

CHAPTER I

1. VECTORS and SCALARS

1.1 Introduction

Mechanics is a physical science which deals with the state of rest or motion of rigid bodies under the action of forces. It is divided into three parts: mechanics of rigid bodies, mechanics of deformable bodies, and mechanics of fluids. Thus it can be inferred that Mechanics is a physical science which deals with the external effects of force on rigid bodies. Mechanics of rigid bodies is divided into two parts: Statics and Dynamics.

Statics: deals with the equilibrium of rigid bodies under the action of forces.

Dynamics: deals with the motion of rigid bodies caused by unbalanced force acting on them. Dynamics is further subdivided into two parts:

- *Kinematics:* dealing with geometry of motion of bodies with out reference to the forces causing the motion, and
- *Kinetics:* deals with motion of bodies in relation to the forces causing the motion.

Basic Concepts:

The concepts and definitions of **Space, Time, Mass, Force, Particle** and **Rigid body** are basic to the study of mechanics.

In this course, the bodies are assumed to be rigid such that what ever load applied, they don't deform or change shape. But translation or rotation may exist. The loads are assumed to cause only external movement, not internal. In reality, the bodies may deform. But the changes in shapes are assumed to be minimal and insignificant to affect the condition of equilibrium (stability) or motion of the structure under load.

When we deal Statics/Mechanics of rigid bodies under equilibrium condition, we can represent the body or system under a load by a particle or centerline. Thus, the general response in terms of other load of the bodies can be spotted easily.

Fundamental Principles

The three laws of Newton are of importance while studying mechanics:

First Law: A particle remains at rest or continues to move in a straight line with uniform velocity if there is no unbalanced force on it.

Second Law: The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.

$$\begin{array}{ccc} \rightarrow & \rightarrow & \rightarrow \\ F = & m \times & a \end{array}$$

Third Law: The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear.

The first and third laws have of great importance for Statics whereas the second one is basic for dynamics of Mechanics.

Another important law for mechanics is the Law of gravitation by Newton, as it usual to compute the weight of bodies. Accordingly:

$$F = G \frac{m_1 m_2}{r^2} \text{ thus the weight of a mass 'm' } W = mg$$

1.2 Scalars and Vectors

1.2.1 Definition and properties

After generally understanding quantities as Fundamental or Derived, we shall also treat them as either Scalars or Vectors.

Scalar quantities: - are physical quantities that can be completely described (measured) by their magnitude alone. These quantities do not need a direction to point out their application (Just a value to quantify their measurability). They only need the magnitude and the unit of measurement to fully describe them.

E.g. Time[s], Mass [Kg], Area [m²], Volume [m³], Density [Kg/m³], Distance [m], etc.

Vector quantities: - Like Scalar quantities, Vector quantities need a magnitude. But in addition, they have a direction, and sometimes point of application for their complete description. Vectors are represented by short arrows on top of the letters designating them.

E.g. Force [N, Kg.m/s²], Velocity [m/s], Acceleration [m/s²], Momentum [N.s, kg.m/s], etc.

1.2.2 Types of Vectors

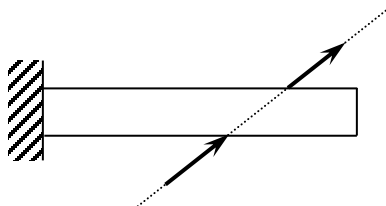
Generally vectors fall into the following three basic classifications:

Free Vectors: are vectors whose action in space is not confined or associated with a unique line in space; hence they are 'free' in space.

E.g. Displacement, Velocity, Acceleration, Couples, etc.

Sliding Vectors: are vectors for which a unique line in space along the action of the quantity must be maintained.

E.g. Force acting on rigid bodies.



NB: From the above we can see that a force can be applied any where along its line of action on a rigid body with out altering its external effect on the body. This principle is known as **Principle of Transmissibility**.

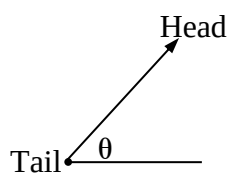
Fixed Vectors: are vectors for which a unique and well-defined point of application is specified to have the same external effect.

E.g. Force acting on non-rigid (deformable) bodies.

1.2.3 Representation of Vectors

A) Graphical representation

Graphically, a vector is represented by a directed line segment headed by an arrow. The length of the line segment is equal to the magnitude of the vector to some predetermined scale and the arrow indicates the direction of the vector.



Length of the line equals, to some scale, the magnitude of the vector and the arrow indicates the direction of the vector

NB: The direction of the vector may be measured by an angle θ from some known reference direction.

B) Algebraic (arithmetic) representation

Algebraically a vector is represented by the components of the vector along the three dimensions.

E.g.:

$\vec{A} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$, Where a_x , a_y and a_z are components of the vector $\vec{A}$ along the x, y and z axes respectively.

NB: The vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the respective axes.

$$a_x = A \cos \theta_x = Al, \quad l = \cos \theta_x$$

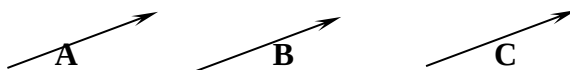
$$a_y = A \cos \theta_y = Am, \quad m = \cos \theta_y$$

$$a_z = A \cos \theta_z = An, \quad n = \cos \theta_z, \text{ where } l, m, n \text{ are the directional cosines of the vector. Thus,}$$

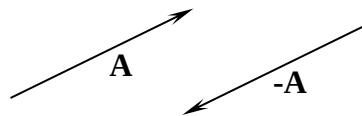
$$A^2 = a_x^2 + a_y^2 + a_z^2 \Rightarrow l^2 + m^2 + n^2 = 1$$

Properties of vectors

Equality of vectors: Two free vectors are said to be equal if and only if they have the same magnitude and direction.



The Negative of a vector: is a vector which has equal magnitude to a given vector but opposite in direction.



Null vector: is a vector of zero magnitude. A null vector has an arbitrary direction.

Unit vector: is any vector whose magnitude is unity.

A unit vector along the direction of a certain vector, say vector **A** (denoted by $\mathbf{u}_A$) can then be found by dividing vector **A** by its magnitude.

$$\vec{u}_A = \frac{\vec{A}}{\|\vec{A}\|}$$

Generally, any two or more vectors can be aligned in different manner. But they may be:

- * Collinear- Having the same line of action.
- * Coplanar- Lying in the same plane.
- * Concurrent- Passing through a common point.

1.3 Operations with Vectors

Scalar quantities are operated in the same way as numbers are operated. But vectors are not and have the following rules:

1.3.1 Vector Addition or Composition of Vectors

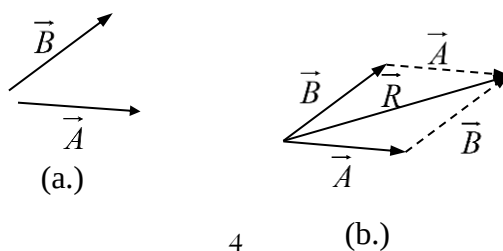
Composition of vectors is the process of adding two or more vectors to get a single vector, a Resultant, which has the same external effect as the combined effect of individual vectors on the rigid body they act.

There are different techniques of adding vectors

A) Graphical Method

I. The parallelogram law

The law states, “if $\vec{A}$ and $\vec{B}$ are two free vectors drawn on scale, the resultant (the equivalent vector) of the vectors can be found by drawing a parallelogram having sides of these vectors, and the resultant will be the diagonal starting from the tails of both vectors and ending at the heads of both vectors.”



Once the parallelogram is drawn to scale, the magnitude of the resultant can be found by measuring the diagonal and converting it to magnitude by the appropriate scale. The direction of the resultant with respect to one of the vectors can be found by measuring the angle the diagonal makes with that vector.

Note: As we can see in the above figure.

$$\vec{A} + \vec{B} = \vec{R} = \vec{B} + \vec{A}, \Rightarrow \text{vector addition is commutative}$$

The other diagonal of the parallelogram gives the difference of the vectors, and depending from which vertex it starts, it represents either $\vec{A} - \vec{B}$ or $\vec{B} - \vec{A}$



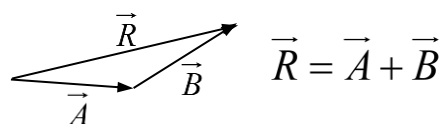
Since the two diagonal vectors in the above figure are not equal, of course one is the negative vector of the other, vector subtraction is not commutative.

i.e. $\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$

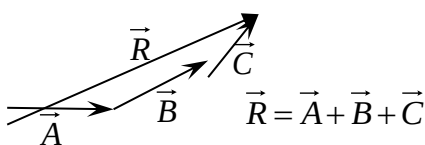
NB. Vector subtraction is addition of the negative of one vector to the other.

II. The Triangle rule

The Triangle rule is a corollary to the parallelogram axiom and it is fit to be applied to more than two vectors at once. It states “If the two vectors, which are drawn on scale, are placed tip (head) to tail, their resultant will be the third side of the triangle which has tail at the tail of the first vector and head at the head of the last.”



Thus the Triangle rule can be extended to more than two vectors as, “If a system of vectors are joined head to tail, their resultant will be the vector that completes the polygon so formed, and it starts from the tail of the first vector and ends at the head of the last vector.”



NB. From the Triangle rule it can easily be seen that if a system of vectors when joined head to tail form a closed polygon, their resultant will be a null vector.

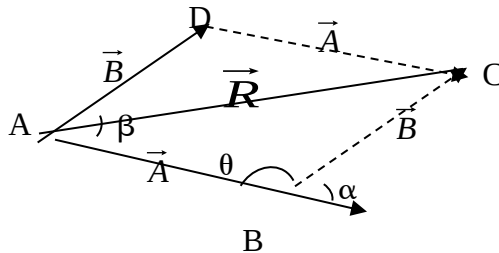
III. Analytic method.

The analytic methods are the direct applications of the above postulates and theorems in which the resultant is found mathematically instead of measuring it from the drawings as in the graphical method.

A. Trigonometric rules:

The resultant of two vectors can be found analytically from the parallelogram rule by applying the cosine and the sine rules.

Consider the following parallelogram. And let θ be the angle between the two vectors



Consider triangle ABC

From cosine law,

$$\Rightarrow \|\vec{R}\|^2 = \|\vec{A}\|^2 + \|\vec{B}\|^2 - 2\|\vec{A}\|\|\vec{B}\|\cos(\theta)$$

$$\Rightarrow \|\vec{R}\| = \sqrt{\|\vec{A}\|^2 + \|\vec{B}\|^2 - 2\|\vec{A}\|\|\vec{B}\|\cos(\theta)},$$

This is the magnitude of the RESULTANT of the two vectors,

Similarly, the inclination, β , of the resultant vector from $\vec{A}$ can be found by using sine law

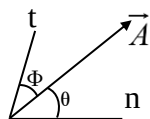
$$\frac{\sin \beta}{\|\vec{B}\|} = \frac{\sin \theta}{\|\vec{R}\|} \Rightarrow \beta = \sin^{-1} \left[\sin \theta * \frac{\|\vec{B}\|}{\|\vec{R}\|} \right],$$

, which is the angle the resultant makes with vector A.

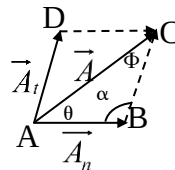
Decomposition of vectors:

Decomposition is the process of getting the components of a given vector along some other different axis. Practically decomposition is the reverse of composition.

Consider the following vector $\vec{A}$. And let our aim be to find the components of the vector along the n and t axes.



(a)



(b)

From Triangle ABC @ (B), $\alpha = 180 - (\theta + \phi)$

From sine law then,

$$\frac{\|\vec{A}_n\|}{\sin \phi} = \frac{\|\vec{A}\|}{\sin \alpha} \Rightarrow \|\vec{A}_n\| = \|\vec{A}\| \frac{\sin \phi}{\sin \alpha}$$

$$\text{Similarly, } \|\vec{A}_t\| = \|\vec{A}\| \frac{\sin \theta}{\sin \alpha}$$

The above are general expressions to get the components of a vector along any axis. In most cases though, components are sought along perpendicular axes, i.e. $\alpha = 180 - (\theta + \phi) = 90$

$$\Rightarrow \sin \alpha = 1$$

$$\Rightarrow \|\vec{A}_n\| = \|\vec{A}\| \sin \phi = \|\vec{A}\| \cos \theta$$

$$\|\vec{A}_t\| = \|\vec{A}\| \sin \theta = \|\vec{A}\| \cos \phi$$

B. Component method of vector addition

This is the most efficient method of vector addition, especially when the number of vectors to be added is large. In this method first the components of each vector along a convenient axis will be calculated. The sum of the components of each vector along each axis will be equal to the components of their resultant along the respective axes. Once the components of the resultant are found, the resultant can be found by parallelogram rule as discussed above.

1.3.2. Vector Multiplication: Dot and Cross products

I) Multiplication of vectors by scalars

Let n be a non-zero scalar and $\vec{A}$ be a vector, then multiplying $\vec{A}$ by n gives as a vector whose magnitude is $n\|\vec{A}\|$ and whose direction is in the direction of $\vec{A}$ if n is positive or is in opposite direction to $\vec{A}$ if n is negative.

Multiplication of vectors by scalars obeys the following rules:

i. Scalars are distributive over vectors.

$$n(\vec{A} + \vec{B}) = n\vec{A} + n\vec{B}$$

ii. Vectors are distributive over scalars.

$$(n + m)\vec{A} = n\vec{A} + m\vec{A}$$

iii. Multiplication of vectors by scalars is associative.

$$(nm)\vec{A} = n(m\vec{A}) = m(n\vec{A})$$

II) Multiplication of vector by a vector

In mechanics there are a few physical quantities that can be represented by a product of vectors.
Eg. Work, Moment, etc

There are two types of products of vector multiplication

III) Dot Product: Scalar Product

The scalar product of two vectors A and B which are θ degrees inclined from each other denoted by

A.B (A dot B) will result in a scalar of magnitude $\|\vec{A}\|\|\vec{B}\|\cos\theta$

$$\text{i.e. } \vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\|\cos\theta$$

If the two vectors are represented analytically as

$$\vec{A} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \quad \text{and} \quad \vec{B} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}, \text{ then}$$

$$\vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z$$

IV) Cross Product: Vector Product

The vector product of two vectors A and B that are θ degrees apart denoted by $\vec{A} \times \vec{B}$ (A cross B) is a

vector of magnitude $\|\vec{A}\|\|\vec{B}\|\sin\theta$ and direction perpendicular to the plane formed by the vectors A and B. The sense of the resulting vector can be determined by the right-hand rule.

$$\text{i.e. } \vec{A} \times \vec{B} = \|\vec{A}\|\|\vec{B}\|\sin\theta, \text{ perpendicular to the plane formed by } \vec{A} \text{ and } \vec{B}$$

If the two vectors are represented analytically as,

$$\vec{A} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \quad \text{and} \quad \vec{B} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$$

then the cross product $\vec{A} \times \vec{B}$ will be the determinant of the three by three matrix as,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y) \vec{i} + (a_z b_x - a_x b_z) \vec{j} + (a_x b_y - a_y b_x) \vec{k}$$

NB. Vector product is not commutative; in fact, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

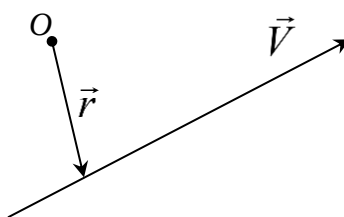
Moment of a Vector

The moment of a vector V about any point O is given by:

$$\vec{M}_O = \vec{r} \times \vec{V}$$

Where: $\vec{r}$ is a position vector from point O to any point on the line of action of the vector.

$$\vec{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ V_x & V_y & V_z \end{vmatrix}$$



Position vector $\vec{r}$ is defined as a fixed vector that locates a point in space relative to another point in space.

