\overline{z})\geq(1-t)\,r\,(r-|w|)-tb\left(1-r^2\right)=:\varphi(r).$$

Since for $t \in [0, 1-\delta]$, $\varphi(1) \geq \delta(1-|w|) > 0$ it follows that for $\varepsilon > 0$ small enough,

$$\operatorname{Re} G_t(z)\,\overline{z} \geq 0, \quad |z| = r,$$

whenever $r \in (1-\varepsilon, 1)$.

Then it follows by the argument principle that for each $t \in [0, 1-\delta]$, the function G_t has the same number of null points in the disk $|z| < r$ as the identity map. Since $\delta \in (0,1)$ is arbitrary, G_t has a unique null point in Δ for each $t \in [0,1)$, and so equation (3) has a unique solution $z := J_t(w) \in \Delta$. In addition, it follows by the logarithmic residue theorem that this solution is holomorphic in $w \in \Delta$, and we are done. □

Remark 10. If F has a continuous extension on $\partial\Delta$, then condition (4) is equivalent to the following boundary condition

$$\operatorname{Re}(F(z)\,\overline{z}) \leq 1,\ z \in \partial\Delta. \tag{10}$$

Indeed, (4) obviously implies (10). The converse implication is a consequence of the following considerations. Writing (10) in the form

$$\operatorname{Re}[(F(z) - F(0))\,\overline{z}] \leq |z|^2 (1 - \operatorname{Re}(F(0)\,\overline{z})), \quad z \in \partial\Delta.$$

we have

$$\operatorname{Re}\frac{(F(z) - F(0))}{z} \leq 1 - F(0)\,\overline{z},\ z \in \partial\Delta$$

Now it follows by the maximum principle for harmonic functions that this inequality holds also for all $z \in \Delta$, $z \neq 0$. Multiplying it back by $|z|^2$ we get (4) with $b = |F(0)|$.

Now let $\Phi \in \operatorname{Hol}(\Delta, \mathbb{C})$ satisfy the conditions

$$\Phi(0) = 0 \text{ with } \Phi'(0) = \varkappa \tag{11}$$

and

$$\Phi(1) = 1 \text{ with } \Phi'(1) = \eta. \tag{12}$$

If Φ is pseudo-contractive, then by the proof of Lemma 9,

$$\operatorname{Re}(\Phi(z)\,\overline{z}) \leq |z|^2 + |\Phi(0)|\left(1 - |z|^2\right) = |z|^2,\ z \in \Delta.$$

Setting $\Phi(z) = zq(z)$, we have by (11) that $q \in \operatorname{Hol}(\Delta, \mathbb{C})$ and satisfies

$$\operatorname{Re} q(z) \leq 1,\ z \in \Delta.$$

Again by the maximum principle for harmonic functions, we get that either $q(z) \equiv 1$ in Δ or

$$\operatorname{Re} q(z) < 1, \quad z \in \Delta, \tag{13}$$

and then we have by (11) that $q(0) = \varkappa$ with $\operatorname{Re}\varkappa < 1$ and by (12), $q(1) = 1$ with

$$q'(1) = \lim_{r\to 1^-} \frac{q(r)-1}{r-1} = \lim_{r\to 1^-} \frac{\frac{1}{r}\Phi(r)-1}{r-1}$$
$$= \lim_{r\to 1^-} \frac{\Phi(r)-r}{(r-1)r} = \lim_{r\to 1^-} \frac{\Phi(r)-1}{r-1} - 1 = \eta - 1.$$

Observe also that (13) implies that $q'(1)$ is a real positive number, so we have that $\eta > 1 > \operatorname{Re}\varkappa$ unless $q(z) = 1$. On the other hand, we will see below that $\operatorname{Re}\varkappa \geq 3 - 2\eta$ and equality holds if and only if Φ is of the form

$$\Phi(z) = z\frac{(1+a)z + 1 - a}{1+z}, \tag{14}$$

where $a = 2(\eta - 1)$.

To see this, consider the function $\Psi \in \operatorname{Hol}(\Delta)$ defined as follows,

$$\Psi(z) = \frac{\alpha - 1 + q(z)}{\overline{\alpha} + 1 - q(z)} = \frac{q(z) - \varkappa}{2 - \overline{\varkappa} - q(z)}, \tag{15}$$

where $\alpha = 1 - q(0) = 1 - \varkappa$. Since $\operatorname{Re} q(z) < 1$, $z \in \Delta$, we have that $|\Psi(z)| < 1$. In addition, $\Psi(0) = 0$ and $\Psi(1) = \frac{\alpha}{\overline{\alpha}} =: \omega \in \partial\Delta$ by (12). Also,

$$\Psi'(1) = \angle\lim_{z\to 1} \frac{\Psi(z) - \frac{\alpha}{\overline{\alpha}}}{z-1} = \angle\lim_{z\to 1} \frac{\frac{\alpha - 1 + q(z)}{\overline{\alpha}+1-q(z)} - \frac{\alpha}{\overline{\alpha}}}{z-1}$$
$$= 2q'(1)\frac{\operatorname{Re}\alpha}{\overline{\alpha}^2} = 2(\eta-1)\frac{\operatorname{Re}\alpha}{\overline{\alpha}^2}.$$

Using now Osserman's inequality [10] and the Schwarz Lemma we write

$$|\Psi'(1)| = 2(\eta-1)\frac{\operatorname{Re}\alpha}{\overline{\alpha}^2} \geq \frac{2}{1+|\Psi'(0)|} \geq 1$$

or

$$2(\eta-1)\operatorname{Re}\alpha \geq |\alpha|^2$$

which is equivalent to

$$|\alpha - (\eta-1)| \leq \eta - 1.$$

Reminding that $\alpha = 1 - q(0) = 1 - \varkappa$ we have

$$|2 - \varkappa - \eta| \leq \eta - 1.$$

This, in turn, implies $\operatorname{Re}\varkappa \geq 3 - 2\eta$. If now $|2 - \varkappa - \eta| = \eta - 1$, then $|\Psi'(1)| = |\Psi'(0)| = 1$, hence again by the Schwarz Lemma $\Psi(z) = \lambda z$ where $|\lambda| = 1$. Since $\Psi(1) = \frac{\alpha}{\overline{\alpha}}$, we have that $\lambda = \frac{\alpha}{\overline{\alpha}} = \frac{1-\varkappa}{1-\overline{\varkappa}}$ and $\Psi(z) = \frac{1-\varkappa}{1-\overline{\varkappa}}z$. Comparing this with (15) one gets

$$q(z) = 1 - |1-\varkappa|^2\frac{1-z}{1-\overline{\varkappa}+(1-\varkappa)z}$$

with

$$\Phi(z) = zq(z).$$

In particular, if $\operatorname{Re}\varkappa = 3 - 2\eta$, then $\varkappa$ is real and we obtain (14) as desired. So we get the following rigidity result.

Theorem 11. *Let $\Phi \in \mathrm{Hol}(\Delta, \mathbb{C})$ be pseudo-contractive on Δ, normalized by the conditions $\Phi(0) = 0$ with $\Phi'(0) = \varkappa$ and $\Phi(1) = 1$ with $\Phi'(1) = \eta$. Then $|2 - \varkappa - \eta| \leq \eta - 1$. Moreover, the equality holds if and only if Φ is of the form*

$$\Phi(z) = z\left(1 - |1-\varkappa|^2 \frac{1-z}{1-\overline{\varkappa} + (1-\varkappa)z}\right).$$

In particular, if $\operatorname{Re}\varkappa = 3 - 2\eta$, then $\varkappa$ is real and

$$\Phi(z) = z\frac{(2-\varkappa)z + \varkappa}{1+z}.$$

Lemma 12. *Let $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ be pseudo-contractive on Δ with*

$$F(1) = F'(1) = 1, \quad \angle\lim_{z\to 1} \frac{F(z) - z}{(z-1)^2} = 0, \quad \text{and} \quad \angle\lim_{z\to 1} \frac{F(z) - z}{(z-1)^3} = \mu.$$

Then the mapping $\Phi \in \mathrm{Hol}(\Delta, \mathbb{C})$, defined by

$$\Phi(z) = z + K(z)(z - F(z)),$$

where $K(z) = \frac{z}{(1-z)^2}$ is the Koebe function, is also pseudo-contractive on Δ and satisfies the conditions of Theorem 11 with

$$\Phi'(0) =: \varkappa = 1 - F(0) \text{ and } \Phi'(1) =: \eta = 1 - \mu. \tag{16}$$

Proof. The function f defined by

$$f(z) = \frac{\Phi(z) - z}{K(z)} = \frac{\Phi(z) - z}{z}(1-z)^2$$

is holomorphic in Δ. On the other hand, $f(z) = z - F(z)$ and by (4),

$$\operatorname{Re} f(z)\,\overline{z} \geq -b\left(1 - |z|^2\right). \tag{17}$$

Denote $h(z) := \frac{\Phi(z)-z}{z}$, $h_s(z) = h(sz)$, $s \in (0,1)$ and $f_s(z) = (1-z)^2 h_s(z)$. It is clear, that for each $z \in \Delta$, $|h_s(z) - h(z)| \to 0$ and $|f_s(z) - f(z)| \to 0$ as $s \to 1^-$. However, the functions h_s and f_s have continuous extensions onto $\partial\Delta$ the boundary of Δ. Therefore, by using (17) we get for $z \in \partial\Delta$:

$$0 \leq \operatorname{Re}(f_s(z)\,\overline{z}) = \operatorname{Re}\left[(1-z)^2 h_s(z)\,\overline{z}\right] = 2(\operatorname{Re} z - 1)\operatorname{Re} h_s(z).$$

Thus, $\sup_{z\in\Delta} \operatorname{Re} h(z) \leq 0$. But according to our notations, $\Phi(z) = z(1 + h(z))$ which implies

$$\operatorname{Re}(\Phi(z)\,\overline{z}) \leq |z|^2.$$

This proves our first claim: Φ is pseudo-contractive on Δ. Initial conditions (16) can be checked by simple direct calculations. □

Now Theorem 8 is a direct consequence of the above lemma and Theorem 11.

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Spectral Convergence of Large Block-Hankel Gaussian Random Matrices

Philippe Loubaton and Xavier Mestre

Dedicated to Prof. Daniel Alpay on occasion of his 60th birthday

Abstract. This paper studies the behaviour of the empirical eigenvalue distribution of large random matrices $\mathbf{W}_N\mathbf{W}_N^H$ where $\mathbf{W}_N$ is a $ML \times N$ matrix, whose M block lines of dimensions $L \times N$ are mutually independent Hankel matrices constructed from complex Gaussian correlated stationary random sequences. In the asymptotic regime where $M \to +\infty$, $N \to +\infty$ and $\frac{ML}{N} \to c > 0$, it is shown using the Stieltjes transform approach that the empirical eigenvalue distribution of $\mathbf{W}_N\mathbf{W}_N^H$ has a deterministic behaviour which is characterized.

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Keywords. Large random matrices, Stieltjes transform of positive matrix-valued measures, Hankel matrices.

1. Introduction

1.1. The addressed problem and summary of the main results

In this paper, we consider a $ML \times N$ block-Hankel matrix $\mathbf{W}_N$ composed of M Hankel matrices gathered on top of each other, namely,

$$\mathbf{W}_N = \left[\begin{array}{ccc} \mathbf{W}_{1,N}^T & \cdots & \mathbf{W}_{M,N}^T \end{array} \right]^T.$$

For each $m = 1, \ldots, M$, $\mathbf{W}_{m,N}$ is a Hankel matrix of dimensions $L \times N$, with (i,j)th entry equal to

$$\{\mathbf{W}_{m,N}\}_{i,j} = w_{m,N}(i+j-1)$$

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for $1 \leq i \leq L$, $1 \leq j \leq N$, where the $(w_{m,N}(n))_{m=1,\ldots,M,n=1,\ldots,N+L-1}$ are zero mean complex Gaussian random variables . The different blocks $\mathbf{W}_{m,N}$ are independent, but we allow for some time invariant correlation structure within each Hankel matrix, namely,

$$\mathbb{E}\left[w_{m,N}(k)w^*_{m',N}(k')\right] = \frac{r_m\left(k-k'\right)}{N}\delta_{m-m'}$$

where $r_m\left(k\right)$, $k \in \mathbb{Z}$, is a sequence of correlation coefficients defined as

$$r_m\left(k\right) = \int_0^1 \mathcal{S}_m\left(\nu\right) \mathrm{e}^{2\mathrm{i}\pi\nu k} d\nu$$

where $(\mathcal{S}_m)_{m=1,\ldots,M}$ are positive functions. We denote by $(\hat{\lambda}_{k,N})_{k=1,\ldots,ML}$ the eigenvalues of random matrix $\mathbf{W}_N\mathbf{W}_N^H$, where $(\cdot)^H$ stands for transpose conjugate. The purpose of this paper is to study the asymptotic properties of the empirical eigenvalue distribution $d\hat{\mu}_N(\lambda) = \frac{1}{ML}\sum_{k=1}^{ML}\delta_{\lambda-\hat{\lambda}_{k,N}}$ of $\mathbf{W}_N\mathbf{W}_N^H$ when $M \to +\infty$, $N \to +\infty$ and L is such that $c_N = \frac{ML}{N}$ converges towards a nonzero constant $c > 0$.

It is well established that the asymptotic behaviour of the empirical eigenvalue distribution of large Hermitian matrices can be evaluated by studying the behaviour of their Stieltjes transforms. In the context of the present paper, the Stieltjes transform of $d\hat{\mu}_N(\lambda)$ is the function $q_N(z)$ defined on $\mathbb{C}\setminus\mathbb{R}^+$ by

$$q_N(z) = \int_{\mathbb{R}^+} \frac{1}{\lambda - z}\, d\hat{\mu}_N(\lambda) = \frac{1}{ML}\sum_{k=1}^{ML}\frac{1}{\hat{\lambda}_{k,N}-z}$$

and also coincides with $q_N(z) = \frac{1}{ML}\mathrm{tr}\mathbf{Q}_N(z)$ where $\mathbf{Q}_N(z)$ is the resolvent of matrix $\mathbf{W}_N\mathbf{W}_N^H$ defined by

$$\mathbf{Q}_N(z) = \left(\mathbf{W}_N\mathbf{W}_N^H - z\mathbf{I}\right)^{-1}.$$

We denote by $\mathcal{S}_{ML}(\mathbb{R}^+)$ the set of all $ML \times ML$ matrix-valued functions defined on $\mathbb{C}\setminus\mathbb{R}^+$ by

$$\mathcal{S}_{ML}(\mathbb{R}^+) = \left\{ \int_{\mathbb{R}^+} \frac{1}{\lambda - z}\, d\boldsymbol{\mu}(\lambda) \right\}$$

where $\boldsymbol{\mu}$ is a positive $ML \times ML$ matrix-valued measure carried by $\mathbb{R}^+$ satisfying $\boldsymbol{\mu}(\mathbb{R}^+) = \mathbf{I}_{ML}$. In this paper, we establish that there exists a function $\mathbf{T}_N(z)$ of $\mathcal{S}_{ML}(\mathbb{R}^+)$, defined as the unique element of $\mathcal{S}_{ML}(\mathbb{R}^+)$ satisfying a certain functional equation that verifies

$$\frac{1}{ML}\mathrm{tr}\left((\mathbf{Q}_N(z) - \mathbf{T}_N(z))\mathbf{A}_N\right) \to 0 \tag{1.1}$$

almost surely for each $z \in \mathbb{C}\setminus\mathbb{R}^+$, where $(\mathbf{A}_N)_{N\geq 1}$ is an arbitrary sequence of deterministic $ML \times ML$ matrices satisfying $\sup_N \|\mathbf{A}_N\| < +\infty$. Particularized to the case where $\mathbf{A}_N = \mathbf{I}$, this property implies that

$$q_N(z) - t_N(z) \to 0 \tag{1.2}$$

almost surely for each $z \in \mathbb{C} \setminus \mathbb{R}^+$, where $t_N(z) = \frac{1}{ML}\mathrm{tr}(\mathbf{T}_N(z))$ is the Stieltjes transform of the probability measure $\mu_N = \frac{1}{ML}\mathrm{tr}(\boldsymbol{\mu}_N)$. In the present context, this turns out to imply that almost surely, for each bounded continuous function ϕ defined on $\mathbb{R}^+$, it holds that

$$\frac{1}{ML}\sum_{k=1}^{ML}\phi(\hat{\lambda}_{k,N}) - \int_{\mathbb{R}^+}\phi(\lambda)\,d\mu_N(\lambda) \to 0. \tag{1.3}$$

It is also useful to study the respective rates of convergence towards 0 of the variance and of the mean of $\frac{1}{ML}\sum_{k=1}^{ML}\phi(\hat{\lambda}_{k,N}) - \int_{\mathbb{R}^+}\phi(\lambda)\,d\mu_N(\lambda)$ when ϕ is a smooth function. For this, it appears sufficient to restrict to the case $\phi(\lambda) = \frac{1}{\lambda - z}$ where $z \in \mathbb{C} \setminus \mathbb{R}^+$, i.e., to study the rate of convergence of $\mathrm{var}(q_N(z)) = \mathbb{E}|q_N(z) - \mathbb{E}(q_N(z))|^2$ and of $\mathbb{E}(q_N(z)) - \int_{\mathbb{R}^+}\phi(\lambda)\,d\mu_N(\lambda)$. More generally, if $(\mathbf{A}_N)_{N\geq 1}$ is any sequence of deterministic $ML \times ML$ matrices satisfying $\sup_N \|\mathbf{A}_N\| < +\infty$, we establish that

$$\mathrm{var}\left[\frac{1}{ML}\mathrm{tr}\left(\mathbf{Q}_N(z)\mathbf{A}_N\right)\right] = \mathcal{O}(\frac{1}{MN}) \tag{1.4}$$

and that, provided $\frac{L^{3/2}}{MN} \to 0$,

$$\frac{1}{ML}\mathrm{tr}\left(\left(\mathbb{E}(\mathbf{Q}_N(z)) - \mathbf{T}_N(z)\right)\mathbf{A}_N\right) = \mathcal{O}(\frac{L}{MN}) \tag{1.5}$$

for each $z \in \mathbb{C} \setminus \mathbb{R}^+$.

In this paper, we concentrate on the characterization of the asymptotic behaviour of the terms $\frac{1}{ML}\mathrm{tr}\left(\mathbf{Q}_N(z)\mathbf{A}_N\right)$, and do not discuss the behaviour of general linear statistics $\frac{1}{ML}\sum_{k=1}^{ML}\phi(\hat{\lambda}_{k,N})$ of the eigenvalues. In order to establish our results, we follow the general approach introduced in [14] and developed in more general contexts in [15]. This approach takes benefit of the Gaussianity of the random variables $w_m(n)$, and use the Poincaré–Nash inequality to evaluate the variance of various terms and the integration by parts formula to evaluate approximations of matrix $\mathbb{E}(\mathbf{Q}_N(z))$.

Apart large random matrix methods, the properties of Stieltjes transforms of positive matrix-valued measures play a crucial role in this paper. The first author (in the alphabetic order) of this paper had the chance to learn these tools from Prof. D. Alpay at the occasion of past collaborations. The authors are thus delighted to dedicate this paper to Prof. D. Alpay on occasion of his 60th birthday.

1.2. Motivations

The present paper is motivated by the problem of testing whether M complex Gaussian zero mean times series $(x_m(n))_{n\in\mathbb{Z}}$ are mutually independent. For each $m = 1, \ldots, M$, x_m is observed from time $n = 1$ to $n = N$, and a relevant statistics depending on $((x_m(n))_{n=1,\ldots,N})_{m=1,\ldots,M}$ has to be designed and studied. A reasonable approach can be drawn by noting that if the M time series

are independent, then, for each integer L, the covariance matrix $\mathbf{R}_L$ of ML-dimensional random vector $\mathbf{x}_L(n) = (x_1(n), \ldots, x_1(n+L-1), x_2(n), \ldots, x_2(n+L-1), \ldots, x_M(n), \ldots, x_M(n+L-1))^T$ is block diagonal, a property implying that

$$\kappa_N = \frac{1}{ML}\left(\log\det(\mathbf{R}_L) - \sum_{m=1}^{M}\log\det(\mathbf{R}_L^{m,m})\right) = 0 \tag{1.6}$$

where $\mathbf{R}_L^{m,m}$ represents the m-th $L \times L$ diagonal block of $\mathbf{R}_L$. Therefore, it seems relevant to approximate matrix $\mathbf{R}_L$ by the standard estimator $\hat{\mathbf{R}}_L$ defined by

$$\hat{\mathbf{R}}_L = \frac{1}{ML}\sum_{n=1}^{N}\mathbf{x}_L(n)\left(\mathbf{x}_L(n)\right)^H$$

so that we can evaluate the term $\hat{\kappa}_N$, obtained by replacing $\mathbf{R}_L$ by $\hat{\mathbf{R}}_L$ in (1.6), and compare it to 0. The present paper is motivated by the study of this particular test under the hypothesis that the series $(x_m)_{m=1,\ldots,M}$ are uncorrelated and assuming that M and N are both large. In this context, a crucial problem is to choose parameter L. On one hand, L should be chosen in such a way that $ML/N << 1$ in order to make the estimation error $\|\hat{\mathbf{R}}_L - \mathbf{R}_L\|$ reasonably low, and thus $\hat{\kappa}_N$ close to 0 under the uncorrelation hypothesis. On the other hand, choosing a small value for L is not satisfying because comparing $\hat{\kappa}_N$ to 0 allows to test that $\mathbb{E}(x_m(l)x_{m'}^*(l')) = 0$ for each (m, m') only for $l, l' \in [0, \ldots, L-1]$, a property that does not imply formally that the time series x_m and $x_{m'}$ are independent. Therefore, choosing L as large as possible is relevant. In this case, the ratio $\frac{ML}{N}$ may no longer be very small, and $\hat{\kappa}_N$ may not converge towards 0. It is thus of fundamental interest to evaluate the behaviour of $\hat{\kappa}_N$ when M and N are large and that $\frac{ML}{N}$ is not negligible. This question is connected to the problem addressed in the present paper because the following results potentially allow to establish that $\frac{1}{ML}\log\det(\hat{\mathbf{R}}_L)$ has a deterministic behaviour that can be characterized. This term can indeed be written as

$$\frac{1}{ML}\log\det(\hat{\mathbf{R}}_L) = \frac{1}{ML}\sum_{k=1}^{ML}\phi(\hat{\lambda}_{k,N})$$

where $(\hat{\lambda}_{k,N})_{k=1,\ldots,ML}$ are the eigenvalues of $\hat{\mathbf{R}}_L$ and where $\phi(\lambda) = \log\lambda$. Moreover, if we denote by $w_{m,N}(n)$ the random variable $w_{m,N}(n) = \frac{1}{\sqrt{N}}x_m(n)$, then it is easily seen that matrix $\hat{\mathbf{R}}_L$ coincides with matrix $\mathbf{W}_N\mathbf{W}_N^H$ where $\mathbf{W}_N$ is constructed from the $w_{m,N}(n)$ as above, up to end effects (because matrix $\mathbf{W}_N$ depends on random variables $(w_m(N+l))_{m=1,\ldots,M,l=1,\ldots,L-1}$ while matrix $\hat{\mathbf{R}}_L$ does not depend on these entries). However, these end effects can be shown to be negligible. Therefore, the asymptotic behaviour of $\frac{1}{ML}\log\det(\hat{\mathbf{R}}_L)$ appears to be a consequence of the results of the present paper. We finally mention that under

some reasonable assumptions, $\|\hat{\mathbf{R}}_L^{m,m} - \mathbf{R}_L^{m,m}\| \to 0$ so that it holds that

$$\frac{1}{ML}\sum_{m=1}^{M} \log\det(\hat{\mathbf{R}}_L^{m,m}) - \frac{1}{ML}\sum_{m=1}^{M} \log\det(\mathbf{R}_L^{m,m}) \to 0.$$

In summary, the asymptotic behaviour of $\hat{\kappa}_N$ appears to be a consequence the study of the empirical eigenvalue distribution of $\mathbf{W}_N\mathbf{W}_N^H$ in the asymptotic regime where $M, N \to +\infty$ in such a way that $\frac{ML}{N}$ converges towards a nonzero constant.

1.3. On the literature

The study of the asymptotic behaviour of large random Gram matrices has a long history. Since the pioneering work of Marcenko–Pastur in 1967 ([13]), a number of random matrix models have been considered (see, e.g., [2], [15] and the references therein). In the following, we mention the previous works that are connected to the present paper. As matrix $\mathbf{W}_N$ is a block line matrix with $L \times N$ blocks, we mention the works of Girko ([7], chapter 16) as well as [4] devoted to the case where the blocks are i.i.d. We however mention that [7] and [4] did not characterize the rates of convergence and that the techniques used in these works do not allow to address the case where $L \to +\infty$. The works devoted to Hankel matrices are also relevant. [3] addressed the case where $M = 1$ and $N, L \to +\infty$ at the same rate, except that in [3], the random variables $w_{1,N}(n)$ are forced to 0 for $N < n \leq N + L - 1$. Using the moments method, [3] showed that the empirical eigenvalue distribution of $\mathbf{W}_N\mathbf{W}_N^*$ converges towards a non-compactly supported limit distribution. The random matrix model considered in [12] is similar to the matrix $\mathbf{W}_N$ of the present paper, except that in [12], for each $m = 1, \ldots, M$, the random variables $(w_m(n))$ are uncorrelated, i.e., the spectral densities $(\mathcal{S}_m(\nu))_{m=1,\ldots,M}$ are reduced to $\mathcal{S}_m(\nu) = 1$ for each ν. Using the Poincaré–Nash inequality and the integration-by-parts formula, [12] studied the asymptotic behaviour of the empirical eigenvalue distribution $\hat{\mu}_N$ in the asymptotic regime $M, N \to +\infty$ and $\frac{ML}{N} \to c$, $c > 0$. It was established that function $t_N(z)$ defined in (1.2) coincides with the Stieltjes transform of the Marcenko–Pastur distribution with parameter c_N, so that $\hat{\mu}_N$ converges weakly almost surely towards the Marcenko–Pastur distribution. The rates of convergence of the variance and of the expectation of $q_N(z) - t_N(z)$ are both characterized. Finally, [12] proved that provided $L = \mathcal{O}(N^\alpha)$ with $\alpha < 2/3$, then the extreme nonzero eigenvalues of $\mathbf{W}_N\mathbf{W}_N^*$ converge almost surely towards the end points of the support of the Marcenko–Pastur distribution. Therefore, the present paper is a partial generalization of [12].

1.4. Assumptions, general notations, and background on Stieltjes transforms of positive matrix-valued measures

Assumptions on L, M, N

Assumption 1.1. • *All along the paper, we assume that L, M, N satisfy $M \to +\infty, N \to +\infty$ in such a way that $c_N = \frac{ML}{N} \to c$, where $0 < c < +\infty$. In*

order to shorten the notations, $N \to +\infty$ should be understood as the above asymptotic regime.

- *In section 6, L, M, N also satisfy $\frac{L^{3/2}}{MN} \to 0$ or equivalently $\frac{L}{M^4} \to 0$.*

Assumptions on sequences $(r_m)_{m=1,\ldots,M}$ and spectral densities $(\mathcal{S}_m)_{m=1,\ldots,M}$
We assume that sequences $(r_m)_{m=1,\ldots,M}$ satisfy the condition

$$\sup_M \sum_{n\in\mathbb{Z}} \left(\frac{1}{M} \sum_{m=1}^{M} |r_m(n)|^2 \right)^{1/2} < +\infty. \tag{1.7}$$

As it holds that $\left(\frac{1}{M}\sum_{m=1}^{M} |r_m(n)|\right)^2 \leq \frac{1}{M}\sum_{m=1}^{M} |r_m(n)|^2$, condition (1.7) implies that

$$\sup_M \frac{1}{M} \sum_{m=1}^{M} \sum_{n\in\mathbb{Z}} |r_m(n)| < +\infty$$

and that for each m, $\sum_{n\in\mathbb{Z}} |r_m(n)| < +\infty$. Therefore, each spectral density $\mathcal{S}_m$ is continuous. We also assume that

$$\sup_M \sup_{m=1,\ldots,M} \max_{\nu\in[0,1]} \mathcal{S}_m(\nu) < +\infty \tag{1.8}$$

$$\inf_M \inf_{m=1,\ldots,M} \min_{\nu\in[0,1]} \mathcal{S}_m(\nu) > 0. \tag{1.9}$$

In the following, we denote by $\mathcal{R}(k)$ the diagonal matrix

$$\mathcal{R}(k) = \mathrm{diag}(r_1(k), \ldots, r_M(k)). \tag{1.10}$$

General notations
In the following, we will often drop the index N, and will denote $\mathbf{W}_N, \mathbf{Q}_N, \ldots$ by $\mathbf{W}, \mathbf{Q}, \ldots$ in order to simplify the notations. The N columns of matrix $\mathbf{W}$ are denoted $(\mathbf{w}_j)_{j=1,\ldots,N}$. For $1 \leq l \leq L$, $1 \leq m \leq M$, and $1 \leq j \leq N$, $\mathbf{W}_{i,j}^{m}$ represents the entry $(i + (m-1)L, j)$ of matrix $\mathbf{W}$.

If $\mathbf{A}$ is a $ML \times ML$ matrix, we denote by $\mathbf{A}_{i_1,i_2}^{m_1,m_2}$ the entry $(i_1 + (m_1 - 1)L$, $i_2 + (m_2 - 1)L)$ of matrix $\mathbf{A}$, while $\mathbf{A}^{m_1,m_2}$ represents the $L \times L$ matrix $(\mathbf{A}_{i_1,i_2}^{m_1,m_2})_{1\leq(i_1,i_2)\leq L}$.

$\mathbb{C}^+$ denotes the set of complex numbers with strictly positive imaginary parts. The conjuguate of a complex number z is denoted by z^*. If $z \in \mathbb{C} \setminus \mathbb{R}^+$, we denote by δ_z the term

$$\delta_z = \mathrm{dist}(z, \mathbb{R}^+). \tag{1.11}$$

The conjugate transpose of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}^H$ while the conjugate of $\mathbf{A}$ (i.e., the matrix whose entries are the conjugates of the entries of $\mathbf{A}$) is denoted by $\mathbf{A}^*$.

$\|\mathbf{A}\|$ represents the spectral norm of matrix $\mathbf{A}$. If $\mathbf{A}$ and $\mathbf{B}$ are 2 matrices, $\mathbf{A} \otimes \mathbf{B}$ represents the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, i.e., the block matrix whose

block (i,j) is $\mathbf{A}_{i,j}\,\mathbf{B}$. If $\mathbf{A}$ is a square matrix, $\mathrm{Im}(\mathbf{A})$ and $\mathrm{Re}(\mathbf{A})$ represent the Hermitian matrices

$$\mathrm{Im}(\mathbf{A}) = \frac{\mathbf{A} - \mathbf{A}^H}{2i},\ \mathrm{Re}(\mathbf{A}) = \frac{\mathbf{A} + \mathbf{A}^H}{2}$$

If $(\mathbf{A}_N)_{N\geq 1}$ (resp. $(\mathbf{b}_N)_{N\geq 1}$) is a sequence of matrices (resp. vectors) whose dimensions increase with N, $(\mathbf{A}_N)_{N\geq 1}$ (resp. $(\mathbf{b}_N)_{N\geq 1}$) is said to be uniformly bounded if $\sup_{N\geq 1}\|\mathbf{A}_N\| < +\infty$ (resp. $\sup_{N\geq 1}\|\mathbf{b}_N\| < +\infty$).

If $\nu \in [0,1]$ and if R is an integer, we denote by $\mathbf{d}_R(\nu)$ the R-dimensional vector $\mathbf{d}_R(\nu) = (1, e^{2\mathrm{i}\pi\nu}, \ldots, e^{2\mathrm{i}\pi(R-1)\nu})^T$, and by $\mathbf{a}_L(\nu)$ the vector $\mathbf{a}_L(\nu) = \frac{1}{\sqrt{R}}\,\mathbf{d}_R(\nu)$.

If x is a complex-valued random variable, the variance of x, denoted by $\mathrm{Var}(x)$, is defined by

$$\mathrm{Var}(x) = \mathbb{E}(|x|^2) - |\mathbb{E}(x)|^2$$

The zero-mean random variable $x - \mathbb{E}(x)$ is denoted x°.

Nice constants and nice polynomials

Definition 1.2. A nice constant is a positive constant independent of the dimensions L, M, N and complex variable z. A nice polynomial is a polynomial whose degree is independent from L, M, N, and whose coefficients are nice constants.

In the following, P_1 and P_2 will represent generic nice polynomials whose values may change from one line to another, and $C(z)$ is a generic term of the form $C(z) = P_1(|z|)P_2(1/\delta_z)$.

Background on Stieltjes transforms of positive matrix-valued measures
In the following, we denote by $\mathcal{S}_K(\mathbb{R}^+)$ the set of all Stieltjes transforms of $K \times K$ positive matrix-valued measures $\boldsymbol{\mu}$ carried by $\mathbb{R}^+$ verifying $\boldsymbol{\mu}_K(\mathbb{R}^+) = \mathbf{I}_K$. The elements of the class $\mathcal{S}_K(\mathbb{R}^+)$ satisfy the following properties:

Proposition 1.3. *Consider an element* $\mathbf{S}(z) = \int_{\mathbb{R}^+} \frac{d\,\boldsymbol{\mu}(\lambda)}{\lambda - z}$ *of* $\mathcal{S}_K(\mathbb{R}^+)$. *Then, the following properties hold true:*

(i) $\mathbf{S}$ *is analytic on* $\mathbb{C} \setminus \mathbb{R}^+$.
(ii) $\mathrm{Im}(\mathbf{S}(z)) \geq 0$ *and* $\mathrm{Im}(z\,\mathbf{S}(z)) \geq 0$ *if* $z \in \mathbb{C}^+$.
(iii) $\lim_{y\to+\infty} -\mathrm{i}y\mathbf{S}(\mathrm{i}y) = \mathbf{I}_K$.
(iv) $\mathbf{S}(z)\mathbf{S}^H(z) \leq \frac{\mathbf{I}_K}{\delta_z^2}$ *for each* $z \in \mathbb{C} \setminus \mathbb{R}^+$.
(v) $\int_{\mathbf{R}^+} \lambda\, d\boldsymbol{\mu}(\lambda) = \lim_{y\to+\infty} \mathrm{Re}\,(-\mathrm{i}y(\mathbf{I}_K + \mathrm{i}y\mathbf{S}(iy))$.

Conversely, if a function $\mathbf{S}(z)$ *satisfy properties (i), (ii), (iii), then* $\mathbf{S}(z) \in \mathcal{S}_K(\mathbb{R}^+)$.

While the reader might not been able to find a paper in which this result is proved, it has been well known for a long time (see, however, [9] for more details on (i), (ii), (iii), (v)), as well as Theorem 3 of [1] from which (iv) follows immediately)
.

2. Toeplitzification operators

In the following derivations, it will be useful to consider the following Toeplitzification operators, which inherently depend on the correlation function $r_m(\cdot)$. Let $\mathbf{J}_K$ denote the $K \times K$ shift matrix with ones in the first upper diagonal and zeros elsewhere, namely $\{\mathbf{J}_K\}_{i,j} = \delta_{j-i-1}$, and let $\mathbf{J}_K^{-1}$ denote its transpose. For a given squared matrix $\mathbf{M}$ with dimensions $R \times R$, we define $\Psi_K^{(m)}(\mathbf{M})$ as an $K \times K$ Toeplitz matrix with (i,j)th entry equal to

$$\left\{\Psi_K^{(m)}(\mathbf{M})\right\}_{i,j} = \sum_{l=-R+1}^{R-1} r_m(i-j-l)\,\tau(\mathbf{M})(l) \tag{2.1}$$

or, alternatively, as the matrix

$$\Psi_K^{(m)}(\mathbf{M}) = \sum_{n=-K+1}^{K-1} \left(\sum_{l=-R+1}^{R-1} r_m(n-l)\,\tau(\mathbf{M})(l) \right) \mathbf{J}_K^{-n} \tag{2.2}$$

where the sequence $\tau(\mathbf{M})(l)$, $-R < l < R$, is defined as

$$\tau(\mathbf{M})(l) = \frac{1}{R}\mathrm{tr}\left[\mathbf{M}\mathbf{J}_R^l\right]. \tag{2.3}$$

We can express this operator more compactly using frequency notation, namely

$$\begin{aligned}\Psi_K^{(m)}(\mathbf{M}) &= \sum_{n=-K+1}^{K-1} \left(\int_0^1 \mathcal{S}_m(\nu)\,\mathbf{a}_R^H(\nu)\,\mathbf{M}\mathbf{a}_R(\nu)\,\mathrm{e}^{2\pi \mathrm{i}\nu n} d\nu \right) \mathbf{J}_K^{-n} \\ &= \int_0^1 \mathcal{S}_m(\nu)\,\mathbf{a}_R^H(\nu)\,\mathbf{M}\mathbf{a}_R(\nu)\,\mathbf{d}_K(\nu)\,\mathbf{d}_K^H(\nu)\,d\nu\end{aligned}$$

where $\mathbf{a}_R(\nu) = \mathbf{d}_R(\nu)/\sqrt{R}$ and $\mathbf{d}_R(\nu) = \left(1, \mathrm{e}^{2\pi \mathrm{i}\nu}, \ldots, \mathrm{e}^{2\pi \mathrm{i}(R-1)\nu}\right)^T$. In particular, when $r_m(k) = \sigma^2\delta_k$(white observations), we have

$$\begin{aligned}\Psi_K^{(m)}(\mathbf{M}) &= \sigma^2 \sum_{n=-K+1}^{K-1} \frac{1}{R}\mathrm{tr}\left[\mathbf{M}\mathbf{J}_R^n\right]\mathbf{J}_K^{-n} = \sigma^2 \sum_{n=-K+1}^{K-1} \tau(\mathbf{M})\,\mathbf{J}_K^{-n} \\ &= \sigma^2 \mathcal{T}_{K,\min(R,K)}(\mathbf{M})\end{aligned}$$

where $\mathcal{T}_{K,R}(\mathbf{X})$ is the classical Toeplitzation operator in [12]. The following properties are easily checked.

- Given a square matrix $\mathbf{A}$ of dimension $K \times K$ and a square matrix $\mathbf{B}$ of dimension $R \times R$, we can write

$$\begin{aligned}\frac{1}{K}\mathrm{tr}\left[\mathbf{A}\Psi_K^{(m)}(\mathbf{B})\right] &= \int_0^1 \mathcal{S}_m(\nu)\,\mathbf{a}_K^H(\nu)\,\mathbf{A}\mathbf{a}_K(\nu)\,\mathbf{a}_R^H(\nu)\,\mathbf{B}\mathbf{a}_R(\nu)\,d\nu \\ &= \frac{1}{R}\mathrm{tr}\left[\Psi_R^{(m)}(\mathbf{A})\,\mathbf{B}\right]\end{aligned} \tag{2.4}$$

- Given the square matrices $\mathbf{B}, \mathbf{C}$ (of dimension $K \times K$) and $\mathbf{D}, \mathbf{E}$ (of dimension $R \times R$), we have

$$\frac{1}{K}\mathrm{tr}\left[\mathbf{B}\Psi_K^{(m)}\left(\mathbf{D}\Psi_R^{(m)}(\mathbf{C})\,\mathbf{E}\right)\right] = \frac{1}{K}\mathrm{tr}\left[\mathbf{C}\Psi_K^{(m)}\left(\mathbf{D}\Psi_R^{(m)}(\mathbf{B})\,\mathbf{E}\right)\right].$$

- Given a square matrix $\mathbf{M}$ and a positive integer K, we have

$$\left\|\Psi_K^{(m)}(\mathbf{M})\right\| \leq \sup_{\nu\in[0,1]}\left|\mathcal{S}_m(\nu)\,\mathbf{a}_R^H(\nu)\,\mathbf{M}\mathbf{a}_R(\nu)\right| \leq \sup_{\nu\in[0,1]}|\mathcal{S}_m(\nu)|\,\|\mathbf{M}\|.$$

- Given a square positive definite matrix $\mathbf{M}$ and a positive integer K, condition (1.9) implies that

$$\Psi_K^{(m)}(\mathbf{M}) > 0. \tag{2.5}$$

We define here two other operators that will be used throughout the paper, which respectively operate on $N \times N$ and $ML \times ML$ matrices. In order to keep the notation as simple as possible, we will drop the dimensions in the notation of these operators.

- Consider an $N \times N$ matrix $\mathbf{M}$. We define $\Psi(\mathbf{M})$ as an $ML \times ML$ block diagonal matrix with mth diagonal block given by $\Psi_L^{(m)}(\mathbf{M})$.
- Consider an $ML \times ML$ matrix $\mathbf{M}$, and let $\mathbf{M}^{m,m}$ denote its mth $L \times L$ diagonal block. We define $\overline{\Psi}(\mathbf{M})$ as the $N \times N$ matrix given by

$$\overline{\Psi}(\mathbf{M}) = \frac{1}{M}\sum_{m=1}^{M}\Psi_N^{(m)}(\mathbf{M}^{m,m}). \tag{2.6}$$

$\overline{\Psi}(\mathbf{M})$ can also be expressed as

$$\overline{\Psi}(\mathbf{M}) = \sum_{n=-(N-1)}^{N-1}\sum_{l=-(L-1)}^{L-1}\tau^{(M)}\left(\mathbf{M}(\mathcal{R}(n-l)\otimes\mathbf{I}_L)\right)(l)\,\mathbf{J}_N^{-n} \tag{2.7}$$

where $\tau^{(M)}(\mathbf{A})(l)$ is defined for any $ML \times ML$ matrix $\mathbf{A}$ by

$$\tau^{(M)}(\mathbf{A})(l) = \frac{1}{ML}\mathrm{tr}\left(\mathbf{A}(\mathbf{I}_M\otimes\mathbf{J}_L^l)\right) = \frac{1}{L}\mathrm{tr}\left[\left(\frac{1}{M}\sum_{m=1}^{M}\mathbf{A}^{m,m}\right)\mathbf{J}_L^l\right]$$

and where $\mathcal{R}(m)$ is defined in (1.10). Note also that $\overline{\Psi}(\mathbf{M})$ can alternatively be written as

$$\overline{\Psi}(\mathbf{M}) = \frac{1}{M}\sum_{m=1}^{M}\int_0^1 \mathcal{S}_m(\nu)\,\mathbf{a}_L^H(\nu)\,\mathbf{M}^{m,m}\mathbf{a}_L(\nu)\,\mathbf{d}_N(\nu)\,\mathbf{d}_N^H(\nu)\,d\nu.$$

Given these two new operators, and if $\mathbf{A}$ and $\mathbf{B}$ are $ML \times ML$ and $N \times N$ matrices, we see directly from (2.4) that

$$\frac{1}{N}\mathrm{tr}\left[\overline{\Psi}(\mathbf{A})\,\mathbf{B}\right] = \frac{1}{ML}\mathrm{tr}\left[\mathbf{A}\Psi(\mathbf{B})\right]. \tag{2.8}$$

3. Variance evaluations

In this section, we provide some estimates on the variance of certain quantities that depend on the resolvent $\mathbf{Q}(z) = \left(\mathbf{W}\mathbf{W}^H - z\mathbf{I}_{ML}\right)^{-1}$ and co-resolvent $\widetilde{\mathbf{Q}}(z) = \left(\mathbf{W}^H\mathbf{W} - z\mathbf{I}_N\right)^{-1}$. We express the result in the following lemma.

Lemma 3.1. *Let $(\mathbf{A}_N)_{N\geq 1}$ be a sequence of deterministic $ML \times ML$ matrices and $(\mathbf{G}_N)_{N\geq 1}$ a uniformly bounded sequence of deterministic $N \times N$ matrices. Then*

$$\mathrm{Var}\left(\frac{1}{ML}\mathrm{tr}\mathbf{A}_N\mathbf{Q}(z)\right) \leq \frac{C(z)}{MN}\frac{1}{ML}\mathrm{tr}(\mathbf{A}_N\mathbf{A}_N^H) \tag{3.1}$$

$$\mathrm{Var}\left(\frac{1}{ML}\mathrm{tr}\mathbf{A}_N\mathbf{Q}(z)\mathbf{W}\mathbf{G}\mathbf{W}^H\right) \leq \frac{C(z)}{MN}\frac{1}{ML}\mathrm{tr}(\mathbf{A}_N\mathbf{A}_N^H) \tag{3.2}$$

where $C(z) = P_1(|z|)P_2(1/\delta_z)$ for two nice polynomials P_1, P_2 (see Definition 1.2).

We devote the rest of this section to proving of this result. In order to short the notations, matrices $\mathbf{A}_N$ and $\mathbf{G}_N$ will be denoted by $\mathbf{A}$ and $\mathbf{G}$. We will be using the Poincaré–Nash inequality ([6], [5]), which, in the present context, can be formulated as follows ([15, 10]).

Lemma 3.2. *Let $\xi = \xi\left(\mathbf{W}, \mathbf{W}^*\right)$ denote a $\mathcal{C}^1$ complex function such that both itself and its derivatives are polynomically bounded. Under the above assumptions, we can write*

$$\mathrm{Var}\xi \leq \mathbb{E}\sum_{m,i_1,i_2,j_1,j_2}\left(\frac{\partial\xi}{\partial\left(\mathbf{W}^m_{i_1,j_1}\right)^*}\right)^*\mathbb{E}\left[\mathbf{W}^m_{i_1,j_1}\left(\mathbf{W}^m_{i_2,j_2}\right)^*\right]\frac{\partial\xi}{\partial\left(\mathbf{W}^m_{i_2,j_2}\right)^*}$$
$$+\mathbb{E}\sum_{m,i_1,i_2,j_1,j_2}\frac{\partial\xi}{\partial\left(\mathbf{W}^m_{i_1,j_1}\right)}\mathbb{E}\left[\mathbf{W}^m_{i_1,j_1}\left(\mathbf{W}^m_{i_2,j_2}\right)^*\right]\left(\frac{\partial\xi}{\partial\mathbf{W}^m_{i_2,j_2}}\right)^*$$

where $\mathbf{W}^m_{i,j}$ is the $((m-1)L+i,j)$th entry of $\mathbf{W}$.

We just check that the first term, denoted β, on the right-hand side of the upper bound of $\mathrm{Var}\xi$ is in accordance with the results claimed in Lemma 3.1 for $\xi = \frac{1}{ML}\mathrm{Tr}(\mathbf{A}\mathbf{Q}(z))$ and $\xi = \frac{1}{ML}\mathrm{Tr}(\mathbf{A}\mathbf{Q}(z)\mathbf{W}\mathbf{G}\mathbf{W}^H))$. For this, we establish that it is possible to be back to the case where the spectral densities $(\mathcal{S}_m(\nu))_{m=1,\ldots,M}$ all coincide with 1 which is covered by the results of [12]. More precisely, given the Hankel structure of the matrices $\mathbf{W}_m$, we can state that

$$\mathbb{E}\left[\mathbf{W}^m_{i_1,j_1}\left(\mathbf{W}^m_{i_2,j_2}\right)^*\right] = \frac{1}{N}r_m\left(i_1 - i_2 + j_1 - j_2\right). \tag{3.3}$$

Using that $r_m(i_1 - i_2 + j_1 - j_2) = \int_0^1 e^{2\pi\mathrm{i}(i_1-i_2+j_1-j_2)\nu}\mathcal{S}_m(\nu)\,d\nu$, we obtain immediately that β can be written as $\beta = \mathbb{E}(\alpha)$ where α is defined by

$$\alpha = \frac{1}{N}\int_0^1\sum_{m=1}^M\mathcal{S}_m(\nu)\left|\sum_{i_2,j_2}\frac{\partial\xi}{\partial\left(\mathbf{W}^m_{i_2,j_2}\right)^*}e^{-2\pi\mathrm{i}(i_2+j_2)\nu}\right|^2 d\nu.$$

Thus, (1.8) implies that $\beta \leq C\tilde{\beta}$, where $\tilde{\beta} = \mathbb{E}(\tilde{\alpha})$ where $\tilde{\alpha}$ is defined by

$$\tilde{\alpha} = \frac{1}{N} \int_0^1 \sum_{m=1}^{M} \left| \sum_{i_2,j_2} \frac{\partial \xi}{\partial \left(\mathbf{W}^m_{i_2,j_2}\right)^*} e^{-2\pi \mathrm{i}(i_2+j_2)\nu} \right|^2 d\nu$$

and where C is a nice constant. It is clear that $\tilde{\alpha}$ coincides α when $\mathcal{S}_m(\nu) = 1$ for each $m = 1, \ldots, M$ and each $\nu \in [0, 1]$. When $\xi = \frac{1}{ML}\mathrm{Tr}(\mathbf{AQ}(z))$, it is proved in [12] that

$$\tilde{\alpha} \leq \frac{1}{MN} \frac{1}{ML} \mathrm{tr}\left(\mathbf{QAQWW}^H\mathbf{Q}^H\mathbf{A}^H\mathbf{Q}^H\right).$$

As it holds that $\mathbf{QWW}^H = \mathbf{I} + z\mathbf{Q}$ and that $\|\mathbf{Q}\| \leq \frac{1}{\delta_z}$, we obtain that

$$\mathbf{QWW}^H\mathbf{Q}^H \leq \frac{1}{\delta_z}\left(1 + \frac{|z|}{\delta_z}\right)\mathbf{I}_{ML}.$$

Therefore,

$$\tilde{\alpha} \leq \frac{1}{\delta_z}\left(1 + \frac{|z|}{\delta_z}\right) \frac{1}{MN} \frac{1}{ML} \mathrm{tr}(\mathbf{QAA}^H\mathbf{Q}^H)$$

and using again $\|\mathbf{Q}\| \leq \delta_z^{-1}$,

$$\tilde{\beta} = \mathbb{E}(\tilde{\alpha}) \leq \frac{1}{\delta_z^3}\left(1 + \frac{|z|}{\delta_z}\right) \frac{1}{MN} \frac{1}{ML} \mathrm{tr}(\mathbf{AA}^H).$$

The conclusion follows from the observation that

$$\frac{1}{\delta_z^3}\left(1 + \frac{|z|}{\delta_z}\right) \leq \left[\frac{1}{\delta_z^3} + \frac{1}{\delta_z^4}\right](|z| + 1).$$

As for the case $\xi = \frac{1}{ML}\mathrm{Tr}(\mathbf{AQ}(z)\mathbf{WGW}^H)$, we refer to upper bound of the term equivalent to $\tilde{\alpha}$ expressed in Eq. (3.12-3.13) in [12], and omit further details.

4. Expectation of resolvent and co-resolvent

In this section, we analyze the expectation of the

resolvent $\mathbf{Q}(z) = \left(\mathbf{WW}^H - z\mathbf{I}_{ML}\right)^{-1}$ and co-resolvent $\widetilde{\mathbf{Q}}(z) = \left(\mathbf{W}^H\mathbf{W} - z\mathbf{I}_N\right)^{-1}$.

As a previous step, we need to ensure the properties of certain useful matrix-valued functions. This is summarized in the following lemma.

Lemma 4.1. *For $z \in \mathbb{C} \setminus \mathbb{R}^+$, the matrix $\mathbf{I}_N + c_N\overline{\Psi}\left(\mathbb{E}\mathbf{Q}(z)\right)$ is invertible, so that we can define*

$$\widetilde{\mathbf{R}}(z) = \frac{-1}{z}\left(\mathbf{I}_N + c_N\overline{\Psi}\left(\mathbb{E}\mathbf{Q}(z)\right)\right)^{-1}. \tag{4.1}$$

On the other hand, the matrix $\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{R}}^T(z)\right)$ is also invertible, and we define

$$\mathbf{R}(z) = \frac{-1}{z}\left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{R}}^T(z)\right)\right)^{-1}. \tag{4.2}$$

Furthermore, $\widetilde{\mathbf{R}}(z)$ *and* $\mathbf{R}(z)$ *are elements of* $\mathcal{S}_N(\mathbb{R}^+)$ *and* $\mathcal{S}_{ML}(\mathbb{R}^+)$ *respectively. In particular, they are holomorphic on* $\mathbb{C} \setminus \mathbb{R}^+$ *and satisfy*

$$\mathbf{R}(z)\mathbf{R}^H(z) \leq \frac{\mathbf{I}_{ML}}{\delta_z^2},\ \widetilde{\mathbf{R}}(z)\widetilde{\mathbf{R}}^H(z) \leq \frac{\mathbf{I}_N}{\delta_z^2}. \tag{4.3}$$

Moreover, there exist two nice constants (see Definition 1.2*)* η *and* $\tilde{\eta}$ *such that*

$$\mathbf{R}(z)\mathbf{R}^H(z) \geq \frac{\delta_z^2}{16(\eta^2+|z|^2)^2}\mathbf{I}_{ML}, \tag{4.4}$$

$$\widetilde{\mathbf{R}}(z)\widetilde{\mathbf{R}}^H(z) \geq \frac{\delta_z^2}{16(\tilde{\eta}^2+|z|^2)^2}\mathbf{I}_N. \tag{4.5}$$

Proof. If $z \in \mathbb{R}^{-*}$, the invertibility of $\mathbf{I}_N + c_N\overline{\Psi}\left(\mathbb{E}\mathbf{Q}(z)\right)$ is obvious. If $z \in \mathbb{C}^+$, it follows from the fact that

$$\mathrm{Im}\left[\mathbf{I}_N + c_N\overline{\Psi}\left(\mathbb{E}\mathbf{Q}(z)\right)\right] = c_N\overline{\Psi}\left(\mathbb{E}\mathrm{Im}\mathbf{Q}(z)\right)$$

and $\mathrm{Im}\mathbf{Q}(z) > 0$. We now establish that $\widetilde{\mathbf{R}}(z)$ and $\mathbf{R}(z)$ are elements of $\mathcal{S}_N(\mathbb{R}^+)$ and $\mathcal{S}_{ML}(\mathbb{R}^+)$. By Proposition 1.3, we only need to prove that $\mathrm{Im}\widetilde{\mathbf{R}}(z) \geq 0$, $\mathrm{Im}z\widetilde{\mathbf{R}}(z) \geq 0$ when $\mathrm{Im}z > 0$, $\lim_{y\to+\infty} -\mathrm{i}y\widetilde{\mathbf{R}}(\mathrm{i}y) = \mathbf{I}_N$, and similar properties for the matrix $\mathbf{R}(z)$. Clearly,

$$\mathrm{Im}\widetilde{\mathbf{R}}(z) = \widetilde{\mathbf{R}}^H(z)\left[\mathrm{Im}z\mathbf{I}_N + c_N\overline{\Psi}\left(\mathrm{Im}\left[z\mathbb{E}\mathbf{Q}(z)\right]\right)\right]\widetilde{\mathbf{R}}(z) > 0,$$
$$\mathrm{Im}z\widetilde{\mathbf{R}}(z) = c_N\left|z\right|^2\widetilde{\mathbf{R}}^H(z)\left[\overline{\Psi}\left(\mathrm{Im}\mathbb{E}\mathbf{Q}(z)\right)\right]\widetilde{\mathbf{R}}(z) > 0,$$

whereas, noting that $\mathbb{E}\mathbf{Q}(\mathrm{i}y) \to 0$ as $y \to +\infty$, we see that

$$-\mathrm{i}y\widetilde{\mathbf{R}}(\mathrm{i}y) = \left(\mathbf{I}_N + c_N\overline{\Psi}\left(\mathbb{E}\mathbf{Q}(\mathrm{i}y)\right)\right)^{-1} \to \mathbf{I}_N$$

as $y \to +\infty$. In order to justify that $\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{R}}^T(z)\right)$ is invertible, we remark that $\mathrm{Im}\left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{R}}^T(z)\right)\right)$ coincides with $\Psi\left(\mathrm{Im}(\widetilde{\mathbf{R}}^T(z))\right)$ which is positive definite because $\mathrm{Im}\widetilde{\mathbf{R}}(z) > 0$ (see (2.5)). Therefore, $\mathrm{Im}\left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{R}}^T(z)\right)\right) > 0$ and $\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{R}}^T(z)\right)$ is invertible. Finally, observing that

$$\mathrm{Im}\mathbf{R}(z) = \mathbf{R}^H(z)\left[\mathrm{Im}z\mathbf{I}_{ML} + \Psi\left(\mathrm{Im}\left[z\widetilde{\mathbf{R}}^T(z)\right]\right)\right]\mathbf{R}(z) > 0,$$
$$\mathrm{Im}z\mathbf{R}(z) = \left|z\right|^2\mathbf{R}^H(z)\left[\Psi\left(\mathrm{Im}\widetilde{\mathbf{R}}^T(z)\right)\right]\mathbf{R}(z) > 0,$$

together with the fact that, since $\widetilde{\mathbf{R}}(\mathrm{i}y) \to 0$ as $y \to \infty$,

$$-\mathrm{i}y\mathbf{R}(\mathrm{i}y) = \left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{R}}^T(\mathrm{i}y)\right)\right)^{-1} \to \mathbf{I}_{ML}.$$

We eventually establish (4.4), and omit the proof of (4.5). For this, we notice that $\mathbf{R}(z)$ is a block-diagonal matrix, and that measure $\boldsymbol{\nu}$ defined by $\mathbf{R}(z) = \int_{\mathbb{R}^+} \frac{d\boldsymbol{\nu}(\lambda)}{\lambda - z}$

is block diagonal as well. In order to establish (4.4), it is thus sufficient to prove that for each unit-norm L-dimensional vector $\mathbf{b}$, it holds that

$$\mathbf{b}^H \mathbf{R}^{m,m}(z)(\mathbf{R}^{m,m}(z))^H \mathbf{b} \geq \frac{\delta_z^2}{16(\eta^2 + |z|^2)^2} \tag{4.6}$$

for some nice constant η (of course independent on m and $\mathbf{b}$). For this, we remark that

$$\mathbf{b}^H \mathbf{R}^{m,m}(z)(\mathbf{R}^{m,m}(z))^H \mathbf{b} \geq \left|\mathbf{b}^H \mathbf{R}^{m,m}(z)\mathbf{b}\right|^2.$$

We denote ξ_m the term $\xi_m(z) = \mathbf{b}^H \mathbf{R}^{m,m}(z)\mathbf{b}$ which can be written as

$$\xi_m(z) = \int_{\mathbf{R}^+} \frac{d\mu_{\xi_m}(\lambda)}{\lambda - z}$$

where probability measure μ_{ξ_m} is defined by $d\mu_{\xi_m}(\lambda) = \mathbf{b}^H d\boldsymbol{\nu}^{m,m}(\lambda)\mathbf{b}$. We claim that

$$|\xi_m(z)| \geq \delta_z \int_{\mathbf{R}^+} \frac{\mathbf{b}^H d\boldsymbol{\nu}^{m,m}(\lambda)\mathbf{b}}{|\lambda - z|^2}. \tag{4.7}$$

To justify this, we first remark that $\delta_z = |\mathrm{Im}(z)|$ if $\mathrm{Re}(z) \geq 0$ and that $\delta_z = |z|$ if $\mathrm{Re}(z) \leq 0$. Next, we note that $|\xi_m(z)| \geq |\mathrm{Im}(\xi_m(z))| = |\mathrm{Im}(z)| \int_{\mathbb{R}^+} \frac{d\mu_{\xi_m}(\lambda)}{|\lambda - z|^2}$ whatever the sign of $\mathrm{Re}(z)$. Therefore, if $\mathrm{Re}(z) \geq 0$, it holds that

$$|\xi_m(z)| \geq \delta_z \int_{\mathbb{R}^+} \frac{d\mu_{\xi_m}(\lambda)}{|\lambda - z|^2}.$$

If $\mathrm{Re}(z) \leq 0$, $\mathrm{Re}(\xi_m(z)) = \int_{\mathbb{R}^+} \frac{\lambda - \mathrm{Re}(z)}{|\lambda - z|^2} d\mu_{\xi_m}(\lambda)$ verifies $\mathrm{Re}(\xi_m(z)) \geq -\mathrm{Re}(z) \int_{\mathbb{R}^+} \frac{d\mu_{\xi_m}(\lambda)}{|\lambda - z|^2}$. Therefore, if $\mathrm{Re}(z) \leq 0$,

$$\begin{aligned} |\xi_m(z)|^2 &= (\mathrm{Im}(\xi_m(z))^2 + (\mathrm{Re}(\xi_m(z))^2 \geq |z|^2 \left(\int_{\mathbb{R}^+} \frac{d\mu_{\xi_m}(\lambda)}{|\lambda - z|^2} \right)^2 \\ &= \delta_z^2 \left(\int_{\mathbb{R}^+} \frac{d\mu_{\xi_m}(\lambda)}{|\lambda - z|^2} \right)^2 \end{aligned}$$

Therefore, (4.7) holds. We now consider the family of probability measures

$$\{(\mathbf{b}_N^H d\boldsymbol{\nu}_N^{m,m}(\lambda)\mathbf{b}_N)_{N \geq 1, m=1,\ldots,M, \|\mathbf{b}_N = 1\|}\}$$

where we have mentioned the dependency of $\mathbf{b}$ and $\boldsymbol{\nu}$ w.r.t. N. Using item (v) of Proposition 1.3 and hypothesis (1.8) , it is easily seen that

$$\int_{\mathbb{R}^+} \lambda \, \mathbf{b}_N^H d\boldsymbol{\nu}_N^{m,m}(\lambda)\mathbf{b}_N = \mathbf{b}_N^H \boldsymbol{\Psi}_L^m(\mathbf{I}_N)\mathbf{b}_N < C$$

for some nice constant C. Therefore, it holds that

$$\sup_{N \geq 1, m=1,\ldots,M, \|\mathbf{b}_N\|=1} \int_{\mathbb{R}^+} \lambda \, \mathbf{b}_N^H d\boldsymbol{\nu}_N^{m,m}(\lambda)\mathbf{b}_N < +\infty.$$

The family of probability measures is thus tight, and it exists a nice constant η such that

$$\inf_{N\geq 1, m=1,\ldots,M, \|\mathbf{b}_N\|=1} \mathbf{b}_N^H \boldsymbol{\nu}_N^{m,m}([0,\eta])\mathbf{b}_N > 1/2. \tag{4.8}$$

We now use the obvious inequality

$$\int_{\mathbf{R}^+} \frac{\mathbf{b}^H d\boldsymbol{\nu}^{m,m}(\lambda)\mathbf{b}}{|\lambda - z|^2} \geq \int_0^\eta \frac{\mathbf{b}^H d\boldsymbol{\nu}^{m,m}(\lambda)\mathbf{b}}{|\lambda - z|^2}.$$

If $\lambda \in [0,\eta]$, it is clear that $|\lambda - z|^2 \leq 2(|z|^2 + \eta^2)$, and that

$$\int_0^\eta \frac{\mathbf{b}^H d\boldsymbol{\nu}^{m,m}(\lambda)\mathbf{b}}{|\lambda - z|^2} \geq \frac{1}{4(|z|^2 + \eta^2)}.$$

(4.7) eventually leads to

$$\mathbf{b}^H \mathbf{R}^{m,m}(z)(\mathbf{R}^{m,m}(z))^H \mathbf{b} \geq |\mathbf{b}^H \mathbf{R}^{m,m}(z)\mathbf{b}|^2 \geq \frac{\delta_z^2}{16(|z|^2 + \eta^2)^2}$$

as expected. □

In order to address the expectation of $\mathbf{Q}(z)$ and $\widetilde{\mathbf{Q}}(z)$, we will apply the integration by parts formula for the expectations of Gaussian functions, which is presented next.

