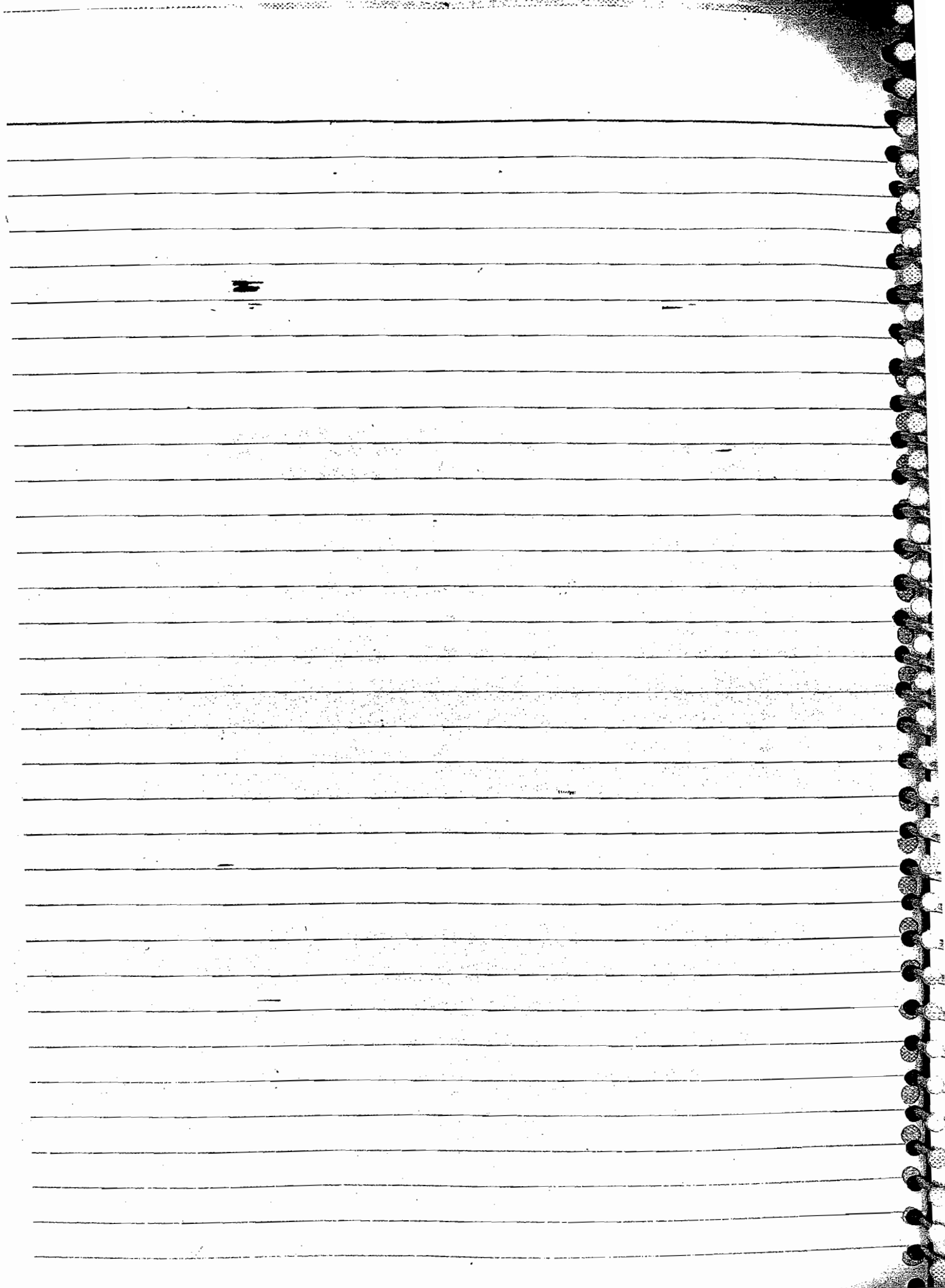


INTEGRAL

CALCULAS

SONAM DANDAL



DEFINITIONS $\rightarrow$ "An Equation is said to be integral Equⁿ in which an unknown function appear under one or more integral Sign."

Eg $\rightarrow \bar{y}(x) = \int_0^1 \sin(x-t) y(t) dt$

function of function is called functional.

$$\lambda(x) I\{y(x)\} = f(x) + \int_a^b K(x,t) I\{y(t)\} dt \quad (1)$$

It is most general form of I.E.

$\lambda(x)$ — known fun^c , $K(x,t)$ — Kernal (known fun^c)
 $f(x)$ — known fun^c , a — limit of integration.
 a is any parameter , b — limit of integration.
 $y(x)$ — Unknown fun^c

Classification Based on $y(x)$ $\rightarrow$

If functional $I\{y(x)\}$ & $I\{y(t)\}$ are linear i.e.

$$\forall a, b \in \mathbb{R} \text{ and } y_1, y_2$$

$$I\{ay_1 + by_2\} = aI\{y_1\} + bI\{y_2\}$$

$$I\{ay_1 + by_2\} = aI\{y_1\} + bI\{y_2\}$$

Then the integral Equⁿ is called Linear Integral Equation.

General form of the linear integral —

$$\lambda(x) \cdot y(x) = f(x) + \int_a^b K(x,t) y(t) dt$$

Classification Based on limit of Integration →

Let $\alpha(x) \cdot y(x) = f(x) + \int_a^x k(x, t) y(t) dt$, $n: a \leq x, t \leq b$
 $a \neq b$ are finite limit

If the upper limit of integration is variable —

$$\alpha(x) \cdot y(x) = f(x) + \int_a^x k(x, t) y(t) dt$$

$a \leq x, t \leq b$

Then it is called "Volterra" Integral Equ.

1) If both limit of integration are fixed —

i.e. $\alpha(x) \cdot y(x) = f(x) + \int_a^b k(x, t) y(t) dt$
 $a \leq x, t \leq b$

Then it is called "Fredholm" Integral Equ.

2) Singular I.E. → If either both | lower limit |
 upper limit | $k(x, t)$ is unbdd
 in domain of I.E. The I.E. is called Singular I.E.

eg → $y(x) = \int_{-\infty}^0 e^{xt} y(t) dt$ —

$$y(x) = \int_{-\infty}^{\infty} \sin(x-t) y(t) dt$$

$$y(x) = x + \int_0^1 \frac{1}{(x-t)^2} y(t) dt, \quad \alpha \in (0, 1)$$

($\alpha \rightarrow \infty$ or $x = t$, unbdd)

($\alpha \rightarrow \infty$ if $-ve$ then limit change then bdd)

3) Classification Based on $\alpha(x)$ →

1) If $\alpha(x) = 0$ i.e. the integral Equⁿ of form —

$$f(x) = \int_a^x k(x, t) y(t) dt$$

It is called 1st Kind I.E.

(ii) If $\alpha(x)$ is Non-zero constant function.

Eg $\rightarrow \frac{1}{2} y(x) = x + \int_0^x e^{|x-t|} y(t) dt$

or $\alpha y(x) = 1$

It is called 2nd kind I.E.

(iii) If $\alpha(x)$ is Non-constant funcⁿ of x .

Sinx, $y(x) = \cos x + \int_0^\pi e^{xt} y(t) dt$

It is called "3rd kind" I.E.

4) Classification Based on $f(x) \rightarrow$

(I) If $f(x) = 0$ Then I.E. is called Homogeneous I.E.

$$\alpha(x) \cdot y(x) = \int_a^b K(x,t) \cdot y(t) dt$$

otherwise Non-Homogeneous I.E.

Classification of $K(x,t) \rightarrow$

(I) Separable / Degenerate Kernel $\rightarrow$ The kernel $K(x,t)$ is called Separable if it is express as sum of finite no. of terms s.t. each term is product of funcⁿ of ' x ' only & funcⁿ of ' t ' only

Eg $\rightarrow K(x,t) = \sum_{i=1}^n \alpha_i(x) \cdot \beta_i(t)$

① $K(x,t) = \sin(x-t) = \overset{\alpha_1(x)}{\sin x} \overset{\beta_1(t)}{\cos t} - \overset{\alpha_2(x)}{\cos x} \overset{\beta_2(t)}{\sin t}$

② $K(x,t) = e^{x+t} = e^x \cdot e^t$

③ $K(x,t) = e^{|x-t|} = 0 \leq x, t \leq 1$

$e^{xt} \rightarrow$ Non Separable

$$4) K(x, t) = \begin{cases} e^{x-t}, & 0 \leq t \leq x \\ e^{t-x}, & x < t \leq 1 \end{cases}$$

Is Non-Separable Equⁿ.

$$5) K(x, t) = \begin{cases} \cos x \sin t & 0 \leq x \leq t \\ \sin x \cos t & t < x \leq \pi \end{cases}$$

Is Non-Separable Equⁿ.

$$6) K(x, t) = (x+t)^n \\ = n c_0 x^n t^0 + n c_1 x^{n-1} t^1 + \dots + n c_n x^0 t^n \\ = \sum_{i=1}^n \alpha_i(x) \cdot \beta_i(t)$$

E Symmetric Kernel | Skew Symmetric $\rightarrow$

$$\text{If } \bar{K}(t, x) = K(x, t)$$

Kernel may be complex | Real valued fun^c is called Symmetric Kernel.

$$\text{eg} \rightarrow \textcircled{1} K(x, t) = \sin(x+t) \\ \bar{K}(t, x) = \sin(t+x) = K(x, t)$$

$$\textcircled{2} K(x, t) = \cos(x-t) \\ \bar{K}(x, t) = \cos[-(x-t)] = K(x, t)$$

$$\textcircled{3} K(x, t) = e^{|x-t|} \\ \bar{K}(x, t) = e^{|x-t|} = K(x, t)$$

Symmetric Kernel may be degenerate | Non-deg.

$$\# \text{ If } \bar{K}(t, x) = -K(x, t)$$

Then $K(x, t)$ is called Skew Symmetric

$$K(x, t) = \sin(x-t)$$

$$\bar{K}(x, t) = -\sin(t-x) = -K(x, t)$$

$$K(x, t) = e^{x-t}$$

$$= e^{t-x} \neq -K(x, t)$$

$$\neq K(x, t)$$

Zero fun^c is only fun^c which is both symmetric & skew symmetric.

$$K(x, t) = \begin{cases} \cos x \sin t & , 0 \leq x < t \\ \sin x \cos t & , t \leq x < \pi \end{cases}$$

$$K(x_1, x_2) = \begin{cases} \cos x_1 \sin x_2 & , 0 \leq x_1 < t \\ \sin x_1 \cos x_2 & , t \leq x_2 < \pi \end{cases}$$

$$x_1 = t, \quad x_2 = x$$

$$K(t, x) = \begin{cases} \cos t \sin x & , 0 \leq t < x \\ \sin t \cos x & , x \leq t < \pi \end{cases}$$

$$t < x \rightarrow \sin x \cos t$$

$$t > x \rightarrow \cos x \sin t$$

It is symmetric.

$$\# \quad K(x, t) = \begin{cases} t(1-x) & , 0 \leq x < t \\ x^2(1-t) & , t \leq x \leq 1 \end{cases}$$

$$K(t, x) = x(1-t)$$

It is neither symmetric nor skew symmetric.

CONVOLUTION Type $\rightarrow$

If $K(x, t)$ is fun^c of $(x-t)$

$$K(x, t) = \phi(x-t)$$

$$\text{Eg} \rightarrow K(x, t) = \sin(x-t)$$

$$K(x, t) = e^{|x-t|}$$

Then corresponding Integral Equation —

$$\alpha(x) \cdot y(x) = f(x) + \int_a^b K(x, t) y(t) dt$$

Is called I.E. of convolution type.

2. CLASSIFICATION

Integral Equⁿ

Linear

$$\alpha(x) \cdot y(x)$$

Non Linear

$$f(x) + d \int_a^b K(x, t) y(t) dt$$

based on $\alpha \rightarrow$

Volterra if upper
lim. is variable

Fradholm if both
lim. are fix

Singular atleast one lim.
 $\infty / -\infty$ or $K(x, t)$ unbdd

based on $\alpha(x) \rightarrow$

1st Kind, $\alpha(x) = 0$

2nd Kind, $\alpha(x) = \text{Non zero con.}$

3rd type.
Non cont funⁿ

based on $f(x) \rightarrow$

$f(x) = 0$ Then homo.

$f(x) \neq 0$, Non homo.

In linear diff. Equ. highest order-coeff. is constant
then 2nd Kind Equⁿ.

$\alpha \rightarrow$ Depends on Initial / Boundary problem.

Rule $\rightarrow$ (I) $F(x, t)$ & $\partial F / \partial x$ continuous & $h(x), g(x)$
are two fun^c s.t. $\partial h / \partial x, \partial g / \partial x$ are two
continuous then -

$$\frac{d}{dx} \int_{h(x)}^{g(x)} F(x, t) dt = \int_{h(x)}^{g(x)} \frac{\partial F}{\partial x} dt + F[x, g(x)] \frac{dg}{dx} - F[x, h(x)] \frac{dh}{dx}$$

(Int. - diff)

$$\int_a^x \int_a^x \dots \int_a^x y(t) dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} y(t) dt$$

(Diff - Int.)

Q. $y'' + y = f(x)$ — (1) defined on $[0, \infty)$
 $y(0) = \alpha_0, \quad y'(0) = \alpha_1$

→ Kernel of $K(x, t)$ is degenerated.

→ $K(x, t)$ is Symm.

→ $K(x, t)$ is of convolution.

→ Homogeneous Integral Equⁿ.

→ 2nd kind I.E.

Solⁿ → $y'' = g(x)$ — (1)

$\int_0^x y''(t) dt = \int_0^x g(t) dt$

$\Rightarrow y'(x) - y'(0) = \int_0^x g(t) dt$

$\Rightarrow y'(x) = \alpha_1 + \int_0^x g(t) dt$

$\Rightarrow y(x) - y_0 = \alpha_1 x + \int_0^x \int_0^x g(t) dt^2$

$\Rightarrow y(x) = \alpha_0 + \alpha_1 x + \int_0^x (x-t) g(t) dt$

$\Rightarrow g(x) + \alpha_0 + \alpha_1 x + \int_0^x (x-t) g(t) dt = f(x)$

$\Rightarrow g(x) = -\alpha_0 - \alpha_1 x + f(x) - \int_0^x (x-t) g(t) dt$ — (2)

Solution of both Equⁿ are not same.

Sol Equ. (2) = 2nd derivative of equ. (i)

IInd method → $y'' = -y + f(x)$

$y'(x) - y'(0) = - \int_0^x y(t) dt + \int_0^x f(t) dt$

$y'(x) = \alpha_1 - \int_0^x y(t) dt + \int_0^x f(t) dt$

$y(x) = \alpha_0 + \alpha_1 x - \int_0^x \int_0^x y(t) dt^2 + \int_0^x \int_0^x f(t) dt^2$

$$y(x) = \alpha_0 + \alpha_1 x - \int_0^x (x-t) y(t) dt + \int_0^x (x-t) f(t) dt$$

$$y(x) = \alpha_0 + \alpha_1 x + \int_0^x (x-t) y(t) dt + \int_0^x (x-t) f(t) dt \quad (3)$$

It is I.E. of corresponding equ. (1)

So both solution are same.

$f(x) \neq 0$, Non homo., skew symmetry,
IInd Kind, Volterra.

#

If all initial condition are zero & given diff. Equⁿ is non homo. then I.E. is Non homo.

If initial condition zero & Equⁿ is homo. then I.E. is homogeneous.

Boundary problem - Fredholm.

Eg $\rightarrow y^{(n)} + y = 0$, $y(0) = 0$, $y'(0) = 0$, $y^{(n-1)}(0) = 0$
Let $y(x) = \int_0^x K(x,t) y(t) dt$ be an I.E. Then -

Solⁿ $\rightarrow$

a) $\checkmark$ $K(x,t)$ Symm. If n even.

b) $\times$ $K(x,t)$ Skew Symm. if n odd.

c) $\checkmark$ $K(x,t)$ - degenerate $\forall n$.

(d) $\checkmark$ $K(x,t)$ convolution type $\forall n$.

Solⁿ $\rightarrow$

$$y^{(n)} = -y$$

$$y^{(n-1)}(x) = - \int_0^x y(t) dt$$

$$y(x) = - \int_0^x \int_0^x \dots \int_0^x y(t) dt^n$$

$$y(x) = - \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} y(t) dt$$

Kernel depends on coeff. of diff. Equ + order of diff. Equ.

Sol If n even then Kernel Skew Symmetric. Kernel also degenerate.

If L.D.E. with constant coefficient then always convolution & degenerate.

#

$$y^{(n)} + y' = 0$$

$$y^{(n)} = -y'$$

$$y^{(n+1)}(x) = - \int_0^x y'(t) dt = y(x)$$

$$\begin{aligned} y(x) &= - \int_0^x \int_0^x \dots \int_0^x y(t) dt^{n-1} \\ &= - \int_0^x \frac{(x-t)^{n-2}}{(n-2)!} y(t) dt \end{aligned}$$

if n even $\rightarrow$ Symmetric

n odd $\rightarrow$ Skew Symmetric

#

$$y^{(n)} + y^{(k)} = 0$$

$$y^{(n)} = -y^{(k)}$$

$$y^{(n-k)} = -y(x)$$

$$y(x) = - \int_0^x \int_0^x \dots \int_0^x y(t) dt^{(n-k)}$$

$$y(x) = - \int_0^x \frac{(x-t)^{n-k-1}}{(n-k-1)!} y(t) dt$$

(a) $K(x, t)$ is Symm. if both n & k even.

(b) $K(x, t)$ is Skew symm. if both n & k odd.

(c) $K(x, t)$ is Symm. if n even & k odd.

(d) $K(x, t)$ is Skew Symm. odd n odd & k even.

#

$$y^{(n)} + y^{(k)} + y^{(l)} = 0, \quad n > k > l$$

Then $K(x, t)$ is —

a) Symm. if $k \neq l$ odd and n even.

b) Skew Symm. if $k \neq l$ odd & n even.

c) Skew Symm. if $k \neq l$ odd & n odd.

d) Neither Symm. Nor Skew Symm. if k odd & l even.

Solⁿ →

$$y^{(n)} = -y^{(k)} - y^{(l)}$$

$$y^{(n-l)} = -y^{(k-l)} - y^{(x)}$$

$$y(x) = \iint \dots \int y(t) dt^{(n+k)} \dots - \int y(t) dt^{(x-t)}$$

$$y(x) = - \left[\int_0^x \frac{(x-t)^{n-k-1}}{(n-k-1)!} y(t) dt + \int_0^x \frac{(x-t)^{n-l-1}}{(n-l-1)!} y(t) dt \right]$$

$$K(x, t) = \frac{(x-t)^{n-k-1}}{(n-k-1)!} + \frac{(x-t)^{n-l-1}}{(n-l-1)!}$$

NOTE → ① If $k \neq l$ Same and opposite to n .

Then always Symmetric.

② If $k \neq l$ opposite then Neither Symmetric Nor skew.

③ If n, k, l Same then Skew Symmetric

Ex → $y^{10} + y^7 + y^4 = 0$ Neither Symm. Nor skew.
 $y^{10} + y^7 + y^3 = 0$, Symmetric

3. $\sum_{i=1}^n a_i y^{(i)} = 0$, $n \geq 10$

⇒ $a_n \neq 0$, $y^{(i)}(0) = 0$, $i = 0, 1, \dots, n-1$

Then $K(x, t)$ of I.E. -

a) Symm. If $a_{2i} = 0$ & n even

b) Skew Symm. if $a_{2i+1} = 0$ & n odd

c) Neither Symm. Nor Skew if $a_{2i} = 0$, n odd

d) None of these.

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Solⁿ → (a) n - odd & n - even
 $n - k - 1 = \text{Even}$

if $a_{2i} = 0$ then Symm.

(b) n - odd & k - Even

$n - k - 1 = \text{Even}$ then Symm.

if $a_{2i+1} = 0$

(c)

$$y^{(11)} + y^{(10)} + y^{(8)} + y^{(7)} + y^{(5)} + y^{(4)} + y'' + y' = 0$$

$$K(x, t) = 1 + \frac{(x-t)^2}{2!} + \frac{(x-t)^3}{3!} + \dots$$

$$K(t, x) = 1 + \frac{(x-t)^2}{2!} - \frac{(x-t)^3}{3!} + \dots$$

Neither Symm. Nor Skew Symm $\leftarrow \begin{cases} a_{3i} = 0, n \text{ odd} \\ \text{one term even, one odd} \end{cases}$

Q → The I.E. corresponding to

$$xy'' + y = 0 \quad y(0) = 1, \quad y'(0) = 0$$

(a) ✓ 3rd kind Volterra.