CHAPTER TWO

2. FORCE SYSTEMS

2.1 Introduction

Defⁿ: A force can be defined as the action of one body on another that changes/tends to changes the state of the body acted on.

A force can be applied on a body as;

Contact force:-Applied by direct mechanical contact of the acting body on the acted one (Created by push and pull).

Remote action (Body force):-Applied by remote action as in gravitational, electrical, Magnetic, etc forces.

The action of a force on a body can be divided as internal and external. Internal force is a force exerted by one part of a body on another part of the same body. External force is a force exerted on a body by some other body. An external force can then be applied on a body as:

- Applied force
- Reactive force

In Engineering mechanics, only external effects of forces, hence external forces are considered.

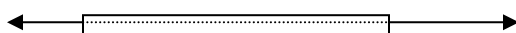
2.1.1. Force systems

A system of forces can be grouped into different categories depending on their arrangement in space.

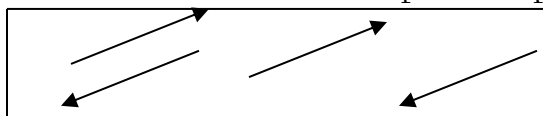
Coplanar Forces:-are forces which act on the same plane.

Depending on their arrangement on the plane too, coplanar forces can further be divided as:

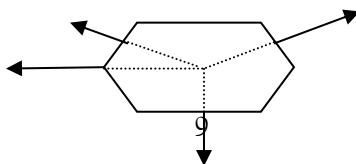
Coplanar collinear forces:-are coplanar forces acting on the same line-collinear.



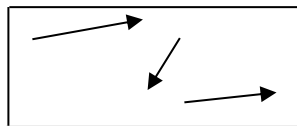
Coplanar parallel forces:-Are forces which are on the same plane and parallel



Coplanar concurrent forces:-Are forces on the same plane whose lines of action intersect at a point.



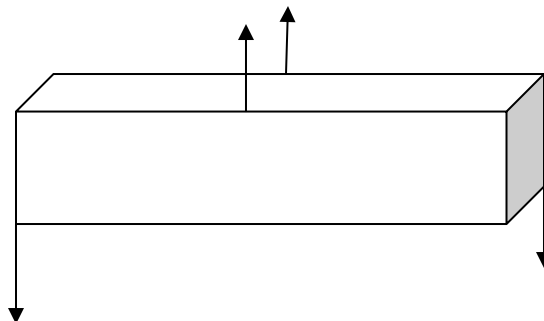
General coplanar forces:



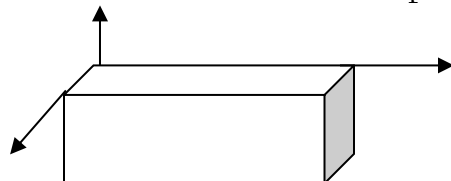
Non coplanar forces:-are
Again it can further be broken as

forces which act on different planes

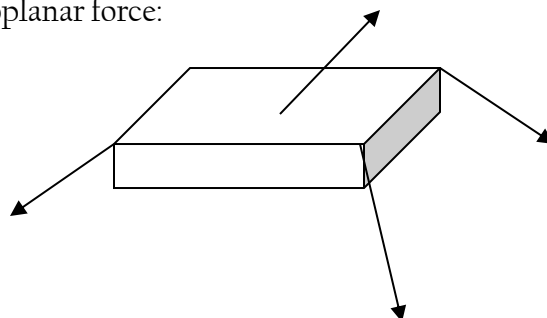
I. Non coplanar parallel forces:-is system of non planar forces but which are parallel.



II. Non-coplanar concurrent forces:-are non planar forces whose lines of action meet at a point.



III. General Non coplanar force:



2.1.2. Composition and Resolution of Forces

2.1.2.1. Composition of forces

Composition of forces is the process of combining two or more forces in to a single resultant force, which has the same external effect as that of the applied system of forces.

In the previous chapter we defined force to be either sliding or fixed vector depending on what type of bodies it acts-rigid or deformable bodies respectively.

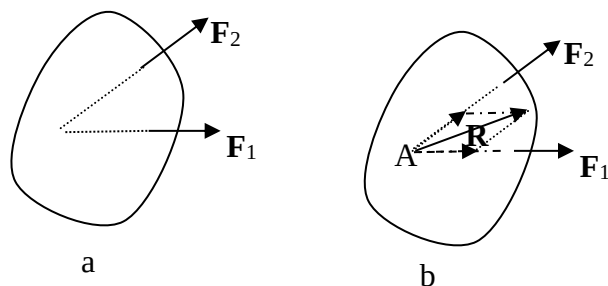
In engineering mechanics we will be considering rigid bodies only; hence we can treat force as sliding vector.

As discussed in the previous chapter, we have two laws of adding vectors:

- The parallelogram rule
- The triangle rule

The parallelogram rule

Consider the following planar force systems acting on the rigid body.



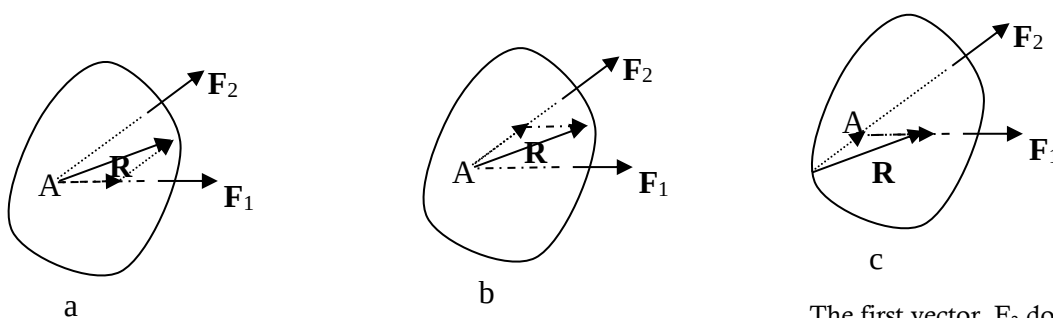
To apply the parallelogram rule of vector addition, the vectors should be placed in such a way that they form a parallelogram and they shouldn't change their external effect. The principle of transmissibility states that "A force may be applied at any point on its line of action, without altering the resultant effects of the force external to the rigid body on which it acts". Thus, by the principle of Transmissibility we can move each force on its line to meet at A without affecting the external effect as shown in (b) above.

Once the parallelogram is formed the resultant can be found as in the previous chapter and its line of action will pass through A .

The triangle rule

The triangle law can also be used to find the resultant of the above forces, but it needs moving of the line of action of one of the forces.

We can change the line of action of either force, but the start of the first vector should coincide with the point of intersection of the line of action of the forces.



F_2 moved out of its line of action but F_1 starts at A

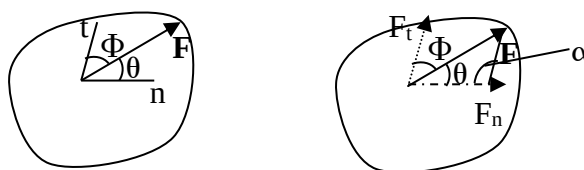
F_1 moved out of its line of action but F_2 starts at A

The first vector, F_2 doesn't start at the intersection of the line of action of the two vectors

As can be seen in the (a) and (b) parts the first force in the combination starts at the intersection of the line of action of the forces. In the (c) part, however, the first force F_2 doesn't start at the intersection. Although the resulting resultant has the same magnitude and direction as the previous ones, its line of action is different; hence its external effect is also different.

2.1.2.2. Resolution of forces

Resolution as defined earlier is the reverse of composition. It is the process of getting the components of a vector along different axes.



$$F_n = \|F\| \frac{\sin \phi}{\sin \alpha}$$

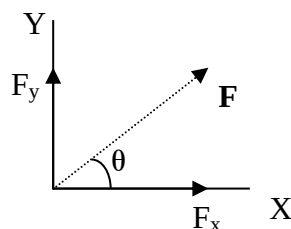
$$F_t = \|F\| \frac{\sin \theta}{\sin \alpha}$$

As derived earlier

2.2. Two-Dimensional Force Systems

2.2.1. Rectangular components

Rectangular components of a force are the components of the force along the rectangular coordinate axes.



As $\alpha=90$,

$$F_x = F \sin(90-\theta) = F \cos \theta$$

$$F_y = F \sin \theta$$

Algebraically, a force is represented by its scalar component along the coordinate axes and a unit vector along that axis.

$$\vec{F} = F_x \vec{i} + F_y \vec{j}$$

Where F_x and F_y are the scalar components of F along the X and Y -axes, and $\vec{i}$ and $\vec{j}$ are unit vectors along the x and y -axes respectively.

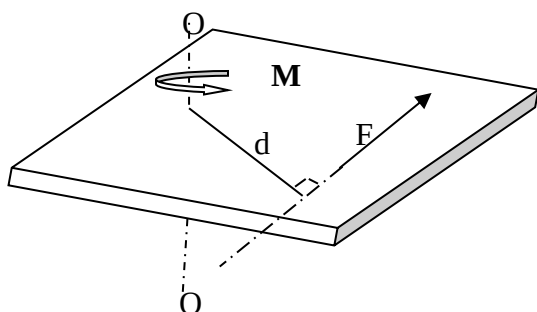
NB. Depending to which quadrant the vector corresponds; the scalar components can be negative.

2.2.2. Moments and Couples

I) Moment

In addition to its tendency to move a body in the direction of its application, a force also tends to rotate the body about any axis which doesn't intersect the line of action of the force and which is not parallel to it. This tendency of a force to rotate a body about a given axis is known as the **moment**, M , of the force. The moment of a force is also known as **torque**.

The fig below shows a two-dimensional body acted upon by a force F in its plane.



The axis of the force to rotate the body about the axis $O-O$ normal to the plane of the body –hence the moment is proportional both to the magnitude of the force and the moment arm d , which is the perpendicular distance from the axis to the line of action of the force.

Hence the magnitude of the moment is defined as:

$$M = F \times d$$

The moment will be a vector perpendicular to the plane of the body-parallel to the axis $o-o$ -and its sense depends on the direction in which the force tends to rotate the body. The right hand rule can be used to identify this sense; curl your fingers in the direction of the tendency to rotate, the thumb will point in the direction of the moment vector.

For a coplanar force system vector representation of the moment is unnecessary as it can be represented as its tendency to rotate that plane-clockwise or counter clockwise.

Sign convention

In representing moment by its tendency to rotate, it is a good practice to assign one of the senses, clockwise or counter clockwise, a positive direction and the other negative.

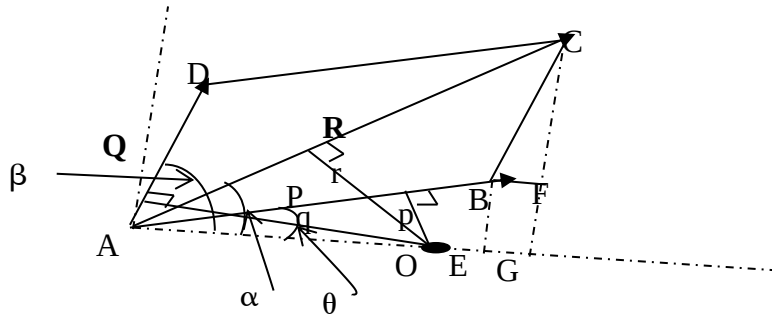
Here we will be treating counter clockwise moment as positive moment and clockwise moment as negative moment.

Note: - One can assign the positive sense to either the clockwise or counterclockwise moments. What is important is in a given problem; he should stick to his assignment.

Principle of moment: - One of the most important principles in mechanics is Varignon's theorem, or the principle of moment, which for coplanar forces states "the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point."

Proof

Let P and Q be the components of R as shown in fig below



Let point O be any arbitrary point in the plane of the forces through which the moments are sought. Draw the x-y axes by coinciding the x-axis with the line joining O with A.

$$CG = GF + CF = R \sin \alpha$$

$$CF = P \sin \beta$$

$$BE = GF = Q \sin \theta$$

$$\Rightarrow R \sin \alpha = P \sin \beta + Q \sin \theta$$

Constructing perpendicular distances (moment arms) to all the forces from O:

$$\sin \theta = \frac{p}{AO}, \sin \alpha = \frac{r}{AO} \text{ \& } \sin \beta = \frac{q}{AO}$$

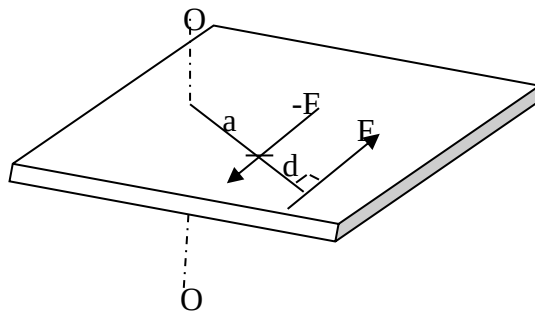
$$\Rightarrow R \frac{r}{AO} = P \frac{p}{AO} + Q \frac{q}{AO}$$

$$\Rightarrow Rr = Pp + Qq, \text{ Which proves Varignon's theorem}$$

In case it is easier to find the moments of the components of a force about an axis through a point, one can then easily determine the moment of the force about an axis through that same point by applying the principle of moment.

II) Couples

The moment produced by two equal and opposite and non-collinear forces is known as couple. Consider the action of equal and opposite forces F and $-F$ a distance d apart.