Lemma 4.2. *Let $\xi = \xi(\mathbf{W}, \mathbf{W}^*)$ denote a $\mathcal{C}^1$ complex function such that both itself and its derivatives are polynomically bounded. Under the above assumptions, we can write*

$$\mathbb{E}\left[\mathbf{W}_{i_1,j_1}^m \xi\right] = \sum_{i_2=1}^{L}\sum_{j_2=1}^{N} \mathbb{E}\left[\mathbf{W}_{i_1,j_1}^m \left(\mathbf{W}_{i_2,j_2}^m\right)^*\right] \mathbb{E}\left[\frac{\partial \xi}{\partial\left(\mathbf{W}_{i_2,j_2}^m\right)^*}\right]$$

where $\mathbf{W}_{i,j}^m$ is the $((m-1)L+i, j)$th entry of $\mathbf{W}$.

Proof. See [15, 10]. □

Consider the resolvent identity

$$z\mathbf{Q}(z) = \mathbf{Q}(z)\mathbf{W}\mathbf{W}^H - \mathbf{I}_{ML}. \tag{4.9}$$

Let $\mathbf{w}_k$ denote the kth column of matrix $\mathbf{W}$, $1 \leq k \leq N$. For an $ML \times ML$ matrix $\mathbf{A}$, we recall that we denote as $[\mathbf{A}]^{m_1,m_2}$ its (m_1, m_2)th block matrix (of size $L \times L$) and as $[\mathbf{A}]_{i_1,i_2}^{m_1,m_2}$ the (i_1, i_2)th entry of its (m_1, m_2)th block. Applying the integration by parts formula in Lemma 4.2 and the identity in (3.3), we are

able to write

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{Q}(z)\mathbf{w}_k\mathbf{w}_j^H\right]_{i_1,i_2}^{m_1,m_2} \\
&= \sum_{m_3,i_3} \mathbb{E}\left[\left[\mathbf{Q}(z)\right]_{i_1,i_3}^{m_1,m_3} \mathbf{W}_{i_3,k}^{m_3} \left(\mathbf{W}_{i_2,j}^{m_2}\right)^*\right] \\
&= \sum_{r=1}^{N}\sum_{i_4=1}^{L}\sum_{m_3,i_3} \mathbb{E}\left[\mathbf{W}_{i_3,k}^{m_3}\left(\mathbf{W}_{i_4,r}^{m_3}\right)^*\right]\mathbb{E}\left[\frac{\partial\left[\mathbf{Q}(z)\right]_{i_1,i_3}^{m_1,m_3}\left(\mathbf{W}_{i_2,j}^{m_2}\right)^*}{\partial\left(\mathbf{W}_{i_4,r}^{m_3}\right)^*}\right] \\
&= -\sum_{r=1}^{N}\sum_{m_3=1}^{M}\sum_{i_3=1}^{L}\sum_{i_4=1}^{L}\frac{r_{m_3}\left(k-r+i_3-i_4\right)}{N}\mathbb{E}\left[\left[\mathbf{Q}(z)\mathbf{w}_r\mathbf{w}_j^H\right]_{i_1,i_2}^{m_1,m_2}\left[\mathbf{Q}(z)\right]_{i_4,i_3}^{m_3,m_3}\right] \\
&\quad + \sum_{i_3=1}^{L}\frac{r_{m_2}\left(k-j+i_3-i_2\right)}{N}\mathbb{E}\left[\mathbf{Q}(z)\right]_{i_1,i_3}^{m_1,m_2}.
\end{aligned}$$

Now, using the change of variable $i = i_4 - i_3$ we can alternatively express

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{Q}(z)\mathbf{w}_k\mathbf{w}_j^H\right]_{i_1,i_2}^{m_1,m_2} \\
&= -L\sum_{r=1}^{N}\sum_{m_3=1}^{M}\sum_{i=-L+1}^{L-1}\frac{r_{m_3}\left(k-r-i\right)}{N}\mathbb{E}\left[\left[\mathbf{Q}(z)\mathbf{w}_r\mathbf{w}_j^H\right]_{i_1,i_2}^{m_1,m_2}\tau\left(\mathbf{Q}^{m_3,m_3}(z)\right)(i)\right] \\
&\quad + \sum_{i_3=1}^{L}\frac{r_{m_2}\left(k-j+i_3-i_2\right)}{N}\mathbb{E}\left[\mathbf{Q}(z)\right]_{i_1,i_3}^{m_1,m_2}
\end{aligned}$$

where we recall that, for a given square matrix $\mathbf{X}$ of size R, the sequence $\tau\left(\mathbf{X}\right)(i)$ is defined in (2.3).

Using the definition of the operator $\Psi_N^{(m)}$ and its averaged counterpart in (2.6), we may reexpress the above equation as

$$\begin{aligned}
\mathbb{E}\left[\mathbf{Q}(z)\mathbf{w}_k\mathbf{w}_j^H\right]_{i_1,i_2}^{m_1,m_2} &= -c_N\mathbb{E}\left[\overline{\Psi}\left(\mathbf{Q}(z)\right)\mathbf{W}^T\mathbf{Q}^T(z)\mathbf{e}_{i_1}^{m_1}\left(\mathbf{e}_{i_2}^{m_2}\right)^T\mathbf{W}^*\right]_{k,j} \\
&\quad + \sum_{i_3=1}^{L}\frac{r_{m_2}\left(k-j+i_3-i_2\right)}{N}\mathbb{E}\left[\left(\mathbf{Q}(z)\right)_{i_1,i_3}^{m_1,m_2}\right]
\end{aligned} \tag{4.10}$$

where we recall that $c_N = \frac{ML}{N}$. From (4.10) and using the definition of $\Psi(\cdot)$, we may generally write, for any $N \times N$ deterministic matrix $\mathbf{A}$

$$\mathbb{E}\left[\mathbf{Q}(z)\mathbf{W}\mathbf{A}\mathbf{W}^H\right] = -c_N\mathbb{E}\left[\mathbf{Q}(z)\mathbf{W}\overline{\Psi}^T\left(\mathbf{Q}(z)\right)\mathbf{A}\mathbf{W}^H\right] + \mathbb{E}\left[\mathbf{Q}(z)\Psi\left(\mathbf{A}^T\right)\right]. \tag{4.11}$$

Let us now consider the co-resolvent, namely $\widetilde{\mathbf{Q}}(z) = \left(\mathbf{W}^H\mathbf{W} - z\mathbf{I}_N\right)^{-1}$, together with the co-resolvent identity

$$z\widetilde{\mathbf{Q}}(z) = \widetilde{\mathbf{Q}}(z)\mathbf{W}^H\mathbf{W} - \mathbf{I}_N.$$

Observe that we can write $\widetilde{\mathbf{Q}}(z)\mathbf{W}^H\mathbf{W} = \mathbf{W}^H\mathbf{Q}(z)\mathbf{W}$ and therefore

$$\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\mathbf{W}^H\mathbf{W}\right]_{j,k} = \mathbb{E}\left[\mathbf{W}^H\mathbf{Q}(z)\mathbf{W}\right]_{j,k} = \mathbb{E}\mathrm{tr}\left[\mathbf{Q}(z)\mathbf{w}_k\mathbf{w}_j^H\right].$$

Hence, using the expression for the expectation of the resolvent in (4.10), we obtain

$$\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\mathbf{W}^H\mathbf{W}\right]_{j,k} = \mathbb{E}\mathrm{tr}\left[\mathbf{Q}(z)\mathbf{w}_k\mathbf{w}_j^H\right] = -c_N\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\mathbf{W}^H\mathbf{W}\overline{\Psi}^T\left(\mathbf{Q}(z)\right)\right]_{j,k} + \sum_{m_1=1}^{M}\sum_{i_1=1}^{L}\sum_{i_3=1}^{L}\frac{r_{m_1}\left(k-j+i_3-i_1\right)}{N}\mathbb{E}\left[\left(\mathbf{Q}(z)\right)_{i_1,i_3}^{m_1,m_1}\right].$$

The second term can be further simplified by applying the change of variables $i_1 = i + i_3$, namely,

$$\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\mathbf{W}^H\mathbf{W}\right] = -c_N\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\mathbf{W}^H\mathbf{W}\overline{\Psi}^T\left(\mathbf{Q}(z)\right)\right] + c_N\overline{\Psi}^T\left(\mathbb{E}\mathbf{Q}(z)\right)$$

and therefore, by the resolvent's identity,

$$\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\right] = -\frac{1}{z}\mathbf{I}_N - c_N\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\overline{\Psi}^T\left(\mathbf{Q}(z)\right)\right].$$

Now, replacing $\mathbf{Q}(z)$ in the above equation by $\mathbf{Q}(z) = \mathbb{E}\mathbf{Q}(z) + \mathbf{Q}^\circ(z)$ (where $X^\circ = X - \mathbb{E}X$) we see that

$$\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\right] = \widetilde{\mathbf{R}}(z) + zc_N\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\overline{\Psi}^T\left(\mathbf{Q}^\circ(z)\right)\widetilde{\mathbf{R}}(z)\right]$$

where $\widetilde{\mathbf{R}}(z)$ is defined in (4.1). On the other hand, particularizing the equation in (4.11) to the case $\mathbf{A} = \widetilde{\mathbf{R}}(z)$ and using the resolvent's identity in (4.9), we also obtain

$$\mathbb{E}\left[\mathbf{Q}(z)\right] = \mathbf{R}(z) - zc_N\mathbb{E}\left[\mathbf{Q}(z)\mathbf{W}\overline{\Psi}^T\left(\mathbf{Q}^\circ(z)\right)\widetilde{\mathbf{R}}(z)\mathbf{W}^H\mathbf{R}(z)\right]$$

where $\mathbf{R}(z)$ is defined in (4.2). With this, we have arrived at the two fundamental equations, which are summarized in what follows:

$$\mathbb{E}\left[\mathbf{Q}(z)\right] - \mathbf{R}(z) = \mathbf{\Delta}(z), \quad \mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\right] - \widetilde{\mathbf{R}}(z) = \widetilde{\mathbf{\Delta}}(z)$$

where the error terms are defined as

$$\mathbf{\Delta}(z) = -zc_N\mathbb{E}\left[\mathbf{Q}(z)\mathbf{W}\overline{\Psi}^T\left(\mathbf{Q}^\circ(z)\right)\widetilde{\mathbf{R}}(z)\mathbf{W}^H\mathbf{R}(z)\right], \tag{4.12}$$

$$\widetilde{\mathbf{\Delta}}(z) = zc_N\mathbb{E}\left[\widetilde{\mathbf{Q}}(z)\overline{\Psi}^T\left(\mathbf{Q}^\circ(z)\right)\widetilde{\mathbf{R}}(z)\right]. \tag{4.13}$$

4.1. Control of the errors

We develop here a control on certain functionals of the error term $\Delta(z)$. In this section, we establish the following result.

Proposition 4.3. *For each deterministic sequence of $ML \times ML$ matrices $(\mathbf{A}_N)_{N\geq 1}$ satisfying $\sup_N \|\mathbf{A}_N\| < a < +\infty$, it holds that*

$$\left|\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\mathbf{\Delta}(z)\right]\right| \leq a\,C(z)\frac{L}{MN} \tag{4.14}$$

where $C(z) = P_1(|z|)P_2(1/\delta_z)$ *for some nice polynomials* P_1 *and* P_2 *(see Definition* 1.2*) that do not depend on sequence* $(\mathbf{A}_N)_{N\geq 1}$*. Moreover, if* $(\mathbf{b}_{1,N})_{N\geq 1}$ *and* $(\mathbf{b}_{2,N})_{N\geq 1}$ *are two sequences of* L*-dimensional vectors such that* $\sup_N \|\mathbf{b}_{i,N}\| < b < +\infty$ *for* $i = 1, 2$*, and if* $((d_{m,N})_{m=1,\ldots,M})_{N\geq 1}$ *are deterministic complex number verifying* $\sup_{N,m} |d_{m,N}| < d < +\infty$*, then, it holds that*

$$\left| \mathbf{b}_{1,N}^H \left(\frac{1}{M} \sum_{m=1}^{M} d_{m,N} \Delta^{m,m}(z) \right) \mathbf{b}_{2,N} \right| \leq d\, b^2\, C(z) \frac{L^{3/2}}{MN} \tag{4.15}$$

where $C(z)$ *is defined as above, where the nice polynomials* P_1 *and* P_2 *do not depend on* $(\mathbf{b}_{1,N})_{N\geq 1}$*,* $(\mathbf{b}_{2,N})_{N\geq 1}$ *and* $((d_{m,N})_{m=1,\ldots,M})_{N\geq 1}$*.*

We first establish (4.14), and denote $\mathbf{A}_N, (\mathbf{b}_{i,N})_{i=1,2}$ and $(d_{m,N})_{m=1,\ldots,M}$ by $\mathbf{A}, (\mathbf{b}_i)_{i=1,2}$ and $(d_m)_{m=1,\ldots,M}$ in order to simplify the notations. We denote by ξ the term $\frac{1}{ML}\mathrm{tr}\,[\mathbf{A}\Delta(z)]$, and express ξ as

$$\xi = \frac{1}{ML} \mathrm{tr} \left(\overline{\Psi}^T (\mathbf{Q}^\circ) \widetilde{\mathbf{R}} \mathbf{W}^H \mathbf{R} \mathbf{A} \mathbf{Q} \mathbf{W} \right).$$

Using (2.7), we obtain immediately that

$$\xi = \sum_{n=-(N-1)}^{N-1} \sum_{l=-(L-1)}^{L-1} \tau^{(M)} \left(\mathbf{Q}^\circ (\mathcal{R}(n-l) \otimes \mathbf{I}_L) \right)(l)\, \tau \left(\widetilde{\mathbf{R}} \mathbf{W}^H \mathbf{R} \mathbf{A} \mathbf{Q} \mathbf{W} \right)(n).$$

Therefore, $\mathbb{E}(\xi)$ is equal to

$$\mathbb{E}(\xi) = \sum_{n=-(N-1)}^{N-1} \sum_{l=-(L-1)}^{L-1} \mathbb{E} \left[\tau^{(M)} \left(\mathbf{Q}^\circ (\mathcal{R}(n-l) \otimes \mathbf{I}_L) \right)(l)\, \tau \left(\widetilde{\mathbf{R}} \mathbf{W}^H \mathbf{R} \mathbf{A} \mathbf{Q} \mathbf{W} \right)^\circ (n) \right].$$

Using the definition of operators τ and $\tau^{(M)}$, the Schwartz inequality, Lemma 3.1, and the inequality $\mathbf{J}_L^l \mathbf{J}_L^{l(H)} \leq \mathbf{I}_L$, we obtain that

$$\left| \mathbb{E} \left[\tau^{(M)} \left(\mathbf{Q}^\circ (\mathcal{R}(n-l) \otimes \mathbf{I}_L) \right)(l)\, \tau \left(\widetilde{\mathbf{R}} \mathbf{W}^H \mathbf{R} \mathbf{A} \mathbf{Q} \mathbf{W} \right)^\circ (n) \right] \right| \leq C(z) \frac{1}{MN} \left(\frac{1}{ML} \mathrm{Tr} (\mathcal{R}(n-l) \mathcal{R}^H (n-l) \otimes \mathbf{I}_L) \right)^{1/2}.$$

Therefore, it holds that

$$|\mathbb{E}(\xi)| \leq \frac{C(z)}{MN} \sum_{n=-(N-1)}^{N-1} \sum_{l=-(L-1)}^{L-1} \left(\frac{1}{M} \sum_{m=1}^{M} |r_m(n-l)|^2 \right)^{1/2}$$

and that

$$|\mathbb{E}(\xi)| \leq C(z) \frac{L}{MN} \sum_{n\in\mathbb{Z}} \left(\frac{1}{M} \sum_{m=1}^{M} |r_m(n)|^2 \right)^{1/2}.$$

Condition (1.7) thus implies (4.14).

In order to establish (4.15), we denote by η the left-hand side of (4.15), which can be written as

$$\eta = \frac{1}{M}\mathrm{tr}\left(\mathbf{\Delta}(\mathbf{D}\otimes \mathbf{b}_2\mathbf{b}_1^H)\right)$$

where $\mathbf{D}$ represents the $M\times M$ diagonal matrix with diagonal entries $d_1,\ldots,d_M$. Using the above calculations, we obtain that $\mathbb{E}(\eta)$ can be expressed as

$$\mathbb{E}(\eta) = L\sum_{n=-(N-1)}^{N-1}\sum_{l=-(L-1)}^{L-1}\mathbb{E}\left[\tau^{(M)}\left(\mathbf{Q}^{\circ}(\mathcal{R}(n-l)\otimes \mathbf{I}_L)\right)(l)\right.$$
$$\left.\tau\left(\widetilde{\mathbf{R}}\mathbf{W}^H\mathbf{R}(\mathbf{D}\otimes \mathbf{b}_2\mathbf{b}_1^H)\mathbf{Q}\mathbf{W}\right)^{\circ}(n)\right].$$

We use again the Schwartz inequality to evaluate

$$\mathbb{E}\left[\tau^{(M)}\left(\mathbf{Q}^{\circ}(\mathcal{R}(n-l)\otimes \mathbf{I}_L)\right)(l)\,\tau\left(\widetilde{\mathbf{R}}\mathbf{W}^H\mathbf{R}(\mathbf{D}\otimes \mathbf{b}_2\mathbf{b}_1^H)\mathbf{Q}\mathbf{W}\right)^{\circ}(n)\right]$$

together with (3.2) for $\mathbf{A} = \mathbf{R}(\mathbf{D}\otimes \mathbf{b}_2\mathbf{b}_1^H)$, and obtain that

$$\mathbb{E}\left|\tau\left(\widetilde{\mathbf{R}}\mathbf{W}^H\mathbf{R}(\mathbf{D}\otimes \mathbf{b}_2\mathbf{b}_1^H)\mathbf{Q}\mathbf{W}\right)^{\circ}(n)\right|^2 \le b^2\frac{C(z)}{MN}\frac{1}{ML}\mathrm{tr}(\mathbf{D}\mathbf{D}^H\otimes \mathbf{b}_2\mathbf{b}_2^H).$$

The term $\frac{1}{ML}\mathrm{tr}(\mathbf{D}\mathbf{D}^H\otimes \mathbf{b}_2\mathbf{b}_2^H)$ can also be written as

$$\frac{1}{ML}\mathrm{tr}(\mathbf{D}\mathbf{D}^H\otimes \mathbf{b}_2\mathbf{b}_2^H) = \frac{1}{L}\left(\frac{1}{M}\sum_{m=1}^{M}|d_m|^2\right)\|\mathbf{b}_2\|^2$$

and is thus bounded by $\frac{b^2d^2}{L}$. Therefore, the Schwartz inequality leads to

$$\left|\mathbb{E}\left[\tau^{(M)}\left(\mathbf{Q}^{\circ}(\mathcal{R}(n-l)\otimes \mathbf{I}_L)\right)(l)\,\tau\left(\widetilde{\mathbf{R}}\mathbf{W}^H\mathbf{R}(\mathbf{D}\otimes \mathbf{b}_2\mathbf{b}_1^H)\mathbf{Q}\mathbf{W}\right)^{\circ}(n)\right]\right|$$
$$\le b^2 d\,C(z)\frac{1}{MN\sqrt{L}}\left(\frac{1}{M}\sum_{m=1}^{M}|r_m(n-l)|^2\right)^{1/2}. \quad (4.16)$$

Using the same approach as above, we immediately obtain (4.15).

Corollary 4.4. *It holds that*

$$\|\overline{\Psi}(\Delta)\| \le C(z)\frac{L^{3/2}}{MN}. \quad (4.17)$$

(4.17) follows immediately from

$$\|\overline{\Psi}(\Delta)\| \le \sup_{\nu\in[0,1]}\frac{1}{M}\sum_{m=1}^{M}\mathcal{S}_m(\nu)|\mathbf{a}_L^H(\nu)\mathbf{\Delta}^{m,m}\mathbf{a}_L(\nu)|$$

and from the application of (4.15) to the case $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{a}_L(\nu)$ and $d_m = \mathcal{S}_m(\nu)$.

5. The deterministic equivalents

As $\mathbb{E}(\mathbf{Q}(z)) - \mathbf{R}(z)$ converges towards zero in some appropriate sense, (4.1) and (4.2) suggest that it is reasonable to expect that $\mathbb{E}(\mathbf{Q}(z))$ behaves as the first component $\mathbf{T}(z)$ of the solution $(\mathbf{T}(z), \widetilde{\mathbf{T}}(z))$ of the so-called canonical equation

$$\mathbf{T}(z) = -\frac{1}{z}\left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{T}}^T(z)\right)\right)^{-1}, \tag{5.1}$$

$$\widetilde{\mathbf{T}}(z) = -\frac{1}{z}\left(\mathbf{I}_N + c_N\overline{\Psi}^T\left(\mathbf{T}(z)\right)\right)^{-1}. \tag{5.2}$$

In the following, we establish that the canonical equation has a unique solution. More precisely:

Proposition 5.1. *There exists a unique pair of functions* $(\mathbf{T}(z), \widetilde{\mathbf{T}}(z)) \in \mathcal{S}_{ML}(\mathbb{R}^+) \times \mathcal{S}_N(\mathbb{R}^+)$ *that satisfy* (5.1), (5.2) *for each* $z \in \mathbb{C} \setminus \mathbb{R}^+$. *Moreover, there exist two nice constants (see Definition* 1.2*)* η *and* $\tilde{\eta}$ *such that*

$$\mathbf{T}(z)\mathbf{T}^H(z) \geq \frac{\delta_z^2}{16(\eta^2 + |z|^2)^2}\mathbf{I}, \tag{5.3}$$

$$\widetilde{\mathbf{T}}(z)\widetilde{\mathbf{T}}^H(z) \geq \frac{\delta_z^2}{16(\tilde{\eta}^2 + |z|^2)^2}\mathbf{I}. \tag{5.4}$$

We devote the rest of this section to proving this proposition. We will first prove existence of a solution by using a standard convergence argument.

Proposition 5.2. *Let* $\mathbf{\Gamma}^m(z)$, $m = 1, \ldots, M$, *be a collection of* $L \times L$ *matrix-valued complex functions belonging to* $\mathcal{S}_L(\mathbb{R}^+)$ *and define* $\mathbf{\Gamma}(z) = \operatorname{diag}\left(\mathbf{\Gamma}^1(z), \ldots, \mathbf{\Gamma}^M(z)\right)$. *Likewise, let* $\widetilde{\mathbf{\Gamma}}(z)$ *be an* $N \times N$ *matrix-valued complex function belonging to* $\mathcal{S}_N(\mathbb{R}^+)$. *Consider the two matrices*

$$\mathbf{\Upsilon}(z) = -\frac{1}{z}\left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{\Gamma}}^T(z)\right)\right)^{-1},$$

$$\widetilde{\mathbf{\Upsilon}}(z) = -\frac{1}{z}\left(\mathbf{I}_N + c_N\overline{\Psi}^T\left(\mathbf{\Gamma}(z)\right)\right)^{-1},$$

and let $\mathbf{\Upsilon}(z) = \operatorname{diag}\left(\mathbf{\Upsilon}^1(z), \ldots, \mathbf{\Upsilon}^M(z)\right)$. *The matrix-valued functions* $\mathbf{\Upsilon}^m(z)$, $m = 1, \ldots, M$ *and* $\widetilde{\mathbf{\Upsilon}}(z)$ *are analytic on* $\mathbb{C}\backslash\mathbb{R}^+$ *and belong to the classes* $\mathcal{S}_L(\mathbb{R}^+)$ *and* $\mathcal{S}_N(\mathbb{R}^+)$, *respectively.*

Proof. The proof follows the lines of [9, Proposition 5.1]. We first need to prove that $\mathbf{\Upsilon}^m(z)$ and $\widetilde{\mathbf{\Upsilon}}(z)$ are analytic on $\mathbb{C}\backslash\mathbb{R}^+$. To see this, observe that if $\mathbf{A}(z)$ is an analytic matrix-valued function, so is $\Psi_K^{(m)}\left(\mathbf{A}(z)\right), m = 1, \ldots, M$. Therefore, we only need to show $\det\left(z\mathbf{I}_L + \Psi_L^{(m)}\left(z\widetilde{\mathbf{\Gamma}}^T(z)\right)\right) \neq 0$ and $\det\left(z\mathbf{I}_N + c_N\overline{\Psi}^T\left(z\mathbf{\Gamma}(z)\right)\right) \neq 0$ when $z \in \mathbb{C}\backslash\mathbb{R}^+$. Let $\mathbf{h}$ denote an arbitrary $L \times 1$ column vector such that

$\left(z\mathbf{I}_L + \Psi_L^{(m)}\left(z\widetilde{\mathbf{\Gamma}}^T(z)\right)\right)\mathbf{h} = 0$. If $z \in \mathbb{C}^+$, we have

$$0 = \left|\mathbf{h}^H\left(z\mathbf{I}_L + \Psi_L^{(m)}\left(z\widetilde{\mathbf{\Gamma}}^T(z)\right)\right)\mathbf{h}\right| \geq \mathrm{Im}\mathbf{h}^H\left(z\mathbf{I}_L + \Psi_L^{(m)}\left(z\widetilde{\mathbf{\Gamma}}^T(z)\right)\right)\mathbf{h}$$
$$= \mathbf{h}^H\left(\mathrm{Im}z\mathbf{I}_L + \Psi_L^{(m)}\left(\mathrm{Im}z\widetilde{\mathbf{\Gamma}}^T(z)\right)\right)\mathbf{h} \geq \mathrm{Im}z\left\|\mathbf{h}\right\|^2 \geq 0$$

where we have used $\mathrm{Im}z\widetilde{\mathbf{\Gamma}}^T(z) \geq \mathbf{0}$ because $\widetilde{\mathbf{\Gamma}}(z) \in \mathcal{S}_N(\mathbb{R}^+)$. From the above chain of inequalities we see that we can only have $\mathbf{h} = \mathbf{0}$. The same argument is valid when $z \in \mathbb{C}^-$. On the other hand, when $z \in \mathbb{R}^-$, we will have $-z\widetilde{\mathbf{\Gamma}}^T(z) \geq 0$ and

$$0 = \mathbf{h}^H\left(-z\mathbf{I}_L + \Psi_L^{(m)}\left(-z\widetilde{\mathbf{\Gamma}}^T(z)\right)\right)\mathbf{h} \geq |z|\left\|\mathbf{h}\right\|^2 \geq 0$$

which also implies that $\mathbf{h} = \mathbf{0}$. A similar argument proves that

$$\det\left(z\mathbf{I}_N + c_N\overline{\Psi}^T\left(z\mathbf{\Gamma}(z)\right)\right) \neq 0$$

when $z \in \mathbb{C}\backslash\mathbb{R}^+$.

Next, we prove that $\mathrm{Im}\mathbf{\Upsilon}^m(z) \geq 0$ and $\mathrm{Im}z\mathbf{\Upsilon}^m(z) \geq 0$ when $z \in \mathbb{C}^+$. Observe that, using the identity $\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{B}^{-1}$, we have

$$\mathrm{Im}\mathbf{\Upsilon}^m(z) = \mathbf{\Upsilon}^m(z)^H\left[\mathrm{Im}z\mathbf{I}_L + \Psi_L^{(m)}\left(\mathrm{Im}\left[z\widetilde{\mathbf{\Gamma}}^T(z)\right]\right)\right]\mathbf{\Upsilon}^m(z) \geq 0$$

because $\mathrm{Im}\left[z\widetilde{\mathbf{\Gamma}}^T(z)\right] \geq 0$ since $\widetilde{\mathbf{\Gamma}}(z) \in \mathcal{S}_N(\mathbb{R}^+)$. On the the other hand, we also have

$$\mathrm{Im}\left[z\mathbf{\Upsilon}^m(z)\right] = \left[z\mathbf{\Upsilon}^m(z)\right]^H\left[\Psi_L^{(m)}\left(\mathrm{Im}\widetilde{\mathbf{\Gamma}}^T(z)\right)\right]\left[z\mathbf{\Upsilon}^m(z)\right] \geq 0$$

because $\mathrm{Im}\widetilde{\mathbf{\Gamma}}^T(z) \geq 0$ since $\widetilde{\mathbf{\Gamma}}(z) \in \mathcal{S}_N(\mathbb{R}^+)$.

Finally, one can readily see that $\lim_{y\to 0}\mathrm{i}y\mathbf{\Upsilon}^m(\mathrm{i}y) = -\mathbf{I}_L$ because $\lim_{y\to 0}\widetilde{\mathbf{\Gamma}}(\mathrm{i}y) = \mathbf{0}$. Consequently, Proposition 1.3 implies that $\mathbf{\Upsilon}^m(z)$, $m = 1, \ldots, M$ belong to the class $\mathcal{S}_L(\mathbb{R}^+)$. We can prove similarly that $\widetilde{\mathbf{\Upsilon}}(z)$ belongs to $\mathcal{S}_N(\mathbb{R}^+)$. □

Let us now define the sequence of functions in $\mathcal{S}_{ML}(\mathbb{R}^+)$ that will lead to a solution. We begin by defining $\mathbf{T}^{(0)}(z) = \left(\Psi(\mathbf{I}_N) - z\mathbf{I}_{ML}\right)^{-1}$, and use the iterative definition

$$\widetilde{\mathbf{T}}^{(p)}(z) = -\frac{1}{z}\left(\mathbf{I}_N + c_N\overline{\Psi}^T\left(\mathbf{T}^{(p)}(z)\right)\right)^{-1}, \tag{5.5}$$

$$\mathbf{T}^{(p+1)}(z) = -\frac{1}{z}\left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{T}}^{(p)}(z)^T\right)\right)^{-1}, \tag{5.6}$$

for $p \geq 0$. By Proposition 5.2 we see that the $L \times L$ diagonal blocks of $\mathbf{T}^{(p)}(z)$ belong to the class $\mathcal{S}_L(\mathbb{R}^+)$, whereas $\widetilde{\mathbf{T}}^{(p)}(z)$ belong to $\mathcal{S}_N(\mathbb{R}^+)$. In order to prove the existence of a solution to the canonical equation, we will first prove that the sequence $\mathbf{T}^{(p)}(z)$ has a limit in the set of $ML \times ML$ diagonal block matrices with blocks belonging to the class $\mathcal{S}_L(\mathbb{R}^+)$. Then, in a second step, we will prove that this limit is a solution to the canonical equation.

Our first objective is to show that, for z belonging to a certain open subset of $\mathbb{C}^+$, we have $\left\|\mathbf{T}^{(p+1)}(z) - \mathbf{T}^{(p)}(z)\right\| \leq K(z)\left\|\mathbf{T}^{(p)}(z) - \mathbf{T}^{(p-1)}(z)\right\|$, $p \geq 0$, where

$0 < K(z) < 1$. This will show that, for each z in this open subset, $\mathbf{T}^{(p)}(z)$ forms a Cauchy sequence and therefore has a limit. Using Montel's theorem, we can establish that convergence is uniform in compact sets and that the limiting matrix function is analytic on $\mathbb{C}^+$. Define, for $p \geq 1$ the error matrices

$$\epsilon^{(p)}(z) = \mathbf{T}^{(p)}(z) - \mathbf{T}^{(p-1)}(z),$$
$$\widetilde{\epsilon}^{(p)}(z) = \widetilde{\mathbf{T}}^{(p)}(z) - \widetilde{\mathbf{T}}^{(p-1)}(z),$$

and note that we have a recurrent relationship through the identity $\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{B}^{-1}$, namely

$$\epsilon^{(p+1)}(z) = \Theta_p\left(\epsilon^{(p)}(z)\right),$$
$$\widetilde{\epsilon}^{(p+1)}(z) = \widetilde{\Theta}_p\left(\widetilde{\epsilon}^{(p)}(z)\right),$$

where we have defined the operators

$$\Theta_p(\mathbf{X}) = c_N z^2 \mathbf{T}^{(p+1)}(z)\Psi\left(\widetilde{\mathbf{T}}^{(p)}(z)\overline{\Psi}(\mathbf{X})\,\widetilde{\mathbf{T}}^{(p-1)}(z)\right)\mathbf{T}^{(p)}(z),$$
$$\widetilde{\Theta}_p(\mathbf{X}) = c_N z^2 \widetilde{\mathbf{T}}^{(p)}(z)\overline{\Psi}^T\left(\mathbf{T}^{(p+1)}(z)\Psi\left(\mathbf{X}^T\right)\mathbf{T}^{(p)}(z)\right)\widetilde{\mathbf{T}}^{(p+1)}(z),$$

for $p \geq 1$. Using the properties of the operators we can obviously establish that

$$\max\left\{\left\|\Theta_p(\mathbf{X})\right\|, \left\|\widetilde{\Theta}_p(\mathbf{X})\right\|\right\} \leq \sup_N c_N \sup_{m,M,\nu} \left|\mathcal{S}_m(\nu)\right|^2 \frac{|z|^2}{(\mathrm{Im}z)^4}\left\|\mathbf{X}\right\|.$$

Consider the domain

$$\mathcal{D} = \left\{z \in \mathbb{C}^+ : \sup_N c_N \sup_{m,M,\nu} \left|\mathcal{S}_m(\nu)\right|^2 \frac{|z|^2}{(\mathrm{Im}z)^4} < \frac{1}{2}\right\}.$$

For $z \in \mathbb{C}^+$ we clearly see that both $\Theta_p(\mathbf{X})$ and $\widetilde{\Theta}_p(\mathbf{X})$ are contractive and therefore the sequences $\left(\mathbf{T}^{(p)}(z)\right)_p$ and $\left(\widetilde{\mathbf{T}}^{(p)}(z)\right)_p$ are both Cauchy and have limits, which will be denoted by $\mathbf{T}(z)$ and $\widetilde{\mathbf{T}}(z)$. Since the sequences $\left(\mathbf{T}^{(p)}(z)\right)_p$ and $\left(\widetilde{\mathbf{T}}^{(p)}(z)\right)_p$ are uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{R}^+$ (because they belong to $\mathcal{S}_{ML}(\mathbb{R}^+)$ and $\mathcal{S}_N(\mathbb{R}^+)$, respectively), Montel's theorem establishes that $\mathbf{T}(z)$ and $\widetilde{\mathbf{T}}(z)$ are analytic on $\mathbb{C}\backslash\mathbb{R}^+$.

It remains to prove that $\mathbf{T}(z)$ and $\widetilde{\mathbf{T}}(z)$, respectively, belong to $\mathcal{S}_{ML}(\mathbb{R}^+)$ and $\mathcal{S}_N(\mathbb{R}^+)$ and that they satisfy the canonical system of equations. From the fact that $\mathrm{Im}\mathbf{T}^{(p)}(z) \geq 0$, $\mathrm{Im}z\mathbf{T}^{(p)}(z) \geq 0$ and $\mathbf{T}^{(p)}(z)\mathbf{T}^{(p)}(z)^H \leq \delta_z^{-1}\mathbf{I}$ for $p \geq 1$ we have $\mathrm{Im}\mathbf{T}(z) \geq 0$, $\mathrm{Im}z\mathbf{T}(z) \geq 0$ and $\mathbf{T}(z)\mathbf{T}(z)^H \leq \delta_z^{-1}\mathbf{I}$. The same argument applies to $\widetilde{\mathbf{T}}(z)$. On the other hand, using the reasoning in the proof of Lemma 4.1 we clearly see that both $\left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{T}}^T(z)\right)\right)$ and $\left(\mathbf{I}_N + c_N\overline{\Psi}^T(\mathbf{T}(z))\right)$ are invertible for $z \in \mathbb{C}\backslash\mathbb{R}^+$, and that they are the limits of the corresponding terms

on right-hand side of (5.5)–(5.6). This shows that the pair $\mathbf{T}(z)$, $\widetilde{\mathbf{T}}(z)$ satisfies the canonical system of equations. Noting that

$$\lim_{y\to\infty} -\mathrm{i}y\mathbf{T}(\mathrm{i}y) = \lim_{y\to\infty} \left(\mathbf{I}_{ML} + \Psi\left(\widetilde{\mathbf{T}}^T(\mathrm{i}y)\right)\right)^{-1} = \mathbf{I}_{ML}$$

(because $\left\|\widetilde{\mathbf{T}}(z)\right\| < \delta_z^{-1}\mathbf{I}$), we can conclude $\mathbf{T}(z)$ belongs to $\mathcal{S}_{ML}(\mathbb{R}^+)$. A similar reasoning shows that $\widetilde{\mathbf{T}}(z)$ belongs to $\mathcal{S}_N(\mathbb{R}^+)$.

Following the proof of (4.4), (4.5), it is easy to check that each solution of (5.1), (5.2) satisfies (5.3), (5.4).

Let us now prove unicity. For this, it would be possible to use arguments based on the analyticity of the solutions and the Montel theorem as in the existence proof. We, however, prefer to use a different approach because the corresponding ideas will be used later, and rather prove that for each z, system (5.1), (5.2) considered as a system in the set of $ML \times ML$ and $N \times N$ matrices, has a unique solution. We fix $z \in \mathbb{C} \setminus \mathbb{R}^+$, and assume that $\mathbf{T}(z), \widetilde{\mathbf{T}}(z)$ and $\mathbf{S}(z), \widetilde{\mathbf{S}}(z)$ are matrices that are solutions of the system (5.1), (5.2) of equations at point z. It is easily seen that

$$\mathbf{T}(z) - \mathbf{S}(z) = c_N z^2 \mathbf{S}(z)\Psi\left(\widetilde{\mathbf{S}}^T(z)\overline{\Psi}\left(\mathbf{T}(z) - \mathbf{S}(z)\right)\widetilde{\mathbf{T}}^T(z)\right)\mathbf{T}(z). \tag{5.7}$$

The above equation can alternatively be written as

$$\mathbf{T}(z) - \mathbf{S}(z) = \Phi_0\left(\mathbf{T}(z) - \mathbf{S}(z)\right)$$

where we have defined the operator $\Phi_0(\mathbf{X})$ as

$$\Phi_0(\mathbf{X}) = c_N z^2 \mathbf{S}(z)\Psi\left(\widetilde{\mathbf{S}}^T(z)\overline{\Psi}(\mathbf{X})\widetilde{\mathbf{T}}^T(z)\right)\mathbf{T}(z) \tag{5.8}$$

where $\mathbf{X}$ is an $ML \times ML$ block-diagonal matrix. We note that operator Φ_0 depends on point z, but we do not mention this dependency in order to simplify the notations. Our objective is to show that the equation $\Phi_0(\mathbf{X}) = \mathbf{X}$ accepts a unique solution in the set of block-diagonal matrices, which is trivially given by $\mathbf{X} = \mathbf{0}$. This will imply that $\mathbf{T}(z) = \mathbf{S}(z)$, contradicting the original hypothesis.

We iteratively define $\Phi_0^{(n)}(\mathbf{X}) = \Phi_0\left(\Phi_0^{(n-1)}\mathbf{X}\right)$ for $n \in \mathbb{N}$, with $\Phi_0^{(1)}(\mathbf{X}) = \Phi_0(\mathbf{X})$. Let $\Phi_0^{(n)}(\mathbf{X})^{m,m}$ denote the $L \times L$ sized mth diagonal block of $\Phi_0^{(n)}(\mathbf{X})$. In the following, we establish that for a block diagonal $ML \times ML$ matrix $\mathbf{X}$, it holds that $\lim_{n\to+\infty} \Phi_0^{(n)}(\mathbf{X}) = 0$. If this property holds, a solution of the equation $\mathbf{X} = \Phi_0(\mathbf{X})$ satisfies $\mathbf{X} = \Phi_0^{(n)}(\mathbf{X})$ for each n, thus leading to $\mathbf{X} = 0$. It is useful to mention that in the following analysis, dimensions L, M, N are fixed. We establish the following proposition, which, of course, implies that each element of matrix $\Phi_0^{(n)}(\mathbf{X})$ converges towards 0, i.e., that matrix $\Phi_0^{(n)}(\mathbf{X})$ converges towards 0.

Proposition 5.3. *For each $m = 1, \ldots, M$, and for all L-dimensional vectors* $\mathbf{a}$ *and* $\mathbf{b}$*, it holds that*

$$\left|\mathbf{a}^H \Phi_0^{(n)}(\mathbf{X})^{m,m}\mathbf{b}\right| \to 0$$

as $n \to \infty$ for each $ML \times ML$ matrix $\mathbf{X}$.

To simplify the notation, in what follows we drop the dependence on z from $\mathbf{T}(z)$, $\widetilde{\mathbf{T}}(z)$, $\mathbf{S}(z)$ and $\widetilde{\mathbf{S}}(z)$. We begin by defining two operators $\Phi_{\mathbf{S}}(\mathbf{X})$ and $\Phi_{\mathbf{T}^H}(\mathbf{X})$ that operate on $ML \times ML$ matrices as

$$\Phi_{\mathbf{S}}(\mathbf{X}) = c_N |z|^2 \mathbf{S}\Psi\left(\widetilde{\mathbf{S}}^T \overline{\Psi}(\mathbf{X}) \widetilde{\mathbf{S}}^*\right)\mathbf{S}^H, \tag{5.9}$$

$$\Phi_{\mathbf{T}^H}(\mathbf{X}) = c_N |z|^2 \mathbf{T}^H\Psi\left(\widetilde{\mathbf{T}}^* \overline{\Psi}(\mathbf{X}) \widetilde{\mathbf{T}}^T\right)\mathbf{T}. \tag{5.10}$$

We remark that $\Phi_{\mathbf{S}}(\mathbf{X}) \geq 0$ and $\Phi_{\mathbf{T}^H}(\mathbf{X}) \geq 0$ if $\mathbf{X} \geq 0$. Matrices $\mathbf{T}, \widetilde{\mathbf{T}}, \mathbf{S}, \widetilde{\mathbf{S}}$ being solutions of equations (5.1), (5.2) are nonsingular. Therefore, $\Phi_{\mathbf{S}}(\mathbf{X})$ and $\Phi_{\mathbf{T}^H}(\mathbf{X})$ are also positive definite as soon as $\mathbf{X}$ is positive definite.

Consider the integral representation of the mth diagonal block of $\Phi_0(\mathbf{X})$, that is,

$$\begin{aligned}\Phi_0(\mathbf{X})^{m,m} = c_N z^2 \int_0^1 \int_0^1 F_m(\nu,\alpha)\mathbf{S}^{m,m}\mathbf{d}_L(\nu)\,\mathbf{a}_N^H(\nu)\,\widetilde{\mathbf{S}}\mathbf{d}_N(\alpha) \\ \times \mathbf{d}_N^H(\alpha)\,\widetilde{\mathbf{T}}\mathbf{a}_N(\nu)\,\mathbf{d}_L^H(\nu)\,\mathbf{T}^{m,m} d\nu d\alpha\end{aligned}$$

where

$$F_m(\nu,\alpha) = \mathcal{S}_m(\nu)\,\frac{1}{M}\sum_{k=1}^{M}\mathcal{S}_k(\alpha)\,\mathbf{a}_L^H(\alpha)\,\mathbf{X}^{k,k}\mathbf{a}_L(\alpha).$$

It turns out that, for each integer $n \geq 0$ and each $m = 1, \ldots, M$ we have

$$\begin{aligned}\Phi_0^{(n+1)}(\mathbf{X})^{m,m}\left(\Phi_{\mathbf{T}^H}^{(n+1)}(\mathbf{I})^{m,m}\right)^{-1}\left(\Phi_0^{(n+1)}(\mathbf{X})^{m,m}\right)^H \\ \leq \Phi_{\mathbf{S}}\left(\Phi_0^{(n)}(\mathbf{X})\left(\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I})\right)^{-1}\Phi_0^{(n)}(\mathbf{X})^H\right)^{m,m}.\end{aligned} \tag{5.11}$$

To see this, we consider the matrix

$$\mathcal{M}_m(\mathbf{X},\mathbf{B}) = \begin{bmatrix} \Phi_{\mathbf{S}}\left(\mathbf{X}\mathbf{B}^{-1}\mathbf{X}^H\right)^{m,m} & \Phi_0(\mathbf{X})^{m,m} \\ \Phi_0^H(\mathbf{X})^{m,m} & \Phi_{\mathbf{T}^H}(\mathbf{B})^{m,m} \end{bmatrix} \tag{5.12}$$

where $\mathbf{B}$ is an arbitrary $ML \times ML$ Hermitian positive definite block-diagonal matrix. It turns out that $\mathcal{M}_m(\mathbf{X},\mathbf{B}) \geq 0$. Indeed, to see this we only need to observe that this matrix can alternatively be expressed as

$$\mathcal{M}_m(\mathbf{X},\mathbf{B}) = \frac{c_N}{M}\sum_{k=1}^{M}\int_0^1\int_0^1 \mathcal{S}_m(\nu)\,\mathcal{S}_k(\alpha)\,\Psi_{m,k}(\mathbf{X},\mathbf{B})\Psi_{m,k}(\mathbf{X},\mathbf{B})^H d\nu d\alpha$$

where

$$\Psi_{m,k}(\mathbf{X},\mathbf{B}) = \begin{bmatrix} z\mathbf{S}^{m,m}\mathbf{d}_L(\nu)\,\mathbf{a}_N^H(\nu)\,\widetilde{\mathbf{S}}^T\mathbf{d}_N(\alpha)\,\mathbf{a}_L^H(\alpha)\,\mathbf{X}^{k,k}\left(\mathbf{B}^{k,k}\right)^{-1/2} \\ z^*\left(\mathbf{T}^{m,m}\right)^H\mathbf{d}_L(\nu)\,\mathbf{a}_N^H(\nu)\,\widetilde{\mathbf{T}}^*\mathbf{d}_N(\alpha)\,\mathbf{a}_L^H(\alpha)\left(\mathbf{B}^{k,k}\right)^{1/2} \end{bmatrix}.$$

Now, since $\mathcal{M}_m(\mathbf{X},\mathbf{B}) \geq 0$, the Schur complement of this matrix will also be positive semidefinite, so that we can state

$$\Phi_0\left(\mathbf{X}\right)^{m,m}\left(\Phi_{\mathbf{T}^H}\left(\mathbf{B}\right)^{m,m}\right)^{-1}\Phi_0^H\left(\mathbf{X}\right)^{m,m} \leq \Phi_{\mathbf{S}}\left(\mathbf{X}\mathbf{B}^{-1}\mathbf{X}^H\right)^{m,m}.$$

Thus, fixing $\mathbf{B} = \Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{I}\right)$ and replacing $\mathbf{X}$ with $\Phi_0^{(n)}\left(\mathbf{X}\right)$ in the above equation, we directly obtain (5.11).

By iterating the inequality in (5.11) and using the positivity of the operator $\Phi_{\mathbf{S}}\left(\cdot\right)$ we obtain

$$\Phi_0^{(n)}\left(\mathbf{X}\right)^{m,m}\left(\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{I}\right)^{m,m}\right)^{-1}\left(\Phi_0^{(n)}\left(\mathbf{X}\right)^{m,m}\right)^H \leq \Phi_{\mathbf{S}}^{(n)}\left(\mathbf{X}\mathbf{X}^H\right)^{m,m}$$

and

$$\Phi_0^{(n)}\left(\mathbf{X}\right)\left(\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{I}\right)\right)^{-1}\left(\Phi_0^{(n)}\left(\mathbf{X}\right)\right)^H \leq \Phi_{\mathbf{S}}^{(n)}\left(\mathbf{X}\mathbf{X}^H\right). \tag{5.13}$$

We can now finalize the proof of Proposition 5.3 by noting that, by the Cauchy-Schwarz inequality and the above inequality

$$\begin{aligned}
&\left|\mathbf{a}^H\Phi_0^{(n)}\left(\mathbf{X}\right)^{m,m}\mathbf{b}\right| \\
&\leq \left[\mathbf{a}^H\Phi_0^{(n)}\left(\mathbf{X}\right)^{m,m}\left(\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{I}\right)^{m,m}\right)^{-1}\left(\Phi_0^{(n)}\left(\mathbf{X}\right)^{m,m}\right)^H\mathbf{a}\right]^{1/2} \\
&\quad\cdot\left(\mathbf{b}^H\left(\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{I}\right)^{m,m}\right)\mathbf{b}\right)^{1/2} \\
&\leq \left[\mathbf{a}^H\Phi_{\mathbf{S}}^{(n)}\left(\mathbf{X}\mathbf{X}^H\right)^{m,m}\mathbf{a}\right]^{1/2}\left[\mathbf{b}^H\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{I}\right)^{m,m}\mathbf{b}\right]^{1/2}.
\end{aligned}$$

Therefore, to conclude the proof we only need to show that both $\Phi_{\mathbf{S}}^{(n)}\left(\mathbf{X}\mathbf{X}^H\right)^{m,m}$ and $\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{I}_N\right)^{m,m}$ converge to zero as $n \to \infty$. This can be shown following the steps in [12], as established in the following proposition.

Lemma 5.4. *Let* $\mathbf{T}(z), \widetilde{\mathbf{T}}(z)$ *be a solution to the canonical equation* (5.1), (5.2) *at point* z*, and let* $\Phi_{\mathbf{T}}^{(n)}\left(\mathbf{B}\right)$ *be defined, for a positive semidefinite* $\mathbf{B}$*, as in* (5.9)*. Then, it holds that*

$$\Phi_{\mathbf{T}}^{(n)}\left(\mathbf{B}\right) \to 0 \tag{5.14}$$

and

$$\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{B}\right) \to 0 \tag{5.15}$$

as $n \to \infty$*. Moreover, the series* $\sum_{n=0}^{+\infty}\Phi_{\mathbf{T}}^{(n)}\left(\mathbf{B}\right)$ *and* $\sum_{n=0}^{+\infty}\Phi_{\mathbf{T}^H}^{(n)}\left(\mathbf{B}\right)$ *are finite.*

Proof. If $\mathbf{T}$ is a solution to the canonical equation, we must have

$$\mathrm{Im}\mathbf{T} = \frac{\mathbf{T}-\mathbf{T}^H}{2\mathrm{i}} = \mathrm{Im}z\mathbf{T}\mathbf{T}^H + \Phi_{\mathbf{T}}\left(\mathrm{Im}\mathbf{T}\right) \tag{5.16}$$

or, equivalently,

$$\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} = \mathbf{T}\mathbf{T}^H + \Phi_{\mathbf{T}}\left(\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z}\right) \tag{5.17}$$

if $\mathrm{Im}(z) \neq 0$. If z belongs to $\mathbb{R}^{-*}$, then $\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z}$ should be interpreted as positive matrix $\mathbf{T}'(z) = \int_{\mathbb{R}^+} \frac{d\boldsymbol{\mu}(\lambda)}{(\lambda - z)^2}$, and the reader may verify that the following arguments are still valid. Iterating the above relationship, we see that for any $n \in \mathbb{N}$,

$$\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} = \sum_{k=0}^{n} \Phi_{\mathbf{T}}^{(k)} \left(\mathbf{T}\mathbf{T}^H \right) + \Phi_{\mathbf{T}}^{(n+1)} \left(\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} \right).$$

Since $\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} \geq 0$, we have $\Phi_{\mathbf{T}}^{(n+1)} \left(\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} \right) \geq 0$. On the other hand, we also have $\Phi_{\mathbf{T}}^{(k)} \left(\mathbf{T}\mathbf{T}^H \right) \geq 0$ and therefore it holds that for each n,

$$\sum_{k=0}^{n} \Phi_{\mathbf{T}}^{(k)} \left(\mathbf{T}\mathbf{T}^H \right) \leq \frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z}.$$

The series $\sum_{k=0}^{+\infty} \Phi_{\mathbf{T}}^{(k)} \left(\mathbf{T}\mathbf{T}^H \right)$ is thus convergent and we must have $\Phi_{\mathbf{T}}^{(n)} \left(\mathbf{T}\mathbf{T}^H \right) \to \mathbf{0}$ as $n \to \infty$. Since matrix $\mathbf{T}$ is invertible, $\mathbf{T}^H\mathbf{T} > \alpha(z)\mathbf{I}$ where $\alpha(z) > 0$. Therefore, $\alpha(z)\, \Phi_{\mathbf{T}}^{(n)}(\mathbf{I}) \leq \Phi_{\mathbf{T}}^{(n)} \left(\mathbf{T}\mathbf{T}^H \right)$, which implies that $\Phi_{\mathbf{T}}^{(n)}(\mathbf{I})$ converges towards 0 and that $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}}^{(n)}(\mathbf{I}) < +\infty$. Now, consider a general positive semidefinite $\mathbf{B}$. Then, $\mathbf{B} \leq \|\mathbf{B}\|\mathbf{I}$ and $\Phi_{\mathbf{T}}^{(n)}(\mathbf{B}) \leq \|\mathbf{B}\|\Phi_{\mathbf{T}}^{(n)}(\mathbf{I})$. Hence, it holds that $\Phi_{\mathbf{T}}^{(n)}(\mathbf{B}) \to 0$ and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}}^{(n)}(\mathbf{B}) < +\infty$. In particular, $\Phi_{\mathbf{T}}^{(n+1)} \left(\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} \right) \to 0$ as $n \to \infty$ and

$$\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} = \sum_{k=0}^{+\infty} \Phi_{\mathbf{T}}^{(k)} \left(\mathbf{T}\mathbf{T}^H \right).$$

In order to establish (5.15), we use the observation that

$$\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} = \mathbf{T}^H\mathbf{T} + \Phi_{\mathbf{T}^H} \left(\frac{\mathrm{Im}\mathbf{T}}{\mathrm{Im}z} \right)$$

and use the same arguments as above. □

Remark 5.5. In the above analysis, L, M, N are fixed parameters. Therefore, $\alpha(z)$ a priori depends on L, M, N as well as the norms of the series $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}}^{(n)}(\mathbf{I})$. In the following, a more precise analysis will be needed, and it will be important to show that such an $\alpha(z)$ can be chosen independent from L, M, N, and that

$$\sup_N \| \sum_{n=0}^{+\infty} \Phi_{\mathbf{T}}^{(n)}(\mathbf{I}) \| < +\infty.$$

6. Convergence towards the deterministic equivalent

If $(\mathbf{A}_N)_{N \geq 1}$ is a sequence of deterministic uniformly bounded $ML \times ML$ matrices, Lemma 3.1 implies that the rate of convergence of $\mathrm{var}\left(\frac{1}{ML} \mathrm{tr}\left[\mathbf{A}_N \mathbf{Q}(z) \right] \right)$ is $\mathcal{O}(\frac{1}{MN})$.

In the absence of extra assumptions on M (e.g., $M = \mathcal{O}(N^\epsilon)$ for $\epsilon > 0$), this does not allow us to conclude that

$$\frac{1}{ML}\mathrm{tr}\left(\mathbf{A}_N\left[\mathbf{Q}(z) - \mathbb{E}(\mathbf{Q}(z))\right]\right) \to 0 \tag{6.1}$$

almost surely. In order to obtain the almost sure convergence, we use the identity

$$\begin{aligned}&\mathbb{E}\left|\frac{1}{ML}\mathrm{tr}\left(\mathbf{A}_N\mathbf{Q}^\circ(z)\right)\right|^4\\ &= \left|\mathbb{E}\left(\frac{1}{ML}\mathrm{tr}(\mathbf{A}_N\mathbf{Q}^\circ(z))\right)^2\right|^2 + \mathrm{var}\left[\left(\frac{1}{ML}\mathrm{tr}(\mathbf{A}_N\mathbf{Q}^\circ(z))\right)^2\right],\end{aligned}$$

note that

$$\left|\mathbb{E}\left(\frac{1}{ML}\mathrm{tr}(\mathbf{A}_N\mathbf{Q}^\circ(z))\right)^2\right|^2 \leq \left[\mathrm{var}\left(\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\mathbf{Q}(z)\right]\right)\right]^2 \leq C(z)\frac{1}{(MN)^2},$$

and establish using the Nash–Poincaré inequality that

$$\mathrm{var}\left[\left(\frac{1}{ML}\mathrm{tr}(\mathbf{A}_N\mathbf{Q}^\circ(z))\right)^2\right] \leq C(z)\frac{1}{(MN)^2}$$

(the proof is straighforward and thus omitted). Markov inequality and Borel–Cantelli's lemma immediately imply that (6.1) holds, and that

$$\frac{1}{ML}\mathrm{tr}\left(\mathbf{A}_N\left[\mathbf{Q}(z) - \mathbb{E}(\mathbf{Q}(z))\right]\right) = \mathcal{O}_P\left(\frac{1}{\sqrt{MN}}\right).$$

In the following, we study the behaviour of $\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\left(\mathbb{E}(\mathbf{Q}(z)) - \mathbf{T}(z)\right)\right]$. In this section, we first establish that for each sequence of deterministic uniformly bounded $ML \times ML$ matrices $(\mathbf{A}_N)_{N\geq 1}$, it holds that

$$\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\left(\mathbb{E}\mathbf{Q}(z) - \mathbf{T}(z)\right)\right] \to 0 \tag{6.2}$$

for each $z \in \mathbb{C}\setminus\mathbb{R}^+$, a property which, by virtue of Proposition 4.3, is equivalent to

$$\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\left(\mathbf{R}(z) - \mathbf{T}(z)\right)\right] \to 0 \tag{6.3}$$

for each $z \in \mathbb{C}\setminus\mathbb{R}^+$. However, (6.2) does not provide any information on the rate of convergence. Under the extra-assumption that

$$\frac{L^{3/2}}{MN} \to 0 \tag{6.4}$$

we establish that

$$\left|\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\left(\mathbb{E}(\mathbf{Q}(z)) - \mathbf{T}(z)\right)\right]\right| \leq C(z)\frac{L}{MN} \tag{6.5}$$

or, equivalently, that

$$\left|\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\left(\mathbf{R}(z)-\mathbf{T}(z)\right)\right]\right| \leq C(z)\frac{L}{MN} \tag{6.6}$$

when z belongs to a set E_N defined by

$$E_N = \{z \in \mathbb{C}\setminus\mathbb{R}^+, \frac{L^{3/2}}{MN}P_1(|z|)P_2(1/\delta_z) < 1\}$$

where P_1 and P_2 are some nice polynomials. When (6.4) holds, each element $z \in \mathbb{C}\setminus\mathbb{R}^+$ belongs to E_N for N greater than a certain integer depending on z. Therefore, (6.6) implies that the rate of convergence of $\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}_N\left(\mathbb{E}(\mathbf{Q}(z))-\mathbf{T}(z)\right)\right]$ towards 0 is $\mathcal{O}(\frac{L}{MN})$ for each $z \in \mathbb{C}\setminus\mathbb{R}^+$.

6.1. Proof of (6.3)

In order to simplify the notations, matrix $\mathbf{A}_N$ is denoted by $\mathbf{A}$. Writing $\mathbf{R}(z)-\mathbf{T}(z)$ as $\mathbf{R}(z)\left(\mathbf{R}^{-1}(z)-\mathbf{T}^{-1}(z)\right)\mathbf{T}(z)$ and $\widetilde{\mathbf{R}}(z)-\widetilde{\mathbf{T}}(z) = \widetilde{\mathbf{T}}(z)\left(\widetilde{\mathbf{T}}^{-1}(z)-\widetilde{\mathbf{R}}^{-1}(z)\right)\widetilde{\mathbf{R}}(z)$, we obtain immediately that

$$\mathbf{R}(z)-\mathbf{T}(z) = z\mathbf{R}(z)\Psi\left(\left(\widetilde{\mathbf{R}}(z)-\widetilde{\mathbf{T}}(z)\right)^T\right)\mathbf{T}(z), \tag{6.7}$$

$$\left(\widetilde{\mathbf{R}}(z)-\widetilde{\mathbf{T}}(z)\right)^T = zc_N\widetilde{\mathbf{R}}^T(z)\overline{\Psi}\left(\mathbb{E}\mathbf{Q}(z)-\mathbf{T}(z)\right)\widetilde{\mathbf{T}}^T(z). \tag{6.8}$$

We introduce the linear operator Φ_1 defined on the set of all $ML\times ML$ matrices by

$$\Phi_1(\mathbf{X}) = z^2\, c_N\, \mathbf{R}(z)\Psi\left(\widetilde{\mathbf{R}}^T(z)\overline{\Psi}(\mathbf{X})\widetilde{\mathbf{T}}^T(z)\right)\mathbf{T}(z). \tag{6.9}$$

The operator Φ_1 is clearly obtained from operator Φ_0 defined by (5.8) by replacing matrices $\mathbf{S}(z)$ and $\widetilde{\mathbf{S}}(z)$ by matrices $\mathbf{R}(z)$ and $\widetilde{\mathbf{R}}(z)$. Then, it holds that

$$\mathbf{R}(z)-\mathbf{T}(z) = \Phi_1\left(\mathbf{R}(z)-\mathbf{T}(z)\right)+\Phi_1\left(\mathbf{\Delta}(z)\right). \tag{6.10}$$

Thus, matrix $\mathbf{R}(z)-\mathbf{T}(z)$ can be interpreted as the solution of the linear equation (6.10). Therefore, in some sense, showing that $\mathbf{R}(z)-\mathbf{T}(z)$ converges towards 0 can be proved by showing that operator $\mathbf{I}-\mathbf{\Phi}_1$ is invertible, and that the action of its inverse on $\Phi_1\left(\mathbf{\Delta}(z)\right)$ still converges towards 0. In this subsection, we implicitely prove that Φ_1 is a contractive operator for z well chosen, obtain that $\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}\left(\mathbf{R}(z)-\mathbf{T}(z)\right)\right] = \mathcal{O}(\frac{L}{MN})$ for such z, and use Montel's theorem to conclude that (6.3) holds for each $z \in \mathbb{C}\setminus\mathbb{R}^+$.

Using (2.8), we remark that for each $ML\times ML$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\frac{1}{ML}\mathrm{tr}(\Phi_1(\mathbf{B})\mathbf{A})) = \frac{1}{ML}\mathrm{tr}(\mathbf{B}\Phi_1^t(\mathbf{A})) \tag{6.11}$$

where Φ_1^t represents the linear operator defined on the $ML\times ML$ matrices by

$$\Phi_1^t(\mathbf{A}) = z^2 c_N\,\Psi\left(\widetilde{\mathbf{T}}^T\overline{\Psi}(\mathbf{TAR})\widetilde{\mathbf{R}}^T\right). \tag{6.12}$$

We note that operator Φ_1^t is related to the adjoint Φ_1^* of Φ_1 w.r.t. the scalar product $< \mathbf{A}, \mathbf{B} >= \frac{1}{ML}\mathrm{tr}(\mathbf{A}\mathbf{B}^*)$ through the relation $\Phi_1^t(\mathbf{A}) = \left(\Phi_1^*(\mathbf{A}^H)\right)^H$.