(b) X kernel $K(x, t)$ Symm.

(c) X Non homo. Fredholm I.E. → Boundary prob.

(d) ✓ Non homo. Volterra I.E. with degenerate kernel.

Solⁿ → Highest coeff. → x variable

∴ Third kind Volterra.

$$xy'' = -y$$

$$\int_0^x ty''(t) dt = - \int_0^x y(t) dt$$

$$\Rightarrow xy'(x) - \int_0^x y'(t) dt = - \int_0^x y(t) dt$$

$$\Rightarrow xy'(x) = y(x) - 1 - \int_0^x y(t) dt$$

$$\Rightarrow \int_0^x ty'(t) dt = \int_0^x y(t) dt - x - \int_0^x \int_0^t y(t) dt^2$$

$$xy(x) - \int_0^x y(t) dt = \int_0^x y(t) dt - x - \int_0^x (x-t) y(t) dt$$

$$xy(x) = \int_0^x [2 - (x-t)] y(t) dt - x$$

$$\Rightarrow \int_0^x t y''(t) dt = \int_0^x y(t) dt$$

$$K(x, t) = 2 - (x-t)$$

$$K(t, x) = 2 + (x-t)$$

$$\neq K(x, t), \neq -K(x, t)$$

It is third kind, Volterra, Non-homo. Equⁿ.

Q. $y(x) = 1 + \frac{1}{6} x^3 - x + \int_0^x [\sin t - (x-t)(e^t + \cos t)] y(t) dt$

The I.F. Equⁿ has $\rightarrow$

1) Unique Sol. if $y(0) = -1, y'(0) = 1$

✓ " " " $y(0) = 1, y'(0) = -1$

1) Unique Sol. $\forall \alpha, \beta \in \mathbb{R}$ s.t. $y(0) = \alpha, y'(0) = \beta$

1) Infinite Sol. if $y(0) = 1, y'(0) = 0$

Solⁿ $\rightarrow$ $y(0) = 1$
 $y'(x) = \frac{1}{2} x^2 - 1 + (-1) \int_0^x (e^t + \cos t) y(t) dt$

$$+ \sin x \cdot y(x) \rightarrow (\text{upper limit})$$

$$\text{By } F \left(x, g(x) \cdot \frac{dg}{dx} = F(x, h(x)) \cdot \frac{dh}{dx} \right)$$

$$y'(0) = -1$$

$$y''(x) = x - (e^x + \cos x) y(x) + \cos x y(x) + \sin x y'(x)$$

$$= x - e^x y(x) + \sin x \cdot y'(x)$$

$$y''(x) - \sin x \cdot y'(x) + e^x y(x) = x$$

$$y(0) = 1, y'(0) = -1$$

$$y''(0) = 1 \Rightarrow y'(0) = 1 \quad (\text{Unique Sol}^n)$$

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$$

$$\text{Sol}^n \rightarrow \begin{cases} y'' = -\lambda y \\ y'(x) - y'(0) = -\lambda \int_0^x y(t) dt \end{cases}$$

$$y'(x) = \alpha - \lambda \int_0^x y(t) dt, \quad \alpha = y'(0)$$

$$y(x) = \alpha x - \lambda \int_0^x (x-t) y(t) dt$$

$$y'(\pi) = 0 \quad \pi$$

$$0 = \alpha - \lambda \int_0^\pi y(t) dt \Rightarrow \alpha = \lambda \int_0^\pi y(t) dt$$

$$y(x) = \lambda \int_0^\pi x y(t) dt - \lambda \int_0^x (x-t) y(t) dt$$

$$y(x) = \lambda \left[\int_0^x x y(t) dt - \int_0^x (x-t) y(t) dt \right] + \lambda \int_x^\pi x y(t) dt$$

$$y(x) = \lambda \int_0^x t y(t) dt + \lambda \int_x^\pi x y(t) dt$$

$$y(x) = \lambda \int_0^\pi K(x, t) y(t) dt$$

$$K(x, t) = \begin{cases} t, & 0 \leq t < x \\ x, & x \leq t \leq \pi \end{cases}$$

It is Homogeneous, Symmetric, Fredholm & 2nd kind
Countably Infinite $\rightarrow$ Non Trivial Solⁿ

Non degenerate never finite

Non degenerate then $\lambda \rightarrow$ Non trivial Solⁿ $\rightarrow$ countably

No. of Eigen func $\rightarrow$ Infinite

Q. Does not have any Non-Trivial solⁿ for uncountable values of λ .

$$Q. \quad y(x) = \lambda \int_0^{\pi} K(x,t) y(t) dt, \quad y'(0)=0, \quad y(\pi)=0$$

$$K(x,t) = \begin{cases} \cos x \sin t, & 0 \leq t < x \\ \sin x \cos t, & x \leq t \leq \pi \end{cases}$$

X Has unique solⁿ $\forall \lambda \rightarrow$ Eigen values $\rightarrow$ Inf. Non Trivial

✓ Has a solution $\forall \lambda$.

✓ Has infinite many solⁿ for infinite many λ .

X Has no-sol. for uncountable value of λ .

$$Q \rightarrow y(x) = \lambda \int_0^x \cos x \sin t y(t) dt + \lambda \int_x^{\pi} \sin x \cos t y(t) dt$$

$$y'(x) = -\lambda \int_0^x \sin x \sin t y(t) dt + \lambda \cos x \sin x y(x)$$

$$+ \lambda \int_x^{\pi} \cos x \cos t y(t) dt - \lambda \sin x \cos x y(x)$$

$$y'(0) = \lambda \int_0^{\pi} \cos t y(t) dt$$

$$y''(x) = -\lambda \int_0^x \cos x \sin t y(t) dt - \lambda \sin^2 x y(x) -$$

$$- \lambda \cos^2 x y(x) - \lambda \int_x^{\pi} \sin x \cos t y(t) dt$$

$$y''(x) = -y'(x) - \lambda y(x)$$

$$y''(x) + (1+\lambda) y(x) = 0$$

≡
Solution of Second Kind Fredholm I.E. with degenerate kernel $\rightarrow$

$$y(x) = \lambda \int_a^b K(x,t) y(t) dt$$

$$K(x,t) = \sum_{i=1}^n \alpha_i(x) \cdot \beta_i(t)$$

The value of d for which I.E. has Non-Trivial solⁿ is called Eigen value.

Ex: $y(x) = d \int_0^{2\pi} \sin(x+t) y(t) dt$

$$K(x, t) = \underbrace{\sin x \cos t}_{\alpha_1 \beta_1} + \underbrace{\cos x \sin t}_{\alpha_2 \beta_2} \quad \left(\begin{matrix} \alpha_1, \beta_1, \alpha_2, \beta_2 \\ \text{L.I.} \end{matrix} \right)$$

$$y(x) = d \sin x \underbrace{\int_0^{2\pi} \cos t y(t) dt}_{c_1} + d \cos x \underbrace{\int_0^{2\pi} \sin t y(t) dt}_{c_2}$$

$$y(x) = d c_1 \sin x + d c_2 \cos x$$

$$y(x) = c_1 d \alpha_1(x) + c_2 d \alpha_2(x) + \dots + c_n d \alpha_n(x)$$

$$c_1 = \int_0^{2\pi} \cos t (d c_1 \sin t + d c_2 \cos t) dt$$

$$c_1 \left[1 - d \int_0^{2\pi} \sin t \cos t dt \right] - d c_2 \int_0^{2\pi} \cos^2 t dt = 0$$

$$c_1 - d \pi c_2 = 0 \quad \text{--- (1)}$$

$$c_2 = \int_0^{2\pi} \sin t (d c_1 \sin t + d c_2 \cos t) dt$$

$$c_2 - d c_1 \pi = 0 \quad \text{--- (2)}$$

System of Equⁿ. -

$$\begin{bmatrix} 1 & -d\pi \\ -d\pi & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$D(d)$

$$\text{If } -D(d) = 0, \quad 1 - d^2 \pi^2 = 0$$

$$d^2 = 1/\pi^2 \Rightarrow d = \pm \frac{1}{\pi}$$

if $D(d) = 0 \rightarrow$ Non Trivial solⁿ exist

at $d = \frac{1}{\pi}, -\frac{1}{\pi}$ Non Trivial solⁿ exist.

Degenerate Equⁿ $\rightarrow$ Always finite.

Put $d = 1/\pi$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 = c_2$$

$$\text{Eigen fun}^c = y_{1/\pi}(x) = \frac{1}{\pi} [c_1 (\sin x + \cos x)]$$

$$y_{1/\pi}(x) = \sin x + \cos x$$

No. of L.I. fun^c at $1/\pi = 1$

If Eigen value Non repeated then no. of L.I. fun^c is always 1.

$$\text{A.M. of } \pm 1/\pi = 1$$

$$\& \text{ g.m. " " } = 1$$

Put $d = -1/\pi$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 = -c_2$$

$$y_{-1/\pi}(x) = -\frac{1}{\pi} [c_1 (\sin x - \cos x)]$$

$$y_{-1/\pi}(x) = (\sin x - \cos x)$$

Q. $y(x) = d \int_a^b K(x, t) y(t) dt$

$$K(x, t) = \alpha_1(x) \beta_1(t) + \alpha_2(x) \beta_2(t) + \alpha_3(x) \beta_3(t)$$

$$y(x) = d c_1 \alpha_1(x) + d c_2 \alpha_2(x) + d c_3 \alpha_3(x) \quad \text{--- (1)}$$

$$c_i = \int_a^b \beta_i(t) y(t) dt \quad \text{--- (2)}$$

$$\int_a^b \underbrace{\beta_1(x)}_{\alpha_1} y(x) dx = d c_1 \int_a^b \underbrace{\alpha_1(x)}_{\alpha_1} \beta_1(x) dx + d c_2 \int_a^b \underbrace{\alpha_2(x)}_{\alpha_2} \beta_1(x) dx + d c_3 \int_a^b \underbrace{\alpha_3(x)}_{\alpha_3} \beta_1(x) dx$$

$$\begin{aligned}(1 - d\alpha_{11})c_1 - d\alpha_{12}c_2 - d\alpha_{13}c_3 &= 0 \\ -d\alpha_{21}c_1 - (1 - d\alpha_{22})c_2 + d\alpha_{23}c_3 &= 0\end{aligned}$$

$$D(\lambda) = \begin{bmatrix} 1 - d\alpha_{11} & -d\alpha_{12} & -d\alpha_{13} \\ -d\alpha_{21} & 1 - d\alpha_{22} & -d\alpha_{23} \\ -d\alpha_{31} & -d\alpha_{32} & 1 - d\alpha_{33} \end{bmatrix}$$

$$\alpha_{ji} = \int_a^b \alpha_i(x) \cdot \beta_j(x) dx$$

Q. $\sin(x+t) = \sin x \cos t + \cos x \sin t$

$$\begin{bmatrix} 1 - d\alpha_{11} & -d\alpha_{12} \\ -d\alpha_{21} & 1 - d\alpha_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\alpha_{11} = \int_0^{2\pi} \sin x \cos x dx = 0$$

$$\alpha_{12} = \int_0^{2\pi} \cos t \cdot \cos x dx = \pi$$

$$\alpha_{21} = \int_0^{2\pi} \sin^2 x dx = \pi$$

$$\alpha_{22} = \int_0^{2\pi} \cos x \sin x dx = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Q.

$$y(x) = \lambda \int_0^1 (x\sqrt{x} - x\sqrt{x}) y(t) dt$$

$$\lambda = 1$$

$$\lambda = -3/4$$

Q

$$y(x) = \lambda \int_1^2 \left(xt + \frac{1}{xt} \right) y(t) dt$$

Find Eigen value's.

$$\lambda = -17\pi \pm 6 \cdot 0$$

A/Oct/14, 3, 13, 21, 35, 37, 40, 41, 42

$$D(A) \cdot C = 0$$

I) $|D(A)| = 0 \Rightarrow$ Infinite Solⁿ.

$$AX = 0$$

$$|A| = 0 \Rightarrow \exists \text{ Infinitely many sol}^n.$$

$$\left[\begin{array}{l} |A| \neq 0 \text{ iff } \rho(A) = n \\ |A| = 0 \text{ iff } \rho(A) < n \end{array} \right]$$

II) $D(A) \neq 0$

$\Rightarrow$ Unique Sol. or only trivial solⁿ.
No solution Does not exist.

$d = d_i \Rightarrow |D(A)| = 0 \Rightarrow$ Infinite solⁿ.

$d \neq d_i \forall i \quad |D(A)| \neq 0 \Rightarrow$ Unique solⁿ.

In homo. case trivial solⁿ exist.

Q. (4)

$$\phi(x) = \frac{5x}{6} + \frac{1}{2} \int_0^1 x \pm \phi(t) dt$$

$$\phi(x) = \frac{5x}{6} + d \int_0^1 -x \pm \phi(t) dt$$

$$\phi(x) = \frac{5x}{6} + dxc$$

where $c = \int_0^1 \pm \phi(t) dt$

$$c = - \int_0^1 t \left[\frac{5t}{6} + dtc \right] dt$$

$$c = \frac{5}{18} + \frac{dc}{3}$$

$$c \left(1 - \frac{d}{3} \right) = \frac{5}{18} \quad \left(\because d = \frac{1}{2} \right)$$

$$c \left(1 - \frac{1}{6} \right) = \frac{5}{18} \Rightarrow c = \frac{1}{3}$$

$$\phi(x) = \frac{5}{6}x + \frac{1}{2} \cdot \frac{1}{3}x = x$$

$$\phi(x) = x$$

it is Fredholm T.E.

(a) (d)

(limits are variable)

$$\phi(0) + \phi(1) = 1 + 0 = 1$$

$$\phi\left(\frac{3}{4}\right) + \phi\left(\frac{1}{4}\right) = \frac{3}{4} + \frac{1}{4} = 1$$

$$\int_0^x e^{x-t} \cdot \phi(t) dt = \sinh x$$

D. w. r. to t (Le

$$\int_0^x e^{x-t} \phi(t) dt + \phi(x) = \cosh x$$

$$\sinh x \Rightarrow \phi(x) = \cosh x - \sinh x$$

$$\phi(x) = e^{-x}$$

$$\int_0^x 3^{x-t} \cdot \phi(t) dt = x$$

$$\log 3 \int_0^x 3^{x-t} \phi(t) dt + \phi(x) = 1$$

$$\phi(x) = 1 - x \log 3$$

7 - (c)

$$\left(\frac{d}{dx} a^x = a^x \log_e a \right)$$

Solution of Non-homo. Fredholm T.E. of Second Kind with degenerate $\rightarrow$

$$y(x) = f(x) + \int_a^b K(x, t) y(t) dt$$

$$K(x, t) = \alpha_1(x) \cdot \beta_1(t) + \alpha_2(x) \cdot \beta_2(t) + \alpha_3(x) \cdot \beta_3(t)$$

$$\int_a^b \alpha_1(x) \cdot \beta_1(x) y(x) dx = \alpha_1(x) \cdot c_1$$

$$y(x) = f(x) + d c_1 \alpha_1(x) + d c_2 \alpha_2(x) + d c_3 \alpha_3(x) \quad \text{--- (1)}$$

$$\int_a^b \textcircled{1} \times \beta_1(x) dx \Rightarrow \int_a^b \beta_1(x) y(x) dx = \int_a^b f(x) \beta_1(x) dx + d c_1 \underbrace{\int_a^b \alpha_1(x) \beta_1(x) dx}_{\alpha_{11}} + d c_2 \underbrace{\int_a^b \alpha_2(x) \beta_1(x) dx}_{\alpha_{12}} + d c_3 \underbrace{\int_a^b \alpha_3(x) \beta_1(x) dx}_{\alpha_{13}}$$

$$c_1 = \gamma_1 + d c_1 \alpha_{11} + d c_2 \alpha_{12} + d c_3 \alpha_{13}$$

$$(1 - d \alpha_{11}) c_1 - d \alpha_{12} c_2 - d \alpha_{13} c_3 = \gamma_1$$

$$-d \alpha_{21} c_1 - (1 - d \alpha_{22}) c_2 - d \alpha_{23} c_3 = \gamma_2$$

$$-d \alpha_{31} c_1 - d \alpha_{32} c_2 - (1 - d \alpha_{33}) c_3 = \gamma_3$$

System of Equation $\rightarrow$

$$\begin{bmatrix} 1 - d \alpha_{11} & -d \alpha_{12} & -d \alpha_{13} \\ -d \alpha_{21} & (1 - d \alpha_{22}) & -d \alpha_{23} \\ -d \alpha_{31} & -d \alpha_{32} & (1 - d \alpha_{33}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

$$\# \quad \underline{AX = b} \quad ; \quad A_{n \times n}$$

$$\text{I) } |A| \neq 0 \Rightarrow \bar{A}^{-1} (AX) = \bar{A}^{-1} b$$

$$\Rightarrow x = \bar{A}^{-1} b$$

$\Rightarrow$ Unique Non-Trivial Solⁿ.

$$\text{II) } |A| = 0 \Rightarrow \rho(A) = \underline{k} < n$$

$$\left[\begin{array}{l} \text{(a)} \quad \rho(A) = \rho(A; b) \end{array} \right.$$

$\Rightarrow$ Infinite Solution (Many)

$$\left[\begin{array}{l} \text{(b)} \quad \rho(A) \neq \rho(A; b) \end{array} \right.$$

$\Rightarrow$ No Solution

Q. $y(x) = f(x) + d \int_0^1 (1-3xt) y(t) dt$

- (a) $d=2$ & $f(x)=1$, I.E. has unique solⁿ. $\rightarrow$ Indiv
 (b) $d=-2$ & $f(x)=\sinh x$, I.E. " " "
 (c) $d=2$ & $f(x)=\cosh x$, I.E. " " "
 (d) $d=1$ & $f(x)=x$, I.E. " " "
 $|D(d)| \neq 0 \rightarrow$ Unique.

first check d 's are Eigen value.

if No Eigen value then never Unique Solⁿ.

Solⁿ $\rightarrow$ $\alpha_1(x) = 1$, $\beta_1(x) = 1$
 $\alpha_2(x) = -3x$, $\beta_2(x) = t$

$$D(d) = \begin{bmatrix} 1-d\alpha_{11} & -d\alpha_{12} \\ -d\alpha_{21} & 1-d\alpha_{22} \end{bmatrix}$$

$$\alpha_{11} = \int_0^1 dt = 1$$

$$\alpha_{12} = \int_0^1 -3x dt = -\frac{3}{2}$$

$$\alpha_{21} = -\int_0^1 x dx = \frac{1}{2}$$

$$\alpha_{22} = \int_0^1 -3x^2 dx = -1$$

$$\Rightarrow \begin{bmatrix} 1-d & \frac{3}{2}d \\ -\frac{1}{2}d & 1+d \end{bmatrix}$$

$$\Rightarrow |D(d)| = 0 \Rightarrow 1-d^2 + \frac{3}{4}d^2 = 0$$

$$\Rightarrow 1 - \frac{d^2}{4} = 0 \Rightarrow d = \pm 2$$

✓ $d = 2$, $f(x) = 1$ has no-Solⁿ.

✗ $d = -2$, $f(x) = 1$ has Infinite Solⁿ.

✓ $d = 2$, $f(x) = x$ has No Solⁿ.

✗ $d = -2$, $f(x) = x$ has Infinite Solⁿ.

Solⁿ →

$$D(d) = \begin{bmatrix} 1-d & 3/2 d \\ -1/2 d & 1+d \end{bmatrix}$$

$$\gamma_1 = \int_0^1 f(x) \beta_1(x) = \int_0^1 dx = 1$$

$$\gamma_2 = \int_0^1 x \cdot dx = \frac{1}{2}$$
$$= \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$d = 2$

$$D(d) = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}$$

Rank = 1

$$[D(d) : \gamma] = \begin{bmatrix} -1 & 3 & 1 \\ -1 & 3 & 1/2 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

$$\rho(A) \neq \rho(A:b)$$

∴ No solution.

