

QUADRATIC EXPRESSIONS AND EQUATIONS (II)

1.1: Factorisation of Quadratic Expressions

In chapter 15 of book 2, we learnt how to factorise a quadratic expression. Study the following quadratic expressions and factorise them:

(i) $x^2 + 6x + 9$

(ii) $x^2 - 5x + 6$

(iii) $x^2 - 8x + 16$

(iv) $x^2 + x - 12$

(v) $4x^2 - 12x + 9$

(vi) $4x^2 + 4x - 3$

(vii) $9x^2 - 6x + 1$

(viii) $3x^2 - 11x - 4$

What do you notice about expressions (i), (iii), (v) and (vii) as compared to (ii), (iv), (vi) and (viii)?

Expressions in the first group, i.e., (i), (iii), (v) and (vii) factorise into two equal factors. Such expressions are called **perfect squares**. The rest are not perfect squares.

Exercise 1.1

Factorise the following expressions and hence identify the perfect squares:

1. (a) $x^2 - 6x + 8$

(b) $x^2 + 12x + 36$

2. (a) $2x^2 + x - 6$

(b) $4x^2 + 12x + 9$

3. (a) $x^2 - 8x - 9$

(b) $x^2 - 14x + 49$

4. (a) $2x^2 + 3x + 1$

(b) $16x^2 - 8x + 1$

5. (a) $9x^2 + 12x + 4$

(b) $5x^2 + 14x - 3$

1.2: Completing the Square

In section 1.1, you factorised the following expressions and got the factors given:

(i) $x^2 + 6x + 9 = (x + 3)^2$

(ii) $4x^2 - 12x + 9 = (2x - 3)^2$

(iii) $x^2 - 8x + 16 = (x - 4)^2$

(iv) $9x^2 - 6x + 1 = (3x - 1)^2$

These expressions are perfect squares. We shall now see ways of making any quadratic expression a perfect square.

Any quadratic equation can be simplified and presented in the form $ax^2 + bx + c$, where a , b and c are constants and $a \neq 0$. We shall first deal with cases where $a = 1$, as in (i) and (iii) above. Identify similar expressions which are perfect squares from exercise 1.1. You will notice that in all cases, $\left(\frac{b}{2}\right)^2 = c$. We shall use this relation to make expressions perfect squares.

Example 1

What must be added to $x^2 + 10x$ to make it a perfect square?

Solution

Let the number to be added be a constant c . Then;

$x^2 + 10x + c$ is a perfect square.

Therefore, $(10 \div 2)^2 = c$

$$\Rightarrow c = 25$$

Thus, 25 must be added to $x^2 + 10x$ to make it a perfect square.

Alternatively;

Let the number to be added be a constant c .

Therefore;

$x^2 + 10x + c = (x + k)^2$, where k is another constant.

So, $x^2 + 10x + c = x^2 + 2kx + k^2$.

Comparing coefficients of x ;

$$2k = 10$$

Therefore, $k = 5$

Comparing constant terms;

$$c = k^2$$

But $k = 5$.

Therefore $c = 5^2$

$$= 25$$

Thus, 25 must be added to $x^2 + 10x$ to make it a perfect square.

Example 2

What must be added to $x^2 + \text{---} + 36$ to make it a perfect square?

Solution

Let the term to be added be bx , where b is a constant.

Then $x^2 + bx + 36$ is a perfect square.

Therefore, $\left(\frac{b}{2}\right)^2 = 36$

Taking square roots of both sides;

$$\frac{b}{2} = \sqrt{36}$$

$$\frac{b}{2} = \pm 6$$

$$b = 12 \text{ or } -12.$$

Thus, the term to be added is $12x$ or $-12x$.

Alternatively;

Let the term to be added be bx , where b is a constant. Then;

$$x^2 + bx + 36 = (x + m)^2, \text{ where } m \text{ is another constant.}$$

$$\text{So, } x^2 + bx + 36 = x^2 + 2mx + m^2$$

Comparing the constants;

$$m^2 = 36$$

$$m = \pm 6$$

Comparing the coefficients of x ;

$$b = 2m$$

$$\text{when } m = 6, b = 12$$

$$\text{and when } m = -6, b = -12$$

The term to be added is therefore $12x$ or $-12x$.

In each of the following expressions, insert the term which will make it a perfect square:

$$(i) \quad x^2 - 16x + \text{---}$$

$$(ii) \quad x^2 + 2x + \text{---}$$

$$(iii) \quad x^2 + \text{---} + 16$$

$$(iv) \quad x^2 - \text{---} + 100$$

$$(v) \quad A^2 - \text{---} + B^2$$

$$(vi) \quad A^2 + \text{---} + B^2$$

We shall now consider situations where $a \neq 1$, and is not zero, as in;

$$4x^2 - 12x + 9 = (2x - 6)^2$$

$$9x^2 + 6x + 1 = (3x + 1)^2$$

Identify similar expressions in exercise 1.1.

You will notice that in all cases, $\left(\frac{b}{2}\right)^2 = ac$.

We shall use this relation to make expressions perfect squares.

Example 3

What must be added to $25x^2 + \text{---} + 9$ to make it a perfect square?

Solution

Let the term to be added be bx .

Then, $25x^2 + bx + 9$ is a perfect square.

$$\text{Therefore, } \left(\frac{b}{2}\right)^2 = 25 \times 9$$

$$\left(\frac{b}{2}\right)^2 = 225$$

$$\frac{b}{2} = \pm 15$$

So, $b = 30$ or -30 .

The term to be added is thus $30x$ or $-30x$.

Alternatively;

Let the term to be added be bx .

Therefore, $25^2 + bx + 9 = (5x + n)^2$, for some constant n .

By expanding the expression on the right hand side and comparing the constants and the coefficients of x , you should find that the term to be added is either $30x$ or $-30x$.

Example 4

What must be added to $-40x + 25$ to make it a perfect square?

Solution

Let the missing term be ax^2 . Then, $ax^2 - 40x + 25$ is a perfect square.

Therefore, $(-40 \div 2)^2 = 25a$

$$400 = 25a$$

$$a = 16$$

Thus, the missing term is $16x^2$

Alternatively;

Let the missing term be ax^2 . Therefore;

$ax^2 - 40x + 25 = (mx + 5)^2$, where m is another constant.

$$ax^2 - 40x + 25 = m^2x^2 + 10mx + 25$$

Comparing terms gives;

$$10m = -40$$

$$m = -4$$

$$\text{Now } a = m^2$$

$$= (-4)^2$$

$$= 16$$

Thus, the term to be added is $16x^2$.

In each of the following expressions, insert the term which will make it a perfect square.

(i) $25x^2 + 40x + \text{---}$

(ii) $9x^2 - 6x + \text{---}$

(iii) $4x^2 + \text{---} + 25$

(iv) $36x^2 - \text{---} + 25$

(v) $\text{---} + 14x + 1$

(vi) $\text{---} - 48x + 9$

The process of adding a term to an expression to make it a perfect square is called **completing the square**.

Exercise 1.2

Add the missing terms to the following expressions to make them perfect squares.

1. (a) $x^2 + 8x + \text{---}$

(b) $x^2 + 14x + \text{---}$

(c) $x^2 - 22x + \text{---}$

2. (a) $\text{---} + 2x + 1$

(b) $\text{---} - 18x + 81$

(c) $16x^2 + \text{---} + 36$

3. (a) $49x^2 - \text{---} + 4$ (b) $\frac{1}{4}x^2 + \text{---} + 1$ (c) $121x^2 + 22x + \text{---}$
 4. (a) $\frac{1}{64}x^2 - \text{---} + \frac{1}{4}$ (b) $a^2x^2 + 6ax + \text{---}$ (c) $x^2y^2 + \text{---} + 4$
 5. (a) $a^2x^2 + 6abx + \text{---}$ (b) $(x - 1)^2 + \text{---} + (y - 1)^2$ (c) $-10a + 1$
 6. (a) $\text{---} - 60x + 25$ (b) $\text{---} - 4x + 1$ (c) $\text{---} + 2k + 4$
 7. (a) $4 - 20x + \text{---}$ (b) $\text{---} + 42x + 9$

1.3: Solution of Quadratic Equations by Completing the Square

Consider the quadratic equation;

$$x^2 + 4x - 12 = 0$$

On adding 12 to both sides, we have;

$$x^2 + 4x = 12.$$

The expression on the left hand side can be made a perfect square by adding 4.