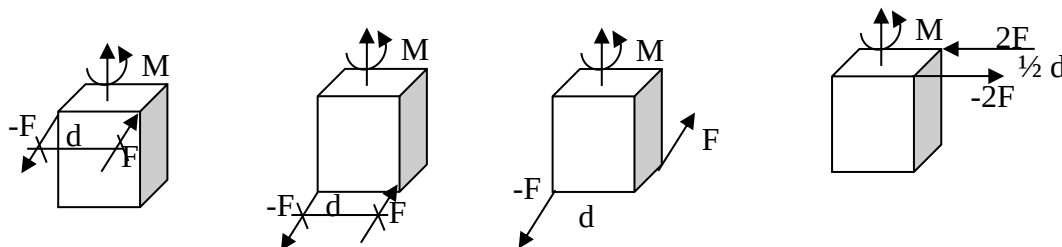
These two forces can't be combined into a single force of the same effect on the body, as their sum in every direction is zero. But the effect of the forces on the body isn't zero.

The combined moment of the two forces about an axis normal to their plane and passing through any point such as o in their plane is the couple, M .

The magnitude of the couple $M = F(a + d) - Fa$
 $= Fd$

It can, therefore, be concluded that the moment of a couple is independent of the moment center selected-hence a couple can be represented as free vector.

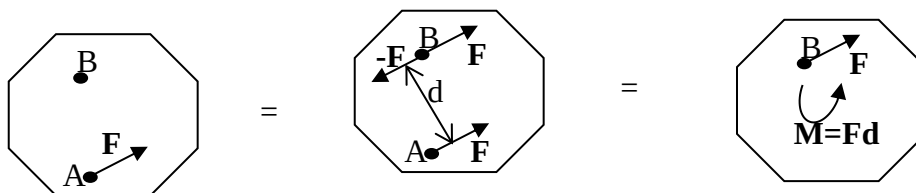
NB. A couple is unchanged as long as the magnitude and direction of its vector remains constant, i.e. a given couple will not be altered by changing the value of F and d as long as their product remains the same. Likewise a couple is not affected by allowing the forces act in any one of parallel planes.



III) Resolution of a force into a force and a couple

The effect of a force on a body has been described in terms of the tendency to push or pull the body in the direction of the force and to rotate the body about any axis which doesn't intersect the line of the force. The representation of this dual effect can be facilitated by replacing the given force by an equal and parallel force at the new point sought and a couple to compensate for the change in the moment of the forces.

The resolution process can best be illustrated by the following figures.



The given force F acting at A is replaced by an equal force at point B and the anti clockwise couple $M = F \times d$. The transfer process can be seen from the middle figure and it involves the following procedure:

- Apply two equal and apposite forces of F and $-F$ at B where F is equal in magnitude and parallel to the force acting at A .
- The forces applied at B will cancel each other -hence they will have no effect on the body.
- The Forces F at A and $-F$ at B form a couple; hence can be replaced by the counter clockwise couple $M = F \times d$

The original force at A can be replaced by an equal and parallel force at B and a corresponding couple as shown in the right figure.

Note: - The transformation described above can be performed in the reverse order. i.e. A force F acting at a point B and a couple M acting on the body can be combined into a single resultant force. This is

performed by moving F until its moment about B becomes equal to the moment M of the couple to be excluded.

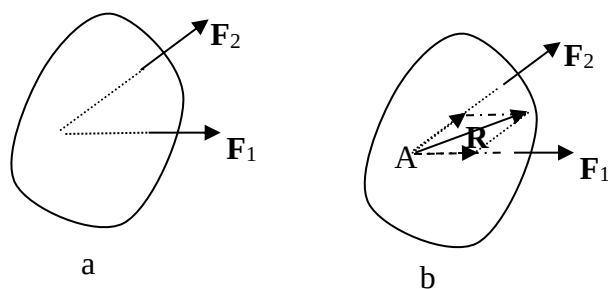
2.2.3. Resultants of general coplanar force systems

The resultant of a force system is the simplest force combination that can replace the original forces without altering the external effect of the system on the rigid body to which the forces are applied.

The equilibrium of a body is the condition where the resultant if all forces that act on it is zero. When the resultant is not zero, the acceleration of the body is described by equating the force resultant to the product of the mass and the acceleration of the body. Thus, the determination of the resultant is basic to both statics and dynamics.

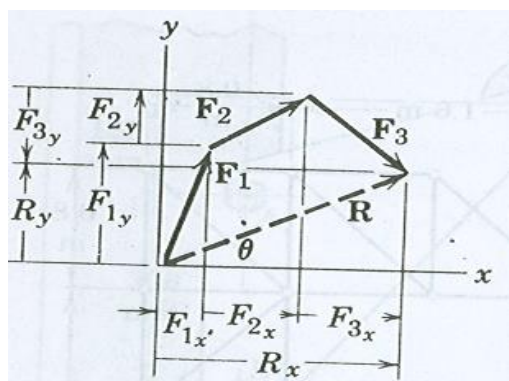
The most common type of force system occurs when the forces all act in a single plane (coplanar forces). The resultant can be computed by using the parallelogram rule or using analytical methods.

A. Parallelogram rule



B.

Analytic Method



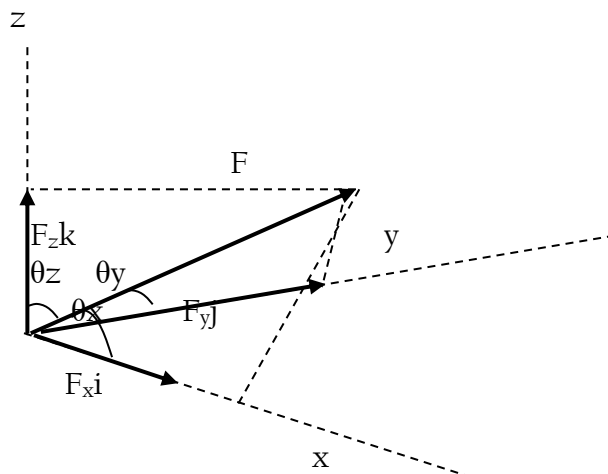
$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \dots = \Sigma \mathbf{F}$$

$$R_x = \Sigma F_x \quad R_y = \Sigma F_y \quad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$\theta = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \frac{\Sigma F_y}{\Sigma F_x}$$

2.3. Three-Dimensional Force Systems

2.3.1. Rectangular Components



$F_x i$, $F_y j$ and $F_z k$ are rectangular components of F . Thus,

$$\vec{F} = F_x i + F_y j + F_z k, F_x = F \cos \theta_x, F_y = F \cos \theta_y, F_z = F \cos \theta_z$$

$$\vec{F} = F \cos \theta_x i + F \cos \theta_y j + F \cos \theta_z k = F (\cos \theta_x i + \cos \theta_y j + \cos \theta_z k)$$

$\underbrace{\hspace{10em}}_{\vec{n} = \text{unit vector in the direction of } F}$

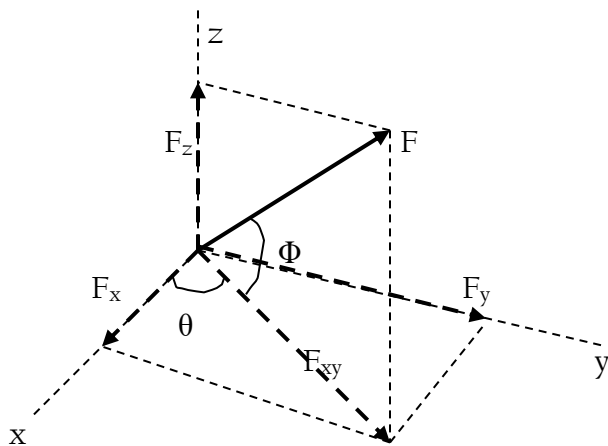
$$\vec{F} = \vec{n} F, F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

In solving three-dimensional problems, one must usually find the x, y and z scalar components of a given or unknown force. In most cases, the direction of a force is described;

- A/ by two points on the line of action of the force, or
- B/ by two angles which orient the line of action.

A. Using two points: If the coordinates of points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ on the line of action of the force are known and the direction of the force is from A to B, the force may be written as;

$$\vec{F} = F \vec{n}_{AB} = F \frac{\vec{AB}}{|\vec{AB}|} = F \frac{(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

B. Using two angles:

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

$$F_{xy} = F \cos \Phi, F_z = F \sin \Phi, F_x = F_{xy} \cos \theta = F \cos \Phi \cos \theta, F_y = F_{xy} \sin \theta = F \cos \Phi \sin \theta$$

Rectangular components of a force $\vec{F}$ may be written with the aid of dot or scalar product operation. If the unit vector $\vec{n} = \alpha \vec{i} + \beta \vec{j} + \gamma \vec{k}$ and $\vec{F} = F(\vec{l} + \vec{m} + \vec{n})$, the projection of $\vec{F}$ in the $\vec{n}$ direction is given by:

i/ As a scalar

$$F_n = \vec{F} \cdot \vec{n} = F(\vec{l} + \vec{m} + \vec{n}) \cdot (\alpha \vec{i} + \beta \vec{j} + \gamma \vec{k}) = F(l\alpha + m\beta + n\gamma)$$

ii/ As a vector

$$\vec{F}_n = (\vec{F} \cdot \vec{n}) \vec{n}$$

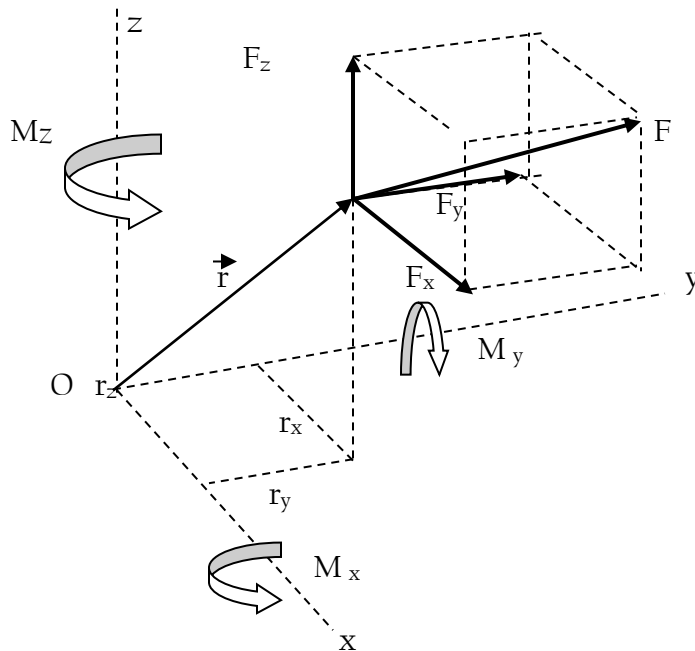
$$\text{If } \theta \text{ is the angle between } \vec{F} \text{ and } \vec{n}, \quad \vec{F} \cdot \vec{n} = F n \cos \theta, \quad \theta = \cos^{-1}((\vec{F} \cdot \vec{n}) / (|\vec{F}| |\vec{n}|))$$

It should be observed that the dot product relationship applies to non intersecting vectors as well as to intersecting vectors.

2.3.2. Moment and Couple

In three dimensions, the determination of the perpendicular distance between a point or line and the line of action of the force can be a tedious computation. The use of a vector approach using cross-product multiplication becomes advantageous.

I) Moment



The moment $\vec{M}_O$ of $\vec{F}$ about an axis through O is given by;

$$\vec{M}_O = \vec{r} \times \vec{F} = (r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k}) \times (F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (r_y F_z - r_z F_y) \mathbf{i} + (r_z F_x - r_x F_z) \mathbf{j} + (r_x F_y - r_y F_x) \mathbf{k}$$

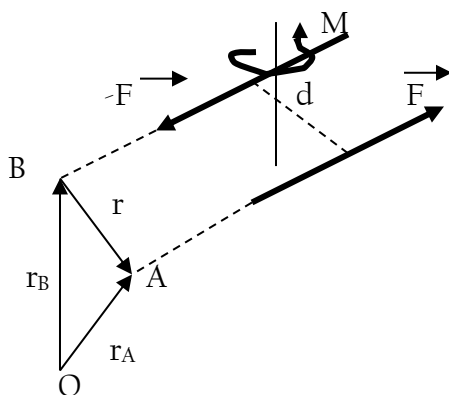
$$= M_x \mathbf{i} + M_y \mathbf{j} + M_z \mathbf{k}$$

If $\vec{n}$ is a unit vector in the λ direction, the moment $\vec{M}_\lambda$ of $\vec{F}$ about any axis λ through O is expressed by;

$$\vec{M}_\lambda = \vec{M}_O \cdot \vec{n} = (\vec{r} \times \vec{F} \cdot \vec{n}) \quad \text{--- which is the scalar magnitude. Or} \quad \rightarrow$$

$$\vec{M}_\lambda = (\vec{M}_O \cdot \vec{n}) \vec{n} = (\vec{r} \times \vec{F} \cdot \vec{n}) \vec{n} \quad \text{--- vector expression for the moment of F about an axis } \lambda \text{ through O.}$$

II) Couple



If the vector $\vec{r}$ joins any point B on the line of action of $-\vec{F}$ to any point A on the line of action of $\vec{F}$. The combined moment (couple) of the two forces about O is;

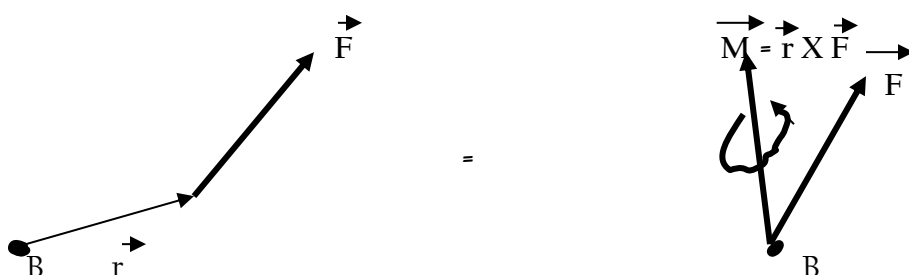
$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times (-\vec{F}) = (\vec{r}_A - \vec{r}_B) \times \vec{F} = \vec{r} \times \vec{F}$$

The moment of the couple, $M = r \times F$. It is the same about all points. Thus,

- The moment of a couple is a free vector, whereas the moment of a force about a point (which is also the moment about a defined axis through the point) is a sliding vector whose direction is along the axis through the point.
- A couple tends to produce a pure rotation of the body about an axis normal to the plane of the forces which constitute the couple.

III) Resolution of a force into a couple and a force.

Force-Couple System



The force $\vec{F}$ at point A is replaced by an equal force $\vec{F}$ at point B and the couple $\vec{M} = \vec{r} \times \vec{F}$.

- Couple vectors obey all of the rules, which govern vector quantities.

