

Chapter 2: Basics of Probability Theory**EXERCISE**

What is the sample space if a coin is tossed four times.

SOLUTION:

Since either coin can turn up Head (H) or Tail (T), are the possible outcomes.

So, when 1 coin is tossed once the sample space = 2

Then,

Coin is tossed 3 times the sample space = $2^4 = 16$

Thus, the sample space is $S = \{HHHH, THHH, HTHH, HHTH, HHHT, TTTT, HTTT, THTT, TTHT, TTTH, TTHH, HHTT, THTH, HTHT, THHT, HTTH\}$

EXERCISE

A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.

SOLUTION:

From the question it is given that, a box contains 1 red and 3 identical white balls.

Let us assume 'R' be the event of red ball is drawn and 'W' be the event of white ball is drawn.

Given in the question that white balls are identical, therefore the event of drawing any one of the three white ball is same.

Then, total number of sample space = $(2^2 - 1) = 3$

∴ Sample space $S = \{WW, WR, RW\}$

EXERCISE

A die is thrown. Describe the following events:

(i) A: a number less than 7 (ii) B: a number greater than 7

(iii) C: a multiple of 3 (iv) D: a number less than 4

(v) E: an even number greater than 4 (vi) F: a number not less than 3

Also find $A \cup B$, $A \cap B$, $B \cup C$, $E \cap F$, $D \cap E$, $A - C$, $D - E$, $E \cap F^I$, F^I

SOLUTION:

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible outcomes when the die is thrown.

So, $S = (1, 2, 3, 4, 5, 6)$

As per the conditions given in the question,

(i) A: a number less than 7

All the numbers in the die are less than 7,

$A = (1, 2, 3, 4, 5, 6)$

(ii) B: a number greater than 7

There is no number greater than 7 on the die

Then,

$B = (\varphi)$

(iii) C: a multiple of 3

There are only two numbers which are multiple of 3.

Then,

$C = (3, 6)$

(iv) D: a number less than 4

$D = (1, 2, 3)$

(v) E: an even number greater than 4

$E = (6)$

(vi) F: a number not less than 3

$F = (3, 4, 5, 6)$

Also we have to find, $A \cup B$, $A \cap B$, $B \cup C$, $E \cap F$, $D \cap E$, $D - E$, $A - C$, $E \cap F'$, F'

So,

$$A \cap B = (1, 2, 3, 4, 5, 6) \cap (\varnothing) = (\varnothing)$$

$$B \cup C = (\varnothing) \cup (3, 6) = (3, 6)$$

$$E \cap F = (6) \cap (3, 4, 5, 6) = (6)$$

$$D \cap E = (1, 2, 3) \cap (6) = (\varnothing)$$

$$D - E = (1, 2, 3) - (6) = (1, 2, 3)$$

$$A - C = (1, 2, 3, 4, 5, 6) - (3, 6) = (1, 2, 4, 5)$$

$$F' = S - F = (1, 2, 3, 4, 5, 6) - (3, 4, 5, 6) = (1, 2)$$

$$E \cap F' = (6) \cap (1, 2) = (\varnothing)$$

EXERCISE

A die is thrown, find the probability of following events:

- (i) A prime number will appear,
- (ii) A number greater than or equal to 3 will appear,
- (iii) A number less than or equal to one will appear,
- (iv) A number more than 6 will appear,
- (v) A number less than 6 will appear.

SOLUTION:

Let us assume that 1, 2, 3, 4, 5 and 6 are the possible out comes when the die is thrown.

Here $S = \{1, 2, 3, 4, 5, 6\}$

$$\therefore n(S) = 6$$

- (i) A prime number will appear,

Let us assume 'A' be the event of getting a prime number,

$$A = \{2, 3, 5\}$$

$$\text{Then, } n(A) = 3$$

$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$

$$\therefore P(A) = n(A)/n(S)$$

$$= 3/6$$

$$= \frac{1}{2}$$

(ii) A number greater than or equal to 3 will appear,

Let us assume 'B' be the event of getting a number greater than or equal to 3,

$$B = \{3, 4, 5, 6\}$$

$$\text{Then, } n(B) = 4$$

$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$

$$\therefore P(B) = n(B)/n(S)$$

$$= 4/6$$

$$= 2/3$$

(iii) A number less than or equal to one will appear,

Let us assume 'C' be the event of getting a number less than or equal to 1,

$$C = \{1\}$$

$$\text{Then, } n(C) = 1$$

$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$

$$\therefore P(C) = n(C)/n(S)$$

$$= 1/6$$

(iv) A number more than 6 will appear,

Let us assume 'D' be the event of getting a number more than 6, then

$$D = \{0\}$$

$$\text{Then, } n(D) = 0$$

$$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$$

$$\therefore P(D) = n(D)/n(S)$$

$$= 0/6$$

$$= 0$$

(v) A number less than 6 will appear.