$d = -2$

$$D(d) = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$$

Rank = 1

$$[D(d) : \gamma] = \begin{bmatrix} 3 & -3 & 1 \\ 1 & -1 & 1/2 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

∴ No Solⁿ

$$y_1 = \int_0^1 x \cdot 1 = \frac{1}{2}$$

$$y_2 = \int_0^1 x \cdot x dx = \frac{1}{3}$$

$$\underline{\underline{=}} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$$

$$d=2$$

$$D(d) = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}, \quad \text{Rank} = 1$$

$$[D(d):d] = \begin{bmatrix} -1 & 3 & 1/2 \\ -1 & 3 & 1/3 \end{bmatrix}, \quad \text{Rank} = 2$$

$\therefore$ No Solⁿ.

Q. $y(x) = \lambda \int_0^1 (1-3xt) y(t) dt$

Eigen value $\lambda_1 = 2, \lambda = -2$

Find Eigen fun^c corresponding to Fredholm Equⁿ.

Solⁿ $\rightarrow y(x) = \lambda c_1 - 3\lambda c_2 x \quad (\alpha_1 = 1, \alpha_2 = -3x)$

$$D(2) = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-c_1 = -3c_2$$

$$c_1 = 3c_2$$

$$y_2(x) = 2(c_1 - c_2 x) = 2c_1 (1-x)$$

$$y_2(x) = \underline{\underline{=}} (1-x)$$

$$D(-2) = \begin{bmatrix} 3 & -3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = c_2$$

$$y_{-2}(x) = -2 [c_1 - 3c_1 x] \\ = -2c_1 (1-3x)$$

$$y_{-2}(x) = 1-3x \Rightarrow 1 \cdot \alpha_1(x) + 1 \cdot \alpha_2(x)$$

$$y(x) = d_1 \alpha_1(x) + d_2 \alpha_2(x) + \dots + d_n \alpha_n(x) \cdot d_n$$

if any fun^c is not in linear combination then never Eigen fun^c .

$\Rightarrow$ Eigen fun^c is l.c. of fun^c .

$$\alpha_1(x) = 1, \quad \alpha_2(x) = -3x$$

$$(a) \quad \lambda = 2, \quad f(x) = 1$$

$$y(x) = 1-x$$

$$\int_0^1 y(x) \cdot f(x) dx = \int_0^1 (1-x) dx = \frac{1}{2} \neq 0$$

No Solⁿ.

Note $\Rightarrow \lambda = d_i, \quad \forall d_i \rightarrow$ Eigen fun^c .

$$\int_a^b y d_i(x) \cdot f(x) \neq 0$$

$\Rightarrow$ No Solⁿ

$$(b) \quad \lambda = -2, \quad y(x) = 1-3x$$

$$\int_0^1 (1-3x) \cdot 1 dx = 1 - \frac{3}{2} = -\frac{1}{2} \neq 0$$

No Solⁿ.

$$(c) \quad \lambda = 2, \quad f(x) = 2, \quad y(x) = 1-x$$

$$\int_0^1 2(1-x) dx = 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right] \Rightarrow \frac{1}{6} \neq 0$$

No Solⁿ.

$$(d) \quad \lambda = -2, \quad f(x) = x, \quad y(x) = 1-3x$$

$$\Rightarrow \int_0^1 x(1-3x) dx = \left(\frac{x^2}{2} - x^3 \right) \Big|_0^1 = \frac{1}{2} - 1 \neq 0$$

No Solⁿ.

Summary

$$y(x) = f(x) + d \int_a^b k(x,t) y(t) dt$$

$$k(x,t) = \sum_{i=1}^n \alpha_i(x) \beta_i(t)$$

$$y(x) = f(x) + \sum_{i=1}^n d \alpha_i(x)$$

$$D(d) = \begin{bmatrix} 1 - d\alpha_1 & \dots & -d\alpha_{1n} \\ -d\alpha_{1n} & \dots & 1 - d\alpha_{nn} \end{bmatrix}$$

$$\alpha_{ji} = \int_a^b \alpha_i(x) \cdot \beta_j(x) dx$$

$$V_j = \int_a^b f(x) \cdot \beta_j(x) dx$$

For Eigen value & Eigen func.

② For uniqueness & Existence $\rightarrow$

$$y(x) = f(x) + d \int_a^b k(x,t) y(t) dt$$

$$|D(d)| \neq 0$$

$$|D(d)| = 0$$

$$f(x) \neq 0$$

$$f(x) = 0$$

$$f(x) \neq 0$$

$$f(x) = 0$$

$$f(x) \neq 0$$

$$V_j = 0$$

$$\beta_j(x) \mid f(x)$$

Unique

Trivial solⁿ

Unique Non-Trivial Solⁿ

Infinite Solⁿ

Infinite Solⁿ

$$e(D(d)) = e(D(d))$$

$$; \gamma$$

$$e(D(d)) \neq 0$$

$$; \gamma$$

Infinite Solⁿ

No solution

Now $\exists j, \gamma_j \neq 0$
 $d = d_i \text{ (} |D(d_i)| = 0 \text{)}$

If $y_{d_i}(x)$ is an eigen fun^c corresponding to d_i
 for Homo. f.o.I.E.

(I) $\int_a^b f(x) \cdot y_{d_i}(x) dx = 0 \Rightarrow \text{Infinite Sol}^n$

II) $\int_a^b f(x) \cdot y_{d_i}(x) dx \neq 0 \Rightarrow \text{No solution}$

$d = d_i, D(d_i)$
 G.M. (Geometry Multiplicity) $d_i = n - \rho[D(d_i)]$
 $n = \rho[D(d_i)] + \eta[D(d_i)]$
 $\rightarrow$ (No. of l.i. Eigen fun^c)

III) If $d = d_i$ is an Eigen value
 & $y_{d_i}, y_{2d_i}, \dots, y_{kd_i}$ are l.i. functions
 Eigen function corresponding to d_i for homogeneous
 finite Integral Equⁿ.

Then if -

(a) $\forall j \int_a^b f(x) \cdot y_{d_{j_i}}(x) dx = 0 \Rightarrow \text{Infinite Sol}^n$

b) $\exists j \text{ s.t. } \int_a^b f(x) \cdot y_{d_{j_i}}(x) dx \neq 0 \Rightarrow \text{No - Sol}^n$

Q. $y(x) = f(x) + d \int_0^{2\pi} \sin(x+t) \cdot y(t) dt$

(a) $f(x) = \sin x$ & $d = \frac{1}{\pi} \Rightarrow \text{f.o.I.E. has No - Sol.}$

$$2\pi - (\sin x) \Big|_0^{2\pi} = 0$$

$$= (\sin x) \Big|_0^{2\pi} = 0$$

$$f(x) = \cos x \quad \& \quad d = \frac{1}{\pi} \Rightarrow f \cdot I \cdot E \text{ has No-Sol}^n.$$

$$f(x) = x \quad \& \quad d = -\frac{1}{\pi} \Rightarrow f \cdot I \cdot E \text{ has Infinite Sol}^n$$

$$f(x) = \cos x \quad \& \quad d = -\frac{1}{\pi} \Rightarrow \text{ " " " " }$$

$$d = +\frac{1}{\pi}$$

$$y = \frac{1}{\pi}, \quad y(x) = \sin x + \cos x$$

$$y = -\frac{1}{\pi}, \quad y(x) = \sin x - \cos x$$

$$\int_0^{2\pi} (\sin x) (\sin x + \cos x) dx = \pi \neq 0$$

No-Solⁿ.

$$\int_0^{2\pi} x (\sin x - \cos x) dx = \int_0^{2\pi} x \sin x dx - \int_0^{2\pi} x \cos x dx$$

$$= [-x \cos x]_0^{2\pi} + \int_0^{2\pi} \cos x dx - [x \sin x]_0^{2\pi} - \int_0^{2\pi} \sin x dx$$

$$= -2\pi \neq 0$$

No Solⁿ.

$$\int_0^{2\pi} (\cos x) (-\sin x + \cos x) dx = \pi \neq 0$$

No Solⁿ.

$$14 \rightarrow d = \frac{4}{\pi+2}, \quad d = \frac{4}{\pi-2} \quad \begin{matrix} \psi(c) d = 4/\pi+2 \\ \psi(d) d = 4/\pi-2 \end{matrix}$$

Unique Sol. for $d \neq \frac{4}{\pi+2}$

$$d = \pm \text{ True, } d = \frac{4}{\pi-2} \quad \text{D.O.}$$

6/10/14.

$$(26) (c) \quad d = \frac{1}{\pi}, \quad d = -\frac{1}{\pi}$$

$$\begin{matrix} \cos x & , & \sin x \\ f(x) = & \sin x + \cos x \end{matrix}$$

$$\text{at } d = \frac{1}{\pi}, \quad |D(d)| \neq 0 \rightarrow \text{Unique Sol}^n.$$

$$(9) \quad y(x) = f(x) + d \int_{-1}^1 \cos[\pi(x-t)] \phi(t) dt$$

$$\begin{aligned} K(x,t) &= \cos \pi(x-t) \\ &= \cos \pi x \cos \pi t + \sin \pi x \sin \pi t \end{aligned}$$

$$\begin{aligned} \alpha_{11} &= \int_{-1}^1 \cos^2 \pi t dt = 2 \int_0^1 \frac{1 + \cos 2\pi x}{2} dx \\ &= \left[x + \frac{\sin 2\pi x}{2\pi} \right]_0^1 = 1 \end{aligned}$$

$$\alpha_{21} = 0$$

$$\alpha_{12} = 0$$

$$\alpha_{22} = \int_{-1}^1 \sin^2 \pi x dx = 1$$

$$D(d) = \begin{bmatrix} 1-d & 0 \\ 0 & 1-d \end{bmatrix}, \quad d = 1, -1$$

Eigen func $\rightarrow$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y(x) = c_1 \cos \pi x + c_2 \sin \pi x$$

$$y(x) = \cos \pi x$$

$$y(x) = \sin \pi x$$

$$\gamma_1 = \int_{-1}^1 x \cos \pi x dx = 0 \quad \text{even odd}$$

$$\gamma_2 = \int_{-1}^1 x \sin \pi x dx \neq 0 \quad \gamma_1 \neq \gamma_2$$

$$f(x) = 1, \quad d = 1$$

$$I_1 = \int_{-1}^1 \cos \pi x \, dx = 0$$

$$I_2 = \int_{-1}^1 \sin \pi x \, dx = 0$$

$\gamma_1 = \gamma_2$ Infinite solⁿ

$$(36) (a) \quad y(x) = \sinh x + \int_0^1 \cos x \cdot \sec t \, y(t) \, dt$$

$$y(x) = d \cos x$$

$$y(x) = \cos x$$

$$\int_0^1 \sinh x \cdot \cos x \, dx \Rightarrow \int_0^1 \frac{e^x - e^{-x}}{2} \cos x \, dx$$

$$= \frac{1}{2} \left[e^x \sin x + e^x \cos x - e^{-x} \sin x + e^{-x} \cos x \right]$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

$$\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$$

Q. $y(x) = 2x + d \int_0^1 \sin(\log x) y(t) dt$

$$y(x) = 2x + dc \sin(\log x)$$

$$c = \int_0^1 y(t) dt$$

$$c = \int_0^1 [2t + dc \sin(\log t)] dt$$

$$= \left[t^2 + dc \cos(\log t) \cdot t \right]_0^1$$

CA) $\frac{1}{1+d} = 1$, $c = \frac{1}{1+d}$

$y(x) = \frac{2}{1+d}$

$$c = \int_0^1 2t dt + dc \int_0^1 \sin(\log t) dt$$

Put $\log t = z$

$t = e^z$, $dt = e^z dz$

$$\left(t^2 \right)_0^1 + dc \int_{-\infty}^0 e^z \sin z dz$$

Put $z = -u$

$dz = -du$

$$c = 1 - dc \int_0^{\infty} e^{-u} \sin u du$$

$$c = 1 - \frac{dc}{2} \Rightarrow c \left(1 + \frac{d}{2} \right) = 1$$

$$c = \frac{1 \times 2}{d+2}$$

$$y(x) = 2x + \frac{2d}{(d+2)} \sin(\log x)$$

$$\left[-e^{-u} \sin u \right]_0^{\infty} + \int_0^{\infty} \cos u e^{-u} du$$

$$- \sin u [0 - 1] + 0$$

$$- \cos u [0 - 1] + \frac{1}{2} c$$

α . $d = 2$

β . $d \neq 2$ $d = -2$ fail $y(x)$

γ . $d = 1$

δ . $d \neq 1$ $d = -1$

Q.

$$y(x) = d \int_0^1 \cos(\log(x^k)) y(t) dt, \quad K \in \mathbb{N}$$

Let $y(x) = f(x) + d \int_a^b k(x, t) y(t) dt$

Iterated Kernel of $k(x, t)$

$$k_1(x, t) = k(x, t)$$

$$k_m(x, t) = \int_a^b k(x, z) k_{m-1}(z, t) dz, \quad m=2,$$

$$= \int_a^b k_{m-1}(x, z) k(z, t) dz, \quad m=2, 3.$$

It is m^{th} Iterated Kernel of $k(x, t)$.

Volterra Equⁿ $\rightarrow$

$$y(x) = f(x) + d \int_a^x k(x, t) y(t) dt$$

$$k_1(x, t) = k(x, t)$$

$$k_m(x, t) = \int_t^x k(x, z) k_{m-1}(z, t) dz =$$

$$\int_t^x k_{m-1}(x, z) \cdot k(z, t) dz$$

$$m=2, 3.$$

Let $y(x) = f(x) + d \int_a^b k(x, t) y(t) dt$

Resolvent Kernel : $R(x, t, d) = \sum_{m=1}^{\infty} d^{m-1} \cdot k_m(x, t)$

Q.

$$y(x) = \frac{5}{6}x + d \int_0^1 xt y(t) dt$$

Solⁿ →

$$K_1(x, t) = K(x, t) = xt$$

$$K_2(x, t) = \int_0^1 K(x, z) K_1(z, t) dz$$

$$= \int_0^1 xz \cdot zt dz = \frac{1}{3}xt$$

$$K_3(x, t) = \int_0^1 K(x, z) \cdot K_2(z, t) dz$$

$$= \frac{1}{3} \int_0^1 x \cdot z \cdot zt dz = \frac{1}{3^2} xt$$

$$R(x, d, t) = \sum_{m=1}^{\infty} d^{m-1} K_m(x, t)$$

$$= \sum_{m=1}^{\infty} \left(\frac{d}{3}\right)^{m-1} xt = xt \cdot \frac{1}{1-d/3}$$

$$\left(1 + \lambda + \lambda^2 + \dots \infty = \frac{1}{1-\lambda}\right) \quad \left|\frac{d}{3}\right| < 1$$

$$|d| < 3$$

$$y(x) = \frac{5}{6}x + d \int_0^1 xt y(t) dt$$

$$y(x) = \frac{5}{6}x + dcx$$

$$c = \int_0^1 \left(\frac{5}{6}t + dct\right) t \cdot dt$$

$$c = \frac{5}{18} + \frac{dc}{3}$$

$$\Rightarrow c \left(1 - \frac{d}{3}\right) = \frac{5}{18}$$

if $d = 3$ then No sol

at $|d| < 3$ it exist unique conti. Solⁿ.
(Resolvent Kernel convergent)

Q. $y(x) = 1 \int_0^1 e^{\min\{x,t\}} y(t) dt$

Then 2nd - Iterate Kernel $K_2(x,t)$ is —

(a)
$$\left\{ \begin{array}{ll} e^{x+t} (2-t) - \frac{e^{2t} + 1}{2} & : 0 \leq x < t \\ e^{x+t} (2-x) - \frac{e^{2x} + 1}{2} & : t \leq x \leq 1 \end{array} \right\}$$

(b)
$$\left\{ \begin{array}{ll} e^{x+t} (2-t) - \frac{e^{2x} + 1}{2} & : 0 \leq x < t \\ e^{x+t} (2-x) - \frac{e^{2t} + 1}{2} & : t \leq x \leq 1 \end{array} \right\}$$

(c)
$$\left\{ \begin{array}{ll} e^{x+t} (2-x) - \frac{e^{2t} + 1}{2} & : 0 \leq x < t \\ e^{x+t} (2-t) - \frac{e^{2x} + 1}{2} & : t \leq x \leq 1 \end{array} \right\}$$

(d)
$$\left\{ \begin{array}{ll} e^{x+t} (2+x) - \frac{e^{2x} + 1}{2} & : 0 \leq x < t \\ e^{x+t} (2-t) - \frac{e^{2t} + 1}{2} & : t \leq x \leq 1 \end{array} \right\}$$

Solⁿ → $K(x,t) = \left\{ \begin{array}{ll} e^x & 0 \leq x < t \\ e^t & t \leq x \leq 1 \end{array} \right\}$

$K_2(x,t) = \int_0^1 K(x,z) K_1(z,t) dz$

$\xrightarrow{x < t}, \int_0^x e^z e^z dz + \int_x^t e^x \cdot z dz + \int_t^1 e^x \cdot e^t dz$

(2 - small)

$z < x < t$

$x / \text{in } (xz) \text{ } x < z \text{ then } e^x \cdot t$

$(x,z) \text{ } x < z$

$$= \frac{1}{2} (e^{2x} - 1) + e^x [e^t - e^x] + e^{x+t} (1-t)$$

$$= \frac{1}{2} e^{2x} - \frac{1}{2} + e^{x+t} - e^{2x} + e^{x+t} (1-t)$$

$$= e^{x+t} (2-t) - \frac{1}{2} (e^{2x} + 1)$$

$$x > t \quad \int_0^t e^z \cdot e^z dz + \int_t^x e^z \cdot e^t dz + \int_x^1 e^x \cdot e^t dz$$

$0 \leq z < t < x \quad \quad \quad t < z < x \quad \quad \quad t < x < z \leq 1$

$$\Rightarrow \frac{1}{2} (e^{2t} - 1) + e^{t+x} - e^{2t} + e^{x+t} (1-x)$$

$$= e^{x+t} (2-x) - \frac{1}{2} (e^{2t} + 1)$$

for Fredholm Equⁿ $\rightarrow$

$$y(x) = f(x) + d \int_a^b k(x,t) y(t) dt$$

let $y_0(x) = f(x)$ (Non homo.)

$$y_1(x) = f(x) + d \int_a^b k(x,t) y_0(t) dt$$

$$y_{n+1}(x) = f(x) + d \int_a^b k(x,t) y_n(t) dt$$

$y(x)$ is exact Solⁿ.

$y_n(x) \rightarrow y(x)$ as $n \rightarrow \infty$ (Unique solⁿ exists)

Only N^y

(I) $K(x,t) \neq 0$ & Real Continuous fun^c

& $|K(x,t)| \leq M$ on $R: a \leq x \leq b, a \leq t \leq b$

only Not

(II) $f(x) \neq 0$, real, continuous
 $\& |f(x)| \leq N, \forall x \in I: a \leq x \leq b$

(III) $|d| < \frac{1}{M(b-a)}$

(I) & (II) are only N^y condition for exactness of uni
 & conti sol. for fredholm Equ.

Condition (III) is not N^y.

If we add (III) then (I), (II), (III) become Suffic

If only (I), (II) condi given then solⁿ is Not unique

If Equation is Volterra $\rightarrow$

(I) & (II) Condition Same.

(III) d is any constant.

Fredholm Integral Equation $\rightarrow$

$$y(x) = f(x) + d \int_a^b k(x,t) y(t) dt$$

(I) & (II) $k(x,t) \neq 0$, cts, real & bdd

(II) $f(x) \neq 0$ " " "

(III) $|d| < \frac{1}{M(b-a)}$

(IV) d is any constant.

F.I.E. $\rightarrow$ I & II N^y for a I.E. have a solution.

I, II & III are Suff. for a I.E. have unique
 cts solution on I.

V.I.E. $\rightarrow$ I, II are N^y & Suff. for existness of unique
 cts solⁿ.

⇒ Every continuous fun^c defined on closed interval is bdd. (In case of single variable).

