Delos Academy

Ordinary Differential Equations

Note→

In Differential Eq^n, we will use open set on Real line, as domain of $y(x)$ in Diff. eq^n as subset of $\mathbb{R}$.

Independent & Dependent Variable →

$f: A \to B$ be a fun given by

$$y = f(x)$$

then the variable to which value is assigned is called Independent variable.

and the variable whose value is to be obtained is called Dependent variable.

Here x is independent variable &
y is dependent variable.

e.g → $A(r)$ → Area of circle with radius r
then Area can be obtained if radius is given
and radius can be obtained if area is given.

So, if radius is given, then r is independent variable
and $A(r)$ is dependent variable.

and if area A is given, then A is independent
variable and r is dependent variable

e.g → $P(t)$ position of earth with time t
then position can be obtained if t is given and
t can be obtained if position is given.

If t is given, then t is I.V and $P(t)$ is D.V
& If $P(t)$ is given, then P is I.V and t is D.V.

Note →

In a differential eq^n of two variables anyone of the variable can be Independent or dependent.

e.g $\frac{dy}{dx} = 1$ (We can choose according to our choice, if not given)

$\Rightarrow dy = dx$

$\Rightarrow \frac{dx}{dy} = 1$

Note → If we want to find derivative of a fu^n then that fu^n must be differentiable in the domain of fu^n.

e.g → $y(x) = x|x|$ is sol^n of nth order Differential Equation, then possible value of n is ?

a) $n \geq 2$

b) $n = 1$

c) $n = 2$

d) None of these

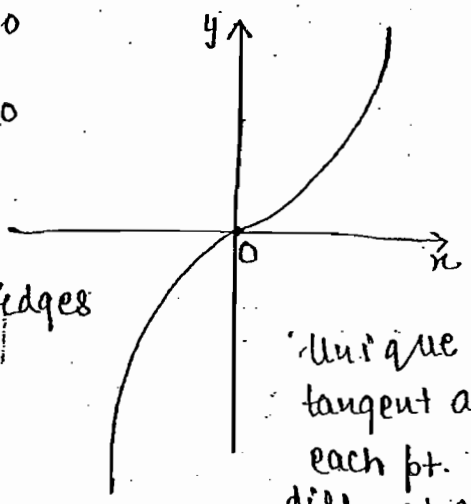
$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Sol^n →

$y(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$

$\lim_{x \to 0} \frac{f(x)}{x} = \frac{x|x|}{x}$

$= |x| = f'(0)$



Its Curve having No Break | No sharp edges

So, first order derivative exist

Unique tangent at each pt. So diff. at each pt of domain

R.H.D $\lim_{x \to 0^+} \frac{x|x| - 0}{x - 0} = \lim_{x \to 0^+} |x|$

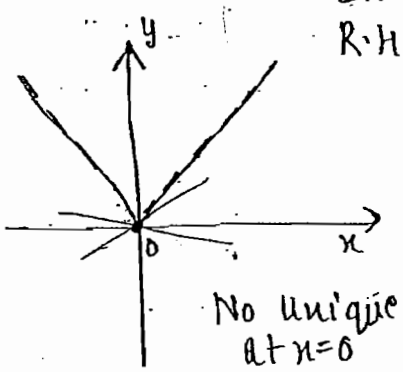
$$y'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

$$= 2|x|$$

$$= \lim_{x \to 0} \frac{y'(x)}{x} \Rightarrow$$

L.H.L $= -2$.

R.H.L $= 2$



No unique tangent at $x=0$

So its derivative does not exist.

$\Rightarrow$ Second order derivative does not exists.

Hence $y(x) = x|x|$ is a sol^n of 1st order diff eq^n.

Hence $n=1$

$\therefore$ option b) is Correct.

In Space →

$z = z(x,t)$

We consider 2 directions x & t.

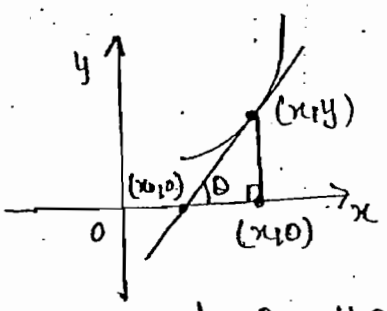
$\frac{\partial z}{\partial x}$ is tangent in x-direction treating t = constant

$\frac{\partial z}{\partial t}$ is tangent in t-direction taking x = constant.