If $\mathbf{A}$ is a $ML \times ML$ deterministic matrix, (6.10) leads to

$$\frac{1}{ML}\mathrm{tr}\left[\mathbf{A}\left(\mathbf{R}(z) - \mathbf{T}(z)\right)\right] = \frac{1}{ML}\mathrm{tr}\left[\Phi_1^t(\mathbf{A})\left(\mathbf{R}(z) - \mathbf{T}(z)\right)\right] + \frac{1}{ML}\mathrm{tr}\left[\Phi_1^t(\mathbf{A})\ \mathbf{\Delta}(z)\right]. \tag{6.13}$$

Obviously, from the properties of the operators $\Psi(\cdot)$ and $\overline{\Psi}(\cdot)$ we can write

$$\begin{aligned}\left\|\Phi_1^t(\mathbf{A})\right\| &\leq |z|^2 c_N \sup_{m,M,\nu} \left|\mathcal{S}_m(\nu)\right|^2 \|\mathbf{R}(z)\| \left\|\widetilde{\mathbf{R}}(z)\right\| \|\mathbf{A}\| \|\mathbf{T}(z)\| \left\|\widetilde{\mathbf{T}}(z)\right\| \\ &\leq \frac{|z|^2}{(\delta_z)^4} c_N \sup_{m,M,\nu} \left|\mathcal{S}_m(\nu)\right|^2 \|\mathbf{A}\|.\end{aligned}$$

Let us consider the domain

$$\mathcal{D} = \left\{ z \in \mathbb{C} \setminus \mathbb{R}^+ : \frac{|z|^2}{(\delta_z)^4} \sup_N c_N \sup_{m,M,\nu} \left|\mathcal{S}_m(\nu)\right|^2 < \frac{1}{2} \right\}$$

and define $\alpha(z)$ as

$$\alpha(z) = \sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{ML}\mathrm{tr}\left(\left(\mathbf{R}(z) - \mathbf{T}(z)\right)\mathbf{B}\right) \right|.$$

Then, we establish that if $z \in \mathcal{D}$, then $\alpha(z) \leq C(z)\frac{L}{MN}$. For this, we consider $\mathbf{A}$ such that $\|\mathbf{A}\| \leq 1$. Using that

$$\left| \frac{1}{ML}\mathrm{tr}\left[\Phi_1^t(\mathbf{A})\left(\mathbf{R}(z) - \mathbf{T}(z)\right)\right] \right| \leq \alpha(z) \left\|\Phi_1^t(\mathbf{A})\right\|$$

and that $\|\Phi_1(\mathbf{A})\| \leq 1/2$ if $z \in \mathcal{D}$, we deduce from (6.13) that

$$\alpha(z) \leq \frac{\alpha(z)}{2} + \sup_{\|\mathbf{A}\| \leq 1} \left| \frac{1}{ML}\mathrm{tr}\left(\Phi_1^t(\mathbf{A})\mathbf{\Delta}(z)\right) \right|.$$

As $\|\Phi_1^t(\mathbf{A})\| \leq 1/2$ if $\|\mathbf{A}\| \leq 1$, (4.14) implies that

$$\sup_{\|\mathbf{A}\| \leq 1} \left| \frac{1}{ML}\mathrm{tr}\left(\Phi_1^t(\mathbf{A})\mathbf{\Delta}(z)\right) \right| \leq C(z)\frac{L}{MN}.$$

This implies that $\alpha(z) \leq C(z)\frac{L}{MN}$ for each $z \in \mathcal{D}$, and that for each uniformly bounded sequence of $ML \times ML$ matrices $\mathbf{A}_N$, (6.3) holds on $\mathcal{D}$. Montel's theorem immediately implies that (6.3) also holds for each $z \in \mathbb{C} \setminus \mathbb{R}^+$.

6.2. Proof of (6.5)

We now establish (6.5) for each $z \in \mathbb{C} \setminus \mathbb{R}^+$ under Assumption (6.4). For this, we establish that the linear equation (6.10) can be solved for each $z \in \mathbb{C} \setminus \mathbb{R}^+$. For this, we first prove the following proposition.

Proposition 6.1. *There exist two nice polynomials P_1 and P_2 (see Definition 1.2) such that for each $ML \times ML$ matrix $\mathbf{X}$, the series $\sum_{n=0}^{\infty} \Phi_1^{(n)}(\mathbf{X})$ is convergent when z belongs to the set E_N defined by*

$$E_N = \{z \in \mathbb{C} \setminus \mathbb{R}^+, \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\delta_z) < 1\}. \tag{6.14}$$

In order to establish Proposition 6.1, we first note that for each matrix $\mathbf{X}$, it holds that

$$\Phi_1^{(n)}(\mathbf{X}) \left(\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I})\right)^{-1} \left(\Phi_1^{(n)}(\mathbf{X})\right)^H \leq \Phi_{\mathbf{R}}^{(n)}(\mathbf{X}\mathbf{X}^H) \tag{6.15}$$

where $\Phi_{\mathbf{T}^H}(\cdot)$ is defined in (5.10) and where $\Phi_{\mathbf{R}}(\cdot)$ is as in (5.9) replacing $\mathbf{S}$ with $\mathbf{R}$.This inequality is proved in the same way as (5.13). It has already been proved that $\sum_{n=0} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I})$ is convergent. Following the proof of the uniqueness in Proposition 5.1 , we will obtain the convergence of the series $\sum_{n=0}^{\infty} \Phi_1^{(n)}(\mathbf{X})$, i.e., that

$$\sum_{n=0}^{+\infty} \left| \mathbf{a}^H \Phi_1^{(n)}(\mathbf{X}) \mathbf{b} \right| < +\infty,$$

if we are able to establish that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}}^{(n)}(\mathbf{X}\mathbf{X}^H) < +\infty. \tag{6.16}$$

When $\operatorname{Im} z \neq 0$, we can write

$$\frac{\operatorname{Im}(\mathbf{R})}{\operatorname{Im}(z)} = \mathbf{R}\mathbf{R}^H + \Phi_{\mathbf{R}} \left(\frac{\operatorname{Im}(\mathbb{E}(\mathbf{Q}))}{\operatorname{Im}(z)} \right).$$

Therefore, it holds that

$$\frac{\operatorname{Im}(\mathbb{E}(\mathbf{Q}))}{\operatorname{Im}(z)} = \mathbf{R}\mathbf{R}^H + \frac{\operatorname{Im}(\mathbf{\Delta})}{\operatorname{Im}(z)} + \Phi_{\mathbf{R}} \left(\frac{\operatorname{Im}(\mathbb{E}(\mathbf{Q}))}{\operatorname{Im}(z)} \right). \tag{6.17}$$

When $z \in \mathbf{R}^{-*}$, we can still interpret $\frac{\operatorname{Im}(\mathbf{R})}{\operatorname{Im}(z)}$ and $\frac{\operatorname{Im}(\mathbb{E}(\mathbf{Q}))}{\operatorname{Im}(z)}$ as $\mathbf{R}'(z)$ and $\mathbb{E}(\mathbf{Q}'(z))$, and the following reasoning holds as well. In order to use the ideas of the proof of the uniqueness in Proposition 5.1, matrix $\mathbf{R}\mathbf{R}^H + \frac{\operatorname{Im}(\mathbf{\Delta})}{\operatorname{Im}(z)}$ should be positive. By (4.4), matrix $\mathbf{R}\mathbf{R}^H$ verifies $\mathbf{R}\mathbf{R}^H \geq \frac{\delta_z^2}{Q_2(|z|)}\mathbf{I}$ for some nice polynomial Q_2. In order to guarantee the positivity of $\mathbf{R}\mathbf{R}^H + \frac{\operatorname{Im}(\mathbf{\Delta})}{\operatorname{Im}(z)}$ on a large subset of $\mathbb{C} \setminus \mathbb{R}^+$, condition $\| \frac{\operatorname{Im}(\mathbf{\Delta})}{\operatorname{Im}(z)} \| \to 0$ should hold. However, it can be shown that the rate of convergence of $\| \frac{\operatorname{Im}(\mathbf{\Delta})}{\operatorname{Im}(z)} \|$ is $\mathcal{O}\left((L/M^3)^{1/2}\right)$, assuming that $L/M^3 \to 0$ is a stronger condition than Assumption (6.4) which is equivalent to $L/M^4 \to 0$. Therefore, we have to modify the proof of the uniqueness in Proposition 5.1. Instead of considering Equation (6.17), we consider

$$\Phi_{\mathbf{R}} \left(\frac{\operatorname{Im}(\mathbb{E}(\mathbf{Q}))}{\operatorname{Im}(z)} \right) = \Phi_{\mathbf{R}} \left(\mathbf{R}\mathbf{R}^H + \frac{\operatorname{Im}(\mathbf{\Delta})}{\operatorname{Im}(z)} \right) + \Phi_{\mathbf{R}}^{(2)} \left(\frac{\operatorname{Im}(\mathbb{E}(\mathbf{Q}))}{\operatorname{Im}(z)} \right). \tag{6.18}$$

This time, we will see that $\Phi_{\mathbf{R}}\left(\mathbf{R}\mathbf{R}^* + \frac{\mathrm{Im}(\mathbf{\Delta})}{\mathrm{Im}(z)}\right) \geq \frac{\delta_z^2}{Q_2(|z|)}\mathbf{I}$ when z belongs to a set E_N defined by (6.14), and this property allows us to prove (6.16) if $z \in E_N$. In order to establish this, we first state the following lemma.

Lemma 6.2. *It holds that*

$$\|\overline{\Psi}\left(\frac{\mathrm{Im}(\mathbf{\Delta})}{\mathrm{Im}(z)}\right)\| \leq C(z)\frac{L^{3/2}}{MN}. \tag{6.19}$$

Proof. Following the proof of Corollary 4.4, it is sufficient to establish that if $(\mathbf{b}_{1,N})_{N\geq 1}$ and $(\mathbf{b}_{2,N})_{N\geq 1}$ are two sequences of L dimensional vectors such that $\sup_N \|\mathbf{b}_{i,N}\| < b < +\infty$ for $i = 1, 2$, and if $((d_{m,N})_{m=1,\ldots,M})_{N\geq 1}$ are deterministic complex number verifying $\sup_{N,m} |d_{m,N}| < d < +\infty$, then it holds that

$$\left|\mathbf{b}_{1,N}^H\left(\frac{1}{M}\sum_{m=1}^{M} d_{m,N}\frac{\mathrm{Im}\left(\mathbf{\Delta}^{m,m}\right)}{\mathrm{Im}(z)}\right)\mathbf{b}_{2,N}\right| \leq d\, b^2\, P_1(|z|)P_2(1/\delta_z)\frac{L^{3/2}}{MN} \tag{6.20}$$

for some nice polynomials P_1 and P_2. (6.20) can be established by adapting in a straightforward way the arguments of the proof of Lemma B-1 of [11]. This completes the proof of the lemma. □

Using the definition of operator $\Phi_{\mathbf{R}}$ and the properties of $\mathbf{R}$ and $\widetilde{\mathbf{R}}$, we deduce from Lemma 6.2 that

$$\left\|\Phi_{\mathbf{R}}\left(\frac{\mathrm{Im}(\mathbf{\Delta})}{\mathrm{Im}(z)}\right)\right\| \leq C(z)\frac{L^{3/2}}{MN}$$

when $z \in \mathbb{E}_N$. As $\mathbf{R}(z)\mathbf{R}^H(z) \geq \frac{\delta_z^2}{Q_2(|z|)}\mathbf{I}_{ML}$, it holds that $\overline{\Psi}(\mathbf{R}(z)\mathbf{R}^H(z)) \geq \frac{\delta_z^2}{Q_2(|z|)}\overline{\Psi}(\mathbf{I}_{ML})$. Since all spectral densities $(\mathcal{S}_m)$ verify (1.9), it is clear that $\overline{\Psi}(\mathbf{I}_{ML}) \geq C\,\mathbf{I}_N$ for some nice constant C. Therefore, it appears that

$$\overline{\Psi}(\mathbf{R}(z)\mathbf{R}^H(z)) \geq C\,\frac{\delta_z^2}{Q_2(|z|)}\mathbf{I}_N.$$

Lemma 6.2 thus implies that

$$\overline{\Psi}\left(\mathbf{R}(z)\mathbf{R}(z)^H + \frac{\mathrm{Im}(\mathbf{\Delta})}{\mathrm{Im}(z)}\right) \geq C/2\,\frac{\delta_z^2}{Q_2(|z|)}\mathbf{I}_N$$

on the set E_N defined by

$$E_N = \left\{z \in \mathbb{C}\setminus\mathbb{R}^+, C(z)\frac{L^{3/2}}{MN} \leq C/2\frac{\delta_z^2}{Q_2(|z|)}\right\}$$

which can be written as in (6.14). Since matrix $\widetilde{\mathbf{R}}^T(z)\widetilde{\mathbf{R}}^*(z)$ is also greater than $\frac{\delta_z^2}{Q_2(|z|)}$ for some nice polynomial Q_2 (see (4.5)), we eventually obtain that

$$\Phi_{\mathbf{R}}\left(\mathbf{R}(z)\mathbf{R}(z)^H + \frac{\mathrm{Im}(\mathbf{\Delta})}{\mathrm{Im}(z)}\right) \geq \frac{\delta_z^6}{Q_2(|z|)}\mathbf{I}_{ML} \tag{6.21}$$

for each $z \in \mathbb{E}_N$. Using the same arguments than in the proof of the uniqueness in Proposition 5.1, we obtain that $\Phi_{\mathbf{R}}^{(n)}\left(\mathbf{R}(z)\mathbf{R}^H(z) + \frac{\mathrm{Im}(\mathbf{\Delta})}{\mathrm{Im}(z)}\right) \to 0$ when $n \to 0$, and that

$$\Phi_{\mathbf{R}}\left(\frac{\mathrm{Im}(\mathbb{E}(\mathbf{Q}))}{\mathrm{Im}(z)}\right) = \sum_{n=0}^{+\infty} \Phi_{\mathbf{R}}^{(n)}\left(\Phi_{\mathbf{R}}\left(\mathbf{R}(z)\mathbf{R}^H(z) + \frac{\mathrm{Im}(\mathbf{\Delta})}{\mathrm{Im}(z)}\right)\right)$$

when $z \in E_N$. Using (6.21) as well as

$$\Phi_{\mathbf{R}}\left(\frac{\mathrm{Im}(\mathbb{E}(\mathbf{Q}))}{\mathrm{Im}(z)}\right) \leq C(z)\,\mathbf{I},$$

we eventually obtain that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}}^{(n)}(\mathbf{I}) \leq C(z)\,\mathbf{I}$$

when $z \in E_N$. Therefore, for each $ML \times ML$ matrix $\mathbf{X}$, it holds that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}}^{(n)}(\mathbf{X}\mathbf{X}^*) \leq \|\mathbf{X}\|^2 \sum_{n=0}^{+\infty} \Phi_{\mathbf{R}}^{(n)}(\mathbf{I}) \leq C(z)\,\|\mathbf{X}\|^2\,\mathbf{I} < +\infty.$$

In order to complete the proof of Proposition 6.1 , we just follow the proof of unicity of Proposition 5.1. We express $\mathbf{a}^H\Phi_1(\mathbf{X})\mathbf{b}$ as

$$\mathbf{a}^H\Phi_1^{(n)}(\mathbf{X})\mathbf{b} = \mathbf{a}^H\Phi_1^{(n)}(\mathbf{X})\left[\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I})\right]^{-1/2}\left[\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I})\right]^{1/2}\mathbf{b}$$

use the Schwartz inequality as well as (6.15), and obtain that

$$|\mathbf{a}^H\Phi_1^{(n)}(\mathbf{X})\mathbf{b}| \leq \left(\mathbf{a}^H\Phi_{\mathbf{R}}^{(n)}(\mathbf{X}\mathbf{X}^*)\mathbf{a}\right)^{1/2}\left(\mathbf{b}^H\Phi_{\mathbf{T}^\mathbf{H}}^{(n)}(\mathbf{I})\mathbf{b}\right)^{1/2}$$

and that

$$\sum_{n=0}^{+\infty} |\mathbf{a}^H\Phi_1^{(n)}(\mathbf{X})\mathbf{b}| \leq \left(\sum_{n=0}^{+\infty} \mathbf{a}^H\Phi_{\mathbf{R}}^{(n)}(\mathbf{X}\mathbf{X}^*)\mathbf{a}\right)^{1/2}\left(\sum_{n=0}^{+\infty} \mathbf{b}^H\Phi_{\mathbf{T}^\mathbf{H}}^{(n)}(\mathbf{I})\mathbf{b}\right)^{1/2} < +\infty$$

as expected.

We are now in position to complete the proof of (6.6). For this, we consider a uniformly bounded sequence of $ML \times ML$ matrices $\mathbf{A}_N$ and evaluate $\frac{1}{ML}\mathrm{tr}\left(\mathbf{A}_N(\mathbf{R}(z) - \mathbf{T}(z))\right)$. As previously, matrix $\mathbf{A}_N$ is denoted by $\mathbf{A}$ in order to short the notations. For this, we take Eq. (6.10) as a starting point, and assume that z belongs to the set E_N defined by (6.14). As $z \in E_N$, the series

$$\sum_{n=0}^{+\infty} \Phi_1^{(n)}\left(\Phi_1(\mathbf{\Delta})\right)$$

is convergent and

$$\mathbf{R} - \mathbf{T} = \sum_{n=0}^{+\infty} \Phi_1^{(n)}\left(\Phi_1(\mathbf{\Delta})\right). \tag{6.22}$$

Therefore,

$$\frac{1}{ML}\mathrm{tr}\left(\mathbf{A}(\mathbf{R}-\mathbf{T})\right) = \sum_{n=0}^{+\infty} \frac{1}{ML}\mathrm{tr}\left(\mathbf{A}\Phi_1^{(n+1)}(\mathbf{\Delta})\right)$$

or equivalently,

$$\frac{1}{ML}\mathrm{tr}\left(\mathbf{A}(\mathbf{R}-\mathbf{T})\right) = \sum_{n=0}^{+\infty} \frac{1}{ML}\mathrm{tr}\left(\mathbf{\Delta}\Phi_1^{t(n+1)}(\mathbf{A})\right)$$

where Φ_1^t is the operator defined by (6.12). The strategy of the proof consists in showing that the series $\sum_{n=0}^{\infty}\Phi_1^{t(n+1)}(\mathbf{A})$ is convergent, and that

$$\sup_N \left\| \sum_{n=0}^{\infty} \Phi_1^{t(n+1)}(\mathbf{A}) \right\| < +\infty. \tag{6.23}$$

If (6.23) holds, (4.14) will imply that

$$\left| \frac{1}{ML}\mathrm{tr}\left(\left(\sum_{n=0}^{\infty} \Phi_1^{t(n+1)}(\mathbf{A}) \right) \mathbf{\Delta} \right) \right| \leq C(z)\,\frac{L}{MN}$$

if $z \in E_N$.

In order to establish the convergence of the series as well as (6.23), we use the following inequality: for each $ML \times ML$ deterministic matrix $\mathbf{A}$, it holds that

$$\Phi_1^{t(n)}(\mathbf{A}) \left[\Phi_{\mathbf{T}}^{t(n)}(\mathbf{I})\right]^{-1} \left(\Phi_1^{t(n)}(\mathbf{A})\right)^H \leq \Phi_{\mathbf{R}^H}^{t(n)}(\mathbf{A}^H\mathbf{A}) \tag{6.24}$$

where the operators $\Phi_{\mathbf{T}}^t$ and $\Phi_{\mathbf{R}^H}^t$ are defined by

$$\begin{aligned} \Phi_{\mathbf{T}}^t(\mathbf{X}) &= c_N\,|z|^2\,\Psi\left(\widetilde{\mathbf{T}}^T\overline{\mathbf{\Psi}}(\mathbf{T}\mathbf{X}\mathbf{T}^H)\widetilde{\mathbf{T}}^*\right), \\ \Phi_{\mathbf{R}^H}^t(\mathbf{X}) &= c_N\,|z|^2\,\Psi\left(\widetilde{\mathbf{R}}^*\overline{\Psi}(\mathbf{R}^H\mathbf{X}\mathbf{R})\widetilde{\mathbf{R}}^T\right) \end{aligned} \tag{6.25}$$

The proof is similar to the proof of (6.15), and is thus omitted. We remark that operators $\Phi_{\mathbf{T}}$ and $\Phi_{\mathbf{T}}^t$ are linked by the formula:

$$\Phi_{\mathbf{T}}^t(\mathbf{X}) = \mathbf{T}^{-1}\Phi_{\mathbf{T}}(\mathbf{T}\mathbf{X}\mathbf{T}^H)\mathbf{T}^{-H} \tag{6.26}$$

which implies that

$$\Phi_{\mathbf{T}}^{t(n)}(\mathbf{X}) = \mathbf{T}^{-1}\Phi_{\mathbf{T}}^{(n)}(\mathbf{T}\mathbf{X}\mathbf{T}^H)\mathbf{T}^{-H}. \tag{6.27}$$

Since $\sum_{n=0}^{+\infty}\Phi_{\mathbf{T}}^{(n)}(\mathbf{I}) < +\infty$, it holds that

$$\sum_{n=0}^{+\infty}\Phi_{\mathbf{T}}^{t(n)}(\mathbf{I}) \leq \|\mathbf{T}^{-1}\|^2\|\mathbf{T}\|^2 \sum_{n=0}^{+\infty}\Phi_{\mathbf{T}}^{(n)}(\mathbf{I}) < +\infty.$$

Moreover, we have already shown that $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}}^{(n)}(\mathbf{I}) \leq C(z)\ \mathbf{I}$, and that $\mathbf{T}\mathbf{T}^H \geq \frac{\delta_z^2}{Q_2(|z|)}\mathbf{I}$, or, equivalently, that $\|\mathbf{T}^{-1}\|^2 \leq C(z)$. Therefore, it holds that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}}^{t(n)}(\mathbf{I}) \leq C(z)\ \mathbf{I}. \tag{6.28}$$

In order to control the series $\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}^H}^{(n)}(\mathbf{A}^H\mathbf{A})$, we remark that

$$\Phi_{\mathbf{R}^H}^{t}(\mathbf{X}) = \mathbf{R}^{-H}\Phi_{\mathbf{R}^H}(\mathbf{R}^H\mathbf{X}\mathbf{R})\mathbf{R}^{-1} \tag{6.29}$$

which implies that

$$\Phi_{\mathbf{R}^H}^{t(n)}(\mathbf{X}) = \mathbf{R}^{-H}\Phi_{\mathbf{R}^H}^{(n)}(\mathbf{R}^H\mathbf{X}\mathbf{R})\mathbf{R}^{-1}. \tag{6.30}$$

Using the same kind of arguments as above, we obtain that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}^H}^{t(n)}(\mathbf{A}^*\mathbf{A}) \leq C(z)\ \mathbf{I} \tag{6.31}$$

if z belongs to a set E_N defined by (6.14). Noting that

$$\left|\mathbf{a}^H \sum_{n=0}^{\infty} \Phi_1^{t(n)}(\mathbf{A})\mathbf{b}\right| \leq \left[\mathbf{a}^H \left(\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}}^{t(n)}(\mathbf{I})\right)\mathbf{a}\right]^{1/2} \left[\mathbf{b}^H \left(\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}^H}^{t(n)}(\mathbf{A}^*\mathbf{A})\right)\mathbf{b}\right]^{1/2}$$

we obtain that

$$\|\sum_{n=0}^{\infty} \Phi_1^{t(n)}(\mathbf{A})\| \leq C(z)$$

as soon as $z \in E_N$. This completes the proof of (6.6).

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Bernstein-type Inequalities for Bicomplex Polynomials

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This paper is dedicated to our friend and colleague Daniel Alpay on the occasion of his 60th birthday.

Abstract. This paper considers the well-known Bernstein and Erdős–Lax inequalities in the case of bicomplex polynomials. We shall prove that the validity of these inequalities depends on the norm in use and we consider the cases of the Euclidean, Lie, dual Lie and hyperbolic-valued norms. In particular, we show that in the case of the Euclidean norm the inequalities holds keeping the same relation with the degree of the polynomial that holds in the classical complex case, but we have to enlarge the radius of the ball. In the case of the dual Lie norm both the relation with the degree and the radius of the ball have to be changed. Finally, we prove that the exact analogs of the two inequalities hold when considering the Lie norm and the hyperbolic-valued norm. In the case of these two norms we also show the validity of the maximum modulus principle for bicomplex holomorphic functions.

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1. Introduction

This paper analyzes Bernstein-type inequalities for bicomplex polynomials. The Bernstein inequality and its various generalizations are very well known for complex polynomials, and have important applications, for example, in approximation theory. Bernstein-type inequalities relate the maximum of polynomials and their derivatives on the unit ball via their degree as follows:

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Theorem 1.1 (Bernstein [2]**, Riesz** [18]**).** *If $P(z)$ is a polynomial of degree n with complex coefficients, then*

$$\max_{|z|\leq 1} |P'| \leq n \cdot \max_{|z|\leq 1} |P|.$$

Remark 1.2. In the complex case, one can introduce the notion of the norm of a polynomial as its maximum on the disc of radius 1, i.e., for a polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ of degree n, where $a_k \in \mathbb{C}$ for any $0 \leq k \leq n$, we define:

$$\|P\| := \max_{|z|\leq 1} |P(z)| = \max_{|z|=1} |P(z)|.$$

With this notation, Bernstein's inequality can be stated as: if $P(z)$ is a polynomial of degree n with complex coefficients, then $\|P'\| \leq n \cdot \|P\|$.

Since for our setting we will need to use maxima on disks of different radii, we will only use the maximum from now on, for simplicity of presentation.

A refinement of Bernstein's inequality, conjectured by Erdős and later proved by Lax [10], can be provided under assumptions on the location of the zeros of a polynomial and states the following:

Theorem 1.3 (Erdős–Lax). *If $P(z)$ is a polynomial of degree n with complex coefficients having no zeros in $|z| < 1$, then*

$$\max_{|z|\leq 1} |P'(z)| \leq \frac{n}{2} \cdot \max_{|z|\leq 1} |P(z)|.$$

The inequality is sharp and equality holds if P has all of its zeros on $|z| = 1$.

For polynomials of a complex variable, we also have the following more general result, due to Malik [14], which will be useful in proving some of our results:

Theorem 1.4 (Generalized Bernstein Theorem). *If $P(z)$ is a polynomial of degree n having no zeros in the disk $|z| < K$, where $K \geq 1$, then*

$$\max_{|z|\leq 1} |P'(z)| \leq \frac{n}{1+K} \max_{|z|\leq 1} |P(z)|.$$

The problem of the validity of the Bernstein and Erdős–Lax inequalities makes sense in higher dimensions and, in particular, it has been investigated in [9] in the case of one-sided polynomials with quaternionic coefficients. It has been shown that Bernstein inequality holds, while the Erdős–Lax inequality is, in general, false, but it may hold in some special subsets of the set of quaternionic polynomials. The variable considered in [9] is in a four-dimensional real space, namely, the space of quaternions, however, a different four-dimensional variable can be considered and this will be the purpose of this paper in which we study the case of a bicomplex variable.

Bicomplex polynomials of a bicomplex variable are interesting from several points of view. For example, as is well known, a bicomplex polynomial of degree n does not necessarily have n zeros: it could have an infinite number of zeros, and even when it has a finite number of zeros, this number does not coincide,

in general, with the degree n, but there may be up to n^2 zeros. The validity of Bernstein-type inequalities and other classical features of polynomials, for example, the validity of the maximum modulus principle, in the bicomplex case depend on the norm under consideration. For this reason, in Section 3 we discuss the various notions of bicomplex norms. Moreover, even before we start our discussion on Bernstein inequalities for bicomplex polynomials, we need to add a rescaling of all the theorems above (Bernstein, Erdős–Lax and generalized Bernstein), to a ball that has a general radius R as follows in the paragraphs below.

In particular, to prove our results we need the following version of the Bernstein theorem (see [2, p. 45]):

Theorem 1.5 (Rescaled Bernstein). *If P is a complex polynomial of degree n, then*

$$\max_{|z|\leq R} |P'(z)| \leq \frac{n}{R} \max_{|z|\leq R} |P(z)|. \tag{1.1}$$

In a similar way, we obtain also a rescaled version of Erdős–Lax theorem:

Theorem 1.6 (Rescaled Erdős–Lax). *Let P be a complex polynomial of degree n having no zeros in the disc of radius K/R where $0 < R \leq K$. Then*

$$\max_{|z|\leq R} |P'(z)| \leq \frac{n}{R+K} \max_{|z|\leq R} |P(z)|. \tag{1.2}$$

Proof. Since P has no zeros in the disc of radius K/R and since $0 < R \leq K$, it follows that $P(Rz)$ has no zeros in the disc of radius K. Using the classical generalized Bernstein theorem and the fact that $\frac{d}{dz}(P(Rz)) = RP'(Rz)$, we obtain:

$$\max_{|z|\leq 1} |P'(Rz)| \leq \frac{n}{R(1+K/R)} \max_{|z|\leq 1} |P(Rz)|.$$

Since $\max_{|z|\leq R} |P(z)| = \max_{|z|\leq 1} |P(Rz)|$ the result follows. □

Remark 1.7. We can use the classical Erdős–Lax result to claim that if P is a complex polynomial of degree n such that $P(Kz)$ has no zeros in the disc of radius $K > 0$, then

$$\max_{|z|\leq 1} |P'(Kz)| \leq \frac{n}{2K} \max_{|z|\leq 1} |P(Kz)|,$$

equivalently:

$$\max_{|z|\leq K} |P'(z)| \leq \frac{n}{2K} \max_{|z|\leq K} |P(z)|. \tag{1.3}$$

Using these results, we prove the analogs of the Bernstein-type inequalities in the bicomplex setting. The plan of the paper is the following: Section 2 contains some preliminaries on the bicomplex algebra and the various representations of bicomplex numbers. Section 3 introduces various norms, namely, the Euclidean norm, the hyperbolic-valued norm, the Lie and dual Lie norms. In Section 4 we recall the notion of bicomplex holomorphy and we prove a maximum modulus principle in the case of Lie and hyperbolic-valued norms. Finally, in Section 5 we

prove the validity of the various versions of Bernstein's and Erdős–Lax inequalities, according to the norms used. In particular, we show that in the case of the Euclidean norm the results can be obtained by enlarging the radius of the ball, but we can keep the same relation with the degree of the polynomial that holds in the classical complex case. In the case of the dual Lie norm both the relation with the degree and the radius of the ball have to be changed. Finally, we prove that the exact analogs of the two inequalities hold when considering the Lie norm and the hyperbolic-valued norm.

2. The Bicomplex Algebra

In this section we recall the notion of a bicomplex algebra, which is a commutative generalization of the complex plane. The algebra

$$\mathbb{C}_{\mathbf{i}} := \{x + \mathbf{i}y \,\big|\, x, y \in \mathbb{R}\}$$

of complex numbers is a real bidimensional algebra, having one imaginary unit $\mathbf{i}$, $\mathbf{i}^2 = -1$. Complexifying $\mathbb{C}_{\mathbf{i}}$ one obtains a four-dimensional real algebra denoted by $\mathbb{BC}$, which stands for bicomplex algebra. The key point of the theory of functions on this algebra is that classical complex holomorphic functions can be extended from one complex variable to $\mathbb{BC}$, and one can therefore develop a theory of hyperholomorphic functions. This analysis has extensively been studied in [4, 6, 7, 11, 12, 17, 19, 25]. The algebra $\mathbb{BC}$ is four-dimensional over $\mathbb{R}$, just like the skew-field of quaternions, but while in the space of quaternions we have three anti-commutative imaginary units, in the case of bicomplex numbers two imaginary units $\mathbf{i}, \mathbf{j}$ commute, and thus $\mathbf{k} = \mathbf{ij}$ is a root of 1, called a hyperbolic unit. A bicomplex number has several forms of writing (see [12] for details):

$$\begin{aligned}
Z &= (x_1 + \mathbf{i}y_1) + \mathbf{j}(x_2 + \mathbf{i}y_2) =: z_1 + \mathbf{j}z_2 \\
&= (x_1 + \mathbf{j}x_2) + \mathbf{i}(y_1 + \mathbf{j}y_2) =: \varphi_1 + \mathbf{i}\varphi_2 \\
&= (x_1 + \mathbf{k}y_2) + \mathbf{i}(y_1 - \mathbf{k}x_2) =: \mathfrak{z}_1 + \mathbf{i}\mathfrak{z}_2 \\
&= (x_1 + \mathbf{k}y_2) + \mathbf{j}\,(x_2 - \mathbf{k}y_1) =: \mathfrak{w}_1 + \mathbf{j}\mathfrak{w}_2 \\
&= (x_1 + \mathbf{i}y_1) + \mathbf{k}(y_2 - \mathbf{i}x_2) =: w_1 + \mathbf{k}w_2 \\
&= (x_1 + \mathbf{j}x_2) + \mathbf{k}(y_2 - \mathbf{j}y_1) =: \omega_1 + \mathbf{k}\omega_2 \\
&= x_1 + \mathbf{i}y_1 + \mathbf{j}x_2 + \mathbf{k}y_2,
\end{aligned}$$

where $z_1, z_2, w_1, w_2 \in \mathbb{C}_{\mathbf{i}}$, $\varphi_1, \varphi_2, \omega_1, \omega_2 \in \mathbb{C}_{\mathbf{j}}$, and $\mathfrak{z}_1, \mathfrak{z}_2$, $\mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{D}$, the algebra of hyperbolic numbers, which we summarize below. There are three natural conjugations introduced on $\mathbb{BC}$, corresponding to the involutions $\mathbf{i} \mapsto -\mathbf{i}$ and $\mathbf{j} \mapsto -\mathbf{j}$:

$$\overline{Z} := \overline{z_1} + \mathbf{j}\overline{z_2} = \varphi_1 - \mathbf{i}\varphi_2, \qquad Z^\dagger := z_1 - \mathbf{j}z_2 = \overline{\varphi_1} - \mathbf{i}\overline{\varphi_2},$$
$$Z^* := \left(\overline{Z}\right)^\dagger = \overline{(Z^\dagger)} = \overline{z_1} - \mathbf{j}\overline{z_2} = \mathfrak{z}_1 - \mathbf{i}\,\mathfrak{z}_2 = \mathfrak{w}_1 - \mathbf{j}\,\mathfrak{w}_2,$$

where $\overline{z}_1$, $\overline{z}_2$ are usual complex conjugates to z_1, $z_2 \in \mathbb{C}_{\mathbf{i}}$. The corresponding (square of the) bicomplex moduli are defined as:

$$|Z|^2_{\mathbf{i}} := Z \cdot Z^\dagger = z_1^2 + z_2^2 \in \mathbb{C}_{\mathbf{i}}, \qquad |Z|^2_{\mathbf{j}} := Z \cdot \overline{Z} = \varphi_1^2 + \varphi_2^2 \in \mathbb{C}_{\mathbf{j}},$$
$$|Z|^2_{\mathbf{k}} := Z \cdot Z^* = \mathfrak{z}_1^2 + \mathfrak{z}_2^2 = \mathfrak{w}_1^2 + \mathfrak{w}_2^2 \in \mathbb{D}.$$

Although these squares of the moduli are not real-valued, they are multiplicative.

The bicomplex algebra is not a division algebra. There are two special zero–divisors:

$$\mathbf{e} := \frac{1+\mathbf{k}}{2} \quad \text{and} \quad \mathbf{e}^\dagger := \frac{1-\mathbf{k}}{2}$$

which have the following properties:

$$\mathbf{e} \cdot \mathbf{e}^\dagger = 0, \qquad \mathbf{e}^2 = \mathbf{e}, \qquad (\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger, \qquad \mathbf{e} + \mathbf{e}^\dagger = 1, \qquad \mathbf{e} - \mathbf{e}^\dagger = \mathbf{k}.$$

Moreover, $\{\mathbf{e}, \mathbf{e}^\dagger\}$ is a basis of the algebra $\mathbb{BC}$ as a complex linear space. The next property has no analogs for complex numbers, and it exemplifies one of the deepest peculiarities of the set of bicomplex numbers. Any bicomplex number $Z = z_1 + \mathbf{j}z_2 \in \mathbb{BC}$ has a unique *idempotent representation* with $\mathbb{C}_{\mathbf{i}}$-coefficients:

$$Z = \zeta_1 \mathbf{e} + \zeta_2 \mathbf{e}^\dagger, \qquad \zeta_1 := z_1 - \mathbf{i}z_2, \quad \zeta_2 := z_1 + \mathbf{i}z_2 \in \mathbb{C}_{\mathbf{i}}. \tag{2.1}$$

The addition and multiplication of bicomplex numbers are realized "term-by-term" in this idempotent representation. A non-zero bicomplex number is invertible if and only if the coefficients ζ_1 and ζ_2 are not simultaneously zero. Whenever this holds, the inverse of Z is given by

$$Z^{-1} = \zeta_1^{-1}\mathbf{e} + \zeta_2^{-1}\mathbf{e}^\dagger.$$

The set of zero-divisors $\mathfrak{S} := \{Z \in \mathbb{BC} \,\big|\, Z \neq 0, |Z|^2_{\mathbf{i}} = z_1^2 + z_2^2 = 0\}$ is also known as the two-complex light cone, and we set $\mathfrak{S}_0 := \mathfrak{S} \cup \{0\}$. Then $\mathfrak{S}$ can be identified with the cotangent bundle of S^1, the real circle, minus its zero section:

$$\mathfrak{S} = T^*S^1 \setminus \sigma_0\,.$$

Bicomplex numbers have also a $\mathbb{C}_{\mathbf{j}}$-idempotent representation, but, for simplicity of our presentation, we omit it here (see [12] for details).

A special subalgebra of $\mathbb{BC}$ is the set of hyperbolic numbers. The algebra and the analysis of hyperbolic numbers have been studied, for example, in [1, 3, 12, 23], we summarize below only the notions relevant for our results. A hyperbolic number can be defined independently of $\mathbb{BC}$, by $\mathfrak{z} = x + \mathbf{k}y$, with $x, y, \in \mathbb{R}$, $\mathbf{k} \notin \mathbb{R}, \mathbf{k}^2 = 1$, and we denote by $\mathbb{D}$ the algebra of hyperbolic numbers with the usual component–wise addition and multiplication. The hyperbolic *conjugate* of $\mathfrak{z}$ is defined by $\mathfrak{z}^\diamond := x - \mathbf{k}y$, and note that

$$\mathfrak{z} \cdot \mathfrak{z}^\diamond = x^2 - y^2 \in \mathbb{R}\,, \tag{2.2}$$

which yields the notion of the square of the *modulus* of a hyperbolic number $\mathfrak{z}$, defined by $|\mathfrak{z}|^2_{\mathbb{D}} := \mathfrak{z} \cdot \mathfrak{z}^\diamond$. Similar to the bicomplex case, hyperbolic numbers have a

unique idempotent representation with real coefficients:

$$\mathfrak{z} = s\mathbf{e} + t\mathbf{e}^{\diamond}, \tag{2.3}$$

where $\mathbf{e} = \frac{1}{2}(1+\mathbf{k})$, $\mathbf{e}^{\diamond} = \frac{1}{2}(1-\mathbf{k})$, and $s := x+y$ and $t := x-y$. Note that $\mathbf{e}^{\diamond} = \mathbf{e}^{\dagger}$ if we consider $\mathbb{D}$ as a subset of $\mathbb{BC}$, as briefly explained below. Observe that

$$|\mathfrak{z}|^2_{\mathbb{D}} = x^2 - y^2 = (x+y)(x-y) = st.$$

Define the set $\mathbb{D}^+$ of nonnegative hyperbolic numbers by:

$$\begin{aligned}\mathbb{D}^+ &= \left\{x + \mathbf{k}y \,\middle|\, x^2 - y^2 \geq 0, x \geq 0\right\} = \left\{x + \mathbf{k}y \,\middle|\, x \geq 0, |y| \leq x\right\} \\ &= \{s\mathbf{e} + t\mathbf{e}^{\dagger} \,|\, s,t \geq 0\}.\end{aligned}$$

The hyperbolic algebra $\mathbb{D}$ is a subalgebra of the bicomplex numbers $\mathbb{BC}$ (see [12] for details). Actually $\mathbb{BC}$ is the algebraic closure of $\mathbb{D}$, and it can also be seen as the complexification of $\mathbb{D}$ by using either of the imaginary unit $\mathbf{i}$ or the unit $\mathbf{j}$.

As studied extensively in [1], one can define a partial order relation defined on $\mathbb{D}$ by:

$$\mathfrak{z}_1 \preceq \mathfrak{z}_2 \qquad \text{if and only if} \qquad \mathfrak{z}_2 - \mathfrak{z}_1 \in \mathbb{D}^+. \tag{2.4}$$

In the following section we will use this partial order to study the *hyperbolic-valued* norm, which was first introduced and studied in [1].

3. Norms on bicomplex numbers

In this section, we elaborate on the several types of norms one can define on the space of bicomplex numbers. Let $|z| = \|(x,y)\|$ be the usual Euclidean norm on $\mathbb{C}_{\mathbf{i}} \simeq \mathbb{R}^2$. The open and closed disks in $\mathbb{C}_{\mathbf{i}}$ (or $\mathbb{C}_{\mathbf{j}}$) are denoted by $B_z(r)$ and $\overline{B}_z(r)$, for $0 < r \leq \infty$.

3.1. The Euclidean Norm

The Euclidean norm $\|Z\|$ on $\mathbb{BC}$ is, when it is seen as $\mathbb{C}^2(\mathbf{i}), \mathbb{C}^2(\mathbf{j})$ or $\mathbb{R}^4$,

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}\,(|Z|^2_{\mathbf{k}})} = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}\,.$$

As studied in detail in [12], in idempotent coordinates $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^{\dagger}$, the Euclidean norm becomes:

$$\|Z\| = \frac{1}{\sqrt{2}}\sqrt{|\zeta_1|^2 + |\zeta_2|^2}. \tag{3.1}$$

This equality will become very important in the discussion of the extension of Bernstein's Theorem. Therefore, the Euclidean ball of radius 1 in the bicomplex space will be a subset of the cartesian product of the complex balls of radius $\sqrt{2}$ in ζ_1 and ζ_2, respectively. Albeit this is the easiest to define, the Euclidean norm fails to capture the complexities of the space and problems we are faced with. We

encounter such problems for our extension of the Bernstein-type theorems as well, and we will see that other norms are better suited to this extension.

It is easy to prove that

$$\|Z \cdot W\| \leq \sqrt{2}\,(\|Z\| \cdot \|W\|)\,, \tag{3.2}$$

and we note that this inequality is sharp since if $Z = W = \mathbf{e}$, one has:

$$\|\mathbf{e} \cdot \mathbf{e}\| = \|\mathbf{e}\| = \frac{1}{\sqrt{2}} = \sqrt{2}\,\|\mathbf{e}\| \cdot \|\mathbf{e}\|.$$

We also have the following results about the multiplication property of the Euclidean norm on bicomplex numbers (see [12] for details):

Proposition 3.1. *If $Z = \zeta_1\mathbf{e}+\zeta_2\mathbf{e}^\dagger$ and $W = \gamma_1\mathbf{e}+\gamma_2\mathbf{e}^\dagger$ are two bicomplex numbers, then $\|Z \cdot W\| = \|Z\| \cdot \|W\|$ if and only if*

$$|\zeta_1| = |\zeta_2| \quad \textit{or} \quad |\gamma_1| = |\gamma_2|.$$

Proposition 3.2. *The Euclidean norm of the product of two bicomplex numbers is equal to the product of their norms if and only if at least one of them is the product of a complex number in $\mathbb{C}_\mathbf{i}$ and of a complex number in $\mathbb{C}_\mathbf{j}$.*

3.2. Hyperbolic-valued Norm

Using the partial order relation (2.4) on $\mathbb{D}$, one can define a *hyperbolic-valued* norm for $\mathfrak{z} = x + \mathbf{k}y = s\mathbf{e} + t\mathbf{e}^\dagger$ by:

$$|\mathfrak{z}|_h := |s|\mathbf{e} + |t|\mathbf{e}^\dagger \in \mathbb{D}^+.$$

It is shown in [1] that this definition obeys the corresponding properties of a norm, i.e., $|\mathfrak{z}|_h = 0$ if and only if $\mathfrak{z} = 0$, it is multiplicative: $|\mathfrak{z}_1\mathfrak{z}_2|_h = |\mathfrak{z}_1|_h \cdot |\mathfrak{z}_2|_h$, for any real number $\lambda \in \mathbb{R}$, we have $|\lambda\mathfrak{z}|_h = |\lambda| \cdot |\mathfrak{z}|_h$, and it respects the triangle inequality with respect to the order introduced above:

$$|\mathfrak{z}_1 + \mathfrak{z}_2|_h \preceq |\mathfrak{z}_1|_h + |\mathfrak{z}_2|_h,$$

for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{D}$. The hyperbolic-valued norm coincides with the modulus $|Z|_\mathbf{k}$ of a hyperbolic number $Z \in \mathbb{D} \subset \mathbb{BC}$. Indeed, recall that the hyperbolic-valued modulus $|Z|_\mathbf{k}$ of a bicomplex number $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger$, given by

$$|Z|_\mathbf{k}^2 = |\zeta_1|^2\mathbf{e} + |\zeta_2|^2\mathbf{e}^\dagger,$$

is a nonnegative hyperbolic number in $\mathbb{D}^+$. Moreover, one can properly define the square root of a hyperbolic number (see [1, 12]), hence the modulus $|Z|_\mathbf{k}$ is well defined and given by:

$$|Z|_\mathbf{k} := |\zeta_1|\mathbf{e} + |\zeta_2|\mathbf{e}^\dagger \in \mathbb{D}^+.$$

In the case $Z \in \mathbb{D} \subset \mathbb{BC}$, i.e., $\zeta_1, \zeta_2 \in \mathbb{R}$, it follows that $|Z|_h = |Z|_\mathbf{k}$.

3.3. Lie and dual Lie Norms

The Lie and dual Lie norms are usually defined and studied in the context of several complex variables, see, for example, [15, 24]. A bicomplex number $Z = z_1 + \mathbf{j}z_2$ can be viewed as a two-complex vector, therefore the cross norm $L(Z)$ on $\mathbb{BC}$ corresponding to $Z = (z_1, z_2) \in \mathbb{C}^2_{\mathbf{i}}$ is the Lie norm given by:

$$\begin{aligned} L(Z) &:= \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1y_1 + x_2y_2)^2}} \\ &= \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2|x_1y_2 - x_2y_1|}\,. \end{aligned} \tag{3.3}$$

Consider now Z written in the form:

$$Z = (x_1 + \mathbf{k}y_2) + \mathbf{i}(y_1 - \mathbf{k}x_2) =: \mathfrak{z}_1 + \mathbf{i}\mathfrak{z}_2,$$

and write $\mathfrak{z}_1$ and $\mathfrak{z}_2$ in their hyperbolic idempotent representations:

$$\begin{aligned} \mathfrak{z}_1 &= (x_1 + y_2)\mathbf{e} + (x_1 - y_2)\mathbf{e}^\dagger =: s_1\mathbf{e} + t_1\mathbf{e}^\dagger, \\ \mathfrak{z}_2 &= (y_1 - x_2)\mathbf{e} + (y_1 + x_2)\mathbf{e}^\dagger =: s_2\mathbf{e} + t_2\mathbf{e}^\dagger\,. \end{aligned}$$

In (3.3), if $x_1y_2 - x_2y_1 \geq 0$, we get:

$$L(Z) = \sqrt{(x_1 + y_2)^2 + (x_2 - y_1)^2} = \sqrt{s_1^2 + s_2^2}\,,$$

and if $x_1y_2 - x_2y_1 \leq 0$, we get:

$$L(Z) = \sqrt{(x_1 - y_2)^2 + (x_2 + y_1)^2} = \sqrt{t_1^2 + t_2^2}\,.$$

Now we write Z in its bicomplex $\mathbb{C}_{\mathbf{i}}$-idempotent representation, thus we have:

$$\begin{aligned} \zeta_1 &= z_1 - \mathbf{i}z_2 = (x_1 + y_2) + \mathbf{i}(y_1 - x_2) = s_1 + \mathbf{i}s_2, \\ \zeta_2 &= z_1 + \mathbf{i}z_2 = (x_1 - y_2) + \mathbf{i}(y_1 + x_2) = t_1 + \mathbf{i}t_2, \end{aligned}$$

therefore

$$|\zeta_1|^2 = s_1^2 + s_2^2, \qquad |\zeta_2|^2 = t_1^2 + t_2^2.$$

Put together, we proved that the Lie norm of a bicomplex number is given by the following.

Proposition 3.3. *For any bicomplex number $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger$ written in the $\mathbb{C}_{\mathbf{i}}$–idempotent representation, its Lie norm is given by:*

$$L(Z) = \max\{|\zeta_1|, |\zeta_2|\}\,. \tag{3.4}$$

The dual Lie norm of bicomplex numbers is computed as follows (see [15] for the general definition in $\mathbb{C}^n$):

$$\begin{aligned} &L^*(Z) := \sup\{\langle Z, W\rangle_{\mathbb{R}} \,\big|\, L(W) \leq 1\} \\ &= \frac{\sqrt{2}}{2}\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2 + \sqrt{\left(x_1^2 + x_2^2 - y_1^2 - y_2^2\right)^2 + 4(x_1y_1 + x_2y_2)^2}} \\ &= \frac{\sqrt{2}}{2}\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2 + \sqrt{(x_1 - y_2)^2 + (x_2 + y_1)^2}\sqrt{(x_1 + y_2)^2 + (x_2 - y_1)^2}}\,. \end{aligned}$$

In idempotent coordinates, we obtain:

Proposition 3.4. *For any bicomplex number $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger$, its dual Lie norm is given by:*

$$L^*(Z) = \frac{|\zeta_1| + |\zeta_2|}{2}. \tag{3.5}$$

Proof. As defined before, $|\zeta_1| = \sqrt{s_1^2 + s_2^2} = \sqrt{x_1 - y_2)^2 + (x_2 + y_1)^2}$ with a similar form for ζ_2. We will begin by squaring $|\zeta_1| + |\zeta_2| = \sqrt{s_1^2 + s_2^2} + \sqrt{t_1^2 + t_2^2}$ to obtain:

$$(x_1 - y_2)^2 + (x_2 + y_1)^2 + (x_1 + y_2)^2 + (x_2 - y_1)^2 + 2\sqrt{(x_1 - y_2)^2 + (x_2 + y_1)^2}\sqrt{(x_1 + y_2)^2 + (x_2 - y_1)^2}\,.$$

Simplifying, one obtains precisely twice the formula under the square root, and the result is proven. □

3.4. Elements of topology on the set of bicomplex numbers

We summarize below the norms we introduced in the last section on the set of bicomplex numbers $\mathbb{BC}$, written in the idempotent representation $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger$, with $\zeta_1, \zeta_2 \in \mathbb{C}_\mathbf{i}$:

Euclidean	$\|Z\| = \frac{1}{\sqrt{2}}\sqrt{\|\zeta_1\|^2 + \|\zeta_2\|^2} \in \mathbb{R}^+$
Lie norm	$L(Z) = \max\{\|\zeta_1\|, \|\zeta_2\|\} \in \mathbb{R}^+$
Dual Lie norm	$L^*(Z) = \frac{1}{2}(\|\zeta_1\| + \|\zeta_2\|) \in \mathbb{R}^+$
Hyperbolic–valued	$\|Z\|_h = \|\zeta_1\|\mathbf{e} + \|\zeta_2\|\mathbf{e}^\dagger \in \mathbb{D}^+$

It is easy to check the following inequalities for any A, B non-negative real numbers:

$$\sqrt{AB} \leq \frac{A+B}{2} \leq \frac{\sqrt{A^2 + B^2}}{\sqrt{2}} \leq \max(A, B).$$

These inequalities yield the following inequalities between the norms:

Lemma 3.5. *For a bicomplex number $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger$ we have:*

$$L^*(Z) \leq \|Z\| \leq L(Z).$$

4. Bicomplex Holomorphy and Maximum Modulus Principle

Consider a bicomplex function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$, which can be written in several forms, depending on the structure of the bicomplex numbers considered (see [12] for complete details). For the purpose of our paper, we are interested only in the following two writings:

$$F(Z) = f_1(z_1, z_2) + \mathbf{j}f_2(z_1, z_2) = g_1(\zeta_1, \zeta_2)\mathbf{e} + g_2(\zeta_1, \zeta_2)\mathbf{e}^\dagger,$$

where $f_1, f_2, g_2, g_2 : \Omega \to \mathbb{C}_{\mathbf{i}}$.

Bicomplex functions admit the notion of a derivative in the sense of a one-variable bicomplex theory, as follows.

Definition 4.1. Let Ω be an open set in $\mathbb{BC}$ and let $Z_0 \in \Omega$. A function $F : \Omega \to \mathbb{BC}$ has a derivative at Z_0 if the limit

$$F'(Z_0) := \lim_{Z \to Z_0} (Z - Z_0)^{-1} (F(Z) - F(Z_0))$$

exists, for all $Z \in \Omega$ such that $Z - Z_0$ is an invertible bicomplex number in Ω (i.e., not a divisor of zero).

Functions which admit bicomplex derivative at each point in their domain are called bicomplex holomorphic, and it can be shown that this is equivalent to require that they admit a power series expansion in Z [17, Definition 15.2]. There are, however, equivalent descriptions (for details, see [12]):

Theorem 4.2. *Let Ω be an open set in $\mathbb{BC}$ and let $F : \Omega \to \mathbb{BC}$ be of class $\mathcal{C}^1(\Omega)$, written as above:*

$$F(Z) = f_1(z_1, z_2) + \mathbf{j} f_2(z_1, z_2) = g_1(\zeta_1, \zeta_2)\mathbf{e} + g_2(\zeta_1, \zeta_2)\mathbf{e}^\dagger.$$

Then F is bicomplex holomorphic if and only if:

1. *f_1 and f_2 are complex holomorphic in both complex $\mathbb{C}_{\mathbf{i}}$ variables z_1 and z_2, and they verify the* complex *Cauchy–Riemann equations on Ω:*

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \qquad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}$$

2. *g_1 and g_2 are one-variable $\mathbb{C}_{\mathbf{i}}$-complex holomorphic functions in ζ_1 and ζ_2, respectively. In this case we have:*

$$F'(Z) = g_1'(\zeta_1)\mathbf{e} + g_2'(\zeta_2)\mathbf{e}^\dagger.$$

4.1. Maximum Modulus principles in $\mathbb{BC}$

It is a fact often overlooked, but there exist no references that explain a maximum modulus principle in the bicomplex case, due to the fact that the Euclidean norm which is used in most books and papers on this subject do not allow for a clear separation of boundaries.

In this section we first show that the maximum modulus principle holds in the Lie norm and in the hyperbolic-valued norm, defined in Section 3, as these norms preserve the boundaries of the complex disks involved.

Definition 4.3. A *bicomplex domain* $\Omega \subseteq \mathbb{BC}$ is a set of $\mathbb{BC}$ such that the complex sets defined by:

$$\Omega_1\mathbf{e} := \Omega\mathbf{e} \in \mathbb{C}_{\mathbf{i}}, \qquad \Omega_2\mathbf{e}^\dagger := \Omega\mathbf{e}^\dagger \in \mathbb{C}_{\mathbf{i}},$$

are complex domains in $\mathbb{C}_{\mathbf{i}}$.

Theorem 4.4 (Maximum Modulus Principle: hyperbolic-valued norm case). *Let $F : \Omega \subseteq \mathbb{BC} \to \mathbb{BC}$ a bicomplex holomorphic function defined on a bicomplex domain Ω. If there exists $Z_0 = \zeta_1^0\mathbf{e} + \zeta_2^0\mathbf{e}^\dagger \in \Omega$ such that*

$$|F(Z_0)|_h \succeq |F(Z)|_h, \qquad \forall Z \in \Omega,$$

then F is constant.

Proof. Let us write

$$F(Z) = g_1(\zeta_1)\mathbf{e} + g_2(\zeta_2)\mathbf{e}^\dagger,$$

where $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger$, and g_1 and g_2 are complex holomorphic functions defined on the domains Ω_1 and Ω_2, respectively. The hyperbolic-valued modulus of F is:

$$|F(Z)|_h = |g_1(\zeta_1)|\mathbf{e} + |g_2(\zeta_2)|\mathbf{e}^\dagger.$$

Assume there exists $Z_0 = \zeta_1^0\mathbf{e} + \zeta_2^0\mathbf{e}^\dagger \in \Omega$ such that

$$|F(Z_0)|_h \succeq |F(Z)|_h, \quad \forall Z \in \Omega.$$

We write:

$$|F(Z_0)|_h = |g_1(\zeta_1^0)|\mathbf{e} + |g_2(\zeta_2^0)|\mathbf{e}^\dagger \succeq |g_1(\zeta_1)|\mathbf{e} + |g_2(\zeta_2)|\mathbf{e}^\dagger = |F(Z)|_h$$

which is equivalent to:

$$|g_1(\zeta_1^0)| \geq |g_1(\zeta_1)|, \quad \forall \zeta_1 \in \Omega_1$$
$$|g_2(\zeta_2^0)| \geq |g_2(\zeta_2)|, \quad \forall \zeta_2 \in \Omega_2.$$

Because g_1 and g_2 are single-variable complex holomorphic functions, this implies that g_1 and g_2 are constant. □

In an analogous way, one can prove:

Theorem 4.5 (Maximum Modulus Principle: Lie norm case). *Let $F : \Omega \subseteq \mathbb{BC} \to \mathbb{BC}$ a bicomplex holomorphic function defined on a bicomplex domain Ω. If there exists $Z_0 = \zeta_1^0\mathbf{e} + \zeta_2^0\mathbf{e}^\dagger \in \Omega$ such that*

$$L(F(Z_0)) \geq L(F(Z)), \qquad \forall Z \in \Omega,$$

then F is constant.

Proof. As before, we write

$$F(Z) = g_1(\zeta_1)\mathbf{e} + g_2(\zeta_2)\mathbf{e}^\dagger,$$

where $Z = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^\dagger$, and g_1 and g_2 are complex holomorphic functions defined on the domains Ω_1 and Ω_2 are such that $\Omega_1\mathbf{e} := \Omega\mathbf{e} \in \mathbb{C}_\mathbf{i}$ and $\Omega_2\mathbf{e}^\dagger := \Omega\mathbf{e}^\dagger \in \mathbb{C}_\mathbf{i}$. Then:

$$L(F(Z)) = \max_{\zeta_1 \in \Omega_1,\ \zeta_2 \in \Omega_2} \{|g_1(\zeta_1)|, |g_2(\zeta_2)|\}.$$

If there exists $Z_0 = \zeta_1^0\mathbf{e} + \zeta_2^0\mathbf{e}^\dagger \in \Omega$ such that

$$L(F(Z_0)) \geq L(F(Z)), \quad \forall Z \in \Omega,$$

we have

$$L(F(Z_0)) = \max\{|g_1(\zeta_1^0)|, |g_2(\zeta_2^0)|\} \geq \max_{\zeta_1 \in \Omega_1,\ \zeta_2 \in \Omega_2} \{|g_1(\zeta_1)|, |g_2(\zeta_2)|\}.$$

This yields:

$$\max\{|g_1(\zeta_1^0)|, |g_2(\zeta_2^0)|\} \geq |g_1(\zeta_1^0)| \geq |g_1(\zeta_1)|, \quad \forall \zeta_1 \in \Omega_1,$$
$$\max\{|g_1(\zeta_1^0)|, |g_2(\zeta_2^0)|\} \geq |g_2(\zeta_2^0)| \geq |g_2(\zeta_2)|, \quad \forall \zeta_2 \in \Omega_2,$$

which implies that g_1 and g_2 are constant complex functions. □

We will use these maximum modulus principles to show that Bernstein's and Erdős–Lax theorems are true analogues of the complex case for the hyperbolic-valued and Lie norms.

It is worth noting that one could easily find examples of polynomials for which the maximum modulus principle does not hold either in the Euclidean or the dual Lie norm, and we will see that the difference is apparent in the next section as well. This is due to the fact that the boundaries of the open balls in these norms are "fuzzy" and do not allow sharp estimates.

5. Bernstein–Erdős–Lax-type theorems for Bicomplex polynomials

In this section we will describe all the forms that Bernstein-type theorems are taking in the bicomplex case for the four types of norms introduced above. We start with the properties of polynomials in a bicomplex variable, then proceed to discuss each case.

5.1. Bicomplex polynomials

Here we introduce the unique perspective that bicomplex polynomials introduce in complex analysis. A *polynomial of a bicomplex variable* has the form $P(Z) = \sum_{k=0}^{n} a_k Z^k$ where $a_n \in \mathbb{BC}$. Using the idempotent representation $Z = \zeta_1 \mathbf{e} + \zeta_2 \mathbf{e}^\dagger$ and $a_k = \alpha_k \mathbf{e} + \beta_k \mathbf{e}^\dagger$ we obtain the following rather obvious remark:

Remark 5.1. In the idempotent representation, a polynomial P in the bicomplex variable $Z = \zeta_1 \mathbf{e} + \zeta_2 \mathbf{e}^\dagger$ can be written as:

$$P(Z) = P_1(\zeta_1)\mathbf{e} + P_2(\zeta_2)\mathbf{e}^\dagger,$$

where $P_1(\zeta_1)$ and $P_2(\zeta_2)$ are complex polynomials in the complex variable ζ_1 and ζ_2, respectively. If either P_1 or P_2 are identically zero, we call the bicomplex polynomial P *degenerate*. For example, if $P_2(\zeta_2) \equiv 0$, then

$$P(Z) = P_1(\zeta_1)\mathbf{e}. \tag{5.1}$$

We remark also that for any complex polynomial $P(Z)$ as in (5.1), it follows that $P(Z) \cdot \mathbf{e}^\dagger \equiv 0$, therefore the word degenerate.

Theorem 5.2 ([12]**).** *The number of zeros of the polynomial $P(Z)$ is either less than or equal to n^2 or, in the degenerate case, infinity.*

In what follows we study bicomplex polynomials P written in the idempotent decomposition above:

$$P(Z) = P_1(\zeta_1)\mathbf{e} + P_2(\zeta_2)\mathbf{e}^\dagger.$$

It turns out that there are different Bernstein-type theorems depending on the norm used on the bicomplex space. Apart from the case of the hyperbolic-valued norm and of the Lie norm, we will see that we need to increase the radius of the ball where we take the maximum of the polynomial, therefore we need to use the generalized Bernstein theorem for complex polynomials.