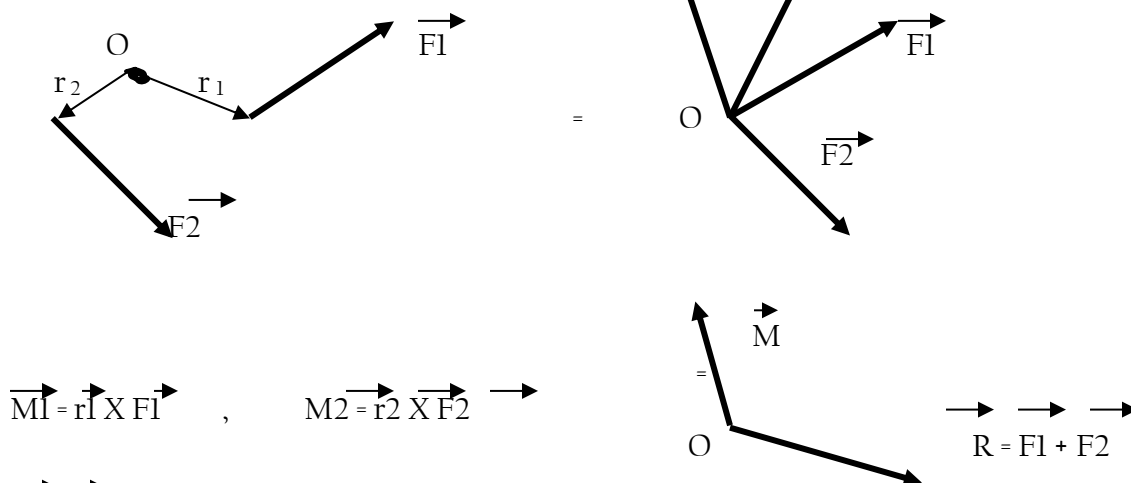
2.3.3. Resultants

Any system of forces may be replaced by its resultant force $\vec{R}$ and the resultant couple $\vec{M}$. For the system of forces $F_1, F_2, F_3, \dots$ acting on a rigid body, the resultant force $\vec{R}$ and the resultant couple $\vec{M}$ is given by;

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots = \sum \vec{F}_i$$

$$\vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \dots = \sum \vec{M}_i$$

Eg.



$\vec{r}_1, \vec{r}_2$ = vectors from O to any point on the line of action of $\vec{F}_1$ and $\vec{F}_2$ respectively. In three dimensions, the magnitudes of the resultants and their components are;

$$R_x = \sum F_x, R_y = \sum F_y, R_z = \sum F_z \quad R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2 + (\sum F_z)^2}$$

$$M_x = \sum (\vec{r} \times \vec{F})_x, M_y = \sum (\vec{r} \times \vec{F})_y, M_z = \sum (\vec{r} \times \vec{F})_z \quad M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

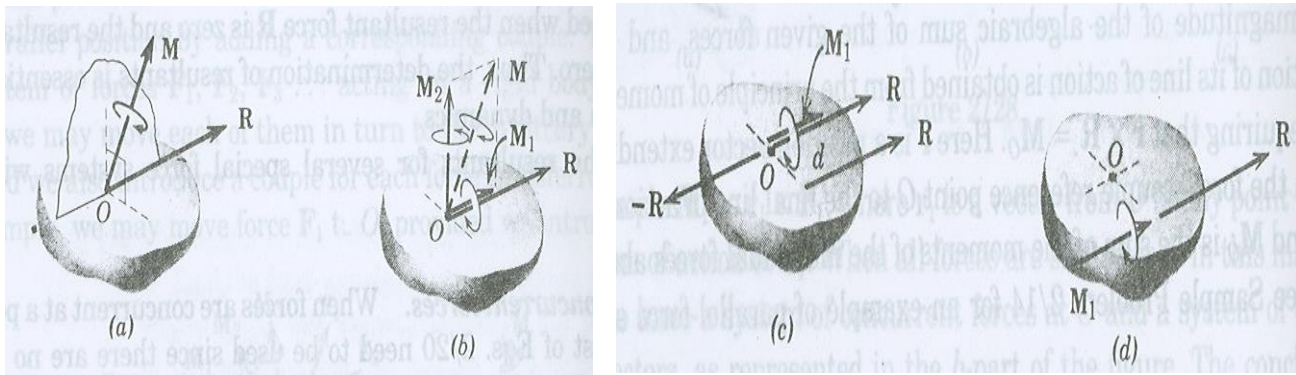
The magnitude and direction of M depends on the particular point selected. The magnitude and direction of R , however, are the same no matter which point is selected.

Wrench Resultant

When the resultant couple vector M is parallel to the resultant force R , as shown in figure below, the resultant is said to be a wrench. A common example of a positive wrench is found with the application of screwdriver.



Any general force system may be represented by a wrench applied along a unique line of action as shown below.



CHAPTER III

EQUILIBRIUM

3.1 Introduction

In the previous chapter, we have seen systems of forces. In this chapter stability of force systems, named as equilibrium of a body. Thus a body is said to be in equilibrium when the resultant of all the forces acting on it is zero. That is, the resultant force vector $\vec{R}$ and the resultant couple vector $\vec{M}$ are both zero.

Expressed mathematically:

$$\vec{R} = \sum \vec{F} = 0 ; \quad \vec{M} = \sum \vec{M} = 0$$

Note that these are both necessary and sufficient conditions for equilibrium.

3.2 Equilibrium in Two-Dimension

3.2.1 Mechanical system isolation and free body diagram (FBD)

Before considering equilibrium conditions, it is very much essential and absolutely necessary to define unambiguously the particular body or mechanical system to be analyzed and represent clearly and completely all forces which act on the body.

Modeling the action of forces in Two – Dimensional Analysis

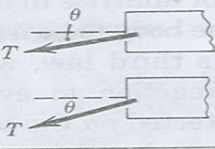
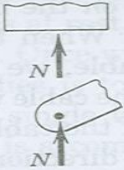
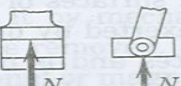
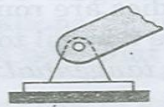
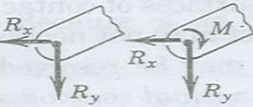
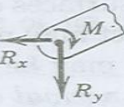
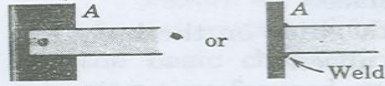
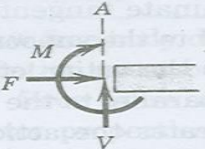
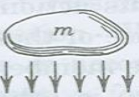
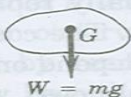
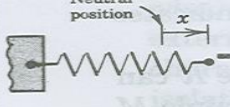
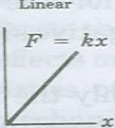
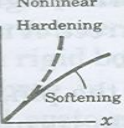
The most important step in drawing the free body diagram will be to show the external forces exerted on the rigid body. One of the external forces will be forces exerted by contacts with supports and reactions. The different support and contact forces are shown in the figure below.

A diagram showing a body/group of bodies considered in the analysis with all forces and relevant dimensions is called free body diagram (FBD). It is after such diagram is clearly drawn that the equilibrium equations be used to determine some of the unknown forces.

Therefore free body diagram is the most important single step in the solution of problems in mechanics.

Steps for the construction of free body diagram

- Decide which body or combination of bodies to be considered.
- The body or combination chosen is isolated by a diagram that represents its complete external boundary.
- All forces that act on the isolated body by the removed contacting and attracting bodies and known forces represented in their proper positions on the diagram of the isolated body.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	Action on Body to be Isolated
1. Flexible cable, belt, chain, or rope Weight of cable negligible  Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.
6. Pin connection 	Pin free to turn  Pin not free to turn  A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the axis; usually shown as two components R_x and R_y. A pin not free to turn may also support a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center of mass G.
9. Spring action Neutral position  Linear  Nonlinear 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

- Each unknown force should be represented by a vector arrow with the unknown magnitude and direction.
- The force exerted on the body to be isolated by the body to be removed is indicated and its sense shall be opposite to the movement of the body which would occur if the contacting or supporting member were removed.

- The choice of coordinate axes should be indicated directly on the diagram and relevant dimensions should be represented.

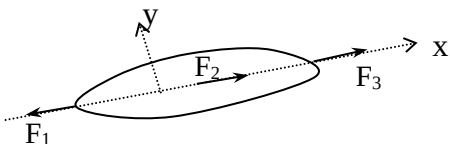
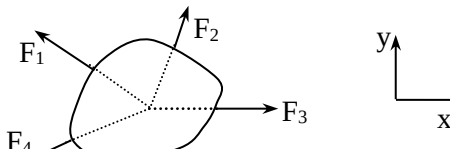
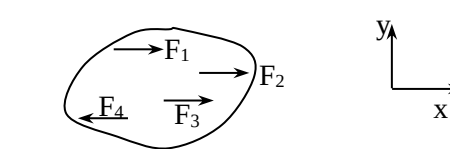
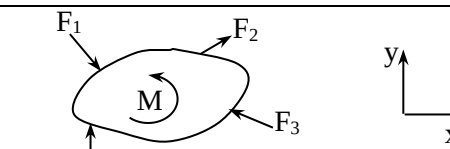
3.2.2 Equilibrium Conditions

It was stated that a body is in equilibrium if the resultant force vector and the resultant couple vector are both zero. These requirements can be stated in the form of vector equation of equilibrium, which in two dimensions can be written as:

$$\Sigma F_X = 0 \quad \Sigma F_Y = 0 \quad \Sigma M_O = \Sigma M_Z = 0$$

Categories of Equilibrium in Two Dimensions

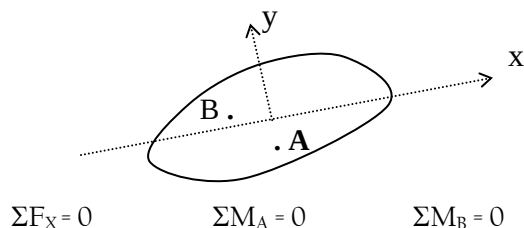
The following categories of equilibrium conditions can be identified due to the nature of forces considered.

<i>Categories of equilibrium in two-dimension</i>		
Force System	Free Body diagram	Independent Equations
1. Collinear		$\Sigma F_X = 0$
2. Concurrent		$\Sigma F_X = 0$ $\Sigma F_Y = 0$
3. Parallel		$\Sigma F_X = 0$ $\Sigma M_Z = 0$
4. General		$\Sigma F_X = 0$ $\Sigma F_Y = 0$ $\Sigma M_Z = 0$

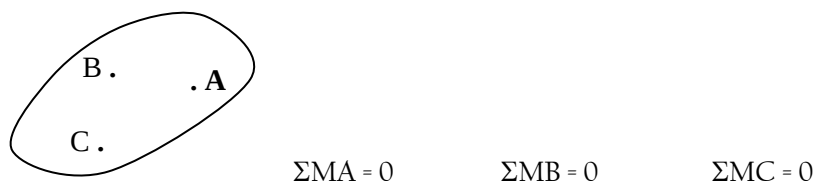
Alternative equilibrium equations

In two-dimensional body, the maximum number of unknown variables is three. And the three equilibrium equations are sufficient to solve the unknown variables. Thus, whatever the combination, three total equations are maximally needed. What we have seen is two forces and one moment equations. But we could have the following combinations.

One force and two moment equations:



Three-moment equation



3.3. Equilibrium in Three-Dimensions

3.3.1. Equilibrium Conditions

The necessary and sufficient conditions for complete equilibrium in three dimensions are;

$$\Sigma \vec{F} = 0 \quad \text{OR} \quad \Sigma F_x = 0, \Sigma F_y = 0 \text{ and } \Sigma F_z = 0$$

$$\text{and } \Sigma \vec{M} = 0 \quad \text{OR} \quad \Sigma M_x = 0, \Sigma M_y = 0 \text{ and } \Sigma M_z = 0$$

Notes; -In applying the vector form of the above equations, we first express each of the forces in terms of the coordinate unit vectors i, j and k .

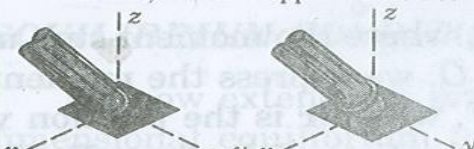
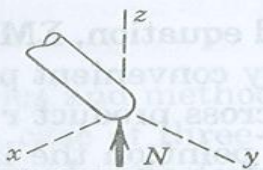
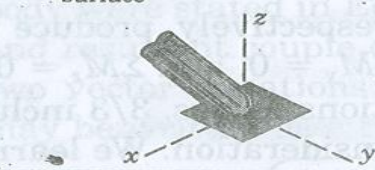
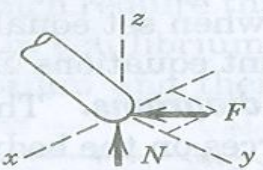
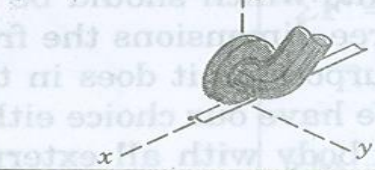
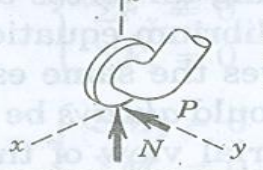
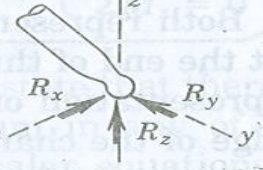
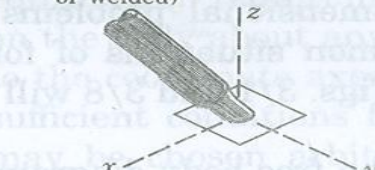
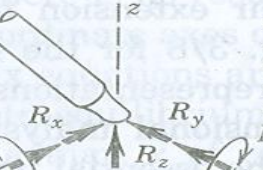
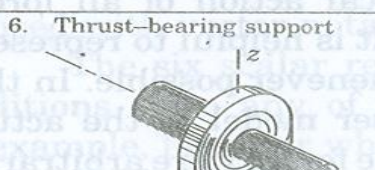
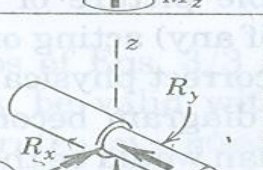
- For the first equation, $\Sigma \vec{F} = 0$, the vector sum will be zero only if the coefficients of i, j and k in the expression are, respectively, zero. These three sums when each is set equal to zero yield precisely the three scalar equations of equilibrium,