Adding 4 to both sides, we get;

$$x^2 + 4x + 4 = 16$$

Factorising the L.H.S. gives;

$$(x + 2)^2 = 16$$

Taking square roots of both sides gives $x + 2 = \sqrt{16}$

$$x + 2 = \pm 4$$

$$x = -2 \pm 4$$

$$x = 2 \text{ or } -6$$

We have solved the quadratic equation above by making use of an expression which was a perfect square. This method of solving a quadratic equation is known as **completing the square**.

Example 5

Solve $x^2 + 5x + 1 = 0$ using completing the square method.

Solution

Subtracting one from both sides and proceeding to complete the square on the left hand side;

$$x^2 + 5x = -1$$

$$x^2 + 5x + \left(\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2 - 1$$

$$x^2 + 5x + \frac{25}{4} = \frac{21}{4}$$

Factorising the L.H.S. gives $\left(x + \frac{5}{2}\right)^2 = \frac{21}{4}$

Taking square roots gives $x + \frac{5}{2} = \pm \sqrt{\frac{21}{4}}$

$$\begin{aligned} x &= -\frac{5}{2} \pm \frac{4.583}{2} \\ &= -\frac{0.417}{2} \text{ or } -\frac{9.583}{2} \end{aligned}$$

Therefore, $x = -0.2085$ or -4.792 .

Note:

The method of completing the square enables us to solve quadratic equations which cannot be solved by factorisation.

Exercise 1.3

Use the method of completing the square to solve the following quadratic equations:

1. (a) $x^2 + 2x - 1 = 0$ (b) $x^2 - 5x + 2 = 0$ (c) $x^2 + 6x + 3 = 0$
2. (a) $x^2 - 8x + 13 = 0$ (b) $x^2 - 3x - 5 = 0$ (c) $x^2 - 22x + 6 = 0$
3. (a) $x^2 + 5x + 3 = 0$ (b) $x^2 - 8x - 30 = 0$ (c) $x^2 + 7x + 3 = 0$
4. $x^2 - \frac{1}{2}x - \frac{1}{3} = 0$

If the coefficient of x^2 is not equal to 1, i.e., $a \neq 1$, proceed as in example 6 below.

Example 6

Solve the equation $2x^2 + 4x + 1 = 0$ using the method of completing the square.

Solution

Subtracting 1 from both sides;

$$2x^2 + 4x = -1$$

Making the coefficient of x^2 one by dividing through by 2;

$$x^2 + 2x = -\frac{1}{2}$$

Adding 1 to complete the square on L.H.S., $x^2 + 2x + 1 = 1 - \frac{1}{2}$

$$x^2 + 2x + 1 = \frac{1}{2}$$

$$(x + 1)^2 = \frac{1}{2}$$

$$x + 1 = \pm \sqrt{\frac{1}{2}}$$

$$\begin{aligned} x &= -1 \pm \sqrt{0.5} \\ &= -1 \pm 0.7071 \end{aligned}$$

Therefore, $x = 0.2929$ or -1.7071 (t 4 s.f.).

Exercise 1.3 (continued)

5. (a) $2x^2 + 3x - 7 = 0$ (b) $3x^2 + 7x - 4 = 0$ (c) $4x^2 + 12x - 9 = 0$
 6. (a) $5x^2 + 6x - 3 = 0$ (b) $4x^2 + 3x - 5 = 0$ (c) $4 - 9x - 3x^2 = 0$
 7. (a) $6x^2 - 5x - 4 = 0$ (b) $2x^2 + 9x + 9 = 0$ (c) $2x(x + 1) = 4$
 8. $(x + 1)(x + 3) = 13$

1.4: The Quadratic Formula

Consider the general quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$

Let us solve this equation using the method of completing the square.

Subtracting c from both sides; $ax^2 + bx = -c$

Dividing through by a ; $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Complete the square on L.H.S. by adding $\left(\frac{b}{2a}\right)^2$ to both sides;

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\begin{aligned} \text{but } \left(\frac{b}{2a}\right)^2 - \frac{c}{a} &= \frac{b^2}{4a^2} - \frac{c}{a} \\ &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

$$\text{Therefore, } x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{Factorising the L.H.S. gives } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{Taking square roots of both sides gives } x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\text{It therefore follows that } x = \frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the solution to the general quadratic equation and is known as the **quadratic formula**.

Example 7

Use the quadratic formula to solve $2x^2 - 5x - 3 = 0$.

Solution

Comparing this equation to the general equation $ax^2 + bx + c = 0$, we get;

$$a = 2$$

$$b = -5$$

$$c = -3$$

Substituting in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

$$x = \frac{-5 \pm \sqrt{25 - 4(2)(-3)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{49}}{4}$$

$$= \frac{5 \pm 7}{4}$$

$$= \frac{12}{4} \text{ or } \frac{-2}{4}$$

Therefore, $x = 3$ or $-\frac{1}{2}$

Exercise 1.4

Use the quadratic formula to solve the following equations:

1. (a) $x^2 + 7x + 3 = 0$ (b) $x^2 - 4x + 3 = 0$ (c) $2x^2 + 11x + 7 = 0$
2. (a) $2x + 7 - 7x^2 = 0$ (b) $3x^2 - 3x - 2 = 0$ (c) $5r^2 - 5r + 1 = 0$
3. (a) $4d^2 + 7d + 3 = 0$ (b) $9p^2 + 24p + 16 = 0$ (c) $6k^2 + 9k + 1 = 0$
4. $1 - 3x - 3x^2 = 0$

From the quadratic formula, we notice that the nature of the roots will depend on the value of $b^2 - 4ac$. There are three possibilities, i.e.:

- (i) $b^2 - 4ac > 0$: Here, the value of $\sqrt{b^2 - 4ac}$ will be real and will take either a positive or negative value, leading to two distinct roots (values of x).
- (ii) $b^2 - 4ac = 0$: In this case, the quadratic formula will give us a single value. The quadratic equation has repeated roots and is a perfect square.
- (iii) $b^2 - 4ac < 0$: Here, $b^2 - 4ac$ will have no real values. The quadratic equation has no real roots.

The expression $b^2 - 4ac$ is known as the **discriminant**. It allows us to determine the nature of the roots of a quadratic equation.

Determine the nature of the roots of the following quadratic equations:

- (i) $x^2 - 3x + 4 = 0$ (ii) $x^2 + 5x + 1 = 0$ (iii) $x^2 + 6x + 9 = 0$
 (iv) $2x^2 - 3x + 1 = 0$ (v) $3x^2 + x + 4 = 0$ (vi) $(x - 4)(3x - 1) = 0$

1.5: Formation of Quadratic Equations

Consider the following examples.

Example 8

Peter travels to his uncle's home, 30 km away from his place. He cycles for two thirds of the journey before the bicycle develops mechanical problems and he has to push it for the rest of the journey. If his cycling speed is 10 km/h faster than his walking speed and he completes the journey in 3 hours 20 minutes, determine his cycling speed.

Solution

Let Peters' cycling speed be x km/h. Then, his walking speed is $(x - 10)$ km/h.

$$\begin{aligned}\text{Time taken in cycling} &= \left(\frac{2}{3} \text{ of } 30\right) \div x \\ &= \frac{20}{x} \text{ h}\end{aligned}$$

$$\begin{aligned}\text{Time taken in walking} &= (30 - 20) \div (x - 10) \\ &= \frac{10}{x - 10} \text{ h}\end{aligned}$$

$$\text{Total time} = \left(\frac{20}{x} + \frac{10}{x - 10}\right) \text{ h}$$

$$\text{Therefore, } \frac{20}{x} + \frac{10}{x - 10} = 3 \frac{1}{3}$$

$$\frac{20}{x} + \frac{10}{x - 10} = \frac{10}{3}$$

$$60(x - 10) + 30(x) = 10(x)(x - 10)$$

$$10x^2 - 190x + 600 = 0$$

$$x^2 - 19x + 60 = 0$$

$$x = \frac{19 \pm \sqrt{361 - 240}}{2}$$

$$x = 15 \text{ or } 4$$

If his cycling speed is 4 km/h, then his walking speed is $(4 - 10)$ km/h, which gives -6 km/h. Thus, 4 is not a realistic answer to this situation.