5.2. Euclidean norm case

First of all, we note that, in general, $\deg(P) = n$ implies that $\deg(P_1) \leq n$ and $\deg(P_2) \leq n$. Since, in the Euclidean case, we have $\|P\|^2 = \frac{1}{2}(|P_1|^2 + |P_2|^2)$, one obtains:

$$\max_{\|Z\|\leq 1} \|P\|^2 \leq \frac{1}{2}\left(\max_{|\zeta_1|\leq\sqrt{2}} |P_1(\zeta_1)|^2 + \max_{|\zeta_2|\leq\sqrt{2}} |P_2(\zeta_2)|^2\right).$$

Theorem 5.3 (Bernstein). *If $P(Z)$ is a polynomial of degree n with bicomplex coefficients, then*

$$\max_{\|Z\|\leq 1} \|P'(Z)\| \leq n \cdot \max_{\|Z\|\leq\sqrt{2}} \|P(Z)\|.$$

Proof. It is immediate that $\|Z\| \leq 1$ implies that $|\zeta_\ell| \leq \sqrt{2}$ for $\ell = 1, 2$, (but the converse is not true). We then have

$$\max_{\|Z\|\leq 1} \|P'(Z)\|^2 \leq \frac{1}{2}\left(\max_{|\zeta_1|\leq\sqrt{2}} |P_1'(\zeta_1)|^2 + \max_{|\zeta_2|\leq\sqrt{2}} |P_2'(\zeta_2)|^2\right) \tag{5.2}$$

and the inequality is, in general, not sharp. Since P_1, P_2 are complex-valued polynomials of a complex variable, we can apply the rescaled Bernstein inequality (1.1) to each of them and we obtain

$$\frac{1}{2}\left(\max_{|\zeta_1|\leq\sqrt{2}} |P_1'(\zeta_1)|^2 + \max_{|\zeta_2|\leq\sqrt{2}} |P_2'(\zeta_2)|^2\right) \leq \frac{n^2}{4}\left(\max_{|\zeta_1|\leq\sqrt{2}} |P_1(\zeta_1)|^2 + \max_{|\zeta_2|\leq\sqrt{2}} |P_2(\zeta_2)|^2\right).$$

Since $|\zeta_\ell| < \sqrt{2}$, $\ell = 1, 2$, we have $\|Z\| = \frac{1}{\sqrt{2}}\sqrt{(|\zeta_1|^2 + |\zeta_2|^2)} < \sqrt{2}$, and thus

$$\max_{|\zeta_\ell|\leq\sqrt{2}} |P_\ell(\zeta_\ell)|^2 \leq 2 \max_{\|Z\|\leq\sqrt{2}} \|P(Z)\|^2. \tag{5.3}$$

It follows that

$$\max_{\|Z\|\leq 1}\|P'(Z)\|^2 \leq \frac{n^2}{4}\left(\max_{|\zeta_1|\leq\sqrt{2}}|P_1(\zeta_1)|^2+\max_{|\zeta_2|\leq\sqrt{2}}|P_2(\zeta_2)|^2\right)$$
$$\leq n^2\max_{\|Z\|\leq\sqrt{2}}\|P(Z)\|^2,$$

and the statement follows. □

Theorem 5.4 (Erdős–Lax). *If $P(Z)$ is a polynomial of degree n with bicomplex coefficients and with no zeros in the Euclidean unit ball, then*

$$\max_{\|Z\|\leq 1}\|P'(Z)\| \leq \frac{n}{2}\max_{\|Z\|\leq\sqrt{2}}\|P(Z)\|.$$

Proof. We follow the lines of the proof of the previous result. We start from (5.2)and we note that if P has no zeros in the unit ball $\|Z\| < 1$, then P_ℓ has no zeros in $|\zeta_\ell| < \sqrt{2}$, $\ell = 1, 2$. Indeed, if P_1 has a zero at the point ζ_1^0 inside the disk of radius $\sqrt{2}$, then $P(\zeta_1^0\mathbf{e} + 0\mathbf{e}^\dagger) = P_1(\zeta_1^0)\mathbf{e} + 0\mathbf{e}^\dagger = 0$ and similarly if P_2 possesses a zero. Thus we can use the rescaled Erdős–Lax theorem (with $R = K = \sqrt{2}$) and (5.3) to obtain

$$\frac{1}{2}\left(\max_{|\zeta_1|\leq\sqrt{2}}|P_1'(\zeta_1)|^2+\max_{|\zeta_2|\leq\sqrt{2}}|P_2'(\zeta_2)|^2\right)$$
$$\leq \frac{1}{2}\frac{n^2}{(2\sqrt{2})^2}\left(\max_{|\zeta_1|\leq\sqrt{2}}|P_1(\zeta_1)|^2+\max_{|\zeta_2|\leq\sqrt{2}}|P_2(\zeta_2)|^2\right)$$
$$\leq \frac{1}{2}\frac{4n^2}{(2\sqrt{2})^2}\max_{\|Z\|\leq\sqrt{2}}\|P(Z)\|^2.$$

We finally obtain

$$\max_{\|Z\|\leq 1}\|P'(Z)\| \leq \frac{n}{2}\max_{\|Z\|\leq\sqrt{2}}\|P(Z)\|.$$ □

5.3. Lie norm case

In the Lie norm case, since $L(z) = \max\{|\zeta_1|, |\zeta_2|\} \leq 1$ if and only if $|\zeta_1| \leq 1$ and $|\zeta_2| \leq 1$ we have:

$$\max_{L(Z)\leq 1}L(P(Z)) = \max_{|\zeta_1|\leq 1,\,|\zeta_2|\leq 1}(|P_1(\zeta_1)|, |P_2(\zeta_2)|),$$

The generalization of Bernstein's result to bicomplex polynomials then becomes:

Theorem 5.5 (Bernstein). *If $P(Z)$ is a polynomial of degree n with bicomplex coefficients, then*

$$\max_{L(Z)\leq 1}L(P'(Z)) \leq n\cdot\max_{L(Z)\leq 1}L(P(Z)). \tag{5.4}$$

Proof. By definition of Lie norm and using the classical Bernstein result, we have

$$\begin{aligned}\max_{L(Z)\leq 1} L(P'(Z)) &\leq \max_{|\zeta_1|\leq 1,\,|\zeta_2|\leq 1} (|P_1'(\zeta_1)|, |P_2'(\zeta_2)\,|) \\ &\leq n \max_{|\zeta_1|\leq 1,\,|\zeta_2|\leq 1} (|P_1(\zeta_1)|, |P_2(\zeta_2)|) = n \max_{L(Z)\leq 1} L(P(Z)),\end{aligned}$$

which gives the desired inequality. □

Theorem 5.6 (Erdős–Lax). *If $P(Z)$ is a polynomial of degree n with bicomplex coefficients with no zeros in the unit ball given by the Lie norm (i.e., $L(Z) \leq 1$), then*

$$\max_{L(Z)\leq 1} L(P'(Z)) \leq \frac{n}{2} \cdot \max_{L(Z)\leq 1} L(P(Z)).$$

Proof. Reasoning as in the proof of the previous result, we have

$$\begin{aligned}\max_{L(Z)\leq 1} L(P'(Z)) &\leq \max_{|\zeta_1|\leq 1,\,|\zeta_2|\leq 1} (|P_1'(\zeta_1)|, |P_2'(\zeta_2)|) \\ &\leq \frac{n}{2} \max_{|\zeta_1|\leq 1,\,|\zeta_2|\leq 1} (|P_1(\zeta_1)|, |P_2(\zeta_2)|) = \frac{n}{2} \max_{L(Z)\leq 1} L(P(Z)).\end{aligned}$$ □

5.4. Dual Lie norm case

In the case of the dual Lie norm, one must recall that:

$$L^*(Z) = \frac{1}{2}\left(|\zeta_1| + |\zeta_2|\right).$$

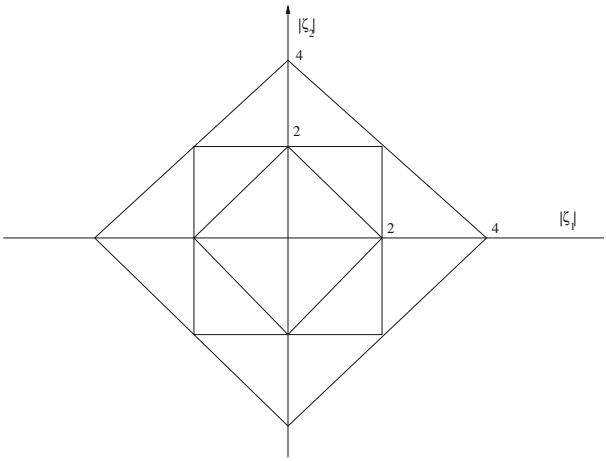


FIGURE 1. Dual Lie norm balls involved in the proof of Bernstein theorem

To obtain a Bernstein theorem one must increase the radius of the ball and we have:

Theorem 5.7 (Bernstein). *If $P(Z)$ is a polynomial of degree n with with bicomplex coefficients, then*

$$\max_{L^*(Z)\leq 1} L^*(P'(Z)) \leq \frac{n}{2} \max_{L^*(Z)\leq 2} L^*(P(Z))\,.$$

Proof. We have:

$$\max_{L^*(Z)\leq 1} L^*(P'(Z)) = \max_{\frac{1}{2}(|\zeta_1|+|\zeta_2|)\leq 1} \left(\frac{1}{2}\left(|P_1'(\zeta_1)| + |P_2'(\zeta_2)|\right)\right),$$

The right-hand side of the inequality is smaller than

$$\frac{1}{2}\left(\max_{|\zeta_1|\leq 2}(|P_1'(\zeta_1)|) + \max_{|\zeta_2|\leq 2}(|P_2'(\zeta_2)|)\right).$$

Using the generalized Bernstein inequality for the polynomials P_1, P_2 we obtain:

$$\max_{L^*(Z)\leq 1} L^*(P'(z)) \leq \frac{n}{4}\left(\max_{|\zeta_1|\leq 2}(|P_1(\zeta_1)|) + \max_{|\zeta_2|\leq 2}(|P_2(\zeta_2)|)\right),$$

here the right-hand side will be smaller than

$$\frac{n}{4}\left(\max_{\frac{1}{2}(|\zeta_1|+|\zeta_2|)\leq 2}(|P_1(\zeta_1)| + |P_2(\zeta_2)|)\right),$$

which yields the result. This increasing sequence of balls in the dual Lie norm is illustrated in Figure 1. □

Theorem 5.8 (Erdős–Lax). *If $P(Z)$ is a polynomial of degree n with bicomplex coefficients and with no zeros in the unit ball given by the dual Lie norm (i.e., $L^*(Z) \leq 1$), then*

$$\max_{L^*(Z)\leq 1} L^*(P'(Z)) \leq \frac{n}{4}\cdot \max_{L^*(Z)\leq 2} L^*(P(Z)).$$

Proof. Just as before, as $P(Z)$ has no zeroes in the unit ball $L^*(Z) \leq 1$, then P_1 and P_2 will have no zeroes on the respective complex unit ball of radius 2 and, for

$$\max_{L^*(Z)\leq 1} L^*(P'(Z)) \leq \frac{1}{2}\left(\max_{|\zeta_1|\leq 2}(|P_1'(\zeta_1)|) + \max_{|\zeta_2|\leq 2}(|P_2'(\zeta_2)|)\right),$$

we can use the rescaled Erdős–Lax Theorem for $K = R = 2$ to obtain that the right-hand side is smaller than:

$$\frac{n}{8}\left(\max_{\frac{1}{2}(|\zeta_1|+|\zeta_2|)\leq 2}(|P_1(\zeta_1)| + |P_2(\zeta_2)|)\right) = \frac{n}{4}\cdot \max_{L^*(Z)\leq 2} L^*(P(Z)).$$ □

5.5. Hyperbolic-valued norm case

For the hyperbolic-valued norm one has the advantage that it is specifically tailored for the decomposition in the zero divisors, therefore unit balls in the bicomplex space using this norm will be true unit balls in the two complex planes. Recall that

$$|P(Z)'|_h = |P_1'(\zeta_1)|\mathbf{e} + |P_2'(\zeta_2)|\mathbf{e}^\dagger.$$

In this case we have the following Bernstein theorem:

Theorem 5.9 (Bernstein). *If $P(Z)$ is a bicomplex polynomial of degree n, we have that:*

$$\max_{|Z|_h \preceq 1} |P'(Z)|_h \preceq n \max_{|Z|_h \preceq 1} |P(Z)|_h.$$

Proof. Just as in the case of the maximum modulus principle, by the definition of the hyperbolic value norm we have that:

$$\max_{|Z|_h \preceq 1} |P'(Z)|_h = \left(\max_{|\zeta_1| \leq 1} |P_1'(\zeta_1)|\right)\mathbf{e} + \left(\max_{|\zeta_2| \leq 1} |P_2'(\zeta_2)|\right)\mathbf{e}^\dagger.$$

Using Bernstein's Theorem for P_1 and P_2 we obtain:

$$\max_{|Z|_h \preceq 1} |P'(Z)|_h \preceq n\left(\max_{|\zeta_1| \leq 1} |P_1(\zeta_1)|\mathbf{e} + \max_{|\zeta_2| \leq 1} |P_2(\zeta_2)\mathbf{e}^\dagger\right) = n \max_{|Z|_h \preceq 1} |P(Z)|_h. \qquad \square$$

One can easily obtain a similar result for the Erdős–Lax Theorem:

Theorem 5.10 (Erdős–Lax). *If $P(Z)$ is a polynomial of degree n with bicomplex coefficients and $P(Z)$ has no zeros in the unit ball $|Z|_h \preceq 1$, then:*

$$\max_{|Z|_h \preceq 1} |P'(Z)|_h \preceq \frac{n}{2} \max_{|Z|_h \preceq 1} |P(Z)|_h.$$

Proof. As in the proof for Bernstein's theorem, in this case we use the fact that the hyperbolic–valued norm preserves the boundaries for ζ_1 and ζ_2.

Since $P(Z)$ has no zeros in the unit ball $|Z|_h \preceq 1$, we have that P_1 and P_2 have no zeroes on the complex unit balls $\zeta_1 \leq 1$ and $\zeta_2 \leq 1$ and the result follows from the usual Erdős–Lax Theorem as follows:

$$\max_{|Z|_h \preceq 1} |P'(Z)|_h \preceq \frac{n}{2}\left(\max_{|\zeta_1| \leq 1} |P_1(\zeta_1)|\mathbf{e} + \max_{|\zeta_2| \leq 1} |P_2(\zeta_2)\mathbf{e}^\dagger\right) = \frac{n}{2} \max_{|Z|_h \preceq 1} |P(Z)|_h.$$

$\square$

Remark 5.11. We can also note that, since a maximum modulus holds in this case, we have a true analogue of Bernstein's Theorem, i.e., the maxima are attained on the boundaries of these sets, i.e., we can re-write Bernstein's inequality as:

$$\max_{|Z|_h = 1} |P'(Z)|_h \preceq n \max_{|Z|_h = 1} |P(Z)|_h.$$

We have shown that a classical version of the Bernstein and Erdős–Lax theorems holds for the hyperbolic-valued and the dual Lie norms as additional assumptions on the zeroes of the bicomplex polynomial are not necessary in these cases. For the Euclidean and dual Lie norms, since the boundaries of the balls in these cases are not well defined, we used the generalized Bernstein Theorem.

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Finite Unions of Intervals, Part I

Ahmed Sebbar

Dedicated to Professor Daniel Alpay on the occasion of his 60th birthday

Abstract. The aim of this work is to investigate the geometric functions theory of plane domains, which are complements of finite unions of intervals. Our primary goal was the computation of the capacity of a finite union of intervals. We show that the spectral curve, given by the Burchnall–Chaundy theory for commuting differences operators and hyperelliptic theta functions play a chief role. This is the first of two papers devoted to this subject.

Keywords. Capacities, transfinite diameter, Thomae's formula, Burchnall–Chaundy theorem.

1. Introduction

Among all the multiply connected domains in the plane, the complements of finite unions of segments are among the simplest. They have some kind of algebraicity. An example of this circumstance is that the complex Green's function is given by an elliptic integral. The main objective of this work is to relate the computation of the capacity, or the transfinite diameter, of a finite union of intervals in the real axis $E = \bigcup_{i=1}^{2g+1}[e_i, e_{i+1}]$ to the inverse scattering method on the one hand and, on the other hand, to the geometry of the hyperelliptic curve

$$y^2 = \prod_{i=1}^{2g+2}(x - e_i).$$

Complementing certain points of [10], [11] and [12], this study will highlight a class of such domains, which we call calibrated, and which form a dense family, in a sense that will be specified. The main new ingredient is the precise study of a conformal mapping. Another remark, revealed in this study, is to point out the important role that Thomae' formula on hyperelliptic theta functions can have in the final computation of capacities.

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Classically the capacity of a compact set is a measure of its size. The capacity of a segment is the fourth of its length and the capacity of a circle is its radius. There are some important analytic and algebraic results directly attached to capacities. Fekete proved that the transfinite diameter of a compact set E is less than 1 if and only if there exists a monic polynomial

$$h(z) = z^r + a_{r-1}z^{r-1} + \cdots + a_0,\ |h(z)| < 1,\ z \in E.$$

If, moreover, the set E is symmetric with respect to the real axis, then the coefficients of $h(z)$ can be chosen integers. Another kind of results is related to the classical Polya–Carlson theorem. First a theorem of Carlson asserts that if $f(z)$ is an entire function of exponential type less than $\log 2$ and is such that $f(\mathbb{N}) \subset \mathbb{Z}$, then $f(z)$ is a polynomial. This result has been extended by Polya in the following way: assume that $E \subset \mathbb{C}$ is a compact set of capacity $\tau(E) < 1$, stable under complex conjugation and whose complement G is connected. Assume, moreover, that $f(z)$ is regular in G and has an expansion near ∞, given by:

$$f(z) = \sum a_n z^{-n}, \quad a_n \in \mathbb{Z},$$

then $f(z)$ is a rational fraction (then with poles in E). Another result in the same vein, due to Fekete and Szegö [38], [39], is the following: let $E \subset \mathbb{C}$ be a compact set, stable under complex conjugation and having logarithmic capacity $\tau(E)$. Then:

1. If $\tau(E) < 1$, there is some neighborhood of E which contains only finitely many complete conjugate sets of algebraic integers.
2. If $\tau(E) < 1$, then every neighborhood of E contains infinitely many such sets.

This is our main motivation to investigate the capacity of a finite union of segments of the real axis. To highlight the strong link between capacities and finite gap potentials we present two main ideas:

One the one hand, we know from [22] and [23] that the spectrum of the Schrödinger operator

$$H = -\frac{\partial^2}{\partial x^2} + g(g+1)\wp(x)$$

where $\wp(x) = \wp(x; \omega_1, \omega_2)$ is the Weierstrass $\wp$-function, of periods $2\omega_1$ and $2\omega_2$, contains g bands of an absolutely continuous spectrum. Moreover, in the case of a one-gap situation, the potential can be recovered by using one of the four Jacobi theta functions $\vartheta_i(\frac{x}{2\omega_1})$, in the following form

$$u(x) = -2\partial_x^2 \log \vartheta_1(\frac{x}{2\omega_1}).$$

An extension to g-gaps potential was given by Its and Matveev, using the Riemann theta function of a hyperelliptic Riemann surface of genus g. It is instructive to consult [28] for a survey and the history of the finite-gap problem, and also [3] and [8].

The important theorem of Burchnall and Chaundy [21] asserts that if P and Q are two commuting linear ordinary differential operators of positive order,

then they satisfy a non-trivial polynomial relation $f(P,Q)=0$, where $f(X,Y)\in\mathbb{C}[X,Y]$ is a nonzero polynomial, with

$$\deg_X f = \operatorname{order} Q, \quad \deg_Y f = \operatorname{order} P.$$

Thus, a pair of commuting operators gives rise to a plane algebraic curve. As an example, we give the following explicit example due to Wallenberg [43]: A necessary and sufficient condition that the two differential operators

$$P=\frac{d^2}{dx^2}, \quad Q=\frac{d^2}{dx^2}-3u\frac{d}{dx}-\frac{3}{2}u'(x)$$

commute is that u satisfies the KdV equation

$$u'''=12uu'.$$

By integrating we obtain

$$u'^2=4u^3-g_2u-g_3$$

where g_2, g_3 are arbitrary constants. This is the classical differential equation of the Weierstrass $\wp$-function. In this case, we have the algebraic relation between P and Q:

$$Q^2=P^3-\frac{g_2}{4}P-\frac{g_3}{4}.$$

This is the first part of a work on the deep connections that the geometry of planar domains can have with the geometric functions theory. This program has already been started in [15]. In the second part [41] we intend to continue with the study of the variations of the Bergman and Szegö kernels, directly related to the function of Green's function by the classical Schiffer's formulas. This second part will take up again the theorems (4.7) and (4.8) considering the inverse problems: Under what conditions a finite union of intervals $E=\bigcup_{i=1}^{2g+1}[e_i,\, e_{i+1}]$ is the spectrum of a Jacobi matrix? This question has been considered already in [12], but we will stress more on the links of this question with the exact solutions of the periodic Toda lattice.

The remaining sections of the paper are organized as follows:

In Section 2 we recall the equivalent definitions of the transfinite diameter and some needed results. In Section 3 we define Jacobi matrices and give a result on the capacity of their spectrum. This uses a discrete version of the Burchnall and Chaundy theorem. In Section 4 we make a link between oscillations, differential equations and Tchebychev polynomials. Section 5 considers the question of the density of calibrated domains and the Caratheodory kernel theorem. In Section 6 we study the variations of the periods in the genus one case, when one of the end point varies and the others are kept fixed. We thus obtain a kind of Picard–Fuchs equation. We give an expression for the capacity of several intervals in Section 7. This section owes much to [10], [11] and [12]. The new point is the mention made of Thomae's formula. These formulas play an important role in the forthcoming paper [41]. Sections 8 and 9 are concerned with the case of three intervals. The interesting point here is that the transfer theorems are interpreted, in some cases, as equivalent to the reduction of the hyperelliptic theta functions to theta functions of a smaller

number of variables. A nice algorithm, due to J.-F Mestre, is also given. We end, in Section 10, by a table given in order to explain the passage from the Rosenhain notations to those of the theta characteristics more commonly used today. We have deliberately given many relations on theta functions, because on the one hand they allow us to give concise expressions of Green's functions and capacities, and on the other hand to enable the reader to realize that the formulas, well known in the case of genus one, have their equivalent in the higher genus hyperelliptic case.

I would like to express my warmest thanks to Thérèse Falliéro for the wonderful collaboration we had over years.

2. General definitions

2.1. Capacity, Transfinite diameter

We recall the classical definitions of capacities and Tchebychev polynomials. We refer to [1] for more details. We fix a compact set $E \subset \mathbb{C}$ and we assume that E contains infinitely many points.

For each $n \in \mathbb{N}^*$, we introduce the diameter $d_n = d_n(E)$, of order n, by:

$$d_n = \max_{z_i \in E} \prod_{1 \leq i < j \leq n} |z_i - z_j|^{\frac{2}{n(n-1)}}.$$

Hence $d_2 = \max_{z,z' \in E} |z - z'|$ is just the geometric diameter of E. Also if $z_1, \ldots, z_n \in E$, $\prod_{1 \leq i < j \leq n} |z_i - z_j|^{\frac{2}{n(n-1)}}$ is the geometric mean of the distances $|z_i - z_j|$, $i \neq j$, and d_n is the supremum of these geometric means. Furthermore if $z_1, \ldots, z_n \in E$, $\prod_{1 \leq i < j \leq n} |z_i - z_j|^{\frac{2}{n(n-1)}}$ is the $\frac{2}{n(n-1)}$-th power of the module of the Vandermonde determinant $\prod_{1 \leq i < j \leq n} (z_i - z_j)$. An important result, due to Fekete, asserts that that the sequence of positive reals $(d_n)_{n \in \mathbb{N}^*}$ is decreasing, its limit is denoted by $\tau(E)$ and is called the capacity or the transfinite diameter of E ([1], Theorem 2-3, p. 28).

Moreover, for each fixed $n \in \mathbb{N}^*$, there exists at least a monic polynomial $T_n(z)$ of degree n such that

$$\max_{z \in E} |T_n(z)| = \inf\{\max_{z \in E} |P(z)|,\ P \in \mathcal{P}_n\}$$

where $\mathcal{P}_n$ is the set of all the monic polynomials, of degree n and with complex coefficients. Actually a result of Tonelli asserts that this polynomial is unique and is called the n-th Tchebychev polynomial. It has two important properties

1. It is of degree n and reaches its supremum m_n at $n+1$ points of E.
2. The sequence (m_n) is submultiplicative, hence the sequence $(m_n^{\frac{1}{n}})$ converges and its limit is $\tau(E)$.

A third approach for capacities is through Green's functions. If E is a compact set in $\mathbb{C}$, that is connected and bounded by a finite number of piecewise analytic Jordan curves, then $D = \mathbb{P}^1 \setminus E$ has a Green's function $G(z)$ with pole at ∞. So $G(z)$ is harmonic in $\mathbb{P}^1 \setminus E$, vanishes on ∂D and has the asymptotic behaviour at ∞, given by:

$$G(z) = \log|z| + \gamma + \epsilon(z), \qquad \lim_{|z|\to\infty} \epsilon(z) = 0.$$

The constant γ is the Robin constant. It is proved in [1] (p. 25–27) that the capacity $\tau(E)$ is related to the Robin constant by

$$\tau(E) = \mathrm{e}^{-\gamma}. \tag{2.1}$$

Finally we will need the following transfer theorem:

Theorem 2.1 (Fekete). *Let $P(z) = z^k + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients and $k > 0$. Let $B \subset \mathbb{C}$ be a compact set and A its inverse image by the map $z \to w = P(z)$. Then*

$$\tau(B) = \tau(A)^k.$$

and an extension of it:

Theorem 2.2 (Robinson). *Let $P(z) = z^k + \cdots + a_1 z + a_0$ and $Q(z) = z^l + \cdots + b_1 z + b_0$ be two coprime polynomials with complex coefficients and $l < k$. Let $B \subset \mathbb{C}$ be a compact set and A its inverse image by the map $z \to w = \dfrac{P(z)}{Q(z)}$. We suppose that $|Q(z)| = 1$ for every $z \in A$. Then*

$$\tau(B) = \tau(A)^k.$$

3. Difference operators

3.1. General definitions

We considered the problem of capacities in [10], [11] and [12] and, in the present work, we would like to emphasize on the link between periods of hyperelliptic integrals and hyperelliptic θ-constants. We say that a matrix $C = (a_{ij})_{i,j\in\mathbb{Z}}$ has support $[-M', M]$ if a_{ij} is zero outside the band $-M' \leq i - j \leq M$. A difference operator A can be defined as follows

$$(Ay)_n = \sum_{k=-M'}^{k=M} a_{n,n+k} y_{n+k} = \left(\sum_{-M'}^{M} a_{n,n+k} D^k \right) y_n.$$

A generalized Jacobi matrix of order $2m$ is an infinite matrix $T = (a_{j,k})_{j,k\in\mathbb{Z}}$ such that $a_{j,k} = 0$ for $|j - k| > 2m$ and $a_{j,k} \neq 0$ for $|j - k| = m$. We follow the presentation of [32] and the notations of [30] where a more general definition is introduced. We consider only the case where T is symmetric and N-periodic, that is $a_{i+N,j+N} = a_{i,j}$ for $i, j \in \mathbb{Z}$. A difference operator A associated to T is defined in the following way.

Let y be a vector $= (\ldots, y_{-1}, y_0, y_1, \ldots)^t$, and D the shift operator acting on y by $(Dy)_k = y_{k+1}$. The operator A is given by

$$(Ay)_n = \sum_{k=-m}^{k=m} a_{n,n+k} y_{n+k} = \left(\sum_{-m}^{m} a_{n,n+k} D^k \right) y_n.$$

For each positive integer n, we introduce the matrix $E_n = (e_{ij})_{i,j\in\mathbb{Z}}$, $e_{ij} = 1$ if $|i-j| = n$ and $e_{ij} = 0$ otherwise. In other words, $E_n = D^n + D^{-n}$, $e_{k,k-n} = e_{k,k+n} = 1$, for $k \in \mathbb{Z}$.

3.2. Periodic Jacobi matrices

If A is a generalized Jacobi matrix of order $2m$ and period N, then it is permutable with the matrix E_N. The discrete version of the theorem of Burchnall and Chaundy on commuting differential operators [21] says that there exists a polynomial $F(X,Y) \in \mathbb{C}[X,Y]$ of degree $2N$ in X and degree $2m$ in Y such that $F(A, E_N) = 0$. This result is given in [32] without proof. The main result of this section is to make explicit the polynomial F when $m = 1$ and A is symmetric, the case of interest for us. We assume in the following that A is a Jacobi matrix with period N, so it is tridiagonal and N-periodic. One of the main results in this work is the following

Theorem 3.1. *Let A be a Jacobi matrix, which is symmetric and N-periodic, then there is a polynomial $\widetilde{P}(X)$ in $\mathbb{C}[X]$ of degree N, such that $\widetilde{P}(A) = E_N$. Furthermore, the spectrum $\sigma(A)$ of A is compact in $\mathbb{C}$ and the polynomial $\widetilde{P}(X)$ is up to a constant factor the Tchebychev polynomial of degree N of $\sigma(A)$. Moreover, if*

$$A = \begin{pmatrix} . & . & . & . & . & . & ... & . & . & . & . \\ . & . & a_0 & b_0 & . & . & ... & . & . & . & . \\ . & . & b_0 & a_1 & b_1 & . & ... & . & . & . & . \\ . & . & . & b_1 & a_2 & b_2 & ... & . & . & . & . \\ . & . & . & . & . & . & ... & .b_{N-1} & . & . & . \\ . & . & . & . & . & . & ... & a_N & b_N & . & . \\ . & . & . & . & . & . & ... & . & . & . & . \end{pmatrix}$$

with $a_{i+N} = a_i$, $b_{i+N} = b_i$, $i \in \mathbb{Z}$ and $b_0 b_1 \cdots b_{N-1} \neq 0$, then the capacity of the spectrum of A is given by

$$\tau(\sigma(A)) = |b_0 b_1 \cdots b_{N-1}|^{\frac{1}{N}}.$$

We thus have a sort of multiplicative Cayley–Hamilton theorem. From now on we assume the "ellipticity" condition: $b_0 b_1 \cdots b_{N-1} \neq 0$. From the commutation relation $AE_N = E_N A$, a common eigenvector of the operators A and E_N , $y = (y_k)_{k\in\mathbb{Z}}$, exists. We denote by h and μ the corresponding eigenvalues. Hence there

exists $2N+2$ simultaneous relations of the form:

$$\begin{aligned}
b_0y_0 + a_1y_1 + b_1y_2 &= hy_1\\
b_1y_1 + a_2y_2 + b_2y_3 &= hy_2\\
&\cdots\\
b_{N-1}y_{N-1} + a_Ny_N + b_Ny_{N+1} &= hy_N\\
b_0y_N + a_1y_{N+1} + b_1y_{N+2} &= hy_{N+1}\\
b_1y_{N+1} + a_2y_{N+2} + b_2y_{N+3} &= hy_{N+2}\\
&\cdots\\
b_{N-1}y_{2N-1} + a_Ny_{2N} + b_Ny_{2N+1} &= hy_{2N}\\
y_0 + y_{2N} &= \mu y_N\\
y_1 + y_{2N+1} &= \mu y_{N+1}.
\end{aligned}$$

In order for these equations to be consistent, the eigenvalues h and μ must satisfy $F(h,\mu)=0$, where $F(h,\mu)$ is the following determinant of order $2N+2$:

$$F(h,\mu) = \begin{vmatrix}
b_0 & a_1-h & a_1 & & & & & & & & & & \\
 & b_1 & a_2-h & a_2 & & & & & & & & & \\
 & & & & & & & & & & & & \\
 & & & & b_{N-1} & a_N-h & a_N & & & & & & \\
 & & & & & b_0 & a_1-h & b_1 & & & & & \\
 & & & & & & b_1 & a_2-h & b_2 & & & & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & a_{N-1} & a_N-h & b_N & \\
1 & & & & & -\mu & & & & & 1 & & \\
 & 1 & & & & & -\mu & & & & & 1 &
\end{vmatrix}$$

The coefficient a_1-h in the center of the determinant is on the $(N+1)$-th line and on the $(N+1)$-th column. It is easy to see that $F(h,\mu)$ is a polynomial of degree 2N in h and of degree 2 in μ. A remarkable fact is, as shown in [12], that this polynomial is a perfect square in $\mathbb{C}[h,\mu]$. More precisely, if:

$$\Delta(i,j) = \begin{vmatrix}
a_i-h & b_i & \cdots & \cdots & \cdots & \cdots \\
b_i & a_{i+1}-h & b_{i+1} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & b_{j-2} & a_{j-1}-h & b_{j-1} \\
\cdots & \cdots & \cdots & \cdots & b_{j-1} & a_j-h
\end{vmatrix},$$

then the following holds:

Proposition 3.2. *If $P(h) = \Delta(2,N) - b_0^2\Delta(2,N-1)$, the polynomial $F(h,\mu)$ is given by*

$$F(h,\mu) = \left(P(h) + (-1)^{N+1}\mu \prod_{0\le i\le N-1} b_i\right)^2.$$

Consequently, if we denote by $\mathcal{B}$ the product $\prod_{i=0}^{N-1} b_i$, the polynomial $\widetilde{P}(X)$ in the statement of the theorem is

$$\widetilde{P}(X) = \frac{(-1)^N}{\mathcal{B}} P(X).$$

We shall refer to $\mathcal{B} = \prod_{i=0}^{N-1} b_i$ as the modulus of the Jacobi matrix A. The ellipticity condition (also called regularity condition in [31]) $\mathcal{B} \neq 0$ is assumed.

From the relation $\widetilde{P}(A) = E_N$ and the fact that the spectrum $\sigma(E_N)$ is the interval $[-2, 2]$, we obtain from the functional calculus that the spectrum $\sigma(A)$ of A is the inverse image of $[-2, 2]$ under $\widetilde{P}$, consisting of N algebraic arcs, which may possibly have common endpoints. It is a compact set in $\mathbb{C}$ and according to transfer theorems (2.2) or (2.1) we have

Theorem 3.3. *The capacity of the spectrum of a N-periodic Jacobi matrix is given as a geometric mean by:*

$$\tau(\sigma(A)) = |\mathcal{B}|^{\frac{1}{N}}.$$

It depends only on the outer diagonal.

Before continuing further in the proof of the theorem, we give here the example $N = 1$ of the classical Tchebychev polynomials. Let A be the matrix

$$A = \begin{pmatrix}
. & \ddots & \ddots & . & . & . & \dots & . & . & . & . \\
. & \ddots & 0 & \frac{1}{2} & . & . & \dots & . & . & . & . \\
. & . & \frac{1}{2} & \ddots & \ddots & . & \dots & . & . & . & . \\
. & . & . & \ddots & \ddots & \ddots & \dots & . & . & . & . \\
. & . & . & . & \ddots & \ddots & \ddots & . & . & . & . \\
. & . & . & . & . & .. & . & . & . & . \\
. & . & . & . & . & \frac{1}{2} & 0 & \frac{1}{2} & . & . \\
. & . & . & . & . & .. & \ddots & \ddots & \ddots & . \\
. & . & . & . & . & .. & . & . & . & .
\end{pmatrix}.$$

The associated operator L is given by $(Lu)_j = \frac{1}{2}u_{j-1} + \frac{1}{2}u_{j+1}$. In the relations

$$\frac{1}{2}u_{j-1} + \frac{1}{2}u_{j+1} = zu_j,$$

we set $z = \cos\theta$, then

$$u_j = \frac{\sin(j+1)\theta}{\sin\theta}, \quad j = 0, \pm 1, \dots$$

If we introduce the variable z:

$$u_j = P_j(z) = \frac{\sin\left((j+1)\arccos z\right)}{\sin\arccos z}, \quad j = 0, \pm 1, \ldots,$$

we obtain the Tchebychev polynomials of the second kind, associated to the Cauchy problem $u_o = 1, u_{-1} = 0$. With the new initial conditions $v_o = 0$, $v_1 = 2$, we get

$$\frac{1}{2}v_{j-1} + \frac{1}{2}v_{j+1} = zv_j, \quad v_o = 0,\ v_1 = 2,$$

$$v_j = Q_j(z) = 2P_{j-1}(z).$$

One can show that

$$\frac{2}{\pi}\int_{-1}^{1} P_j(\lambda)P_k(\lambda)\sqrt{1-\lambda^2}d\lambda = \delta_{jk},$$

and

$$\sigma(L) = [-1, 1], \quad \sigma(E_1) = [-2, 2].$$

Furthermore, the capacity of the spectrum of L is related to the outer diagonal in the matrix A by:

$$\tau(\sigma(L)) = \frac{1}{2} = b_i,$$

in accordance with the general theorem. Here the matrix A is 1-periodic, the outer diagonal contains only $\frac{1}{2}$ as element. These polynomials are solutions of the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0. \tag{3.1}$$

We now turn to the last statement of our theorem. We will use, in a crucial way, a result of Ostrovskii, Pakovitch and Zaidenberg [34]. We first recall some definitions.

Let $D = D(a, r)$ be a closed disc centered at $a \in \mathbb{C}$ and of radius r. We say that a compact $K \subset D$ supports D if D is the (unique) disc of the least radius which contains K. We will call D the Tchebychev disc of the compact K and a its Tchebychev center.

For a given compact $K \subset \mathbb{C}$, a monic polynomial $p(z) \in \mathbb{C}[z]$ of degree $n \geq 0$ is called the n-th polynomial of least deviation (from zero) or the Tchebychev polynomial of degree n if $||p||_K \leq ||q||_K$ for any monic polynomial $q(z) \in \mathbb{C}[z]$ of degree n, where $||p||_K = \max_{z\in K} |p(z)|$. The classical inequality of Jung [25] asserts that each compact convex K has a Tchebychev disc.

Theorem 3.4. *Let $K \subset \mathbb{C}$ be a compact set. Then:*

1. *There exist a smallest closed disc $D(a, \rho)$ containing K. This disc is unique and a belongs to the closed convex hull* conv$K$ *of K.*
2. *Every compact convex K of diameter δ is contained in a closed disc $D(a, \rho)$, with*

$$\frac{\delta}{2} \leq \rho \leq \frac{\sqrt{3}}{3}\delta.$$

The second inequality is optimal. It is an equality in the case of an equilateral triangle.

Remark 3.5. It is important for us to observe that the various proofs of Jung's theorem reveal that if $D(a,\rho)$ is the Tchebychev disc of a compact K, then every closed half-circle of the circle $\{|z-a|=\rho\}$ meets K at at least one point.

The following theorem is the main result in [34].

Theorem 3.6. *Let $\Delta_r = \Delta(0,r) \subset \mathbb{C}$ be the disc of radius r centered at the origin, $K \subset \Delta_r$ be a supporting compact of Δ_r, and $p \in \mathbb{C}[z]$ be a monic polynomial of degree n. Then p is the unique n-th polynomial of least deviation on $K_p = p^{-1}(K)$.*

This theorem shows clearly that the polynomial $\tilde{P}(X)$ in Theorem (3.1) is

$$\tilde{P}(X) = \frac{(-1)^N}{\mathcal{B}} P(X).$$

The polynomial $(-1)^N P(X)$ is the Tchebychev polynomial of $E = \sigma = \sigma(A)$, the diameter $[-|\mathcal{B}|^{\frac{1}{N}}, |\mathcal{B}|^{\frac{1}{N}}]$ supports the disc centered at the origin, with radius $|\mathcal{B}|^{\frac{1}{N}}$.

We thus have established, under the preceding conditions, a relation between the theorem of Burchnall and Chaundy and the problem of the explicit determination of a Tchebychev polynomial of some degree and the capacity of some compact sets.

It is perhaps worthwile noticing that Tchebychev polynomials appear in a different kind of commutation problem also related to discrete integrable systems which is the problem of Fatou and Julia on commuting polynomials [40] and [42].

Remark 3.7. Let H be a Hilbert space and $X \subset H$ a bounded set, then X has a Tchebychev center: there exists a unique element $h_0 \in H$ such that the function $h \mapsto \sup_{x\in X} |h-x|$ reaches its minimum at h_0. In [26], Kadets and Zamyatin showed that the bounded sets in the Banach space $\mathcal{C}([a,b],\mathbb{R})$ of real-valued continuous functions on $[a,b]$ admit Tchebychev centers. However this result is not true for an arbitrary Banach space [44].

4. Tchebychev Polynomials and Oscillations

4.1. Green's functions

The complex Green's function with pole at infinity in $\widehat{\mathbb{C}} \setminus E$ is given by the abelian integral [10] (and the references cited there):

$$G(z) = \int_{E_{2n}}^{z} h(\zeta) q(\zeta)^{-\frac{1}{2}} d\zeta \tag{4.1}$$

where

$$q(\zeta) = (\zeta - E_1)(\zeta - E_2)\cdots(\zeta - E_{2n})$$

and $h(\zeta)$ is a monic polynomial of degree $n-1$ of the form

$$h(\zeta) = \zeta^{n-1} + h_{n-2}\zeta^{n-2} + \cdots h_0$$

with coefficients satisfying the following system of equations (a Jacobi Inversion Problem)

$$\sum_{0\le i\le n-1} h_i \int_{E_{2j}}^{E_{2j+1}} \zeta^i |q(\zeta)|^{-\frac{1}{2}}\, d\zeta = 0, \quad 1 \le j \le n-1. \tag{4.2}$$

It can be seen that the roots $c_1, c_2, \ldots, c_{n-1}$ of the polynomial $h(\zeta)$ (equilibrium points) have the following distribution:

$$E_2 < c_1 < E_3 < c_2 \cdots E_{2n-2} < c_{n-1} < E_{2n-1}.$$

This is just another version of the classical theorem of Gauss–Lucas.

The system (4.2) is linear in h_i. Once this system is solved, we get from (2.1) the Green's function and the capacity. Indeed, from (4.1) we have:

Proposition 4.1.

$$G(z) = \log\{z - (e_1+1)\} + \int_{e_1}^{z} \left(\frac{h(\zeta)}{f(\zeta)^{1/2}} - \frac{1}{\zeta - (e_1+1)} \right) d\zeta, \tag{4.3}$$

$$\tau(E) = \exp\left(\int_{-\infty}^{e_1} \left(\frac{h(\zeta)}{f(\zeta)^{1/2}} - \frac{1}{\zeta - (e_1+1)} \right) d\zeta \right). \tag{4.4}$$

There is a deep connection between oscillations, critical points, critical values and Tchebychev polynomials. This connection can be traced back to the following statement, due to Polya [18, p. 46]:

Theorem 4.2 (Polya). *Let $k(x), g(x)$ denote two positive and continuously differentiable functions on an interval $I \subset \mathbb{R}$. Consider a solution y, on I, of the linear second-order differential equation*

$$\frac{d}{dx} k(x) y' + g(x) y = 0.$$

If $k(x)g(x)$ is non-increasing(resp., non-decreasing), then the relative maxima of y^2 are non-decreasing (resp., non-increasing).

Applied to the equation (3.1), written in the form

$$\frac{d}{dx}\left((1-x^2)^{1/2} y' \right) + n^2 (1-x^2)^{-1/2} y = 0,$$

we find that $kg = n^2$ and then the relative maxima of $T_n(x)^2$ must assume the same value. As remarked in [18, p. 46], this property on critical values may be used as a defining property. As it will be fundamental for us in our study of the case of several intervals, we give here the details.

Proposition 4.3. *Let $f_n(x)$ be a polynomial of degree n such that $f_n^2(x_0) = A^2$ whenever $f_n'(x_0) = 0$, then up to an affine change of variables*

1. $(1-x^2) f_n''(x) - x f_n'(x) + n^2 f_n(x) = 0$,
2. $f_n(x) = A \cos[n \arccos x]$.

The idea of the proof goes as follows:

$$\frac{f_n^2(x) - A^2}{(f_n'(x))^2} \tag{4.5}$$

is a rational function without poles at finite distance. So it is a quadratic polynomial that we may assume of the form $k(x^2 - 1)$. In particular $f_n^2(\pm 1) = A^2$. Hence

$$\frac{f_n^2(x) - f_n^2(\pm 1)}{(f_n'(x))^2} = n^2(x^2 - 1). \tag{4.6}$$

It is natural to state the following

Definition 4.4. A real polynomial P of degree n is called g-hyperelliptic if and only if it satisfies a differential equation of the form

$$P^2(z) - \left[\frac{P'(z)}{n(z - x_1)\cdots(z - x_g)}\right]^2 R_{2g+2}(z) = 1 \tag{4.7}$$

with $x_1, \ldots, x_g \in \mathbb{C}$ and R is a polynomial of degree $2g + 2$ with simple and real roots.

From (4.7), we see that $x_1, \ldots, x_g$ are zeros of the derivative P'. There is a deep link between the extremal properties of the Tchebychev polynomial and their oscillations properties. In the case of one interval, this is contained in the Equal Ripple Theorem, essentially due to Tchebychev and concerns the quantity $\mu = \min_{\{p\}} \max_{-1\leq x\leq 1} |f(x) - p(x)|$, where f is continuous on the interval $[-1\ ,\ 1]$ and $p(x)$ is an arbitrary polynomial of degree less or equal to N.

Theorem 4.5 (The Equal Ripple Theorem). *There is a unique polynomial of best approximation to $f(x)$ in $[-1\ ,\ 1]$ and it is characterized by the existence of a sequence of at least $N + 2$ points at which the error function $e(x) = f(x) - p(x)$ assumes the values $\pm\mu$, the signs alternating.*

We wish to explain briefly what happens for several intervals. We first need to make precise the notion of oscillation and we refer to [12] and [36] for the details.. Suppose E is a union of closed segments in $\mathbb{R}$. A monic polynomial $p(z) \in \mathbb{R}[z]$ of degree n oscillates n-times between M_1 and M_2 on E if

1. $p(z)$ is real for all $z \in E$ with $M_1 \leq p(z) \leq M_2$;
2. at each endpoint of a segment in E, $p(z) = M_1$ or $p(z) = M_2$;
3. there are n points of E (counted with multiplicity) where $p(z)$ takes the value M_1, and likewise for M_2.

If $E = [E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2g+1}, E_{2g+2}]$, we can no longer say that the Tchebychev polynomial of degree N oscillates N-times on E between $\pm M$, but we can find $N + 1$ special set of points of E where $T_N(X)$ takes the values $\pm M$ with alternating signs. If two consecutive points of this special set are contained in the same component of E, then $T_N(X)$ oscillates from $+M$ to $-M$ or from $-M$ to $+M$, but if they are in different components, then various behaviours are possible on the finite gaps (bounded components of $\mathbb{R} \setminus E$). If we require that in

$\mathbb{R} \setminus E$, $|T_N(x)| > M$, then $T_N(X)$ will be the N-th Tchebychev polynomial of E because it oscillates N times between $+M$ and $-M$. These remarks motivate an analogue of (4.6). In fact the roots of $T_N(X) = \pm M$ are all double roots except for the ends E_i, $1 \leq i \leq 2g+2$. There are $N-g-1$ of these double roots which are also roots of $T'_N(X)$ and this last equation has g additional roots $c_1, c_2, \ldots, c_g$ located in the gaps, so that $E_1 < E_2 < c_1 < E_3 < E_4 < c_2 < \cdots < c_g < E_{2g+1} < E_{2g+2}$. This leads to the identity

$$\frac{N^2[T_N(x)^2 - M^2]}{(x-E_1)(x-E_2)\cdots(x-E_{2g+1})(x-E_{2g+2})} = \frac{T'_N(x)^2}{(x-c_1)^2(x-c_2)^2\cdots(x-c_g)^2}.$$

This is the analog of equation(4.6) for several intervals and it has the form (4.7). It is a nonlinear differential equation of first order and admits the solution

$$f(z) = M \cosh\left(N \int_{E_{2g+2}}^{z} \sqrt{Q(t)}dt\right) \tag{4.8}$$

where for $E_{2g+2} \leq z = x \in \mathbb{R}$, the positive square root is used and

$$Q(t) = \frac{(t-c_1)^2(t-c_2)^2\cdots(t-c_g)^2}{(t-E_1)(t-E_2)\cdots(t-E_{2g+1})(t-E_{2g+2})}.$$

Proposition 4.6. *In order for the function $f(z)$ given by* (4.8) *to be single valued, it is necessary that*

$$\int_{E_{2k}}^{E_{2k+1}} \sqrt{Q(t)}dt = 0, \quad k = 1, 2, \ldots g. \tag{4.9}$$

And in order for $f(z)$ to oscillate n_k times between $\pm M$ in the k-th interval, it must satisfy

$$\int_{E_{2k-1}}^{E_{2k}} \sqrt{Q(t)}dt = \pm\frac{n_k \pi i}{N}, \quad k = 1, 2, \ldots g+1, \quad n_1+n_2+\cdots+n_{g+1} = N. \tag{4.10}$$

If these conditions are satisfied, the fonction $f(z) = T_N(z)$ is the N-th Tchebychev polynomial of $E = [E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2g+1}, E_{2g+2}]$.

For the sake of completeness we give, without proof, the statements of two important theorems and we refer to [12] for their proofs.

Theorem 4.7. *Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ a g-hyperelliptic real polynomial such that $R(z) = (z-E_1)(z-E_2)\cdots(z-E_{2g+2})$. Then the polynomial $\tilde{P}(z) = \frac{1}{|a_n|}P(z)$ is of least deviation from zero among all the monic polynomials of degree n. The modulus of this deviation is $\frac{1}{|a_n|}$ and $\tilde{P}$ is the Tchebychev polynomial of degree n of $E = [E_1, E_2] \cup [E_3, E_4] \cup \ldots \cup [E_{2g+1}, E_{2g+2}]$.*

and

Theorem 4.8. *Let $E = [E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2g+1}, E_{2g+2}]$ be a finite union of closed intervals Then there exists a Jacobi matrix A, real, symmetric and N-periodic, $g+1 \leq N$ with the spectrum $\sigma(A)$ equal to E if and only if the N-th Tchebychev polynomial of E oscillates N times over E.*

5. Caratheodory kernel theorem

In order to make precise the forcoming statements we recall some definitions that will be needed. Given a bounded set $\mathcal{E}$ in the complex plane, we denote by $\mathcal{E}(r)$ the r-neighborhood of $\mathcal{E}$.

Definition 5.1. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two bounded sets in $\mathbb{C}$, we define the difference between $\mathcal{E}_1$ and $\mathcal{E}_2$ as the smallest r such that $\mathcal{E}_1(r)$ contains $\mathcal{E}_2$ and $\mathcal{E}_2(r)$ contains $\mathcal{E}_1$.

We will denote this difference by $\delta(\mathcal{E}_1, \mathcal{E}_2)$, it is small when $\mathcal{E}_1$ and $\mathcal{E}_2$ are almost superposable. If $\mathcal{E}$ is a compact set and $(\mathcal{E}_\nu)$ a family of sets such that $\delta(\mathcal{E}, \mathcal{E}_\nu)$ tends to zero as ν tends to ∞, we will say simply that $(\mathcal{E}_\nu)$ tends to $\mathcal{E}$.

Theorem 5.2 (G. Calugareanu). *If $(\mathcal{E}_\nu)$ tends to $\mathcal{E}$, then for a fixed n, the Tchebychev polynomials $T_n(z, \mathcal{E}_\nu)$ tend to the Tchebychev polynomials $T_n(z, \mathcal{E})$ as ν tends to ∞, uniformly on $\mathcal{E}$.*

On the one hand, the most powerful result to study variations of domains is the Carathéodory kernel theorem [14].

Definition 5.3. Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of domains in the Riemann sphere $\hat{\mathbb{C}}$ such that a fixed point z_0 belongs to all Ω_n excluding eventually a finite number of them. A domain Ω is said to be the kernel of $\{\Omega_n\}_{n=1}^{\infty}$ and we write $\Omega = \mathrm{Ker}_{z=z_0}\{\Omega_n\}$ if Ω satisfies:

1. $z_0 \in \Omega$;
2. any compact set $K \subset \Omega$ belongs to all Ω_n for $n \geq n(K)$;
3. any domain $\tilde{\Omega}$ satisfying the preceding two conditions is a subset of Ω.

If the point z_0 belongs to all Ω_n for $n \geq n(z_0)$ but there is no neighbourhood of z_0 that is contained in all all Ω_n for $n \geq N$ then we say that $\mathrm{Ker}_{z=z_0}\{\Omega_n\} = \{z_0\}$. A sequence $\{\Omega_n\}_{n=1}^{\infty}$ converges to the kernel Ω with respect to z_0 if every subsequence $\{\Omega_{n_k}\}_{k=1}^{\infty}$ has Ω as its kernel.

An important situation is when the sequence $\{\Omega_n\}_{n=1}^{\infty}\}$ is decreasing. We denote by Ω° the set of interior points of $\Omega = \bigcap_{n=1}^{\infty} \Omega_n$ and we assume $0 \in \Omega$. Then $\{\Omega_n\}$ converges to the component of Ω° that contains 0 if $0 \in \Omega^\circ$ and to $\{0\}$ if $0 \notin \Omega^\circ$. The Caratheodory kernel theorem is:

Theorem 5.4. *Let $f_n(z)$ be analytic and univalent in the unit disc $\mathbb{D}$ with $f_n(0) = 0$, $f_n'(0) > 0$ and let $\Omega_n = f_n(\mathbb{D})$. Then the sequence (f_n) converges locally uniformly in $\mathbb{D} = \mathbb{D}_1$ if and only of the sequence $\{\{\Omega_n\}_{n=1}^{\infty}\}$ converges to its kernel Ω, $\Omega \neq \mathbb{C}$ with respect to the origin. If $\mathrm{Ker}\,\Omega_n \neq \{0\}$, then the limiting function is a univalent map of $\mathbb{D}$ onto Ω. If $\mathrm{Ker}\,\{\Omega_n = \{0\}\}$, then $\lim_{n\to\infty} f_n(z) \equiv 0$.*

Definition 5.5. A compact set $E = [E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2n-1}, E_{2n}]$, $n = g+1$ of the real line is calibrated if the complex Green's function $G(z)$ of $\widehat{\mathbb{C}} \setminus E$ with

pole at infinity given by (4.1) satisfies the conditions (4.9) and (4.10)

$$\int_{E_{2k}}^{E_{2k+1}} \sqrt{Q(t)}dt = 0, \quad k = 1, 2, \dots n-1$$

and

$$\int_{E_{2k-1}}^{E_{2k}} \sqrt{Q(t)}dt = \pm\frac{r_k \pi i}{N}, \quad k = 1, 2, \dots n,$$

where

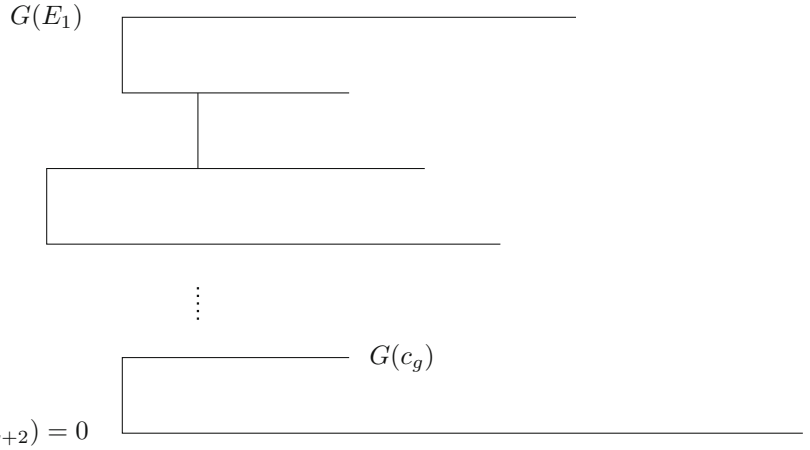
$$Q(t) = \frac{(t-c_1)^2(t-c_2)^2\cdots(t-c_{n-1})^2}{(t-E_1)(t-E_2)\cdots(t-E_{2n-1})(t-E_{2n})} = h(t)^2$$

and $r_1, \dots r_n, N \in \mathbb{N}$, $r_1 + \cdots + r_n = N$.

The following result enables us to see the relevence of this classe of domains:

Theorem 5.6. *Let $E = [E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2g+1}, E_{2g+2}]$ be a finite union of closed intervals. There exists a family of calibrated sets $E_\nu = [E_{1,\nu}, E_{2,\nu}] \cup [E_{3,\nu}, E_{4,\nu}] \cup \ldots \cup [E_{2g+1,\nu}, E_{2g+2,\nu}]$ tending to E.*

For a given $E = [E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2g+1}, E_{2g+2}]$ and $c_j \in [E_{2j},\, E_{2j+1}]$, $1 \le j \le g$, the function $z \mapsto w = G(z)$ maps the upper half plane onto a polygon of the following type:



As we already said, according to the solution of the Jacobi inversion problem there is a unique set

$$\{c_j \in [E_{2j},\, E_{2j+1}],\, 1 \le j \le g\}$$

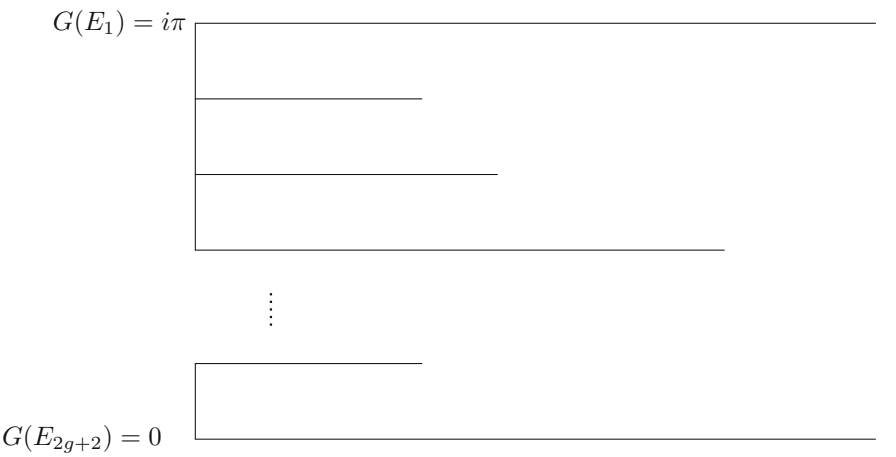
such that the polynomial

$$h(\zeta) = \zeta^g + h_{g-1}\zeta^{g-1} + \cdots + h_0 = (\zeta - c_1)(\zeta - c_2)\cdots(\zeta - c_g)$$

verifies the system (4.2). In this situation,

$$G(E_{2j}) = G(E_{2j+1}), \quad 1 \le j \le g, \quad G(E_1) = i\pi, \quad G(E_{2g+2}) = 0.$$

The map $z \mapsto w = G(z)$ sends the upper half plane onto the polygon Γ of the form:



The calibration of E means that the slits have heights of the form $q/r\pi$, $q, r \in \mathbb{N}^*$.

If a finite union of intervals $E = [E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2g+1}, E_{2g+2}]$ is given, its image Γ by $z \mapsto w = G(z)$ with $G(z)$ satisfying (4.2) does not necessarily satisfy the second condition in the definition of the calibration. But we can find a sequence (Γ_ν) of polygons tending to Γ as ν tends to ∞ in the sense that $\lim_{\nu\to\infty} \delta(\Gamma, \Gamma_\nu) = 0$ where (Γ_ν) is bounded horizontally by the two half lines $\Im w = 0$, $\Im w = i\pi$ and vertically by the imaginary axis and finally has g horizontal slits with heights equal to rational multiples of π. From the definition of the difference $\delta(\Gamma, \Gamma_\nu)$ we see that Γ is the kernel of the family (Γ_ν) in the sense of definition (5.3). By the Riemann mapping theorem there is a unique conformal mapping g_ν of the slit domain Γ_ν onto the upper half plane which, by another theorem of Caratheodory, can be extended continuously and homeomorphically up to the boundary. The image $g_\nu([0, i\pi])$ is a calibrated set $E_\nu = [E_{1,\nu}, E_{2,\nu}] \cup [E_{3,\nu}, E_{4,\nu}] \cup \cdots \cup [E_{2g+1,\nu}, E_{2g+2,\nu}]$ tending to E as ν tends to ∞ by the Caratheodory kernel theorem (5.4). We have thus proved that any finite union E of closed intervals can be approximated in the difference topology by calibrated finite union E_ν of closed intervals. By (5.2), any Tchebychev polynomial $T_n(z, E)$ is a limit of a uniformly convergent sequence on E of Tchebychev polynomials $T_n(z, E_\nu)$ of calibrated E_ν.

To conclude this proof, we add one more remark. The universal covering of $\widehat{\mathbb{C}} \setminus E$ and $\widehat{\mathbb{C}} \setminus E_\nu$ is the upper half plane $\mathcal{H}$. From [17] (Theorem 1), the universal covering maps $\eta_\nu : \mathcal{H} \mapsto \mathbb{C} \setminus E_\nu$ converge uniformly on compact sets of $\mathcal{H}$ to the universal covering map $\eta : \mathcal{H} \mapsto \mathbb{C} \setminus E$.

6. Genus one case

6.1. Differential relations and variations

From the relations (4.1) and (4.2) we see that it is natural to seek the dependence of the Green's function and the capacity on the variation of $E = \bigcup_{i=1}^{2g+1}[e_i,\, e_{i+1}]$, that is on the e_i's. This is a highly interesting problem. It amounts to some differential equations on the periods, like the Picard–Fuchs differential equations. To give an idea we restrict ourselves to the genus one case, that is, $E = [e_1,\, e_2]\cup[e_3,\, e_4]$, $e_1 < e_2 < e_3 < e_4$. It is remarkable that Thomae's formula that will be used later uses also some variational expressions. First we recall some well-known facts from the theory of elliptic functions. An elliptic curve, in the Weierstrass model, is given by an equation of the form

$$y^2 = 4(x-e_1)(x-e_2)(x-e_3) = 4x^3 - g_2x - g_3 = 0$$

where

$$e_1+e_2+e_3 = 0,\ g_2 = -4(e_1e_2+e_2e_3+e_3e_1),\ g_3 = 4e_1e_2e_3.$$

An equation of an elliptic curve of the form

$$y^2 = (x-e_1)(x-e_2)(x-e_3)(x-e_4)$$

can be reduced to the Weierstrass model. We define the abelian integrals of the first and second kind, respectively, by

$$\omega = \int_{e_2}^{e_3}\frac{dx}{dy},\quad \omega' = \int_{e_1}^{e_2}\frac{dx}{dy}, \tag{6.1}$$

$$\eta = \int_{e_3}^{e_2}\frac{xdx}{dy},\quad \eta' = \int_{e_2}^{e_1}\frac{xdx}{dy}. \tag{6.2}$$

We will assume that $e_i, i = 1,2,3$ are real, with $e_1 > e_2 > e_3$. Between half periods ω, ω' and half pseudo-periods η, η' there is the Legendre relation

$$\eta\omega' - \eta'\omega = \frac{i\pi}{2} \tag{6.3}$$

and there is also a relation, given in [16] (formula 36, p. 307):

$$D\omega := 12g_3\frac{\partial\omega}{\partial g_2} + \frac{2}{3}g_2^2\frac{\partial\omega}{\partial g_3} = -2\eta. \tag{6.4}$$

Before proceeding further, we recall the classical definitions of the four theta functions of Jacobi, adopting the notations of Halphen [16].