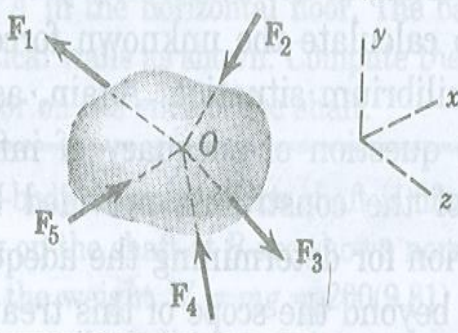
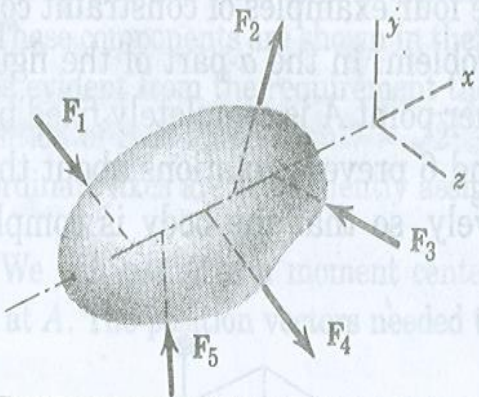
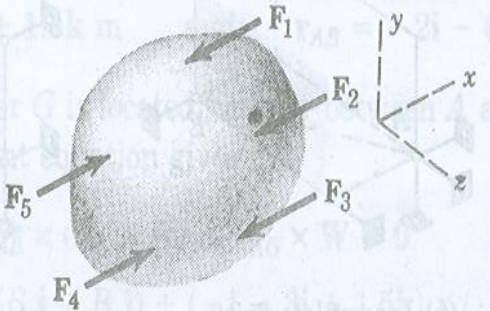
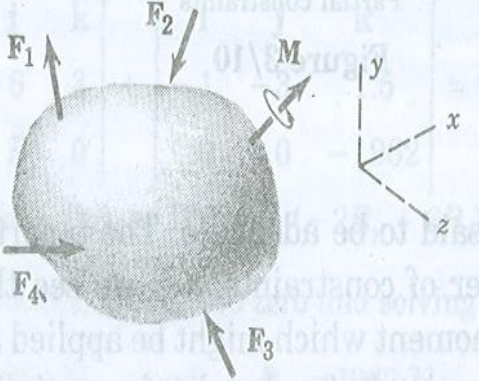
$$\Sigma F_x = 0, \Sigma F_y = 0 \text{ and } \Sigma F_z = 0$$

- For the second equation, $\Sigma \vec{M} = 0$, where the moment sum may be taken about any convenient point o , we express the moment of each force as the cross product $\vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector from o to any point on the line of action of the force F .

Thus, $\Sigma \vec{M} = \Sigma (\vec{r} \times \vec{F}) = 0$. The coefficients of i, j and k in the resulting moment equation when set equal to zero, respectively, produce the three scalar moment equations $\Sigma M_x = 0, \Sigma M_y = 0$ and $\Sigma M_z = 0$.

Free Body Diagram (FBD) shall always be drawn before analysis of the force system. Usually either pictorial view or orthogonal projects of the FBD are used.

MODELING THE ACTION OF FORCES IN THREE-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	Action on Body to be Isolated
1. Member in contact with smooth surface, or ball-supported member 	 Force must be normal to the surface and directed toward the member.
2. Member in contact with rough surface 	 The possibility exists for a force F tangent to the surface (friction force) to act on the member, as well as a normal force N.
3. Roller or wheel support with lateral constraint 	 A lateral force P exerted by the guide on the wheel can exist, in addition to the normal force N.
4. Ball-and-socket joint 	 A ball-and-socket joint free to pivot about the center of the ball can support a force R with all three components.
5. Fixed connection (embedded or welded) 	 In addition to three components of force, a fixed connection can support a couple M represented by its three components.
6. Thrust-bearing support 	 Thrust bearing is capable of supporting axial force R_y as well as radial forces R_x and R_z. Couples M_x and M_z must, in some cases, be assumed zero in order to provide statical determinacy.

CATEGORIES OF EQUILIBRIUM IN THREE DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$
2. Concurrent with a line		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$ $\Sigma M_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$

CHAPTER IV

4. ANALYSIS OF SIMPLE STRUCTURES

4.1 Introduction

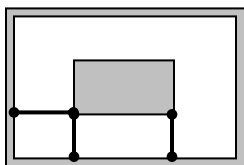
- An engineering structure is any connected system of members built to support or transfer forces and to safely withstand the loads applied to it.
- In this chapter we shall analyze the internal forces acting in several types of structures, namely, trusses, frames and simple machines.

Constraints and Statical Determinacy

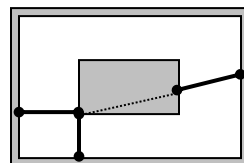
Equilibrium equations, once satisfied, are both necessary and sufficient conditions to establish the equilibrium of a body. However they don't necessarily provide all the information that is required to determine all the unknown forces that may act on a body in equilibrium.

If the number of unknown forces is more than the number of independent equilibrium equations, the equilibrium equations alone are not enough to determine the unknown forces, possibly reaction forces at the constraints.

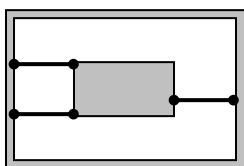
The adequacy of the constraints to prevent possible movement of the body depends on the number, arrangement and characteristics of the constraints.



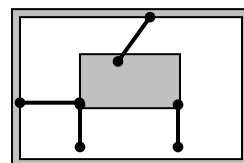
a) Complete fixity adequate constraints



b) Incomplete fixity partial constraints



c) Incomplete fixity partial constraints



d) Excessive fixity redundant constraints

Problem Solution

It is found important to develop a logical and systematic approach in the solution of problems of mechanics, which includes the following steps:

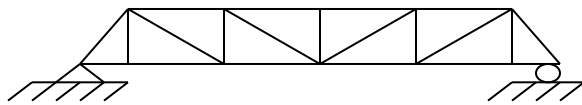
- Identify clearly the quantities that are known and unknown.
- Make an unambiguous choice of the body/group of bodies/ to be isolated and draw its complete FBD, labeling all external known and unknown forces and couples which act on it.
- Designate a convenient set of axes and choose moment centers with a view to simplifying the calculations.

- Identify and state the applicable force and moment principles or equations which govern the equilibrium condition of problem.
- Match the number of independent equations with the number of unknowns in each problem.
- Carry out the solution and check the results.

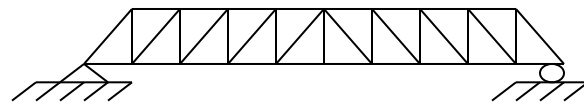
4.2 Plane Trusses

A truss is a framework composed of members joined at their ends to form a rigid structure. When the members of the truss lie essentially in a single plane, the truss is known as a plane truss. Examples of commonly used trusses that can be analyzed as plane as plane trusses are; -

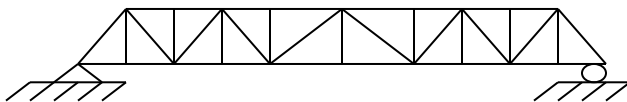
i/ Bridge Trusses



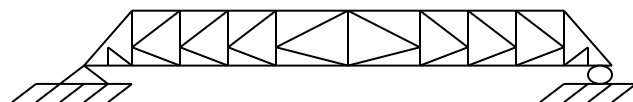
Pratt



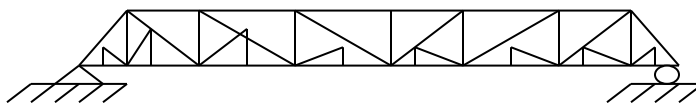
Howe



Warren

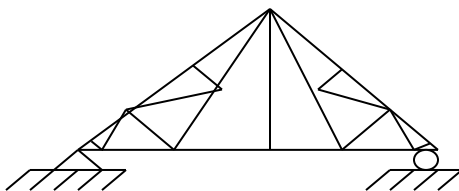


K

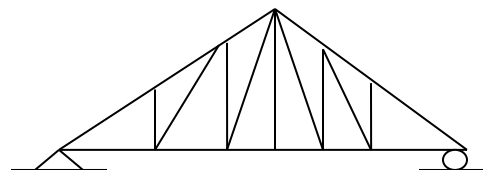


Baltimore

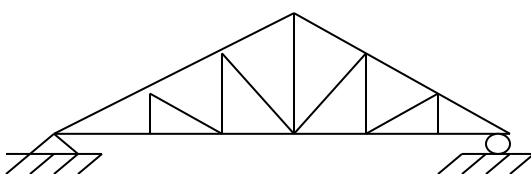
ii/ Roof Trusses



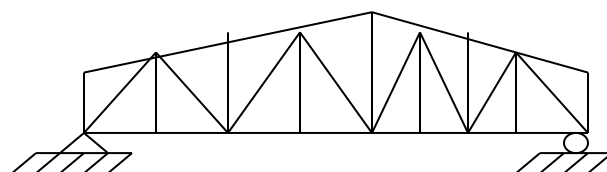
Fink



Pratt



Howe

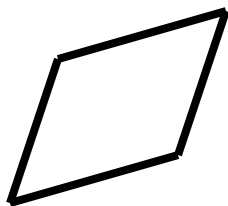


Warren

The basic element of a plane truss is the triangle. Structures that are built from a basic triangle in the manner described are known as simple trusses. When more members are present than are needed to prevent collapse, the truss is statically indeterminate. A statically indeterminate truss cannot be analyzed by the equations of equilibrium alone. Additional members or supports that are not necessary for maintaining the equilibrium position are called redundant.



-Three bars joined by pins at their ends constitute a rigid frame.

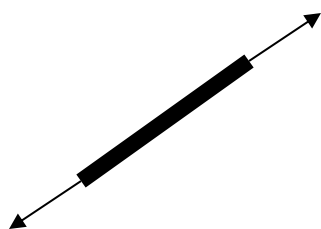


-Four or more bars pin-jointed to form a polygon of as many sides constitute a non rigid frame.

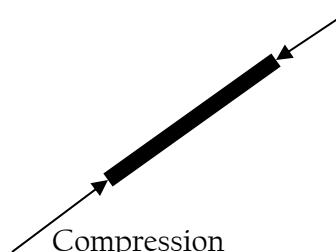
- We can make the non rigid frame stable or rigid by adding diagonal bars.

-The term rigid is used in the sense of non-collapsible and also in the sense that deformation of the members due to induced internal strains is negligible.

All members in a simple truss are assumed to be two-force members. The members may be in tension (T) or in compression (C).



Tension



Compression

Fig. Two-force members

The weight of truss members is assumed small compared with the force it supports. If it is not, or if the small effect of the weight is to be accounted for, the weight W of the member may be replaced by two forces, each $W/2$ if the member is uniform, with one force acting at each end of the member. These forces, in effect, are treated as loads externally applied to the pin connections. Accounting for the weight of a member in this way gives the correct result for the average tension or compression along the member but will not account for the effect of bending of the member.

- When welded or riveted connections are used to join structural members, the assumption of a pin-jointed connection is usually satisfactory if the centerlines of the members are concurrent at the joint.
- We also assume in the analysis of simple trusses that all external forces are applied at the pin connections. This condition is satisfied in most trusses. In bridge trusses the deck is usually laid on cross beams that are supported at the joints.

Force analysis of plane trusses

Two methods for the force analysis of simple trusses will be given. The external reactions are usually determined by computation from the equilibrium equations applied to the truss as a whole before the force analysis of the remainder of the truss is begun.

4.2.1 Method of joints

This method for finding the forces in the members of a simple truss consists of satisfying the conditions of equilibrium for the forces acting on the connecting pin of each joint.

- The method deals with the equilibrium of concurrent forces, and only two independent equilibrium equations are involved. ($\sum F_x = 0$ and $\sum F_y = 0$ for each joint)
- We begin the analysis with any joint where at least one known load exists and where not more than two unknown forces are present. Taking free body diagram of a joint, tension will always be indicated by an arrow away from the pin, and compression will always be indicated by an arrow toward the pin.
- In some instances it is not possible to initially assign the correct direction of one or both of the unknown forces acting on a given pin. In this event we may make an arbitrary assignment. A negative value from the computation indicates that the assumed direction is incorrect.

4.2.2 Method of sections

On the analysis of plane trusses by the method of joints, we took advantage of only two of the three equilibrium equations, since the procedures involve concurrent forces at each joint.

We may take advantage of the third or moment equation of equilibrium by selecting an entire section of the truss for the free body in equilibrium under the action of a non-concurrent system of forces. This method of sections has the basic advantage that the force in almost any desired member may be found directly from an analysis of a section, which has cut that member. Thus it is not necessary to proceed with the calculation from joint to joint until the member in question has been reached.

-In choosing a section of the truss, we note that, in general, not more than three members whose forces are unknown may be cut, since these are only three available equilibrium relations which are independent.

- It is essential to understand that in the method of sections an entire portion of the truss is considered a single body in equilibrium. Thus, the forces in members internal to the section are not involved in the analysis of the section as a whole.
- To classify the free body and the forces acting externally on it, the section is preferably passed through the members and not the joints.
- In some cases the methods of sections and joints can be combined for an efficient solution.

. The moment equations are used to great advantage in the method of sections. One should choose a moment center, either on or off the section, through which as many unknown forces as possible pass.

. It is not always possible to assign the proper sense of an unknown force when the free-body diagram of a section is initially drawn. With an arbitrary assignment made, a positive answer will verify the assumed sense and a negative result will indicate that the force is in the sense opposite to that assumed.

4.3 Frames and Simple Machines

A structure is called a frame or machine if at least one of its individual members is a multiforce member. A multiforce member is defined as one with three or more forces acting on it or one with two or more forces and one or more couples acting on it.

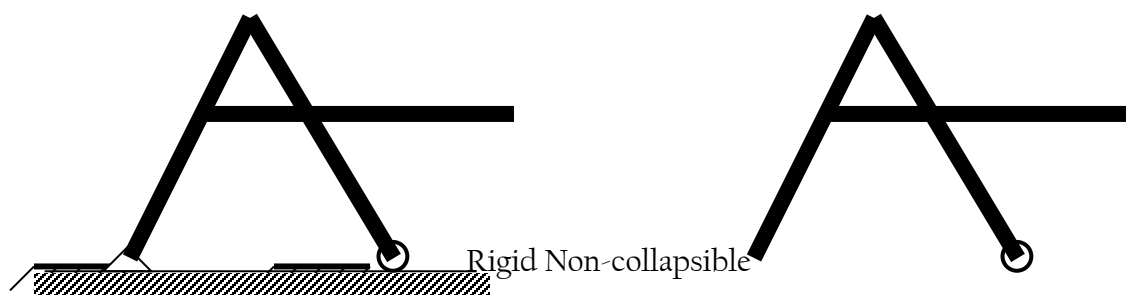
- Frames are structures which are designed to support applied loads and are usually fixed in position.
- Machines are structures which contain moving parts and are designed to transmit forces or couples from input values to output values.