Therefore, his cycling speed is 15 km/h.

Example 9

A positive two-digit number is such that the product of the digits is 24. When the digits are reversed, the number formed is greater than the original number by 18. Find the number.

Solution

Let the ones digit of the number be y and the tens digit be x .

Then, $xy = 24$ (1)

When the number is reversed, the ones digit is x and the tens digit is y .

Therefore;

$$(10y + x) - (10x + y) = 18$$

$$9y - 9x = 18$$

$$y - x = 2$$

$$y = x + 2$$
 (2)

Substituting (2) in equation (1) gives;

$$x(x + 2) = 24$$

$$x^2 + 2x - 24 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 96}}{2}$$

$$= 4 \text{ or } -6$$

Since the required number is positive, $x = 4$ and $y = 4 + 2 = 6$.

Therefore, the number is 46.

Exercise 1.5

1. Solve the following quadratic equations using any suitable method.

(a) $x^2 - x - 6 = 0$

(b) $6x^2 + 11x + 3 = 0$

(c) $6x^2 + 13x + 6 = 0$

(d) $2x^2 + x = 0$

(e) $\frac{1}{2}x^2 - 2x - 2 = \frac{3}{4}(x - 2)$

(f) $4 - 25x^2 = 0$

(g) $4y^2 + \frac{1}{2}y = 0$

(h) $k^2 + 1.5 = 3k$

(i) $3y^2 - 2y - 5 = 0$

(j) $t^2 - 15t + 8 = 0$

(k) $2 - b - 10b^2 = 0$

(l) $\frac{x-1}{2} + \frac{x+3}{4} = \frac{1}{x-1}$

(m) $-1 - 5r + 7r^2 = 0$

(n) $\frac{2}{y} + 1 = \frac{y-2}{3}$

(p) $\frac{t}{6} + \frac{1}{2-t} = \frac{1}{3}$

(q) $\frac{y-3}{4} - \frac{1}{y} = \frac{4}{3} - y$

(r) $\frac{1}{x-1} + 3 = \frac{1}{x+2} - \frac{1}{4}$

2. ABCD is a rectangle whose area is 170 cm^2 . The internal rectangle is drawn with dimensions as shown in figure 1.1.

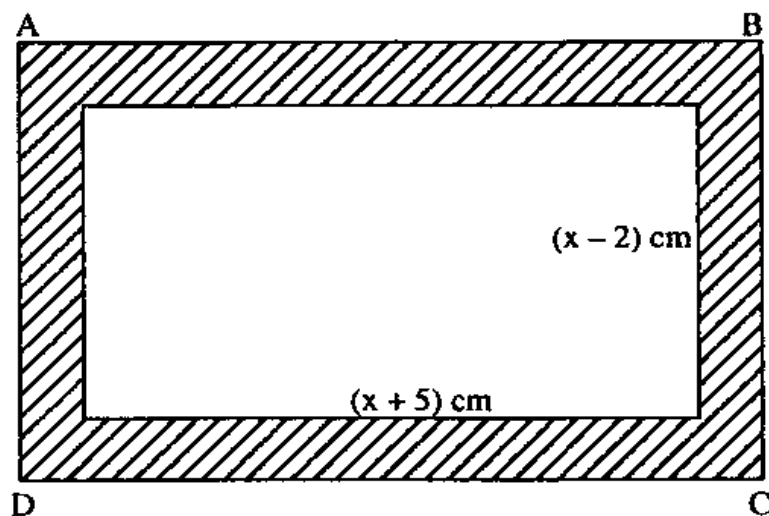


Fig. 1.1

Determine the area of the shaded part if its thickness is $\frac{1}{10}x \text{ cm}$.

3. A group of young men decided to raise sh. 480 000 to start a business. Before the actual payment was made, four of the members pulled out and each of those remaining had to pay an additional sh. 20 000. Determine the original number of members.
4. A piece of wire 90 cm long is bent into the pattern ABCDAEFD, in that order, which consists of two equal rectangles, see figure 1.2.

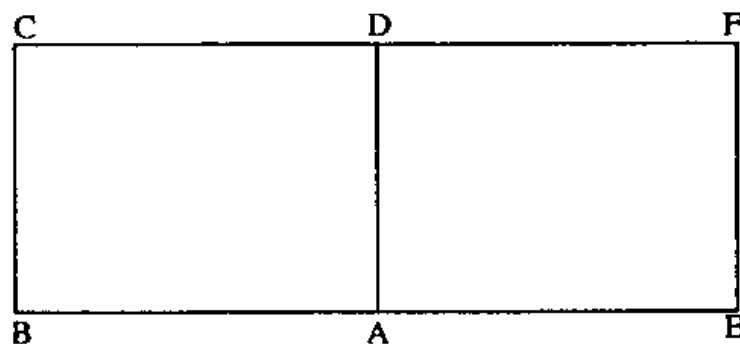


Fig. 1.2

If the total area enclosed is 300 cm^2 , determine possible lengths of:

- (a) BC.
 - (b) CF.
5. A lawn is in the shape of a right-angled triangle. The lengths of the two shorter sides are $(x + 2) \text{ m}$ and $(4x + 4) \text{ m}$, while the length of the hypotenuse is $5x \text{ m}$. Form and solve an equation in x and hence find the area of the lawn in m^2 .
6. A boat's speed in still water is 4 km/h . The boat cruises from A to B along

- a river flowing at an average speed of x kmh in the direction A to B. If the distance AB is 5 km and the boat takes 2 hours more on its return journey, determine x . Hence, find the total time taken for the whole journey.
7. The length of a rectangle is three times its breadth. If the breadth is decreased by 2 m and the length increased by 4 m, the area of the rectangle is decreased by a third. Find the breadth of the original rectangle. Hence, find its area.
8. A two-digit number is such that the product of its digits is 12. When the digits are reversed, the number formed exceeds the original by 9. Find the original number.

1.6: Graphs of Quadratic Functions

We have drawn graphs of linear equations of the form $y = mx + c$ and noticed that each value of x gives a specific value of y . For example, in the equation $y = 2x + 1$, if $x = 2$, then $y = 5$.

Generally, if y depends on x , y is said to be a **function** of x provided that for each value of x , there is a unique value of y .

In this section, we shall take a look at graphs of quadratic functions and their applications to the solution of equations.

Example 10

Draw the graph of $y = x^2$ for values of x from -4 to 4 .

Solution

Table 1.1 below shows the corresponding values of x and y .

Table 1.1

x	-4	-3	-2	-1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Using the values of x and y in the table, we choose suitable scales. In this case we choose: 10 small squares to represent 1 unit along the x -axis and 10 small squares to represent 5 units along the y -axis.

We plot the points and join them by a smooth curve.