Definition 6.1. With $q = e^{i\pi\tau}$, the four theta functions are

$$\vartheta_1(v,\tau) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+1/2)^2}e^{i(2n+1)v} = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)v,$$

$$\vartheta_2(v,\tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}e^{i(2n+1)v} = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)v,$$

$$\vartheta_3(v,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{2inv} = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nv,$$

$$\vartheta_0(v,\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2inv} = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nv.$$

Hence, as function of v, $\vartheta_1(v,\tau)$ is odd and the three others are even. They all satisfy the heat equation:

$$4i\pi\frac{\partial\vartheta_j}{\partial\tau} = \frac{\partial^2\vartheta_j}{\partial^2 v},\ j=0,1,2,3.$$

The θ-constants are defined by $\vartheta_1' = \vartheta_1'(0,\tau)$, $\vartheta_2 = \vartheta_2(0,\tau)$, $\vartheta_3 = \vartheta_3(0,\tau)$, $\vartheta_4 = \vartheta_0(0,\tau)$. Between the theta constants, there is two fundamental and important formulas:

$$\vartheta_3^4 = \vartheta_0^4 + \vartheta_2^4, \tag{6.5}$$

$$\vartheta_1' = \vartheta_0\vartheta_2\vartheta_3, \tag{6.6}$$

where $\vartheta_1' = \dfrac{\partial\vartheta_1}{\partial v}(0,\tau)$. The product $\eta\omega$, as we will see in the next section, has a modular properties and is given in terms of the odd theta function by

$$\eta\omega = -\frac{1}{12}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)}, \tag{6.7}$$

see [16, p. 248], [24, p. 416, p. 418]. In terms of the even theta functions we have a relation of a completely different nature:

Proposition 6.2. *The period 2ω and the quasi-period 2η are related to the θ-constants ϑ_2, ϑ_3, ϑ_4 and the branch points by*

$$2\omega = \frac{\pi\vartheta_2^2}{\sqrt{e_2-e_3}} = \frac{\pi\vartheta_3^2}{\sqrt{e_1-e_3}} = \frac{\pi\vartheta_4^2}{\sqrt{e_1-e_2}}, \tag{6.8}$$

$$2\eta = -2\omega e_1 - \frac{1}{2\omega}\frac{\vartheta_2''}{\vartheta_2} = -2\omega e_2 - \frac{1}{2\omega}\frac{\vartheta_3''}{\vartheta_3} = -2\omega e_3 - \frac{1}{2\omega}\frac{\vartheta_4''}{\vartheta_4}. \tag{6.9}$$

The proof of (6.9) is a consequence of (6.7) and the following relation, valid for $\alpha = 1,2,3$ [16, p. 257]:

$$e_\alpha = -\frac{1}{(2\omega)^2}\left(\frac{\vartheta_{\alpha+1}''}{\vartheta_{\alpha+1}} - \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'}\right). \tag{6.10}$$

An immediate consequence of the relations (6.8) is that branch points and theta constants are related by

$$\frac{e_2-e_3}{e_1-e_3}=\frac{\vartheta_2^4}{\vartheta_3^4},\quad \frac{e_1-e_2}{e_1-e_3}=\frac{\vartheta_2^4}{\vartheta_3^4}. \tag{6.11}$$

The branch points are given by theta constants [24]:

$$\begin{aligned} 3e_1&=(\frac{\pi}{2\omega_1})^2\left[\vartheta_2^4(0)+\vartheta_3^4(0\right],\\ 3e_2&=(\frac{\pi}{2\omega_1})^2\left[-\vartheta_3^4(0)-\vartheta_1^4(0\right],\\ 3e_3&=(\frac{\pi}{2\omega_1})^2\left[\vartheta_1^4(0)-\vartheta_2^4(0\right]. \end{aligned} \tag{6.12}$$

Now from the first equation in (6.8) we deduce, as in [9]:

$$\frac{\partial\omega}{\partial e_1}=\frac{2\omega}{\vartheta_2}\frac{\partial\vartheta_2}{\partial e_1}=-\frac{1}{2}\frac{\eta+e_1\omega}{(e_1-e_2)(e_1-e_3)}. \tag{6.13}$$

In the same way

$$\frac{\partial\eta}{\partial e_1}=\frac{1}{2}\frac{e_1\eta-(e_1^2+e_2e_3)\omega}{(e_1-e_2)(e_1-e_3)}. \tag{6.14}$$

As consequence we obtain for $\tau=\dfrac{\omega'}{\omega}$:

$$\frac{\partial\tau}{\partial e_1}=\frac{\partial}{\partial e_1}\frac{\omega'}{\omega}=\frac{1}{4\omega^2}\frac{i\pi}{(e_1-e_2)(e_1-e_3)}. \tag{6.15}$$

The first-order differential equations (6.13) and (6.14) can be put in a differential system:

$$\begin{pmatrix}\dfrac{\partial\omega}{\partial e_1}\\[2ex]\dfrac{\partial\eta}{\partial e_1}\end{pmatrix}=\begin{pmatrix}-\dfrac{1}{2}\dfrac{e_1}{(e_1-e_2)(e_1-e_3)} & -\dfrac{1}{2}\dfrac{1}{(e_1-e_2)(e_1-e_3)}\\[2ex] -\dfrac{1}{2}\dfrac{(e_1^2+e_2e_3)}{(e_1-e_2)(e_1-e_3)} & \dfrac{1}{2}\dfrac{e_1}{(e_1-e_2)(e_1-e_3)}\end{pmatrix}\begin{pmatrix}\omega\\[2ex]\eta\end{pmatrix}. \tag{6.16}$$

We need the following elementary lemma:

Lemma 6.3. *Let $(f(t),g(t)$ be two twice differentiable functions on some interval I, $a(t),b(t),c(t),d(t)$ be differentiable on I. We assume that $b(t)$ and $c(t)$ do not vanish identically on I and*

$$\begin{pmatrix}f'(t)\\[2ex]g'(t)\end{pmatrix}=\begin{pmatrix}a(t) & b(t)\\[2ex] c(t) & d(t)\end{pmatrix}\begin{pmatrix}f(t)\\ g(t)\end{pmatrix}, \tag{6.17}$$

then $f(t)$ and $g(t)$ satisfy the second order differential equations

$$\frac{d^2f(t)}{dt^2}+\left(-a-d-\frac{b'}{b}\right)\frac{df(t)}{dt}+\left(ad-bc+\frac{a}{b}b'-a'\right)f(t)=0 \tag{6.18}$$

and

$$\frac{d^2g(t)}{dt^2}+\left(-a-d-\frac{c'}{c}\right)\frac{dg(t)}{dt}+\left(ad-bc+\frac{d}{c}c'-d'\right)g(t)=0. \tag{6.19}$$

With Lemma 6.3 we obtain a second-order differential equation describing the variation of ω and η when e_1 varies and e_2, e_3 are kept fixed. We give, as example, the equation for ω where t stands for e_1:

$$\frac{d^2\omega(t)}{dt^2}+\frac{2t-e_2-e_3}{(t-e_2)(t-e_3)}\frac{d\omega(t)}{dt}+\frac{-2te_2-2te_3+e_2e_3}{4(t-e_2)^2(t-e_3)^2}\omega(t)=0. \tag{6.20}$$

6.2. Eisenstein series

In this short subsection, we introduce the Eisenstein series for two reasons. On the one hand, they have a link with the invariants g_2 and g_3 introduced through:

$$y^2=4(x-e_1)(x-e_2)(x-e_3)=4x^3-g_2x-g_3=0$$

and, on the other hand, they can also be defined in higher genus. In the case of genus one, this link has been exploited for example in [15] but in the higher genus case this link is only in its beginnings.

For $q=e^{2i\pi\tau}$, $\Im\tau>0$ we define:

$$E_4(\tau)=1+240\sum_{n\geq1}\frac{n^3q^n}{1-q^n}=1+240e^{2i\pi\tau}+2160e^{4i\pi\tau}+\cdots,$$

$$E_6(\tau)=1-504\sum_{n\geq1}\frac{n^5q^n}{1-q^n}=1-504e^{2i\pi\tau}-16632e^{4i\pi\tau}+\cdots$$

and for the discriminant function:

$$\Delta(\tau)=\frac{1}{1728}\left(E_4^3(\tau)-E_6^2(\tau)\right).$$

The functions E_4, E_6 and $\Delta(\tau)$ are modular forms of weight 4, 6 and 12, respectively, for the modular group $\mathrm{SL}_2(\mathbb{Z})$. Another important Eisenstein series is

$$E_2(\tau)=1-24\sum_{n\geq1}\frac{nq^n}{1-q^n}=1-24e^{2i\pi\tau}-72e^{4i\pi\tau}-96e^{6i\pi\tau}\cdots.$$

We have, finally, as in [24], for example:

$$E_2(\tau)=3\frac{\omega'}{\pi^2}\eta',\quad E_4(\tau=\frac{3}{4}\left(\frac{\omega'}{\pi}\right)^4g_2,\quad E_6(\tau)=\frac{27}{8}\left(\frac{\omega'}{\pi}\right)^6g_3.$$

7. The capacity of a finite union of p intervals

7.1. General considerations on curves

Let X be an irreducible, smooth, projective curve of genus $g\geq2$ defined over $\mathbb{C}$. We denote the moduli space of genus g curves by $\mathcal{M}_g$ and the hyperelliptic locus in $\mathcal{M}_g$ by $\mathcal{N}_g$. It is known that the dimension of $\mathcal{M}_g$ is $3g-3$ and $\mathcal{N}_g$ is a $(2g-1)$-dimensional subvariety of $\mathcal{M}_g$. Choose a symplectic homology basis

for X, $\{A_1, \dots A_g, B_1, \dots B_g\}$, such that the intersection products satisfy $A_i.A_j = B_i.B_j = 0$ and $A_i.B_j = \delta_{ij}$. We choose a basis $\{\omega_1, \dots, \omega_g\}$ for the space of holomorphic 1-forms such that $\int_{A_i} \omega_j = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. The matrix $\Omega = \left[\int_{B_i} \omega_j\right]$ is the period matrix of X . The columns of the matrix $[I, \Omega]$ form a lattice L in $\mathbb{C}^g$ and the Jacobian of X is $\mathrm{Jac}(X) = \mathbb{C}^g/L$. The Siegel upper-half space is

$$\mathcal{H}_g = \{\tau : \tau \text{ is a symmetric } g \times g\text{-matrix with positive definite imaginary part}\}$$

The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is the subgroup of $\mathrm{GL}_{2g}(\mathbb{Z})$ of matrices M verifying

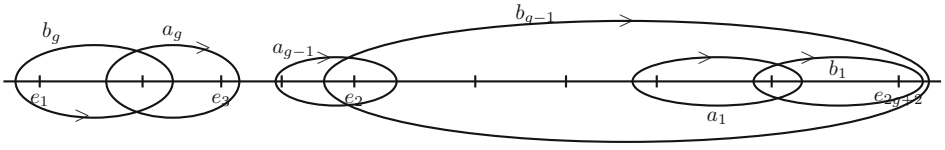
$$M^t J M = J, \quad J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on the Siegel half-plane in the following way:

$$\text{If } M = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}) \text{ and if } \tau \in \mathcal{H}_g \text{ we set } M\tau = (R\tau + S)(T\tau + U)^{-1}.$$

7.2. Hyperelliptic curves and their theta functions

An hyperelliptic curve M, defined over $\mathbb{C}$, is a cover of order two of the projective line $\mathbb{P}^1$. Let z be the generator (the hyperelliptic involution) of the Galois group $\mathrm{Gal}(M/\mathbb{P}^1)$. It is known that $< z >$ is a normal subgroup of the $\mathrm{Aut}(M)$ and z is in the center of $\mathrm{Aut}(M)$. A hyperelliptic curve is ramified in $2g + 2$ places $e_1, \dots, e_{2g+2}$. This sets up a bijection between isomorphism classes of hyperelliptic genus g curves and unordered distinct $(2g + 2)$-tuples $e_1, \dots, e_{2g+2} \in \mathbb{P}^1$ modulo automorphisms of $\mathbb{P}^1$. An unordered $(2g+2)$-tuple $e_1, \dots, e_{2g+2}$ can be described by a binary form (i.e., a homogenous equation $f(X, Z)$ of degree $2g + 2$). To describe $\mathcal{H}_g$, we need rational functions of the coefficients of a binary form $f(X, Z)$, invariant under linear substitutions in X and Z. Such functions are called absolute invariants for $g = 2$. The absolute invariants are $\mathrm{GL}_2(\mathbb{C})$ invariants under the natural action of $\mathrm{GL}_2(\mathbb{C})$ on the space of binary forms of degree $2g+2$. Two genus g- hyperelliptic curves are isomorphic if and only if they have the same absolute invariants. The locus of genus g hyperelliptic curves with an extra involution is an irreducible g-dimensional subvariety of $\mathcal{H}_g$ which is denoted by $\mathcal{L}_g$. Finding an explicit description of $\mathcal{L}_g$ means finding explicit equations in terms of absolute invariants. Such equations are computed only for $g = 2$. To the surface M we associate the homology basis ${}^t\{a, b\} = \{a_1, \dots, a_g, b_1, \dots, b_g\}$:



We denote by $\{\zeta\} =^t \{dv_1, \dots, dv_g\}$ the normalized canonical basis in $\mathcal{H}^1(M)$ dual of the basis $\{a, b\}^t$. We denote by τ the periods matrix $\tau = (\tau_{i,j})$, $\tau_{i,j} =$

$\int_{b_j} dv_i$. The matrix τ is a $g\times g$ symmetric matrix, of definite and positive imaginary part, that is it belongs to the Siegel upper half-space $\mathcal{H}_g$. The Riemann theta function, associated to τ is:

$$\theta(z)=\theta(z,\tau)=\sum_{N\in\mathbb{Z}^g}\exp\pi i({}^tN\tau N+2\ {}^tNz),\ z\in\mathbb{C}^g.$$

It has the transformation properties:

$$\theta(z+I\mu'+\tau\mu,\tau)=\exp(i\pi[-2\ {}^t\mu z-{}^t\mu\tau\mu])\theta(z,\tau)$$

for every $\mu,\ \mu'\in\mathbb{Z}^g$. For the sequel we need more notations and we refer to ([13]) for more details.

Definition 7.1. A theta characteristics is a $2\times g$ matrix whose entries are all integers:

$$\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}=\begin{bmatrix}\epsilon_1, & \dots, & \epsilon_g\\ \epsilon'_1, & \dots, & \epsilon'_g\end{bmatrix}$$

The characteristics is said to be even or odd depending on whether $\sum_{i=1}^g\epsilon_i\epsilon'_i$ is even or odd. A characteristics is called reduced when ϵ_i, ϵ'_i are 0 or 1.

To each characteristics $\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$ we associate a theta function, defined for each $z\in\mathbb{C}^g$ by:

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau)=\sum_{N\in\mathbb{Z}^g}\exp\pi i({}^t(N+\epsilon/2)\tau(N+\epsilon/2)+2^t(N+\epsilon/2)(z+\epsilon'/2)) \tag{7.1}$$

$$=\exp(2\pi i[\frac{1}{8}\ {}^t\epsilon\tau\epsilon+\frac{1}{2}\ {}^t\epsilon z+\frac{1}{4}\ {}^t\epsilon\epsilon'])\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau).$$

The Riemann theta function is $\theta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau)$ and the theta constant are $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)$, or simply $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$.

The Riemann theta function associated to the the surface M and to the homology basis ${}^t\{a,b\}$ is $\theta\circ\phi$ where ϕ is the Abel map, a holomorphic map from M to its Jacobian $J(M)=\mathbb{C}^g/L(M)$, $L(M)$ being the lattice of periods, the span of the $2g$ columns of the $g\times 2g$ matrix (I,τ), $\phi(P)=\int_{e_1}^P\zeta$. In the hyperelliptic case, the vector of Riemann constant in a given basis has a simple form: If e_1 is a Weierstrass point $K_{e_1}=\sum_{k=1}^g\int_{e_1}^{e_{2k+1}}\zeta$, it is a half-period:

$$K_{e_1}=\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\ \epsilon,\epsilon'\in\mathbb{Z}^g. \tag{7.2}$$

7.3. Thomae's formula

There is a bijection between isomorphism classes of hyperelliptic genus g curves and unordered distinct $2g+2$-tuples $e_1, e_2, \dots e_{2g+2} \in \mathbb{P}^1$ modulo automorphisms of the Riemann sphere $\mathbb{P}^1$. An unordered $(2g+2)$-tuple can be described by a binary form, a homogenous polynomial of degree $2g+2$, to which we can associate the invariants [33]. The invariants of a binary sextic form an algebra, generated by invariants of degree $2, 4, 6, 10, 15$. The generators of degrees $2, 4, 6, 10$ generate a polynomial ring, which contains the square of the generator of degree 15 ([33, p. 40]). The ring of covariants is generated by 26 covariants. The configuration of three intervals is very special in the sense that it can be described by the genus two theta functions. So we restrict ourselves to the genus $g = 2$ case and we recall the definitions.

Let $f(x, z)$ be a binary sextic:

$$f(x, z) = u_0 x^6 + u_1 x^5 z + \cdots + u_6 z^6,$$

that is a homogeneous polynomial in two variables, of degree 6 and with complex coefficients. Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{C})$ and let

$$x = \alpha x' + \beta z',\ z = \gamma x' + \delta z'.$$

By substituting these expressions fot x, z and expanding we obtain

$$f'(x', z') = u'_0 {x'}^6 + u'_1 {x'}^5 z' + \cdots + u_6 {z'}^6.$$

A polynomial $J = J(u_0, \dots, u_6)$ in the variables $u_0, \dots, u_6$ is called an invariant of index k if

$$J(u'_0, \dots, u'_6) = \det(M)^k J(u_0, \dots, u_6).$$

If $f(x) = u_0 x^6 + u_1 x^5 + \cdots + u_6$ and $f'(x) = u'_0 x^6 + u'_1 x^5 + \cdots + u'_6$, then the two hyperelliptic curves

$$\mathcal{C} : y^2 = f(x), \quad \mathcal{C}' : {y'}^2 = f'(x)$$

are isomorphic by the map

$$(x', y') \mapsto \left(\frac{ax' + b}{cx' + d}, \frac{y'}{(cx' + d)^3} \right)$$

due to the fact that $\left(\dfrac{y'}{(cx' + d)^3} \right)^2 = f\left(\dfrac{ax' + b}{cx' + d} \right)$. The invariants of the sextics are given by the Igusa–Clebsch formulae I_2, I_4, I_6, I_{10}. So every sextic gives a vector (I_2, I_4, I_6, I_{10}) with $I_{10} \neq 0$ and conversely, every such vector comes from a sextic. Two such curves over $\mathbb{C}$ are isomorphic if and only if their respective vectors are (I_2, I_4, I_6, I_{10}) and $(r^2 I_2, r^4 I_4, r^6 I_6, r^{10} I_{10})$ for some $r \in \mathbb{C}^*$. To give the Igusa–Clebsch invariants in compact form we denote by (ij) the difference $(\alpha_{\sigma(i)} - \alpha_{\sigma(i)})$, for any permutation $\sigma \in S_6$, the group of permutations of six elements. The

invariants are then given by

$$
\begin{aligned}
I_2 &= u_0^2 \sum_{15} (12)^2 (34)^2 (56)^2, \\
I_4 &= u_0^4 \sum_{10} (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2, \\
I_6 &= u_0^6 \sum_{60} (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 (14)^2 (25)^2 (36)^2, \\
I_{10} &= u_0^{10} \prod_{i<j} (\alpha_i - \alpha_j)^2.
\end{aligned}
$$

The sums are taken over all distinct expressions (in the roots of f) that are obtained when σ ranges over S_6. The subscript indicates the number of expressions encountered. More precisely

- There are 15 ways of partitioning the six roots of f into three subsets of two. Each yields a triple f_1, f_2, f_3 of monic quadratic polynomials with complex coefficients, and the summand in I_2 is the product of their discriminants.
- Similarly, for I_4 there are 10 ways of partitioning the six roots of f into two subsets of three, and each yields a summand that is the product of two cubic discriminants. For each of the 10 ways of partitioning the six roots of f into two subsets of three, there are 6 ways of giving a bijection between those two subsets, and each gives a summand for I_6.
- Finally, I_{10} is simply the discriminant of f which is nonzero as f is separable.

These invariants were introduced by Igusa [19]. They were based on much earlier work on invariants by Clebsch [7]. To compute the absolute Igusa invariants corresponding to a point $Z \in \mathcal{H}_2$, the Siegel upper half-space $\mathcal{H}_2$, which is the 2-fold cartesian product of the upper half plane $\mathcal{H}$. For theta characteristic $c \in \{0, \frac{1}{2}\}^4$, $c = (c_1, c_2, c_3, c_4)$, $c' = (c_1, c_2), c'' = (c_3, c_4)$, we consider the theta constant of characteristic c

$$
\theta[c](z) = \sum_{n \in \mathbb{Z}^2} e^{i\pi z\left((n+c')^t + 2(n+c')c''^t\right)}.
$$

We call a theta characteristic and the corresponding theta constant even or odd depending on whether $4c'c''^t$ is even or odd. The odd theta constants are zero by the anti-symmetry in the definition, and there are exactly 10 even theta constants

$$
\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_6, \theta_8, \theta_9, \theta_{12}, \theta_{15}.
$$

Another way of defining the invariants I_k is through the use of Eisenstein series on the 2×2 Siegel upper half-space $\mathcal{H}_2$, defined by

$$
\psi_k(Z) = \sum_{C,D} \det(CZ + D)^{-k} \tag{7.3}
$$

where the sum is taken over the set of bottom halves (C, D) of elements of $\mathrm{Sp}_4(\mathbb{Z})$ up to the left multiplication by $\mathrm{SL}_2(\mathbb{Z})$. Now we define

$$\chi_{10} = -43867(2^{12}3^5 5^2 7.53)^{-1})(\psi_4\psi_6 - \psi_{10}), \tag{7.4}$$

$$\chi_{12} = 131.593(2^{13}3^7 5^3 7^2 337)^{-1}(3^2 7^2 \psi_4^3 + 2.5^3\psi_6^2 - 691\psi_{12}). \tag{7.5}$$

Then we have [19]:

$$I_4 = 2^2\psi_4, \quad I_6 = 2^2\psi_6,$$
$$I_{10} = -2^{14}\chi_{10}, \quad I_{12} = -2^{17}\chi_{12}.$$

A relationship between theta constants and the branch points of the hyperelliptic curve is given by Thomae's formula, see [31, p. 120, p. 128] for the notations and for the proof. Using Thomae's formula we express the branch points of the hyperelliptic curves in terms of even theta constants, as in (6.8) and (6.12) when $g = 1$:

A relationship between theta constants and the branch points of the hyperelliptic curve is given by Thomae's formula.

Theorem 7.2 (Thomae). *For all sets of branch points $B = \alpha_1, \alpha_2, \ldots, \alpha_{2g+1}$, there is a constant A such that for all $T \subset B, \sharp T$ is even,*

$$\vartheta[\eta_T](0;\tau)^4 = (-1)^{\sharp T\cap U} A \prod_{i<j,\, i,j\in T\triangle U} (\alpha_i - \alpha_j) \prod_{i<j,\, i,j\notin T\triangle U} (\alpha_i - \alpha_j), \tag{7.6}$$

where η_T is a non singular even half-integer characteristic corresponding to the subset T of branch points.

See [31, p. 128] for the description of A and [31, p. 120] for the proof. Using Thomae's formula we express the branch points of the hyperelliptic curves in terms of even theta constants.

We shall denote by $\mathfrak{H}_n$ the class of all domains in the plane which possess a hyperelliptic double, and by Σ_n the class of all domains which are the exterior of a system of n slits taken from the real axis. A general theorem of Koebe asserts that any multiply connected domain in $\mathbb{C}$, bounded by n continua (in particular any domain in Σ_n) can be mapped conformally on a circular domain, bounded by n circles. In the case of the complement of two segments this conformal mapping can be given explicitely [10] (Subsection 1.3.3).

The following result links the two classes $\mathfrak{H}_n$ and Σ_n:

Theorem 7.3. *The following are equivalent*

1. $D \in \mathfrak{H}_n$.
2. *D can be mapped one-to-one conformally onto a domain in Σ_n.*
3. *D admits an anticonformal involution possessing precisely $2n$ fixed points on the boundary of D.*
4. *D can be mapped one-to-one conformally onto a domain D bounded by circles $C_1, \ldots, C_n$ such that there exists a circle C orthogonal to all C_k, $1 \le k \le n$.*

The equivalences

$$1 \Longleftrightarrow 2 \Longleftrightarrow 3$$

result from [2]. The equivalence with the point 4. results from Koebe's theorem and the fact that the Schottky double of a domain D bounded by circles $C_1, \ldots, C_n$ is hyperelliptic if and only if there exists a circle C orthogonal to all C_k, $1 \leq k \leq n$ [27].

Remark 7.4. Each genus two Riemann surface is hyperelliptic as does the Schottky double of a domain D bounded by circles C_1, C_2, C_3; The point 4 in Theorem 7.3 is just another formulation of the classical Monge problem: It is always possible to find a circle orthogonal to three given circles. Indeed if D_{12}, D_{12} are the radical axis of C_1, C_2 and C_2, C_3, respectively, and if O is the common point of D_{12} and D_{12}, then O has equal power with respect to the three given circles. Let OT be the tangent to one of the circles. The circle with center O and radius OT is orthogonal to all C_i, $1 \leq i \leq 3$. If the radical axis are parallel, the orthogonal circle is the line of centers. The question of finding an orthogonal circle to more than three given circles is less simple.

For each $p = g + 1 \in \mathbb{N}$, we denote by X_p the set

$$X_p = \{\mathbf{e} = (e_1, e_2, \ldots, e_n) \in \mathbb{R}^n,\ e_1 < e_2 < \cdots e_{2p}\}$$

and

$$\Sigma_p^\infty = \{D \in \Sigma_p,\ \infty \in D\}\,.$$

We have a natural map

$$\begin{array}{rl} \mathfrak{F}: & X_p \longrightarrow \Sigma_p^\infty \\ & \mathbf{e} \longrightarrow \mathbb{P}^1 \setminus \bigcup_{k=1}^p E_k,\ \ E_k = [e_{2k-1}, e_{2k}] \end{array}$$

Proposition 7.5. *The map $\mathfrak{F}$ is continuous and commute with the natural action of the group $PSL_2(\mathbb{R})$ on both X_p and Σ_p^∞*

Let $E = \bigcup_{k=1}^p E_k$ be a finite union of $p = g + 1$ intervals of the real axis, $E_k = [e_{2k-1}, e_{2k}]$, $k = 1, \ldots, p$, $e_{2k-1} < e_{2k+1}$. Let G be the Green function of $\widehat{E} = \mathbb{P}^1 \setminus E$ with pole at infinity. The capacity $\tau(E)$ of E is given, as we saw in (2.1), by :

$$G(z) = \log|z| - \log\tau(E) + o(1), \quad |z| \to \infty.$$

We introduce the hyperelliptic curve M, of genus $g \geq 1$

$$w^2 = f(z) = \prod_1^{2g+2} (z - e_i). \tag{7.7}$$

We have the expansion

$$w^2 = z^{2g+2} - \sigma_1 z^{2g+1} + \sigma_2 z^{2g} - \cdots + \sigma_{2g+2}$$

where σ_i are symmetric functions in $e_1, e_2, \ldots, e_{2g+2}$. These can be expressed in terms of power sum symmetric functions by using the Newton-Girard identities.

The curve M is the set of (z, w) satisfying (7.7) to which we add the points at infinity. It realizes a cover of degree two of the Riemann sphere with ramification points at e_i, $1 \leq i \leq 2g+2$. In general, if M is the Riemann surface with boundaries

$$M = \mathbb{P}^1 \setminus \bigcup_{i=1}^{2g+2} [a_i, b_i], \quad [a_i, b_i] \subset \mathbb{R}$$

its Schottky double, $\widehat{E}$ is the hyperelliptic Riemann surface of equation [10]:

$$w^2 = \prod_{1}^{2g+2} (z - a_i)(z - b_i).$$

We denote by J the holomorphic involution

$$(z, \sqrt{f(z)}) \longrightarrow (z, -\sqrt{f(z)})$$

where $\sqrt{f(z)}$ is fixed by $\lim_{z\to\infty} \dfrac{\sqrt{f(z)}}{z^{g+1}} = 1$, which exchanges the two copies of M_1, M_2 in the construction of the Schottky double. We denote by ∞_i, $i = 1, 2$ the point at infinity in M_i. A local parametrization (z, w) in a neighborhood of (z_0, w_0) is given by:

$$z = z_0 + \begin{cases} t \text{ at a regular point } (z, w), \\ t^2 \text{ at a branch point } (e_i, 0), \\ t^{-1} \text{ at the regular point } \infty_i, \quad i = 1, 2. \end{cases}$$

The Green's function of $\widehat{E}$ extends to M by the Schwarz reflexion principle as an harmonic function. This harmonic extension is again denoted by G, hence

$$\begin{aligned} G(\mathcal{I}(P)) &= -G(P) \\ &= G(\overline{J(P)}) \end{aligned}$$

In all the rest of the paper we set $G(z) = G((z, \sqrt{f(z)}))$.

The following proposition gives the Green's function of $\widehat{E}$ in terms of the theta function of M.

Proposition 7.6. *The Green's function with pole at ∞ is given by*

$$G(z) = \Re\Omega(z)$$

where

$$\Omega(z) = -\log \frac{\theta(\int_{e_1}^{z} \zeta - \int_{e_1}^{\infty_1} \zeta + K_{e_1}, \tau)}{\theta(\int_{e_1}^{z} \zeta - \int_{e_1}^{\infty_2} \zeta + K_{e_1}, \tau)}. \tag{7.8}$$

To prove the formula (7.8) of the proposition 7.6, we write the Green's function G as:

$$G(z) = \Re\Omega(z)$$

where Ω is a normalized abelian integral of the third kind, that is:

$$\Omega = \int d\Omega$$

with $d\Omega$ a meromorphic differential having two simple poles of residues $+1$ and -1, and vanishing a-periods:

$$\int_{e_{2k}}^{e_{2k+1}} d\Omega = 0,\ k = 1, \cdots, g.$$

We write $d\Omega(z) = \dfrac{h(z)}{w} dz$ with $h(z) = h_0 + h_1 z + \ldots + h_{g-1} z^{g-1} + z^g$. The vanishing of the a-periods gives the equations:

$$\int_{e_{2j}}^{e_{2j+1}} h(z) |f(z)|^{-1/2}\, dz = 0, \quad j = 1, \ldots, g, \tag{7.9}$$

which is the linear system (4.2).
In the case $g = 2$, the Riemann vanishing theorem [13] shows that

$$z \to \theta\Big(\int_{e_1}^{z} \zeta - \int_{e_1}^{\infty_i} \zeta + K_{e_1}, \tau\Big)$$

has exactly the zeros e_1 et ∞_i. In the general case one can show that the numerator of (7.8) vanishes at $z = \infty_1$ and the denominator vanishes at $z = \infty_2$ and, moreover, they have $p - 1$ other common zeros. This means that Ω gets the important form

$$\Omega(z) = -\log \frac{\theta\big(\int_{e_1}^{z} \zeta - \int_{e_1}^{\infty_1} \zeta + K_{e_1}, \tau\big)}{\theta\big(\int_{e_1}^{z} \zeta - \int_{e_1}^{\infty_2} \zeta + K_{e_1}, \tau\big)}$$

and we obtain the desired properties

$$\Omega(z) = \log(z) + \text{holomorphic function near } \infty_1,$$
$$\Omega(z) = -\log(z) + \text{holomorphic function near } \infty_2,$$

or

$$d\Omega = \frac{dz}{z} + \text{holomorphic differential near } \infty_1,$$
$$d\Omega = -\frac{dz}{z} + \text{holomorphic differential near } \infty_2.$$

This means that $d\Omega = w_{\infty_1 \infty_2}$.

7.4. The expression of the capacity

According to the expression of the Riemann vector constants, given previously:

$$\Omega(z) = -\log \frac{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(z) - v_j(\infty_1)), \tau)}{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(z) - v_j(\infty_2)), \tau)}$$

where $v_j(z) = \int_{e_1}^{z} dv_j$. As $v_j(\infty_2) = v_j(J(\infty_1)) = -v_j(\infty_1)$, we get:

$$\Omega(z) = -\log \frac{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(z) - v_j(\infty_1)), \tau)}{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(z) + v_j(\infty_1)), \tau)}.$$

Consider:

$$\Re\Omega(z) = -\log \left| \frac{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(z) - v_j(\infty_1)), \tau)}{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(z) + v_j(\infty_1)), \tau)} \right|$$

and introduce $z = 1/t$, then we obtain:

$$\Re\Omega(z) - \log z = -\log \frac{\left| \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(1/t) - v_j(\infty_1)), \tau) \right|}{\left| t\ \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(v_j(1/t) + v_j(\infty_1)), \tau) \right|}$$

so that:

$$\lim_{z \to \infty_1} \Re\Omega(z) - \log z = -\log \left| \frac{\sum_j \left| \frac{\partial}{\partial t} v_j(1/t) \right|_{t=0} \partial_j \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} ({}^t(2v_j(\infty_1)), \tau)} \right|.$$

We change the right-hand side by introducing a basis of holomorphic differentials, more adapted to the hyperelliptic structure of M. This basis is:

$$\frac{dz}{\sqrt{\prod_1^{2g+2}(z - e_i)}}, \ldots, \frac{z^{g-1} dz}{\sqrt{\prod_1^{2g+2}(z - e_i)}}.$$

Hence for $1 \leq j \leq g$:

$$dv_j(z) = \sum_{k=0}^{g-1} \alpha_{j,k} \frac{z^k dz}{\sqrt{\prod_1^{2g+2}(z - e_i)}}, \quad \alpha_{j,k} \in \mathbb{C},$$

and

$$\left| \frac{\partial}{\partial t} v_j(1/t) \right|_{t=0} = -\left| \frac{1}{t^2} \sum_{k=0}^{g-1} \alpha_{j,k} \frac{t^{g+1-k}}{\sqrt{\prod_1^{2g+2}(1 - te_i)}} \right|_{t=0} = -\alpha_{j,g-1}.$$

Finally we obtain the formula for the capacity of an union of $p = g + 1$ given intervals:

Theorem 7.7.

$$\tau(E) = \left| \frac{\sum_j \alpha_{j,g-1} \partial_j \theta \left[\begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (0,\tau)}{\theta \left[\begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] ({}^t(2v_j(\infty_1)),\tau)} \right|.$$

Here, as in (7.2), ϵ et ϵ' are related to the Riemann constants vector by $K_{e_1} = \dfrac{\epsilon'}{2} + \tau\dfrac{\epsilon}{2}$, $\epsilon, \epsilon' \in \mathbb{Z}^g$.

8. Capacity of an union of three intervals

We now give a formula for the capacity (transfinite diameter). In the case of 3 intervals, that is to say in genus 2 situation. We will exploit the fact that we have a canonical form for the algebraic equation. To this end, we use the formulas in Rosenhain [29], from which the notations are taken.

Any genus 2 curve is isomorphic to a curve with a Rosenhain model. More precisely, let M be the hyperelliptic curve of equation

$$w^2 = (z-e_1)(z-e_2)(z-e_3)(z-e_4)(z-e_5)(z-e_6).$$

Consider the Möbius transformation of z :

$$t = \frac{e_3 - e_1}{e_3 - e_2}\left(1 - \frac{e_2 - e_1}{z - e_1}\right). \tag{8.1}$$

The map t defines another covering of degree 2 of M and

$$z = e_1 + \frac{1}{\frac{(e_3-e_2)}{(e_3-e_1)(e_2-e_1)}\left(\frac{e_3-e_1}{e_3-e_2} - t\right)}.$$

In particular if $y = \Im z$ then $y\Im t \geq 0$. Furthermore:

$$t(e_1) = \infty,\ t(e_2) = 0,\ t(e_3) = 1,\ t(e_4) = 1/\kappa^2,\ t(e_5) = 1/\lambda^2,\ t(e_6) = 1/\mu^2,$$

where we have set for the cross ratios:

$$\kappa^2 = \frac{e_3 - e_2}{e_3 - e_1}\frac{e_4 - e_1}{e_4 - e_2} = (e_4, e_1; e_3, e_2)$$
$$\lambda^2 = \frac{e_3 - e_2}{e_3 - e_1}\frac{e_5 - e_1}{e_5 - e_2} = (e_5, e_1; e_3, e_2)$$
$$\mu^2 = \frac{e_3 - e_2}{e_3 - e_1}\frac{e_6 - e_1}{e_6 - e_2} = (e_6, e_1; e_3, e_2).$$

The constants $\kappa^2, \lambda^2, \mu^2$ are called Rosenhain invariants.

From now on we work with the Rosenhain model:

$$s^2 = t(1-t)(1-\kappa^2 t)(1-\lambda^2 t)(1-\mu^2 t),$$

and according to [37, p. 388] we introduce the trigonometric series

$$\phi_{3,3}(v,w,p,q,A) = \sum_{m,n\in\mathbb{Z}} e^{m^2\log p + n^2 \log q + 4mnA + 2mv + 2nw},$$

where p, q, A are some constants that will be made precise later. Rosenhain [37] considers also 15 other trigonometric series, all obtained from $\phi_{3,3}(v, w, p, q, A)$ that also will be made precise later .

The following proposition is fundamental. It gives the Weierstrass points (branch points) in terms of theta constants and the series $\phi_{3,3}$, seen as Riemann theta function. This is prior formulation to Thomae's formulas.

Proposition 8.1 (G. Rosenhain ([37]), Rauch–Farkas ([35])). *The Rosenhain invariants are given by:*

$$\kappa^2 = \frac{\phi_{2,2}^2\phi_{2,3}^2}{\phi_{3,2}^2\phi_{3,3}^2}, \quad \lambda^2 = \frac{\phi_{2,2}^2\phi_{2,0}^2}{\phi_{3,2}^2\phi_{3,0}^2}, \quad \mu^2 = \frac{\phi_{2,3}^2\phi_{2,0}^2}{\phi_{3,3}^2\phi_{3,0}^2}$$

if and only if

$$\theta((z_1, z_2)|\tau) = \phi_{3,3}(i\pi z_2, -i\pi z_1, p, q, A)$$

where $\theta(z, \tau)$ is the Riemann theta function of the hyperelliptic curve

$$s^2 = t(1-t)(1-\kappa^2 t)(1-\lambda^2 t)(1-\mu^2 t).$$

The proof can be found in the cited references above. For the convenience of the reader we give in the appendix a table explaining the correspondence between $\phi_{r,s}$ and $\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ where the matrix τ used in the definition of $\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ is:

$$\tau = \begin{pmatrix} \dfrac{\log q}{i\pi} & -\dfrac{2A}{i\pi} \\ -\dfrac{2A}{i\pi} & \dfrac{\log p}{i\pi} \end{pmatrix}.$$

Using this table, the Rosenhain invariants of the proposition (8.1) can also be written as:

$$\kappa^2 = \frac{\theta\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}(\tau)^2\,\theta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}(\tau)^2}{\theta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(\tau)^2\,\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(\tau)^2},$$

$$\lambda^2 = \frac{\theta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(\tau)^2\,\theta\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}(\tau)^2}{\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(\tau)^2\,\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(\tau)^2},$$

$$\mu^2 = \frac{\theta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(\tau)^2\,\theta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}(\tau)^2}{\theta\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}(\tau)^2\,\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(\tau)^2}.$$

Now we compute the periods matrix and the vector of Riemann constants. The differentials du, dv form a basis of the vector space of holomorphic differentials.

$$\begin{array}{llll} \int_{a_2} du = & 2\int_{e_2}^{e_3} du = & 2\int_0^1 du(t) = & \int_{A_2} du(t) = 0, \\ \int_{a_1} du = & 2\int_{e_4}^{e_5} du = & 2\int_{1/\kappa^2}^{1/\lambda^2} du(t) = & \int_{A_1} du(t) = 1. \end{array}$$

Similarly:

$$\begin{array}{llll} \int_{a_2} dv = & 2\int_{e_2}^{e_3} dv = & 2\int_0^1 dv(t) = & \int_{A_2} dv(t) = 0, \\ \int_{a_1} dv = & 2\int_{e_4}^{e_5} dv = & 2\int_{1/\kappa^2}^{1/\lambda^2} dv(t) = & \int_{A_1} dv(t) = 1. \end{array}$$

We then deduce:

$$du(t) = -\frac{1}{i\pi}\left(B'\frac{dt}{s} - C'\frac{tdt}{s}\right),$$
$$dv(t) = \frac{1}{i\pi}\left(B\frac{dt}{s} - C\frac{tdt}{s}\right),$$

where the constants B, B', C, C' are determined ([37], p.433) by:

$$2B = i\frac{\partial_2\phi_{3,1}(0)}{\phi_{3,0}\phi_{3,2}\phi_{3,3}}, \qquad 2B' = -i\frac{\partial_1\phi_{3,1}(0)}{\phi_{3,0}\phi_{3,2}\phi_{3,3}},$$

$$2C = i\sqrt{\kappa\lambda\mu}\frac{\partial_2\phi_{2,1}(0)}{\phi_{3,0}\phi_{3,2}\phi_{3,3}}, \quad 2C' = -i\sqrt{\kappa\lambda\mu}\frac{\partial_1\phi_{2,1}(0)}{\phi_{3,0}\phi_{3,2}\phi_{3,3}}.$$

We compute the periods of the differential forms du, dv, we have:

$$\begin{array}{llll} \int_{b_2} du = & 2\int_{e_2}^{e_1} du = & 2\int_0^{\infty} du(t) = & \int_{B_2} du(t), \\ \int_{b_1} du = & 2\int_{e_5}^{e_6} du = & 2\int_{1/\lambda^2}^{1/\mu^2} du(t) = & \int_{B_1} du(t), \end{array}$$

and similarly for dv. The periods matrix is then:

$$\tau = \begin{pmatrix} \dfrac{\log q}{i\pi} & -\dfrac{2A}{i\pi} \\ -\dfrac{2A}{i\pi} & \dfrac{\log p}{i\pi} \end{pmatrix}.$$

To find the vector of Riemann constants $K_{e_1} = {}^t(K^1_{e_1}, K^2_{e_1})$, we use

$$\int_{e_1}^{e_3} du = -\frac{1}{2}\int_{b_2} du + \frac{1}{2}\int_{a_2} du,$$
$$\int_{e_3}^{e_5} du = \frac{1}{2}\int_{a_1} du + \frac{1}{2}\int_{b_2} du - \frac{1}{2}\int_{b_1} du,$$

and we obtain:

$$K^1_{e_1} = \int_{e_1}^{e_3} du + \int_{e_1}^{e_5} du = \frac{1}{2} + \frac{A}{i\pi} - \frac{1}{2}\frac{\log q}{i\pi},$$
$$K^2_{e_1} = \int_{e_1}^{e_3} dv + \int_{e_1}^{e_5} dv = \frac{A}{i\pi} - \frac{1}{2}\frac{\log p}{i\pi}.$$

The following lemma gives a precise link between $\phi_{3,3}$ and $\phi_{2,1}$:

Lemma 8.2. *The following identity holds:*

$$\phi_{3,3}(v + i\pi K^2_{e_1}, w - i\pi K^1_{e_1}, p, q, A) = \phi_{3,3}(v + A - \frac{\log p}{2}, w - \frac{i\pi}{2} - A + \frac{\log q}{2})$$
$$= \frac{1}{e^{w+\log q/4} e^{v-\log p/4}} \phi_{2,1}(v, w, p, q, A).$$

Proof. This can be seen by

- using [37] (p. 409, (79), second relation):

$$\phi_{3,3}(v + A - \frac{\log p}{2}, w - \frac{i\pi}{2} - A + \frac{\log q}{2}) = \phi_{3,0}(v + A - \frac{\log p}{2}, w - A + \frac{\log q}{2});$$

- using [37] (p. 410, (80), third relation):

$$\phi_{3,0}(v + A - \frac{\log p}{2}, w - A + \frac{\log q}{2}) = \frac{e^{\frac{w^2}{\log q}}}{e^{\frac{(w+1/2\log q)^2}{\log q}}} \phi_{3,1}(v - \frac{\log p}{2}, w - A, p, q, A)$$
$$= \frac{e^{\frac{w^2}{\log q}}}{e^{\frac{(w+1/2\log q)^2}{\log q}}} \phi_{1,3}(w - A, v - \frac{\log p}{2}, q, p, A);$$

- finally using [37] (p. 410, (80) first relation), we find:

$$\phi_{1,3}(w - A, v - \frac{\log p}{2}, q, p, A) = \frac{e^{\frac{v^2}{\log p}}}{e^{\frac{(v-1/2\log p)^2}{\log p}}} \phi_{1,2}(w, v, q, p, A)$$
$$= \frac{e^{\frac{v^2}{\log p}}}{e^{\frac{(v-1/2\log p)^2}{\log p}}} \phi_{2,1}(v, w, p, q, A).$$

Hence the result. □

According to Proposition 7.8 and Lemma 8.2, the complex Green's function takes the more symmetric form:

$$\Omega = -\log \frac{\theta(\int_{e_1}^z \zeta - \int_{e_1}^{\infty_1} \zeta + K_{e_1}, \tau)}{\theta(\int_{e_1}^z \zeta - \int_{e_1}^{\infty_2} \zeta + K_{e_1}, \tau)}$$
$$= -\log \frac{e^{W+V}}{e^{w+v}} \frac{\phi_{2,1}(v, w)}{\phi_{2,1}(V, W)}$$

where

$$v = i\pi \int_{\frac{e_3-e_1}{e_3-e_2}}^t dv, \quad w = -i\pi \int_{\frac{e_3-e_1}{e_3-e_2}}^t du$$

and

$$V = V(t) = i\pi z_2(t) = i\pi \left(\int_{-\infty}^{t} dv + \int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}} dv \right),$$

$$W = W(t) = -i\pi z_1(t) = -i\pi \left(\int_{-\infty}^{t} du + \int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}} du \right).$$

Now we return to the central question of giving the capacity $\tau(E)$. We consider the function

$$R(t) = \Re\Omega(z) - \log|z|, \quad t = \frac{e_3-e_1}{e_3-e_2}\left(1 - \frac{e_2-e_1}{z-e_1}\right),$$

we have:

$$\lim_{t\to\frac{e_3-e_1}{e_3-e_2}} R(t) = -\log \lim_{t\to\frac{e_3-e_1}{e_3-e_2}} \left| \frac{\phi_{2,1}(v,w)}{\frac{e_3-e_2}{(e_3-e_1)(e_2-e_1)}\left[\frac{e_3-e_1}{e_3-e_2} - t\right]\phi_{2,1}(V,W)} \right|.$$

We set $\xi = \frac{e_3-e_1}{e_3-e_2}$, then:

$$\lim_{t\to\xi} R(t) = -\log \left| \frac{\frac{B-C\xi}{s(\xi)}\partial_1\phi_{2,1}(0,0) + \frac{B'-C'\xi}{s(\xi)}\partial_2\phi_{2,1}(0,0)}{\frac{e_3-e_2}{(e_3-e_1)(e_2-e_1)}\phi_{2,1}(V(\xi),W(\xi))} \right|.$$

An essential step now is the extension to higher genus of the Jacobi derivative formula. In genus one case, this formula relates theta constants. It gives the derivative of the odd theta function, evaluated at the origin in terms of the even theta constants:

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{\pi}\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0) = 2\eta^3(\tau)$$

whre η is the Dedekind eta function :

$$\eta(\tau) = e^{i\pi\tau/12}\prod_{m=1}^{\infty}(1-e^{2i\pi m\tau}), \quad \Im\tau > 0.$$

An extension to the genus two case is stated without proof by Rosenhain, and proved by Thomae and Weber [20]. When the genus is two, given two odd theta characteristics α_1 and α_2, there are four even theta characteristics β_i, $i = 1,\ldots,4$, that depend on α_1 and α_2 such that:

$$\begin{vmatrix} \partial_1\theta[\alpha_1](0) & \partial_2\theta[\alpha_1](0) \\ \partial_1\theta[\alpha_2](0) & \partial_2\theta[\alpha_2](0) \end{vmatrix} = \pm\pi^2\prod_{i=1}^{4}\theta[\beta_i](0).$$

We then use the following relation, obtained from [37, p. 433] which is the extension to higher genus of (6.6):

$$\partial_2\phi_{3,1}(0,0)\partial_1\phi_{2,1}(0,0) - \partial_1\phi_{3,1}(0,0)\partial_2\phi_{2,1}(0,0) = \phi_{1,1}\phi_{0,0}\phi_{0,2}\phi_{0,3},$$

where the right-hand sides are evaluated at $(0,0)$. We thus obtain:

$$B\partial_1\phi_{2,1}(0) + B'\partial_2\phi_{2,1}(0) = \frac{i}{2}\frac{\phi_{1,1}\phi_{0,0}\phi_{0,2}\phi_{0,3}}{\phi_{3,0}\phi_{3,2}\phi_{3,3}}$$

and

$$C\partial_1\phi_{2,1}(0) + C'\partial_2\phi_{2,1}(0) = 0.$$

Hence

$$\tau(E) = \frac{1}{2}\left|\frac{\sqrt{(e_3-e_1)(e_4-e_2)(e_5-e_2)(e_6-e_2)}}{e_2-e_1}\frac{\phi_{1,1}\phi_{0,0}\phi_{0,2}\phi_{0,3}}{\phi_{2,1}(V(\xi),W(\xi))\phi_{3,0}\phi_{3,2}\phi_{3,3}}\right|.$$

On the other hand, we have the correspondences:

$$\phi_{0,0} = \theta\begin{bmatrix}0&0\\1&1\end{bmatrix},\quad \phi_{0,3} = \theta\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \phi_{3,0} = \theta\begin{bmatrix}0&0\\1&0\end{bmatrix},$$
$$\phi_{0,2} = i\theta\begin{bmatrix}-1&0\\0&1\end{bmatrix},\quad \phi_{3,2} = i\theta\begin{bmatrix}-1&0\\0&0\end{bmatrix}$$

and

$$\left|\frac{\phi_{1,1}}{\phi_{2,1}(V(\zeta),W(\zeta))}\right| = \left|\frac{\theta\begin{bmatrix}-1&1\\1&1\end{bmatrix}}{\theta\begin{bmatrix}-1&1\\1&0\end{bmatrix}\left(2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}du, 2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}dv\right)}\right|,$$

and by using the following transformation property, which is a consequence of the definition:

$$\theta\begin{bmatrix}\epsilon+2\nu\\ \epsilon'+2\nu'\end{bmatrix}(z,\tau) = exp(2i\pi^t\epsilon\nu')\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau)$$

it is possible to replace in the characteristics of the theta functions above -1 by $+1$ each time it appears to finally get one of the main results of this work:

Theorem 8.3.

$$\tau(E)$$
$$= \frac{1}{2}\frac{\sqrt{(e_3-e_1)(e_4-e_2)(e_5-e_2)(e_6-e_2)}}{e_2-e_1}$$
$$\times\left|\frac{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}\left(2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}du, 2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}dv\right)\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}}\right|.$$

Here, as before, $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$ stands for $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)$ (see formula (7.1)). The Thomae formula (also [37, p. 53]) enables us to express the fourth powers of theta

constants in terms of the cross-ratio of the branch points:

$$\left|\frac{\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}}\right| = \sqrt{1-\lambda^2} = \left[\frac{(e_5-e_3)(e_2-e_1)}{(e_3-e_1)(e_5-e_2)}\right]^{1/2}$$

and ([37, p. 417f]):

$$\left|\frac{\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}}\right|^2 = \frac{\sqrt{1-\kappa^2}\sqrt{1-\mu^2}}{\sqrt{1-\lambda^2}}.$$

Consequently, the capacity $\tau(E)$ gets the more simplified form:

Corollary 8.4.

$$\tau(E) = \frac{1}{2}\mathcal{S}(e_2)\mathcal{S}(e_3)\left|\frac{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}\left(2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}du, 2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}dv\right)}\right|$$

where:

$$\mathcal{S}^4(x) = \frac{(e_4-x)(e_5-x)(e_6-x)}{(x-e_1)}.$$

9. Symmetry and Reduction

We have seen that for a finite union of segments E of the real axis (hence invariant under complex conjugation), the associated Schottky double M is a hyperelliptic Riemann surface. We would like to see in this section that if E has a symmetry with respect to the origin, then a reduction of the capacity of E can be obtained either by using a transfer theorem or by a reduction of the Riemann theta function of the Riemann surface M. This is a part of the important theory of transformations of abelian functions for which we refer the reader to [3]. To see an example, we consider the situation of three intervals:

$$E = [-\frac{1}{a}, -\frac{1}{b}] \cup [-1, +1] \cup [+\frac{1}{b}, +\frac{1}{a}], \quad 0 < a < b < 1.$$

If $P(x) = 2a^2(x^2 - 1/2a^2)$, then $P(E) = [-1, 2a^2 - 1] \cup [2\frac{a^2}{b^2} - 1, 1]$ and by the transfer theorem (2.2):

$$\begin{aligned}\tau(E) &= \left[\frac{1}{2a^2}\tau(P(E))\right]^{1/2}\\ &= \frac{1}{\sqrt{2}a}\tau(P(E))^{1/2}.\end{aligned}$$

We have, by [10, Section 1.3.3] and by introducing characteristics, even in genus one case, to unify the presentation:

$$-\theta\left[\begin{array}{c}1\\1\end{array}\right](u/2K,\tau)=2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)\pi u/2K,$$

$$\theta\left[\begin{array}{c}1\\0\end{array}\right](u/2K,\tau)=2\sum_{n=0}^{\infty}q^{(n+1/2)^2}\cos(2n+1)\pi u/2K,$$

$$\theta\left[\begin{array}{c}0\\0\end{array}\right](u/2K,\tau)=\Theta_1(u,\tau)=1+2\sum_{n=1}^{\infty}q^{n^2}\cos n\pi u/K,$$

$$\theta\left[\begin{array}{c}0\\1\end{array}\right](u/2K,\tau)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos n\pi u/K.$$

We also have the Legendre *sn* function:

$$sn\,u=\frac{1}{\sqrt{k}}\frac{-\theta\left[\begin{array}{c}1\\1\end{array}\right](u/2K,\tau)}{\theta\left[\begin{array}{c}1\\0\end{array}\right](u/2K,\tau)},\quad cn^2u=1-sn^2u,$$

$$k^2=\frac{1-b^2}{1-a^2},$$
$$sn^2M=1-a^2,$$

so that

$$\tau(P(E))=\frac{1}{2a^2}\frac{1}{2}\left[\frac{\theta\left[\begin{array}{c}0\\1\end{array}\right](0)\;\theta\left[\begin{array}{c}0\\0\end{array}\right](0)}{\theta\left[\begin{array}{c}0\\1\end{array}\right](\frac{M}{2K})\;\theta\left[\begin{array}{c}0\\0\end{array}\right](\frac{M}{2K})}\right],$$

and

$$\tau(E)=\frac{1}{2a}\left[\frac{\theta\left[\begin{array}{c}0\\1\end{array}\right](0)\;\theta\left[\begin{array}{c}0\\0\end{array}\right](0)}{\theta\left[\begin{array}{c}0\\1\end{array}\right](\frac{M}{2K})\;\theta\left[\begin{array}{c}0\\0\end{array}\right](\frac{M}{2K})}\right]. \tag{9.1}$$

This being said, we are now going to investigate the reduction problem of the genus 2 theta function, underlying the calculation of $\tau(E)$. We follow [3] and [10].

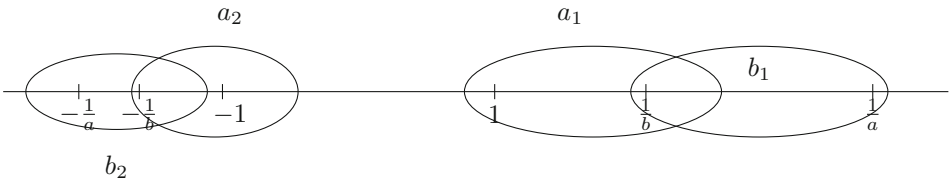
Theorem 9.1 (Weierstrass). *A Riemann surface M of genus 2 is an N-sheeted covering of an elliptic curve if and only if there is an element $\sigma\in Sp_4(\mathbb{Z})$ and a point $B\in\mathcal{H}_2$ such that*

$$\sigma B=\left|\begin{array}{cc}a' & 1/N\\ 1/N & b'\end{array}\right|,\quad N\in\mathbb{N},\ \ N>1.$$

We illustrate this reduction in a simple case. We naturally associate to the compact set $E = [-\frac{1}{a}, -\frac{1}{b}] \cup [-1, +1] \cup [+\frac{1}{b}, +\frac{1}{a}]$, the genus 2 hyperelliptic curve

$$w^2 = (z^2 - 1)(z^2 - \frac{1}{b^2})(z^2 - \frac{1}{a^2}) \tag{9.2}$$

with the homology basis:



We take the positive value of

$$w = \sqrt{(z^2 - 1)(z^2 - \frac{1}{b^2})(z^2 - \frac{1}{a^2})}$$

at $+\infty$ and we use the following notations:

$$K = \int_1^{1/b} \frac{dz}{|w|}, \quad H = \int_1^{1/b} \frac{zdz}{|w|},$$
$$K' = \int_{1/b}^{1/a} \frac{dz}{|w|}, \quad H' = \int_{1/b}^{1/a} \frac{zdz}{|w|}.$$

We denote by $\{du,\ dv\}$ the dual basis:

$$du = -\frac{1}{4K}\frac{dz}{w} - \frac{1}{4H}\frac{zdz}{w},$$
$$dv = \frac{1}{4K}\frac{dz}{w} - \frac{1}{4H}\frac{zdz}{w}.$$

The period matrix is then:

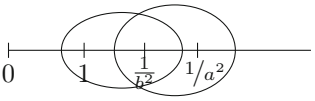
$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \frac{i}{2}\frac{K'}{K} + \frac{i}{2}\frac{H'}{H} & -\frac{i}{2}\frac{K'}{K} + \frac{i}{2}\frac{H'}{H} \\ -\frac{i}{2}\frac{K'}{K} + \frac{i}{2}\frac{H'}{H} & \frac{i}{2}\frac{K'}{K} + \frac{i}{2}\frac{H'}{H} \end{pmatrix}.$$

The curve (9.2) possesses the automorphism T_1 of order 2:

$$T_1 : (z, w) \to (-z, -w).$$

Hence, the algebraic curve $M_1 = M/T_1$ is defined by the equation:

$$\Gamma_1 : \ w_1^2 = \zeta(\zeta - 1)(\zeta - \frac{1}{b^2})(\zeta - \frac{1}{a^2}).$$



The normalized holomorphic differential of Γ_1 is:

$$du_1 = du - dv = -\frac{1}{4K}\frac{d\zeta}{w_1(\zeta)},$$

to which we associate:

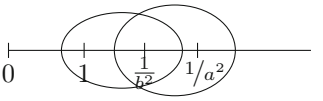
$$\tau_1 = \int_b du_1 = i\frac{K'}{K}.$$

The curve (9.2) has another automorphism T_2 of order 2:

$$T_1 : (z, w) \to (-z, w).$$

The algebraic curve $M_1 = M/T_2$ is then defined by the equation:

$$\Gamma_2 : \ w_2^2 = (\eta - 1)(\eta - \frac{1}{b^2})(\eta - \frac{1}{a^2}).$$



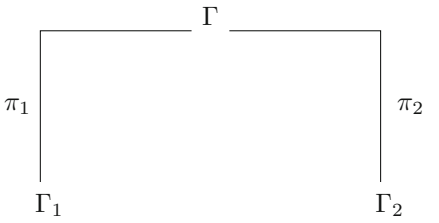
The holomorphic normalized differential Γ_2 is:

$$du_2 = du + dv = -\frac{1}{4H}\frac{d\eta}{w_2(\eta)}$$

to which we associate:

$$\tau_2 = \int_\beta du_2 = i\frac{H'}{H}.$$

Thus we have two 2-sheeted covers



and

$$\tau = \begin{pmatrix} \frac{\tau_2 + \tau_1}{2} & \frac{\tau_2 - \tau_1}{2} \\ \frac{\tau_2 - \tau_1}{2} & \frac{\tau_2 + \tau_1}{2} \end{pmatrix}.$$

This particular form of the periods matrix enables us to reduce the genus 2 theta function into genus one theta functions [3, p. 11]:

Lemma 9.2. *We have:*

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z|\tau)=\theta\begin{bmatrix}\frac{\epsilon_1-\epsilon_2}{2}\\ \epsilon'_1-\epsilon'_2\end{bmatrix}(z_1-z_2|2\tau_1)\theta\begin{bmatrix}\frac{\epsilon_1+\epsilon_2}{2}\\ \epsilon'_1+\epsilon'_2\end{bmatrix}(z_1+z_2|2\tau_2)$$
$$+\theta\begin{bmatrix}\frac{\epsilon_1-\epsilon_2}{2}+1\\ \epsilon'_1-\epsilon'_2\end{bmatrix}(z_1-z_2|2\tau_1)\theta\begin{bmatrix}\frac{\epsilon_1+\epsilon_2}{2}+1\\ \\ \epsilon'_1+\epsilon'_2\end{bmatrix}(z_1+z_2|2\tau_2).$$

Proof.

$$\sum_{(n_1,n_2)\in\mathbb{Z}}\exp(i\pi(n_1+\alpha_1)^2\frac{\tau_1+\tau_2}{2}+i\pi(n_2+\alpha_2)^2\frac{\tau_1+\tau_2}{2}$$
$$+2i\pi(n_1+\alpha_1)(n_2+\alpha_2)\frac{\tau_2-\tau_1}{2}$$
$$+2i\pi(n_1+\alpha_1)(z_1+\beta_1)+2i\pi(n_2+\alpha_2)(z_2+\beta_2))$$
$$=\sum_{(n_1+n_2)\text{ even}}+\sum_{(n_1+n_2)\text{ odd}}e^{i\pi(n_1+n_2+\alpha_1+\alpha_2)^2\frac{\tau_2}{2}+i\pi(n_1-n_2+\alpha_1-\alpha_2)^2\frac{\tau_1}{2}}$$
$$\times e^{i\pi(n_1+n_2+\alpha_1+\alpha_2)(z_1+\beta_1+z_2+\beta_2)+i\pi(n_1-n_2+\alpha_1-\alpha_2)(z_1+\beta_1-z_2-\beta_2)}$$
$$=\sum_{(n_1+n_2)\text{even}}e^{i\pi(\frac{n_1+n_2}{2}+\frac{\alpha_1+\alpha_2}{2})^2 2\tau_2+2i\pi(\frac{n_1+n_2}{2}+\frac{\alpha_1+\alpha_2}{2})(z_1+z_2+\beta_1+\beta_2)}$$
$$\times e^{i\pi(\frac{n_1-n_2}{2}+\frac{\alpha_1-\alpha_2}{2})^2 2\tau_1+2i\pi(\frac{n_1-n_2}{2}+\frac{\alpha_1-\alpha_2}{2})(z_1-z_2+\beta_1-\beta_2)}$$
$$+\sum_{(n_1+n_2)\text{odd}}e^{i\pi(\frac{n_1-1+n_2}{2}+\frac{\alpha_1+\alpha_2+1}{2})^2 2\tau_2+2i\pi(\frac{n_1-1+n_2}{2}+\frac{\alpha_1+\alpha_2+1}{2})(z_1+z_2+\beta_1+\beta_2)}$$
$$\times e^{i\pi(\frac{n_1-1-n_2}{2}+\frac{\alpha_1-\alpha_2+1}{2})^2 2\tau_1+2i\pi(\frac{n_1-1-n_2}{2}+\frac{\alpha_1-\alpha_2+1}{2})(z_1-z_2+\beta_1-\beta_2)}.\qquad\square$$

We apply this lemma to the general formula of the capacity $\tau(E)$ of theorem (8.3)). From the preceding formula we get:

$$\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}(U,V|\tau)=\theta\begin{bmatrix}0\\1\end{bmatrix}(U-V|2\tau_1)\theta\begin{bmatrix}1\\1\end{bmatrix}(U+V|2\tau_2)$$
$$+\theta\begin{bmatrix}1\\1\end{bmatrix}(U-V|2\tau_1)\theta\begin{bmatrix}0\\1\end{bmatrix}(U+V|2\tau_2).$$

As

$$2(z_1+z_2)=2\int_{\frac{1}{b^2}}^{\infty}du+dv=\int_\beta du_2=\tau_2$$

which is purely imaginary, we deduce that $2\Re(z_1+z_2)=0$ and

$$\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}(2\Re z_1,2\Re z_2)=\theta\begin{bmatrix}1\\1\end{bmatrix}(2\Re(z_1-z_2)|2\tau_1)\theta\begin{bmatrix}0\\1\end{bmatrix}(0|2\tau_2).$$

Similarly with $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} = \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)$:

$$\theta\begin{bmatrix}0&0\\0&1\end{bmatrix} = \theta\begin{bmatrix}0\\1\end{bmatrix}(0|2\tau_1)\theta\begin{bmatrix}0\\1\end{bmatrix}(0|2\tau_2).$$

We then arrive at the equality:

$$\frac{\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}(2\Re z_1, 2\Re z_2)} = \frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(0)\,\theta\begin{bmatrix}0\\0\end{bmatrix}(0)}{\theta\begin{bmatrix}0\\1\end{bmatrix}(\frac{M}{2K})\,\theta\begin{bmatrix}0\\0\end{bmatrix}(\frac{M}{2K})}\frac{b}{a\sqrt{1-b^2}}.$$

It remains to analyze:

$$A = \left|\frac{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}}\right|$$

and

$$B = \frac{1}{2}\frac{\sqrt{(e_3-e_1)(e_4-e_2)(e_5-e_2)(e_6-e_2)}}{e_2-e_1}.$$

By using [10], [37] we have $A = \sqrt{\frac{\mu_\lambda\lambda_\kappa\lambda_1}{\lambda}}$, thus $\lambda_1^2 = 1-\lambda^2$, $\mu_\lambda^2 = \lambda^2-\mu^2$, $\lambda_\kappa^2 = \kappa^2-\lambda^2$. and we find that

$$A^2 = (1-\frac{a}{b})^2[\frac{1-b}{2(1-a)(1+\frac{a}{b})}],$$
$$B^2 = \frac{(1-a)(1+b)(1+\frac{a}{b})}{2b^2(1-\frac{a}{b})^2},$$

so $AB = \frac{\sqrt{1-b^2}}{2b}$ and:

$$\tau(E) = \frac{1}{2a}\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(0)\,\theta\begin{bmatrix}0\\0\end{bmatrix}(0)}{\theta\begin{bmatrix}0\\1\end{bmatrix}(\frac{M}{2K})\,\theta\begin{bmatrix}0\\0\end{bmatrix}(\frac{M}{2K})},$$

in accordance with (9.1).