NEUMANN SERIES →

$$R(x, t, d) = \sum_{m=1}^{\infty} d^{m-1} K_m(x, t)$$

$$y(x) = f(x) + d \int_a^b R(x, t, d) f(t) dt$$

$$|d| < \frac{1}{B}$$

$$B^2 = \int_a^b \int_a^b |K(x, t)|^2 dx dt \rightarrow$$

Norm of Kernel $K(x, t)$

Ex. 3 $K(x, t) = xt$

(c) $B^2 = \int_0^1 \int_0^1 x^2 t^2 dx dt$

$$B^2 = \frac{1}{9}, \quad B^2 = \frac{1}{9}, \quad B = \frac{1}{3}$$

$$|d| < 1/1/9 \Rightarrow |d| < 3$$

$$-3 < d < 3$$

$y(x) = x^2 + dx c$

$$c = \int_0^1 t [t^2 + dct] dt \Rightarrow$$

$$c = \frac{1}{4} + \frac{dc}{3}$$

$$c \left(\frac{1-d}{3} \right) = \frac{1}{4}$$

$d = 3$ it has no solution

Resolvent Kernel

$d \neq 3$ it has unique Solⁿ.

(29) (d) $y_0(x) = x$ (Given)

$$y_1(x) = x - \int_0^x t(x-t) dt$$

$$y_1(x) = x - \left[\frac{xt^2}{2} - \frac{t^3}{3} \right]_0^x$$

$$y_1(x) = x - \frac{x^3}{2} + \frac{x^3}{3} = x - \frac{x^3}{6}$$

$$y_2(x) = x - \int_0^x (x-t) \left(\frac{t-t^3}{6} \right) dt$$

$$y_2(x) = x - \left[\frac{xt^2}{2} - \frac{xt^4}{24} - \frac{t^3}{3} + \frac{t^5}{30} \right]_0^x$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^3}{3} - \frac{x^5}{30}$$

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$y_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

(23) (a) $K(x,t) = \begin{cases} x+t & 0 \leq t \leq x \\ 0 & \text{otherwise} \end{cases}$

$$y(x) = 1 + \lambda \int_0^x (x+t) y(t) dt$$

$$0 \leq x \leq 1 \Rightarrow 0 \leq t \leq 1$$

(35) (a) $| \lambda | < \frac{1}{M}$

$$-\frac{1}{M} < \lambda < \frac{1}{M}$$

Uncountable $\left(-\frac{1}{M}, 0 \right)$, $\left(0, \frac{1}{M} \right) \rightarrow$ Uncountable +ve value of λ
 -ve λ value

$$\neq y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt$$

$$y(x) = f(x) + \lambda \int_a^b K(x, t) \left[f(t) + \lambda \int_a^b K(t, z) y(z) dz \right] dt$$

$$y(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x, t) K(t, z) y(z) dz dt$$

$$y(x) = f(x) + \lambda \int_a^b K_1(x, t) f(t) dt + \lambda^2 \int_a^b \left[K(x, t) \int_a^b K(t, z) f(z) dz \right] dt$$

$$= \int_a^b f(z) \left[\int_a^b K(x, t) K(t, z) dt \right] dz \quad \text{--- (1)}$$

$$y(x) = f(x) + \lambda \int_a^b K_1(x, t) f(t) dt + \lambda^2 \int_a^b K_2(x, z) f(z) dz$$

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x, t) f(t) dt$$

$$y(x) = f(x) + \lambda \int_a^b \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) f(t) dt$$

$$y(x) = f(x) + \lambda \int_a^b R(x, t, \lambda) f(t) dt$$

This method not apply on first kind Equⁿ.

Fredholm First Kind can't be obtain Second Kind.

$$f(x) = \int_a^b k(x, t) y(t) dt$$

$$f'(x) = \int_a^b k(x, t) y(t) dt \quad (\text{first kind})$$

Volterra ~~Eq. (1)~~ $\rightarrow$

$$f(x) = \int_a^x k(x, t) y(t) dt$$

$$f'(x) = \int_a^x k_x(x, t) y(t) dt + k(x, x)$$

$$y(x) = \frac{f'(x) - \int_a^x k_x(x, t) y(t) dt}{k(x, x)} \quad (2)$$

First Kind Volterra convert into Second Kind if

1) $k(x, x) \neq 0$

2) $\frac{\partial k}{\partial x}$ exists.

Equ. (2) is called Second Kind Volterra Eqⁿ
then we find resolvent kernel.

Q. 40 $\frac{x^2}{2} = \int_0^x e^{x-t} y(t) dt + y(x)$

$$x = \int_0^x e^{x-t} y(t) dt + y(x)$$

$$y(x) = x - \int_0^x e^{x-t} y(t) dt$$

$$y(x) = x - \frac{x^2}{2}$$

Q. let $y(x) = -x - \int_0^x e^{x-t} y(t) dt \quad 0 \leq x \leq 1$

$R(x, t, d)$ is Resolvent Kernel then —

- a) $R(1, t, 1)$ is $\uparrow \forall t \in [0, 1]$
 b) $R(x, 1, -1)$ is $\downarrow \forall t \in [0, 1]$
 c) $R(x, 1, -1)$ Bounded & periodic
 d) $R(x, 1, 1) \uparrow \forall x \in [0, 1]$

Solⁿ →

$$K_2(x, t) = \int_t^x e^{x-z} \cdot e^{z-t} dz$$

$$= e^{x-t} (x-t)$$

$$K_3(x, t) = \int_t^x e^{x-z} (z-t) \cdot e^{z-t} dz$$

$$= e^{x-t} \int_t^x (z-t) dz$$

$$= e^{x-t} \cdot \frac{(x-t)^2}{2}$$

$$K_m(x, t) = e^{x-t} \cdot \frac{(x-t)^{m-1}}{(m-1)!}$$

$$R(x, t, d) = \sum_{m=1}^{\infty} d^{m-1} K_m(x, t)$$

$$= e^{x-t} \sum_{m=1}^{\infty} \frac{d^{m-1}}{(m-1)!} (x-t)^{m-1}$$

$$= e^{x-t} \cdot e^{d(x-t)}$$

$$R(x, t, d) = e^{(x-t)(1+d)}$$

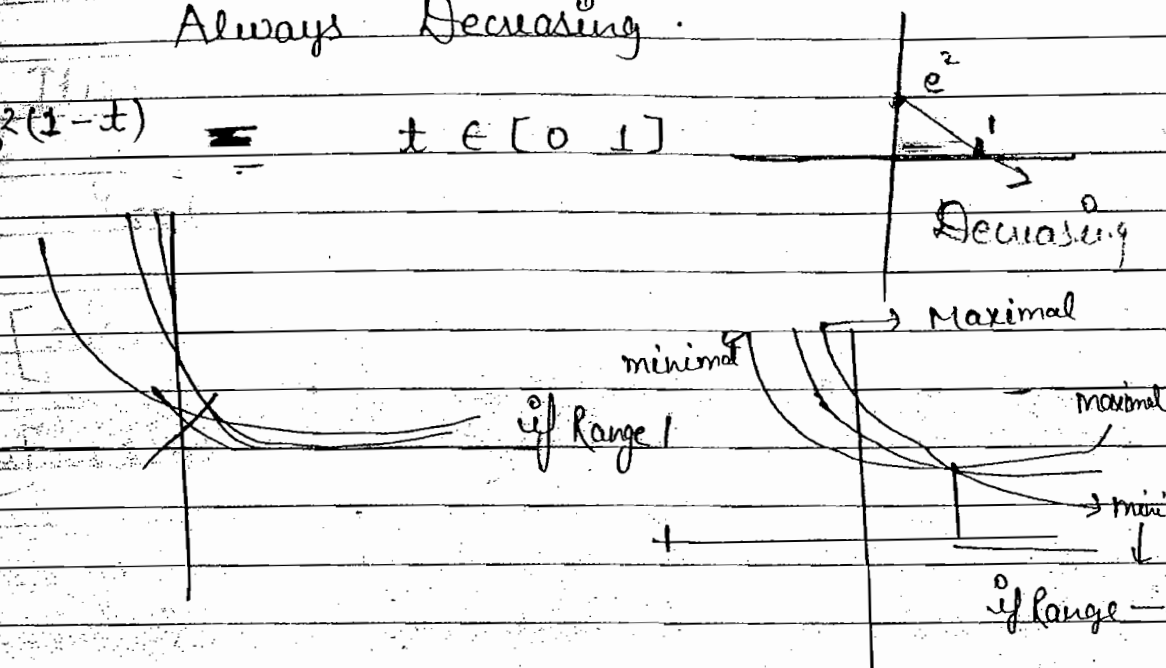
- (a) $d=1 \therefore R(1, t, 1) = e^{2(1-t)} \quad t \in [0, 1]$
 (b) $R(x, 1, -1) = e^{(x-1)(1-1)} = 1$
 Constant func

$$R(x, t, 1) = e^{2(x-1)}, \quad x \in [0, 1]$$

$$\text{or } e^{2(x-1)} < 0$$

Always Decreasing.

$$e^{2(1-t)} \quad t \in [0, 1]$$



Find Resolvent Kernel $R(x, \xi; 1)$

$$\phi(x) = x + \int_0^{1/2} \phi(\xi) d\xi$$

$$K(x, t) = 1 = K_1(x, t)$$

$$K_2(x, t) = \int_0^{1/2} K_1(x, z) \cdot K(z, t) dz$$

$$= \int_0^{1/2} 1 dz = [z]^{1/2} = \frac{1}{2}$$

$$K_3(x, t) = \int_0^{1/2} -K_2(x, z) \cdot K(z, t) dz$$

$$= \int_0^{1/2} \frac{1}{2} dz \rightarrow \frac{1}{2^2}$$

Resolvent Kernel $R(x, \xi; 1)$

$$= \sum_{n=0}^{\infty} K_{x+1} d^n$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \dots \infty = \frac{1}{1-1/2} = 2$$

27/Oct/2014

Let $R(x, t, d)$ is Resolvent Kernel of
 $y(x) = f(x) + d \int_a^b K(x, t) y(t) dt$

Then $\rightarrow$

$$K_1(x, t) = K(x, t)$$

$$K_m(x, t) = \int_a^b K(x, z) \cdot K_{m-1}(z, t) dz$$

$$= K(x, t) + \sum_{n=1}^{\infty} d^n \left[\int_a^b K(x, z) K_n(z, t) dz \right]$$

$$= K(x, t) + d \int_a^b K(x, z) \left[\sum_{n=1}^{\infty} d^{n-1} K_n(z, t) \right] dz$$

$$R(x, t, d) = \sum_{m=1}^{\infty} d^{m-1} K_m(x, t)$$

$$= K_1(x, t) + \sum_{m=2}^{\infty} d^{m-1} K_m(x, t)$$

$$= K_1(x, t) + \sum_{n=1}^{\infty} d^n \cdot K_{n+1}(x, t) \quad \left(\text{put } n=m-1 \right)$$

Properties $\rightarrow$

$K(x, t)$ Symmetric Kernel :

I) The Eigen fun^c Corresponding to distinct Eigen value are orthogonal.

i.e. $d_1 \neq d_2$ are two Eigen value's. &

y_{d_1} and y_{d_2} are corresponding Eigen fun^c then -

$$\langle y_1, y_2 \rangle = \int_a^b y_{d_1}(x) \cdot \overline{y_{d_2}(x)} dx = 0 \quad [\text{Fredholm}]$$

II) Eigen values of Symmetric Kernel are real.

(III) If $K(x, t)$ Symmetric then $K_m(x, t)$ Symmetric $\forall m$.

(IV) Iterative kernel of Symmetric kernel is Symm.

$$y(x) = d \int_a^b K(x, t) y(t) dt$$

$$y_1(x) = d_1 \int_a^b K(x, t) y_1(t) dt$$

$$y_2(x) = d_2 \int_a^b K(x, t) y_2(t) dt$$

$$\int_a^b y_1(x) \cdot \bar{y}_2(x) dx = \int_a^b \bar{y}_2(x) \left[d_1 \int_a^b K(x, t) y_1(t) dt \right] dx$$

$$= d_1 \int_a^b y_1(t) \left[\int_a^b K(x, t) \bar{y}_2(x) dx \right] dt$$

$$\bar{y}_2(x) = \bar{d}_2 \int_a^b \bar{K}(x, t) \bar{y}_2(t) dt$$

$$\frac{\bar{y}_2(x)}{\bar{d}_2} = \int_a^b K(t, x) \bar{y}_2(t) dt$$

$$= \frac{d_1}{\bar{d}_2} \int_a^b y_1(t) \bar{y}_2(t) dt$$

$$\Rightarrow \left(1 - \frac{d_1}{\bar{d}_2} \right) \int_a^b y_1(t) \bar{y}_2(t) dt = 0$$

$$\Rightarrow \int_a^b y_1(t) \bar{y}_2(t) dt = 0$$

$$y_1 \perp y_2$$

$$\text{II)} \quad \left(1 - \frac{\lambda_1}{\lambda_2}\right) \int_a^b y_1(t) \overline{y_2(t)} dt = 0$$

$$\lambda_1 = \alpha + i\beta \rightarrow y_1(x) \text{ eigen fun}^c$$

$$\lambda_2 = \overline{\alpha} - i\beta \rightarrow \overline{y_1(x)}$$

$$\Rightarrow \left(1 - \frac{\lambda_1}{\lambda_2}\right) \int_a^b y_1(t) \overline{y_1(t)} dt = 0$$

$$\Rightarrow \left(1 - \frac{\lambda_1}{\lambda_1}\right)$$

In Skew Symmetric Kernel Eigen values are either purely imaginary or zero.

→ In Herm. Skew Symm. Kernel $\Rightarrow$ purely img. e-value.

→ In case of Skew Symm. eigen fun^c corresponding to distinct eigen values are orthogonal.

III)

$$K(x, t) = \overline{K(t, x)}$$

$$K_1(x, t) = k(x, t)$$

$$K_2(x, t) = \int_a^b k(x, z) K_1(z, t) dz$$

$$= \int_a^b \overline{K(z, x)} \overline{K_1(t, z)} dz$$

$$= \int_a^b \overline{K_1(t, z)} \overline{K(z, x)} dz = \overline{K_2(t, x)}$$

$$m = n$$

$$K_n(x, t) \text{ Symm.}$$

$$K_{n+1}(x, t) = \int_a^b k(x, z) K_n(z, t) dz$$

$$= \int_a^b \overline{K(z, x)} \overline{K_n(t, z)} dz$$

$$\int_a^b K_n(t, z) \cdot \bar{K}(z, x) dz = \bar{K}_{n+1}(t, x)$$

$$K_m(x, t) = \begin{cases} \text{Symm.} & \text{if } m \text{ - even} \\ \text{Skew-Symm} & \text{if } m \text{ - odd} \end{cases}$$

L_2 - Function $\rightarrow$ $K(x, t)$ is called L_2 - fun^c —

(I) If $\int_a^b \int_a^b |K(x, t)|^2 dx dt$ is finite or $< \infty$

(II) If $\int_a^b |K(x, t)| dx$ finite.

(III) If $\int_a^b |K(x, t)| dt$ finite.

If Kernel is one-variable then condition (II) & (III) Satisfied.

$K(x, t)$ Symm. & L_2 - fun^c —

(I) Then Multiplicity of each eigen value is finite.

(II) The Seq. ~~$\langle \lambda_n \rangle$~~ of eigen value's does not have any finite limit point.

$$\boxed{\frac{1}{\lambda_n} \rightarrow 0 \text{ or } \lambda_n \rightarrow \infty} \quad \left(\text{for Non-degenerate} \right)$$

$K(x, t)$ be any kernel. λ is an eigen value of $K(x, t)$
 iff λ^n is an eigen value of $K^n(x, t)$

$K(x, t)$ has distinct eigen value $\Rightarrow K_2(x, t)$
 has distinct eigen value (Incorrect)

$$K_3(x, t) = \int_0^{2\pi} \sin(x-t) = i\pi, -i\pi$$

$$K(x, t) = \frac{1}{\pi}, -\frac{1}{\pi}$$

$$K_2(x, t) = \frac{1}{\pi^2}$$

$$K_3(x, t) = i\pi^3, -i\pi^3$$

All are distinct e-value

Picard Method $\rightarrow$

$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

$$\int_{x_0}^x \frac{dy}{dx} \cdot dx = \int_{x_0}^x f(x, y) dx$$

$$y(x) = y_0 + \int_{x_0}^x f(x, y) dx$$

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$y_n(x) = y_0 + \int_{x_0}^x f(x, y_{n-1}(x)) dx$$

$$y_n(x) \rightarrow y(x) \quad (\text{a sol. of diff. Equ}^n)$$

If $f(x, y)$ continuous on R $|x - x_0| \leq a$

& Bdd on R $|y - y_0| \leq b$

Q. Find the Second Successive diff. Equ of —

$$\frac{dy}{dx} = e^x + y^2(x), \quad y(0) = 0$$

Solⁿ Any $f(x, y) = e^x + y^2$
 $y_0(x) = 0$

$$y_1(x) = y_0 + \int_0^x f(t, y_0) dt$$

$$y_0 + \int_0^x e^t dt = e^x - 1$$

$$y_2(x) = y_0 + \int_0^x f(t, y_1) dt$$

$$y_2(x) = \int_0^x e^t + (e^t - 1)^2 dt$$

$$= \int_0^x (e^t + e^{2t} + 1 - 2e^t) dt$$

$$= \left[-e^t + \frac{1}{2}e^{2t} + t \right]_0^x$$

$$y_2(x) = -e^x + \frac{1}{2}e^{2x} + x + \frac{1}{2}$$

Lipschitz function $\rightarrow$ $f(x, y)$ is Lipschitz
funcⁿ of y if —

$$\exists K \in \mathbb{R}^+ \text{ s.t.}$$

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$$

$$\forall (x, y_1), (x, y_2) \in R$$

$$R: |x - x_0| \leq a, \quad |y - y_0| \leq b$$

Q. $f(x, y) = xy^2$

$R_1: |x| \leq 1, |y| \leq 1$

$R_2: |x| \leq 1, |y| < \infty$

Then $f(x, y)$ is -

- (a) Lipschitz ~~on~~ R_1 but not on R_2 .
 (b) Lipschitz on R_2 but not on R_1 .
 (c) " " both $R_1 + R_2$
 (d) Neither Lipschitz on R_1 Nor on R_2 .

Solⁿ $\rightarrow |f(x, y_1) - f(x, y_2)|$

$$\begin{aligned} &= |xy_1^2 - xy_2^2| \leq |x| \cdot |y_1 + y_2| \cdot |y_1 - y_2| \\ &\leq |x| \cdot |y_1| + |y_2| \cdot |y_1 - y_2| \\ &\leq 2 \cdot 1 \cdot |y_1 - y_2| \\ &\leq 2 |y_1 - y_2| \end{aligned}$$

where $k = 2$, $k \in \mathbb{R}^+$ is Lipschitz const.

$$\begin{aligned} |f(x, y_1) - f(x, 0)| &\leq k |y_1 - 0| \\ &\leq k |y_1| \end{aligned}$$

$$|xy_1^2| \leq k |y_1|$$

$$|x| \cdot |y_1| \leq k$$

k - finite

✗

$$|x| \leq 1$$

$$|y| < \infty$$

$|y_1| \rightarrow$ Infinite

$\therefore$ This func is not bounded on Lipschitz on R_2 .

==

$f(x, y) = y^{2/3}$

$$|x| \leq 1, |y| \leq 1$$

$$|f(x, y_1) - f(x, 0)| \leq k |y_1|$$

$$|y^{2/3}| \leq k |y_1|$$

$$|y'| \leq k, \quad \alpha < 0$$

If $y \rightarrow 0 \Rightarrow y^\alpha \rightarrow \infty$ Unbdd
 $-1 \leq y \leq 1$

$\therefore$ Not Lipschitz.

Any funⁿ y^α , $y < 1$ & domain containing '0' then
 Not Lipschitz.

If fun^c is continuous, bdd & Lipschitz then always has a unique solution.

Q. 14.

I. Let $f(x, y)$ be a function defined on domain D subse of xy -plane.

Let $(x, y_1), (x, y_2) \in D$ (arbitrary point) : $P_1 \neq P_2$ are two points lying on.

Surface $z = f(x, y)$, α be the angle b/w chord joining P_1 & P_2 with xy plane. then choose the correct-

(I) If $0 \leq \alpha < \frac{\pi}{2} \Rightarrow f(x, y)$ Lipschitz fun^c.

II) If $f(x, y)$ is Lipschitz fun^c then chord joining P_1 & P_2 not $\perp$ to xy plane.

III) If $\frac{\pi}{2} \leq \alpha \leq \pi \Rightarrow f(x, y)$ is Lipschitz fun^c.

(IV) None of these.

