

Chapter Five

FURTHER LOGARITHMS

We have learnt that when a number is written in the form $C = a^b$, a is the **base**, while b is the **index**. This is the Index notation of a number. There is also logarithmic notation of a number. For example, a statement such as:

$25 = 5^2$ can be written as;

$\log_5 25 = 2$ and read as; the 'logarithm of 25 to base 5 is 2'. We therefore interpret the logarithm as the power to which 5 must be raised to give 25.

Consider the following cases:

(i) $2^3 = 8$

(ii) $3^2 = 9$

(iii) $\left(\frac{1}{5}\right)^{-2} = 25$

The three statements can be written in logarithmic notation as;

(a) $\log_2 8 = 3$

(b) $\log_3 9 = 2$

(c) $\log_{\frac{1}{5}} 25 = -2$

In general, if $a^b = c$, then $\log_a c = b$.

Note:

$$a^0 = 1$$

$$\therefore \log_a 1 = 0$$

$$\text{Also, } a^1 = a$$

$$\therefore \log_a a = 1$$

That is the logarithm of 1 to any base is zero and the logarithm of any number to the base equal to the number is 1.

Logarithms of numbers can be read directly from the tables of logarithms or by using a calculator, which gives answers to base 10.

From tables, $\log_{10} 5 = 0.6990$, which is written as $\log 5 = 0.6990$. (4 s.f.)

When using a calculator to get $\log_{10} 5$, the procedure is as follows:

Step I Key in 5.

Step II Press the log button.

The answer 0.6990 (to 4 s.f.) will be displayed on the screen.

Use a calculator to get logarithms of the following numbers:

(i) $\log 4$

(ii) $\log 12$

(iii) Log 20

Express the following in logarithmic notation:

(i) $5^3 = 125$

(ii) $8^4 = 4\,096$

(iii) $3^3 = 27$

(iv) $10^{-1} = 0.1$

5.1: Laws of Logarithms

We saw earlier that:

(i) $a^m \times a^n = a^{(m+n)}$

(ii) $a^m \div a^n = a^{(m-n)}$

(iii) $(a^m)^n = a^{mn}$

Considering the above;

(i) $a^m \times a^n = a^{(m+n)}$

If $x = a^m$ and $y = a^n$, then;

$$\log_a x = m \text{ and } \log_a y = n \dots\dots\dots(1)$$

$$\text{Therefore, } a^m \times a^n = xy \dots\dots\dots(2)$$

$$a^{(m+n)} = xy$$

$$\text{Hence, } \log_a xy = m + n \dots\dots\dots(3)$$

But from equation (1) $\log_a x = m$ and $\log_a y = n$.

Substituting this in equation (3), we get;

$$\log_a x = \log_a x + \log_a y \dots\dots\dots(4)$$

(b) $a^m \div a^n = a^{(m-n)}$

If $x = a^m$ and $y = a^n$, then;

$$\log_a x = m \text{ and } \log_a y = n \dots\dots\dots(1)$$

$$\text{Therefore, } a^m \div a^n = x \div y$$

$$\frac{a^m}{a^n} = \frac{x}{y}$$

$$a^{(m-n)} = \frac{x}{y} \dots\dots\dots(2)$$

$$\text{Hence, } \log_a \frac{x}{y} = m - n \dots\dots\dots(3)$$

In equation (1), $\log_a x = m$ and $\log_a y = n$.

Substituting in (3);

$$\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y \dots\dots\dots(4)$$

(c) Suppose $\log_a x^n = y \dots\dots\dots(1)$

In index notation, this can be written as;

$$a^y = x^n.$$

Finding the n^{th} root on both sides;

$$a^{\frac{y}{n}} = x \dots\dots\dots (2)$$

$$\text{Hence, } \log_a x = \frac{y}{n}$$

Multiplying both sides by n ;

$$n \log_a x = y \dots\dots\dots (3)$$

From equation (1) $\log x^n = y$

Substituting for y in equation (iii);

$$n \log_a x = \log_a x^n \text{ or } \log_a x^n = n \log_a x \dots\dots\dots (4)$$