9.1. An algorithm of Mestre

In [29], J.-F. Mestre gives an algorithm, built on the geometric-arithmetic mean, giving the capacity of two intervals of the form $A = [-1,a]\cup[b,1],\ -1 \le a < b \le 1$.

We set

$$a_0 = \sqrt{(1+b)(1-a)}, \quad b_0 = \sqrt{(1+a)(1-b)},$$
$$a_{n+1} = \frac{a_n+b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n},$$

and we define the sequence (c_n) by:

$$c_n = \frac{(a_n - b_n)^2}{2}$$

and

$$x_0 = \frac{1}{2}, \quad x_{n+1} = \frac{x_n - c_n + \sqrt{x_n(x_n - 2c_n)}}{2}.$$

Then the capacity of $A = [-1, a] \cup [b, 1]$ is

$$\tau(A) = \lim_{n \to +\infty} \frac{x_{n+1}^{2^n}}{x_1 x_2^2 \cdots x_n^{2^{n-1}}}$$

As an example, J.-F. Mestre considers $A = [-1, -0.3] \cup [0.4, 1]$ for which he gives

$$x_1 = 0.468784430022729886755858612263562457961417682414552671929551,$$
$$x_2 = 0.468532110831975033096826687874608265872826009617103897008842,$$
$$x_3 = 0.468532093799463941802087628056809942212749432070019836600296,$$
$$x_4 = 0.468532093799463864168010269170037970165116049266755803066383,$$
$$x_5 = 0.468532093799463864168010269170036357293366379302670646214299,$$
$$x_6 = 0.468532093799463864168010269170036357293366379302670646214299,$$

and the capacity of A is

$$\tau(A) = 0.468279859356485325504599890968206130383542084405775474694334.$$

10. Appendix

(r,s)	$\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$	
(3,0)	$\begin{matrix}0&0\\1&0\end{matrix}$	$\phi_{3,0}(i\pi z_2,-i\pi z_1,p,q,A)=\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(0,3)	$\begin{matrix}0&0\\0&1\end{matrix}$	$\phi_{0,3}(i\pi z_2,-i\pi z_1,p,q,A)=\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(0,0)	$\begin{matrix}0&0\\1&1\end{matrix}$	$\phi_{0,0}(i\pi z_2,-i\pi z_1,p,q,A)=\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(1,0)	$\begin{matrix}0&1\\1&1\end{matrix}$	$\phi_{1,0}(i\pi z_2,-i\pi z_1,p,q,A)=e^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(0,1)	$\begin{matrix}-1&0\\1&1\end{matrix}$	$\phi_{0,1}(i\pi z_2,-i\pi z_1,p,q,A)=e^{\frac{i\pi}{4}\tau_{1,1}-i\pi z_1}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(1,3)	$\begin{matrix}0&1\\0&1\end{matrix}$	$\phi_{1,3}(i\pi z_2,-i\pi z_1,p,q,A)=e^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(3,1)	$\begin{matrix}1&0\\-1&0\end{matrix}$	$\phi_{3,1}(i\pi z_2,-i\pi z_1,p,q,A)=e^{\frac{i\pi}{4}\tau_{1,1}-i\pi z_1}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(0,2)	$\begin{matrix}-1&0\\0&1\end{matrix}$	$\phi_{0,2}(i\pi z_2,-i\pi z_1,p,q,A)=ie^{\frac{i\pi}{4}\tau_{1,1}-i\pi z_1}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(2,0)	$\begin{matrix}0&1\\1&0\end{matrix}$	$\phi_{2,0}(i\pi z_2,-i\pi z_1,p,q,A)=ie^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(2,2)	$\begin{matrix}-1&1\\0&0\end{matrix}$	$\phi_{2,2}(i\pi z_2,-i\pi z_1,p,q,A)=ie^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(2,1)	$\begin{matrix}-1&1\\1&0\end{matrix}$	$\phi_{2,1}(i\pi z_2,-i\pi z_1,p,q,A)=-e^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(1,2)	$\begin{matrix}-1&1\\0&1\end{matrix}$	$\phi_{1,2}(i\pi z_2,-i\pi z_1,p,q,A)=-e^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(1,1)	$\begin{matrix}-1&1\\-1&1\end{matrix}$	$\phi_{1,1}(i\pi z_2,-i\pi z_1,p,q,A)=-ie^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(3,2)	$\begin{matrix}-1&0\\0&0\end{matrix}$	$\phi_{3,2}(i\pi z_2,-i\pi z_1,p,q,A)=ie^{\frac{i\pi}{4}\tau_{1,1}-i\pi z_1}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$
(2,3)	$\begin{matrix}0&1\\0&0\end{matrix}$	$\phi_{2,3}(i\pi z_2,-i\pi z_1,p,q,A)=ie^{\frac{i\pi}{4}\tau_{2,2}+i\pi z_2}\theta(z+I\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau)$

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Directing Functionals and De Branges Space Completions in Almost Pontryagin Spaces

Harald Woracek

Dedicated to Daniel Alpay on the occasion of his 60th birthday

Abstract. The following theorem holds: Let $\mathcal{L}$ be a – not necessarily nondegenerated or complete – positive semidefinite inner product space carrying an anti-isometric involution, and let S be a symmetric operator in $\mathcal{L}$. If S possesses a universal directing functional $\Phi : \mathcal{L} \times \mathbb{C} \to \mathbb{C}$ which is real w.r.t. the given involution, and the closure of S in the completion of $\mathcal{L}$ has defect index $(1, 1)$, then there exists a de Branges (Hilbert-) space $\mathcal{B}$ such that $x \mapsto \Phi(x, \cdot)$ maps $\mathcal{L}$ isometrically onto a dense subspace of $\mathcal{B}$ and the multiplication operator in $\mathcal{B}$ is the closure of the image of S under this map.

In this paper we consider a version of universal directing functionals defined on an open set $\Omega \subseteq \mathbb{C}$ instead of the whole plane, and inner product spaces $\mathcal{L}$ having finite negative index. We seek for representations of S in a class of reproducing kernel almost Pontryagin spaces of functions on Ω having de Branges-type properties. Our main result is a version of the above stated theorem, which gives conditions making sure that Φ establishes such a representation. This result is accompanied by a converse statement and some supplements.

As a corollary, we obtain that if a de Branges-type inner product space of analytic functions on Ω has a reproducing kernel almost Pontryagin space completion, then this completion is a de Branges-type almost Pontryagin space. This is an important fact in applications. The corresponding result in the case that $\Omega = \mathbb{C}$ and $\mathcal{L}$ is positive semidefinite is well known, often used, and goes back (at least) to work of M. Riesz in the 1920s.

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Keywords. Directing functional, almost Pontryagin space, reproducing kernel space, de Branges space, completion.

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1. Introduction

The method of directing functionals was orginally developed by M.G. Krein in the 1940s. Nowadays it comes in various flavours and generalities: scalar- or operator-valued, defined locally around the real line or on the whole complex plane, for positive semidefinite inner product spaces or indefinite ones leading to Pontryagin or Krein spaces, for symmetric or isometric operators, etc.

One feature of directing functionals, which is the mainly exploited one, is that they can be used to establish existence of spectral functions μ for a symmetric operator S which is given on a *non-complete* inner product space. They provide *explicit* representations of S as a part of the multiplication operator in $L^2(\mu)$. Another feature of the method of directing functionals is that it can be combined with the theory of entire operators, and then leads to *explicit* representations of S as the multiplication operator in a *de Branges space* of entire functions.

In many concrete situations directing functionals are naturally present and lead in a most elegant way to interesting and deep results. Surprisingly, despite its power and ease in application, the method of directing functionals seems to be not widely known.

In the present paper we develop a generalisation of the second mentioned feature to the setting of almost Pontryagin spaces and spaces of analytic functions on arbitrary open base sets $\Omega \subseteq \mathbb{C}$. Our main result is a theorem which constructs in presence of an "Ω-directing functional" a representation of S as the multiplication operator in an "Ω-space".

We continue with a rather detailed discussion of the known theory and available literature. Only after that, in §1.2, we explain the contribution of the present paper in detail, since then it will be much clearer how it relates to the existing theory and what our major achivements are[1].

1.1. Review of the existing theory

Krein considered in his pioneering work [33] finite families of directing functionals, in other words, directing mappings into a finite-dimensional space $\mathbb{C}^p$. For the reason of simplicity, we shall restrict ourselves to the scalar-valued case. Moreover, at least in the subsequent informal discussion, we always think of densely defined linear operators (rather than linear relations).

Directing functionals in positive semidefinite inner product spaces. The power of the method of directing functionals can – to the taste of the author – be best understood in the context of models for symmetric operators. Hence, let us start with a short digression about representations of symmetric operators as the operator of multiplication by the independent variable in certain classes of spaces.

Generally speaking, it is an important problem in spectral theory to find such representations. The first result in this direction which comes to ones mind is, of course, the spectral theorem providing a normal form for a self-adjoint operator.

[1] We compiled the following §1.1 also with the idea to give a very brief survey of the presently available theory of directing functionals.

A few concrete situations where such results play a role are: the spectral theory of differential operators where one speaks of a Fourier transform or – perhaps slightly old-fashioned – of eigenfunction expansions; power moment problems where representations of a certain operator yield measures having a given sequence of numbers as moments; the theory of continuous positive definite functions on the real line or an interval where a proof of Bochner's theorem or certain continuation results can be obtained using representations of an associated self-adjoint or symmetric operator.

We will now state three theorems which provide representations. To do that some notation is required:

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space and T a closed symmetric operator in $\mathcal{H}$.

(i) We call $\beta_+ := \dim[\mathcal{H}/\operatorname{ran}(T-i)]$ and $\beta_- := \dim[\mathcal{H}/\operatorname{ran}(T+i)]$ the *upper- and lower deficiency indices* of T, and the pair (β_+, β_-) the *defect index* of T.

(ii) T is *completely non-selfadjoint* if $\bigcap_{\zeta\in\mathbb{C}\setminus\mathbb{R}} \operatorname{ran}(T-\zeta) = \{0\}$.

(iii) T is *regular* if its set of points of regular type equals $\mathbb{C}$.

Moreover:

(iv) If $\mathcal{K}$ is a Hilbert space of functions, we denote by $S(\mathcal{K})$ the *operator of multiplication by the independent variable* in $\mathcal{K}$ defined on its natural maximal domain.

Remember in this context that every regular operator is completely non-selfadjoint.

The first theorem is a simple consequence of the spectral theorem applied to self-adjoint extensions of T.

Theorem 1.2. *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$. Assume that T is completely non-selfadjoint and that the deficiency indices of T do not exceed* 1. *Then there exists a positive Borel measure μ on $\mathbb{R}$ and an isometric operator $\mathcal{F} : \mathcal{H} \to L^2(\mu)$ such that $\mathcal{F} \circ T \subseteq S(L^2(\mu)) \circ \mathcal{F}$.*

The other two theorems deal with the defect $(1,1)$ case. These results go back at least to work of M.S. Livšic [45, 44] on the characteristic function (when transferred via Caley transforms), or to work of M.G. Krein [32, 31] on the Q-function of a symmetric operator (when combined with reproducing kernel models for Q-functions). There is a vast literature on the topic from many different points of view; for instance, let us also mention the more function-theoretic viewpoint taken up in [47].

A *Herglotz space* $\mathcal{B}$ is a reproducing kernel Hilbert space of functions analytic on $\mathbb{C}\setminus\mathbb{R}$ with certain additional properties. For details see [2, p. 1042], from where we also borrowed the terminology, and [11, Theorems 5,6].

Theorem 1.3. *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$. Assume that T has defect index $(1,1)$ and is completely non-selfadjoint. Then there exists a Herglotz space $\mathcal{B}$ and a unitary operator $\mathcal{F} : \mathcal{H} \to \mathcal{B}$ such that $\mathcal{F} \circ T = S(\mathcal{B}) \circ \mathcal{F}$.*

A *de Branges (Hilbert-) space* $\mathcal{B}$ is a reproducing kernel Hilbert space of entire functions with certain additional properties. For details see [11, §19, Problem 50, Theorem 23].

Theorem 1.4. *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$. Assume that T has defect index $(1,1)$ and is regular. Then there exists a de Branges space $\mathcal{B}$ and a unitary operator $\mathcal{F}:\mathcal{H}\to\mathcal{B}$, such that $\mathcal{F}\circ T = S(\mathcal{B})\circ\mathcal{F}$.*

References which contain explicit proofs are [2, Theorem 6.3] for Theorem 1.3 and [46, Theorem 5.0.7] for Theorem 1.4. In these papers reproducing kernel space theory is systematically exploited, and proofs are based on Livšic' characteristic functions.

Usually, in concrete situations one is given a linear space $\mathcal{L}$ with a positive semidefinite inner product and a symmetric operator S in this space. However, this explicitly accessible space $\mathcal{L}$ is (usually) not complete. For instance, when dealing with differential operators one has a linear space of compactly supported C^∞-functions, or, when dealing with power moments, one has the linear space of polynomials. In order to apply general results as the ones mentioned above, one first has to pass to the Hilbert space completion $\mathcal{H}$ of $\mathcal{L}$ and the closure T of S in this completion. Whereas $\mathcal{L}$ and S are given directly by the data of the problem under consideration, the space $\mathcal{H}$ and the operator T are often not anymore easily and explicitly accessible, simply by the inconstructive nature of the completion process. And this is the point where the method of directing functionals comes into play: It can be used to produce *explicit* representations of S and to draw conclusions *about the closure* T of S.

In the sequel we explain two aspects related to representations as in Theorem 1.2 and Theorem 1.4 in more detail.

Aspect 1. Representations in spaces $L^2(\mu)$

At this point recall the definition of a directing functional, cf. [33, Definition 1] (see also [16, Definition II.8.1]).

Definition 1.5. Let $\mathcal{L}$ be a positive semidefinite inner product space (not necessarily nondegenerated or complete) and let S be a symmetric operator (not necessarily everywhere defined) in $\mathcal{L}$. A function $\Phi:\mathcal{L}\times\mathbb{R}\to\mathbb{C}$ is called a *directing functional* for S, if:

- **(df1)** For each $x\in\mathcal{L}$ the map $\zeta\mapsto\Phi(x,\zeta)$, $\zeta\in\mathbb{R}$, has a continuation to an analytic function defined on some open set Ω_x containing $\mathbb{R}$.
- **(df2)** For each $\zeta\in\mathbb{R}$ the map $x\mapsto\Phi(x,\zeta)$, $x\in\mathcal{L}$, is linear.
- **(df3)** For each $x\in\mathcal{L}$ and $\zeta\in\mathbb{R}$ we have $x\in\operatorname{ran}(S-\zeta)$ if and only if $\Phi(x,\zeta)=0$.
- **(df4)** There exists an element $u\in\mathcal{L}$ such that the function $\Phi(u,\cdot)$ does not vanish identically.

Regardless of the properties of the closure T of S in the Hilbert space completion $\mathcal{H}$ of $\mathcal{L}$, presence of a directing functional for S yields an explicit representation of S in a space $L^2(\mu)$.

Theorem 1.6. *Let $\mathcal{L}$ be a positive semidefinite inner product space, let S be a symmetric operator in $\mathcal{L}$, and let Φ be a directing functional for S. Then there exists a positive Borel measure μ on $\mathbb{R}$, such that the map $\mathcal{F} : x \mapsto [\Phi(x, \cdot)]_\mu$ (here $[\cdot]_\mu$ denotes the equivalence class μ-almost everywhere) is an isometry of $\mathcal{L}$ into $L^2(\mu)$ and satisfies $\mathcal{F} \circ S \subseteq S(L^2(\mu)) \circ \mathcal{F}$.*

A proof can be found in the original reference [33, Theorem 1] or in [16, Theorem II.8.1][2].

As far as we understand history and literature, validity of this result was considered the main feature of directing functionals. However, we believe that the features explained below as "Aspect 2" are equally valuable.

Aspect 2. Representations in reproducing kernel spaces

Recall the definition of a universal directing functional, cf. [16, Definition II.8.2] (we do not know where this notion appeared for the first time).

Definition 1.7. Let $\mathcal{L}$ be a positive semidefinite inner product space (not necessarily nondegenerated or complete) and let S be a symmetric operator (not necessarily everywhere defined) in $\mathcal{L}$. A function $\Phi : \mathcal{L} \times \mathbb{C} \to \mathbb{C}$ is called a *universal directing functional* for S, if:

- **(udf1)** For each $x \in \mathcal{L}$ the map $\zeta \mapsto \Phi(x, \zeta)$, $\zeta \in \mathbb{C}$, is analytic.
- **(udf2)** For each $\zeta \in \mathbb{C}$ the map $x \mapsto \Phi(x, \zeta)$ is linear.
- **(udf3)** For each $x \in \mathcal{L}$ and $\zeta \in \mathbb{C}$ we have $x \in \operatorname{ran}(S - \zeta)$ if and only if $\Phi(x, \zeta) = 0$.
- **(udf4)** There exists an element $u \in \mathcal{L}$ such that the function $\Phi(u, \cdot)$ does not vanish identically.

Note that if S has a universal directing functional, the deficiency indices of the closure T of S cannot exceed 1.

Presence of a universal directing functional plus assuming existence of an entire gauge (using the terminology of [16]) plus assuming that T (the closure of S in the Hilbert space completion of $\mathcal{L}$) has defect $(1, 1)$ invokes Krein's theory of entire operators, and hence leads to an explicit representation in a de Branges

[2]When referring to the treatment of directing functionals in [16] one has to be careful. Theorem II.8.1 in this reference is correctly proved, but Theorems II.8.4 and II.8.5 are not. For Theorem II.8.4 the mistake is obvious: what if $\Phi(v, \cdot)$ vanishes identically (compare with the correct statement [18, Behauptung 3.5]). The mistake in the proof of Theorem II.8.5 is more subtle: In the course of the proof Theorem II.3.3 is employed. This result, however, requires that T is completely non-selfadjoint, which is not known at that point. Adding the additional assumption that T is completely non-selfadjoint implies correctness of the proof of Theorem II.8.5. Despite this mistake in the proof, we do not know whether the actual assertion of Theorem II.8.5 is true or false. However, compare with Proposition 1.9.

space (this is probably a classical fact, an explicit mentioning can be found in [56]).

Theorem 1.8. *Let $\mathcal{L}$ be a positive semidefinite inner product space, let S be a symmetric operator in $\mathcal{L}$, and let Φ be a universal directing functional for S. Assume that T has defect index $(1,1)$, and that there exists an element $u \in \mathcal{L}$ with $\Phi(u,\cdot) = 1$. Then there exists a de Branges space $\mathcal{B}$ such that the map $\mathcal{F} : x \mapsto \Phi(x,\cdot)$ is an isometry of $\mathcal{L}$ onto a dense subspace of $\mathcal{B}$ and satisfies $\mathcal{F} \circ S \subseteq S(\mathcal{B}) \circ \mathcal{F}$ where (thinking of operators in terms of their graphs) $S(\mathcal{B}) = \mathrm{Clos}_{\mathcal{B}\times\mathcal{B}}[(\mathcal{F}\times\mathcal{F})(S)]$. In particular, T is regular.*

It is worth to compare the notions of directing and universal directing functionals in greater detail. Apparently, the conditions (udf1) and (udf3) in Definition 1.7 are strengthenings of Definition 1.5, (df1) and (df3). The fact in (udf3) that the description of $\mathrm{ran}(S-\zeta)$ holds for nonreal points, together with analyticity along $\mathbb{R}$ in (udf1) already implies existence of a representation in a de Branges space (combine the below Proposition 1.9 with Theorem 1.4). Having entire functions in (udf1) in conjunction with existence of an entire gauge (the element u) is responsible for one representation in a de Branges space to be actually given by $x \mapsto \Phi(x,\cdot)$. For completeness, let us note that (df2) together with analyticity on all of $\mathbb{C}$ readily implies (udf2).

Proposition 1.9. *Let $\mathcal{L}$ be a positive semidefinite inner product space and let S be a symmetric operator in $\mathcal{L}$ whose domain is dense in $\mathcal{L}$ w.r.t. the seminorm induced by the inner product. Let $\Omega \subseteq \mathbb{C}$ be an open neighbourhood of $\mathbb{R}$, and let $\Phi : \mathcal{L} \times \mathbb{R} \to \mathbb{C}$ be a function which satisfies:*

- **(df1$^+$)** *For each $x \in \mathcal{L}$ the map $\zeta \mapsto \Phi(x,\zeta)$, $\zeta \in \mathbb{R}$, has a continuation to an analytic function on Ω (which we shall again denote by $\Phi(x,\cdot)$).*
- **(df2$^+$)** *For each $\zeta \in \Omega$ the map $x \mapsto \Phi(x,\zeta)$, $x \in \mathcal{L}$, is linear.*
- **(df3$^+$)** *For each $x \in \mathcal{L}$ and $\zeta \in \Omega$ we have $x \in \mathrm{ran}(S-\zeta)$ if and only if $\Phi(x,\zeta) = 0$.*
- **(df4$^+$)** *There exists an element $u \in \mathcal{L}$ such that the function $\Phi(u,\cdot)$ does not vanish identically on $\mathbb{R}$.*

Assume that T has defect index $(1,1)$. Then T is regular.

The proof of this fact will be given in §4.

It is noteworthy that Theorem 1.8 has a partial converse; one half of (udf3) should be replaced by an approximative variant of the condition.

Proposition 1.10. *Let $\mathcal{L} \neq \{0\}$ be a positive semidefinite inner product space and let S be a symmetric operator in $\mathcal{L}$. Assume that there exists a de Branges space $\mathcal{B}$ and an isometric map $\mathcal{F} : \mathcal{L} \to \mathcal{B}$ such that $\mathrm{ran}\,\mathcal{F}$ is dense in $\mathcal{B}$ and $S(\mathcal{B}) = \mathrm{Clos}_{\mathcal{B}\times\mathcal{B}}[(\mathcal{F}\times\mathcal{F})(S)]$. Then the map*

$$\Phi(x,\zeta) := [\mathcal{F}(x)](\zeta), \quad x \in \mathcal{L}, \zeta \in \mathbb{C},$$

satisfies (udf1), (udf2), (udf4), *and*

(udf3′) *If* $\zeta \in \mathbb{C}$ *and* $x \in \operatorname{ran}(S-\zeta)$, *then* $\Phi(x,\zeta)=0$.

(udf3″) *If* $x \in \mathcal{L}$, $\zeta \in \mathbb{C}\setminus\mathbb{R}$, *and* $\Phi(x,\zeta)=0$, *then there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *with* $x_n \in \operatorname{ran}(S-\zeta)$, $n\in\mathbb{N}$, *such that*

$$\lim_{n\to\infty}(x_n-x,x_n-x)_{\mathcal{L}}=0, \quad \lim_{n\to\infty}\Phi(x_n-x,\eta)=0, \ \ \eta\in\mathbb{C}.$$

The proof is straightforward (using verbatim the same argument as in the proof of Proposition 4.6, (v)).

Directing functionals in indefinite inner product spaces. Let us turn to the presently available versions for indefinite inner product spaces. These are [18, §3] treating the Pontryagin space case and [59] for the Krein space case.

The two aspects of directing functionals discussed above now become indistinct. The reason for this is that self-adjoint operators in a Pontryagin or Krein space may have nonreal spectrum. When seeking representations in spaces of the type $L^2(\mu)$, μ should be substituted by a suitable distribution ϕ and $L^2(\mu)$ by the model space $\Pi(\phi)$ as introduced in [20] for the Pontryagin space case and in [29] for the Krein spaces case. Analyticity of the functions $\Phi(x,\cdot)$ is needed on the spectrum of self-adjoint extensions of T. Thus, contrasting the Hilbert space case, analyticity on the whole plane must be required even if only interested in "Aspect 1".

Perhaps it is for the reason just explained that the definitions of a directing functional in [18] and [59] (when written out for the scalar-valued case) read exactly as Definition 1.7. The results establishing the analogues of Theorem 1.6 are [18, Behauptung 3.1] and [59, Theorem 5.1]. These statements have a different form, but can be reformulated in a way that matches Theorem 1.6. For the analogue of Theorem 1.8 we do not know an explicit reference, but the results of [18, §3] certainly go in that direction. In fact, it can be obtained by combining the "defect $(1,1)$" case in [18, §3] with the indefinite version of entire operators developed a bit later in [35, §6].

About the literature. Our research for literature about theory and applications of directing functionals produced the following list of references which seems to be exhaustive at this point in time. If the reader knows about further references, we would be grateful to learn about them.

`(1) Positive definite theory:` First of all, of course, the pioneering work [33] of M.G. Krein (announced in [30]). There he considered directing mappings into a finite-dimensional space $\mathbb{C}^p$. The basic theorem is a vector-valued version of Theorem 1.6, cf. [33, Theorem 1]. The paper contains two applications, namely a matrix-valued version of Bochner's theorem and a Fourier transform for a first-order differential operator. An overview (including more details and also other topics) can be found in [9].

Soon after that the theory was supplemented by A.Ya. Povzner in [50] who proved inversion formulas, and a bit later F.S. Rofe-Beketov implicitly used directing mappings into infinite-dimensional spaces in his consideration [54] of differential operators with an operator-valued potential. A systematic treatment of the theory of directing mappings into infinite-dimensional spaces was given by H. Langer in [39]. A generalisation to linear relations, including applications to differential expressions and an operator-valued moment problem, was given by H. Langer and B. Textorius in [40, 41, 42]. These results were somewhat refined and transferred to isometric operators by V.A. Yavryan and A.V. Yavryan in [61], where also some more applications are found.

A further generalisation can be found in [8, Chapter V,§5]. A version of directing functionals for isometric operators (instead of symmetric ones) was investigated by M.B. Bekker in [5]. In [56] universal directing functionals and the theory of entire operators is applied to obtain representations in de Branges spaces.

Finally we mention the book [1] where a treatment (somewhat incomplete and bound to a differential operator) can be found in Appendix II, §7, and the book [16] where it is attempted to elaborate the scalar-valued case (remember footnote 2).

`(2) Indefinite theory:` A treatment of directing functionals in the Pontryagin space setting was given by M. Großmann and H. Langer in [18], where the theory was applied to a continuation problem for hermitian indefinite functions on an interval. It is noteworthy that in this paper the method of directing functionals was also combined with ideas from the theory of entire operators.

A version of the basic Theorem 1.6 in the setting of Krein spaces was given by B. Textorius in [59].

`(3) Applications:` The following is a list of references where typical applications of the method of directing functionals are given, i.e., where existence of spectral functions for various problems is established by applying an appropriate variant of Theorem 1.6. In chronological order: [17], [48], [22, 21] (announced in [23]), [6] (where moreover the method is combined with the theory of entire operators), [43], [4], [15] (this paper is remarkable since it deals with an equation having two singular endpoints). Most recently, an application to the transfer function of a Sturm–Liouville equation was elaborated by H. Langer in [38, §5.2,5.3].

1.2. The contribution of the present paper

In this paper we investigate "Aspect 2" and establish a variant of Theorem 1.8. Our results generalise this theorem in different directions:

(1) *Geometrically:* We allow $\mathcal{L}$ to be indefinite with finite index of negativity, and ask for an explicit representation in reproducing kernel almost Pontryagin spaces $\mathcal{B}$ with certain additional properties. At this point it should be said that such representations may exist even if there are none in reproducing kernel Pontryagin spaces.

(2) *Function-theoretically:* We neither require the elements of $\mathcal{B}$ to be analytic on $\mathbb{C} \setminus \mathbb{R}$ nor to be entire. We ask for analyticity on some pregiven subset Ω

of $\mathbb{C}$. In particular, $\Omega \cap \mathbb{R}$ may be a proper – but nonempty – subset of the real line.

(3) *Operator-theoretically:* We allow S to be a linear relation. Of course, this is only a minor point.

We improve upon the conditions given in Theorem 1.8:

(4) Using an approximate version as in Proposition 1.10, an "if and only if" statement is obtained.

(5) The condition that T has defect index $(1, 1)$ is replaced by a pair of conditions which are more directly formulated in terms of $\mathcal{L}$ and Φ. Admittedly, these conditions are still not easy to check (but we believe that this is in the nature of the problem).

The version of directing functionals used in this paper – we call them Ω-directing functionals – is introduced in Definition 4.1. The type of spaces $\mathcal{B}$ in which we seek for representations of S – we call them Ω-spaces – is introduced in Definition 2.19. The main result is Theorem 4.3, where we show that presence of an Ω-directing functional together with some further conditions yields an explicit representation in an Ω-space. This theorem is accompanied by the converse result Proposition 4.6, and a supplement taking care of an additional algebraic structure, cf. Proposition 4.8. Matching the presently used conditions with the ones more familiar from the literature is done in the second part of §4.

Besides what we already mentioned above, a striking difference between Definition 4.1 and Definition 1.7 is that Definition 4.1 does not pose any assumptions on $\operatorname{ran}(S - \eta)$ when η is real. However, this does not mean that such assumptions are not necessary; they just take a different form, namely, they appear as the separate assumption (ii) in Theorem 4.3.

1.3. Organisation of the manuscript

Section 2 is of preliminary nature. However, we do not only set up notation and recall some basics, we also introduce and study some essential notions: *sets of Φ-containement*, cf. Definitions 2.7 and 2.8, and *Ω-spaces*, cf. Definition 2.19. Moreover, we formulate an appropriate version of a theorem of Krein about preservation of analyticity, cf. Theorem 2.15.

Section 3 contains the core arguments needed for the proof of our main result Theorem 4.3. These are arranged in the form of eight lemmata; an outline is given in the beginning of Section 3. Concerning proofs and technique, in this section the major part of the work is done.

In Section 4 we define *Ω-directing functionals*, cf. Definition 4.1, and establish our main theorem. Its proof is obtained by applying the lemmata from the previous section. Moreover, we establish the aforementioned converse and supplement.

In Section 5 we introduce a noncomplete variant of Ω-spaces, cf. Definition 5.1, and show that these geometric properties are inherited when passing to a reproducing kernel completion, cf. Proposition 5.2. This includes the case of *de Branges space completions*, cf. Corollary 5.8, a fact which is of importance in

many concrete situations (and which is well known for the positive definite case, at least going back to work of M. Riesz from the 1920s). One such application, namely, a treatment of (an indefinite version of) the Hamburger power moment problem and the index of determinacy of a measure, will be presented in forthcoming work.

The paper closes with an appendix where we elaborate on the proof of two results on preservation of analyticity for which we cannot appoint an explicit reference. The intention is to make our work more easily accessible also to those readers who are not fully familiar with the subject.

Finally, some standard references: for the geometry of Pontryagin spaces [19], for indefinite inner product spaces in general [10, 3], for almost Pontryagin spaces [25], [58], and [60, Appendix], for complex analysis, e.g., [7, 12, 51], and for linear relation in indefinite inner product spaces [13].

2. Preliminaries

2.1. Some general notation

In the following items we introduce some notation which is used throughout the paper.

1° For a subset Ω of $\mathbb{C}$, set $\Omega^{\#} := \{\eta \in \mathbb{C} : \overline{\eta} \in \Omega\}$, and for a complex-valued function f on Ω, set

$$f^{\#} : \left\{ \begin{array}{ccc} \Omega^{\#} & \to & \mathbb{C} \\ \eta & \mapsto & \overline{f(\overline{\eta})}. \end{array} \right. \tag{2.1}$$

2° Let Ω be an open and nonempty subset of $\mathbb{C}$. Then we denote by $\mathbb{H}(\Omega)$ the set of all complex-valued analytic functions on Ω.

We denote by $\mathbb{C}_\infty$ the *Riemann sphere* considered as a Riemann surface in the usual way, and by $\mathbb{H}(\Omega, \mathbb{C}_\infty)$ the set of all analytic functions of Ω into $\mathbb{C}_\infty$. In other words, $\mathbb{H}(\Omega, \mathbb{C}_\infty)$ is the set of all meromorphic functions on Ω.

The spaces $\mathbb{H}(\Omega)$ and $\mathbb{H}(\Omega, \mathbb{C}_\infty)$ are always endowed with the compact-open topology, that is, the topology of locally uniform convergence.

3° For a function $f \in \mathbb{H}(\Omega, \mathbb{C}_\infty)$, we denote by $\mathfrak{d}_f : \Omega \to \mathbb{Z} \cup \{\pm\infty\}$ its *divisor*. This is the function defined as $\mathfrak{d}_f(\eta) := -\infty$ if $f = \infty$ on the connected component of Ω containing η, as $\mathfrak{d}_f(\eta) := +\infty$ if $f = 0$ on the component of Ω containing η, and as $\mathfrak{d}_f(\eta)$ being the power of the first nonvanishing term in the Laurent expansion of f at η if none of the previous cases occurs.

For a nonempty family $\mathcal{F} \subseteq \mathbb{H}(\Omega, \mathbb{C}_\infty)$, we set

$$\mathfrak{d}_{\mathcal{F}}(\eta) := \inf\{\mathfrak{d}_f(\eta) : f \in \mathcal{F}\} \in \mathbb{Z} \cup \{\pm\infty\}, \quad \eta \in \Omega.$$

4° For two nonempty sets Ω and X we denote by $\chi_\eta : X^\Omega \to X$ the *point evaluation functional* at the point $\eta \in \Omega$, i.e.,

$$\chi_\eta : \left\{ \begin{array}{ccc} X^\Omega & \to & X \\ f & \mapsto & f(\eta), \end{array} \right. \qquad \eta \in \Omega.$$

If Ω is an open and nonempty subset of $\mathbb{C}$, we denote

$$\chi_\eta^{(l)} : \left\{ \begin{array}{ccc} \mathbb{H}(\Omega) & \to & \mathbb{C} \\ f & \mapsto & f^{(l)}(\eta), \end{array} \right. \qquad \eta \in \Omega, l \in \mathbb{N}_0. \tag{2.2}$$

5° Let $\mathcal{X}, \mathcal{Y}$ be linear spaces. A linear subspace of V of $\mathcal{X} \times \mathcal{Y}$ is called a *linear relation*. We set

$$\begin{aligned} \operatorname{dom} V &:= \big\{x \in \mathcal{X} : \ \exists\, y \in \mathcal{Y} \text{ with } (x;y) \in V\big\}, \\ \operatorname{ran} V &:= \big\{y \in \mathcal{Y} : \ \exists\, x \in \mathcal{X} \text{ with } (x;y) \in V\big\}, \\ \ker V &:= \big\{x \in \mathcal{X} : \ (x;0) \in V\big\}, \\ \operatorname{mul} V &:= \big\{y \in \mathcal{Y} : \ (0;y) \in V\big\}, \end{aligned}$$

and speak of the *domain*, *range*, *kernel*, and *multivalued part* of V. Moreover, set

$$V^{-1} := \{(y;x) : (x;y) \in V\}.$$

If $v : \mathcal{L} \subseteq \mathcal{X} \to \mathcal{Y}$ is a linear map defined on some linear subspace $\mathcal{L}$ of $\mathcal{X}$, then its *graph*

$$\operatorname{graph} v := \big\{(x; v(x)) : x \in \mathcal{L}\big\} \subseteq \mathcal{X} \times \mathcal{Y}$$

is a linear relation. A linear relation is the graph of some linear map if and only if its multivalued part equals $\{0\}$.

6° Let $\mathcal{X}$ be a linear space, and let $V \subseteq \mathcal{X} \times \mathcal{X}$ be a linear relation. For $\eta \in \mathbb{C}$ set

$$V - \eta := \big\{(x; y - \eta x) : (x;y) \in V\big\}$$

and

$$\begin{aligned} \sigma_p(V) &:= \big\{\eta \in \mathbb{C} : \ker(V - \eta) \neq \{0\}\big\} \\ &= \big\{\eta \in \mathbb{C} : \ \exists\, x \in \mathcal{X} \setminus \{0\} \text{ with } (x; \eta x) \in V\big\}. \end{aligned}$$

7° Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and let $V \subseteq \mathcal{X} \times \mathcal{Y}$ be a linear relation. Then we say that V is *closed*, if it is closed as a subset of $\mathcal{X} \times \mathcal{Y}$ w.r.t. the product topology.

Let $\mathcal{X}$ be a normed space, and let $V \subseteq \mathcal{X} \times \mathcal{X}$ be a linear relation. We say a point $\eta \in \mathbb{C}$ is a *point of regular type* of V, if $(V - \eta)^{-1}$ is (the graph of) a bounded linear operator (with domain and range not necessarily equal to all of $\mathcal{X}$). The set of all points of regular type is denoted by $r(V)$. If $\mathcal{X}$ is a Banach space and V is closed, then $\eta \in r(V)$ if and only if $\operatorname{ran}(V - \eta)$ is closed and $\eta \notin \sigma_p(V)$.

8° Let $\mathcal{X}$ be a linear space. A map $[\cdot,\cdot]_\mathcal{X}$ is called an *inner product*, if it is linear in the first argument and $[x,y]_\mathcal{X} = \overline{[y,x]_\mathcal{X}}$, $x, y \in \mathcal{X}$. We call $\langle \mathcal{X}, [\cdot,\cdot]_\mathcal{X} \rangle$ an *inner product space*. Note that we do not assume any definiteness properties.

Let $\langle \mathcal{X}, [\cdot,\cdot]_\mathcal{X} \rangle$ be an inner product space. Then we put

$$\begin{aligned} \operatorname{ind}_+ \mathcal{X} &:= \sup\big\{ \dim \mathcal{L} : \mathcal{L} \text{ positive definite subspace of } \mathcal{X}\big\}, \\ \operatorname{ind}_- \mathcal{X} &:= \sup\big\{ \dim \mathcal{L} : \mathcal{L} \text{ negative definite subspace of } \mathcal{X}\big\}. \end{aligned}$$

We write $x \perp y$ if $[x, y]_{\mathcal{X}} = 0$, set $M^{\perp} := \{x \in \mathcal{X} : x \perp y, y \in M\}$ for $M \subseteq \mathcal{X}$, and set $\mathcal{X}^{\circ} := \mathcal{X}^{\perp}$ and $\operatorname{ind}_0 \mathcal{X} := \dim \mathcal{X}^{\circ}$.

Let us point out that we do not distinguish between different cardinalities of infinity. All dimensions, as well as $\operatorname{ind}_{\pm}$ and ind_0, are understood as elements of $\mathbb{N}_0 \cup \{\infty\}$. In particular, if we write expressions like "$d < \dim \mathcal{L}$", then this implies that d is finite.

2.2. Almost Pontryagin spaces

Almost Pontryagin spaces are a type of complete topological inner product spaces which generalise Pontryagin spaces. We recall only the definition; precise references to the literature for geometric and topological properties will be given during the presentation.

Definition 2.1. We call a triple $\langle \mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \mathcal{O} \rangle$ an *almost Pontryagin space*, if $\mathcal{A}$ is a linear space, $[\cdot, \cdot]_{\mathcal{A}}$ is an inner product on $\mathcal{A}$, and $\mathcal{O}$ is a topology on $\mathcal{A}$, such that the following axioms hold.

- **(aPs1)** The topology $\mathcal{O}$ is a Hilbert space topology on $\mathcal{A}$, i.e., it is induced by some inner product which turns $\mathcal{A}$ into a Hilbert space.
- **(aPs2)** The inner product $[\cdot, \cdot]_{\mathcal{A}}$ is $\mathcal{O}$-continuous, i.e., it is continuous as a map of $\mathcal{A} \times \mathcal{A}$ into $\mathbb{C}$ where $\mathcal{A} \times \mathcal{A}$ carries the product topology $\mathcal{O} \times \mathcal{O}$ and $\mathbb{C}$ the euclidean topology.
- **(aPs3)** There exists an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension in $\mathcal{A}$, such that $\langle \mathcal{M}, [\cdot, \cdot]_{\mathcal{A}}|_{\mathcal{M} \times \mathcal{M}} \rangle$ is a Hilbert space.

If $\langle \mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \mathcal{O} \rangle$ and $\langle \mathcal{B}, [\cdot, \cdot]_{\mathcal{B}}, \mathcal{T} \rangle$ are two almost Pontryagin spaces, then we call a map $\psi : \mathcal{A} \to \mathcal{B}$ an *isomorphism* if ψ is a linear and isometric homeomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Here isometry is understood w.r.t. the inner products $[\cdot, \cdot]_{\mathcal{A}}$ and $[\cdot, \cdot]_{\mathcal{B}}$, and bicontinuity w.r.t. $\mathcal{O}$ and $\mathcal{T}$.

When no confusion is possible, we drop explicit notation of inner product and topology, and shortly speak of an almost Pontryagin space $\mathcal{A}$.

An almost Pontryagin space $\mathcal{A}$ is a Pontryagin space if and only if $\operatorname{ind}_0 \mathcal{A} = 0$. If $\operatorname{ind}_0 \mathcal{A} = 0$, the topology $\mathcal{O}$ is uniquely determined by the linear space $\mathcal{A}$ and the inner product $[\cdot, \cdot]_{\mathcal{A}}$ on $\mathcal{A}$. If $\operatorname{ind}_0 \mathcal{A} > 0$, this is not the case, see, e.g., [25, Lemma 2.8].

Our focus in the present paper lies on a special kind of almost Pontryagin spaces, namely such whose elements are complex-valued functions and which have the property that point evaluation functionals are continuous.

Definition 2.2. Let Ω be a nonempty set. We call an almost Pontryagin space $\langle \mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \mathcal{O} \rangle$ a *reproducing kernel almost Pontryagin space on* Ω, if the following axioms hold.

- **(RKS1)** The elements of $\mathcal{A}$ are complex-valued functions on Ω, and the linear operations of $\mathcal{A}$ are given by pointwise addition and scalar multiplication.

(RKS2) For each $\eta \in \Omega$ the point evaluation functional $\chi_\eta|_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ is continuous w.r.t. the topology $\mathcal{O}$ on $\mathcal{A}$.

A systematic treatment of reproducing kernel almost Pontryagin spaces is given in [60]. Among other things, it is shown in this paper that for a reproducing kernel almost Pontryagin space an analogue of the reproducing kernel of a reproducing kernel Pontryagin space exists (which also justifies the choice of terminology).

If $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}} \rangle$ is an inner product space subject to (RKS1), then there exists at most one topology on $\mathcal{A}$ such that $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O} \rangle$ is a reproducing kernel almost Pontryagin space. This is due to the presence of the point separating family $\{\chi_\eta|_{\mathcal{A}} : \eta \in \Omega\}$ of continuous functionals, cf. [25, Proposition 2.9].

2.3. Almost Pontryagin space completions

Given an inner product space $\mathcal{L}$, the question appears naturally whether $\mathcal{L}$ can be embedded into an almost Pontryagin space as a dense subspace. This leads to the notion of completions which was formally introduced in [25], with some ideas going back to [20]. A more systematic study was undertaken in [58] and supplemented in [60].

Again we recall only some definitions; precise references to the literature are provided in course of the presentation.

Definition 2.3. Let $\mathcal{L}$ be an inner product space. We call a pair $\langle \iota, \mathcal{A} \rangle$ an *almost Pontryagin space completion of* $\mathcal{L}$, if $\mathcal{A}$ is an almost Pontryagin space, and ι is a linear and isometric map of $\mathcal{L}$ onto a dense subspace of $\mathcal{A}$.

Let $\langle \iota_i, \mathcal{A}_i \rangle$, $i = 1, 2$, be two almost Pontryagin space completions of $\mathcal{L}$. We say that $\langle \iota_1, \mathcal{A}_1 \rangle$ and $\langle \iota_2, \mathcal{A}_2 \rangle$ are *isomorphic*, if there exists an isomorphism φ of $\mathcal{A}_1$ onto $\mathcal{A}_2$ with $\varphi \circ \iota_1 = \iota_2$.

It is easy to see that an inner product space $\mathcal{L}$ has an almost Pontryagin space completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$. If $\operatorname{ind}_- \mathcal{L} < \infty$, the totality of all completions of $\mathcal{L}$ corresponds to certain spaces of linear functionals on $\mathcal{L}$. The relevant notion in this context is given in Definition 2.4 below. We state this definition in the intrinsic form given in [60, Lemma A.14]. The description of isomorphy classes of completions was given in [58, Theorem 6.8].

Definition 2.4. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. We denote by $\mathcal{L}'$ the set of all linear functionals φ on $\mathcal{L}$ with

$$\forall\, (x_n)_{n\in\mathbb{N}}, x_n \in \mathcal{L} : \quad \left.\begin{array}{l} \lim_{n\to\infty} [x_n, x_n]_{\mathcal{L}} = 0 \\ \lim_{n\to\infty} [x_n, y]_{\mathcal{L}} = 0, y \in \mathcal{L} \end{array}\right\} \implies \lim_{n\to\infty} \varphi(x_n) = 0.$$

The choice of notation $\mathcal{L}'$, resembling the standard notation for the topological dual of a topological vector space, is motivated by the fact that indeed $\mathcal{L}'$ is the topological dual of $\mathcal{L}$ w.r.t. a certain intrinsically contructed seminorm. However,

for a quick approach, it is simpler to just write out the explicit conditions as done above.

It is an important fact that $\mathcal{L}'$ can be topologised in a canonical way (different than with a weak topology); for a proof of the following see [60, Lemma A.13, Lemma A.17].

Remark 2.5 (The topology $\mathcal{T}(\mathcal{L}')$ on $\mathcal{L}'$). Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$ and let $\langle \iota, \mathcal{A} \rangle$ be an almost Pontryagin space completion of $\mathcal{L}$. Then the restriction $\iota^*|_{\mathcal{A}'}$ of the algebraic dual map ι^* of ι to the topological dual $\mathcal{A}'$ of $\mathcal{A}$ maps the closed subspace

$$\mathcal{A}^\iota := \{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\}$$

of $\mathcal{A}'$ onto $\mathcal{L}'$. We denote by $\mathcal{T}(\mathcal{L}')$ the final topology on $\mathcal{L}'$ w.r.t. the map $\iota^*|_{\mathcal{A}^\iota}$ and the norm topology on $\mathcal{A}^\iota$ inherited from $\mathcal{A}'$. The topology $\mathcal{T}(\mathcal{L}')$ does not depend on the choice of $\langle \iota, \mathcal{A} \rangle$. It can be constructed using only the Pontryagin space completion of $\mathcal{L}$.

Let us return to our focus: reproducing kernel almost Pontryagin spaces on a set Ω. In this context the following question occurs:

> *Given an inner product space $\mathcal{L}$ whose elements are complex-valued functions on a set Ω, does there exist a reproducing kernel almost Pontryagin space $\mathcal{A}$ which contains $\mathcal{L}$ isometrically as a dense subspace?*

Definition 2.6. Let Ω be a nonempty set, $\mathcal{L} \subseteq \mathbb{C}^\Omega$ an inner product space, and $\mathcal{A}$ a reproducing kernel almost Pontryagin space. We say that $\mathcal{A}$ is a *reproducing kernel space completion* of $\mathcal{L}$, if $\mathcal{A}$ contains $\mathcal{L}$ isometrically as a dense subspace.

Observe that, if $\mathcal{A}$ is a reproducing kernel space completion of $\mathcal{L}$, then $\langle \subseteq, \mathcal{A} \rangle$ is an almost Pontryagin space completion of $\mathcal{L}$.

The obvious necessary condition on $\mathcal{L}$ that a reproducing kernel space completion exists, namely, that $\operatorname{ind}_- \mathcal{L} < \infty$, is far from sufficient. In [60, Theorem 4.1] a characterisation of existence was given. Moreover, it is shown that a reproducing kernel space completion is unique (provided there exists one).

2.4. Sets of Φ-containement

The following sets play a central role in the present paper.

Definition 2.7. Let $\mathcal{L}$ be a linear space, let $V \subseteq \mathcal{L} \times \mathcal{L}$ be a linear relation, let $\Omega \subseteq \mathbb{C}$, and $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$. Then we denote

$$\begin{aligned} r_{\subseteq}(V, \Phi) &:= \big\{\eta \in \Omega : \operatorname{ran}(V - \eta) \subseteq \ker \Phi(\cdot, \eta)\big\}, \\ r_{\supseteq}(V, \Phi) &:= \big\{\eta \in \Omega : \operatorname{ran}(V - \eta) \supseteq \ker \Phi(\cdot, \eta)\big\}, \\ r_{=}(V, \Phi) &:= r_{\subseteq}(V, \Phi) \cap r_{\supseteq}(V, \Phi). \end{aligned}$$

Moreover, we introduce an approximative variant of $r_{\supseteq}(\cdot, \cdot)$.

Definition 2.8. Let $\langle \mathcal{L}, [\cdot,\cdot]_{\mathcal{L}} \rangle$ be an inner product space, let $V \subseteq \mathcal{L} \times \mathcal{L}$ be a linear relation, let $\Omega \subseteq \mathbb{C}$, let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$, and $M \subseteq \Omega$. Then we write

$$r_{\supseteq}^{\mathrm{app}}(V,\Phi;M) := \Big\{ \eta \in \Omega : \ \forall x \in \ker \Phi(\cdot,\eta)\ \exists (x_n)_{n\in\mathbb{N}} \text{ with } x_n \in \operatorname{ran}(V-\eta), \\ \lim_{n\to\infty} [x_n,x_n]_{\mathcal{L}} = [x,x]_{\mathcal{L}}, \quad \lim_{n\to\infty} [x_n,y]_{\mathcal{L}} = [x,y]_{\mathcal{L}},\ y \in \mathcal{L}, \\ \lim_{n\to\infty} \Phi(x_n,\zeta) = \Phi(x,\zeta),\ \zeta \in M \Big\}.$$

Clearly, the inclusion $r_{\supseteq}(V,\Phi) \subseteq r_{\supseteq}^{\mathrm{app}}(V,\Phi;M)$ holds; for given $x \in \ker \Phi(\cdot,\eta)$ simply use the constant sequence $x_n := x$.

For later reference let us explicitly state the following fact.

Remark 2.9. Assume that $\mathcal{A}$ is an almost Pontryagin space, $V \subseteq \mathcal{A} \times \mathcal{A}$ is a linear relation, $\Omega \subseteq \mathbb{C}$, and $\Phi : \mathcal{A} \times \Omega \to \mathbb{C}$. Let $M \subseteq \Omega$ be a subset with the properties

(i) $\Phi(\cdot,\eta) \in \mathcal{A}'$, $\eta \in M$;
(ii) the family $\{\Phi(\cdot,\eta) : \eta \in M\}$ is *point separating on* $\mathcal{A}^\circ$ which means that

$$\mathcal{A}^\circ \cap \bigcap_{\eta\in M} \ker \Phi(\cdot,\eta) = \{0\}.$$

Then

$$r_{\supseteq}(V,\Phi) \ \supseteq \ r_{\supseteq}^{\mathrm{app}}(V,\Phi;M) \cap \{\eta \in \mathbb{C} : \operatorname{ran}(V-\eta) \text{ closed}\}.$$

To see this, choose a sequence $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\supseteq}^{\mathrm{app}}(V,\Phi;M)$. By [60, Proposition A.5] it holds that $\lim_{n\to\infty} x_n = x$ in the norm of $\mathcal{A}$, whence $x \in \operatorname{ran}(V-\eta)$.

2.5. Symmetric relations in almost Pontryagin spaces

Definition 2.10. Let $\langle \mathcal{L}, [\cdot,\cdot]_{\mathcal{L}} \rangle$ be an inner product space, and let $S \subseteq \mathcal{L} \times \mathcal{L}$ be a linear relation in $\mathcal{L}$. We say that S is *symmetric*, if

$$[y_1,x_2]_{\mathcal{L}} = [x_1,y_2]_{\mathcal{L}}, \quad (x_1;y_1),(x_2;y_2) \in S.$$

For a linear map (considered as a linear relation via its graph) this condition reduces to the usual symmetry condition $[Sx,y]_{\mathcal{L}} = [x,Sy]_{\mathcal{L}}$, $x,y \in \operatorname{dom} S$.

The following basic result on symmetric linear relations in a Pontryagin space is shown in [13].

Theorem 2.11. *Let $\mathcal{P}$ be a Pontryagin space and let S be a closed symmetric relation in $\mathcal{P}$. Then the following statements hold.*

(i) $\operatorname{ran}(S-\eta)$ *is closed for* $\eta \in \mathbb{C} \setminus \mathbb{R}$.
(ii) *The codimension of* $\operatorname{ran}(S-\eta)$ *is constant on connected components of* $r(S)$.

(iii) *There exist numbers* $\alpha, \beta_+, \beta_- \in \mathbb{N}_0$ *with* $\alpha \le \operatorname{ind}_- \mathcal{P}$*, and an exceptional set* $\Upsilon \subseteq \mathbb{C} \setminus \mathbb{R}$*, such that*

$$\max\big\{|\Upsilon \cap \mathbb{C}^+|, |\Upsilon \cap \mathbb{C}^-|\big\} \le \operatorname{ind}_- \mathcal{P} - \alpha, \tag{2.3}$$

$$\operatorname{codim}_{\mathcal{P}} \operatorname{ran}(S-\eta) = \beta_\pm, \ \dim\ker(S-\eta) = \alpha, \quad \eta \in \mathbb{C}^\pm \setminus \Upsilon, \tag{2.4}$$

$$\operatorname{codim}_{\mathcal{P}} \operatorname{ran}(S-\eta) - \beta_\pm = \dim\ker(S-\eta) - \alpha > 0, \quad \eta \in \mathbb{C}^\pm \cap \Upsilon, \tag{2.5}$$

$$\alpha \le \dim \operatorname{mul} S. \tag{2.6}$$

(iv) *The numbers* $\beta_\pm$ *can be evaluated as*

$$\beta_\pm = \min\big\{ \operatorname{codim}_{\mathcal{A}} \operatorname{ran}(S-\eta) : \eta \in L\big\}, \quad L \subseteq \mathbb{C}^\pm, \ |L| > \operatorname{ind}_- \mathcal{P}. \tag{2.7}$$

Item (i) is [13, Proposition 4.3, (ii)], item (ii) is [13, Corollary to Theorem 2.4], item (iii) is [13, Proposition 4.4 and Theorem 2.4], and item (iv) follows immediately by combining (2.4) and (2.5).

Using [25, Proposition 3.2], Theorem 2.11 transfers immediately to the almost Pontryagin space situation.

Corollary 2.12. *Let* $\mathcal{A}$ *be an almost Pontryagin space and let* S *be a closed symmetric relation in* $\mathcal{A}$*. Then the assertions of Theorem* 2.11 *remain true when* $\operatorname{ind}_- \mathcal{P}$ *is everywhere replaced by* $\operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$*.*

Proof. Due to [25, Proposition 3.2], we can choose a Pontryagin space $\mathcal{P}$ which contains $\mathcal{A}$ as a closed subspace with codimension $\operatorname{ind}_0 \mathcal{A}$. It holds that $\operatorname{ind}_- \mathcal{P} = \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A}$. □

We refer to β_+ and β_- as the *upper-* and *lower defect indices* of S in $\mathcal{A}$, and to the pair (β_+, β_-) as the *defect index* of S in $\mathcal{A}$.

The concept of a completely non-selfadjoint symmetric relation in an almost Pontryagin space is defined in the usual way (only taking care of possible unsymmetrically located spectral points).

Definition 2.13. Let $\mathcal{A}$ be an almost Pontryagin space and let S be a closed symmetric relation in $\mathcal{A}$ with $r(S) \ne \emptyset$. We say that S is *completely non-selfadjoint*, if

$$\bigcap_{\eta \in r(S) \cap r(S)^\#} \operatorname{ran}(S-\eta) = \{0\}.$$

The usual consequence of analyticity holds true (an explicit proof can be found in [57]):

Lemma 2.14. *Let* $\mathcal{A}$ *be an almost Pontryagin space and let* S *be a closed symmetric relation in* $\mathcal{A}$ *with* $r(S) \ne \emptyset$*. Let* Ω *be an open and nonempty subset of* $r(S) \cap r(S)^\# \cap \mathbb{C}^+$*, and set*

$$U := \begin{cases} r(S) \cap r(S)^\# & , \quad r(S) \cap \mathbb{R} \ne \emptyset, \\ r(S) \cap r(S)^\# \cap \mathbb{C}^+ & , \quad r(S) \cap \mathbb{R} = \emptyset. \end{cases}$$

Then

$$\bigcap_{\eta\in\Omega} \operatorname{ran}(S-\eta) = \bigcap_{\eta\in U} \operatorname{ran}(S-\eta).$$

The same holds with $\mathbb{C}^+$ *replaced by* $\mathbb{C}^-$.

2.6. Krein's Theorem on preservation of analyticity

A central result in the theory of entire operators and directing functionals is a theorem on preservation of analyticity which goes back to M.G. Krein. The below Theorem 2.15 is a variant of this result formulated in a sufficiently general way to serve our present needs. The crux of the proof can be extracted from [34], see also the argument leading to [16, Ch. 2, Theorem 3.2] and the argument indicated in [35] leading to Items (i)–(iii) on p. 430 of this reference. The reader who is not familiar with this type of arguments can find a fully elaborated proof in Appendix A of this paper.

Theorem 2.15 ([34], [35]). *Let* $\mathcal{K}$ *be a Krein space and* $A \subseteq \mathcal{K}\times\mathcal{K}$ *a definitisable self-adjoint linear relation. Let* $a_0 \in \mathcal{K}$, $\zeta_0 \in \rho(A)$, *and set*

$$a(\zeta) := \big(I + (\zeta-\zeta_0)(A-\zeta)^{-1}\big)a_0, \quad \zeta\in\rho(A).$$

Moreover, assume that $y \in \mathcal{K}$ *with the property that* $[a(\cdot), y]_{\mathcal{K}}$ *does not vanish identically on any component of* $\rho(A)$. *Then the following statements hold.*

(i) *Let* $x \in \mathcal{K}$. *Consider the set of all open subsets* $O \subseteq \mathbb{C}$ *with the property that the function*

$$\begin{cases} \{\zeta\in\rho(A) : [a(\zeta),y]_{\mathcal{K}} \neq 0\} & \to & \mathbb{C} \\ \zeta & \mapsto & \frac{[a(\zeta),x]_{\mathcal{K}}}{[a(\zeta),y]_{\mathcal{K}}}, \end{cases} \tag{2.8}$$

has a meromorphic continuation to O. *This set has a largest element.*

We denote this largest element by $\Omega_{x,y}$, *and the meromorphic continuation of the function* (2.8) *to* $\Omega_{x,y}$ *by* $\Theta_{x,y}$.

(ii) *Let* $\Omega \subseteq \mathbb{C}$ *be open and let* $\mathfrak{d} : \Omega \to \mathbb{Z}$ *be a function with discrete support. Choose a function* $W \in \mathbb{H}(\Omega, \mathbb{C}_\infty)$ *with* $\mathfrak{d}_W = -\mathfrak{d}$. *Then the family* ($\|\cdot\|_{\mathcal{K}}$ *denotes some norm induced by a fundamental decomposition of* $\mathcal{K}$)

$$\mathcal{F} := \big\{W\Theta_{x,y} : O_{x,y} \supseteq \Omega, \mathfrak{d}_{\Theta_{x,y}} \geq \mathfrak{d}, \|x\|_{\mathcal{K}} \leq 1\big\}$$

is normal in $\mathbb{H}(\Omega)$.

(iii) *Let* $\Omega \subseteq \mathbb{C}$ *be open and let* $\mathfrak{d} : \Omega \to \mathbb{Z}$ *be a function with discrete support. Then the linear subspace*

$$\mathcal{M} := \big\{x\in\mathcal{K} : O_{x,y} \supseteq \Omega, \mathfrak{d}_{\Theta_{x,y}}|_\Omega \geq \mathfrak{d}\big\} \subseteq \mathcal{K}$$

is closed.

We will also make use of the following lemma which depends on analyticity. This fact is folklore, however, we cannot appoint an explicit reference. For completeness, a proof is provided in Appendix A.

Lemma 2.16. *Let $\mathcal{X}$ be a complete metrisable topological vector space, let $\Omega \subseteq \mathbb{C}$ be open, nonempty, and connected, and let $\Phi : \mathcal{X} \times \Omega \to \mathbb{C}$. Assume that*

(i) *for each $x \in \mathcal{X}$, the map $\Phi(x, \cdot)$ is analytic;*
(ii) *the set $L := \{\eta \in \Omega : \Phi(\cdot, \eta) \in \mathcal{X}'\}$ has an accumulation point in Ω.*

Then $\Phi(\cdot, \eta)$ is linear for all $\eta \in \Omega$, and the map

$$\Phi_{\mathcal{X}} : \begin{cases} \mathcal{X} & \to & \mathbb{H}(\Omega) \\ x & \mapsto & \Phi(x, \cdot) \end{cases}$$

is continuous. In particular, we have $\frac{\partial^l}{\partial \eta^l}\Phi(\cdot, \eta) \in \mathcal{X}'$, $\eta \in \Omega$. For every bounded subset M of $\mathcal{X}$, the image $\{\Phi(x, \cdot) : x \in M\}$ of M under $\Phi_{\mathcal{X}}$ is a normal family.

2.7. The multiplication operator

In this subsection we introduce the type of spaces of analytic functions which we are going to deal with, and study the operator of multiplication by the independent variable in such spaces.

Definition 2.17. Let $\Omega \subseteq \mathbb{C}$ be open and nonempty, and let $\mathcal{B}$ be a reproducing kernel almost Pontryagin space of analytic functions on Ω. Then we denote by $S(\mathcal{B})$ the *multiplication operator in $\mathcal{B}$*. Written as a linear relation this is

$$S(\mathcal{B}) := \big\{(f(\zeta); \zeta f(\zeta)) : f(\zeta), \zeta f(\zeta) \in \mathcal{B}\big\}.$$

The operator $S(\mathcal{B})$ is closed and (remember the notation (2.2))

$$\sigma_p(S(\mathcal{B})) = \emptyset, \quad \operatorname{mul} S(\mathcal{B}) = \{0\}, \tag{2.9}$$

$$\operatorname{ran}(S(\mathcal{B}) - \eta) \subseteq \ker \chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}, \ \eta \in \Omega, \mathfrak{d}_{\mathcal{B}}(\eta) < \infty. \tag{2.10}$$

Moreover, for each subset $L \subseteq \Omega$ which has accumulation points in each connected component of Ω, it holds that $\bigcap_{\eta \in L} \operatorname{ran}(S(\mathcal{B}) - \eta) = \{0\}$. The above properties give rise to the following fact.

Remark 2.18. Assume that $S(\mathcal{B})$ is symmetric. Then $\mathbb{C} \setminus \mathbb{R} \subseteq r(S(\mathcal{B}))$ and $S(\mathcal{B})$ is completely non-selfadjoint.

The following class of spaces is an analogue of de Branges spaces for arbitrary open subsets Ω of $\mathbb{C}$.