Solⁿ $\rightarrow P_1 [(x, y_1), f(x, y_1)],$
 $P_2 [(x, y_2), f(x, y_2)]$

$$|f(x, y_1) - f(x, y_2)| \leq k |y_1 - y_2|$$

$$\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| = |\tan \alpha| < k$$

$$y_1 \neq y_2$$

$$\forall (x, y_1), (x, y_2) \in D$$

$$0 < \alpha < \frac{\pi}{2}$$

$$, \frac{\pi}{2} < \alpha < \pi$$

Line on xy plane not be $\perp$ i.e. $\alpha \neq \pi/2$

Tangent line of Surface not be $\perp$ to xy -plane.

Note $\rightarrow$

Q. (1) Let $f(x, y)$ be a funcⁿ defined on Domain D Subset of xy -plane if $\frac{\partial f}{\partial y}$ exist & Bdd on $D \Rightarrow$

$f(x, y)$ is Lipschitz on D .

$$\left| \frac{\partial f}{\partial y} \right| < k \quad \forall y$$

$$\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| < k$$

$$\Rightarrow |f(x, y_1) - f(x, y_2)| < k |y_1 - y_2|$$

Conversely,

$$f(x) = x|y|$$

$$D: |x| \leq 1, |y| \leq 1$$

$$\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| = \left| \frac{x|y_1| - x|y_2|}{y_1 - y_2} \right|$$

$$< |x| \cdot |y_1 - y_2|$$

$$f(x, y) = \begin{cases} xy & y \geq 0 \\ -xy & y < 0 \end{cases}$$

$$\leq |y_1 - y_2|$$

$f'(x)$ exist at $x = a$

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \text{exist}$$

$$\lim_{y_1 \rightarrow y_2} \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}$$

$$(x, y_2) = (x, 0)$$

$$\Rightarrow \frac{f(x, y_1) - 0}{y_1} = \begin{cases} x & y_1 \geq 0 \\ -x & y_1 < 0 \end{cases}$$

$$= \frac{xy_1}{y_1} = x$$

e. $f(x, y)$ is Lipschitz on $D \nRightarrow \frac{\partial f}{\partial y}$ exist.

(II) $f(x, y)$ Lipschitz $\nRightarrow \frac{\partial f}{\partial y}$ exist

(III) If $f(x, y)$ Lipschitz + $\frac{\partial f}{\partial y}$ exist $\Rightarrow \frac{\partial f}{\partial y}$ bdd.

(IV) If $\frac{\partial f}{\partial y}$ exist But not bdd on $D \Rightarrow f(x, y)$ not Lipschitz.

Lipschitz $\text{fun}^c \nRightarrow$ continuous (In two variable)

$$f(x, y) = y + [x] \quad D: |x| \leq 5, |y| \leq 5$$

$$|f(x, y_1) - f(x, y_2)| = |y_1 + [x] - y_2 - [x]|$$

$$= |y_1 - y_2|$$

$\Rightarrow f(x, y)$ Lipschitz fun^c

$$\lim_{(x, y) \rightarrow (x, y)} = \text{Does not exist}$$

$x \in \mathbb{Z} \Rightarrow f(x, y)$ not continuous at (x, x)

Lipschitz $\text{fun}^c \Rightarrow$ conti. (Single variable)

Q. Let $f(x, y)$ be a Lipschitz fun^c defined on domain D then —

- a. $f(x, y)$ conti^c for any fixed x .
- b. $f(x, y)$ " " " " y —
- c. $f(x, y)$ " " " $\forall (x, y)$
- d. None of these.

Note $\rightarrow$
$$\begin{aligned} |f(y_1) - f(y_2)| &\leq K |y_1 - y_2|^{1+\alpha} \\ \alpha > 0 &\Rightarrow f(x) \text{ is conti.} \\ \alpha > 0 &\Rightarrow f(x) \text{ is diff.} \end{aligned}$$

* Let $f(x, y)$ be any fun^c defined on D s.t. $\partial f / \partial y$ exists & Bdd.

Then Lipschitz constant is $L.U.B \left| \frac{\partial f}{\partial y} \right|$
 $(x, y) \in D$

Q. Which of following is Lip. fun^c —

- a. $f(x, y) = x^2 + y^2$ $R: |x| \leq a$
- b. $f(x, y) = x \sin y + y \cos x$ $|y| \leq b$
- c. $f(x, y) = x^2 e^{x+y}$ $a, b \in \mathbb{R}^+$
- d. $f(x, y) = A(x) \cdot y^2 + B(x) \cdot y + c(x)$
 A, B, c are conti. on $|x| \leq a$

∂^n_a $|f(x, y_1) - f(x, y_2)|$

$\partial f / \partial y = 2y$ bdd fun^c

$\partial f / \partial y = x \cos y + y \sin x$ bdd

$\partial f / \partial y = x^2 e^{x+y}$ bdd.

A, B, c conti. on closed interval $\Rightarrow$ Bdd

$\frac{\partial f}{\partial y} = 2y \cdot A(x) + B(x) = \text{Bdd.}$

Ans. a, b, c, d.

And if $R: |x-x_0| \leq a, |y-y_0| \leq b$

$$R \subset D$$

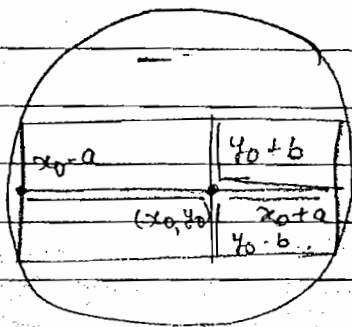
then $h = \min \left\{ a, \frac{b}{M} \right\}$

$$-a \leq x - x_0 \leq a$$

$$x_0 - a \leq x \leq x_0 + a$$

$$-b \leq y - y_0 \leq b$$

$$y_0 - b \leq y \leq y_0 + b$$



(III) $f(x, y)$ is l.p. func on $D \Rightarrow$ Unique Solⁿ on $|x-x_0| \leq h$ guarantee
 $|x-x_0| < h, h > 0$ This Interval is Local Int.

The Interval in which picard's thm guarantee the existence of the solⁿ is Local Interval.

2) The diff. Equ. is Soluble then the largest interval in which solution is defined is called "local Interval".

Suff. condition for existence I, II, III
 $\vdash$ I + II are Necessary condition.

Q. $\frac{dy}{dx} = 1 + y^2, y(0) = 0$

The largest Interval in which picard's Diff. Equ. has a solution —

✓ ✓ $-\frac{1}{2} < x < \frac{1}{2}$

∴ $\frac{3}{4} < x < \frac{5}{4}$

$\frac{3}{4}, \frac{5}{4} \notin (0, 0)$ Picard's fail

✗ ✓ $-\pi/2 < x < \pi/2$

∴ $-\pi/2 \leq x \leq \pi/2$

Solⁿ $R: |x-x_0| \leq a, |y-y_0| \leq b \quad (y(0)=0)$
 $|x| \leq a, |y| \leq b, a, b \in \mathbb{R}^+$

$$f = \frac{dy}{dx} = 1+y^2$$

$f' = 2y$ it is odd, conti. + Lipschitz.
 $\exists h > 0$ s.t.

Diff. Equⁿ has unique sol. on $|x| \leq h$

$h = \min \left\{ a, \frac{b}{M} \right\}$

$$|f(x,y)| = |1+y^2| = 1+b^2$$

$$h = \min \left\{ a, \frac{b}{1+b^2} \right\}$$

($h \rightarrow$ Largest interval)

$$f(b) = \frac{b}{1+b^2}$$

$$f'(b) = \frac{(1+b^2) \cdot 1 - 2b^2}{(1+b^2)^2} = 0$$

$$= \frac{1-b^2}{(1+b^2)^2} = 0 \Rightarrow b^2 = 1$$

$$b = \pm 1$$

$$f(1) = 1/2, \quad f(-1) = -1/2$$

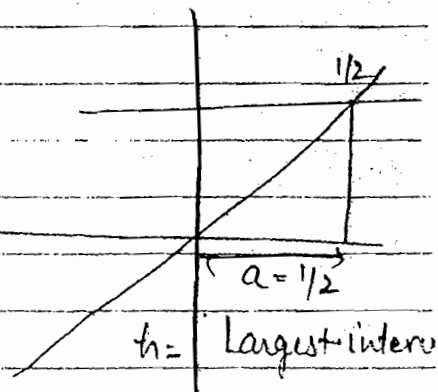
maxima at $1/2$

$$h = \min \left\{ a, 1/2 \right\}$$

$$a < \frac{1}{2}, \quad \min \left\{ a, \frac{1}{2} \right\} = a = h$$

$$a > \frac{1}{2}, \quad \min \left\{ a, \frac{1}{2} \right\} = \frac{1}{2} = h$$

Equate to $a = 1/2$



Largest Interval of solⁿ of diff. Equⁿ. —

$$\frac{dy}{1+y^2} = dx$$

$$\tan^{-1}(y) = x + c$$

$$y(x) = \tan x$$

$$y(0) = \tan 0 = 0$$

If in Que. Largest Interval of solⁿ of diff. Equⁿ
in which diff. Equⁿ has unique solⁿ —

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

Ans: c correct.

Unique Solⁿ exist then — (a)

has a solⁿ — (a), (c)

Largest interval — (c)

I. The largest Interval in which diff. Equⁿ has a solⁿ —

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

II) The largest Interval in which picard's thm holds
local interval $-\frac{1}{2} < x < \frac{1}{2}$

III) The Interval in which diff. Equ has a sol.

$$I \subseteq -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Q. $\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 0$

The local Interval in which diff. Equⁿ has unique.

Q. $|x| \leq \frac{1}{\sqrt{2}}$

$$(b) \quad |x| \leq \frac{1}{2}$$

$$(c) \quad |x| \leq \frac{1}{4}$$

$$(d) \quad |x| \leq \frac{\sqrt{2}}{2}$$

Solⁿ, $f(x, y) = x^2 + y^2$
 $R: |x| \leq a, |y| \leq b, a, b \in \mathbb{R}^+$

$$h = \min \left\{ a, \frac{b}{M} \right\} = \min \left\{ a, \frac{b}{a^2 + b^2} \right\}$$

$$f(b) = \frac{b}{a^2 + b^2}$$

$$f'(b) = \frac{a^2 + b^2 - 2b^2}{(a^2 + b^2)^2} = 0$$

$$b = \pm a$$

$$f(a) = \frac{a}{2a^2} = \frac{1}{2a}, \quad f(-a) = \frac{-a}{2a^2} = -\frac{1}{2a}$$

$$\text{Max. } b = a$$

$$h = \min \left\{ a, \frac{1}{2a} \right\}$$

$$a = \frac{1}{2a} \Rightarrow a = \frac{1}{\sqrt{2}}$$

Q. $\frac{dy}{dx} = \frac{1+2x+3y}{1+x^2+y^2} \quad |x| \leq 2, |y| \leq 1$

The interval in which Picard's thm guarantees the existence of Solⁿ -

$$(a) \quad |x| \leq 1/2$$

$$(b) \quad |x| \leq 1/4$$

$$(c) \quad |x| \leq 3/4$$

$$(d) \quad \text{None of these.}$$

Sol. $h = \left\{ a, \frac{b}{M} \right\}$

$$M = |f(x, y)| = \left| \frac{1+2x+3y}{1+x^2+y^2} \right|$$

$$= \left| \frac{(1+2x+3y)}{1+x^2+y^2} \right| \leq \frac{1+2|x|+3|y|}{1+x^2+y^2}$$

$$= \frac{1+2 \cdot 2+3 \cdot 1}{1} = 8 \quad \left(\begin{array}{l} \text{Deno.} \rightarrow \text{Minimize} \\ \text{Nume.} \rightarrow \text{Maxo} \end{array} \right)$$

Q. $\frac{dy}{dx} = y^2, \quad y(1) = -1$

find the local interval in which diff. Eqnⁿ has unique solⁿ.

Solⁿ

$$f = y^2$$

$$f' = 2y, \quad \text{odd, conti, Lipschitz.}$$

$$|f(x, y)| = b^2$$

$$h = \text{Min} \left\{ a, \frac{b}{M} \right\}$$

$$\begin{array}{l} -b < y+1 < b \\ -(1+b) \leq y < b-1 \end{array}$$

$$R: |x-1| \leq a,$$

$$|y+1| \leq b, \quad a, b \in \mathbb{R}^+$$

$$h = \text{Min} \left\{ a, \frac{b}{(b+1)^2} \right\}$$

$$f(b) = b / (b+1)^2 \Rightarrow$$

$$f'(b) = \frac{(b+1)^2 \cdot 1 - 2b(b+1)}{(b+1)^4} = 0$$

$$\Rightarrow (b+1) [b+1-2b] = 0$$

$$\Rightarrow -b = -1 \Rightarrow b = +1$$

$$f(1) = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$-x \leq \frac{5}{4} \quad (c) \quad |x| \leq \frac{1}{4}$$

$$\frac{3}{4} \leq x \leq \frac{5}{4} \quad (d) \quad \text{None of these.}$$

$$h = \min \left\{ a, \frac{1}{4} \right\} \Rightarrow a = \frac{1}{4}$$

$$|x-1| \leq \frac{1}{4}$$

$$-\frac{1}{4} \leq x-1 \leq \frac{1}{4} \Rightarrow \frac{3}{4} \leq x \leq \frac{5}{4}$$

$$Q. D_1: \frac{dy}{dx} = y^{1/3}, \quad y(0) = 0$$

$$D_2: \frac{dy}{dx} = y^2, \quad y(1) = -1$$

a. X D_1 has unique ^{infinite} solⁿ on every bdd Interval I .

b. D_2 " " " on $|x-1| \leq \frac{1}{4}$

c. D_1 has Infinite many solⁿ on $\mathbb{R}^+$

d. D_2 has Infinite Sol on $|x-1| \leq \frac{1}{4}$

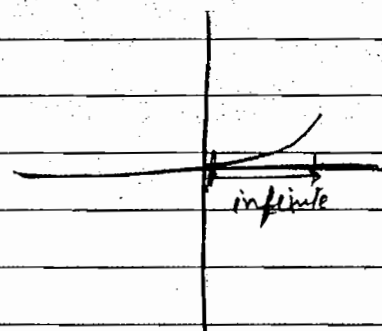
$$\text{Sol}^n \rightarrow y^{-1/3} dy = dx$$

$$\frac{y^{-1/3+1}}{-1/3+1} = x+C$$

$$\Rightarrow -\frac{3}{2} y^{2/3} = x$$

$$y(x) = \left(\frac{2}{3}\right)^{3/2} \cdot x^{3/2}$$

$$y(x) = \begin{cases} (x-r)^{3/2} & x > r \\ 0 & x \leq r \end{cases}$$



Q. $\frac{dy}{dx} = e^{y^2} - 1$, $y(0) = 0$

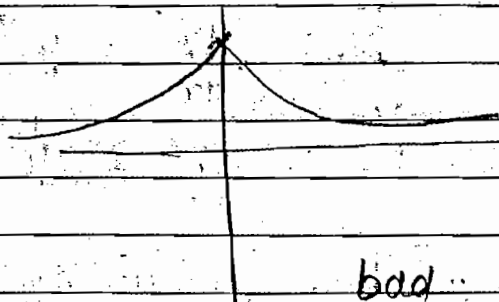
a. $y(x) > 0 \quad \forall x < 0$

b. $y(x) < 0 \quad \forall x > 0$

c. $y(x)$ changes its sign in $[0, 1]$

~~d.~~ $y(x) \equiv 0 \quad \forall x > 0$

Solⁿ $\rightarrow x \rightarrow -\infty, y(x) \rightarrow 1$
 $x \rightarrow \infty, y(x) \rightarrow \infty$



$\forall -\infty < y < \infty$

$-\infty < x < \infty$

at $x \rightarrow \infty, e^{y^2}(-2y) \rightarrow 0$

$x \rightarrow -\infty, e^{y^2}(-2y) \rightarrow 0$

It is Lipschitz funcⁿ on $\mathbb{R}$.

It has trivial Solⁿ on $\mathbb{R}$.

$y(0) = e^0 - 1 = 1 - 1 = 0$

Integral Equation Assignment - 1

(1) $g(x) y(x) = f(x) + \lambda \int_a^b K(x, t) y(t) dt$

(b) Linear Equⁿ of Fredholm Type

(2) $\lambda = \lambda_1$
 $|D(\lambda)| = 0 \rightarrow \text{Infinite Sol}^n$
 $\lambda \neq \lambda_1$
 $|D(\lambda)| \neq 0 \rightarrow \text{Unique Sol}^n$

Ans $\rightarrow$ (d)

(3) $y(x) = x^2 + \lambda \int_0^1 x t y(t) dt$

$y(x) = x^2 + \lambda c x$
 where $c = \int_0^1 t y(t) dt$

$c = \int_0^1 t [t^2 + \lambda c t] dt$

$c = \left[\frac{t^4}{4} + \frac{\lambda c t^3}{3} \right]_0^1$

$c \left(1 - \frac{\lambda}{3} \right) = \frac{1}{4}$

$\lambda \neq 3$

Laying in interval $-3 < \lambda < 3$

Ans $\rightarrow$ (c)

(4) $\phi(x) = \frac{5x}{6} + \frac{\lambda}{2} \int_0^1 x t \phi(t) dt$

$\phi(x) = \frac{5x}{6} + \lambda c x$

where $c = \int_0^1 t \phi(t) dt$

$$d = \int_0^1 t \left[\frac{5t}{6} + d t \right] dt$$

$$c = \left[\frac{5t^3}{18} + \frac{d t^3}{3} \right]_0^1$$

$$c \left[\frac{1}{3} \right] = \frac{5}{18}$$

$$c \left(1 - \frac{1}{6} \right) = \frac{5}{18} \Rightarrow c = \frac{1}{3}$$

$$\phi(x) = \frac{5x}{6} + \frac{1}{2} x \times \frac{1}{3} = x$$

$$\phi(0) + \phi(1) = 0 + 1 = 1$$

$$\phi\left(\frac{3}{4}\right) + \phi\left(\frac{1}{4}\right) = \frac{3}{4} + \frac{1}{4} = 1$$

Ans $\rightarrow$ (a), (d)

$$(5) \quad \int_0^x e^{x-t} \phi(t) dt = \sinh x \quad \text{--- (1)}$$

$$= \int_0^x e^{x-t} \phi(t) dt + \phi(x) = \cosh x \quad (\text{Diff.})$$

$$\text{by (1)} \quad \sinh x + \phi(x) = \cosh x$$

$$\phi(x) = \cosh x - \sinh x = e^{-x}$$

Ans $\rightarrow$ (a)

$$(6) \quad K(x, t) = \sum_{n=1}^{\infty} x^n t^n \rightarrow \text{Non-degenerate} \\ \text{Infinite terms}$$

$$K(x, t) = x e^{|x|t} \rightarrow \text{Degenerate}$$

$$K(x, t) = e^{|x-t|} \rightarrow \text{Non Degenerate}$$

$$K(x, t) = e^{xt} \rightarrow \text{Non Degenerate}$$

Ans $\rightarrow$ (b)

$$(7) \int_0^x \frac{x-t}{3^{x-t}} \phi(t) dt = x$$

Diff.

$$\int_0^x \frac{x-t}{3^{x-t}} \log 3 \phi(t) dt + \log 3 \phi(x) = 1$$

$$x \log 3 + \log 3 \phi(x) = 1$$

$$\phi(x) = 1 - x \log 3.$$

Ans $\rightarrow$ (c)

$$(8) (d) u(x) = f(x) + \int_a^x K(x,t) u(t) dt$$

Fredholm's Linear I.E. of Second Kind.