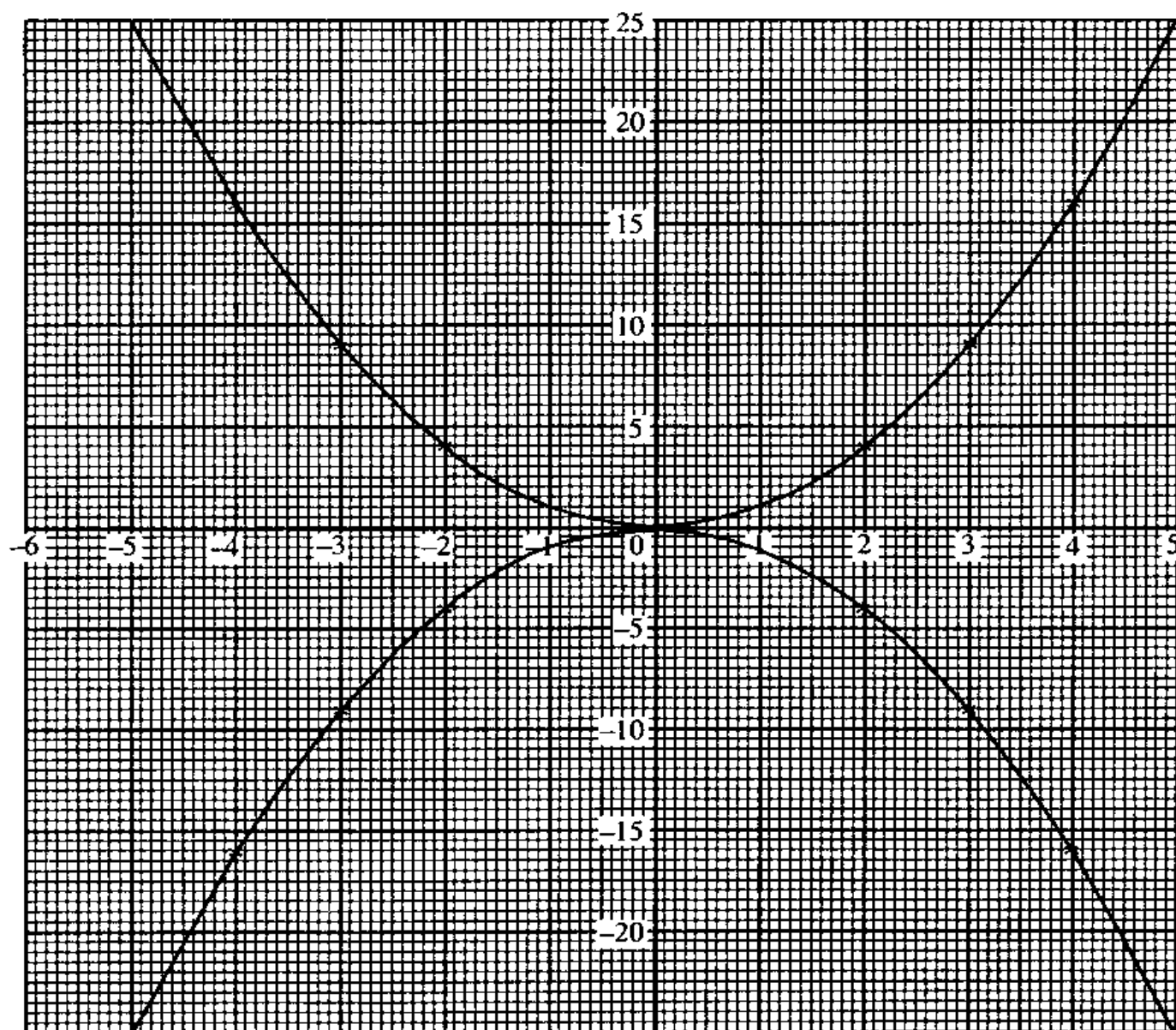


Fig. 1.3

The graph of $y = x^2$ is shown in figure 1.3.

On the same axes and using the same range, draw the graphs of $y = x^2$ and $y = -x^2$.

What is the relationship between the two curves? State the lines of symmetry of the two curves.

Note:

For the graph of $y = x^2$, the minimum value of the function is zero and for the graph of $y = -x^2$, the maximum value of the function is zero.

Exercise 1.6

- Copy and complete tables 1.2(a) and (b) below for the functions $y = x^2 - 2x - 8$ and $y = 8 + 2x - x^2$ respectively. Hence draw the graphs of the given quadratic functions on the same axes.

Table 1.2 (a): $y = x^2 - 2x - 8$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
x^2	25								9				
$-2x$	10								-6				
-8	-8								-8				
$y = x^2 - 2x - 8$	27								-5				

Table 1.2 (b): $y = 8 + 2x - x^2$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
8		8				8							
$+2x$		-8				0							
x^2		-16				0							
$y = 8 + 2x - x^2$		-16				8							

State the equation of the line of symmetry between the two curves.

- Draw the graphs of the following quadratic functions for the given range of values of x . In each case, state the equation of the line of symmetry and the minimum/maximum value of the function.
 - $y = x^2 + 2x - 3$ $-6 \leq x \leq 4$
 - $y = x^2 + 6x + 9$ $-7 \leq x \leq 1$
 - $y = x^2 + 5x + 7$ $-7 \leq x \leq 2$
 - $y = 12 - 4x - x^2$ $-8 \leq x \leq 4$
 - $y = -x^2 + 2x - 5$ $-3 \leq x \leq 5$
- Using the same axes, draw the graphs of:
 - $y = x^2$
 - $y = x^2 + 10$
 - $y = x^2 - 10$ for $-2 \leq x \leq 4$.
 - Identify the minimum points.
 - What maps $y = x^2$ onto each of the other functions?
 - What maps $y = x^2 + 10$ onto $y = x^2 - 10$?
- By choosing a suitable range of values of x , draw the graph of $y = (3 - x)(x - 4) + 1$. Using the same axes, draw the graph of its reflection in the x -axis. What is the equation of the line of symmetry for the two curves?
- Copy and complete table 1.3 below and hence draw the graph of $y^2 = 4x$.

Table 1.3

y	-4	-3	-2	-1	0	1	2	3	4
x		$\frac{9}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{9}{4}$	4

1.7: Graphical Solutions of Quadratic Equations

Consider the quadratic equation $x^2 - x - 6 = 0$, whose solution is $x = 3$ or $x = -2$.

Using table 1.4 below, the graph of the quadratic function $y = x^2 - x - 6$ is drawn as shown in figure 1.4.

Table 1.4

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	24	14	6	0	-4	-6	-6	-4	0	6	14

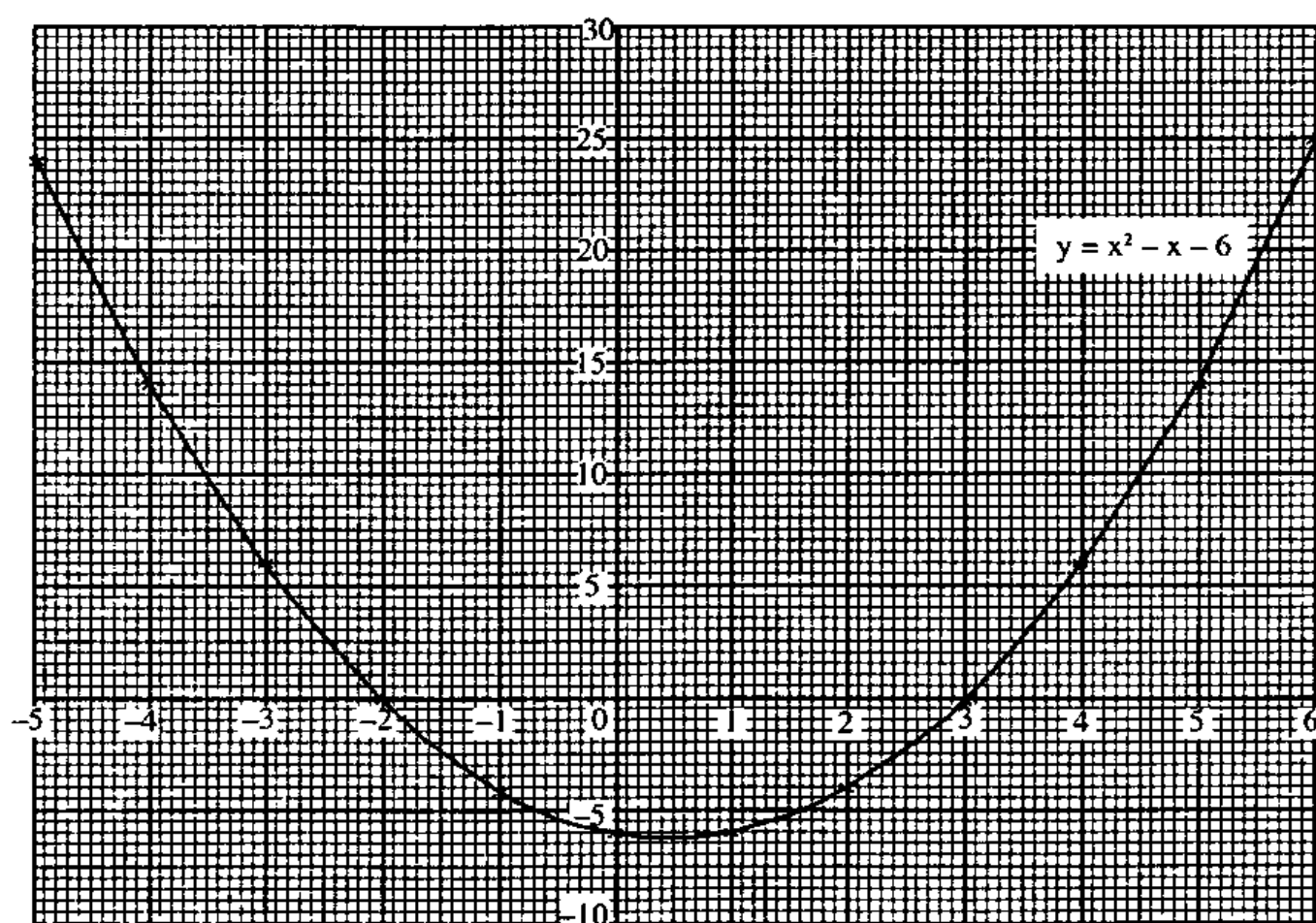


Fig. 1.4

We notice that the graph cuts the x-axis (the line $y = 0$) at $x = 3$ and $x = -2$, which are the roots of the equation $x^2 - x - 6 = 0$

In general, if the graph of $y = a, x^2 + bx + c$ cuts the x -axis, the values of x at the points of intersection of the curve and the x -axis are the roots of the equation $ax^2 + bx + c = 0$.