Definition 2.19. Let Ω be an open and nonempty subset of $\mathbb{C}$, and let $\mathcal{B}$ be a reproducing kernel almost Pontryagin space on Ω. We call $\mathcal{B}$ an *Ω-space*, if it satisfies the following axioms.

(Ω1) The elements of $\mathcal{B}$ are analytic functions on Ω.

(Ω2) $\operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ is a discrete subset of Ω (in particular, thus, $\mathcal{B} \neq \{0\}$ and $\mathfrak{d}_{\mathcal{B}}(\eta) < \infty$, $\eta \in \Omega$).

(Ω3) $\forall \xi \in \Omega \ \forall f \in \mathcal{B}, f^{(\mathfrak{d}_{\mathcal{B}}(\xi))}(\xi) = 0 : \quad \dfrac{f(\zeta)}{\zeta - \xi} \in \mathcal{B}.$

(Ω4) $\forall \xi \in \Omega \ \forall f, g \in \mathcal{B}, f^{(\mathfrak{d}_{\mathcal{B}}(\xi))}(\xi) = g^{(\mathfrak{d}_{\mathcal{B}}(\xi))}(\xi) = 0 :$

$$\Big[\frac{\zeta - \overline{\xi}}{\zeta - \xi} f(\zeta), \frac{\zeta - \overline{\xi}}{\zeta - \xi} g(\zeta)\Big]_{\mathcal{B}} = \big[f(\zeta), g(\zeta)\big]_{\mathcal{B}}.$$

An Ω-space has several operator theoretic properties which follow immediately from the definition.

Remark 2.20. Let $\mathcal{B}$ be an Ω-space. Then $S(\mathcal{B})$ is symmetric, completely nonselfadjoint, and

$$\operatorname{ran}(S(\mathcal{B})-\eta)=\ker\chi_\eta^{(\mathfrak{d}_\mathcal{B}(\eta))},\ \eta\in\Omega,\quad \Omega\cup(\mathbb{C}\setminus\mathbb{R})\subseteq r(S(\mathcal{B})). \tag{2.11}$$

If $\Omega\cap\mathbb{C}^+\neq\emptyset$, then the upper defect index β_+ of $S(\mathcal{B})$ equals 1. The same holds with $\mathbb{C}^+$ and β_+ replaced by $\mathbb{C}_-$ and β_-.

Using Lemma 2.14 and analyticity, we obtain the following fact: For each $f\in\mathcal{B}$, the set $\operatorname{supp}\mathfrak{d}_f\cap\mathbb{C}^+$ is either a discrete subset of Ω or all of $\Omega\cap\mathbb{C}^+$. The same holds with $\mathbb{C}^+$ replaced by $\mathbb{C}^-$. As a consequence, if $\Omega\cap\mathbb{R}\neq\emptyset$, then $\operatorname{supp}\mathfrak{d}_f$ is a discrete subset of Ω whenever $f\in\mathcal{B}\setminus\{0\}$.

These operator-theoretic properties are characteristic for Ω-spaces.

Proposition 2.21. *Let $\Omega\subseteq\mathbb{C}$ be open and nonempty, and let $\mathcal{B}$ be a reproducing kernel almost Pontryagin space on Ω. Assume that $\mathcal{B}$ satisfies* (Ω1), (Ω2), *and*

(i) *$S(\mathcal{B})$ is symmetric;*
(ii) *if $\Omega\cap\mathbb{C}^+\neq\emptyset$, then $\beta_+=1$, and if $\Omega\cap\mathbb{C}^-\neq\emptyset$, then $\beta_-=1$.*

Then $\mathcal{B}$ is an Ω-space.

Proof. We need to show that the condition in (Ω3), i.e., invariance under division, holds for all $\xi\in\Omega$. Since $\operatorname{supp}\mathfrak{d}_\mathcal{B}$ is discrete, for no $\eta\in\Omega$ the functional $\chi_\eta^{(\mathfrak{d}_\mathcal{B}(\eta))}$ vanishes identically. If $\xi\in\Omega\cap r(S(\mathcal{B}))$, then by (ii) we have $\operatorname{codim}_\mathcal{B}\operatorname{ran}(S(\mathcal{B})-\eta)=1$. Hence, in the inclusion (2.10) equality must hold. This means that the condition in (Ω3) holds for ξ. Since $\mathbb{C}\setminus\mathbb{R}\subseteq S(\mathcal{B})$, the condition in (Ω3) holds for all $\xi\in\Omega\setminus\mathbb{R}$.

Assume that $\Omega\cap\mathbb{R}\neq\emptyset$. Then Ω certainly intersects both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, whence by (ii) the defect index of $S(\mathcal{B})$ is $(1,1)$. We are going to show that $\Omega\cap\mathbb{R}\subseteq r(S(\mathcal{B}))$. This is achieved by reducing to the Hilbert space case.

Fix $\xi\in\Omega\cap\mathbb{R}$, and choose a nonempty interval $(a,b)\subseteq\Omega$ with $\xi\in(a,b)$. Let $f\in\mathcal{B}$, and assume that f vanishes on (a,b). Then it vanishes identically on the component Ω_0 of Ω which contains (a,b). Using Lemma 2.14, it follows that

$$f\in\bigcap_{\eta\in(\Omega_0\cap\Omega_0^\#)\setminus\mathbb{R}}\operatorname{ran}\big(S(\mathcal{B})-\eta\big)=\bigcap_{\eta\in r(S(\mathcal{B}))\cap r(S(\mathcal{B}))}\operatorname{ran}\big(S(\mathcal{B})-\eta\big)=\{0\}.$$

In other words, we may say that the family $\{\chi_\eta:\eta\in(a,b)\}$ is a point separating subset of $\mathcal{B}'$. From [60, Proposition A.9] we obtain $\gamma\in\mathbb{R}$ and $\eta_1,\dots,\eta_n\in(a,b)$, such that the inner product

$$(f,g)_\mathcal{B}:=[f,g]_\mathcal{B}+\gamma\sum_{l=1}^n f(\eta_l)\overline{g(\eta_l)},\quad f,g\in\mathcal{B},$$

turns $\mathcal{B}$ into a Hilbert space and induces the topology of $\mathcal{B}$.

Since the points $\eta_1,\dots,\eta_n$ are real, it is straightforward to check that $S(\mathcal{B})$ is also symmetric w.r.t. $(\cdot,\cdot)_\mathcal{B}$. All algebraic and topological properties of $S(\mathcal{B})$

are independent of the inner product, and hence remain valid. In particular, $S(\mathcal{B})$ is completely non-selfadjoint and has defect index $(1,1)$ when considered as an operator in the Hilbert space $\langle \mathcal{B}, (\cdot,\cdot)_{\mathcal{B}} \rangle$. Moreover, $r(S(\mathcal{B}))$ is the same whether considered in the almost Pontryagin space $\mathcal{B}$ or in the Hilbert space $\langle \mathcal{B}, (\cdot,\cdot)_{\mathcal{B}} \rangle$.

Choose $u \in \mathcal{B} \setminus \{0\}$. Then $M := \{\eta \in \Omega_0 : u(\eta) = 0\}$ is a discrete subset of Ω_0. We have

$$f - \frac{f(\eta)}{u(\eta)} u \in \ker \chi_\eta = \operatorname{ran}(S(\mathcal{B}) - \eta), \quad f \in \mathcal{B}, \eta \in \Omega_0 \setminus (\mathbb{R} \cup M).$$

Hence, for each $\eta \in \Omega_0 \setminus (\mathbb{R} \cup M)$, the projection of f onto $\operatorname{span}\{u\}$ according to the decomposition $\mathcal{B} = \operatorname{ran}(S(\mathcal{B}) - \eta) \dot{+} \operatorname{span}\{u\}$ is equal to $\frac{f(\eta)}{u(\eta)} u$. Since M is discrete, it holds that $\mathfrak{d}_u(\eta) < \infty$, $\eta \in \Omega_0$, and we see that the function $p_f := \frac{f(\eta)}{u(\eta)}$ has an extension $P_f \in \mathbb{H}(\Omega, \mathbb{C}_\infty)$ with $\mathfrak{d}_{P_f} \geq -\mathfrak{d}_u$. Applying [16, Ch. 2, Theorem 3.3] yields $\xi \in r(S(\mathcal{B}))$.

Now it follows that the condition in (Ω3) holds also for all $\xi \in \Omega \cap \mathbb{R}$. Finally, the condition in (Ω4) is void if $\xi \in \Omega \cap \mathbb{R}$, and for $\xi \in \Omega \setminus \mathbb{R}$ it is just the isometry property of the Caley transform of $S(\mathcal{B})$. □

3. The core arguments

In this section the core arguments needed to establish Theorem 4.3 are proved. We present the matters in the form of eight lemmata. In order to point out "what is needed where", we tried to give minimal assumptions in each of them, rather than proving "just" what is needed somewhere afterwards.

We start with explaining the setting which will mainly be present.

Remark 3.1 (Frequently considered setup). Data $\mathcal{L}, S, \langle \iota, \mathcal{A} \rangle, \Omega, \Phi, M$ has the meaning

(1) $\mathcal{L}$ is an inner product space and S is a symmetric linear relation in $\mathcal{L}$;
(2) $\langle \iota, \mathcal{A} \rangle$ is an almost Pontryagin space completion of $\mathcal{L}$;
(3) Ω is a subset of $\mathbb{C}$ and Φ is a map $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ with the property that for each $\eta \in \Omega$ the map $\Phi(\cdot, \eta) : \mathcal{L} \to \mathbb{C}$ is linear;
(4) M is a subset of Ω with the property that

$$\iota^*(\mathcal{A}') = \mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot, \eta) : \eta \in M\big\}. \tag{3.1}$$

Whenever we are given data $\mathcal{L}, S, \langle \iota, \mathcal{A} \rangle$ according to Remark $3.1_{(1,2)}$, we set

$$T := \operatorname{Clos}_{\mathcal{A}} \big[(\iota \times \iota)(S)\big]. \tag{3.2}$$

Since ι is isometric, T is a closed symmetric relation in $\mathcal{A}$.

Whenever we are given data according to Remark $3.1_{(1-3)}$, we set

$$\tilde{\Phi}_\eta := (\iota^*|_{\mathcal{A}'})^{-1} \Phi(\cdot, \eta) \text{ whenever } \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}').$$

The above described situation arises in particular when we already start with an almost Pontryagin space.

Remark 3.2 (Inclusion of the complete case). Assume we are given data $\mathcal{A}, T, \Omega, \Psi, M$ such that

- $(1')$ $\mathcal{A}$ is an almost Pontryagin space, T is a closed symmetric relation in $\mathcal{A}$;
- $(2')$ Ω is a subset of $\mathbb{C}$ and $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ is a map with the property that for each $\eta \in \Omega$ the map $\Psi(\cdot, \eta) : \mathcal{A} \to \mathbb{C}$ is linear;
- $(3')$ M is a subset of Ω, such that $\Psi(\cdot, \eta) \in \mathcal{A}'$, $\eta \in M$, and $\{\Psi(\cdot, \eta) : \eta \in M\}$ is point-separating on $\mathcal{A}^\circ$.

Set

$$\mathcal{L} := \mathcal{A}, \quad S := T, \quad \iota := \mathrm{id}_{\mathcal{A}}, \quad \Phi := \Psi.$$

Then $\mathcal{L}, S, \langle \iota, \mathcal{A} \rangle, \Omega, \Phi, M$ satisfies Remark $3.1_{(1\text{–}4)}$. Moreover, the relation (3.2) equals the given relation T, and $\tilde{\Phi}_\eta = \Psi(\cdot, \eta)$ whenever $\Psi(\cdot, \eta) \in \mathcal{A}'$.

If we are given $\mathcal{A}, T, \Omega, \Psi$ with $(1'), (2')$, then the above-defined data $\mathcal{L}, S, \langle \iota, \mathcal{A} \rangle, \Omega, \Phi$ satisfies Remark $3.1_{(1\text{–}3)}$.

Now let us give a brief outline of what happens in the lemmata of this section.

(1) `Lemma` 3.3: Some basic relations among $r_\subseteq(S, \Phi)$, $r_\supseteq^{\mathrm{app}}(S, \Phi; M)$, $\mathrm{ran}(T - \eta)$, and $\ker \tilde{\Phi}_\eta$.
(2) `Lemma` 3.4: Data as in Remark 3.1. Conclusion about the defect indices of T, in particular a way to conclude that T has defect indices equal to 1; properties of points in $r_\supseteq^{\mathrm{app}}(S, \Phi; M)$.
(3) `Lemma` 3.5: Data as in Remark 3.1. Conclusion towards complete non-selfadjointness of T; conclusion about the size of $r(T)$.
(4) `Lemma` 3.6: Data as in Remark 3.2. Construction of a reproducing kernel almost Pontryagin space $\mathcal{B}$; comparison of T and $S(\mathcal{B})$; properties of $r_\subseteq(S, \Phi)$.
(5) `Lemma` 3.7 `(cont. of Lemma` 3.6`)`: Data as in Remark 3.2. More properties of $S(\mathcal{B})$; identification with T; properties of $r_\supseteq(S, \Phi)$.
(6) `Lemma` 3.8: Data as in Remark 3.1. Construction of a lifting of Φ to the completion $\mathcal{A}$. The central assumption is that T has defect index $(1, 1)$. This is the point where the connection with Krein's representation by analytic functions is made and where preservation of analyticity is applied.
(7) `Lemma` 3.9 `(cont. of Lemma` 3.8`)`: Data as in Remark 3.1. Conclusion towards complete non-selfadjointness of T; properties of $r_\subseteq(T, \Psi)$, $r_=(T, \Psi)$.
(8) `Lemma` 3.10 `(cont. of Lemmata` 3.8 `and` 3.9`)`: Data as in Remark 3.1. Showing that $\mathcal{B}$ is an Ω-space.

Lemma 3.3. *Let data $\mathcal{L}, S, \langle \iota, \mathcal{A} \rangle, \Omega, \Phi$ be given according to Remark $3.1_{(1\text{–}3)}$. Then it holds that (remember the definitions of Φ-containement sets from Definition 2.7 and Definition 2.8)*

$$\eta \in r_\subseteq(S, \Phi), \ \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}') \quad \Longrightarrow \quad \mathrm{ran}(T - \eta) \subseteq \ker \tilde{\Phi}_\eta. \tag{3.3}$$

If, in addition, a set M is given according to Remark $3.1_{(4)}$*, then it holds that*

$$\eta \in r_{\supseteq}^{\mathrm{app}}(S,\Phi;M),\ \operatorname{ran}(T-\eta)\ \textit{closed} \quad\Longrightarrow\quad \operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)\le 1, \tag{3.4}$$

$$\begin{aligned}\eta \in r_{\supseteq}^{\mathrm{app}}(S,\Phi;M),\ \operatorname{ran}(T-\eta)\ \textit{closed},\ \operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)=1\\ \Longrightarrow\quad \eta\in r_{\subseteq}(S,\Phi),\ \Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\setminus\{0\},\ \operatorname{ran}(T-\eta)=\ker\tilde{\Phi}_\eta.\end{aligned} \tag{3.5}$$

Proof. 1° *The implication* (3.3)*:* We have

$$\operatorname{ran}(T-\eta)\subseteq \operatorname{Clos}_{\mathcal{A}}\big[\iota\big(\operatorname{ran}(S-\eta)\big)\big]\subseteq \operatorname{Clos}_{\mathcal{A}}\big[\iota\big(\ker\Phi(\cdot,\eta)\big)\big]\subseteq\ker\tilde{\Phi}_\eta.$$

Here the last but one inclusion holds since $\eta\in r_{\subseteq}(S,\Phi)$, and the last inclusion because $\ker\tilde{\Phi}_\eta$ is closed and $\tilde{\Phi}_\eta\circ\iota=\Phi(\cdot,\eta)$.

2° *The implication* (3.4)*:* Choose $y_0\in\mathcal{L}$ with $\ker\Phi(\cdot,\eta)+\operatorname{span}\{y_0\}=\mathcal{L}$. Since $\iota(\mathcal{L})$ is dense in $\mathcal{A}$, it follows that

$$\operatorname{Clos}_{\mathcal{A}}\big[\iota\big(\ker\Phi(\cdot,\eta)\big)\big]+\operatorname{span}\{\iota y_0\}=\mathcal{A},$$

and hence that $\operatorname{codim}_{\mathcal{A}}\operatorname{Clos}_{\mathcal{A}}[\iota(\ker\Phi(\cdot,\eta))]\le 1$.

Let $x\in\ker\Phi(\cdot,\eta)$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\supseteq}^{\mathrm{app}}(S,\Phi;M)$. By (3.1) the family of functionals $\mathcal{F}:=\{\tilde{\Phi}_\eta:\eta\in M\}$ is point-separating on $\mathcal{A}^\circ$, cf. [60, Remark A.4, Lemma A.13]. We apply [60, Proposition A.5] with the dense set $\iota(\mathcal{L})$ and the family $\mathcal{F}$. This yields that $\lim_{n\to\infty}\iota x_n=\iota x$ in the norm of $\mathcal{A}$. Since $\iota x_n\in\operatorname{ran}(T-\eta)$ and $\operatorname{ran}(T-\eta)$ is closed, it follows that $\iota x\in\operatorname{ran}(T-\eta)$. We conclude that $\operatorname{Clos}_{\mathcal{A}}[\iota(\ker\Phi(\cdot,\eta))]\subseteq\operatorname{ran}(T-\eta)$, and hence that $\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)\le 1$.

3° *The implication* (3.5)*:* Since $\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)=1$, we can choose $y\in\mathcal{A}$ with

$$\mathcal{A}=\operatorname{ran}(T-\eta)\dot{+}\operatorname{span}\{y\}.$$

Denote by $\varphi:\mathcal{A}\to\mathbb{C}$ the linear functional defined by

$$x-\varphi(x)y\in\operatorname{ran}(T-\eta),\quad x\in\mathcal{A}.$$

Clearly, φ does not vanish identically. Since $\operatorname{ran}(T-\eta)$ is closed, φ is continuous. If $x\in\operatorname{ran}(S-\eta)$, then $\iota x\in\operatorname{ran}(T-\eta)$, and hence $\varphi(\iota x)=0$. This shows that

$$\iota\big[\operatorname{ran}(S-\eta)\big]\subseteq\ker\varphi. \tag{3.6}$$

Let $x\in\ker\Phi(\cdot,\eta)$ and choose $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\supseteq}^{\mathrm{app}}(S,\Phi;M)$. Again, we have $\lim_{n\to\infty}\iota x_n=\iota x$ in $\mathcal{A}$. Moreover, $\iota x_n\in\ker\varphi$ by (3.6). Since $\ker\varphi$ is closed, it follows that $\iota x\in\ker\varphi$. We conclude that

$$\ker\Phi(\cdot,\eta)\subseteq\ker(\iota^*\varphi). \tag{3.7}$$

Since $\iota^*|_{\mathcal{A}'}$ is injective, we have $\iota^*\varphi\ne 0$. This implies that in (3.7) equality holds, and hence that $\Phi(\cdot,\eta)$ is a nonzero scalar multiple of $\iota^*\varphi$. In particular, $\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\setminus\{0\}$. Now (3.6) says that $\eta\in r_{\subseteq}(S,\Phi)$, and the already proved implication (3.3) yields $\operatorname{ran}(T-\eta)\subseteq\ker\tilde{\Phi}_\eta$. Since $\operatorname{codim}_{\mathcal{A}}\operatorname{ran}(T-\eta)=1$ and $\tilde{\Phi}_\eta\ne 0$, equality must hold. □

Lemma 3.4. *Let data $\mathcal{L}, S, \langle \iota, \mathcal{A}\rangle, \Omega, \Phi, M$ be given according to Remark* $3.1_{(1\text{–}4)}$. *Assume that*

$<1>$ *the set $M_0 := M \cap r_{\subseteq}(S,\Phi) \cap \{\eta \in \Omega : \Phi(\cdot,\eta) \neq 0\}$ satisfies*

$$\dim\Big(\big[\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M\}\big]\Big/\mathcal{L}'\Big) + \operatorname{ind}_-\mathcal{L} < |M_0 \cap \mathbb{C}^+|.$$

Then the upper defect index β_+ of T is larger or equal to 1. *We have*

$$r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+ \subseteq r_{\subseteq}(S,\Phi) \cap \big\{\eta \in \mathbb{C} : \Phi(\cdot,\eta) \neq 0\big\} \cap \\ \cap \big\{\eta \in \mathbb{C} : \Phi(\cdot,\eta) \in \iota^*(\mathcal{A}'),\ \operatorname{ran}(T-\eta) = \ker\tilde{\Phi}_\eta\big\}. \quad (3.8)$$

In particular, $\Upsilon \cap (r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+) = \emptyset$ where Υ is the exceptional set for T as in Theorem 2.11*/Corollary* 2.12, *and $r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+ \neq \emptyset$ implies $\beta_+ = 1$.*

The same statement holds with $\mathbb{C}^+, \beta_+$ replaced by $\mathbb{C}^-, \beta_-$.

Proof. Let $\alpha, \beta_\pm, \Upsilon$ be the data obtained by applying Corollary 2.12 with T in $\mathcal{A}$. If $\eta \in M_0$, then $\Phi(\cdot,\eta) \in \iota^*(\mathcal{A}')$, $\Phi(\cdot,\eta) \neq 0$, and $\eta \in r_{\subseteq}(S,\Phi)$. The implication (3.3) yields $\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(T-\eta) \geq 1$. Remembering that

$$\operatorname{ind}_0 \mathcal{A} = \dim\big(\iota^*(\mathcal{A}')\big/\mathcal{L}'\big),$$

our assumption $<1>$ ensures that M_0 contains sufficiently many points to evaluate β_+ by means of (2.7). Doing so yields $\beta_+ \geq 1$.

Let $\eta \in r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+$. Then $\operatorname{ran}(T-\eta)$ is closed and (3.4) implies that $\operatorname{codim}_{\mathcal{A}} \operatorname{ran}(T-\eta) \leq 1$. Since $\beta_+ \geq 1$, this codimension (and hence also β_+ itself) must be equal to 1 and the point η cannot belong to the exceptional set Υ. Applying (3.5) yields $\eta \in r_{\subseteq}(S,\Phi)$, $\Phi(\cdot,\eta) \in \iota^*(\mathcal{A}') \setminus \{0\}$, $\operatorname{ran}(T-\eta) = \ker\tilde{\Phi}_\eta$. □

Lemma 3.5. *Let data $\mathcal{L}, S, \langle \iota, \mathcal{A}\rangle, \Omega, \Phi, M$ be given according to Remark* $3.1_{(1\text{–}4)}$. *If*

$<2>$ *a set $M' \subseteq r_{\subseteq}(S,\Phi) \cap \{\eta \in \Omega : \Phi(\cdot,\eta) \in \iota^*(\mathcal{A}')\}$ is given such that $\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M'\}$ is dense in $\mathcal{L}'$ (w.r.t. $\mathcal{T}(\mathcal{L}')$, cf. Remark* 2.5*),*

then

$$\bigcap_{\eta\in M'} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\eta\in M'} \ker\tilde{\Phi}_\eta \subseteq \mathcal{A}^\circ.$$

If, in addition to $<2>$,

$<3>$ *a set $M'' \subseteq r_{\subseteq}(S,\Phi) \cap \{\eta \in \Omega : \Phi(\cdot,\eta) \in \iota^*(\mathcal{A}')\}$ is given such that $\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M''\} = \iota^*(\mathcal{A}')$,*

then

$$\bigcap_{\eta\in M'\cup M''} \operatorname{ran}(T-\eta) = \bigcap_{\eta\in M'\cup M''} \ker\tilde{\Phi}_\eta = \{0\}, \quad \operatorname{mul} T = \{0\}.$$

If, in addition to $<2>$ and $<3>$, the hypothesis $<1>$ of Lemma 3.4 *is fulfilled, then*

$$r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^+ \subseteq r(T). \quad (3.9)$$

The same holds with $\mathbb{C}^+$ replaced by $\mathbb{C}^-$ in $<1>$ and (3.9).

Proof. Assume that a set M' with the properties stated in $<2>$ is given. We know that $(\iota^*|_{\mathcal{A}'})^{-1}$ maps $\mathcal{L}'$ homeomorphically onto the closed subspace $\mathcal{A}^\wr$ of $\mathcal{A}'$ whose annihilator is equal to $\mathcal{A}^\circ$, cf. [60, Lemma A.17, Proposition A.1]. Using (3.3) it follows that (here $\perp$ denotes the annihilator w.r.t. the usual duality between $\mathcal{A}$ and $\mathcal{A}'$)

$$\bigcap_{\eta\in M'} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\eta\in M'} \ker\tilde{\Phi}_\eta$$
$$\subseteq \big[(\iota^*|_{\mathcal{A}'})^{-1}\big(\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta\in M'\}\big)\big]^\perp = \big[(\iota^*|_{\mathcal{A}'})^{-1}(\mathcal{L}')\big]^\perp = \mathcal{A}^\circ.$$

Assume that, in addition, a set M'' with the properties stated in $<3>$ is given. Then $\{\tilde{\Phi}_\eta : \eta\in M''\}$ is point-separating on $\mathcal{A}^\circ$, cf. [60, Remark A.4, Lemma A.13]. Hence,

$$\bigcap_{\eta\in M'\cup M''} \operatorname{ran}(T-\eta) \subseteq \bigcap_{\eta\in M'\cup M''} \ker\tilde{\Phi}_\eta \subseteq \mathcal{A}^\circ \cap \bigcap_{\eta\in M''} \ker\tilde{\Phi}_\eta = \{0\}.$$

In particular, $\operatorname{mul} T = \{0\}$.

Since $\operatorname{mul} T = \{0\}$, the constant α from Theorem 2.11/Corollary 2.12 equals 0. This implies that $r(T)\setminus\mathbb{R} = \mathbb{C}\setminus(\mathbb{R}\cup\Upsilon)$.

Assume that, in addition, also $<1>$ holds. If $\eta \in r^{\mathrm{app}}_{\supseteq}(S,\Phi;M)\cap\mathbb{C}^+$, then we already saw in Lemma 3.4 that $\eta\notin\Upsilon$. □

Lemma 3.6. *Let data $\mathcal{A}, T, \Omega, \Psi$ be given according to Remark $3.2_{(1',2')}$. Assume that*

$<4>$ $\displaystyle\bigcap_{\eta\in\Omega} \ker\Psi(\cdot,\eta) \subseteq \mathcal{A}^\circ$*;*

$<5>$ *the set Ω is open and for each $x\in\mathcal{A}$ the function $\Psi(x,\cdot)$ is analytic;*

$<6>$ *both sets $\{\eta\in\Omega : \Psi(\cdot,\eta)\in\mathcal{A}'\}$ and $r_\subseteq(T,\Psi)$ have accumulation points in each connected component of Ω;*

Set $\Psi_{\mathcal{A}} : x\mapsto\Psi(x,\cdot)$ and

$$\mathcal{B} := \operatorname{ran}\Psi_{\mathcal{A}} = \big\{\Psi(x,\cdot) : x\in\mathcal{A}\big\}.$$

Then $\mathcal{B}$ can be endowed with a unique almost Pontryagin space structure such that $\Psi_{\mathcal{A}} : \mathcal{A}\to\mathcal{B}$ becomes isometric, continuous and open. In this way $\mathcal{B}$ becomes a reproducing kernel almost Pontryagin space of analytic functions on Ω. We have (here $S(\mathcal{B})$ denotes the multiplication operator in $\mathcal{B}$)

$$\Psi(\cdot,\eta)\in\iota^*(\mathcal{A}'), \eta\in\Omega, \quad (\Psi_{\mathcal{A}}\times\Psi_{\mathcal{A}})(T)\subseteq S(\mathcal{B}), \quad r_\subseteq(T,\Psi)=\Omega. \tag{3.10}$$

Proof. First, existence of an almost Pontrygain space structure on $\mathcal{B}$ such that $\Psi_{\mathcal{A}}$ becomes isometric, continuous and open follows from [60, Proposition A.7]. The hypothesis necessary to apply this result is satisfied by our assumption $<4>$. Uniqueness is clear.

The topology of $\mathcal{B}$ is the final topology w.r.t. the map $\Psi_{\mathcal{A}}$. Applying Lemma 2.16 with each component of Ω separately (which is possible by the first

half of <6>), it follows that it is finer than the topology of locally uniform convergence. In particular, all point evaluation functionals are continuous, i.e., $\mathcal{B}$ is a reproducing kernel almost Pontryagin space. Also Lemma 2.16 already gives $\Psi(\cdot,\eta)\in\mathcal{A}'$, $\eta\in\Omega$.

To show that $(\Psi_{\mathcal{A}}\times\Psi_{\mathcal{A}})(T)\subseteq S(\mathcal{B})$, let $(x;y)\in T$. Then

$$y-\eta x\in\operatorname{ran}(T-\eta)\subseteq\ker\Psi(\cdot,\eta),\quad \eta\in r_{\subseteq}(T,\Psi),$$

and hence

$$\Psi(y,\eta)-\eta\Psi(x,\eta)=0,\quad \eta\in r_{\subseteq}(T,\Psi).$$

By the second half of <6> and analyticity, this identity holds for all $\eta\in\Omega$. Hence,

$$(\Psi_{\mathcal{A}}(x);\Psi_{\mathcal{A}}(y))=(\Psi(x,\cdot);\Psi(y,\cdot))\in S(\mathcal{B}).$$

It remains to show that $r_{\subseteq}(T,\Psi)=\Omega$. However, the just proved inclusion in (3.10) yields

$$\Psi_{\mathcal{A}}\big(\operatorname{ran}(T-\eta)\big)=\operatorname{ran}\big((\Psi_{\mathcal{A}}\times\Psi_{\mathcal{A}})(T)-\eta\big)\subseteq\operatorname{ran}\big(S(\mathcal{B})-\eta\big)\subseteq\ker(\chi_\eta|_{\mathcal{B}}).$$

Hence, for each $x\in\operatorname{ran}(T-\eta)$,

$$\Psi(x,\eta)=\big[\Psi_{\mathcal{A}}(x)\big](\eta)=\chi_\eta\big(\Psi_{\mathcal{A}}(x)\big)=0.\qquad\square$$

Lemma 3.7. *Let data $\mathcal{A},T,\Omega,\Psi$ be given according to Remark 3.2$_{(1',2')}$, and assume that the hypothesis <4>, <5>, <6> are fulfilled. Assume, in addition, that*
<7> $r_{\supseteq}(T,\Psi)\neq\emptyset$.
Then $S(\mathcal{B})$ is symmetric,

$$(\Psi_{\mathcal{A}}\times\Psi_{\mathcal{A}})(T)=S(\mathcal{B}),\quad r_{\supseteq}(T,\Psi)\cup(\mathbb{C}\setminus\mathbb{R})\subseteq r(S(\mathcal{B})),$$

and

$$\Big(\forall\,\eta\in r_{\supseteq}(T,\Psi)\cap\mathbb{C}^+\colon\ \Psi(\cdot,\eta)\neq 0\Big)\ \text{or}\ \Big(\forall\,\eta\in r_{\supseteq}(T,\Psi)\cap\mathbb{C}^+\colon\ \Psi(\cdot,\eta)=0\Big).\quad(3.11)$$

Assume, in addition to <4>–<7>, that $r_{\supseteq}(T,\Psi)\cap\mathbb{C}^+\neq\emptyset$. If the first case in (3.11) takes place, then the upper defect index β_+ of $S(\mathcal{B})$ is equal to 1. In the second case $\beta_+=0$ and $\Psi|_{\mathcal{A}\times(\Omega\cap\mathbb{C}^+)}=0$. The same statement holds true when $\mathbb{C}^+,\beta_+$ are replaced by $\mathbb{C}^-,\beta_-$.

If $r_{\supseteq}(T,\Psi)\cap\mathbb{R}\neq\emptyset$, then either Ψ vanishes identically or the defect index of $S(\mathcal{B})$ is $(1,1)$.

Proof. Using the inclusion in (3.10), the fact that $\Psi(\cdot,\eta)=\chi_\eta\circ\Psi_{\mathcal{A}}$, and that $\Psi_{\mathcal{A}}$ is surjective (onto $\mathcal{B}$), we obtain that for each $\eta\in r_{\supseteq}(T,\Psi)$

$$\begin{aligned}\ker(\chi_\eta|_{\mathcal{B}})&=\Psi_{\mathcal{A}}\big(\ker\Psi(\cdot,\eta)\big)\subseteq\Psi_{\mathcal{A}}\big(\operatorname{ran}(T-\eta)\big)\\&=\operatorname{ran}\big((\Psi_{\mathcal{A}}\times\Psi_{\mathcal{A}})(T)-\eta\big)\subseteq\operatorname{ran}\big(S(\mathcal{B})-\eta\big)\subseteq\ker(\chi_\eta|_{\mathcal{B}}),\end{aligned}$$

and hence

$$\operatorname{ran}\big((\Psi_{\mathcal{A}}\times\Psi_{\mathcal{A}})(T)-\eta\big)=\operatorname{ran}\big(S(\mathcal{B})-\eta\big)=\ker(\chi_\eta|_{\mathcal{B}}),\quad\eta\in r_{\supseteq}(T,\Psi).\quad(3.12)$$

By <7> the set $r_{\supseteq}(T,\Psi)$ is nonempty. Since $\sigma_p(S(\mathcal{B}))=\emptyset$, cf. (2.9), the known inclusion of relations and equality of ranges implies $(\Psi_{\mathcal{A}}\times\Psi_{\mathcal{A}})(T)=S(\mathcal{B})$. Now

isometry of $\Psi_{\mathcal{A}}$ implies that $S(\mathcal{B})$ is symmetric. Moreover, we see from (3.12) that $\operatorname{ran}(S(\mathcal{B}) - \eta)$ is closed for all $\eta \in r_{\geq}(T, \Psi)$, and that (by surjectivity of $\Psi_{\mathcal{A}}$ we have $\chi_\eta|_{\mathcal{B}} = 0$ if and only if $\Psi(\cdot, \eta) = 0$)

$$\operatorname{codim}_{\mathcal{B}} \operatorname{ran}(S(\mathcal{B}) - \eta) = \begin{cases} 1\,, & \Psi(\cdot, \eta) \neq 0 \\ 0\,, & \Psi(\cdot, \eta) = 0 \end{cases}, \qquad \eta \in r_{\geq}(T, \Psi). \tag{3.13}$$

Remembering again that $\sigma_p(S(\mathcal{B})) = \emptyset$, we conclude that $r_{\geq}(T, \Psi) \subseteq r(S(\mathcal{B}))$. Moreover, the exceptional set Υ for $S(\mathcal{B})$ is empty. Hence, $\mathbb{C} \setminus \mathbb{R} \subseteq r(S(\mathcal{B}))$ and the codimension of $\operatorname{ran}(S(\mathcal{B}) - \eta)$ is constant on each component of $r(S(\mathcal{B}))$.

If $r_{\geq}(T, \Psi) \cap \mathbb{C}^+ = \emptyset$, (3.11) is void. Hence, assume that $r_{\geq}(T, \Psi) \cap \mathbb{C}^+ \neq \emptyset$. Then (3.13) shows that $\beta_+ \in \{0, 1\}$ and, depending whether $\beta_+ = 1$ or $\beta_+ = 0$, the first or second alternative in (3.11) takes place. Let us further consider the case $\beta_+ = 0$. Then $\operatorname{ran}(S(\mathcal{B}) - \eta) = \mathcal{B}$, $\eta \in \mathbb{C}^+$. However, $\operatorname{ran}(S(\mathcal{B}) - \eta) \subseteq \ker(\chi_\eta|_{\mathcal{B}})$, $\eta \in \Omega$, and we conclude that each element of $\mathcal{B}$ (i.e., each function of the form $\Psi(x, \cdot)$) vanishes on $\Omega \cap \mathbb{C}^+$.

The case of the lower half-plane is treated in the same way. If $r_{\geq}(T, \Psi) \cap \mathbb{R} \neq \emptyset$, the set $r(S(\mathcal{B}))$ contains a real point, and hence is connected. Thus the defect numbers of $S(\mathcal{B})$ are equal, and they are either both equal to 1 or both equal to 0. In the latter case, it follows that each element of $\mathcal{B}$ vanishes on $\Omega \setminus \mathbb{R}$ and by continuity everywhere. □

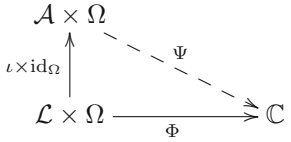
Lemma 3.8. *Let data $\mathcal{L}, S, \langle \iota, \mathcal{A} \rangle, \Omega, \Phi$ be given according to Remark 3.1$_{(1–3)}$. Assume that*

- <8> *the set Ω is open and for each $x \in \mathcal{L}$ the function $\Phi(x, \cdot)$ is analytic;*
- <9> *the set $\{\eta \in \Omega : \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')\} \cap r_{\subseteq}(S, \Phi)$ has accumulation points in each connected component of $\Omega \setminus \mathbb{R}$;*
- <10> *the relation $T := \operatorname{Clos}_{\mathcal{A} \times \mathcal{A}} \big[(\iota \times \iota)(S)\big]$ has defect index $(1, 1)$ and $r(T) \neq \emptyset$.*

Then there exists a unique function $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ with

$$\Psi(\cdot, \eta) \in \mathcal{A}', \ \eta \in \Omega, \qquad \Psi(x, \cdot) \in \mathbb{H}(\Omega), \ x \in \mathcal{A},$$

which lifts Φ:



It holds that

$$\dim \Big(\mathcal{L}' + \operatorname{span}\{\Phi(\cdot, \eta) : \eta \in \Omega\} \Big/ \mathcal{L}' \Big) \leq \operatorname{ind}_0 \mathcal{A}, \quad r_{\subseteq}(S, \Phi) \subseteq r_{\subseteq}(T, \Psi).$$

Proof. Choose a Pontryagin space $\mathcal{P}$ which contains $\mathcal{A}$ as a closed subspace with codimension $\operatorname{ind}_0 \mathcal{A}$, and choose $\zeta_0 \in r(T)$. The relation T, considered in the Pontryagin space $\mathcal{P}$, has finite and equal defect numbers, namely equal to $1 +$

$\operatorname{ind}_0 \mathcal{A}$. Hence, we can choose a self-adjoint extension A_0 of T in $\mathcal{P}$ with $\zeta_0 \in \rho(A_0)$, see, e.g., [26, Lemma 2.1]. Since $\operatorname{ran}(T-\zeta_0) \subseteq \mathcal{A}$, there exists $y_0 \in \mathcal{P}$ with

$$\mathcal{P}[-]\operatorname{ran}(T-\zeta_0) = \mathcal{A}^\circ \dot{+} \operatorname{span}\{y_0\}. \tag{3.14}$$

We use y_0 to generate a family of defect elements of T: set

$$y(\zeta) := \big(I + (\zeta - \overline{\zeta_0})(A_0-\zeta)^{-1}\big)y_0, \quad \zeta \in \rho(A_0).$$

Then $y(\zeta) \perp \operatorname{ran}(T-\overline{\zeta})$, $\zeta \in \rho(A_0)$. Since $y(\overline{\zeta_0}) = y_0 \notin \mathcal{A}^\circ$, the set $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ intersects the connected component of $\rho(A_0)$ which contains $\overline{\zeta_0}$ only in a discrete set.

In the subsequent items 1°–3° we carry out some reductions, in 4° we comment on uniqueness, and in 5° we provide an additional preliminary observation.

1° We show that the choice of y_0 can be made such that $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ intersects each component of $\rho(A_0)$ only in a discrete set. To this end assume that $\rho(A_0)$ has two components and that $y(\zeta) \in \mathcal{A}^\circ$ for all ζ in the component which does not contain $\overline{\zeta_0}$. Fix a point ζ_1 in this component. The map $(I+(\zeta_1-\overline{\zeta_0})(A_0-\zeta_1)^{-1})$ is a bijection of $\mathcal{P}[-]\operatorname{ran}(T-\zeta_0) = \mathcal{A}^\circ \dot{+} \operatorname{span}\{y_0\}$ onto $\mathcal{P}[-]\operatorname{ran}(T-\overline{\zeta_1})$. Since $\rho(A_0) \subseteq r(T)$, the latter space contains $\mathcal{A}^\circ$ as a proper subset. Since $y(\zeta_1) \in \mathcal{A}^\circ$, there must exist $a_0 \in \mathcal{A}^\circ$ with $(I + (\zeta_1 - \overline{\zeta_0})(A_0-\zeta_1)^{-1})a_0 \notin \mathcal{A}^\circ$. Set $\tilde{y}_0 := y_0 + a_0$, and let $\tilde{y}(\zeta)$, $\zeta \in \rho(A_0)$, be the correspondingly defined defect family. Note here that, clearly, (3.14) holds also for $\tilde{y}_0$ in place of y_0. Then $\tilde{y}(\overline{\zeta_0}) = \tilde{y}_0 \notin \mathcal{A}^\circ$ and also

$$\tilde{y}(\zeta_1) = y(\zeta_1) + (I + (\zeta_1 - \overline{\zeta_0})(A_0-\zeta_1)^{-1})a_0 \notin \mathcal{A}^\circ.$$

Thus we may assume for the rest of the proof that $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ intersects each component of $\rho(A_0)$ only in a discrete set.

2° In order to define a lifting $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ we may define Ψ separately for each set $\mathcal{A} \times \Omega'$ where Ω' is a component of Ω. This holds because each component of $\Omega' \setminus \mathbb{R}$ is a component of $\Omega \setminus \mathbb{R}$, and hence our assumption $<9>$ remains valid when Ω is replaced by Ω'. Thus we may assume for the rest of the proof that Ω is connected.

3° If Φ vanishes identically, the map $\Psi(x,\eta) := 0$, $x \in \mathcal{A}$, $\eta \in \Omega$, trivially satisfies all requirements. Thus we may assume for the rest of the proof that Φ does not vanish identically.

4° Uniqueness of a lifting Ψ is clear since $\iota(\mathcal{L})$ is dense in $\mathcal{A}$ and $\Psi(\cdot,\eta)$ is required to be continuous.

5° We show that $\{\eta \in \rho(A_0) : y(\eta) \in \mathcal{A}^\circ\}$ has no accumulation point in $\mathbb{C} \setminus \mathbb{R}$. Assume on the contrary that this set would accumulate in $\mathbb{C}^+$ (the case of $\mathbb{C}^-$ is treated in the same way). Nonreal spectral points of A_0 are poles of the resolvent. Hence, we find a polynomial p having $\sigma(A_0) \cap \mathbb{C}^+$ as its zero set, such that $p(\zeta)y(\zeta)$ has an analytic continuation to $\mathbb{C}^+$, say $z(\zeta)$. The set $\{\eta \in \mathbb{C}^+ : z(\eta) \in \mathcal{A}^\circ\}$ has an accumulation point in $\mathbb{C}^+$, and hence $z(\zeta) \in \mathcal{A}^\circ$ for all $\zeta \in \mathbb{C}^+$. Thus $y(\zeta) \in \mathcal{A}^\circ$ for all $\zeta \in \rho(A_0) \cap \mathbb{C}^+$, and we have reached a contradiction.

Now we come to the actual construction of Ψ. The facts that $y(\zeta) \perp \operatorname{ran}(T - \overline{\zeta})$, $\zeta \in \rho(A_0)$, and that T has defect index $1 + \operatorname{ind}_0 \mathcal{A}$ in $\mathcal{P}$, lead to

$$\mathcal{P}[-] \operatorname{ran}(T - \overline{\zeta}) = \mathcal{A}^\circ \dotplus \operatorname{span}\{y(\zeta)\}, \quad \zeta \in \rho(A_0),\ y(\zeta) \notin \mathcal{A}^\circ.$$

We conclude that (note here that $\rho(A_0)$ is symmetric w.r.t. the real line)

$$x \in \operatorname{ran}(T - \zeta) \ \Leftrightarrow\ x \perp y(\overline{\zeta}), \qquad x \in \mathcal{A}, \zeta \in \rho(A_0),\ y(\overline{\zeta}) \notin \mathcal{A}^\circ. \tag{3.15}$$

Choose $b_0 \in \mathcal{L}$ with $\Phi(b_0, \cdot) \neq 0$, and consider the set

$$\begin{aligned} N := r_\subseteq(S, \Phi) \cap \big\{\eta \in \Omega : \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')\big\} \cap (\mathbb{C} \setminus \mathbb{R}) \\ \cap \rho(A_0) \cap \big\{\eta \in \Omega : \Phi(b_0, \eta) \neq 0\big\} \cap \big\{\eta \in \rho(A_0) : y(\overline{\eta}) \notin \mathcal{A}^\circ\big\}. \end{aligned}$$

From our assumption $<9>$ and 5° it follows that this set has accumulation points in each component of $\Omega \setminus \mathbb{R}$.

If $\eta \in r_\subseteq(S, \Phi) \cap \{\eta \in \Omega : \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')\}$, then (3.3) gives $\operatorname{ran}(T - \eta) \subseteq \ker \tilde{\Phi}_\eta$. For $\eta \in N$ we have $\tilde{\Phi}_\eta(\iota b_0) = \Phi(b_0, \eta) \neq 0$ and hence $\iota b_0 \notin \operatorname{ran}(T - \eta)$. The equivalence (3.15) yields

$$[\iota b_0, y(\overline{\eta})]_\mathcal{P} \neq 0, \quad \eta \in N,$$

and

$$x - \frac{[x, y(\overline{\eta})]_\mathcal{P}}{[\iota b_0, y(\overline{\eta})]_\mathcal{P}} \iota b_0 \in \operatorname{ran}(T - \eta), \quad x \in \mathcal{A},\ \eta \in N.$$

We conclude that

$$\tilde{\Phi}_\eta(x) = \frac{[x, y(\overline{\eta})]_\mathcal{P}}{[\iota b_0, y(\overline{\eta})]_\mathcal{P}} \cdot \Phi(b_0, \eta), \quad x \in \mathcal{A},\ \eta \in N. \tag{3.16}$$

For $a \in \mathcal{L}$ and $x \in \mathcal{A}$ consider the quotients

$$p_a := \Big(\frac{\Phi(a, \cdot)}{\Phi(b_0, \cdot)}\Big)^\# \quad \text{and} \quad q_x := \frac{[y(\cdot), x]_\mathcal{P}}{[y(\cdot), \iota b_0]_\mathcal{P}}.$$

Then $p_a \in \mathbb{H}(\Omega^\#, \mathbb{C}_\infty)$ and the divisor of p_a can be estimated from below by

$$\mathfrak{d}_{p_a} = \mathfrak{d}_{\Phi(a, \cdot)^\#} - \mathfrak{d}_{\Phi(b_0, \cdot)^\#} \geq -\mathfrak{d}_{\Phi(b_0, \cdot)^\#}.$$

Since the nonreal spectral points of A_0 are poles of the resolvent, we have $q_x \in \mathbb{H}(\mathbb{C} \setminus \mathbb{R}, \mathbb{C}_\infty)$. Denoting

$$\mathfrak{e}_{A_0}(\eta) := \begin{cases} -\text{pole order of } (A_0 - \eta)^{-1} \text{ at } \eta\,, & \eta \in \sigma(A_0) \setminus \mathbb{R}, \\ 0 \qquad\qquad\qquad\qquad\quad\ , & \eta \in \rho(A_0) \setminus \mathbb{R}, \end{cases}$$

we can estimate the divisor of q_x from below by

$$\mathfrak{d}_{q_x} = \mathfrak{d}_{[y(\cdot), x]_\mathcal{P}} - \mathfrak{d}_{[y(\cdot), \iota b_0]_\mathcal{P}} \geq \mathfrak{e}_{A_0} - \mathfrak{d}_{[y(\cdot), \iota b_0]_\mathcal{P}}.$$

Let $a \in \mathcal{L}$. By (3.16) the functions p_a and $q_{\iota a}$ coincide on $N^\#$. Since $N^\#$ accumulates in each component of $\Omega^\# \cap (\mathbb{C} \setminus \mathbb{R})$, they coincide on all of $\Omega^\# \cap (\mathbb{C} \setminus \mathbb{R})$, i.e., are analytic extensions of each other. This shows that the quotient $q_{\iota a}$ has

an extension $Q_{\iota a} \in \mathbb{H}(\Omega^{\#} \cup (\mathbb{C} \setminus \mathbb{R}), \mathbb{C}_\infty)$. The divisor of this extension can be estimated from below by[3]

$$\mathfrak{d}_{Q_{\iota a}} \geq \mathbb{1}_{(\mathbb{C}\setminus\mathbb{R})\setminus\Omega^{\#}} \cdot \big(\mathfrak{e}_{A_0} - \mathfrak{d}_{[y(\cdot),\iota b_0]_{\mathcal{P}}}\big) - \mathbb{1}_{\Omega^{\#}} \cdot \mathfrak{d}_{\Phi(b_0,\cdot)^{\#}}. \tag{3.17}$$

Note that the right-hand side is independent of $a \in \mathcal{L}$.

Now Theorem 2.15, (iii), applies and yields that for each $x \in \mathcal{A}$ the quotient q_x has an extension $Q_x \in \mathbb{H}(\Omega^{\#} \cup (\mathbb{C} \setminus \mathbb{R}), \mathbb{C}_\infty)$ whose divisor is bounded from below by the right side of (3.17). Set

$$\Psi(x,\cdot) := Q_x^{\#} \cdot \Phi(b_0,\cdot), \quad x \in \mathcal{A}.$$

Then $\Psi(x,\cdot) \in \mathbb{H}(\Omega,\mathbb{C})$, $x \in \mathcal{A}$, and

$$\Psi(\iota a,\eta) = \big[p_a^{\#} \cdot \Phi(b_0,\cdot)\big](\eta) = \Phi(a,\eta), \quad a \in \mathcal{L},\ \eta \in \Omega.$$

Remember here that $\Phi(b_0,\cdot)$ vanishes only on a discrete subset of Ω. By (3.16) we have

$$\Psi(x,\eta) = \big[q_x^{\#} \cdot \Phi(b_0,\cdot)\big](\eta) = \tilde{\Phi}_\eta(x), \quad x \in \mathcal{A},\ \eta \in N,$$

and hence $\Psi(\cdot,\eta) \in \mathcal{A}'$, $\eta \in N$. Since N accumulates in Ω, we may apply Lemma 2.16 to conclude that $\Psi(\cdot,\eta) \in \mathcal{A}'$, $\eta \in \Omega$. This implies that

$$\dim\Big(\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in \Omega\}\Big/\mathcal{L}'\Big) \leq \dim\Big(\iota^*(\mathcal{A}')\Big/\mathcal{L}'\Big) = \operatorname{ind}_0 \mathcal{A}.$$

The inclusion $r_{\subseteq}(S,\Phi) \subseteq r_{\subseteq}(T,\Psi)$ now follows from (3.3). □

Lemma 3.9. *Let data $\mathcal{L}, S, \langle \iota, \mathcal{A}\rangle, \Omega, \Phi$ be given according to Remark 3.1$_{(1\text{–}3)}$, and assume that the hypothesis <8>,<9>,<10>, are fulfilled. Let Ψ be the lifting of Φ constructed in Lemma 3.8. If*

<11> $\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in \Omega\}$ *is dense in* $\mathcal{L}'$,

then the following statements hold.

(i) *For each subset $L \subseteq \Omega$ which has accumulation points in each connected component of Ω, we have*

$$\bigcap_{\eta\in L} \ker \Psi(\cdot,\eta) \subseteq \mathcal{A}^\circ. \tag{3.18}$$

(ii) $$r_{\subseteq}(T,\Psi) = \Omega, \quad r_{=}(T,\Psi) \supseteq r(T) \cap \big\{\eta \in \Omega : \Phi(\cdot,\eta) \neq 0\big\}. \tag{3.19}$$

(iii) *For each subset $L' \subseteq \mathbb{C}$ which has accumulation points in each of the half-planes $\mathbb{C}^{\pm}$ intersecting Ω, we have*

$$\bigcap_{\eta\in L'} \operatorname{ran}(T-\eta) \subseteq \mathcal{A}^\circ. \tag{3.20}$$

If, in addition to <8>–<11>,

<12> $\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in \Omega\} = \iota^*(\mathcal{A}')$,

then $\sigma_p(T) = \emptyset$, $\operatorname{mul} T = \{0\}$, and for sets L and L' as above

$$\bigcap_{\eta\in L} \ker \Psi(\cdot,\eta) = \{0\}, \quad \bigcap_{\eta\in L'} \operatorname{ran}(T-\eta) = \{0\}. \tag{3.21}$$

[3]Here $\mathbb{1}_E$ denotes the characteristic function of the set E.

Proof. Assume that $<11>$ holds. Using analyticity and the same argument as in Lemma 3.5, it follows that

$$\begin{aligned}\bigcap_{\eta\in L}\ker\Psi(\cdot,\eta)=\bigcap_{\eta\in\Omega}\ker\Psi(\cdot,\eta)&=\big[\operatorname{span}\{\Psi(\cdot,\eta):\eta\in\Omega\}\big]^{\perp}\\&\subseteq\big[(\iota^*|_{\mathcal{A}'})^{-1}\big(\mathcal{L}'\cap\operatorname{span}\{\Phi(\cdot,\eta):\eta\in\Omega\}\big)\big]^{\perp}\\&=\big[(\iota^*|_{\mathcal{A}'})^{-1}(\mathcal{L}')\big]^{\perp}=\mathcal{A}^{\circ}.\end{aligned}$$

By means of Lemma 3.8 and the just proved (3.18), all assumptions of Lemma 3.6 are fulfilled. Applying this lemma yields $r_{\subseteq}(T,\Psi)=\Omega$. If $\eta\in r(T)\cap\Omega$, the codimension of $\operatorname{ran}(T-\eta)$ is equal to 1. As we just showed, this range is contained in $\ker\Psi(\cdot,\eta)$. Hence, provided that $\Psi(\cdot,\eta)\neq 0$, we must have $\operatorname{ran}(T-\eta)=\ker\Psi(\cdot,\eta)$. This is (3.19).

It remains to prove (3.20). Assume that L' has an accumulation point in $\mathbb{C}^+$. Then (remember here that the resolvent of any self-adjoint extension is meromorphic in $\mathbb{C}^+$ and hence we may apply the identity theorem in $\mathbb{H}(\mathbb{C}^+,\mathbb{C}_\infty)$, see, e.g., [52, p.319f])

$$\begin{aligned}\bigcap_{\eta\in L'\cap\mathbb{C}^+}\operatorname{ran}(T-\eta)\subseteq\bigcap_{L'\cap r(T)\cap\mathbb{C}^+}\operatorname{ran}(T-\eta)&=\bigcap_{r(T)\cap\mathbb{C}^+}\operatorname{ran}(T-\eta)\\&\subseteq\bigcap_{\Omega\cap r(T)\cap\mathbb{C}^+}\operatorname{ran}(T-\eta)\subseteq\bigcap_{\Omega\cap r(T)\cap\mathbb{C}^+}\ker\Psi(\cdot,\eta).\end{aligned}\tag{3.22}$$

In the same way, we obtain $\bigcap_{\eta\in L'\cap\mathbb{C}^-}\operatorname{ran}(T-\eta)\subseteq\bigcap_{\Omega\cap r(T)\cap\mathbb{C}^-}\ker\Psi(\cdot,\eta)$ provided that L' has an accumulation point in $\mathbb{C}^-$. Our assumption ensures that we may put together these inclusions and apply (3.18) with the set $L:=\Omega\cap r(T)\cap(\mathbb{C}^+\cup\mathbb{C}^-)$.

Assume that in addition $<12>$ holds. Then $\{\Psi(\cdot,\eta):\eta\in\Omega\}$ is point-separating on $\mathcal{A}^\circ$. By analyticity, the family $\{\Psi(\cdot,\eta):\eta\in L\}$ has the same property. Using the already proved inclusion (3.18), it follows that $\bigcap_{\eta\in L}\ker\Psi(\cdot,\eta)=\{0\}$. The argument which led to (3.22) thus also gives $\bigcap_{\eta\in L'}\operatorname{ran}(T-\eta)=\{0\}$. From this we get $\operatorname{mul}T=\{0\}$. Moreover, if $\eta\in\mathbb{C}$ and $x\in\ker(T-\eta)$, then $x\in\operatorname{ran}(T-\zeta)$, $\zeta\in\mathbb{C}\setminus\{\eta\}$, and it follows that $x=0$. □

Lemma 3.10. *Let data $\mathcal{L},S,\langle\iota,\mathcal{A}\rangle,\Omega,\Phi$ be given according to Remark* $3.1_{(1\text{–}3)}$*, and assume that the hypothesis $<8>$–$<12>$ are fulfilled. Let Ψ be the lifting of Φ constructed in Lemma* 3.8*, and let $\mathcal{B}:=\operatorname{ran}\Psi_{\mathcal{A}}$ (remember that, by* (3.21)*, $\Psi_{\mathcal{A}}$ is injective). Assume, in addition to $<8>$–$<12>$ that*

$<13>$ there is no nonempty open subset O of Ω, such that $\Phi|_{\mathcal{L}\times O}=0$.

Then $\mathcal{B}$ is an Ω-space. In particular, $\Omega\subseteq r(T)$ and $r_{=}(T,\Psi)=\{\eta\in\Omega:\Phi(\cdot,\eta)\neq 0\}$.

Proof. Our aim is to apply Proposition 2.21. We know from the proof of Lemma 3.9 that the assumptions $<4>$–$<6>$ of Lemma 3.6 are fulfilled. Thus, $\mathcal{B}$ is an reproducing kernel almost Pontryagin space of analytic functions on Ω. Our present assumption $<13>$ yields that $\operatorname{supp}\mathfrak{d}_{\mathcal{B}}$ is a discrete subset of Ω.

Since T has no eigenvalues, we have $\mathbb{C} \setminus \mathbb{R} \subseteq r(T)$. Relation (3.19) together with $<13>$ shows that the assumption $<7>$ of Lemma 3.7 is fulfilled. In fact, if $\Omega \cap \mathbb{C}^+ \neq \emptyset$ then $r_=(T, \Psi) \cap \mathbb{C}^+ \neq \emptyset$, and the same for the lower half-plane. Lemma 3.7 implies that $S(\mathcal{B})$ is symmetric and isomorphic to T via $\Psi_{\mathcal{A}}$. If $\Omega \cap \mathbb{C}^+ \neq \emptyset$, the second alternative in (3.11) is ruled out by $<13>$, and it follows that $\beta_+ = 1$. The same holds for the lower half-plane.

Now Proposition 2.21 applies and yields that $\mathcal{B}$ is an Ω-space. Since $S(\mathcal{B})$ and T are isomorphic, the statements for T follow. □

4. Embeddings into spaces of analytic functions

The following version of directing functionals should be thought of as a generalisation of universal directing functionals. Remember the definition of $r_\subseteq(S, \Phi)$ and $r_\supseteq^{\mathrm{app}}(S, \Phi; \Omega)$ from Definitions 2.7 and 2.8.

Definition 4.1. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$. We call Φ an Ω-*directing functional* for S, if:

- **(Ωdf1)** The set Ω is open and nonempty. For each $x \in \mathcal{L}$ the function $\Phi(x, \cdot) : \Omega \to \mathbb{C}$ is analytic.
- **(Ωdf2)** For each $\zeta \in \Omega$ the function $\Phi(\cdot, \zeta) : \mathcal{L} \to \mathbb{C}$ is linear.
- **(Ωdf3′)** The set $r_\subseteq(S, \Phi)$ has accumulation points in each connected component of $\Omega \setminus \mathbb{R}$.
- **(Ωdf3″)** The set $r_\supseteq^{\mathrm{app}}(S, \Phi; \Omega)$ has nonempty intersection with both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$.
- **(Ωdf4)** There is no nonempty open subset O of Ω, such that $\Phi|_{\mathcal{L} \times O} = 0$.

Note that (Ωdf3″) implicitly contains that Ω intersects both half-planes $\mathbb{C}^\pm$.

Remark 4.2. Let us pause and revisit the discussions in §1.1.

- We seek for representations in almost Pontryagin spaces. Hence, we restrict from the start to spaces $\mathcal{L}$ with finite index of negativity.
- We want to have the representations of S to be established explicitly by Φ. Hence, $\Phi(x, \cdot)$ is defined and analytic on all of Ω.
- We aim towards an "if and only if" statement. Hence, we use the approximative version (Ωdf3″).
- The conditions known from (udf3′) and (udf3″) are assumed only on sufficiently large sets. Weakening the first is possible by analyticity, weakening the second by constancy of defect on half-planes.

Our aim is to determine when presence of an Ω-directing functional for S gives rise to a representation of S in an Ω-space (remember Definition 2.19). The next theorem is the main result of this paper. We give conditions which ensure that such a representation exists and is established by Φ. It is not difficult to show that these conditions are also necessary. This is deferred to Proposition 4.6 below, where a slightly refined converse statement is given.

Theorem 4.3. *Let $\mathcal{L}$ be an inner product space with* $\operatorname{ind}_- \mathcal{L} < \infty$*, let S be a symmetric linear relation in $\mathcal{L}$, and let $\Omega \subseteq \mathbb{C}$. Assume that $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ is an Ω-directing functional for S, and that there exists a subset $M \subseteq r_{\subseteq}(S, \Phi)$ which has accumulation points in each connected component of $\Omega \setminus \mathbb{R}$, such that (notation $\mathcal{L}'$ and $\mathcal{T}(\mathcal{L}')$ as in Definition* 2.4 *and Remark* 2.5*)*

(i) $\dim \Big(\big[\mathcal{L}' + \operatorname{span}\{\Phi(\cdot, \eta) : \eta \in M\} \big] \Big/ \mathcal{L}' \Big) < \infty$,

(ii) $\mathcal{L}' \cap \operatorname{span} \big\{ \Phi(\cdot, \zeta) : \ \zeta \in r_{\subseteq}(S, \Phi), \ \Phi(\cdot, \zeta) \in \mathcal{L}' + \operatorname{span}\{\Phi(\cdot, \eta) : \eta \in M\} \big\}$ *is dense in $\mathcal{L}'$ w.r.t. $\mathcal{T}(\mathcal{L}')$.*

Then there exists a unique reproducing kernel almost Pontryagin space $\mathcal{B}$, such that the assignment $\Phi_{\mathcal{L}} : x \mapsto \Phi(x, \cdot)$ maps $\mathcal{L}$ isometrically onto a dense subspace of $\mathcal{B}$. This space $\mathcal{B}$ is an Ω-space and $\operatorname{Clos}_{\mathcal{B}} \big[(\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S) \big] = S(\mathcal{B})$*. Moreover,*

$$[\Phi_{\mathcal{L}}]^*(\mathcal{B}') = \mathcal{L}' + \operatorname{span} \big\{ \Phi(\cdot, \eta) : \eta \in M \big\} = \mathcal{L}' + \operatorname{span} \big\{ \Phi(\cdot, \eta) : \eta \in \Omega \big\}, \tag{4.1}$$

$$\operatorname{ind}_0 \mathcal{B} = \dim \Big(\big[\mathcal{L}' + \operatorname{span}\{\Phi(\cdot, \eta) : \eta \in \Omega\} \big] \Big/ \mathcal{L}' \Big). \tag{4.2}$$

Remark 4.4. In the proof we will not use the full strength of (Ωdf3"). It is only needed that $r_{\supseteq}^{\mathrm{app}}(S, \Phi; M)$ has nonempty intersection with both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$.

Remark 4.5. Notice the balance between the conditions (i) and (ii): Condition (i) gets stronger when M gets larger, whereas (ii) gets stronger when M gets smaller. Hence, it is important that (i) and (ii) hold with a common set M.