$$(9) \frac{d^2 y}{dx^2} + y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

$$\frac{d^2 y}{dx^2} = -y$$

$$\frac{dy}{dx} = 1 - \int y \phi(xy) dy$$

$$(11) \quad \phi(x) = x + 2 \int_0^x e^{x-\xi} (\xi) d\xi$$

$$K(x, \xi) = e^{x-\xi} = K_1(x, \xi)$$

$$K_2(x, \xi) = \int_0^x K(x, z) \cdot K_1(z, \xi) dz$$

$$= \int_0^x e^{x-z} \cdot e^{z-\xi} dz$$

$$K_2(x, \xi) = x \cdot e^{x-\xi}$$

$$K_3(x, \xi) = \int_0^x K(x, z) \cdot K_2(z, \xi) dz$$

$$= \int_0^x e^{x-z} \cdot z \cdot e^{z-\xi} dz$$

$$= \frac{e^{x-\xi} \cdot x^2}{2!}$$

$$R(x, t, d) = \sum_{m=1}^{\infty} d^{m-1} K_m(x, t)$$

$$= (e^{x-\xi}) + 2x e^{(x-\xi)} + \frac{2^2 x^2}{2!} e^{(x-\xi)}$$

$$= e^{(x-\xi)} \left[1 + 2x + \frac{(2x)^2}{2!} + \dots \right]$$

$$= e^{(x-\xi)} \cdot e^{2(x-\xi)} = e^{3(x-\xi)}$$

$$Q.12 \quad \phi(x) = \cos x + d \int_0^\pi \cos(x+t) \phi(t) dt$$

$$\cos(x+t) = \underbrace{\cos x}_{\alpha_1} \underbrace{\cos t}_{\beta_1} + \underbrace{\sin x}_{\alpha_2} \underbrace{\sin t}_{\beta_2}$$

$$\alpha_{11} = \int_0^\pi \cos x \cdot \cos x dx = \frac{\pi}{2}$$

$$\alpha_{12} = \int_0^{\pi} -\sin x \cos x dx = 0$$

$$\alpha_{21} = \int_0^{\pi} \cos x \sin x dx = 0$$

$$\alpha_{22} = \int_0^{\pi} \sin x \cdot \cos x \sin x dx = -\frac{\pi}{2}$$

$$D(d) = \begin{vmatrix} 1-d\alpha_{11} & -d\alpha_{12} \\ -d\alpha_{21} & 1-d\alpha_{22} \end{vmatrix} = \begin{vmatrix} 1-d\pi/2 & 0 \\ 0 & 1+d\pi/2 \end{vmatrix}$$

$$1-d^2\pi^2/4 = 0 \Rightarrow d = \pm \frac{\pi}{2}$$

$$y(x) = dC_1 \cos x + dC_2 \sin x$$

$$D_{2/\pi}(d) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$D_{-2/\pi}(d) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\gamma_1 = \int_0^{\pi} f(x) \cdot \beta_1 dx = \int_0^{\pi} \cos x \cdot \cos x dx = \frac{\pi}{2}$$

$$\gamma_2 = \int_0^{\pi} \cos x \cdot \sin x dx = 0$$

$\gamma_1 \neq \gamma_2$ (No solution)

$$d = -\frac{2}{\pi}$$

$$\text{Rank of } D(d) = 1$$

$$\begin{bmatrix} 2 & 0 & \pi/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(D:e) = 1$$

Infinite many solⁿ.

$$d = -1, \quad D(-1) = \begin{vmatrix} 1+d/2 & 0 \\ 0 & 1-d/2 \end{vmatrix}$$

$$D(d) \neq 0$$

At $d = -1$, Unique Solⁿ.

$$15) \quad \phi(x) = x + \int_0^{1/2} \phi(\xi) d\xi$$

$$k(x, t) = k_1(x, t) = 1$$

$$k_2(x, t) = \int_0^{1/2} k(x, z) k_1(z, t) dz$$

$$= \int_0^{1/2} 1 \cdot dz \Rightarrow (z)_0^{1/2} = \frac{1}{2}$$

$$k_3(x, t) = \int_0^{1/2} \frac{1}{2} dz = \left(\frac{1}{2}\right)^2$$

$$R(x, t, 1) = \sum d^{m+1} k_m(x, t)$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \dots \infty$$

$$= \frac{1}{1-1/2} = 2 //$$

I.E. Assignment

(2) $y(x) = 1 + \int_0^1 x(e^{xt} - 1) y(t) dt = e^x - x$

At $y(x) = 1$

$$1 + \int_0^1 x(e^{xt} - 1) dt$$

$$= 1 + x \int_0^1 (e^{xt} - 1) dt = 1 + x \left(\frac{e^{xt} - t}{x} \right)_0^1$$

$$= 1 + e^x - x - 1$$

$$= e^x - x$$

Ans $\rightarrow$ (a)

(4) $y(x) = 1 + \int_0^x (x+t) y(t) dt$

$$y'(x) = \int_0^x y(t) dt + 2xy(x)$$

$$y''(x) = 3y(x) + 2xy'(x)$$

$$y'' - 2xy' - 3y = 0$$

Ans $\rightarrow$ (b)

(3) $\sum_{i=0}^{35} a_i y^{(i)}(x) = 0, \quad n > 5, \quad 0 \leq x \leq 1$

$$y^{(i)}(0) = 0$$

(a) Skew if $a_{2i} = 0$

$$n = 35, \quad K \text{ odd}$$

$$n - k - 1 = \text{odd}$$

Skew Symm.

(b) $a_i = 0 \quad \forall \quad i \nmid 35$

i.e. $a_1, a_5, a_7, a_{35} = 0$

$$2 \nmid 35, \quad 3 \nmid 35$$

$$n - k - 1 = 34 - 2 - 1 \text{ odd}$$

$$n - k - 1 = 34 - 3 - 1 \text{ even}$$

Neither Symm. nor Skew

$$(8) \quad y(x) = \sin x + \lambda \int_0^1 (x^2 + t^2) y(t) dt$$

$$\alpha_1 = x^2, \quad \beta_1 = 1, \quad \alpha_2 = 1, \quad \beta_2 = t^2$$

$$\alpha_{11} = \int_0^1 \sin x \cdot x^2 dx = \frac{1}{3}$$

$$\alpha_{12} = \int_0^1 1 \cdot 1 dx = 1$$

$$\alpha_{21} = \int_0^1 x^2 \cdot x^2 dx = \frac{1}{5}$$

$$\alpha_{22} = \int_0^1 1 \cdot t^2 dx = \frac{1}{3}$$

$$D(\lambda) = \begin{bmatrix} 1 - \frac{\lambda}{3} & -\lambda \\ -\frac{1}{5}\lambda & 1 - \frac{1}{3}\lambda \end{bmatrix}$$

$$\Rightarrow 1 + \frac{\lambda^2}{9} - \frac{2\lambda}{3} - \frac{\lambda^2}{5} = 0$$

$$\Rightarrow 4\lambda^2 + 30\lambda - 45 = 0$$

Kernel Symmetric, $\lambda \in \mathbb{R}$

2 Eigen value exist. Both are Real

Kernel Skew Symmetric, $\lambda \in \mathbb{R}$

only imaginary Eigen value.

Kernel Neither Skew Nor Symmetric, $\lambda \in \mathbb{R}$

$$B^2 - 4AC > 0$$

Then Real Eigen value Exist.

$R = \{\lambda_1, \lambda_2\} \rightarrow$ No solⁿ / Infinite.

$\lambda = 0$ Not an Eigen value. But at $\lambda = 0$

$|D(\lambda)| \neq 0$ So unique solⁿ.

If Eqn is homo. then $\lambda = 0$ Not exist

Only Trivial Solⁿ Exist.

OR $|\lambda| < \frac{1}{\beta} = \alpha$

$\Rightarrow (-\alpha, \alpha)$, $\frac{(-\alpha, 0)}{\text{Unique}}$ $\frac{(0, \alpha)}{\text{Unique}}$

Ans $\rightarrow$ (a) (b) (c)

(a) $1 + \int_0^x e^{(x-t)} \lambda (dt) dt = 0$

$1 + \lambda e^x \int_0^x t e^{-t} dt = 0$

$1 + \lambda e^x \left[-t e^{-t} - e^{-t} \right]_0^x = 0$

$\Rightarrow 1 + \lambda e^x (-x e^{-x} - e^{-x} + 1) = 0$

$\Rightarrow 1 - \lambda x - \lambda + \lambda e^x = 0$

$\Rightarrow \lambda (-x - 1 + e^x) = -1$

$\Rightarrow \lambda (1 + x - e^x) = 1$

$\lambda = \frac{1}{1+x-e^x}$ Constant $\neq$ Variable

It is fun^c of x and

But λ is any constant value

So it has no solution of an integral Eqnⁿ for any value of λ .

Ans $\rightarrow$ (d)

(12)

$y'' - f(\lambda) y = 0$

$y(0) = y'(0)$

$y(1) = y'(1)$

(a) $f(\lambda) < 0$, $f(\lambda) = -u^2$

$f(\lambda) > 0$, $f(\lambda) = u^2$

$$y'' - \mu^2 y = 0$$

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$y'(x) = \mu (c_1 e^{\mu x} - c_2 e^{-\mu x})$$

$$y(0) = y'(0)$$

$$c_1 + c_2 = \mu (c_1 - c_2)$$

$$c_1(1 - \mu) + c_2(1 + \mu) = 0$$

$$c_1 e^{\mu} + c_2 e^{-\mu} = \mu (c_1 e^{\mu} - c_2 e^{-\mu})$$

$$c_1 e^{\mu}(1 - \mu) + c_2 e^{-\mu}(1 + \mu) = 0$$

$$c_1(1 - \mu) + c_2(1 + \mu) = 0$$

$$\mu = 1,$$

$$2c_2 e^{-\mu} = 0, \quad 2c_2 = 0 \Rightarrow c_2 = 0$$

$$c_1 = \text{arbitrary}$$

$$\mu = -1,$$

$$c_1 = 0$$

$$c_2 = \text{arbitrary}, \quad f(1) = 1$$

$$f(1) = 1, \quad \text{Non Trivial Sol}^n$$

$$f(1) < 0 \quad \text{Infinite no. of -ve Eigen values.}$$

$$\text{UnBounded below, bdd. above.}$$

$$(c) \quad \text{Eigen fun}^c = e^x \text{ or } \cos x \text{ or } \sin x$$

$$\text{So bdd.}$$

$$\text{Ans} \rightarrow (c), (d)$$

$$(13) \quad \int_t^x e^{x-t} y(t) dt$$

$$\log_2 - K_1(x, t) = e^{x-t}$$

$$\Rightarrow K_2(x, t) = \int_t^x e^{x-z} \cdot e^{z-t} dz$$

$$= e^{x-t} (x-t)$$

$$K_3(x, t) = \int_t^x e^{x-z} \cdot e^{z-t} (z-t) dz$$

$$= e^{x-t} \frac{(x-t)^2}{2}$$

$$K_m(x, t) = e^{x-t} \frac{(x-t)^{m-1}}{(m-1)!}$$

If domain given then m finite
Ans $\rightarrow$ (c), (d)

$$(14) \quad \int_0^x e^{x-t} y(t) dt = \cos x$$

$$\Rightarrow \int_0^x e^{x-t} y(t) dt + y(x) = -\sin x$$

$$\Rightarrow y(x) = -(\cos x + \sin x)$$

It is periodic

Ans $\rightarrow$ (d)

$$(17) \quad y''(x) - f(d) y(x) = 0$$

$$y(0) = 0, \quad y'(\pi) = 0$$

It is Regular SLP, for any d , Trivial Solⁿ

Except highest coeff. all are coeff. constant then Degenerate kernel.

$$(20) \quad (d) \quad \underline{L_4}$$

$$(21) \quad B^2 = 9 \int_0^1 \int_0^1 (x^2 t)^2 dx dt$$

$$B^2 = 9 \int_0^1 x^4 dx \int_0^1 t^2 dt$$

$$= 9 \cdot \frac{1}{5} \cdot \frac{1}{3} = \frac{9}{15}$$

$$B = \frac{3}{\sqrt{15}}$$

Remove 3 from Que. then Ans - (B)

$$22) (b) \quad K(x, t) = |t| \cdot e^{|x|}$$

$$= f(t) \cdot h(x)$$

So it is Degenerate

$$(c) \quad K(x, t) = \min \{x, t\} e^{|x|}$$

$$= \begin{cases} x e^x & x < t \\ t e^{|x|} & x > t \end{cases} \rightarrow x \text{ depend on } t$$

Not degenerate Both fun^c different (Infinite)

$$23) (c) \quad y(x) = f(x) + \sum_{i=1}^n 4c_i d_i(x)$$

$$D(d) \neq 0$$

So unique Solution

$$(26) \quad \|K(x, t)\|^2 = \int_a^b \int_a^b |K(x, t)|^2 dx dt \neq 0$$

$\Rightarrow$ $\exists$ at least one Eigen value.

Eigen value of Volterra Hom. I.E. eqn^n may or may not exist.

(II) Eigen value of Fredholm I.E. always exist's if -

$$\int_a^b \int_a^b |K(x, t)|^2 dx dt \neq 0$$

(III) Eigen value of Fredholm I.E. may or may not exist

$$\text{if } \int_a^b \int_a^b |K(x, t)|^2 dx dt = 0$$

A be any matrix $d \in F$ (field)

is an eigen value d is root of C.E. of A .

$$|A - dI| = 0$$

S.t. $PAXI = b$ has non solⁿ. Trivial Solⁿ.

$$(27) \quad g(x) = d \int_0^1 (\sin \pi x \cdot \cos \pi t) g(t) dt$$

$$g(x) = d \sin \pi x \int_0^1 \cos \pi t g(t) dt$$

$$g(x) = dc \sin \pi x$$

$$c = \int_0^1 \cos \pi t g(t) dt$$

$$c = \int_0^1 \cos \pi t \cdot dc \sin \pi t dt$$

$$c = \frac{dc}{2} \int_0^1 \sin(2\pi t) dt$$

$$A = \frac{dc}{2} \left[\frac{-\cos(2\pi t)}{2\pi} \right]_0^1$$

$$= C = \frac{dc}{4\pi} [-\cos 2\pi + \cos 0] \Rightarrow C = 0$$

$y(x) = 0$ Unique solⁿ
No Eigen value.

(30) Has no solⁿ $\Rightarrow \int_a^b f(x) y_1(x) dx \neq 0$

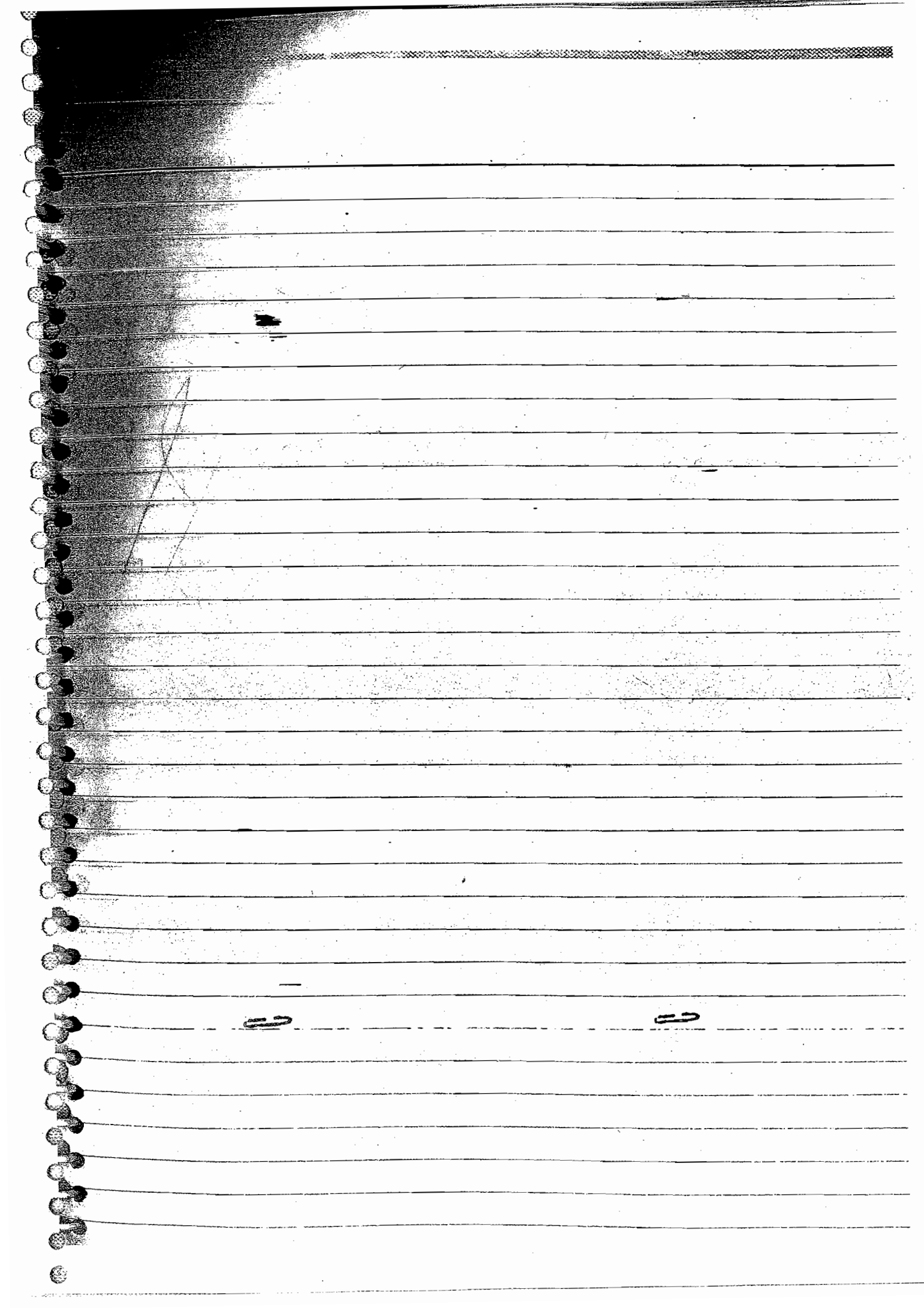
$y_1 \neq y_2$ more than one L.I. vector

$$\int_a^b f(x) y_1(x) = 0$$

$$\int_a^b f(x) y_2(x) \neq 0$$

Ans $\rightarrow$ (b) No sol $\Rightarrow \int_a^b y_1(x) f(x) \neq 0$

it may be $\int_a^b y_2(x) f(x) \neq 0$



Dear Sir

3

$$2x_1 + 3x_2 + 4x_3 = 8$$

$$3x_1 + 4x_2 + 6x_3 = 100$$

If determinant is non zero then Unique solⁿ.

if $x_3 = 0$

$$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \neq 0 \quad \text{Basic sol}^n$$

if $x_2 = 0$

$$\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0 \quad \text{Non Basic}$$

Non Degenerate: $\rightarrow$ if $x_1, x_2, x_3, \dots, x_n \neq 0$

This Basic solⁿ is called Non-degenerate.

if

Degenerate: $\rightarrow$ if $\exists$ any $x_1, \dots, x_n = 0$ This Basic solⁿ is called Degenerate solⁿ.

Q.

$$x_1 + 2x_2 + 3x_3 + x_4 = 8$$

$$2x_1 + x_2 + 6x_3 + x_4 = 16$$

$$x_1 = 0, x_2 = 0$$

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 6 & 0 \end{bmatrix} = 2(6) \neq 0 \quad \text{Basic}$$

$$x_2 = 0, x_4 = 0$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \neq 0 \quad \text{Non Basic}$$

$$x_3 = 0, x_4 = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \neq 0 \quad \text{Basic}$$

$$x_2 = 0, x_3 = 0$$

$$x_1 = 0, x_3 = 0, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq 0 \quad \text{Basic} \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \neq 0$$

$$x_1, x_4 = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix} \neq 0$$

No. of Basic = 5

Degenerate solⁿ $\rightarrow$

$$x_1, x_2 \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix} \quad x_2 = 0 \text{ Deg.}$$

$$x_1, x_3 \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 6 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 16 \end{bmatrix} \rightarrow \text{Deg.}$$

$$x_1, x_4 \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix} \rightarrow \text{Deg.}$$

$$x_2, x_3 \quad \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix} \rightarrow \text{Non Deg. } \lambda \text{ (Non Basic)}$$

$$x_2, x_4 \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix} \rightarrow \text{Non Deg.}$$

$$x_3, x_4 \quad \begin{bmatrix} 3 & 1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$$

$$3x_1 + x_2 = 0$$

$$6x_1 + x_2 = 16$$

$$3x_1 = 0 \Rightarrow x_1 = 0/3$$

$$-0 + x_2 = 0 \Rightarrow x_2 = 0 \rightarrow \text{Deg.}$$

No. of Deg. = 4

No. of Non Deg. = 1

Graphical Solⁿ:-

Q. Max $z = 6x_1 + 7x_2$

$$2x_1 + 3x_2 \leq 12$$

$$-2x_1 + x_2 \leq 8$$

B = $2x_1 + 3x_2 = 12$

$2x_1 + x_2 = 8$

$$2x_2 = 4 \Rightarrow x_2 = 2$$

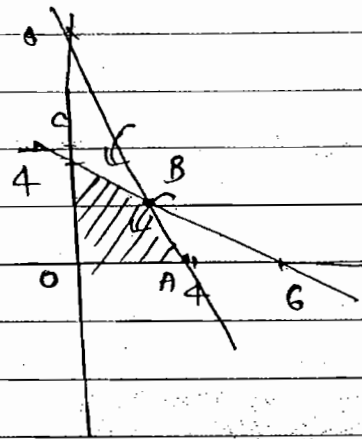
$$2x_1 = 6 \Rightarrow x_1 = 3$$

B = $(-2, 3) (3, 2)$, O = $(0, 0)$

A = $(4, 0)$ C = $(0, 4)$

Max $z = 18 + 28 = 38$

Unique Optimal Solⁿ.