Figure 1.5 shows sketches of graphs of quadratic functions. Where possible, state.

- (i) the roots of the quadratic equation.
- (ii) the corresponding quadratic equation.

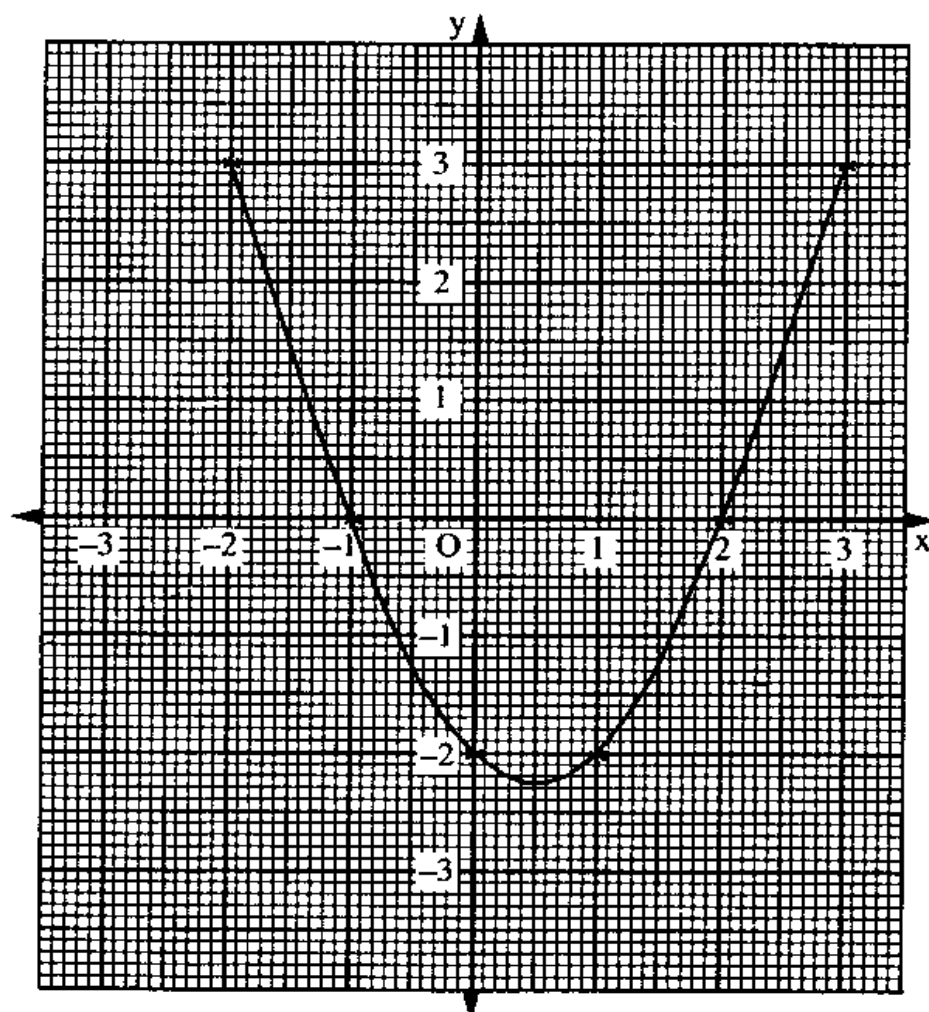


Fig. 1.5 (i)

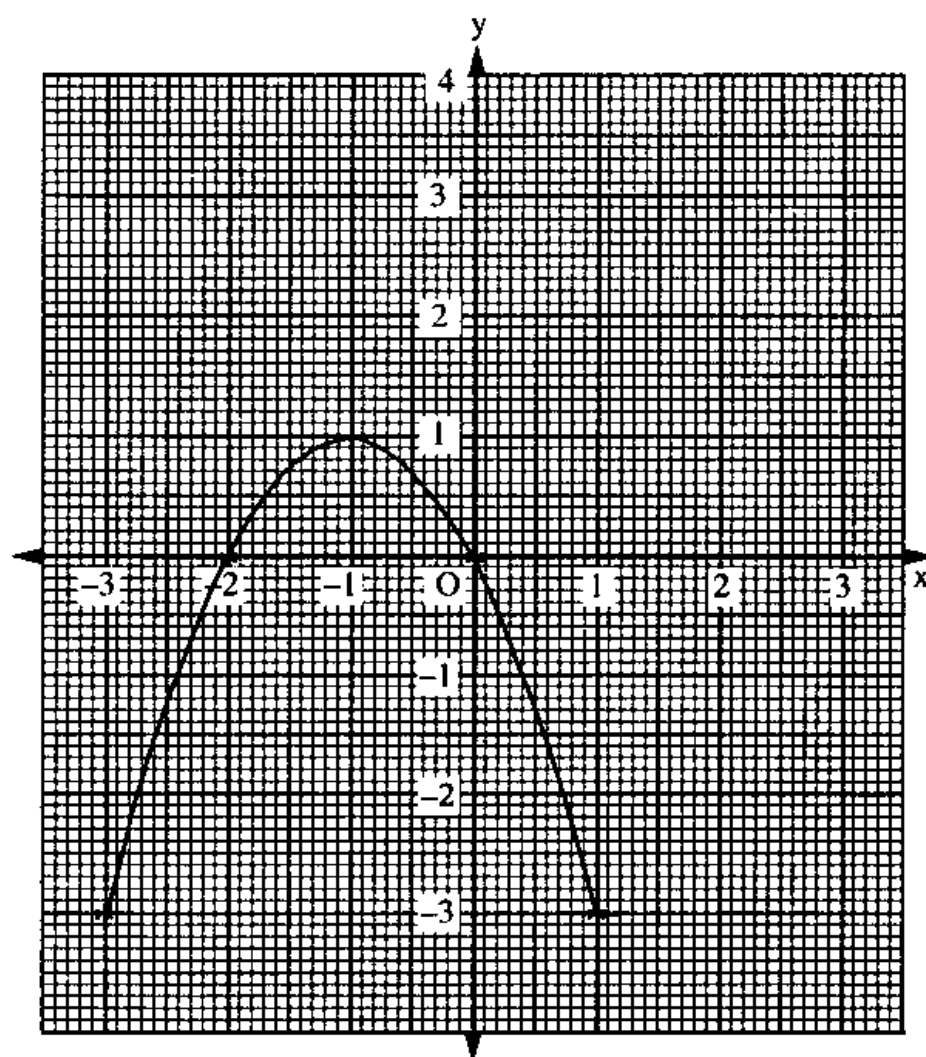


Fig. 1.5 (ii)