Equation (4) in each of the above cases form the three **laws of logarithms**, stated as;

$$(i) \log_a xy = \log_a x + \log_a y$$

$$(ii) \log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$$

$$(iii) \log_a x^n = n \log_a x$$

Example 1

Express the following in terms of $\log a$, $\log b$ and $\log c$.

$$(a) \log ab$$

$$(b) \log abc$$

$$(c) \log \frac{a}{b}$$

$$(d) \log \frac{a}{bc}$$

Solution

$$(a) \log ab = \log a + \log b$$

$$(b) \log abc = \log a + \log b + \log c$$

$$(c) \log \frac{a}{b} = \log a - \log b$$

$$(d) \log \frac{a}{bc} = \log a - (\log b + \log c) \\ = \log a - \log b - \log c$$

Example 2

(a) Express in terms of $\log a$, $\log b$ and $\log c$;

$$\log \left(\frac{a^2}{b^3} \right)$$

(b) Write $\log 100 - 2 \log 50$ as a logarithm of a single number.

Solution

$$(a) \log \left(\frac{a^2}{b^3} \right) = \log a^2 - \log b^3 \\ = 2 \log a - 3 \log b$$

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$$\begin{aligned}
 \text{(b)} \quad \log 100 - 2 \log 50 &= \log 100 - \log 50^2 \\
 &= \log \left(\frac{100}{50^2} \right) \\
 &= \log \left(\frac{100}{2\,500} \right) \\
 &= \log \frac{1}{25}
 \end{aligned}$$

Example 3

Given that $\log 2 = 0.3010$, and $\log 3 = 0.4771$, simplify:

(a) $\log 6$ (b) $\log 1.5$ (c) $\log 54$

Solution

$$\begin{aligned}
 \text{(a)} \quad \log 6 &= \log (3 \times 2) \\
 &= \log 3 + \log 2 \\
 &= 0.4771 + 0.3010 \\
 &= 0.7781 \\
 \text{(b)} \quad \log 1.5 &= \log (3 \div 2) \\
 &= \log 3 - \log 2 \\
 &= 0.4771 - 0.3010 \\
 &= 0.1761 \\
 \text{(c)} \quad \log 54 &= \log (27 \times 2) \\
 &= \log (3^3 \times 2) \\
 &= 3 \log 3 + \log 2 \\
 &= 3(0.4771) + 0.3010 \\
 &= 1.4313 + 0.3010 \\
 &= 1.7323
 \end{aligned}$$

Exercise 5.1

1. Express the following in logarithmic notation:

$$\begin{array}{lll}
 \text{(a)} \quad 7^2 = 49 & \text{(b)} \quad 4^{\frac{1}{2}} = 32 & \text{(c)} \quad 1\,331 = 121^{\frac{3}{2}} \\
 \text{(d)} \quad 10^{-2} = 0.01 & \text{(e)} \quad 5^0 = 1 & \text{(f)} \quad 8^{-\frac{1}{3}} = \frac{1}{2} \\
 \text{(g)} \quad 125 = \left(\frac{1}{5}\right)^{-3} & \text{(h)} \quad 9^{-\frac{3}{2}} = \frac{1}{27} & \text{(i)} \quad 1 = a^0
 \end{array}$$

2. Express in index notation:

$$\begin{array}{lll}
 \text{(a)} \quad \log_5 625 = 4 & \text{(b)} \quad \log_{10} 1000 = 3 & \text{(c)} \quad \log_3 27 = 3 \\
 \text{(d)} \quad \log 1 = 0 & \text{(e)} \quad \log_{\frac{1}{2}} 4 = -2 & \text{(f)} \quad \log_{25} 5 = \frac{1}{2} \\
 \text{(g)} \quad \log_x y = 2
 \end{array}$$

3. Express in terms of a , b and c , given that $\log x = a$, $\log y = b$ and $\log z = c$:

- (a) $\log \frac{1}{x}$ (b) $\log x^2y$ (c) $\log \left(\frac{x}{y}\right)$
 (d) $\log \left(\frac{x^2}{y}\right)$ (e) $\log \left(\frac{z^3}{10}\right)$ (f) $\log \frac{1}{10x}$
 (g) $\log \frac{100}{\sqrt{z}}$