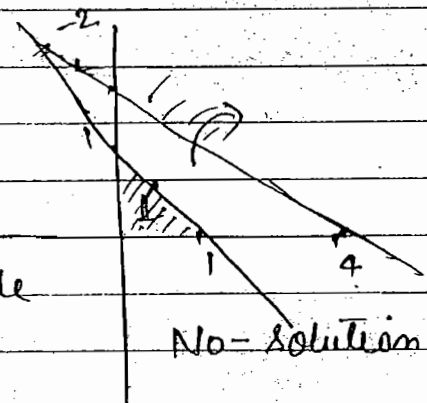
Q.

Max $z = 2x_1 + x_2$

$$x_1 + x_2 \leq 1$$

$$x_1 + 2x_2 \geq 4$$

No-feasible



Q.

Max $z = x_1 + 4x_2$

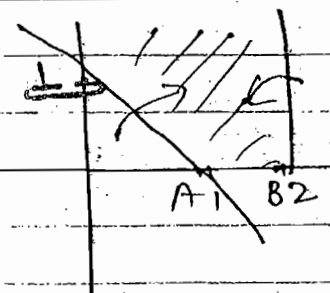
$$x_1 + x_2 \geq 1$$

$$-x_1 \leq 2$$

A = $(1, 0) \rightarrow 1$

B = $(0, 1) \rightarrow 4$

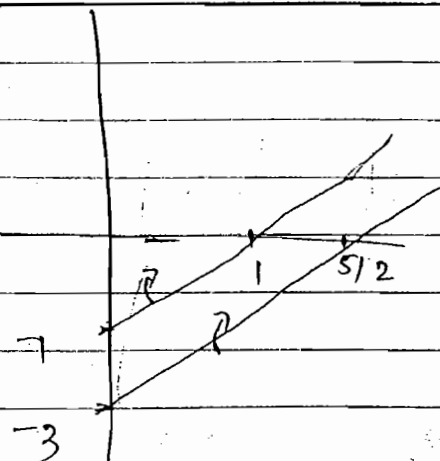
C = $(2, 0) \rightarrow 2$



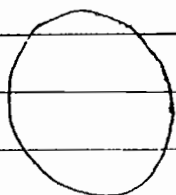
$$\text{Max } z = 12x_1 - 10x_2$$

$$x_1 - x_2 \leq 1$$

$$6x_1 - 5x_2 \geq 15$$



16/11/2014



Circle is not convex

Disk is convex set

$$\{(x, y) : x^2 + y^2 \leq 1\}$$

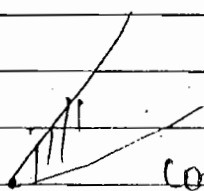
Convex hull =

Line segment

A set which contains smallest convex set

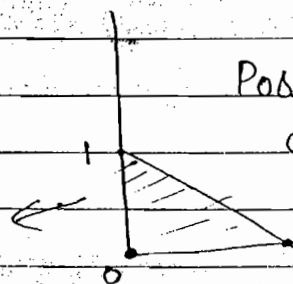
Affine Set :-

Line



Convex hull

Conic convex



Positive multiple of any no.

$$\forall x \in C$$

$$\forall \lambda \geq 0$$

$$c_1x_1 + \dots + c_nx_n = 3$$

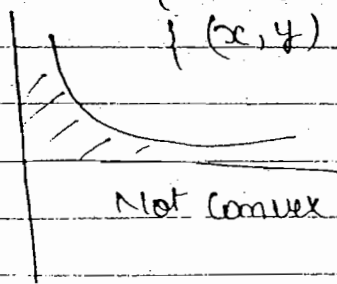
In n -dim it is called Hyper plane.

$$C = \{(x, y) : xy \leq 1\}$$

Not convex

$$\{(x, y) : xy \geq 1\}$$

Convex



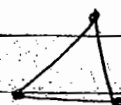
Not convex

Polyhedron :- Intersection of a finite ^{no} set of closed half spaces is called polyhedron.

If polyhedron is bounded is called polytop.

Open ~~dash~~ and line has no vertex.

n-dimension polyhedron has $(n+1)$ vertices is called Simplex.

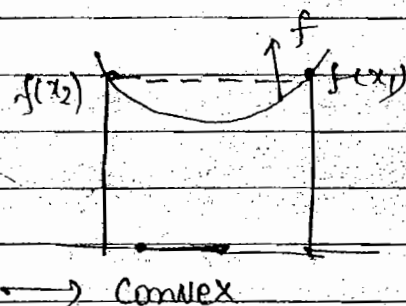


$2D \rightarrow 3$ vertices

A set C is bounded, $C \neq \emptyset$ & closed, C is convex then it has atleast 1 vertex.

$$A \rightarrow \text{func}^c$$

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$



Thm:- Every extreme point of a convex region of all feasible solⁿ region of an L.P.P. correspond to a Basic feasible solution of an L.P.P.
Converse also true.

Q- Min $z = x_1 - 3x_2 + 2x_3$

$$3x_1 - x_2 + 3x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

$$\text{Max } z = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$$

$$3x_1 - x_2 + 3x_3 + s_1 = 7$$

$$-2x_1 + 4x_2 + s_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$

			G							
			-1	3	-2	0	0	0		
B.V.	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3		
s_1	0	7	3	-1	3	1	0	0	-7	
s_2	0	12	-2	(4)	0	0	1	0	3	
s_3	0	10	-4	(3)	0	0	0	1	10/3	
$Z=0$			Δx_1	1	-3	2	0	0	0	

s_1	0	7							
s_2	0	12							
x_2	3	10/3	-4/3	1	0/3	0	0	1/3	

s_1	0	10	5/2	0	3	1	1/4	0	4
x_2	3	3	-1/2	1	0	0	1/4	0	
s_3	0	1	-5/2	0	8	0	-3/4	1	
$z=9$			Δx_1	-1/2	0	2	3/4	0	

x_1	-1	4	1	0	6/5	2/5	1/10	0	
x_2	3	10	0	2/2	6/5	2/5	6/10	0	
s_3	0	11	0	0	11	1	-1/2	1	
z			0	0	13/5	1/5	0/10	0	

Ans:- $x_1 = 4, x_2 = 5, x_3 = 0$

$\text{Max } z = 11$

2)

Q.

$\text{Max } z = 3x_1 + 2x_2$

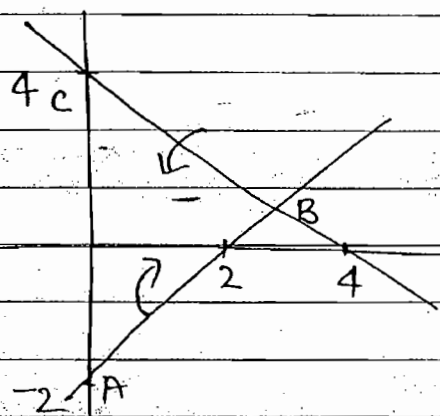
$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$B = 2x_1 = 6 \therefore x_1 = 3$

$x_2 = 1$

$\text{Max } z = 9 + 2 = 11$



A page of lined paper with a large, abstract, scribbled drawing in black ink. The drawing consists of several overlapping loops and lines, resembling a stylized 'S' or a series of connected curves. There are also some small, isolated marks and a small, dark, irregular shape near the bottom left corner.

Exp. Find the no. of significant digits in the following -

1) $10.0005603000 \Rightarrow 5.603 \times 10^3$

4 significant digits i.e. 5603

2) $23.00006 \Rightarrow 7$ significant digits

2.300006×10^5

1) Truncation Method $\rightarrow$

Let $X = 23.718928$, then what will be its truncation value, if it is truncated by 5 significant digits.

$X_1 = 23.718928$

$= 23.718$ (5 significant digit).

2) Round-off Method $\rightarrow$

Case-a) If remove portion $< \frac{1}{2} \times$ place value of n^{th} place. Then add nothing to be taken approximate value.

Case-b) If remove portion $> \frac{1}{2} \times$ place value of n^{th} place.

Then add one at the n^{th} place in the taken value.

Case-c) If remove portion $= \frac{1}{2} \times$ place value of n^{th} place then n^{th} place in taken value has even digit (add nothing).

If n^{th} place in taken value is odd digit (add 1) at the n^{th} place.

Exp. Round off the following no. up to 4 significant

1) $41.1551002 \quad 41.16$

2) $41.1549999 \quad 41.15$

3) $41.155 \quad 41.16$

4) $41.165 \quad 41.16$

even

first take 4 digit from left & then take truncated value & then find place value of last digit i.e. 4th digit.

$$1) \quad 41.15 \mid 51002$$

$$X = 51002, \quad X_1 = 41.15 + 1$$

$$= 51002, \quad = 41.16$$

$$41.15 \rightarrow \text{place value} = \frac{10}{2} = 5$$

$$\text{remove portion} = \frac{1}{2} \times \text{place value}$$

$$5 = 5 \quad (5 \text{ is odd so add } 1)$$

$$2) \quad 41.15 \mid 49999$$

$$X = 49999, \quad X_1 = 41.15$$

$$\text{Place Value} = \frac{1}{2} \times 10 = 5$$

$$4 < 5$$

add nothing.

$$\text{Ans: } 41.15$$

$\Rightarrow$ If $\bar{x}$ is true value of a number & x_1 is approximate value of the no. then absolute error in approximation is equal to true value appx.

$$E_A = X - x_1$$

$$*) \quad \text{Relative Error} \rightarrow E_R = \frac{E_A}{X}$$

$$*) \quad \text{Percentage Error} \rightarrow E_P = E_R \times 100 \%$$

Exp. If 24.5 is approximated at 24 then what is absolute error in approximation?

$$E_A = 1.5, \quad E_R = \frac{1.5}{24.5}$$

$$\text{or } E_p = \frac{1.5}{24.5} \times 100 \%$$

* If we have any fun^c whose value is to be approx'd any pt then in expansion of that fun^c by Taylor thm about any pt we take upto certain terms which involves finite powers of x and remaining terms are truncated thus appro. of every fun^c in numerical analysis is a polynomial fun^c .

Note: $\rightarrow$ If we round-off a number upto n -places after decimal then error in round-off has its magnitude

$$f(x) = |\text{Round-off error}| \leq \frac{1}{2} \times 10^{-n}$$

Q. If we want to appro. value of e^x at $x=1$ correct upto 2 places after decimal. then how many terms in Taylor expansion of e^x about $x=0$ will be taken

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

Let us assume that we are going to n -terms.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$$

$$\left| \frac{x^n}{n!} \right| \leq \frac{1}{2} \times 10^{-2}$$

Remainder after n -terms will be approximately

$$\left| \frac{x^n}{n!} \right| \leq \frac{1}{2} \times 10^{-2}$$

Which at $x=1$ will be $\left| \frac{1}{n!} \right| \leq \frac{1}{2} \times 10^{-2}$

$$\frac{1}{n!} \leq \frac{1}{200}$$

$$(5! = 120, 6! = 720 > 200)$$

$n! \geq 200$ it is true for $n=6$

upto 3 digit $\rightarrow n! \geq 2000 \quad 7! = 5040$

It is true for $n=7$

upto 1 digit $\rightarrow$

$$n! \geq 200$$

$$3! = 6, 4! = 24$$

It is true for $n=4$

Q:- If we want to find approximate value of $1/\sqrt{1+x}$ for every very small value of x then which formula will be used for this purpose for appro.

McLaurine | Taylor

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}x^2 + \dots$$

and the best appro. formula for $1/\sqrt{1+x}$ for

v.v. small value of x is -

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$$

Q. If e^x is approximated by Taylor expansion as polynomial fun^c at $x = 1/2$ then what must be degree of polynomial so that error in appro. does not exceed 10^{-2}

Let us assume that appro. is done by polynomial of order 'n' then error

$$\left(1 + \frac{x}{1!} + \dots + \frac{x^n}{n!}\right) \rightarrow \left(\text{upto 'n' degree only } \frac{x^{n+1}}{(n+1)!} \text{ left}\right)$$

$$\frac{x^{n+1}}{(n+1)!} \Big|_{x=1/2} < \frac{1}{100}$$

$$\frac{n+1}{2^{n+1}} \geq 100 \quad \frac{24}{16 \times 24} \geq 100$$

$$\frac{1}{2^{n+1}} \leq \frac{1}{100}$$

$$\text{at } n=3, \frac{1}{2^{4 \cdot 4!}} \leq \frac{1}{100} = \frac{1}{16 \times 24} \leq \frac{1}{100}$$

$$\Rightarrow \text{Ans} \rightarrow n=3$$

* If we want to evaluate $\sqrt{1+x} - 1$ at $x = 0.123456789 \times 10^8$ then which formula will be taken to find its approximation value with how much accuracy.

$$\begin{aligned} \sqrt{1+x} - 1 &= (1+x)^{1/2} - 1 \\ \Rightarrow 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)x^2 + \dots - 1 \\ &= \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots \end{aligned}$$

* Iteration Method : $\rightarrow$

$$\begin{aligned} x &= \phi(x) \\ x_1 &= \phi(x_0) \\ &\vdots \end{aligned}$$

$$x_n = \phi(x_{n-1})$$

$$\text{Exp. (1)} \quad x^2 - x - 1 = 0 \quad \text{at } x_0 = 2$$

$$\Rightarrow x = x^2 - 1 \quad \text{or} \quad x = (1+x)^{1/2}$$

$$\Rightarrow \text{or} \quad x = (1+x)^{1/2}$$

$$x_1 = (1+2)^{3/2} = (3)^{3/2}$$

$$\text{or} \quad x_1 = (2)^2 - 1 = 3$$

$$x_2 = x_1^2 - 1 = 8$$

Thm:- If $f(x) = 0$ is written as $x = \phi(x)$ where $\phi(x)$ & $\phi'(x)$ both are continuous in an interval I containing initial approx. value x_0 and if $|\phi'(x)| < 1$ in I then $x = \phi(x)$ will converge to root of $f(x) = 0$.

Q:- If we want to find approx. solⁿ of eqnⁿ $x^3 - 3x + 1 = 0$ around $1/3$ then which of the following iterative formula will converge to the root.

✓ $x_{n+1} = \frac{x_n^3 + 1}{3}$ 3) $x_{n+1} = (3x_n - 1)^{1/3}$

2) $x_{n+1} = x_n^3 - 2x_n + 1$ 4) $x_{n+1} = \frac{1}{3 - x_n^2}$

2) $\phi(x) = x^3 - 3x + 1$
 $\phi'(x) = (3x^2 - 2)$ at $1/3$
 $= \left| \frac{1}{3} - 2 \right| > 1$ not conv.

1) $\phi(x) = \frac{x^3 + 1}{3}$
 $\phi'(x) = \frac{1}{3}(3x^2)$ at $1/3$
 $= \left| \frac{1}{9} \right| < 1$ conv.

3) $\phi(x) = (3x - 1)^{1/3}$
 $\phi'(x) = \frac{1}{3}(3x - 1)^{-2/3}$ at $1/3$
 $= \infty$ not conv.

4) $\phi(x) = \frac{1}{3 - x^2}$
 $\phi'(x) = \frac{1 \times 2x}{(3 - x^2)^2}$ at $x = \frac{1}{3}$
 $\left| \frac{2/3}{(3 - \frac{1}{9})^2} \right| < 1$ conv.

Q. $x^2 - x - 2 = 0$, iterative formula is $x_{n+1} = \sqrt{x_n + 2}$
 will this formula converges to find root at 1.

$$\phi(x) = \sqrt{x+2}$$

$$|\phi'(x)| = \left| \frac{1}{2\sqrt{x+2}} \right| < 1, \quad x = 1$$

Ans 14

Q. This formula (Iterative) will converge to solution of formula —

- (a) $\sqrt{2}$ (b) -1 (c) 2 (d) $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$

Let us assume that it is converge to l i.e. $x_n = l$

$$\lim x_n = l$$

$$\lim x_{n+1} = l$$

$$x^2 = x + 2 \Rightarrow l^2 = l + 2$$

$$l = -1, 2 \quad (\text{only +ve root})$$

(d) $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$

take $x = \sqrt{2+x}$

$$l^2 = l + 2, \quad l = -1, 2$$

(c) & (d) are possible limit of iteration $f(x) = \sqrt{2+x}$

2) NEWTON - RAPHSON METHOD $\rightarrow$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$\Rightarrow$

* If we want iterative formula to approximate +ve square root of a number N then the formula will be $\rightarrow$

$$\frac{1}{2} \left(x_n + \frac{N}{x_n} \right)$$

To find $\sqrt[3]{N}$,

$$\frac{1}{3} \left(2x_n + \frac{N}{x_n^2} \right)$$

Exp. If we want to find 1st & 2nd approximate value of $\sqrt{2}$ by Newton Raphson method when $x_0 = 1.5$ then $x_1 = ?$, $x_2 = ?$

Newton Raphson formula to find iteration of $\sqrt{2}$ is

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{2}{x_n} \right)$$

1st iteration: $\rightarrow x_1 = \frac{1}{2} \left(x_0 + \frac{2}{x_0} \right) \quad \left(n=0, x_0 = 1.5 = \frac{3}{2} \right)$

$$= \frac{17}{12} = 1.416 \dots$$

2nd iteration: $\rightarrow x_2 = \frac{1}{2} \left(\frac{17}{12} + \frac{12 \times 2}{17} \right)$

$$= \frac{1}{2} \left(\frac{17}{12} + \frac{24}{17} \right)$$

Exp. Find 1st & 2nd app. value of $\sqrt[3]{7}$ by taking initial value $x_0 = 2$

$$f(x) = x^3 - N$$

$$x^3 = 7$$

$$x_{n+1} = x_n - \frac{(x_n^3 - N)}{3x_n^2}$$

$$x_{n+1} = x_n - \frac{x_n^3 - 7}{3x_n^2} = \frac{2x_n^3 + 7}{3x_n^2}$$

$$x_1 = \frac{2 \times 8 + 7}{3 \times 4} = 1.916$$

$$x_2 = \frac{2 \left(\frac{23}{12} \right)^3 + 7}{3 \times \left(\frac{23}{12} \right)^2}$$

Exp. find x_1 and x_2 for equ. $x^3 - 5x + 3 = 0$ by using Newton's - Raphson method?

$$f(x) = x^3 - 5x + 3, \quad f'(x) = 3x^2 - 5$$

$$f(0) = 3, \quad f(1) = -1$$

$$f(0) \cdot f(1) < 0$$

Then root lies b/w (0, 1)

Here ~~f~~ $f(1) = -1$ is nearer to zero

$$\text{then } f(0) = 3$$

$$\Rightarrow x_0 = 1$$

$$\text{Now, } f(x_0) = f(1) = -1$$

$$f'(x_0) = f'(1) = -2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-1)}{(-2)} = 0.5$$

$$\text{Now } f(0.5) = 0.625$$

$$f'(0.5) = -4.25$$

Exp. Using N-R method, find an iterative formula to compute the reciprocal of a natural no.

$$\text{let } x = \frac{1}{N} \quad \Rightarrow \quad x - \frac{1}{N} = 0$$

$$f(x) = x - \frac{1}{N} = 0$$

$$f'(x) = \frac{1}{x^2}$$

Using N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(N - \frac{1}{x_n}\right)}{\left(\frac{1}{x_n^2}\right)}$$

$$= x_n - N \cdot x_n^2 + x_n$$

$$= \boxed{x_n (2 - N \cdot x_n)}$$

$$\boxed{x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \left(\frac{f(x_0)}{f'(x_0)} \right)^2 \frac{f''(x_0)}{f'(x_0)}}$$

which is required extension formula.