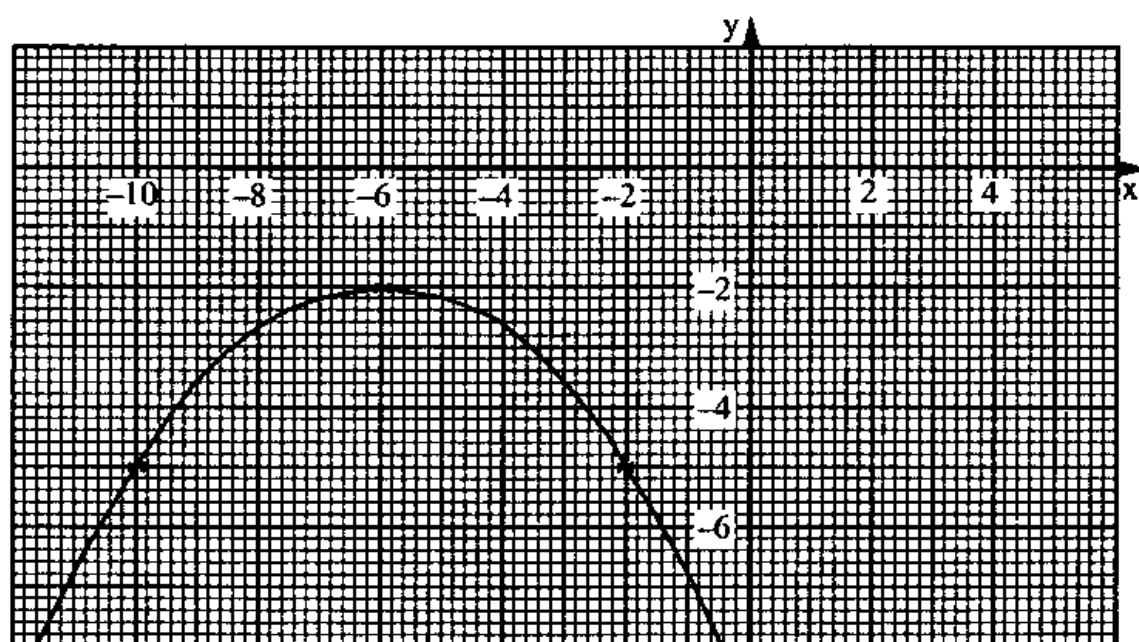


Fig. 1.5 (iii)

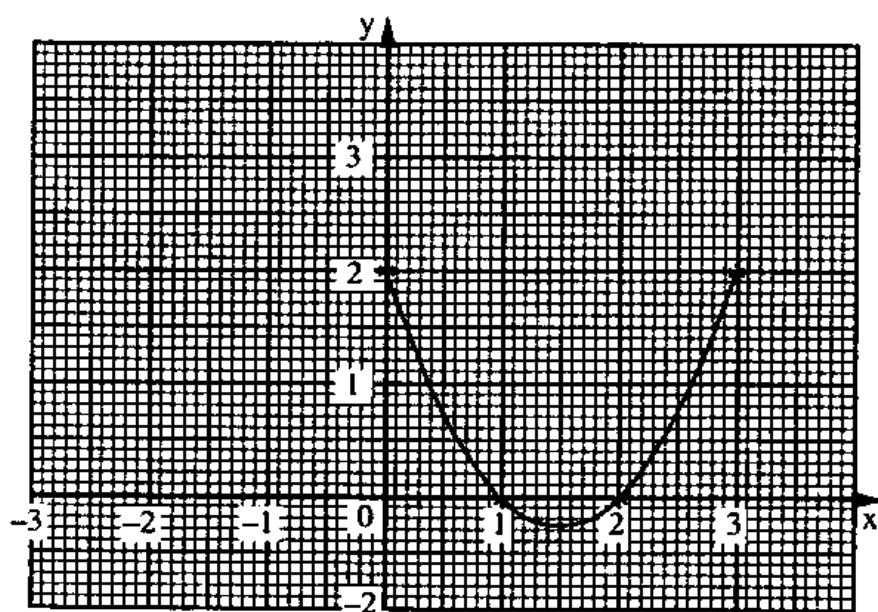


Fig. 1.5 (iv)

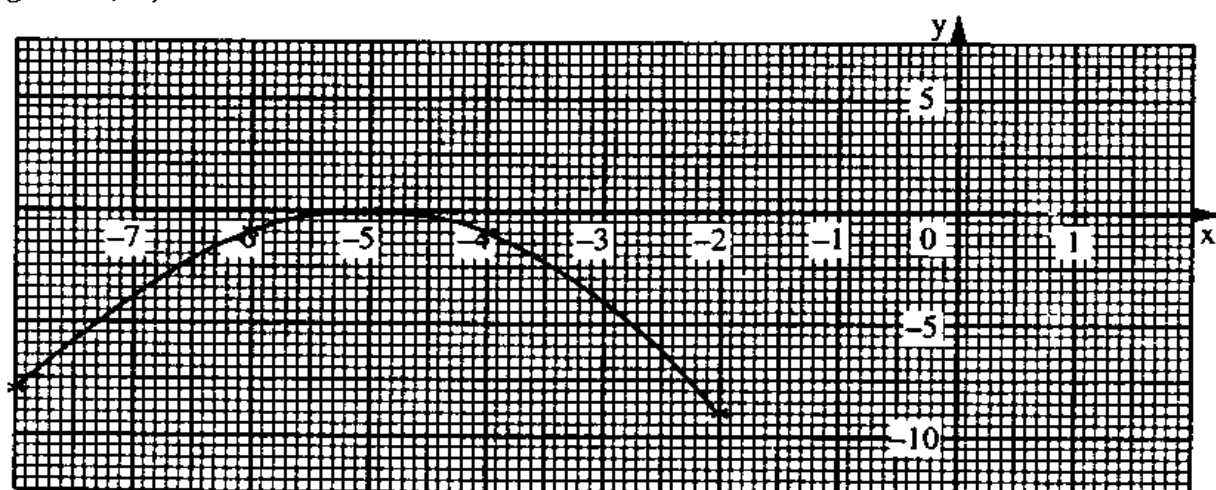


Fig. 1.5 (v)

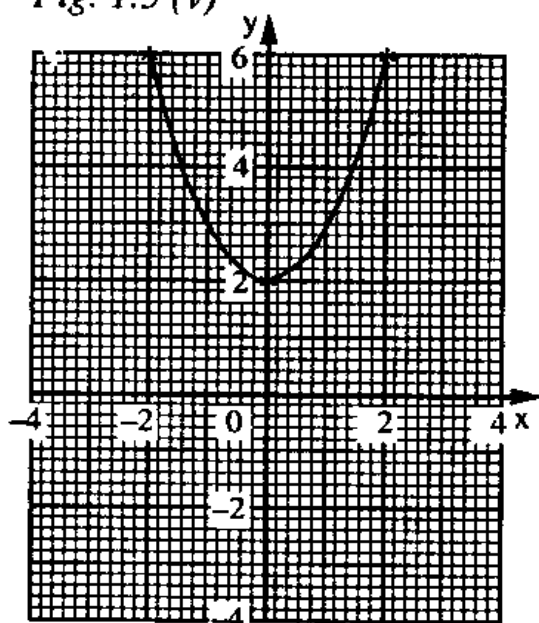


Fig. 1.5 (vi)

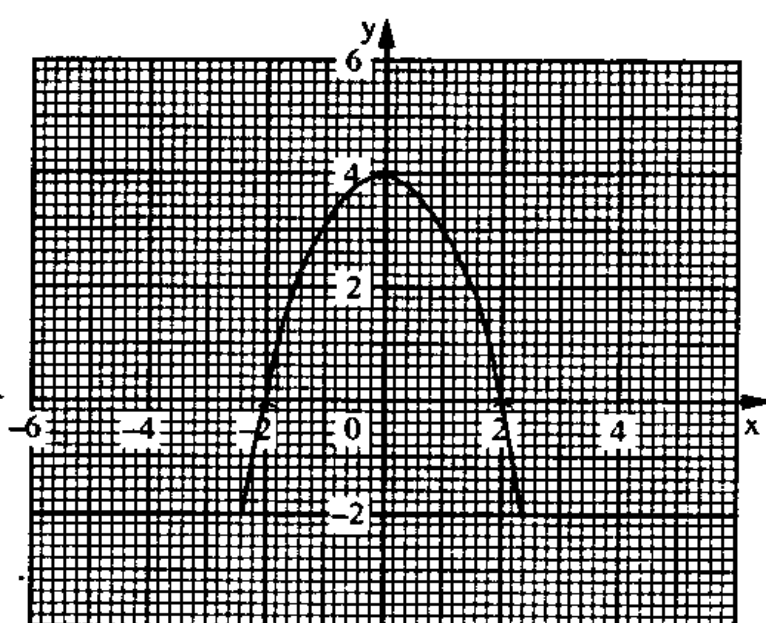


Fig. 1.5 (vii)