In this article attention is focused on the equilibrium of interconnected rigid bodies which contain multiforce members. The forces acting on each member of a connected system are found by isolating the member with a free-body diagram and applying the established equations of equilibrium.

The principle of action and reaction must be carefully observed when we represent the forces of interaction on the separate free-body diagrams.

If the frame or machine constitutes a rigid unit by itself when removed from its supports, the analysis is best begun by establishing all the forces external to the structure considered as a single rigid body. We then dismember the structure and consider the equilibrium of each part separately.

The equilibrium equations for the several parts will be related through the terms involving the forces of interaction.



-If the structure is not a rigid unit by itself but depends on its external supports for rigidity, as in the figure below, then the calculation of the external support reactions cannot be completed until the structure is dismembered and the individual parts are analyzed.

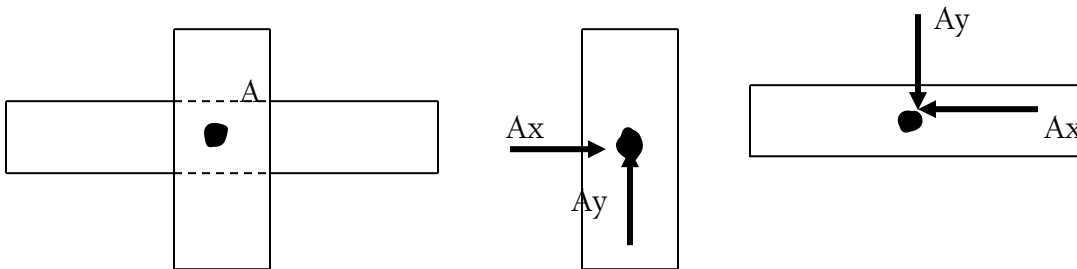


Non rigid Collapsible

In most cases we find that the analysis of frames and machines is facilitated by representing the forces in terms of their rectangular components.

It is not always possible to assign every force or its components in the proper sense when drawing the free body diagrams and it becomes necessary for us to make an arbitrary assignment.

-In any event it is absolutely necessary that a force be consistently represented on the diagrams for interacting bodies, which involve the force in question. For example, for two bodies connected by the pin in the figure below the force components must be consistently represented in opposite directions on the separate free-body diagrams.



If we choose to use vector notation in labeling the forces, then we must be careful to use a plus sign for an action and a minus sign for the corresponding reaction.

-Situations occasionally arise where it is necessary to solve two or more equations simultaneously in order to separate the unknowns. In most instances, however, we may avoid simultaneous solutions by careful choice of the member or group of members for the free-body diagram and by a careful choice of moment axes which will eliminate undesired terms from the equations.

CHAPTER V

5. CENTROIDS

5.1. Introduction

Actually, “concentrated” forces do not exist in the exact sense, since every external force applied mechanically to a body is **distributed** over a finite contact area however small.

- When forces are applied over a region whose dimensions are not negligible compared with other pertinent dimensions, then we must account for the actual manner in which the force is distributed by summing up the effects of the distributed force over the entire region. We carry out this process by using the procedures of mathematical integration. For this purpose we need to know the intensity of the force at any location. There are three categories into which such problems fall;

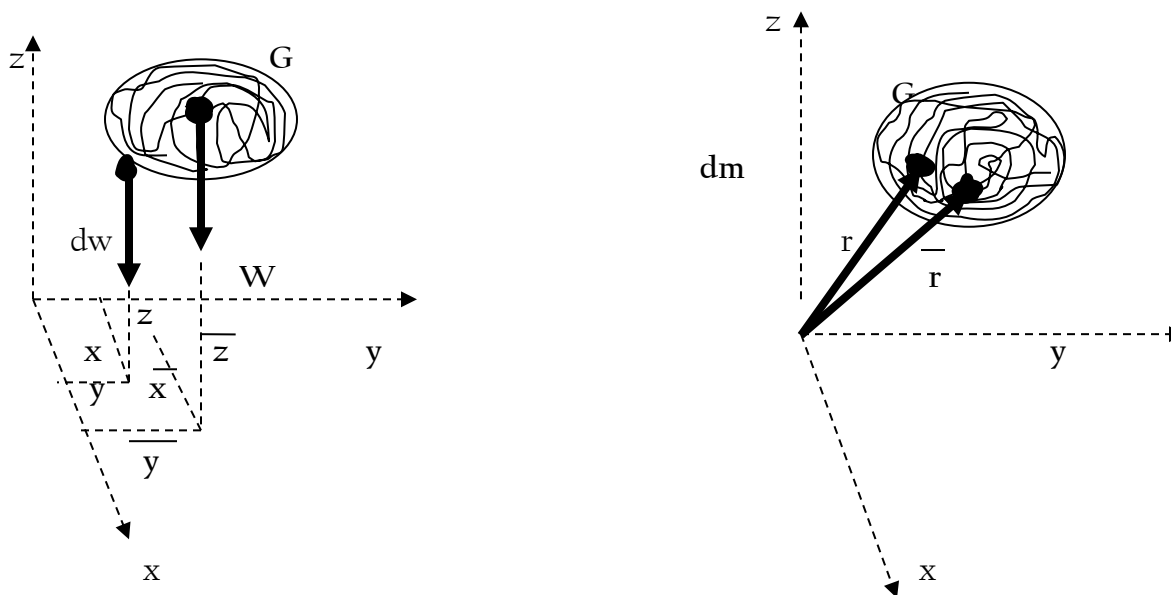
i/ Line Distribution :- when a force is distributed along a line. The loading is expressed as force per unit length of line (N/m).

ii/ Area Distribution :- when a force is distributed over an area. The loading is expressed as force per unit area (N/m^2).

iii/ Volume Distribution :- when a force is distributed over the volume of a body (body force). (N/m^3).

- The body force due to the earth’s gravitational attraction (weight) is by far the most commonly encountered distributed force. The following sections of the chapter deal with the determination of the point in a body through which the resultant gravitational forces acts and the associated geometrical properties of lines, areas and volumes.

5.2. Center of Mass or Center of Gravity



Applying the principle of moments, the moment of the resultant gravitational force W about any axis equals the sum of the moments about the same axis of the gravitational forces dw acting on all particles treated as infinitesimal elements of the body.

$$W = \int dw$$

Applying moment principle about the y -axis, the moment about the y -axis of the elemental weight, dw is $x \cdot dw$. The sum of these moments for all elements of the body is $\int xdw$.

$$\bar{x} \cdot W = \int xdw \quad \bar{x} = \frac{\int xdw}{W}$$

$$\text{Similarly,} \quad \bar{y} = \frac{\int ydw}{W}$$

$$\bar{z} = \frac{\int zdw}{W}$$

Substituting $W = mg$ and $dw = gdm$

$$\bar{x} = \frac{\int xdm}{m}, \quad \bar{y} = \frac{\int ydm}{m}, \quad \bar{z} = \frac{\int zdm}{m}$$

The above equations may be expressed in vector form;

$$\bar{r} = \frac{\int \vec{r} dm}{m}, \quad \vec{r} = xi + yj + zk \quad (\text{position vector for the elemental mass, } dm). \quad \bar{r} = \bar{x}i + \bar{y}j + \bar{z}k$$

(Position vector for G)

If the density of the body is not constant, $dm = \rho dv$

$$\bar{x} = \frac{\int x\rho dv}{\int \rho dv}, \quad \bar{y} = \frac{\int y\rho dv}{\int \rho dv}, \quad \bar{z} = \frac{\int z\rho dv}{\int \rho dv}$$

The point $(\bar{x}, \bar{y}, \bar{z})$ is known as the center of mass, and coincides with the center of gravity as long as the gravity field is treated as uniform and parallel.

-The center of mass has a special significance in calculating the dynamic response of a body to unbalanced forces.

In most problems the calculation of the position of the center of mass may be simplified by an intelligent choice of reference axes. In general the axes should be placed so as to simplify the equations of the boundaries as much as possible. Thus polar coordinates will be useful for bodies having circular boundaries.

Another important clue may be taken from considerations of symmetry. Whenever there exists a line or plane of symmetry in a homogenous body, a coordinate axis or plane should be chosen to coincide with this line or plane. The center of mass will always lie on such a line or plane, since the moments due to symmetrically located elements will always cancel, and the body can be considered composed of pairs of these elements.

-The location of the center of mass is always facilitated by the observation of symmetry when it exists.

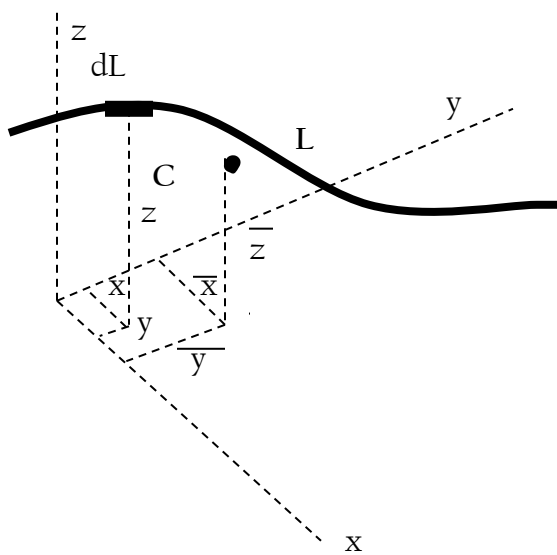
5.3. Centroids of Lines, Areas and Volumes

- When speaking of an actual physical body, we use the term center of mass. If the density is uniform throughout the body, the positions of the centroid and the center of mass are identical, whereas if the density varies, these two points will, in general, not coincide.

The term centroid is used when the calculation concerns a geometrical shape only.

The calculation of centroids fall within three distinct categories, depending on whether the shape of the body involved can be modeled as a line, an area, or a volume.

1/ Centroids of Lines



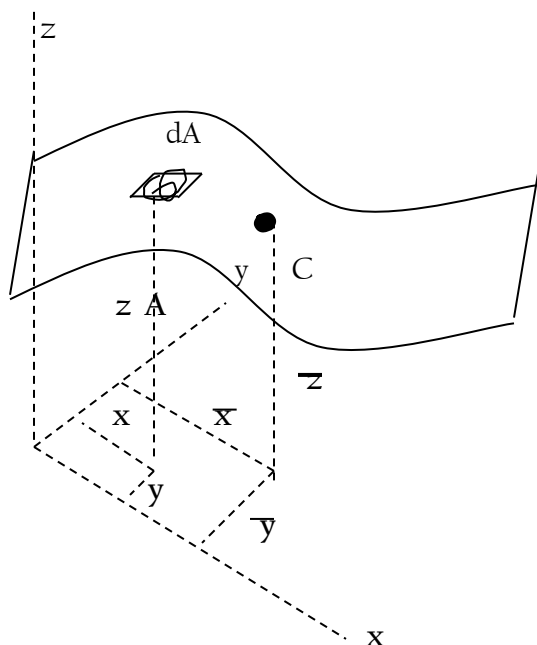
$$\bar{x} = \frac{\int x dL}{L}$$

$$\bar{y} = \frac{\int y dL}{L}$$

$$\bar{z} = \frac{\int z dL}{L}$$

It should be noted that, in general, the centroid C will not lie on the line. If the rod lies in a single plane, such as the x-y plane, only two coordinates will require calculation.

2/ Centroids of Areas



- If ρ and t are constant over the entire area, the coordinates of the center of mass of the body also become the coordinates of the Centroid C of the surface area.

$$\bar{x} = \frac{\int x dA}{A}$$

$$\bar{y} = \frac{\int y dA}{A}$$

$$\bar{z} = \frac{\int z dA}{A}$$

The centroid C for the curved surface will in general not lie on the surface. If the area is a flat surface, say, the x - y plane, only the coordinates of C in that plane will be unknown.

3/ Centroids of Volumes

-For a general body of volume V and constant density, the coordinates of the center of mass also become the coordinates of the centroid C of the body.

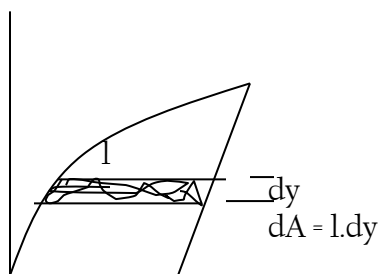
$$\bar{x} = \frac{\int x dV}{V}, \quad \bar{y} = \frac{\int y dV}{V}, \quad \bar{z} = \frac{\int z dV}{V}$$

Choice of Element for Integration

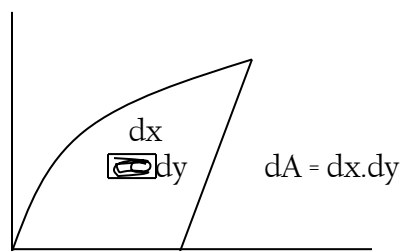
-With mass centers and centroids the concept of the moment principle is simple enough; the difficulties reside primarily with the choice of the differential element and with setting up the integrals. In particular there are five guidelines to be specially observed.

i/ **Order of element** ; - whenever possible, a first order differential element should be selected in preference to a higher order element so that only one integration will be required to cover the entire figure.

Eg.

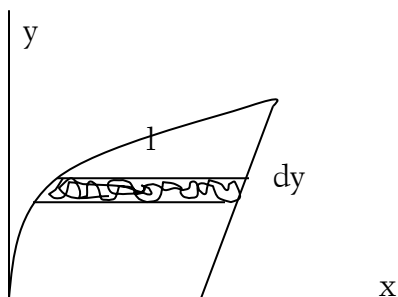


Selected

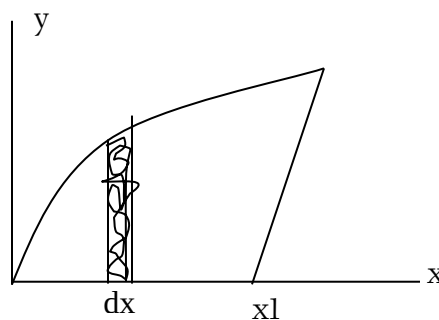


ii/ **Continuity** ; - whenever possible, we choose an element which can be integrated in one continuous operation to cover the figure.