4. Simplify:

- (a) $\log 3 + \log 4$ (b) $\log 6 - \log 2$
 (c) $\log 30 - \log 3$ (d) $\log 2 + \log 6 - \log 4$
 (e) $\frac{1}{2} \log 4 - \log 6$ (f) $2 - 2 \log 5$
 (g) $\log_2 16 - \log_2 8$ (h) $\log_3 2.7 + \log_3 10$

5. Simplify:

- (a) $\log (x+1) - \log (x^2-1)$ (b) $2 \log_a 5 + \log_a 4 - 2 \log_a 10$
 (c) $\log x^4 + \log x$ (d) $\log_9 3 + \log_9 27$
 (e) $\log 3 + \log 15 - \log 4.5$ (f) $\log_8 72 - \log_8 \frac{9}{8}$

6. Given that $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find:

- (a) $\log 12$ (b) $\log 5$ (c) $\log 108$

5.2: Logarithmic Equations and Expressions

Consider the following equations;

- (i) $\log_3 81 = x$ (ii) $\log_x 8 = 3$

The value of x in each case is established as below:

- (i) $\log_3 81 = x$
 $\therefore 3^x = 81$
 $3^x = 3^4$
 $x = 4$
 (ii) $\log_x 8 = 3$
 $x^3 = 8$
 $x^3 = 2^3$
 $x = 2$

Example 4

Solve for x in $\log x + \log 5 = \log 30$.

Solution

$$\log x + \log 5 = \log 30$$

$$\therefore \log x = \log 30 - \log 5$$

$$\log x = \log \left(\frac{30}{5} \right)$$

$$\log x = \log 6$$

$$\therefore x = 6$$

$$\text{or} \quad \log x + \log 5 = \log 30$$

$$\therefore \log 5x = \log 30$$

$$5x = 30$$

$$x = 6$$

Example 5

Find y without using tables if $2 + \log_2 3 + \log_2 y = \log_2 5 + 1$

Solution

$$2 + \log_2 3 + \log_2 y = \log_2 5 + 1$$

$$-1 + 2 + \log_2 3 + \log_2 y = \log_2 5$$

$$1 + \log_2 3 + \log_2 y = \log_2 5$$

$$\text{But } 1 = \log_2 2$$

Substituting;

$$\log_2 2 + \log_2 3 + \log_2 y = \log_2 5$$

$$\log_2 (2 \times 3) + \log_2 y = \log_2 5$$

$$\log_2 6 + \log_2 y = \log_2 5$$

$$\log_2 y = \log_2 5 - \log_2 6$$

$$\log_2 y = \log_2 \left(\frac{5}{6} \right)$$

$$\therefore y = \frac{5}{6}$$

Example 6

Simplify without using tables;

$$\frac{\log 25 + \log 625}{\log 5}$$

Solution

$$\frac{\log 25 + \log 625}{\log 5} = \frac{\log 5^2 + \log 5^4}{\log 5}$$

$$= \frac{2 \log 5 + 4 \log 5}{\log 5}$$

$$= \frac{6 \log 5}{\log 5}$$

$$= 6$$

Simplify without using tables:

$$(i) \quad \frac{\log 8}{\log 2}$$

$$(ii) \quad \frac{\log x^4}{\log x}$$

$$(iii) \quad \frac{\log \sqrt{6}}{\log 6}$$

5.3: Further Computation using Logarithms

Consider the expression $\log_8 3$

The solution to $\log_8 3$ cannot be read from tables, which give logarithms to base 10 only. However, if we let;

$$\log_8 3 = x, \text{ then } 8^x = 3.$$

To solve for x , we introduce logarithms to base 10 on both sides. That is;

$$\log 8^x = \log 3$$

$$x \log 8 = \log 3$$

$$x = \frac{\log 3}{\log 8}$$

From tables the logarithms of 3 and 8 are 0.4771 and 0.9031 respectively. Substituting this is;