Let us assume 'E' be the event of getting a number less than 6, then

$$E = (1, 2, 3, 4, 5)$$

$$\text{Then, } n(E) = 5$$

$$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$$

$$\therefore P(E) = n(E)/n(S)$$

$$= 5/6$$

EXERCISE

A card is selected from a pack of 52 cards.

- (a) How many points are there in the sample space?
- (b) Calculate the probability that the card is an ace of spades.
- (c) Calculate the probability that the card is (i) an ace (ii) black card

SOLUTION:

From the question it is given that, there are 52 cards in the deck.

- (a) Number of points in the sample space = 52 (given)

$$\therefore n(S) = 52$$

(b) Let us assume 'A' be the event of drawing an ace of spades.

$$A = 1$$

$$\text{Then, } n(A) = 1$$

$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$

$$\therefore P(A) = n(A)/n(S)$$

$$= 1/52$$

(c) Let us assume 'B' be the event of drawing an ace. There are four aces.

$$\text{Then, } n(B) = 4$$

$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$

$$\therefore P(B) = n(B)/n(S)$$

$$= 4/52$$

$$= 1/13$$

(d) Let us assume 'C' be the event of drawing a black card. There are 26 black cards.

$$\text{Then, } n(C) = 26$$

$P(\text{Event}) = \text{Number of outcomes favorable to event} / \text{Total number of possible outcomes}$

$$\therefore P(C) = n(C)/n(S)$$

$$= 26/52$$

$$= \frac{1}{2}$$

EXERCISE

There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?

SOLUTION:

From the question it is given that, there are four men and six women on the city council.

Here total members in the council = $4 + 6 = 10$,

Hence, the sample space has 10 points

$$\therefore n(S) = 10$$

Number of women are 6 ... [given]

Let us assume 'A' be the event of selecting a woman

$$\text{Then } n(A) = 6$$

$P(\text{Event}) = \text{Number of outcomes favourable to event} / \text{Total number of possible outcomes}$

$$\therefore P(A) = n(A)/n(S)$$

$$= 6/10 \dots [\text{divide both numerator and denominators by 2}]$$

$$= 3/5$$

EXERCISE

Three coins are tossed once. Find the probability of getting

(i) 3 heads (ii) 2 heads (iii) at least 2 heads

(iv) at most 2 heads (v) no head (vi) 3 tails

(vii) Exactly two tails (viii) no tail (ix) at most two tails

SOLUTION:

Since either coin can turn up Head (H) or Tail (T), are the possible outcomes.

But, now three coin is tossed so the possible sample space contains,

$$S = \{HHH, HHT, HTH, THH, TTH, HTT, TTT, THT\}$$

Where s is sample space and here $n(S) = 8$

(i) 3 heads

Let us assume 'A' be the event of getting 3 heads

$$n(A) = 1$$

$$\therefore P(A) = n(A)/n(S)$$

$$= 1/8$$

(ii) 2 heads

Let us assume 'B' be the event of getting 2 heads

$$n(B) = 3$$

$$\therefore P(B) = n(B)/n(S)$$

$$= 3/8$$

(iii) at least 2 heads

Let us assume 'C' be the event of getting at least 2 heads

$$n(C) = 4$$

$$\therefore P(C) = n(C)/n(S)$$

$$= 4/8$$

$$= 1/2$$

(iv) at most 2 heads

Let us assume 'D' be the event of getting at most 2 heads

$$n(D) = 7$$

$$\therefore P(D) = n(D)/n(S)$$

$$= 7/8$$

(v) no head

Let us assume 'E' be the event of getting no heads

$$n(E) = 1$$

$$\therefore P(E) = n(E)/n(S)$$

$$= 1/8$$

(vi) 3 tails

Let us assume 'F' be the event of getting 3 tails

$$n(F) = 1$$

$$\therefore P(F) = n(F)/n(S)$$

$$= 1/8$$

(vii) Exactly two tails

Let us assume 'G' be the event of getting exactly 2 tails

$$n(G) = 3$$

$$\therefore P(G) = n(G)/n(S)$$

$$= 3/8$$

(viii) no tail

Let us assume 'H' be the event of getting no tails

$$n(H) = 1$$

$$\therefore P(H) = n(H)/n(S)$$

$$= 1/8$$

(ix) at most two tails

Let us assume 'I' be the event of getting at most 2 tails

$$n(I) = 7$$

$$\therefore P(I) = n(I)/n(S)$$

$$= 7/8$$

EXERCISE

If $2/11$ is the probability of an event, what is the probability of the event 'not A'.