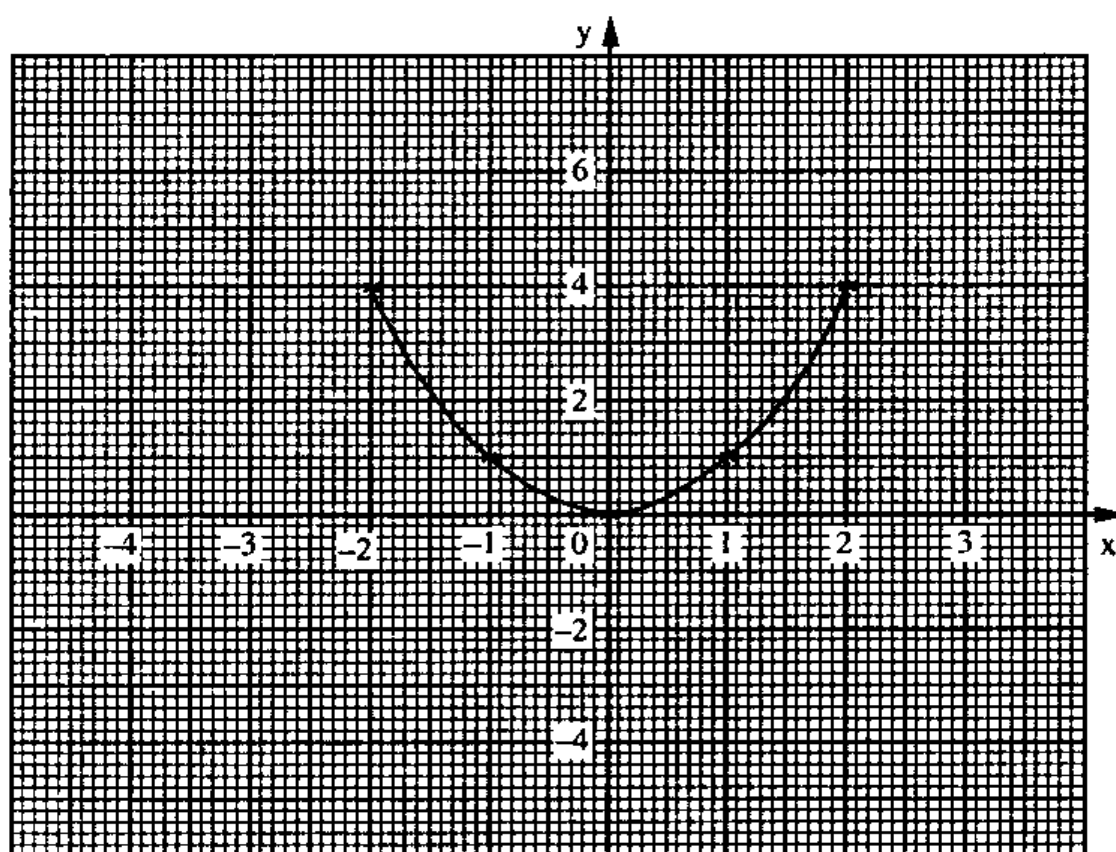


Fig. 1.5 (viii)

We notice that:

- (i) If the graph cuts the x-axis at two distinct points, then the equation has two distinct roots, e.g., figure 1.5 (i).
- (ii) If the graph touches the x-axis at only one point, then the equation has a repeated root at that point, e.g., figure 1.5 (viii).
- (iii) The graphs in figure 1.5(iii) and (vi) neither touch nor cross the x-axis. We say that their equations have no real roots.

Exercise 1.7

1. Draw the graphs of the following quadratic functions. Where possible, estimate the roots from your graph.

(a) $y = 2x^2 - 4x + 1$	(b) $y = x^2 - x + 1$	(c) $y = x^2 + x + 1$
(d) $y = x^2 + 4x + 4$	(e) $y = 5 - 3x - x^2$	(f) $y = -3 + x - x^2$

1.8: Graphical Solutions of Simultaneous Equations

Graphical methods may be used to solve simultaneous equations. In this section, we shall consider simultaneous equations, one of which is linear and the other quadratic.

Example 11

Solve the following simultaneous equations graphically:

$$y = x^2 - 2x + 1$$

$$y = 5 - 2x$$

Solution

Table 1.5 gives corresponding values of x and y for the equation $y = x^2 - 2x + 1$

Table 1.5

x	-2	-1	0	1	2	3	4
y	9	4	1	0	1	4	9

We use the table to draw the graph of $y = x^2 - 2x + 1$, as shown in figure 1.6. On the same axes, the line $y = 5 - 2x$ is drawn.

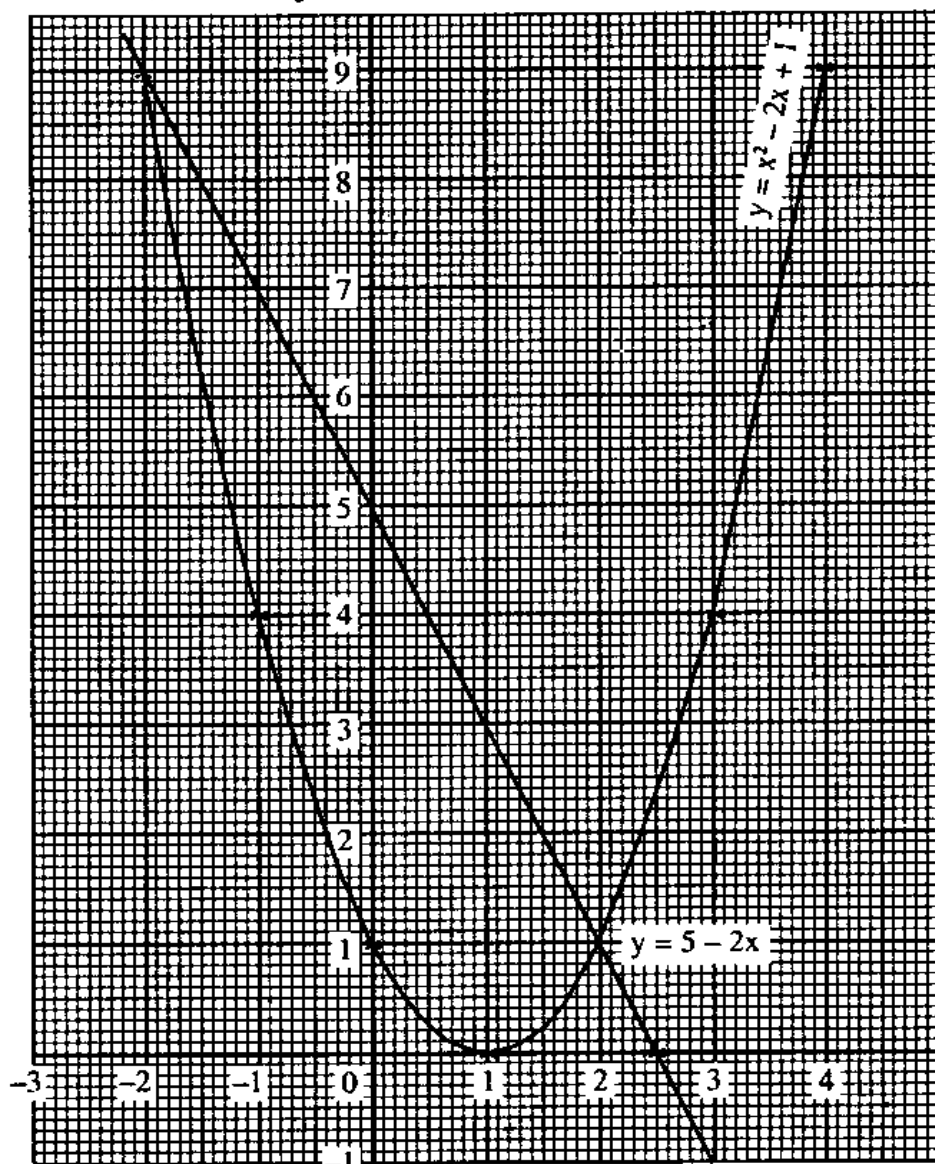


Fig. 1.6

The points where the line $y = 5 - 2x$ and the curve $y = x^2 - 2x + 1$ intersect give the solution. The points are $(-2, 9)$ and $(2, 1)$. Therefore, when $x = -2$, $y = 9$ and when $x = 2$, $y = 1$.