Proof of Theorem 4.3. Choose a subset M according to the assumption of the theorem, and choose an almost Pontryagin space completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$ with

$$\iota^*(\mathcal{A}') = \mathcal{L}' + \operatorname{span} \big\{ \Phi(\cdot, \eta) : \eta \in M \big\}. \tag{4.3}$$

This is possible by (i) and [60, Theorem A.15]. Then the data $\mathcal{L}, S, \langle \iota, \mathcal{A} \rangle, \Omega, \Phi, M$ qualifies according to Remark $3.1_{(1\text{–}4)}$.

1° *Applying Lemma* 3.4*:* We have $M \subseteq r_{\subseteq}(S, \Phi)$, and by ($\Omega$df4) and analyticity the set $\{\eta \in \Omega : \Phi(\cdot, \eta) = 0\}$ is a discrete subset of Ω. Therefore, the set

$$M_0 := M \cap r_{\subseteq}(S, \Phi) \cap \big\{ \eta \in \Omega : \Phi(\cdot, \eta) \neq 0 \big\}$$

has accumulation points in each component of $\Omega \setminus \mathbb{R}$. Since Ω intersects both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, the set M_0 in particular contains infinitely many points of each of these half-planes. Thus $<1>$ is satisfied for both half-planes and Lemma 3.4 is applicable. Since $r_{\supseteq}^{\mathrm{app}}(S, \Phi; M) \cap \mathbb{C}^{\pm} \neq \emptyset$, the relation $T := \operatorname{Clos}_{\mathcal{A}}[(\iota \times \iota)(S)]$ has defect index $(1, 1)$. For later reference, note that Lemma 3.4 also gives

$$\operatorname{ran}(T - \eta) = \ker \tilde{\Phi}_\eta \neq \mathcal{A}, \quad \eta \in r_{\supseteq}^{\mathrm{app}}(S, \Phi; M) \setminus \mathbb{R}. \tag{4.4}$$

2° *Applying Lemma* 3.5*:* Set

$$M' := r_{\subseteq}(S, \Phi) \cap \big\{ \eta \in \Omega : \Phi(\cdot, \eta) \in \iota^*(\mathcal{A}') \big\}, \quad M'' := M.$$

Then $<2>$ holds by assumption (ii) of the theorem, and $<3>$ by the choice $\langle \iota, \mathcal{A}\rangle$, cf. (4.3). It follows that (note here that $M'' \subseteq M'$)

$$\bigcap_{\eta\in M'} \ker \tilde{\Phi}_\eta = \{0\}, \quad \emptyset \neq r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \cap \mathbb{C}^{\pm} \subseteq r(T). \tag{4.5}$$

3° *Applying Lemma* 3.8*:* The required hypothesis $<8>$ holds by (Ωdf1), $<9>$ since $M \subseteq M'$, and $<10>$ by what we showed above. Hence, we find a lifting $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ of Φ with the properties stated in Lemma 3.8. We have $\Phi(\cdot,\eta) = \iota^*\Psi(\cdot,\eta)$, $\eta \in \Omega$, and hence

$$\mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in \Omega\big\} \subseteq \iota^*(\mathcal{A}') = \mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in M\big\}. \tag{4.6}$$

It follows that equality holds throughout.

4° *Applying Lemma* 3.6*:* Clearly the data $\mathcal{A}, T, \Omega, \Psi$ qualifies according to Remark $3.2_{(1',2')}$. Let us check the hypothesis $<4>$, $<5>$, $<6>$. First, by (4.5),

$$\bigcap_{\eta\in\Omega} \ker \Psi(\cdot,\eta) \subseteq \bigcap_{\eta\in M'} \ker \underbrace{\Psi(\cdot,\eta)}_{\tilde{\Phi}_\eta} = \{0\}.$$

Second, $\Psi(x,\cdot) \in \mathbb{H}(\Omega)$ by construction. Third, the first set in $<6>$ equals Ω and the second contains M.

It follows that the space

$$\mathcal{B} := \big\{\Psi(x,\cdot) : x \in \mathcal{A}\big\}$$

becomes a reproducing kernel almost Pontryagin space and that $\Psi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is an isomorphism; remember here that $\ker \Psi_{\mathcal{A}} = \bigcap_{\eta\in\Omega} \ker \Psi(\cdot,\eta) = \{0\}$.

We may view $\mathcal{B}$ as a completion of $\mathcal{L}$ via $\Phi_{\mathcal{L}}$. To be precise, $\langle \Psi_{\mathcal{A}} \circ \iota, \mathcal{B}\rangle$ is an almost Pontryagin space completion of $\mathcal{L}$ which is isomorphic to $\langle \iota, \mathcal{A}\rangle$. Now (4.1) follows from (4.6) since $\Psi_{\mathcal{A}} \circ \iota = \Phi_{\mathcal{L}}$, and (4.2) follows from (4.1) and [60, Theorem A.15].

5° *Applying Lemma* 3.7*:* As we have seen in step 1°, cf. (4.4), it holds that

$$\emptyset \neq r_{\supseteq}^{\mathrm{app}}(S,\Phi;M) \setminus \mathbb{R} \subseteq r_{=}(T,\Psi) \cap \{\eta \in \Omega : \Psi(\cdot,\eta) \neq 0\}.$$

In particular, $<7>$ is fulfilled. We conclude that

$$\begin{aligned} S(\mathcal{B}) &= (\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})(T) = (\Psi_{\mathcal{A}} \times \Psi_{\mathcal{A}})\big(\operatorname{Clos}_{\mathcal{A}}[(\iota \times \iota)(S)]\big) \\ &= \operatorname{Clos}_{\mathcal{A}}\big[((\Psi_{\mathcal{A}} \circ \iota) \times (\Psi_{\mathcal{A}} \circ \iota))(S)\big] = \operatorname{Clos}_{\mathcal{A}}\big[(\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S)\big]. \end{aligned}$$

Moreover, being isomorphic to T, the operator $S(\mathcal{B})$ has defect index $(1,1)$.

6° *Applying Lemma* 3.10*:* Hypothesis $<11>$ holds by assumption (ii) of the theorem, $<12>$ holds by (4.6), and $<13>$ by (Ωdf4). It follows that $\mathcal{B}$ is an Ω-space.

7° *Uniqueness:* Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are reproducing kernel almost Pontryagin spaces such that $\Phi_{\mathcal{L}}$ maps $\mathcal{L}$ isometrically onto dense subspaces of each of them. Then the inner products $[\cdot,\cdot]_{\mathcal{B}_1}$ and $[\cdot,\cdot]_{\mathcal{B}_2}$ coincide on the linear space

$$\mathcal{M} := \big\{\Phi_{\mathcal{L}}(x) : x \in \mathcal{L}\big\}.$$

Hence, an inner product is well-defined on $\mathcal{M}$ by setting $[f,g]_{\mathcal{M}} := [f,g]_{\mathcal{B}_i}$, $f,g \in \mathcal{M}$, $i \in \{1,2\}$. The spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ both contain $\langle \mathcal{M}, [\cdot,\cdot]_{\mathcal{M}} \rangle$ isometrically as a dense linear subspace and hence, by the uniqueness part of [60, Theorem 4.1], are equal. □

Next we show the promised converse to Theorem 4.3.

Proposition 4.6. *Let $\mathcal{L}$ be an inner product space. Let $\Omega \subseteq \mathbb{C}$ be open and nonempty, $\mathcal{B}$ a reproducing kernel almost Pontryagin space of analytic functions on Ω, and $\iota : \mathcal{L} \to \mathcal{B}$ a linear and isometric map whose range is dense in $\mathcal{B}$. Define a map $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ by (χ_η denotes point evaluation)*

$$\Phi(x,\eta) := \chi_\eta\big(\iota(x)\big), \quad x \in \mathcal{L},\ \eta \in \Omega.$$

Then:

(i) *Φ satisfies* (Ωdf1) *and* (Ωdf2).

(ii) *Assume that $M \subseteq \Omega$ has accumulation points in each component of Ω. Then $\mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in M\big\} = \iota^*(\mathcal{B}')$, in particular,*

$$\dim\Big(\big[\mathcal{L}' + \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M\}\big] \Big/ \mathcal{L}'\Big) = \operatorname{ind}_0 \mathcal{B} < \infty,$$

and the set $\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M\}$ is dense in $\mathcal{L}'$ w.r.t. $\mathcal{T}(\mathcal{L}')$.

Let, in addition, S be a linear relation in $\mathcal{L}$. Then:

(*iii*) *If $(\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S) \subseteq S(\mathcal{B})$, then $r_{\subseteq}(S,\Phi) = \Omega$, in particular, Φ satisfies* (Ωdf3′).

(*iv*) *If $\operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ is discrete, then Φ satisfies* (Ωdf4).

(*v*) *If $\mathcal{B}$ is an Ω-space and $\operatorname{Clos}_{\mathcal{B}}\big[(\Phi_{\mathcal{L}} \times \Phi_{\mathcal{L}})(S)\big] = S(\mathcal{B})$, then*

$$r^{\mathrm{app}}_{\supseteq}(S,\Phi;\Omega) = \Omega \setminus \operatorname{supp} \mathfrak{d}_{\mathcal{B}}.$$

Provided that Ω intersects both half-planes, Φ is an Ω-directing functional.

Proof. Item (i) is obvious. For the proof of (ii), let $M \subseteq \Omega$ be given and assume that M accumulates in each component of Ω. By analyticity,

$$\bigcap_{\eta \in M} \ker\big(\chi_\eta|_{\mathcal{B}}\big) = \{0\}, \tag{4.7}$$

whence in particular $\{\chi_\eta|_{\mathcal{B}} : \eta \in M\}$ is point separating on $\mathcal{B}^\circ$. By its definition, $\Phi(\cdot,\eta) = \iota^*(\chi_\eta|_{\mathcal{B}})$, and we can invoke [60, Proposition A.3, Lemma A.17] to conclude that

$$\iota^*(\mathcal{B}') = \mathcal{L}' + \operatorname{span}\big\{\Phi(\cdot,\eta) : \eta \in M\big\}.$$

To show the asserted density property we repeat an argument which already appeared in the proof of [60, Proposition 4.3]. The relation (4.7) shows that $\operatorname{span}\{\chi_\eta|_{\mathcal{B}} : \eta \in M\}$ is w^*-dense in $\mathcal{B}'$. By reflexivity and convexity it is thus also dense w.r.t. the norm of $\mathcal{B}'$. Choose $\eta_1,\dots,\eta_m$ such that $\{\chi_{\eta_i}|_{\mathcal{B}} : i = 1,\dots,m\}$ is linearly independent and $\mathcal{B}' = \mathcal{B}^\flat \dot{+} \operatorname{span}\{\chi_{\eta_i}|_{\mathcal{B}} : i = 1,\dots,m\}$, and denote by P the corresponding projection of $\mathcal{B}'$ onto $\mathcal{B}^\flat$. Then

$$\mathcal{B}^\flat \cap \operatorname{span}\big\{\chi_\eta|_{\mathcal{B}} : \eta \in M\big\} = P\Big(\operatorname{span}\big\{\chi_\eta|_{\mathcal{B}} : \eta \in M\big\}\Big),$$

and continuity of P implies that this space is norm-dense in $\mathcal{B}^\flat$. Applying the homeomorphism ι^* yields that $\mathcal{L}' \cap \operatorname{span}\{\Phi(\cdot,\eta) : \eta \in M\}$ is $\mathcal{T}(\mathcal{L}')$-dense in $\mathcal{L}'$.

Item (iii) is again obvious. For (iv) assume that $\operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ is discrete, and let $O \subseteq \Omega$ be open and nonempty. Choose $\eta \in O \setminus \operatorname{supp} \mathfrak{d}_{\mathcal{B}}$, and $f \in \mathcal{B}$ with $f(\eta) \neq 0$. Since $\iota(\mathcal{L})$ is dense in $\mathcal{B}$, we find $x \in \mathcal{L}$ with $\Phi(x,\eta) = (\iota x)(\eta) \neq 0$.

We come to the proof of (v). Let $\eta \in \Omega \setminus \operatorname{supp} \mathfrak{d}_{\mathcal{B}}$ and $x \in \ker \Phi(\cdot,\eta)$. Then $(\iota x)(\eta) = 0$, and hence $\iota x \in \operatorname{ran}(S(\mathcal{B}) - \eta) = \operatorname{Clos}_{\mathcal{B}} \iota[\operatorname{ran}(S-\eta)]$. Choose $x_n \in \operatorname{ran}(S-\eta)$ with $\lim_{n\to\infty} \iota x_n = \iota x$ in the norm of $\mathcal{B}$, then

$$\begin{aligned}
\lim_{n\to\infty} [x_n, x_n]_{\mathcal{L}} &= \lim_{n\to\infty} [\iota x_n, \iota x_n]_{\mathcal{B}} = [\iota x, \iota x]_{\mathcal{B}} = [x,x]_{\mathcal{L}},\\
\lim_{n\to\infty} [x_n, y]_{\mathcal{L}} &= \lim_{n\to\infty} [\iota x_n, \iota y]_{\mathcal{B}} = [\iota x, \iota y]_{\mathcal{B}} = [x,y]_{\mathcal{L}}, \quad y \in \mathcal{L},\\
\lim_{n\to\infty} \Phi(x_n,\eta) &= \lim_{n\to\infty} (\iota x_n)(\eta) = (\iota x)(\eta) = \Phi(x,\eta), \quad \eta \in \Omega,
\end{aligned}$$

and we see that $\eta \in r_{\geq}^{\mathrm{app}}(S,\Phi;\Omega)$.

Assume that $\eta \in r_{\geq}^{\mathrm{app}}(S,\Phi;\Omega) \cap \operatorname{supp} \mathfrak{d}_{\mathcal{B}}$. Let $x \in \mathcal{L}$, then $\Phi(x,\eta) = (\iota x)(\eta) = 0$. Choose a sequence $(x_n)_{n\in\mathbb{N}}$ according to the definition of $r_{\geq}^{\mathrm{app}}(S,\Phi;\Omega)$. Then $\lim_{n\to\infty} \iota x_n = \iota x$ in the norm of $\mathcal{B}$, and we conclude that $\iota x \in \operatorname{ran}(S(\mathcal{B}) - \eta) = \ker \chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}$. This contradicts the fact that $\ker \chi_\eta^{(\mathfrak{d}_{\mathcal{B}}(\eta))}$ is not dense in $\mathcal{B}$. □

In many applications, the space $\mathcal{L}$ carries an additional algebraic structure.

Definition 4.7. Let $\mathcal{L}$ be an inner product space and let $.^{\#}$ be a conjugate-linear involution on $\mathcal{L}$ which is *anti-isometric*, i.e., satisfies

$$[x^{\#}, y^{\#}]_{\mathcal{L}} = [y,x]_{\mathcal{L}}, \quad x,y \in \mathcal{L}.$$

If $\Omega \subseteq \mathbb{C}$ and $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$, then we call Φ *real w.r.t.* $.^{\#}$, if[4]

$$\Omega = \Omega^{\#} \quad \text{and} \quad \Phi(x^{\#},\cdot) = [\Phi(x,\cdot)]^{\#}, \quad x \in \mathcal{L}.$$

The following result is the supplement to Theorem 4.3 taking care of such involutions.

Proposition 4.8. *In the situation and under the hypothesis of Theorem* 4.3*, assume, in addition, that $\mathcal{L}$ carries a conjugate linear and anti-isometric involution $.^{\#}$, and that Φ is real w.r.t. this involution. Then the space $\mathcal{B}$ constructed in Theorem* 4.3 *is invariant under $.^{\#}$, and $.^{\#}|_{\mathcal{B}}$ is anti-isometric.*

Proof. The family $\mathcal{F} := \{\chi_\eta|_{\mathcal{B}} : \eta \in \Omega\}$ is a point-separating subfamily of $\mathcal{B}'$. Let $f \in \mathcal{B}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{L}$ with $\lim_{n\to\infty} \Phi_{\mathcal{L}}(x_n) = f$ in the norm of $\mathcal{B}$. Then

$$\begin{aligned}
&\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m), \Phi_{\mathcal{L}}(y)\big]_{\mathcal{B}} = 0, \quad y \in \mathcal{L},\\
&\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m), \Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m)\big]_{\mathcal{B}} = 0,\\
&\lim_{n,m\to\infty} \chi_\eta\big(\Phi_{\mathcal{L}}(x_n) - \Phi_{\mathcal{L}}(x_m)\big) = 0, \quad \eta \in \Omega.
\end{aligned}$$

[4]Observe the double meaning of the symbol $.^{\#}$. One, it is the given involution on the space $\mathcal{L}$, two, it is the natural involution present on $\mathbb{C}^\Omega$, cf. (2.1). This note will apply throughout.

Using that $.^{\#}$ is anti-isometric, that $\mathcal{L}^{\#} = \mathcal{L}$, and that $\Omega^{\#} = \Omega$, we obtain

$$\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#}), \Phi_{\mathcal{L}}(y)\big]_{\mathcal{B}} = 0, \quad y \in \mathcal{L},$$
$$\lim_{n,m\to\infty} \big[\Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#}), \Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#})\big]_{\mathcal{B}} = 0,$$
$$\lim_{n,m\to\infty} \chi_\eta\big(\Phi_{\mathcal{L}}(x_n^{\#}) - \Phi_{\mathcal{L}}(x_m^{\#})\big) = 0, \quad \eta \in \Omega.$$

By [60, Proposition A.5], this implies that $(\Phi_{\mathcal{L}}(x_n^{\#}))_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm of $\mathcal{B}$. Thus it converges to some element of $\mathcal{B}$, say, $h := \lim_{n\to\infty} \Phi_{\mathcal{L}}(x_n^{\#})$. Continuity of point-evaluations implies that $h = f^{\#}$, and we see that $f^{\#} \in \mathcal{B}$.

Let $f, g \in \mathcal{B}$ be given, and choose approximating sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ as above. Then

$$[f,g]_{\mathcal{B}} = \lim_{n\to\infty} \big[\Phi_{\mathcal{L}}(x_n), \Phi_{\mathcal{L}}(y_n)\big]_{\mathcal{B}} = \lim_{n\to\infty} [x_n, y_n]_{\mathcal{L}} = \lim_{n\to\infty} [y_n^{\#}, x_n^{\#}]_{\mathcal{L}} = [g^{\#}, f^{\#}]_{\mathcal{B}}.$$

□

Matching with the commonly used conditions

The results in the literature providing representations in spaces of entire functions use two kinds of assumptions: one, that the closure T of S has defect index $(1,1)$ and, two, that the directing functional Φ is also defined on $\mathbb{R}$ and characterises the range of $(S-\eta)$ also for real points.

In the following we deduce a result based on similar assumptions, instead of (i) and (ii) of Theorem 4.3. This viewpoint also explains very clearly the different roles of assumptions on real and nonreal points η, cf. Remark 4.14. In order to point out the essentials and directly match the results from the literature, we use characterisation of ranges as in (udf3) instead of the pair of conditions (Ωdf3′) and (Ωdf3″).

Definition 4.9. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$. We call Φ a *strong Ω-directing functional* for S, if $\Omega \cap \mathbb{C}^{\pm} \neq \emptyset$, Φ satisfies (Ωdf1), (Ωdf2), (Ωdf4) and:

(Ωdf3) $\quad \Omega \setminus \mathbb{R} \subseteq r_=(S, \Phi)$.

Theorem 4.10. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, let S be a symmetric linear relation in $\mathcal{L}$, let $\Omega \subseteq \mathbb{C}$, and let $\Phi : \mathcal{L} \times \Omega \to \mathbb{C}$ be a strong Ω-directing functional for S. Assume that there exists an almost Pontryagin space completion $\langle \iota, \mathcal{A}\rangle$ of $\mathcal{L}$, such that*

(i) *$T := \operatorname{Clos}_{\mathcal{A}\times\mathcal{A}} \big[(\iota \times \iota)(S)\big]$ has defect index $(1,1)$;*
(ii) *$\bigcap_{\eta\in\mathbb{C}\setminus\mathbb{R}} \operatorname{ran}(T-\eta) \subseteq \mathcal{A}^{\circ}$.*

Then the function $\Phi_{\mathcal{L}} : x \mapsto \Phi(x, \cdot)$ establishes a representation of S in an Ω-space.

Clearly, the first condition is an assumption on nonreal points. Interestingly, the second condition corresponds to the assumption on real points in the classical case, see Remark 4.14 below.

In the proof we use two lemmata. In the first one we rewrite condition (i) of Theorem 4.10.

Lemma 4.11. *Let Φ be a strong Ω-directing functional for S, let $\langle \iota, \mathcal{A}\rangle$ be an almost Pontryagin space completion of $\mathcal{L}$, and set $T := \mathrm{Clos}_{\mathcal{A}\times\mathcal{A}}\big[(\iota\times\iota)(S)\big]$. Then T has defect index $(1,1)$ if and only if $\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')$, $\eta\in\Omega\setminus\mathbb{R}$.*

Proof. The proof is based on a non-approximative variant of (3.4) and (3.5):

$$\eta\in r_{\supseteq}(S,\Phi),\ \mathrm{ran}(T-\eta)\text{ closed}\quad\Longrightarrow\quad \mathrm{codim}_{\mathcal{A}}\,\mathrm{ran}(T-\eta)\le 1, \tag{4.8}$$

$$\eta\in r_{\supseteq}(S,\Phi),\ \mathrm{ran}(T-\eta)\text{ closed},\ \mathrm{codim}_{\mathcal{A}}\,\mathrm{ran}(T-\eta)=1$$
$$\Longrightarrow\quad \eta\in r_{\subseteq}(S,\Phi),\ \Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\setminus\{0\},\ \mathrm{ran}(T-\eta)=\ker\tilde{\Phi}_\eta. \tag{4.9}$$

Their proof is similar as in Lemma 3.3, even simpler: For (4.8) note that the inclusion $\mathrm{ran}(S-\eta)\supseteq\ker\Phi(\cdot,\eta)$ implies $\mathrm{ran}(T-\eta)\supseteq\mathrm{Clos}_{\mathcal{A}}\,\iota[\ker\Phi(\cdot,\eta)]$ and the right-hand side has codimension at most 1. For (4.9) consider again the projection φ introduced in the proof of (3.4), and observe that

$$\iota\big[\ker\Phi(\cdot,\eta)\big]\subseteq\mathrm{ran}(T-\eta)=\ker\varphi,$$

whence $\Phi(\cdot,\eta)=\iota^*\varphi$.

We come to the actual proof of the present lemma. Assume first that T has defect index $(1,1)$. By (4.8) we have $\mathrm{codim}_{\mathcal{A}}\,\mathrm{ran}(T-\eta)\le 1$, $\eta\in\Omega\setminus\mathbb{R}$. Hence, the exceptional set from Theorem 2.11 does not intersect $\Omega\setminus\mathbb{R}$ and $\mathrm{codim}_{\mathcal{A}}\,\mathrm{ran}(T-\eta)=1$, $\eta\in\Omega\setminus\mathbb{R}$. Now (4.9) implies

$$\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')\setminus\{0\},\ \mathrm{ran}(T-\eta)=\ker\tilde{\Phi}_\eta,\quad \eta\in\Omega\setminus\mathbb{R}. \tag{4.10}$$

Conversely, assume that $\Phi(\cdot,\eta)\in\iota^*(\mathcal{A}')$, $\eta\in\Omega\setminus\mathbb{R}$. By (3.3) we have $\ker(T-\eta)\subseteq\ker\tilde{\Phi}_\eta$, and by ($\Omega$df4) the set $\{\eta\in\Omega\setminus\mathbb{R}:\tilde{\Phi}_\eta=0\}$ is discrete. We conclude that $\mathrm{codim}_{\mathcal{A}}\,\mathrm{ran}(T-\eta)\ge 1$ with possible ecxception of a discrete set. However, by (4.8), for such points in fact $\mathrm{codim}_{\mathcal{A}}\,\mathrm{ran}(T-\eta)=1$. Now we evaluate the defect indices of T with (2.7). □

From this lemma, we have an immediate corollary.

Corollary 4.12. *In the situation of Theorem* 4.10 *there exists $\langle\iota,\mathcal{A}\rangle$ such that the condition Theorem* 4.10, (i)*, holds, if and only if*

$$\dim\Big(\big[\mathcal{L}'+\mathrm{span}\{\Phi(\cdot,\eta):\eta\in\Omega\setminus\mathbb{R}\}\big]\Big/\mathcal{L}'\Big)<\infty. \tag{4.11}$$

If (4.11) *holds, the set of all completions satsfying Theorem* 4.10, (i)*, has a smallest element (w.r.t. the partial order* [58, Definition 6.2(ii)]*), namely, the completion $\langle\iota_0,\mathcal{A}_0\rangle$ with*

$$\iota_0^*(\mathcal{A}_0')=\mathcal{L}'+\mathrm{span}\{\Phi(\cdot,\eta):\eta\in\Omega\setminus\mathbb{R}\}. \tag{4.12}$$

In the second lemma, we investigate condition (ii) of Theorem 4.10.

Lemma 4.13. *Assume that* (4.11) *holds. Then the following are equivalent.*

(i) *There exists a completion $\langle\iota,\mathcal{A}\rangle$ which satisfies Theorem* 4.10, (i) *and* (ii).

(ii) *Every completion that satisfies Theorem* 4.10, (i)*, also satisfies* (ii).
(iii) *The smallest completion with Theorem* 4.10, (i)*, satisfies*

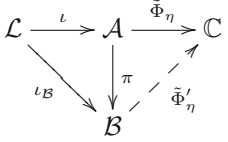
$$\bigcap_{\eta\in\Omega\setminus\mathbb{R}} \operatorname{ran}(T-\eta) = \{0\}.$$

Proof. We need to show "(i)⇒(ii),(iii)". Hence, assume that a completion $\langle \iota, \mathcal{A}\rangle$ is given which satisfies Theorem 4.10, (i) and (ii). Set

$$\mathcal{D} := \bigcap_{\eta\in\Omega\setminus\mathbb{R}} \operatorname{ran}(T-\eta), \quad B := \mathcal{A}/_{\mathcal{D}}, \quad \iota_{\mathcal{B}} := \pi\circ\iota,$$

where $\pi : \mathcal{A} \to \mathcal{B}$ is the canonical projection. Consider $\mathcal{B}$ as an almost Pontryagin space being endowed with the inherited inner product and the factor topology.

The tuple $\langle \iota_{\mathcal{B}}, \mathcal{B}\rangle$ is a completion of $\mathcal{L}$. Due to the choice of $\mathcal{D}$ there exist linear functionals $\tilde{\Phi}'_\eta : \mathcal{B} \to \mathbb{C}$, $\eta \in \Omega \setminus \mathbb{R}$, with



Since $\mathcal{B}$ carries the final topology, we have $\tilde{\Phi}'_\eta \in \mathcal{B}'$. Moreover,

$$\iota_{\mathcal{B}}^*(\tilde{\Phi}'_\eta) = \tilde{\Phi}'_\eta\circ\iota_{\mathcal{B}} = \tilde{\Phi}'_\eta\circ\pi\circ\iota = \tilde{\Phi}_\eta\circ\iota = \Phi(\cdot,\eta),$$

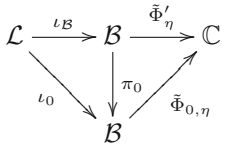
whence $\Phi(\cdot,\eta) \in \iota_{\mathcal{B}}^*(\mathcal{B}')$. Thus $\langle \iota_{\mathcal{B}}, \mathcal{B}\rangle$ satisfies Theorem 4.10, (i). Remembering (4.10), we have

$$\pi^{-1}\Big(\bigcap_{\eta\in\Omega\setminus\mathbb{R}} \ker\tilde{\Phi}'_\eta\Big) = \bigcap_{\eta\in\Omega\setminus\mathbb{R}} \ker\tilde{\Phi}_\eta = \mathcal{D} = \ker\pi,$$

and hence

$$\bigcap_{\eta\in\Omega\setminus\mathbb{R}} \ker\tilde{\Phi}'_\eta = \{0\}.$$

Now let $\langle \iota_0, \mathcal{A}_0\rangle$ be the completion with (4.12). The fact that $\langle \iota_0, \mathcal{A}_0\rangle$ is the smallest w.r.t. the order [58, Definition 6.2], provides a surjective, continuous, and open map $\pi_0 : \mathcal{B} \to \mathcal{A}_0$ with $\iota_0 = \pi_0\circ\iota_{\mathcal{B}}$. The right half of the diagram



commutes on $\operatorname{ran}\iota_{\mathcal{B}}$ since

$$\tilde{\Phi}'_\eta\circ\iota_{\mathcal{B}} = \Phi(\cdot,\eta) = \tilde{\Phi}_{0,\eta}\circ\iota_0 = (\tilde{\Phi}_{0,\eta}\circ\pi_0)\circ\iota_{\mathcal{B}}. \tag{4.13}$$

By continuity it thus commutes on all of $\mathcal{B}$. This yields

$$\ker \pi_0 \subseteq \bigcap_{\eta\in\Omega\setminus\mathbb{R}} \ker \tilde{\Phi}'_\eta = \{0\}$$

and we see that $\langle \iota_{\mathcal{B}}, \mathcal{B}\rangle$ and $\langle \iota_0, \mathcal{A}_0\rangle$ are isomorphic. Thus, item (iii) of the present lemma holds.

To show item (ii), let $\langle \iota_{\mathcal{C}}, \mathcal{C}\rangle$ be an arbitrary completion with Theorem 4.10, (i), and let $\pi_{\mathcal{C}} : \mathcal{C} \to \mathcal{A}_0$ be surjective, continuous, and open with $\iota_{\mathcal{C}} = \iota_0 \circ \pi_{\mathcal{C}}$. Then, again using (4.13) and continuity, we obtain $\tilde{\Phi}_{\mathcal{C},\eta} = \tilde{\Phi}_{0,\eta} \circ \pi_{\mathcal{C}}$. It follows that

$$\bigcap_{\eta\in\Omega\setminus\mathbb{R}} \ker \tilde{\Phi}_{\mathcal{C},\eta} = \pi_{\mathcal{C}}^{-1}\Big(\bigcap_{\eta\in\Omega\setminus\mathbb{R}} \ker \tilde{\Phi}_{0,\eta}\Big) = \pi_{\mathcal{C}}^{-1}(\{0\}) \subseteq \mathcal{C}^\circ. \qquad \square$$

It is now easy to deduce Theorem 4.10.

Proof of Theorem 4.10. Clearly, a strong Ω-directing functional is also an Ω-directing functional. We are going to apply Theorem 4.3 with $M := \Omega \setminus \mathbb{R}$. By means of Corollary 4.12, the present assumption (i) implies that Theorem 4.3, (i), is satisfied.

To show that Theorem 4.3, (ii), holds, we use the argument from the proof of Proposition 4.6, (ii). Consider the completion $\langle \iota_0, \mathcal{A}_0\rangle$ with (4.12). Choose $\eta_1, \dots, \eta_m \in \Omega \setminus \mathbb{R}$ such that $\{\Phi(\cdot, \eta_i) : i = 1, \dots, m\}$ is linearly independent and

$$\mathcal{A}'_0 = \mathcal{A}_0^\wr \dot{+} \operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : i = 1, \dots, m\},$$

and denote by P the corresponding projection of $\mathcal{A}'_0$ onto $\mathcal{A}_0^\wr$. Then

$$\mathcal{A}_0^\wr \cap \operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : \eta \in \Omega \setminus \mathbb{R}\} = P\big(\operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : \eta \in \Omega \setminus \mathbb{R}\}\big).$$

The fact that $\bigcap_{\eta\in\Omega\setminus\mathbb{R}} \ker \tilde{\Phi}_{0,\eta} = \{0\}$ means that $\operatorname{span}\{\tilde{\Phi}_{0,\eta_i} : \eta \in \Omega\setminus\mathbb{R}\}$ is dense in $\mathcal{A}'_0$, whence its image under P is dense in $\mathcal{A}_0^\wr$. Passing to $\mathcal{L}'$ with the isomorphism ι_0^* yields that indeed Theorem 4.3, (ii), is satisfied. $\square$

Remark 4.14. Condition (i) in Theorem 4.10 is an assumption on nonreal points η. Contrasting this, (ii) actually corresponds to assumptions on real points (also if it does not look like this): instead of (ii) we could equally well use assumptions on real points in the definition of a directing functional similar as in Definition 1.5.

Proving this fact would require a sufficiently general version of Theorem 1.6, namely, for linear relations in almost Pontryagin spaces. Such a version is not yet available, and establishing it is beyond the scope of the present paper. However, for the positive definite case the necessary machinery would be available.

The above remark leads to a proof of Proposition 1.9.

Proof of Proposition 1.9. Let $\langle \iota, \mathcal{A}\rangle$ be the Hilbert space completion of $\mathcal{L}$. By our present assumption the relation $T := \operatorname{Clos}_{\mathcal{A}\times\mathcal{A}}(\iota \times \iota)(S)$ has defect index $(1, 1)$ and $\operatorname{mul} T = \{0\}$. By Lemma 4.11, therefore, $\Phi(\cdot, \eta) \in \iota^*(\mathcal{A}')$, $\eta \in \Omega \setminus \mathbb{R}$. Since $\mathcal{A}$ is positive definite we have $\mathbb{C}\setminus\mathbb{R} \subseteq r(T)$. The assumptions of Lemma 3.8 are fulfilled,

and we find a lifting $\Psi : \mathcal{A} \times \Omega \to \mathbb{C}$ with the properties stated there. By (4.9) we have $r_=(T, \Psi) \supseteq \Omega \setminus \mathbb{R}$.

Now use Theorem 1.6: Let μ be a positive Borel measure on $\mathbb{R}$ such that

$$[x, y]_{\mathcal{L}} = \int_{\mathbb{R}} \Phi(x, t)\overline{\Phi(y, t)}\, d\mu(t), \quad x, y \in \mathcal{L}.$$

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}$ such that $\lim_{n\to\infty} \iota x_n = x$ in $\mathcal{A}$, then $(\Phi(x_n, \cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu)$. However, by continuity of $\Psi(\cdot, \eta)$,

$$\lim_{n\to\infty} \Phi(x_n, \eta) = \lim_{n\to\infty} \Psi(\iota x_n, \eta) = \Psi(x, \eta), \quad \eta \in \Omega.$$

Thus $\lim_{n\to\infty} \Phi(x_n, \cdot) = \Psi(x, \cdot)$ in $L^2(\mu)$. It follows that

$$[x, y]_{\mathcal{A}} = \int_{\mathbb{R}} \Psi(x, t)\overline{\Psi(y, t)}\, d\mu(t), \quad x, y \in \mathcal{A}. \tag{4.14}$$

Now consider $x \in \bigcap_{\eta\in\Omega\setminus\mathbb{R}} \operatorname{ran}(T - \eta)$. Then $\Psi(x, \eta) = 0$, $\eta \in \Omega \setminus \mathbb{R}$, and hence also for $\eta \in \mathbb{R}$. The relation (4.14) yields $x = 0$.

The proof is finished by referring to Theorem 4.10 and (2.11) (or to the correct version of [16, Theorem II.8.5]).

□

5. Ω-space completions

The purely geometric version of an Ω-space reads as:

Definition 5.1. Let Ω be an open and nonempty subset of $\mathbb{C}$. We call an inner product space $\mathcal{L}$ an Ω-*inner product space*, if it satisfies the following axioms.

- **(Ω-IP1)** The elements of $\mathcal{L}$ are analytic functions on Ω.
- **(Ω-IP2)** $\operatorname{supp} \mathfrak{d}_{\mathcal{L}}$ is a discrete subset of Ω.
- **(Ω-IP3)** There exists $\eta \in \mathbb{C} \setminus \mathbb{R}$ such that $\eta, \overline{\eta} \in \Omega$, $\mathfrak{d}_{\mathcal{L}}(\eta) = \mathfrak{d}_{\mathcal{L}}(\overline{\eta}) = 0$, and
$$\forall\, \xi \in \{\eta, \overline{\eta}\}\ \forall\, f \in \mathcal{L}, f(\xi) = 0 : \quad \frac{f(\zeta)}{\zeta - \xi} \in \mathcal{L}.$$
- **(Ω-IP4)** There exists $\eta \in \mathbb{C} \setminus \mathbb{R}$ such that (Ω-IP3) holds and
$$\forall\, f, g \in \mathcal{L}, f(\eta) = g(\eta) = 0 :$$
$$\Big[\frac{\zeta - \overline{\eta}}{\zeta - \eta} f(\zeta), \frac{\zeta - \overline{\eta}}{\zeta - \eta} g(\zeta)\Big]_{\mathcal{L}} = \big[f(\zeta), g(\zeta)\big]_{\mathcal{L}}.$$

For an Ω-inner product space $\mathcal{L}$ the following question appears naturally:

Provided $\mathcal{L}$ has a reproducing kernel space completion, is this completion an Ω-space ?

It is a consequence of Theorem 4.3 that the answer is affirmative.

Proposition 5.2. *Let $\Omega \subseteq \mathbb{C}$ be open and nonempty, and let $\mathcal{L}$ be an Ω-inner product space with* $\operatorname{ind}_- \mathcal{L} < \infty$. *Then $\mathcal{L}$ has a reproducing kernel space completion if and only if there exists a subset M of Ω which has accumulation points in each component of $\Omega \setminus \mathbb{R}$ and is such that* (i), (ii) *of Theorem* 4.3 *hold.*

If $\mathcal{L}$ has a reproducing kernel space completion, then this completion is an Ω-space.

Proof. By (Ω-IP3) and (Ω-IP4) the multiplication operator $S(\mathcal{L})$ is the inverse Caley-transform of an isometry, hence symmetric. We show that the map

$$\Phi(f, \eta) := f(\eta), \quad f \in \mathcal{L}, \eta \in \Omega, \tag{5.1}$$

is an Ω-directing functional for the multiplication operator $S(\mathcal{L})$. The properties (Ωdf1), (Ωdf2) are obvious, and (Ωdf3$'$) holds since $r_{\subseteq}(S(\mathcal{L}), \Phi) = \Omega$. From ($\Omega$-IP3) we have $\eta, \overline{\eta} \in r_{\supseteq}(S(\mathcal{L}), \Phi)$, in particular, ($\Omega$df3$''$) holds. Finally, ($\Omega$df4) holds by ($\Omega$-IP2).

The map $\Phi_{\mathcal{L}}$ acts as the identity: $\Phi_{\mathcal{L}}(f) = \Phi(f, \cdot) = f$. The assertion now follows at once from Theorem 4.3 and Proposition 4.6. □

With each space which possesses a reproducing kernel space completion a structural constant is associated.

Definition 5.3. Let Ω be a nonempty set, $\mathcal{L} \subseteq \mathbb{C}^{\Omega}$ an inner product space, and assume that $\mathcal{L}$ has a reproducing kernel space completion (say $\mathcal{A}$). Then we write

$$\Delta(\mathcal{L}) := \operatorname{ind}_0 \mathcal{A}.$$

In general, it is a difficult task to compute $\Delta(\mathcal{L})$, and no effective algorithm is known. Recall [60, Proposition 4.7]: *The number $\Delta(\mathcal{L})$ is the minimum of all numbers $n \in \mathbb{N}$ such that there exists an n-element subset $L \subseteq \Omega$ with*

$$\forall\, (f_n)_{n\in\mathbb{N}}, f_n \in \mathcal{L}: \quad \left.\begin{array}{l} \lim_{n\to\infty}[f_n, f_n]_{\mathcal{L}} = 0 \\ \lim_{n\to\infty}[f_n, g]_{\mathcal{L}} = 0, g \in \mathcal{L} \\ \lim_{n\to\infty} f_n(\eta) = 0, \eta \in L \end{array}\right\} \implies \lim_{n\to\infty} f_n(\eta) = 0, \eta \in \Omega. \tag{5.2}$$

In general, it may happen that (5.2) holds for some subsets with $\Delta(\mathcal{L})$ elements, but does not hold for some others with the same cardinality. Ω-inner product spaces are in this respect better behaved.

Proposition 5.4. *Let $\Omega \subseteq \mathbb{C}$ be open and assume that $\Omega \cap \mathbb{R} \neq \emptyset$. Let $\mathcal{L}$ be an Ω-inner product space and assume that $\mathcal{L}$ has a reproducing kernel space completion. Then there exists a discrete subset $\mathcal{Z}$ of Ω which is symmetric w.r.t. the real line and contains at most* $2 \operatorname{ind}_- \mathcal{L}$ *nonreal points, such that* (5.2) *holds for all $L \subseteq \mathbb{C} \setminus \mathcal{Z}$, $|L| \geq \Delta(\mathcal{L})$.*

Remark 5.5. This result has some computational significance. Let $n \in \mathbb{N}$ be given. Then we can decide whether or not $\Delta(\mathcal{L}) \leq n$ by testing (5.2) for $(\operatorname{ind}_- \mathcal{L} + 1)$

many pairwise disjoint n-element subsets of $\mathbb{C}^+$. Of course, $\Delta(\mathcal{L})$ remains far from effectively computable, simply because (5.2) itself involves the universal quantifiers over the infinite sets of all sequences $(f_n)_{n\in\mathbb{N}}$ and all points $\eta \in \Omega$.

In the proof of Proposition 5.4 we use the following fact.

Lemma 5.6. *Let $\Omega \subseteq \mathbb{C}$ be open and assume that $\Omega \cap \mathbb{R} \neq \emptyset$. Let $\mathcal{B}$ be an Ω-space and assume that $\mathcal{B}^\circ \neq \{0\}$. Then there exists a function $g \in \mathcal{B}$, such that*

$$\mathcal{B}^\circ = \big\{p(\zeta)g(\zeta) : p \in \mathbb{C}[\zeta],\ \deg p < \operatorname{ind}_0 \mathcal{B}\big\}, \quad |\operatorname{supp} \mathfrak{d}_g \cap \mathbb{C}^\pm| \leq \operatorname{ind}_- \mathcal{B}.$$

If $\Omega = \Omega^\#$, $\mathcal{B} = \mathcal{B}^\#$, and $.^\#$ is anti-isometric, then g can be chosen such that $g = g^\#$.

Proof. Our aim is to apply [28, Proposition 1] with the multiplication operator $S(\mathcal{B})$. This requires to check the regularity assumptions [28, (2.3),(2.4)]. The first is obvious. To show the second choose $f \in \mathcal{B}^\circ \setminus \{0\}$. By Remark 2.20, $\operatorname{supp} \mathfrak{d}_f$ is a discrete subset of Ω. Thus we find $\eta_\pm \in \Omega \cap \mathbb{C}^\pm$ with $f(\eta_\pm) \neq 0$. However, $\operatorname{ran}(S(\mathcal{B}) - \eta_\pm) = \ker \chi_{\eta_\pm}|_{\mathcal{B}}$, and it follows that

$$\operatorname{ran}(S(\mathcal{B}) - \eta_\pm) + \operatorname{span}\{f\} = \mathcal{B}.$$

Now [28, Proposition 1] furnishes us with a basis $\{g_1, \dots, g_{\operatorname{ind}_0 \mathcal{B}}\}$ of $\mathcal{B}^\circ$ having the property that $(g_i; g_{i+1}) \in S(\mathcal{B})$, $i = 1, \dots, \operatorname{ind}_0 \mathcal{B} - 1$. This just means that $g_{i+1}(\zeta) = \zeta g_i(\zeta)$, whence

$$\mathcal{B}^\circ = \big\{p(\zeta)g_1(\zeta) : p \in \mathbb{C}[\zeta],\ \deg p < \operatorname{ind}_0 \mathcal{B}\big\}. \tag{5.3}$$

If η is a zero of g, then $\mathcal{B}^\circ \subseteq \operatorname{ran}(S(\mathcal{B}) - \eta)$, and hence $\operatorname{ran}(S(\mathcal{B})/\mathcal{B}^\circ - \eta) \neq \mathcal{B}/\mathcal{B}^\circ$. Thus, each zero of g belongs to the spectrum of $S(\mathcal{B})/\mathcal{B}^\circ$. However, $\sigma(S(\mathcal{B})/\mathcal{B}^\circ)$ may contain at most $\operatorname{ind}_- \mathcal{B}/\mathcal{B}^\circ = \operatorname{ind}_- \mathcal{B}$ many points in the open upper or lower half-plane, cf. [13, p. 162, Corollary].

Assume now that $\mathcal{B}^\circ$ is invariant under the involution $.^\#$ and that this involution is anti-isometric. Then also $\mathcal{B}^\circ$ is invariant under $.^\#$, and we find a polynomial q with $g_1^\# = q g_1$. However, $\left|\frac{g_1^\#(\xi)}{g_1(\xi)}\right| = 1$, $\xi \in \mathbb{R}$, and it follows that q is identically equal to some unimodular constant. Passing from g_1 to a suitable (unimodular) scalar multiple, say h, we can thus achieve that $h^\# = h$. Clearly, (5.3) remains valid with h instead of g_1. □

Proof of Proposition 5.4. Let $\mathcal{B}$ be the reproducing kernel space completion of $\mathcal{L}$. Then $\mathcal{B}$ is an Ω-space and $\operatorname{ind}_- \mathcal{B} = \operatorname{ind}_- \mathcal{L}$. Observe that (5.2) is getting stronger when L gets smaller. Hence, it is enough to construct $\mathcal{Z}$ such that (5.2) holds for each $\Delta(\mathcal{L})$-element subset of $\Omega \setminus \mathcal{Z}$.

Choose $g \in \mathcal{B}^\circ$ according to Lemma 5.6, and set $\mathcal{Z} := \{\eta \in \Omega : g(\eta) = 0\}$. Let $\eta_1, \dots, \eta_{\Delta(\mathcal{L})}$ be pairwise different points of $\Omega \setminus \mathcal{Z}$. Due to the description of $\mathcal{B}^\circ$ in Lemma 5.6, the family $\{\chi_{\eta_i}|_{\mathcal{B}} : i = 1, \dots, \Delta(\mathcal{L})\}$ is point-separating on $\mathcal{B}^\circ$. By [60, Proposition A.5] the hypothesis in (5.2) implies that $\lim_{n\to\infty} f_n = 0$ in the norm of $\mathcal{B}$. Continuity of point evaluations now yields $\lim_{n\to\infty} f_n(\eta) = 0$, $\eta \in \Omega$. □

Finally, we turn to the case that $\Omega = \mathbb{C}$ which deserves particular attention. As in the classical case of universal directing functionals we are led to embeddings into de Branges spaces.

Recall the definition of a de Branges space, including the purely geometric variant:

Definition 5.7. We call an inner product space $\mathcal{L}$ a *de Branges inner product space*, if it is a $\mathbb{C}$-inner product space, is invariant under the involution $f \mapsto f^{\#}$, and

$$\left[f^{\#}, g^{\#}\right]_{\mathcal{L}} = [g, f]_{\mathcal{L}}, \quad f, g \in \mathcal{L}.$$

If $\mathcal{L}$ is actually a $\mathbb{C}$-space subject to the above, then we speak of a *de Branges space*.

Combining Proposition 5.2 with Proposition 4.8 yields:

Corollary 5.8. *Let $\mathcal{L}$ be a de Branges inner product space. If $\mathcal{L}$ has a reproducing kernel space completion, then this completion is a de Branges space.*

Proof. The $\mathbb{C}$-directing functional (5.1) is obviously real w.r.t. to $.^{\#}$. □

For the positive definite case, i.e., when $\mathcal{L}$ and its completion $\mathcal{B}$ are required to be positive definite, this is well known. In fact, it is a classical fact going back (at least) to work of M. Riesz on the power moment problem, cf. [53]; a rather general approach is given by L. Pitt in [49].

In [60, Proposition 4.8] we gave a sufficient condition for an inner product space of functions to have a reproducing kernel space completion. For de Branges inner product spaces this condition is also necessary, in fact, one can say a bit more.

Proposition 5.9. *Let $\mathcal{L}$ be a de Branges inner product space which has a reproducing kernel space completion. Then there exists a positive Borel measure μ on $\mathbb{R}$ with discrete support, a nonnegative integer n, points $\eta_1, \ldots, \eta_n \in \mathbb{R}$ and $\gamma > 0$, such that*

(i) *each element $f \in \mathcal{L}$ is square integrable w.r.t. μ, and*

$$[f, g]_{\mathcal{L}} = \int_{\mathbb{R}} f(\zeta)\overline{g(\zeta)}\, d\mu - \gamma \sum_{i=1}^{n} f(\eta_i)\overline{g(\eta_i)}, \quad f, g \in \mathcal{L};$$

(ii) *for each $\eta \in \mathbb{C}$ the point-evaluation functional $\chi_\eta|_{\mathcal{L}}$ is continuous w.r.t. the $L^2(\mu)$-norm on $\mathcal{L}$.*

Proof. Let $\mathcal{B}$ be the reproducing kernel space completion of $\mathcal{L}$. By Corollary 5.8, $\mathcal{B}$ is a de Branges space. The family $\{\chi_\eta : \eta \in \mathbb{R}\}$ is point-separating, hence [60, Proposition A.9] provides points $\eta_1, \ldots, \eta_n \in \mathbb{R}$ and $\gamma > 0$, such that the inner product

$$(f, g)_{\mathcal{L}} := [f, g]_{\mathcal{L}} + \gamma \sum_{i=1}^{n} f(\eta_i)\overline{g(\eta_i)}, \quad f, g \in \mathcal{B},$$

turns $\mathcal{B}$ into a Hilbert space and induces the topology of $\mathcal{B}$. Moreover, $\langle \mathcal{B}, (\cdot,\cdot)_{\mathcal{B}} \rangle$ is a de Branges–Hilbert space (which is seen by elementary computation or reference to [27, Lemma 3.2]).

Due to [11, Theorem 22], there exists a positive Borel measure μ on $\mathbb{R}$ with discrete support, such that

$$(f,g)_{\mathcal{B}} = \int_{\mathbb{R}} f(\lambda)\overline{g(\lambda)}\, d\mu, \quad f,g \in \mathcal{B}.$$

By the definition of $(\cdot,\cdot)_{\mathcal{B}}$ the formula required in (i) holds. Since $\mathcal{B}$ is a de Branges space if endowed with the $L^2(\mu)$-norm, point evaluations are continuous w.r.t. this norm. □

6. Appendix. Preservation of analyticity

In this appendix we provide explicit proofs for Theorem 2.15 and Lemma 2.16.

Proof of Theorem 2.15(i). This is a general analyticity argument which works since the set $\Lambda := \{\zeta \in \rho(A) : [a(\zeta), y]_{\mathcal{K}} \neq 0\}$ is open and dense in $\mathbb{C}$. We give the general argument.

Let $\Lambda \subseteq \mathbb{C}$ be open and dense and let $F \in \mathbb{H}(\Lambda, \mathbb{C}_\infty)$. Consider the set

$$\mathcal{O} := \big\{O \subseteq \mathbb{C} :\ O \text{ open},\ O \supseteq \Lambda,\ \exists\, F_O \in \mathbb{H}(O, \mathbb{C}_\infty) : F_O|_\Lambda = F\big\},$$

and set $\Lambda^* := \bigcup\{O : O \in \mathcal{O}\}$. If $O_1, O_2 \in \mathcal{O}$ with corresponding extensions F_{O_1} and F_{O_2}, then Λ is dense in $O_1 \cap O_2$ and by continuity $F_{O_1}|_{O_1\cap O_2} = F_{O_2}|_{O_1\cap O_2}$. Thus a function $F^* : \Lambda^* \to \mathbb{C}_\infty$ exists with $F^*|_O = F_O$, $O \in \mathcal{O}$. Since all $O \in \mathcal{O}$ are open and $F^*|_O \in \mathbb{H}(O, \mathbb{C}_\infty)$, it follows that $F^* \in \mathbb{H}(\Lambda^*, \mathbb{C}_\infty)$. Thus the set Λ^* belongs to $\mathcal{O}$. Clearly, it is the largest element of $\mathcal{O}$. □

Item (ii) is the crucial part of Theorem 2.15. It relies on H.Langer's spectral theorem for definitisable self-adjoint operators and estimates of subharmonic functions.

Proof of Theorem 2.15(ii).

Step 1: The essence is to estimate expressions $[(A-\zeta)^{-1}a, b]_{\mathcal{K}}$ where $\zeta \in \rho(A)$ and $a, b \in \mathcal{K}$. To this end we employ the functional calculus for A as developed in [37, §I.7] (see [36], and [14] for an extention to linear relations). For practical reasons, we use the formulation given in [24]. Let p be a real definitising polynomial for A, $\zeta_0 \in \rho(A) \cap \mathbb{C}^+$ and set

$$q(\zeta) := p(\zeta)(\zeta - \zeta_0)^{-\deg p}(\zeta - \overline{\zeta_0})^{-\deg p}.$$

Now let $\mathcal{V}$, and $T : \mathcal{V} \to \mathcal{K}$, and E, be the Hilbert space, the bounded operator, and the $\mathcal{V}$-spectral measure constructed in [24, Theorem 7.19]. The identity

$$\frac{1}{z-\zeta} = \frac{-1}{q(\zeta)}\frac{q(z)-q(\zeta)}{z-\zeta} + q(z)\cdot\frac{1}{q(\zeta)(z-\zeta)}$$

together with the identity in [36, top of p. 26] yields the representation of the resolvent of A as

$$(A-\zeta)^{-1} = \frac{1}{p(\zeta)} \sum_{j=0}^{2\deg p-1} D_j \zeta^j + \frac{1}{q(\zeta)} \int_{\mathbb{R}} \frac{1}{\xi-\zeta} dE_{T^+a,T^+b}(\xi), \quad \zeta \in \rho(A), \quad (6.1)$$

where D_j are appropriate bounded operators, T^+ denotes the Krein space adjoint of T, and E_{T^+a,T^+b} is the measure

$$E_{T^+a,T^+b}(\Delta) := \big(E(\Delta)T^+a, T^+b\big)_{\mathcal{V}}, \quad \Delta \subseteq \mathbb{R} \text{ Borelset.}$$

The formula (6.1) allows us to estimate

$$\Big|\log\big|[(A-\zeta)^{-1}a,b]_{\mathcal{K}}\big|\Big| \le \big|\log|p(\zeta)|\big| + \left|\log\left|\sum_{j=0}^{2\deg p-1} [D_j a,b]_{\mathcal{K}}\zeta^j + (\zeta-\zeta_0)^{\deg p}(\zeta-\overline{\zeta_0})^{\deg p}\int_{\mathbb{R}} \frac{1}{\xi-\zeta} dE_{T^+a,T^+b}(\xi)\right|\right|.$$

The integral on the right-hand side is a function of bounded type in both half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, see, e.g., [16, Lemma I.4.4], and polynomials also have this property. Thus [16, Lemma I.4.3] yields (λ_2 denotes the 2-dimensional Lebesgue measure)

$$\int_{\mathbb{C}} \frac{\big|\log|[(A-\zeta)^{-1}a,b]_{\mathcal{K}}|\big|}{(1+|\zeta|)^4}\, d\lambda_2(\zeta) < \infty. \quad (6.2)$$

Note here that $\mathbb{C}\setminus\rho(A)$ is a Lebesgue zero set.

A more explicit estimate can be given for $\log^+|[(A-\zeta)^{-1}a,b]_{\mathcal{K}}|$. We have

$$\begin{aligned}\log^+\big|[(A-\zeta)^{-1}a,b]_{\mathcal{K}}\big| \le{}& \big|\log|p(\zeta)|\big| + \sum_{j=0}^{2\deg p-1}\Big(\log^+\big|[D_j a,b]_{\mathcal{K}}\big| + j\log^+|\zeta|\Big)\\ &+ \log^+\big|(\zeta-\zeta_0)^{\deg p}(\zeta-\overline{\zeta_0})^{\deg p}\big| + \log^+\left|\int_{\mathbb{R}}\frac{1}{\xi-\zeta}dE_{T^+a,T^+b}(\xi)\right|\\ &+ (2\deg p+1)\log 2.\end{aligned}$$

We have $|[D_j a,b]_{\mathcal{K}}| \le \|D_j\|\cdot\|a\|_{\mathcal{K}}\cdot\|b\|_{\mathcal{K}}$, where $\|\cdot\|$ is the operator norm corresponding to $\|\cdot\|_{\mathcal{K}}$. The total variation of E_{T^+a,T^+b} does not exceed $\|T^+\|^2\cdot\|a\|_{\mathcal{K}}\cdot\|b\|_{\mathcal{K}}$. Using [16, Lemma I.4.5] to estimate the area integral of the term involving the integral, provides us with constants $c_1, c_2 > 0$ such that

$$\int_{\mathbb{C}} \frac{\log^+|[(A-\zeta)^{-1}a,b]_{\mathcal{K}}|}{(1+|\zeta|)^4}\, d\lambda_2(\zeta) \le c_1 \log^+\big(\|a\|_{\mathcal{K}}\cdot\|b\|_{\mathcal{K}}\big) + c_2. \quad (6.3)$$

Step 2: We show that the family $\mathcal{F}$ is locally bounded in Ω. Let $\eta_0 \in \Omega$, and choose a closed disk $U_r(\eta_0) = \{\zeta \in \mathbb{C} : |\zeta-\eta_0| \le r\}$ which is entirely contained in Ω. The

function $W\Theta_{x,y}$ is analytic in Ω, and hence $\log^+|W\Theta_{x,y}|$ is subharmonic. This implies that for $\eta \in D_{\frac{r}{2}}(\eta_0)$

$$\begin{aligned}\log^+\big|(W\Theta_{x,y})(\eta)\big| &\le \frac{4}{\pi r^2}\int\limits_{D_{\frac{r}{2}}(\eta)} \log^+\big|(W\Theta_{x,y})(\zeta)\big|\,d\lambda_2(\zeta)\\ &\le \frac{4(1+r+|\eta_0|)^4}{\pi r^2}\int\limits_{D_{\frac{r}{2}}(\eta)} \frac{\log^+|(W\Theta_{x,y})(\zeta)|}{(1+|\zeta|)^4}\,d\lambda_2(\zeta)\\ &\le \frac{4(1+r+|\eta_0|)^4}{\pi r^2}\Bigg[\int\limits_{D_{\frac{r}{2}}(\eta)} \frac{\log^+|W(\zeta)|}{(1+|\zeta|)^4}\,d\lambda_2(\zeta)\\ &\quad+\int\limits_{\mathbb{C}} \frac{\big|\log|[a(\zeta),y]_{\mathcal{K}}|\,\big|}{(1+|\zeta|)^4}\,d\lambda_2(\zeta)+\int\limits_{\mathbb{C}} \frac{\log^+|[a(\zeta),x]_{\mathcal{K}}|}{(1+|\zeta|)^4}\,d\lambda_2(\zeta)\Bigg].\end{aligned}$$

Note here that $\mathbb{C}\setminus\{\zeta\in\rho(A) : [a(\zeta),y]_{\mathcal{K}}\neq 0\}$ is a Lebesgue zero set. Plugging (6.2), (6.3), and further estimating yields constants $c_1', c_2' > 0$ such that

$$\log^+\big|(W\Theta_{x,y})(\eta)\big| \le c_1'\log^+\|x\|_{\mathcal{K}} + c_2', \quad \eta\in D_{\frac{r}{2}}(\eta_0).$$

We see that the functions in $\mathcal{F}$ are uniformly bounded on the disk $D_{\frac{r}{2}}(\eta_0)$. □

Item (iii) is a simple consequence of (ii).

Proof of Theorem 2.15(iii). Choose W as in (ii). Let $x_n\in\mathcal{M}$, $n\in\mathbb{N}$, $x\in\mathcal{K}$, and assume that $\lim_{n\to\infty} x_n = x$. Then

$$\lim_{n\to\infty}(W\Theta_{x_n,y})(\eta) = (W\Theta_{x,y})(\eta), \quad \eta\in\rho(A),\ [a(\eta),y]_{\mathcal{K}}\neq 0. \tag{6.4}$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in the norm of $\mathcal{K}$, and by item (ii) the family $\{W\Theta_{x_n,y} : n\in\mathbb{N}\}\subseteq\mathbb{H}(\Omega)$ is normal. Vitali's theorem yields that $\lim_{n\to\infty} W\Theta_{x_n,y}$ exists in $\mathbb{H}(\Omega)$; let us denote this limit by G. By (6.4) we have

$$G(\eta) = (W\Theta_{x,y})(\eta), \quad \eta\in\rho(A),\ [a(\eta),y]_{\mathcal{K}}\neq 0,$$

and hence $\frac{G}{W}$ is a meromorphic continuation of $\Theta_{x,y}$ to Ω. Clearly, $\mathfrak{d}_{\frac{G}{W}} \ge -\mathfrak{d}_W = \mathfrak{d}$. Thus $x\in\mathcal{M}$. □

Finally, we give a proof of Lemma 2.16.

Proof of Lemma 2.16. Linearity of $\Phi(\cdot,\eta)$ is easy to see: For each $x,y\in\mathcal{X}$ and $\alpha,\beta\in\mathbb{C}$, the analytic functions

$$\Phi(\alpha x+\beta y,\cdot) \quad\text{and}\quad \alpha\Phi(x,\cdot)+\beta\Phi(y,\cdot)$$

coincide on L. Since L has an accumulation point in Ω and Ω is connected, they coincide everywhere.

We come to the proof of continuity of $\Phi_{\mathcal{X}}$ (which is the main assertion). Consider the map

$$\kappa : \left\{ \begin{array}{ccc} \mathcal{X} & \to & \mathcal{X} \times \mathbb{H}(\Omega) \\ x & \mapsto & \big(x, \Phi(x, \cdot)\big). \end{array} \right.$$

As we just saw, this map is linear. The space $\mathbb{H}(\Omega)$, and hence also $\mathcal{X} \times \mathbb{H}(\Omega)$, is a complete metrisable topological vector space. We are going to apply the Closed Graph Theorem, cf. [55, III.2.3 Theorem]. Let $x_n \in \mathcal{X}$, $x \in \mathcal{X}$, with $\lim_{n\to\infty} x_n = x$, and assume that $\lim_{n\to\infty} \Phi(x_n, \cdot) = f$ in $\mathbb{H}(\Omega)$. Then

$$f(\eta) = \lim_{n\to\infty} \Phi(x_n, \eta) = \Phi(x, \eta), \quad \eta \in L.$$

Since L accumulates, this implies that $f = \Phi(x, \cdot)$. Thus the graph of κ is closed, and we infer that κ is continuous. The map $\Phi_{\mathcal{X}}$ equals $\pi_2 \circ \kappa$, where $\pi_2 : \mathcal{X} \times \mathbb{H}(\Omega)$ is the projection onto the second component. Hence, $\Phi_{\mathcal{X}}$ is continuous. Since point-evaluations of derivatives $\chi_\eta^{(l)}$ are continuous on $\mathbb{H}(\Omega)$ and $\frac{\partial^l}{\partial \eta^l} \Phi(\cdot, \eta) = \chi_\eta^{(l)} \circ \Phi_{\mathcal{X}}$, it follows that $\frac{\partial^l}{\partial \eta^l} \Phi(\cdot, \eta) \in \mathcal{X}'$.

Let K be a compact subset of Ω. The family $\{\Phi(\cdot, \eta) : \eta \in K\}$ is a pointwise bounded subfamily of $\mathcal{X}'$. By the Principle of Uniform Boundedness, cf. [55, III.4.2 Theorem], it is equicontinuous. Thus we find $\epsilon > 0$ such that ($d_{\mathcal{X}}$ denotes a metric on $\mathcal{X}$ which establishes that $\mathcal{X}$ is a complete metrisable topological vector space)

$$|\Phi(x, \eta)| = |\Phi(x, \eta) - \Phi(0, \eta)| \le 1, \quad d_{\mathcal{X}}(x, 0) \le \epsilon, \quad \eta \in K.$$

This shows that $\sup_{x\in M} \sup_{\eta\in K} |\Phi(x, \eta)| \le \frac{1}{\epsilon} \sup_{x\in M} d_{\mathcal{X}}(x, 0)$. □

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