SOLUTION:

From the question it is given that, $2/11$ is the probability of an event A,

$$\text{i.e. } P(A) = 2/11$$

Then,

$$P(\text{not } A) = 1 - P(A)$$

$$= 1 - (2/11)$$

$$= (11 - 2)/11$$

$$= 9/11$$

EXERCISE

Given $P(A) = 5/3$ and $P(B) = 1/5$. Find $P(A \text{ or } B)$, if A and B are mutually exclusive events.

SOLUTION:

From the question it is given that,

$$P(A) = 5/3 \text{ and } P(B) = 1/5$$

Then, $P(A \text{ or } B)$, if A and B are mutually exclusive

$$P(A \cup B) \text{ or } P(A \text{ or } B) = P(A) + P(B)$$

$$= (3/5) + (1/5)$$

$$= 4/5$$

EXERCISE

In Class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology.

SOLUTION:

Let us assume 'A' be the event that the student is studying mathematics and 'B' be the event that the student is studying biology.

So, $P(A) = 40/100$

$$= 2/5$$

And, $P(B) = 30/100$

$$= 3/10$$

Then, $P(A \cap B) = (10/100)$

$= 1/10$, $P(A \cap B)$ is probability of studying both mathematics and biology.

Here, Probability of studying mathematics or biology will be given by $P(A \cup B)$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= (2/5) + (3/10) - (1/10)$$

$$= 6/10$$

$$= 3/5$$

Hence, $(3/5)$ is the probability that the student will studying mathematics or biology

EXERCISE

In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing at least one of them is 0.95. What is the probability of passing both?

SOLUTION:

Let us assume probability of a randomly chosen student passing the first examination is 0.8 be $P(A)$.

And also assume the probability of passing the second examination is 0.7 be $P(B)$

Then,

$P(A \cup B)$ is probability of passing at least one of the examination

Now,

$$P(A \cup B) = 0.95, P(A) = 0.8, P(B) = 0.7$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.95 = 0.8 + 0.7 - P(A \cap B)$$

Transposing $-P(A \cap B)$ to LHS and it becomes $P(A \cap B)$ and 0.95 to RHS and it becomes

$$-0.95$$

$$P(A \cap B) = 1.5 - 0.95$$

$$= 0.55$$

Hence, 0.55 is the probability that student will pass both the examinations.

EXERCISE

A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that

(i) all will be blue? (ii) at least one will be green?

SOLUTION:

From the question it is given that,

Number of red marbles in the box = 10

Number of blue marbles in the box = 20

Number of green marbles in the box = 30

So, Total number of marbles in the box = $10 + 20 + 30 = 60$

Number of ways of drawing 5 marbles from 60 marbles = ${}^{60}C_5$

(i) All the drawn marbles will be blue if we draw 5 marbles out of 20 blue marbles.

We have,

Number of ways of drawing 5 blue marbles from 20 blue marbles = ${}^{20}C_5$

Then,

Probability that all marbles will be blue = ${}^{20}C_5 / {}^{60}C_5$

(ii) Number of ways in which the drawn marble is not green = ${}^{(20+10)}C_5$

We have,

Probability that no marble is green = ${}^{30}C_5 / {}^{60}C_5$

Then,

Probability that at least one marble is green = $1 - {}^{30}C_5 / {}^{60}C_5$

EXERCISE

4 cards are drawn from a well – shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?

SOLUTION:

From the question it is given that,

4 cards are drawn from a well – shuffled deck of 52 cards

Number of ways of drawing 4 cards from 52 cards = ${}^{52}C_4$

In a deck of 52 cards, there are 13 diamonds and 13 spades.

Number of ways of drawing 3 diamonds and one spade = ${}^{13}C_3 \times {}^{13}C_1$

Therefore, the probability of obtaining 3 diamonds and one spade

$$= ({}^{13}C_3 \times {}^{13}C_1) / {}^{52}C_4$$

Conditional Probability

EXERCISE

Two dies are thrown simultaneously and the sum of the numbers obtained is found to be 7. What is the probability that the number 3 has appeared at least once?

SOLUTION:

The sample space S would consist of all the numbers possible by the combination of two dies. Therefore S consists of 6×6 i.e. 36 events.

Event A indicates the combination in which 3 has appeared at least once.

Event B indicates the combination of the numbers which sum up to 7.

$$A = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$$

$$B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

$$P(A) = 11/36$$

$$P(B) = 6/36$$

$$A \cap B = 2$$

$$P(A \cap B) = 2/36$$

Applying the conditional probability formula we get,

$$P(A|B) = P(A \cap B) / P(B) = (2/36) / (6/36) = 1/3$$

EXERCISE

A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it. What is the quality of the parts that make it through the inspection machine and get shipped?