Eg.



a/

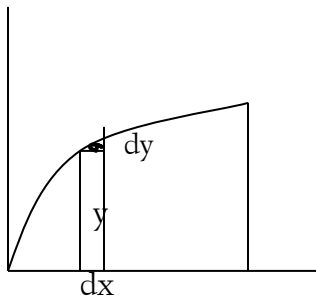


b/

The horizontal strip in fig (a), would be preferable to the vertical strip in fig (b), which, if used, would require two separate integrals because of the discontinuity in the expression for the height of the strip at $x = x_1$.

iii/ Discarding higher-order terms

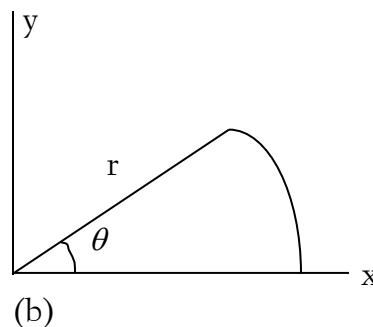
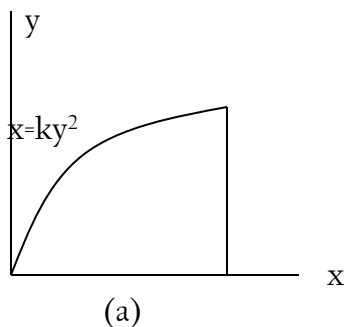
Higher-order terms may always be dropped compared with the lower-order terms.



-The vertical strip of area under the curve is given by the first order term $dA = ydx$, and the 2nd order triangular area $\frac{1}{2}dx \cdot dy$ is discarded. In the limit, of course, there is no error.

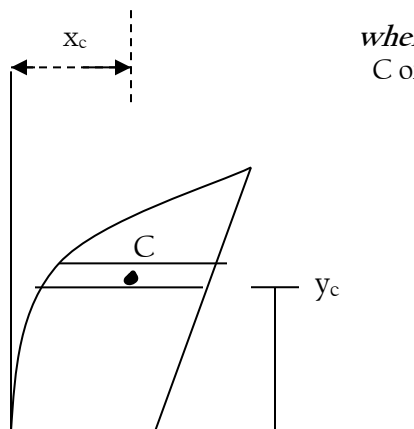
iv/ Choice of Coordinates ; -choose the coordinate system which best matches the boundaries of the figure.

Eg.



-The boundaries of the area in (a) are most easily described in rectangular coordinates, whereas the boundaries of the circular sector in (b) are best suited to polar coordinates.

V/ Centroidal Coordinate of Element:- When a 1st or 2nd order differential element is adopted, it is essential to use the coordinate of the centroid of the element for the moment arm in setting up the moment of the differential element.



-The moment of da about the y -axis is $x_c \cdot dA$,
where x_c is the x -coordinate of the centroid
 C of the element.

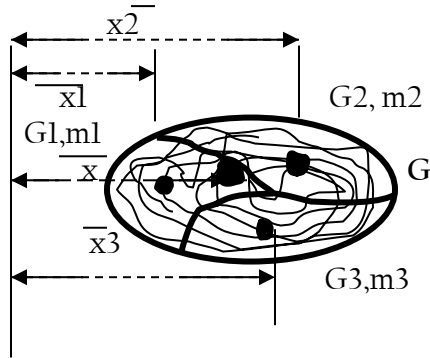
$$\bar{x} = \frac{\int x_c dA}{A}, \quad \bar{y} = \frac{\int y_c dA}{A}, \quad \bar{z} = \frac{\int z_c dA}{A}$$

$$\bar{x} = \frac{\int x_c dV}{V}, \quad \bar{y} = \frac{\int y_c dV}{V}, \quad \bar{z} = \frac{\int z_c dV}{V}$$

It is essential to recognize that the subscript C serves as a reminder that the moment arms appearing in the numerators of the integral expressions for moments are always the coordinates of the centroids of the particular elements chosen.

5.4. Centroids of composite bodies

Consider a body whose parts have masses m_1, m_2, m_3 with the respective mass center coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$ in the x-direction;



The moment principle gives;

$$(m_1 + m_2 + m_3) \bar{x} = m_1 \bar{x}_1 + m_2 \bar{x}_2 + m_3 \bar{x}_3$$

$$\sum m_i \bar{x}_i = \sum (m_i \bar{x}_i)$$

$$\bar{x} = \frac{\sum (m_i \bar{x}_i)}{\sum m_i}$$

$\bar{x}$ is the x-coordinate of the center of mass of the whole.

$$\text{Similarly, } \bar{y} = \frac{\sum (m_i \bar{y}_i)}{\sum m_i}, \quad \bar{z} = \frac{\sum (m_i \bar{z}_i)}{\sum m_i}$$

Analogous relations hold for composite lines, areas and volumes, where the m's are replaced by L's, A's and V's respectively.

It should be pointed out that if a hole or cavity is considered in one of the component parts of a composite body or figure, the corresponding mass represented by the cavity or hole is treated as a negative quantity.

CHAPTER VI

6. AREA Moments of Inertia

6.1. Introduction

- When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area. The intensity of the force (pressure or stress) is proportional to the distance of the force from the moment axis.
- The elemental force acting on an element of area, then, is proportional to distance times differential area, and the elemental moment is proportional to distance squared times differential area. We see, therefore that the total moment involves an integral that has the form $\int (\text{distance})^2 d(\text{area})$. This integral is known as the moment of inertia or the second moment of the area.

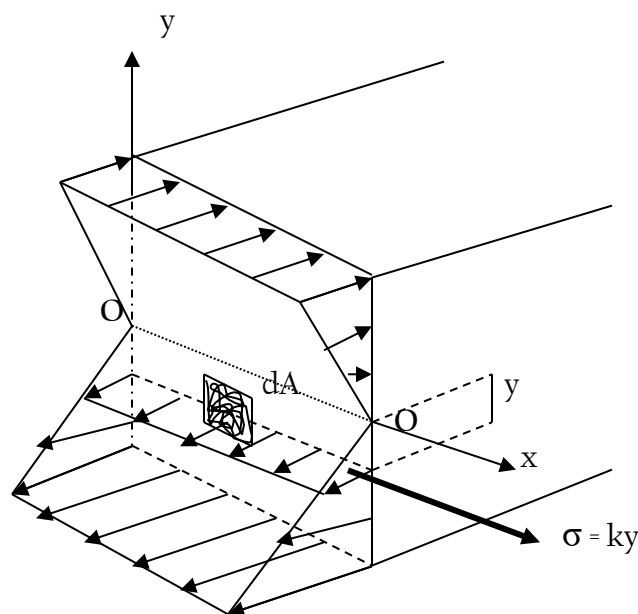


Fig. Stress distribution in cross section of a bending beam.

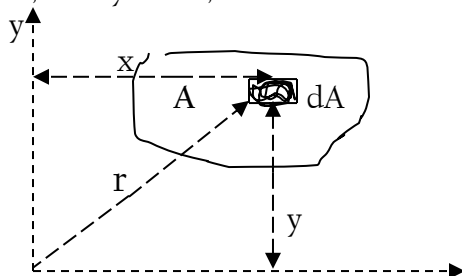
$$dF = \sigma \cdot dA = ky \cdot dA$$

$$dM = dF \cdot y = ky^2 dA$$

$$M = \int dM = \int ky^2 dA = k \int y^2 dA$$

The term $\int y^2 dA$ is called second moment of area or area moment of inertia of the cross-section. It appears so frequently in design formulas that it needs a separate treatment.

In general, for any area A, it will be found as follows.



$$dI_x = y^2 dA$$

$$\longrightarrow I_x = \int_A y^2 \cdot dA$$

$$dI_y = x^2 \cdot dA$$

$$O \quad x \quad \longrightarrow \quad I_y = \int_A x^2 \cdot dA$$

The second moment of area about the z-axis (pole O) is;

$$dI_z = r^2 \cdot dA \quad \longrightarrow \quad I_z = \int_A r^2 \cdot dA$$

$$\text{Or } dI_z = r^2 \cdot dA = (x^2 + y^2) \cdot dA = x^2 \cdot dA + y^2 \cdot dA$$

$$I_z = \int_A (x^2 + y^2) \cdot dA = I_y + I_x$$

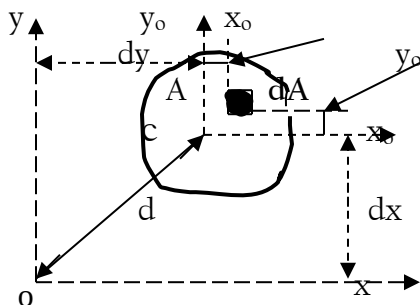
$$I_z = I_x + I_y, \quad - I_x \text{ and } I_y \text{ are called rectangular moments of inertia.}$$

- I_z is called polar moment of inertia and is the sum of rectangular moments of inertia about axes passing through the point.

The choice of elements of integration is similar to that of Centroids.

The parallel axes Theorem (Transfer Axes)

- It is often necessary to get moment of inertia of an area about axes parallel to centroidal axes. So this theorem provides relationship between centroidal moments of inertia and moments of inertia about parallel axes.



----- fig.*

By definition:

$$\begin{aligned} dI_x &= (dx + y_0)^2 \cdot dA \\ &= (dx^2 + 2dx \cdot y_0 + y_0^2) \cdot dA \end{aligned}$$

$$\begin{aligned} I_x &= \int_A y_0^2 \cdot dA + 2dx \int_A y_0 \cdot dA + dx^2 \int_A dA \\ &= I_{x0} + dx^2 \cdot A \end{aligned}$$

$$I_x = I_{x0} + dx^2 \cdot A, \quad \text{Similarly} \quad I_y = I_{y0} + dy^2 \cdot A$$

Where ; - I_{x0} and I_{y0} are centroidal rectangular moments of inertia and ;

- I_x and I_y are rectangular moments of inertia about the x- and y-axes.

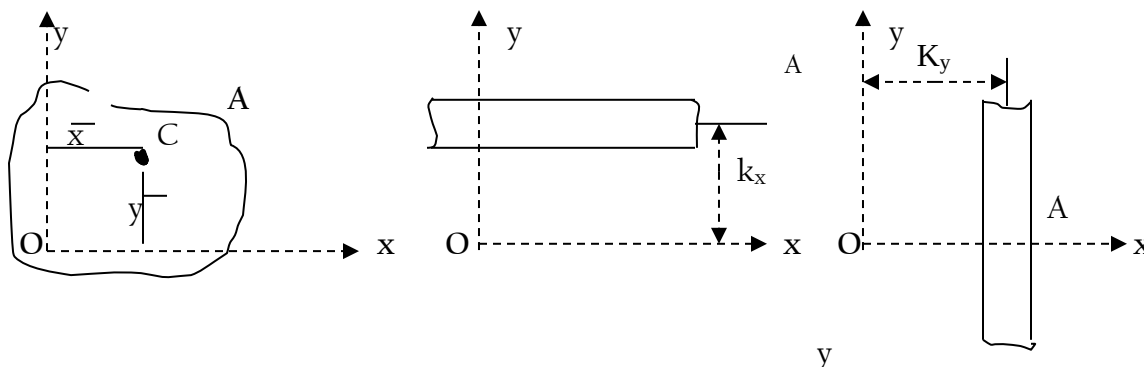
$$I_z = I_{z0} + d^2.A$$

Hence, the parallel axis theorem can be stated as; - The moment of inertia of an area with respect to any axis is equal to the moment of inertia about the parallel axis through the centroid of the area plus the product of the area and the square of the distance between the two axes. Two things to note are;

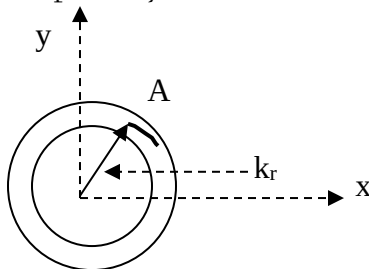
- i/ The axes should be parallel
- ii/ One of the axes should be centroidal

Radius of Gyration

- The radius is a measure of the distribution of the area from the axis in question.



- The distance k_x and k_y are known as the radius of gyration of the area about the x - and y -axis respectively.



- A rectangular or polar moment of inertia may be expressed by specifying the radius of gyration and the area.

$$\begin{aligned} I_x &= k_x^2.A \\ I_y &= k_y^2.A \\ I_z &= k_z^2.A \end{aligned}$$

Or

$$\begin{aligned} k_x &= \sqrt{I_x/A} \\ k_y &= \sqrt{I_y/A} \\ k_z &= \sqrt{I_z/A} \end{aligned}$$

$$k_z^2 = k_x^2 + k_y^2$$

The parallel-axis theorem also hold for radii of gyration. The transfer relation becomes;

$k^2 = \bar{k}^2 + d^2$, Where ;- k is the radius of gyration about a centroidal axis parallel to the axis about which k applies and d is the distance between the two axes. The axes may be either in the plane or normal to the plane of the area.

6.2. Composite Areas

The moment of inertia of a composite area about a particular axis is simply the sum of the moments of inertia of its component parts about the same axis.

-It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

-When a composite area is composed of a large number of parts, it is convenient to tabulate the results for each of the parts in terms of its area A , its centroidal moment of inertia I , the distance d from its centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2 .

.For a composite area in the x - y plane, for eg, and with the notation of fig.*, where I_x is the same as I_{x0} and I_y is the same as I_{y0} , the tabulation would include;

Part	Area, A	dx	dy	Adx^2	Ady^2	$\bar{I}_x$	$\bar{I}_y$
1							
2							
Sums				ΣAdx^2	ΣAdy^2	$\Sigma \bar{I}_x$	$\Sigma \bar{I}_y$

-From the sums of the four columns, then the moments of inertia for the composite area about the x - and y -axes become;

$$I_x = \Sigma \bar{I}_x + \Sigma Adx^2, \quad I_y = \Sigma \bar{I}_y + \Sigma Ady^2$$

Note; - Although we may add the moments of inertia of the individual parts of a composite area about a given axis, we may not add their radii of gyration. The radius of gyration for the composite area about the axis in question is given by $k = \sqrt{I/A}$, where I is the total moment of inertia and A is the total area of the composite figure. Similarly, the radius of gyration k about a polar axis through some point equals $\sqrt{I_z/A}$, where, $I_z = I_x + I_y$ for x - y axis through that point.

6.3. Products of Inertia & Transfer of Axes

6.3.1. Products of Inertia

In certain problems involving unsymmetrical cross sections and in the calculation of moments of inertia about rotated axes, an expression $dI_{xy} = xydA$ occurs, which has the integrated form ;

$$I_{xy} = \int x.y.dA$$

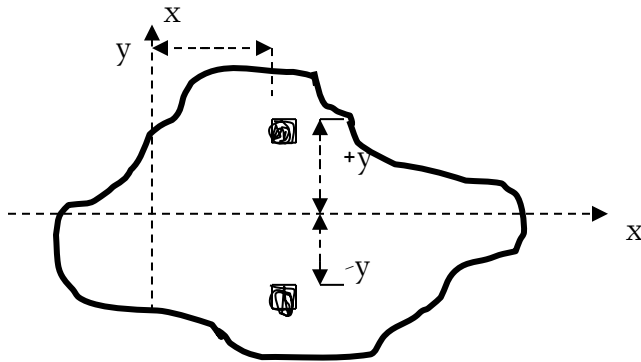
Where ; x and y are the coordinates of the element of area $dA = dx.dy$.