Exercise 1.8

1. Use the graphical method to solve the following pairs of simultaneous equations:

(a) $y = x^2 + 3x - 6$

$y = x + 1$

(c) $y = 2x^2 + 3x - 5$

$y = 2 - x$

(e) $y = \frac{x^2}{3} - \frac{2}{3}x + 2$

$y = \frac{4}{3}x + 1$

(b) $y = x^2 + 4x + 1$

$y = 2x + 1$

(d) $y = 3x^2 - 4x - 5$

$y = 2 - 3x$

2. Re-arrange each of the following quadratic equations to form a pair of simultaneous equations and hence draw a graph to solve the equation. (Hint: Rewrite $x^2 - 3x + 1 = 0$ as $x^2 = 3x - 1$, then draw the graphs of $y = x^2$ and $y = 3x - 1$).

(a) $x^2 - 3x + 2 = 0$

(b) $3x^2 - 7x + 4 = 0$

(c) $\frac{x-2}{4} = \frac{x+2}{x}$

(d) $\frac{x^2}{4} - \frac{7}{8}x = 1$

(e) $x^2 - 22x + 6 = 0$

(f) $x^2 + 7x + 3 = 0$

Further Graphical Solutions

A graph of a given quadratic function can be used to solve other related quadratic equations.

Example 12

Use the graph of the function $y = x^2 + 3x + 1$ to solve:

(a) $x^2 + 3x + 1 = 10$

(b) $x^2 + 3x - 4 = 0$

(c) $x^2 + 2x - 2 = 0$

Solution

Table 1.6 below gives corresponding values of x and y for the function $y = x^2 + 3x + 1$.

Table 1.6

x	-5	-4	-3	-2	-1	0	1	2
y	11	5	1	-1	-1	1	5	11

Figure 1.7 shows the graph of $y = x^2 + 3x + 1$.

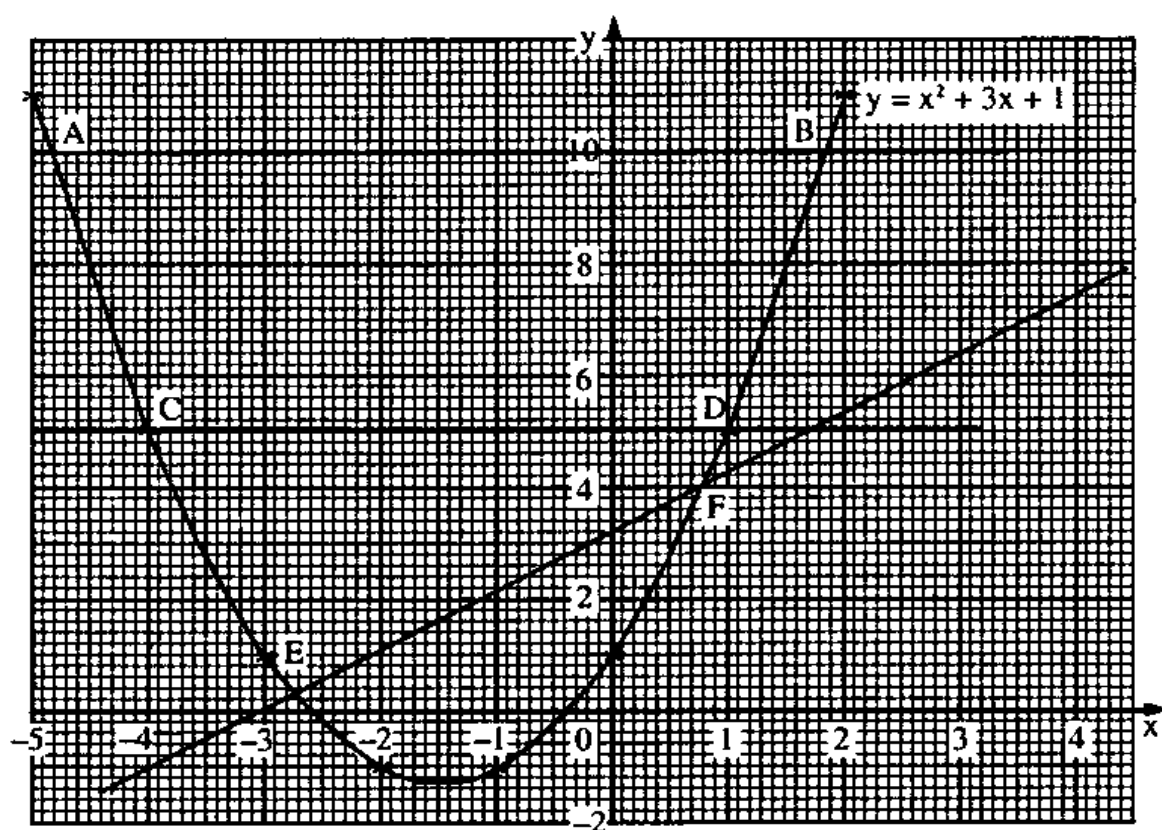


Fig. 1.7

- (a) $x^2 + 3x + 1 = 10$ is the equation we want to solve. But $y = x^2 + 3x + 1$ has been drawn. Comparing the two, $y = 10$.

We draw the line $y = 10$ and read off the values of x at the points A and B on the graph where the line intersects the curve. At these points; $x = -4.9$ and $x = 1.9$.

These are the solutions of $x^2 + 3x + 1 = 10$

- (b) $x^2 + 3x - 4 = 0$

Comparing this equation with the curve already drawn, i.e.,

$$y = x^2 + 3x + 1$$

$$0 = x^2 + 3x - 4$$

$$\underline{y = 5}$$

By subtracting we can eliminate the part having x^2 and remain with a linear equation. This is the line that we will include in our curve in order to solve the quadratic equation.

Draw the line $y = 5$ and read off the values of x at the points C and D on the graph where the line intersects the curve, i.e., $x = -4$ or $x = 1$

- (c) $x^2 + 2x - 2 = 0$.

Comparing this equation with the curve already drawn, we get;

$$y = x^2 + 3x + 1$$

$$0 = x^2 + 2x - 2$$

$$\underline{y = x + 3}$$

We need to include the line $y = x + 3$ on the curve already drawn so that we can solve the quadratic equation $x^2 + 2x - 2 = 0$

We draw the line $y = x + 3$ and read off the values of x at points E and F on the graph. Thus, $x = -2.7$ or 0.8 .

Exercise 1.9

For each of the following, draw the graph of the given function over the given range and use it to solve the given equations.

1. $y = x^2 + 5x + 4$ for $-7 \leq x \leq 3$
 - (a) $x^2 + 5x + 4 = 0$
 - (b) $x^2 + 5x + 2 = 0$
 - (c) $x^2 + 4x + 3 = 0$
2. $y = 2x^2 + x - 1$ for $-4 \leq x \leq 4$
 - (a) $2x^2 + x - 1 = 0$
 - (b) $2x^2 + x = 0$
 - (c) $2x^2 - 5 = 0$
3. $y = 4 + 6x - x^2$ for $-2 \leq x \leq 8$
 - (a) $4 + 6x - x^2 = 0$
 - (b) $x^2 - 4x = 0$
 - (c) $x^2 - \frac{14}{3}x + 1 = 0$
4. $y = \frac{x^2}{2} - \frac{2}{3}x - 1$ for $-4 \leq x \leq 4$
 - (a) $\frac{x^2}{2} - \frac{2}{3}x - 1 = 0$
 - (b) $\frac{x^2}{2} - \frac{2}{3}x - 3 = 0$
 - (c) $\frac{x^2}{2} - \frac{x}{2} - 2 = 0$
5. $y = 2x^2 + 5x - 12$ for $-8 \leq x \leq 4$
 - (a) $2x^2 + 5x - 12 = 0$
 - (b) $x^2 + x - 6 = 0$
 - (c) $3 - 7x - 3x^2 = 0$