$$x = \frac{0.4771}{0.9031}$$

$$x = 0.5283 \text{ (4 s.f.)}$$

$$\therefore \log_8 3 = 0.5283.$$

Example 7

Solve $\log_6 2$

Solution

Let $\log_6 2 = t$. Then, $6^t = 2$

Introducing logarithms to base 10 on both sides;

$$\log 6^t = \log 2$$

$$t \log 6 = \log 2$$

$$t = \frac{\log 2}{\log 6}$$

$$t = \frac{0.3010}{0.7782}$$

$$t = 0.3868$$

$$\therefore \log_6 2 = 0.3868$$

Consider the equation $2^{2x} + 3(2^x) - 4 = 0$.

Taking logs on both sides cannot assist in getting the value of x , since

$2^{2x} + 3(2^x)$ cannot be combined into a single expression. However, if we let $2^x = y$, then the equation becomes quadratic in y .

Thus, let $2^x = y$ (1)

$$\therefore y^2 + 3y - 4 = 0 \text{(2)}$$

$$(y + 4)(y - 1) = 0$$

$$y = -4 \text{ or } y = 1$$

Substituting for y in equation (1);

$$2^x = -4 \text{ or } 2^x = 1$$

There is no real value of x for which $2^x = -4$.

$$\text{Hence, } 2^x = 1$$

$$x = 0$$

Example 8

Solve for x in $(\log_{10} x)^2 = 3 - \log_{10} x^2$

Solution

$$(\log_{10} x)^2 = 3 - 2 \log_{10} x$$

$$\text{Let } \log_{10} x = t \text{(1)}$$

$$\therefore t^2 = 3 - 2t$$

$$t^2 + 2t - 3 = 0$$

$$t^2 + 3t - t - 3 = 0$$

$$t(t + 3) - 1(t + 3) = 0$$

$$(t - 1)(t + 3) = 0$$

$$t = 1 \text{ or } t = -3$$

Substituting for t in equation (1).

$$\log_{10} x = 1 \text{ or } \log_{10} x = -3$$

$$\therefore 10^1 = x \text{ or } 10^{-3} = x$$

$$x = 10 \text{ or } \frac{1}{1000}$$

Note:

$$\log_b a = \frac{1}{\log_a b}$$

Exercise 5.2

Simplify:

1. (a) $\frac{\log 125}{\log 5}$ (b) $\log_a a^2 \div \log_a \sqrt{a}$ (c) $(\log a^2 + \log b^2) - \log ab$

2. Evaluate:

(a) $\log_3 4$ (b) $\log_7 17$ (c) $\log_5 14$ (d) $\log_6 28$
 (e) $\log_{12} 41$ (f) $\log_2 10$ (g) $\log_7 5$

3. Find the value of x in the equation:

(a) $4^x = 3$

(b) $7^x = 11$

(c) $10^x = 7$

(d) $6^x = 15$

(e) $2^{2x} = 5$

(f) $3^{(x-1)} = 7$

(g) $(5^x) 5^{(x-1)} = 10$

(h) $2^{x+1} = 3^x$

4. Express y in terms of x in the equation:

$$\log_6 x + \log_6 y^3 = 1.$$

5. If $\log_2 x + \log_x 2 = 2$, find x .

6. Solve the simultaneous equations below:

(a) $\log_x y = 2$

$$xy = 8$$

(b) $2 \log y = \log 2 + \log x$

$$2^y = 4^x$$

(c) $\text{Log}(x + y) = 0$

$$2 \log x = \log(y + 1).$$

7. Solve the following:

(a) $\log x = 4 \log 2$

(b) $\log 12 + 3 \log x = \log 96$

(c) $1 + \log_5 x = \log_5 12$

(d) $\log_3 y - 4 = \log_3 5$

(e) $\log_4 P^2 = 2 \log_4 5$

(f) $\log_3 4 + \log_3 x + \log 3 - \log_3 5 = \log_3 2$

(g) $\log_2 y = \log_2 3 + \log_2 7 + 2 \log_2 y$

(h) $2^2 + \log_2 x^2 + 5 \log_2 2 = 9$

8. Solve for x in $\log(3x + 4) - \log(3 - x) = 1$