- I_{xy} is called the product of inertia of the area A with respect to the x - y axis.

-Unlike moments of inertia, which are always positive for positive areas, the product of inertia may be positive, negative or zero.

The product of inertia is zero whenever either one of the reference axes is an axis of symmetry.

Eg.



Here we see in the fig. that the sum of the terms $x(-y)dA$ and $x(+y)dA$ due to symmetrically placed elements vanishes. Since the entire area may be considered to be composed of pairs of such elements, it follows that the product of inertia I_{xy} for the entire area is zero.

Transfer of Axes

By definition the product of inertia of the area A in fig.* with respect to the x - and y -axes in terms of the coordinates x_o, y_o to the centroidal axes is;

$$I_{xy} = \int (x_o + dy)(y_o + dx).dA \\ = \int x_o.y_o.dA + dx.\int x_o.dA + dy.\int y_o.dA + dx.dy.\int dA$$

The 1st integral is by definition the product of inertia about the centroidal axes, which we write I_{xy} . The middle two integrals are both zero since the 1st moment of the area about its own centroid is necessarily zero. The third integral is merely $dx.dy.A$. Thus, the transfer-of-axis theorem for products of inertia becomes;

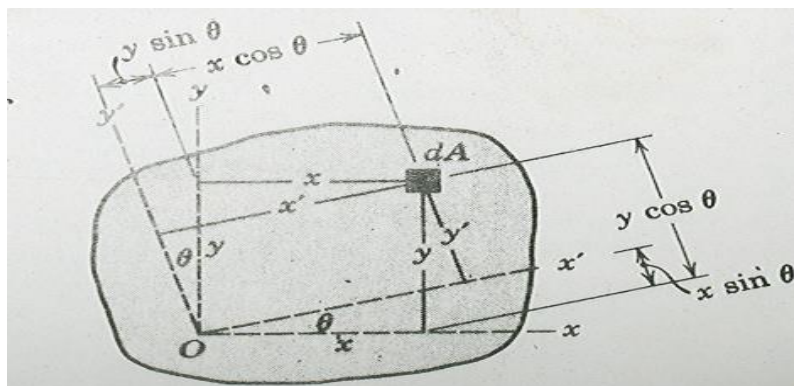
$$I_{xy} = I_{xy} + dx.dy.A$$

6.3.2. Rotation of Axes:

The product of inertia is useful when we need to calculate the moment of inertia of an area about inclined axes. This consideration leads directly to the important problem of determining the axes about which the moment of inertia is a maximum and a minimum. In the figure below, the moments of inertia of the area about the x' - and y' -axes are

$$I_{x'} = \int y'^2 dA = \int (y \cos \theta - x \sin \theta)^2 dA$$

$I_{y'} = \int x'^2 dA = \int (y \sin \theta + x \cos \theta)^2 dA$, where x' and y' have been replaced by their equivalent expressions as seen from the geometry figure.



Expanding and substituting the trigonometric identities and defining relations for I_x , I_y , I_{xy} give us

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$

In a similar manner the product of inertia about the inclined axes as

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

Adding the above equations gives $I_x + I_y = I_{x'} + I_{y'} = I_z$.

The angle that makes $I_{x'}$ and $I_{y'}$ a maximum or a minimum may be determined by setting the derivatives of either $I_{x'}$ or $I_{y'}$ with respect to θ equal to zero.

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

The above equation gives two values for 2α which differ by Π . Consequently the two solutions for α will differ by $\Pi/2$. One value of defines the axis of maximum moment of inertia, and the other value defines the axis of minimum moment of inertia. These two rectangular axes are known as the principal axes of inertia. Thus the maximum and minimum moments of inertia become:

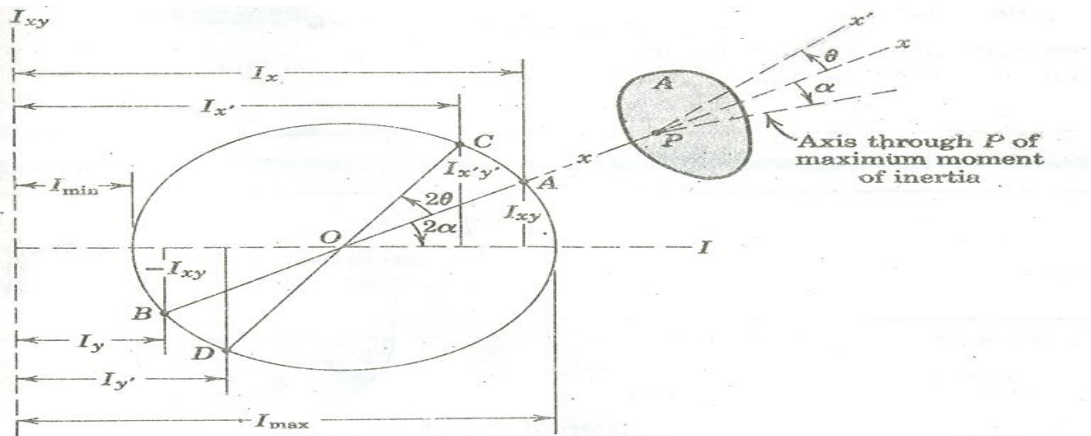
$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

Mohr's Circle of Inertia:

The relationships stated above can be presented graphically by a diagram known as Mohr's Circle as shown below. For given values of I_x , I_y , I_{xy} the corresponding values of $I_{x'}$, $I_{y'}$, $I_{x'y'}$ may be determined from the diagram for any desired angle θ . The coordinates of any point C are $(I_{x'}, I_{x'y'})$, and those of the

corresponding point D are $(I_y, I_{x'y'})$. Also the angle between OA and OC is 2θ or twice the angle from the x - axis to the x' - axis.



CHAPTER VII

7. FRICTION

7.1 Introduction

Whenever the surfaces of two bodies are in contact and an attempt is made to slide or tend to slide one of them relative to the other, there is a limited amount of resistance to sliding and a force tangential to the surfaces has to be overcome before a relative motion can take place between the bodies. This resistance is said to be due to frictional force. **Friction** may therefore be defined as the tangential force generated between contacting surfaces which one body exerts on another, when one body moves or tends to move with respect to the other.

The magnitude of **Friction** depends on the nature of the surfaces in contact. It can be neglected if the contact surface is perfectly smooth.

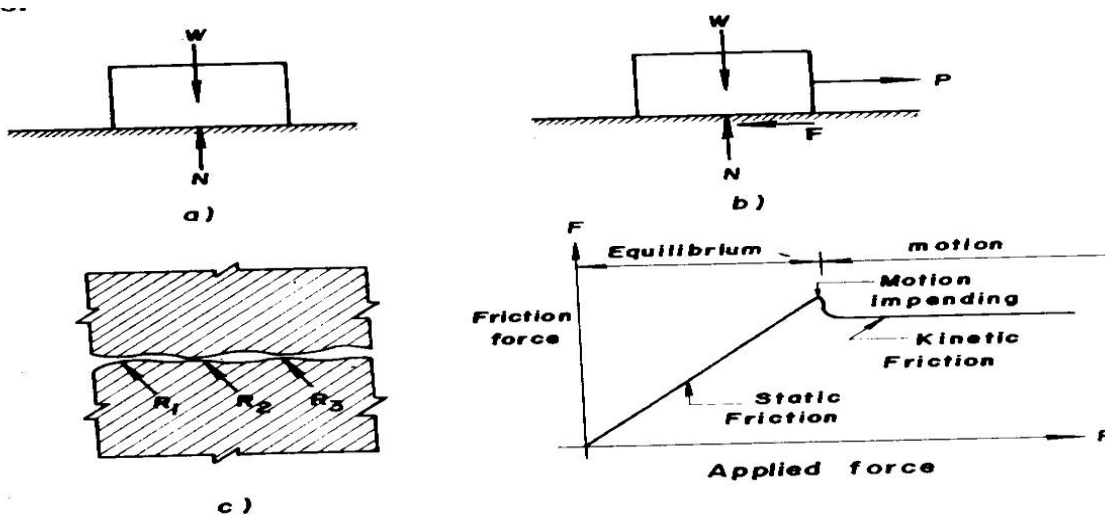
7.2 Types of Friction

Depending on the material, friction can be classified as:

- A) Dry friction – When two un-lubricated surfaces are in contact and one is trying to slide over the other the friction between the surfaces is called Dry friction.
- B) Fluid Friction – is the friction that exists between surfaces of fluids (liquid or gas).
- C) Internal Friction – friction in materials subjected to cyclic loading.

7.3 Dry Friction

Consider the following experiment for a better understanding of the principles of dry friction. Let a block of weight W resting on a horizontal dry un-lubricated plane surface be acted upon by the horizontal force P as shown in Fig. (a). When P is zero, the friction resistance is zero. When P is increased to some small value the block does not move, thus there must be a resisting force equal to P . This resisting force must be exerted by the supporting body at the contact surfaces by F as shown in Fig. (b); this resisting force is called the Dry/Static friction force, which is actually the resultant of great number of forces acting over the entire surface of contact between the block and the support.



Now consider the situation for larger values of the applied force P . If the force P is increased the friction force F also increases until it reaches a definite maximum limit $(F_s)_{\max}$ and a motion is said to impend. If P is further increased

beyond this amount, the block will move and the friction force drops slightly and abruptly to a lower value F_k . This is because the friction resistance is lower when the contact surfaces are in relative motion. The observation discussed is plotted graphically in Fig.(d). The graph illustrates that in the regions up to the point of impending motion, known as the range of static friction, the value of friction F is determined from the equations of equilibrium. Experimental evidence shows that this value may have any value from zero up to the maximum value $(F_s)_{\max}$ which is observed to be proportional to the normal force N . Hence

$$(F_s)_{\max} = f_s N$$

Where f_s is proportionality constant called the coefficient of static friction. It must be noted that the above equation only applies only when sliding is about to occur, or in the case of impending motion.

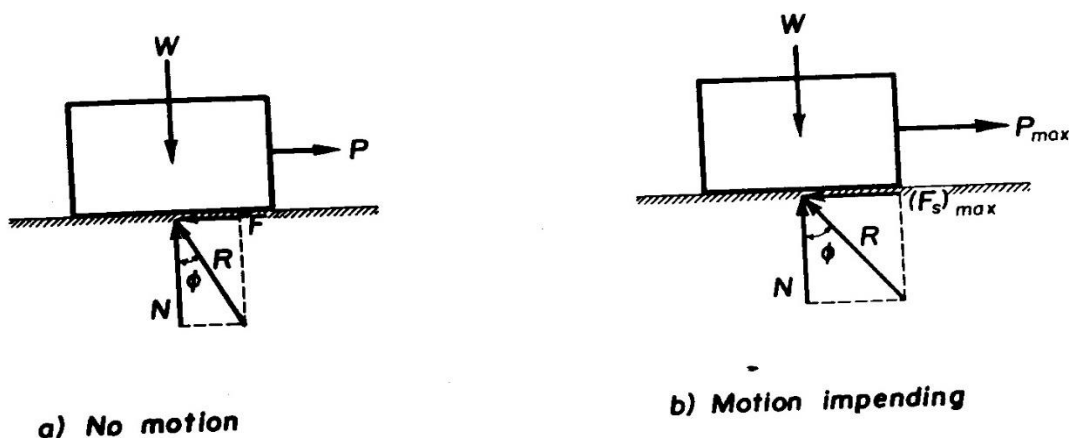
After the motion occurs, the kinetic friction force denoted by F_k is found also to be proportional to the normal force. Hence

$$F_k = f_k N$$

Where f_k is proportionality constant called the coefficient of kinetic friction.

7.4 Angle of Friction

It is sometimes found convenient to replace the normal force N and the friction force F by their resultant R as shown in the figure below.



The angle is determined from the equation of equilibrium requiring that the relation between the components F and N must be

$$\tan \phi = \frac{F}{N}$$

If P is increased until motion becomes impending, the angle between R and N increases until it reaches a maximum angle ϕ_s . This angle ϕ_s is known as the Angle of Static Friction. Which is

$$\tan \phi_s = \frac{(F_s)_{\max}}{N}$$

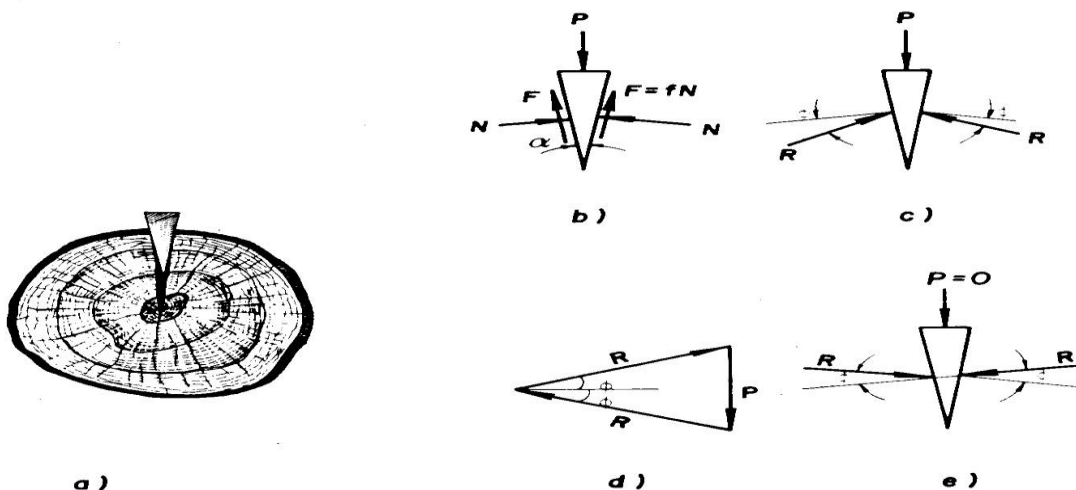
Using the relationship defined for the static friction above:

$$f_s = \tan \phi_s$$

7.5 Application of Friction in Machines

7.5.1 Wedges

A wedge is one of the simplest and most useful of machines which are used as a means of applying large forces or as a means of producing small adjustments in position of heavy objects. A wedge has two flat surfaces that form a small angle relative to each other. A wedge is largely dependant on the friction existing between the surfaces in contact. To study the effect of friction one must distinguish between the situation where the tendency is for the wedge to be forced into the object, and the case where the tendency is for the wedge to be pushed out of the object.



7.5.2 Screws

Screws are used for fastenings and for transmitting power or motion. In each case the friction developed in the threads largely determines the action of the screw.