Gauss - Elimination Method : $\rightarrow$

Exp.

$$x + y - z = 1 \quad \text{--- (1)}$$

$$x - y + z = 1 \quad \text{--- (2)}$$

$$x + y + z = 1 \quad \text{--- (3)}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array} \right]$$

$$\Rightarrow R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 2 & 2 \end{array} \right]$$

$$x = 1, \quad y = 1, \quad z = 1 \quad \times$$

'OR'

$$x + y - z = 1 \quad \text{--- (4)}$$

by (2)

$$1 - y + z - y + z = 1$$

$$-2y + 2z = 0 \Rightarrow y - z = 0 \quad \text{--- (5)}$$

$$\Rightarrow -1 + y - 1 + y + 1 = 1$$

$$y = 1 \quad \text{--- (6)}$$

$$z = 1, \quad x = 1 + 1 - 1 = 1$$

$$\text{Ans. } x = 1, \quad y = 1, \quad z = 1$$

* Gauss - Seidal method : $\rightarrow$

It is an iterative method of solving of linear algebraic equations.

Q.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

$$x = \frac{1}{20} (17 - y + 2z), \quad y = \frac{1}{20} (-18 - 3x + z)$$

$$z = \frac{1}{20} (25 - 2x + 3y)$$

(i) put $y = y_0 = 0, \quad z = z_0 = 0$

$$x = \frac{1}{20} (17 - 0 + 0) = x = 0.85$$

$$y = \frac{1}{20} (-18 - 3 \times \frac{17}{20} + 0) = -1.0275$$

$$z = \frac{1}{20} (25 - 2 \times 0.85 - 3 \times 1.0275) = 1.0109$$

* Finite Differences : $\rightarrow$

Let $y = f(x)$ is tabulated for the equally spaced values of $x = x_0, x_1, x_2, \dots, x_n$ where $x_i = x_0 + ih$

$$\Rightarrow x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \dots$$

1) Forward difference operator : $\rightarrow$

$$\Delta y_i = y_{i+1} - y_i$$

In general,

$$\Delta^n f(x_i) = \sum_{k=0}^n (-1)^k \cdot {}^n C_k \cdot f_{i+n-k}$$

2) Backward difference operator : $\rightarrow$

$$\nabla y_i = y_i - y_{i-1}$$

In general,

$$\nabla^n f(x_i) = \sum_{k=0}^n (-1)^k \cdot {}^n C_k \cdot f_{i-k}$$

3) Central difference operator : $\rightarrow$

$$\delta y_{i+\frac{1}{2}} = y_{i+1} - y_i$$

In general,

$$\delta^n f(x_i) = \sum_{k=0}^n (-1)^k \cdot {}^n C_k \cdot f_{i+\frac{n}{2}-k}$$

Shift operator $E \rightarrow Ey_i = y_{i+1}$

E is Shift operator

$$E^k y_i = y_{i+k}$$

$$(y_{i+1} - y_i) = (E - 1) y_i$$

Relation b/w E & $\Delta \quad \therefore \Delta = E - 1$

$$\Delta^k = (E - 1)^k$$

Exp. Find $\Delta^2 x^3$ at $x = 0$

Here $n = 2, m = 3$

$$2^3 - 2C_1(2-1) + 2C_2(2-2)$$

$$8 - 2 + 0 = 6$$

Exp.

$$\Delta \tan^{-1} \left(\frac{n-1}{n} \right)$$

$$\tan^{-1} \left[\frac{(n+1)-1}{(n+1)} \right] - \tan^{-1} \left[\frac{n-1}{n} \right]$$

$$\Rightarrow \tan^{-1} \left[\frac{n}{n+1} - \frac{n-1}{n} \right] = \frac{x^2 - x^2 + 1 \cdot (n+1)}{n(n+1) \cdot 2n}$$

$$\left[\frac{1 + \frac{n}{n+1} \cdot \frac{n-1}{n}}{2n} \right] = \tan^{-1} \frac{1}{2n^2}$$

$$\text{Ans:- } \tan^{-1} \left(\frac{1}{2n^2} \right)$$

Exp.

$$\Delta^2 \cos 2x$$

$$\Delta [\cos 2(x+h) - \cos 2x]$$

$$\Rightarrow \Delta \cos 2(x+h) - \Delta \cos 2x$$

$$\Rightarrow [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$$

$$\Rightarrow \cos 2 \sin(2x+3h) \cdot \sin h + 2 \sin(2x+h) \sin h$$

$$\Rightarrow -2 \sin h [\sin(2x+3h) - \sin(2x+h)]$$

$$\Rightarrow -2 \sin h [2 \cos(2x+2h) \cdot \sin h]$$

$$= -4 \sin^2 h \cdot \cos(2x+2h)$$

Ex: Prepare the operator difference of the table gives to us -

x	0	1	2	3
y	-1	2	-1	3

x	y	Δ/∇	Δ/∇	Δ/∇	
0	-1	3			$-(-1-2) = 3$
1	2		-6		
2	-1	-3	7		
3	3	4			

Divided difference: $\rightarrow$

x	2	4	9	10
f(x)	4	56	711	980

Sol ⁿ :	x	f(x)	$f(x_0, x_1)$	$f(x_0, x_1, x_2)$	$f(x_0, x_1, x_2, x_3)$
	2	4	$\frac{56-4}{4-2} = 26$ ①	$\frac{131-26}{9-2} = 15$	$\frac{23-15}{10-2} = 1$
	4	56			
	9	711	$\frac{980-711}{10-9} = 269$ ③	$\frac{269-131}{10-4} = 23$	
	10	980			
			$\frac{711-56}{9-4} = 131$ ②		

* Newton Forward Interpolation formula: $\rightarrow$

$$y_p(x) = y_0 + P \Delta y_0 + \frac{P(P-1)}{2!} \Delta^2 y_0 + \dots + \frac{P(P-1) \dots (P-n+1)}{n!} \Delta^n y_0$$

* Newton Backward Interpolation formula: $\rightarrow$

$$y_p(x) = y_n + P \nabla y_n + \frac{P(P+1)}{2!} \nabla^2 y_n + \dots + \frac{P(P+1) \dots (P+n-1)}{n!} \nabla^n y_n$$

Exp.	Consider	x	0	1	2
		y	-1	0	1

By newton Backward formula -

x	y
0	-1
1	0
2	1

$$h = x_1 - x_0$$

$$= 1 - 0 = 1$$

$$x_0 = 0, x_n = 2$$

By Newton forward I. formula.

$$y = -1 + P(1) + \frac{P(P-1)}{2!} \cdot 0$$

$$P = \frac{x - x_0}{h} = \frac{x - 0}{1}$$

$$x = 0 + P(1) \Rightarrow P = x$$

$$y = x - 1$$

By Newton Backward formula -

$$y_p = y_n + P \nabla y_n + \frac{P(P+1)}{2!} \nabla^2 y_n + \dots$$

$$x = 2 + p \cdot 1$$

$$P = x - 2$$

$$y = 1 + (x - 2) = x - 1$$

$$\begin{cases} x = x_n + Ph \\ P = \frac{x_n - x}{h} = \frac{x - 2}{1} \end{cases}$$

Q. The set of data (1,2) (2,5) (3,10) (4,17) (5,26) & (6,37) will interpolate curve of degree -

x	y	Δ	Δ^2
1	2	3	
2	5	3	2
3	10	5	2
4	17	7	2
5	26	9	2
6	37	11	2

Interpolate of degree = 2

$$h = 1$$

$$p = \frac{x - x_0}{h} = x - 1$$

$$y = y_0 + P \Delta y_0 + \frac{P(P-1)}{2!} \Delta^2 y_0$$

$$= 2 + 3P + \frac{2}{2} P(P-1)$$

Ex: If we want to find the values at ??

x	1	2	3	4
y	7	11	?	19

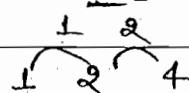
Then we can use ?

1) Newton B.I.F.

2) N.F.I.F.

3) L.I.F.

4) All



1 ≠ 2 diff. not same.

Not N.I.F.

(1, 7) (2, 11) (3, ??) (4, 19)

Newton formula not valid, tabulated pt given.

on 1, 2, 4 only L.I.F. valid here.

Ex: which interpolation formula will be valid ?

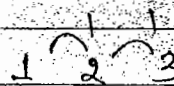
x	1	2	2 1/2	3
y	7	11	?	19

1) N.B.I.F.

2) N.F.I.F.

3) L.I.F.

4) All



diff are same

Lagrange's Interpolation Formula :→

x	0	1	2
y	-1	0	1

$$y = \frac{(x-1)(x-2)}{(x_0-1)(x_0-2)} \times (-1) + \frac{(x-0)(x-2)}{(0-1)(0-2)} \times 0 +$$

$$\frac{(x-0)(x-1)}{(2-0)(2-1)} \times (1)$$

$$y = -\left(\frac{x^3 - 3x + 2}{2}\right) + \left(\frac{x^2 - x}{2}\right)$$

$$= x - 1 //$$

Hermite Interpolation formula : $\rightarrow$

If we have set of tabulated points in which value at independent variables along with corresponding value of dependent variable as well as its derivative is given

★ Spline Interpolation : $\rightarrow$

- (i) $f(x)$ is linear polynomial outside the interval $[x_0, x_n]$
- (ii) on each sub interval $[x_{i-1}, x_i]$, $f(x)$ is a third degree polynomial.
- (iii) $f(x)$, $f'(x)$, $f''(x)$ are continuous on (x_0, x_n) .

Exp. Consider the set of points $(1, 2)$, $(2, 5)$ & $(4, 7)$ by using linear spline, find the value of

- i) $y(3)$ ii) $y(1.5)$

x	1	2	4
y	2	5	7

$1 \leq x \leq 2$
 $2 \leq x \leq 4$

Equⁿ of line $(1, 2)$ & $(2, 5)$

$$(y - y_1) = m(x - x_1)$$

$$(y - 2) = 3(x - 1)$$

$$y = 3x - 1$$

Equⁿ of line $(2, 5)$ & $(4, 7)$

$$(y - 5) = \frac{7-5}{4-2}(x-2) \Rightarrow y - 5 = x - 2$$

$$y = x + 3$$

$$y(3) = 3 + 3 = 6, \Rightarrow y(1.5) = 3 \times 1.5 - 1 = 3.5$$

$$S(x) = \begin{cases} 3x - 1 & 1 \leq x \leq 2 \\ x + 3 & 2 \leq x \leq 4 \end{cases}$$

Q.1. Consider the set of points $(1,0)$ $(2,1)$ $(4,-1)$
 By using quadratic spline interpolation formula $\rightarrow$
 find (i) $y(3)$ (ii) $y(1.5)$

x	1	2	4
y	0	1	-1

Assume

$$S(x) = \begin{cases} a+bx+cx^2, & 1 \leq x \leq 2 \\ d+ex+fx^2, & 2 \leq x \leq 4 \end{cases}$$

$$(1,0) \text{ at } x=1, \quad a+b+c=0 \quad \text{--- (1)}$$

$$(2,1) \text{ at } x=2, \quad a+2b+4c=1 \quad \text{--- (2)}$$

$$(2,1) \text{ at } x=2, \quad d+2e+4f=1 \quad \text{--- (3)}$$

$$(4,-1) \text{ at } x=4, \quad d+4e+16f=-1 \quad \text{--- (4)}$$

$$\text{Continuous in } a+2b+4c = d+2e+4f \quad \text{--- (5)}$$

Whole interval & derivative

$$2cx+b = 2fx+e$$

$$x=2, \quad 4c+b = 4f+e \quad \text{--- (6)}$$

$$(1) + (2) \Rightarrow b+3c=1 \quad (3) + (4) \Rightarrow e = -2-12f$$

$$b = 1-3c$$

$$a = 2c-1$$

$$e = -1-6f$$

$$d = 2+8f$$

$$\text{from (6)}, \quad c+1 = -1-2f$$

$$c+2f = -2$$

* Numerical differentiation $\Leftrightarrow$ integration $\rightarrow$

$$\left| \frac{dy}{dx} \right|_{x=x_n} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n \dots \right]$$

$\Rightarrow$ If $\nabla \rightarrow 0$

Q → If in a company production of footballs in lakhs where observed as :-

Year	2010	2011	2012	2013	2014
production	5	10	8	7	9

What is the annual rate of production of football in year 2010 & year 2014 approximated by data.

x	y	Δ	Δ^2	Δ^3	Δ^4
2010	5	5			
2011	10		-7	8	-6
2012	8	-2			
2013	7	-1	1	2	
2014	9	2	3		

$$\frac{dy}{dx} \Big|_{x=2010} = \frac{1}{h} \left[\Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \dots \right]$$

$$\frac{dy}{dx} \Big|_{x=2010} = \frac{1}{1} \left[5 - \frac{1}{2}(-7) + \frac{1}{3}(8) - \frac{1}{4}(-6) \right] = \frac{30}{3}$$

$$\frac{dy}{dx} \Big|_{x=2014} = \frac{1}{1} \left[2 + \frac{1}{2}(3) + \frac{1}{3}(2) + \frac{1}{4}(-6) \right] = \frac{8}{3}$$

$$\star \quad f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} \Rightarrow$$

Trapezoidal Rule $\rightarrow$

Total Error in trapezoidal rule is given by -

$$E_t = -\frac{h^3}{12} [y_0'' + y_1'' + \dots + y_{n-1}'']$$

$$|E_t| \leq \frac{h^3 \cdot n \cdot y''_{\max}}{12}$$

$$E_t \approx -\frac{(b-a) h^2 y''_{\max}}{12}$$

By Trapezoidal rule,

$$\int_{x_0}^{x_0+nh} y \cdot dx = \frac{h}{2} [y_0 + 2(y_1 + y_2 + \dots + y_{n-1})]$$

Exp. If we want to find approximate value of integral $\int_0^1 x^2 dx$, by using trapezoidal rule and taking step length $1/3$ then its approximate value will be ?, also find error in approximate ?

x	0	1/3	2/3	1
$x^2 = y$	0	1/9	4/9	1

$$\int_0^1 x^2 dx \approx \frac{h}{2} [y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$$

$$\approx \frac{1/3}{2} [0 + 2\left(\frac{1}{9} + \frac{4}{9}\right) + 1]$$

$$= \frac{1}{6} \left(\frac{19}{9}\right) = \frac{19}{54} = \text{approximate value}(x=1)$$

$$\text{Actual value } x = \frac{18}{54}$$

$$\text{Error } E = -1/54$$

$$\% E = \frac{-100}{18} \%$$

Note $\rightarrow$ ① If by trapezoidal rule we find approximate integral of a curve in $[x_0, x_n]$ in which curve is convex then approximate value of the integral.

$\Rightarrow$ If we approximate $\int_0^1 x^4 dx$, by trapezoidal rule of step length 0.01 then approximate value of integral will be .

- 1) $\frac{1}{5}$ 3) $< \frac{1}{5}$
 2) $> \frac{1}{5}$ 4) None

$$\int_0^1 x^4 dx = 12x > 0 \text{ (convex)}$$

* Simpson's $1/3^{\text{rd}}$ Rule $\rightarrow$

$$\int_{x_0}^{x_0+h} y dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots)]$$

Note $\rightarrow$ Simpson's $1/3^{\text{rd}}$ Rule will be appo applicable over the set of data's which are —

- 1) $M(2) + 1$

Exp. By taking step length $h = 1/4$ find the app. value of $\int_0^1 x^2 dx$ by simpsons $1/3^{\text{rd}}$ rule. also find error in approximation.

Sol ⁿ $\rightarrow$	x	0	1/4	2/4	3/4	1	
	y	0	1/16	4/16	9/16	1	

$\underbrace{\hspace{10em}}_{\substack{\text{2 times} \\ \text{4 time}}} \quad h = 1/4$

By Simpsons $1/3^{\text{rd}}$ rule

$$\int_0^1 x^2 dx = \frac{1}{12} \left[(0+1) + 4 \left(\frac{1}{16} + \frac{9}{16} \right) + 2 \left(\frac{4}{16} \right) \right]$$

$$= \frac{1}{3}$$

$$\text{Error} = 0$$

Note: $\rightarrow$ As Simpsons $1/3^{\text{rd}}$ rule taken 3 tabulated pts in each sub-integral so if actual curve is quadratic then approximate value & true value of the integral will coincide.

* Simpson $3/8^{\text{rd}}$ Rule: $\rightarrow$

$$\int_{x_0}^{x_0+n h} y dx = \frac{3h}{8} \left[(y_0 + y_n) + 3(y_1 + y_2 + \dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots) \right]$$

Exp. By Simpsons $3/8$ rule approximate value of integral $\int_0^{10} x^3 + x^2 - x + 1 dx$ by taking step length $1/10$ will be ?

$$1) \quad \frac{(10)^4}{4} + \frac{(10)^3}{3} - \frac{(10)^2}{2} + (10)$$

$$2) \quad 10$$

$$3) \quad 0$$

$$4) \quad \text{None.} \quad \Rightarrow$$

$$\text{Sol}^n \quad \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x = \frac{(10)^4}{4} + \frac{(10)^3}{3} - \frac{(10)^2}{2} + 10 //$$

1. Picard's method $\rightarrow$

$$\frac{dy}{dx} = f(x, y)$$

$$y_n = y_0 + \int_{x_0}^{x_1} f(x, y_{n-1}) dx$$

Exp. Consider $y' = x + y$ & $y(0) = 1$

$$\begin{aligned} y' &= x + y, & y(0) &= 1 \\ y'' &= 1 + y', & y'(0) &= 1 \\ y''' &= y'', & y''(0) &= 2 \\ & & y'''(0) &= 2 \end{aligned}$$

$$y(x) \approx y_n$$

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y^{(0)}) dx$$

$$= y_0 + \int_0^x f(x, y_0) dx$$

$$= 1 + \int_0^x (x+1) dx \Rightarrow 1 + x + \frac{x^2}{2}$$

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)}) dx$$

$$= y_0 + \int_0^x \left(2x + 1 + \frac{x^2}{2} \right) dx$$

$$= 1 + x^2 + x + \frac{x^3}{6}$$

2) Euler's Method $\rightarrow$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y) dx$$

$$y_1 \approx y_0 + h f(x_0, y_0)$$

Modified Euler method :-

$$y_1^{(n)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(n-1)})]$$

Exp. $\frac{dy}{dx} = x+y$, $x_0 = 0$, $y_0 = 1$ with $h = 1$

Compute y for $x = 2$

Since Euler's formula is $y_n = y_{n-1} + hf(x_n, y_{n-1})$

$$\bar{y}_1 = x_0 + 2y_0 = 0 + 2 = 2$$

$$y_1 = y_0 + \frac{1}{2} [f(x_0, y_0) + f(x_1, \bar{y}_1)]$$

$$y_1 = \frac{1}{2} [x_0 + y_0 + x_1 + \bar{y}_1]$$

$$= \frac{1}{2} [0 + 1 + 1 + 2] \quad (\because x_1 = x_0 + h)$$
$$= 3$$

Value of $y(2)$,

$$\bar{y}_2 = x_1 + 2y_1 = 1 + 2 \times 3 = 7$$

$$y_2 = y_1 + \frac{1}{2} [f(x_1, y_1) + f(x_2, \bar{y}_2)]$$

$$= 3 + \frac{1}{2} [1 + 3 + 2 + 7] = \frac{19}{2}$$

* Runge-Kutta Method :- (fourth-order)

$$\frac{dy}{dx} = f(x, y) \Rightarrow y(x_0) = y_0$$

$$y_1 = y_0 + h$$

$$K = \frac{h}{6} [K_1 + 2(K_2 + K_3) + K_4]$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Q. Apply Runge-Kutta method to find an approximate value of y when $x = 0.1$ given?

$$\text{Here } x_0 = 0, y_0 = 1$$

$$\frac{dy}{dx} = x + y, \quad y = 1 \text{ when } x = 0$$

$$h = 0.1$$

$$k_1 = hf(x_0, y_0) = 0.1 [x_0 + y_0]$$

$$= 0.1 [0 + 1] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.1) \left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2} \right] = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.12105$$

$$K = \frac{h}{6} [k_1 + 2(k_2 + k_3) + k_4]$$

$$y_1 = y_0 + K$$

$$x_1 = x_0 + h$$