SOLUTION:

Let G (resp., SD , OD) be the event that a randomly chosen shipped part is good (resp., slightly defective, obviously defective). We are told that $P(G) = .90$, $P(SD) = 0.02$, and $P(OD) = 0.08$.

We want to compute the probability that a part is good *given* that it passed the inspection machine (i.e., it is *not* obviously defective), which is

$$P(G|OD^c) = \frac{P(G \cap OD^c)}{P(OD^c)} = \frac{P(G)}{1 - P(OD)} = \frac{.90}{1 - .08} = \frac{90}{92} = .978$$

EXERCISE

Your neighbor has 2 children. You learn that he has a son, Joe. What is the probability that Joe's sibling is a brother?

SOLUTION:

The "obvious" answer that Joe's sibling is equally likely to have been born male or female suggests that the probability the other child is a boy is $1/2$. This is not correct!

Consider the experiment of selecting a random family having two children and recording whether they are boys or girls. Then, the sample space is $S = \{BB, BG, GB, GG\}$, where, e.g., outcome "BG" means that the first-born child is a boy and the second-born is a girl. Assuming boys and girls are equally likely to be born, the 4 elements of S are equally likely.

The event, E , that the neighbor has a son is the set $E = \{BB, BG, GB\}$. The event, F , that the neighbor has two boys (i.e., Joe has a brother) is the set $F = \{BB\}$.

We want to compute

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{P(\{BB\})}{P(\{BB, BG, GB\})} = \frac{1/4}{3/4} = \frac{1}{3}.$$

EXERCISE

Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

SOLUTION:

Let D_1 (resp., D_2) be the event that we find a defective fuse in the first (resp., second) test. We want to compute

$$P(D_1 \cap D_2) = P(D_1)P(D_2|D_1) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21}.$$

Total Probability**EXERCISE**

Two cards from an ordinary deck of 52 cards are missing. What is the probability that a random card drawn from this deck is a spade?

SOLUTION:

Let E be the event that the randomly drawn card is a spade. Let F_i be the event that i spades are missing from the 50-card (defective) deck, for $i = 0, 1, 2$.

We want $P(E)$, which we compute by conditioning on how many spades are missing from the original (good) deck:

$$P(E) = P(E|F_0)P(F_0) + P(E|F_1)P(F_1) + P(E|F_2)P(F_2) = \frac{13}{50} \frac{\binom{13}{0}\binom{39}{2}}{\binom{52}{2}} + \frac{12}{50} \frac{\binom{13}{1}\binom{39}{1}}{\binom{52}{2}} + \frac{11}{50} \frac{\binom{13}{2}\binom{39}{0}}{\binom{52}{2}} = \frac{1}{4}$$

Bayes Theorem/ Formula

EXERCISE

Urn 1 contains 5 white balls and 7 black balls. Urn 2 contains 3 whites and 12 black. A fair coin is flipped; if it is Heads, a ball is drawn from Urn 1, and if it is Tails, a ball is drawn from Urn 2. Suppose that this experiment is done and you learn that a white ball was selected. What is the probability that this ball was in fact taken from Urn 2? (i.e., that the coin flip was Tails)

SOLUTION:

Let T be the event that the coin flip was Tails. Let W be the event that a white ball is selected. From the given data, we know that $P(W|T) = 3/15$ and that $P(W|T^c) = 5/12$. Since the coin is fair, we know that $P(T) = P(T^c) = 1/2$.

We want to compute $P(T|W)$, which we do using the definition (and the same simple manipulation that results in Bayes Formula):

$$P(T|W) = \frac{P(T \cap W)}{P(W)} = \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|T^c)P(T^c)} = \frac{(3/15)(1/2)}{(3/15)(1/2) + (5/12)(1/2)} = \frac{12}{37}$$

EXERCISE

One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time. (a). What is the probability that Joe (a random person) tests positive? (b). Joe just got the bad news that the test came back positive; what is the probability that Joe has the disease?

SOLUTION:

Let D be the event that Joe has the disease. Let T be the event that Joe's test comes back positive.

We are told that $P(D) = 0.005$, since 1/2% of the population has the disease, and Joe is just an average guy. We are also told that $P(T|D) = .98$, since 2% of the time a person having the disease is missed ("false negative"). We are told that $P(T|D^c) = .03$, since there are 3% false positives.

(a). We want to compute $P(T)$. We do so by conditioning on whether or not Joe has the disease:

$$P(T) = P(T|D)P(D) + P(T|D^c)P(D^c) = (.98)(.005) + (.03)(.995)$$

(b). We want to compute

$$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} = \frac{(.98)(.005)}{(.98)(.005) + (.03)(.995)} \approx .14$$

What a relief! There is only a 14% chance Joe has the disease, even though the test came back positive! The issue here is that the false-positive and false-negative percentages are in fact high, relative to the occurrence of the disease.