Note →

$$\frac{dy}{dx} = \tan\theta$$

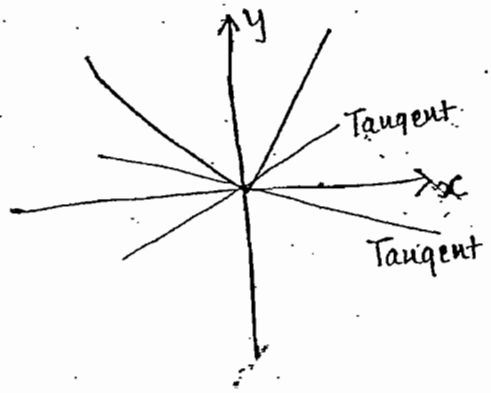
i.e Derivative of fu^n at any pt represents slope of the curve at that pt.



$$\tan\theta = \frac{y-0}{x-x_0} = \frac{dy}{dx}$$

Thus, for a fu^n to be diff. at a pt, it must have Unique tangent at that point.

If fu^n doesn't have Unique Tangent at a pt, then fu^n will not be diff. at that point.



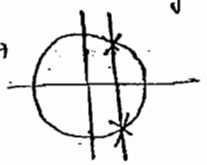
fu^n with sharp Edge → Not Diff.

Function →

Let A and B are two non-empty sets
i.e $A \neq \phi$, $B \neq \phi$

$f: A \to B$ is a rule in which each element of A assign to unique element of B.

Note → Every fu^n is a curve but every curve must not be a fu^n. e.g →

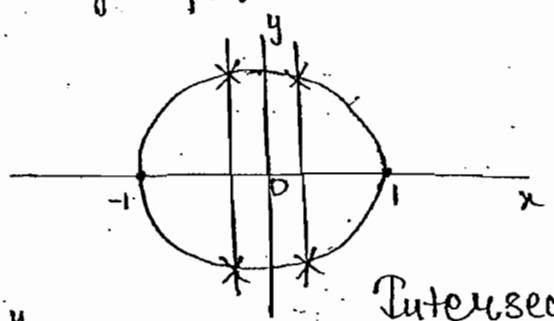


Curve → but not a fu^n.

Graphically,

A curve represents a fu^n if every vertical line passing through (domain) must intersect the curve exactly once.

e.g → (1) $f: [-1, 1] \to R$
where $y = f(x)$ whose curve is

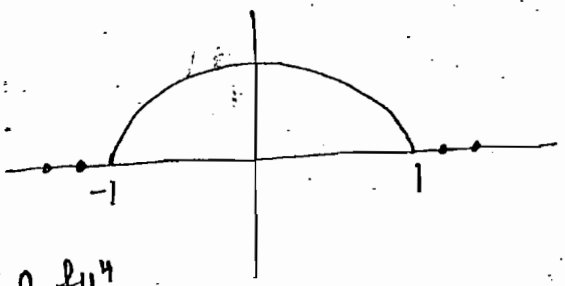


Uniqueness fails.

Not a fu^n.

Intersects at 2 pts.
∃ pts in domain having 2 images.

(2) $f: \mathbb{R} \to \mathbb{R}^+$ given by



is not a fu^n.

Since, there are some points in the domain which have no image. (As domain $\to \mathbb{R}$)

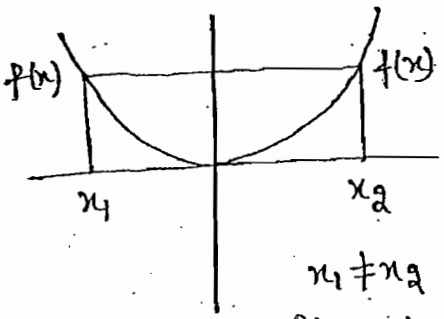
One-One function →

A fu^n $f: A \to B$ is 1-1 if $\forall\, x_1 \neq x_2$, $f(x_1) \neq f(x_2)$

or $f(x) = f(y)$

$\Downarrow$

$x = y$.



$x_1 \neq x_2$

But $f(x_1) = f(x_2) = f(x)$

Here → two diff. pts have same images.

or. $f(x_1) = f(x_2) = f(x)$

$\Downarrow$

$x_1 \neq x_2$ But pts in A are distinct.

Images are same

Graphically,

If line passing through the (co-domain) i.e ||r to x-axis intersects the curve $y = f(x)$ atmost once then fu^n is called 1-1 fu^n.

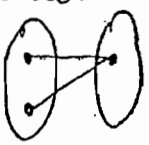
Atmost → We may have some elements in co-domain which does not have pre-image.

Onto fun^n

A fun^n $f: A \to B$ is called Onto fun^n if $\forall\, y \in B\ \exists$ ~~Unique~~ $x \in A$ s.t

$$f(x) = y.$$

Graphically, every line passing through (Co-domain) must intersect $y = f(x)$ atleast once. then fun^n is onto.



Example → Consider $y = f(x)$

where $f(x) = x^2$ then for following cases, discuss whether it is a fun^n, 1-1, onto or not.

a) $f: R^+ \to R$

b) $f: R^+ \to R^+$

c) $f: R \to R^+$

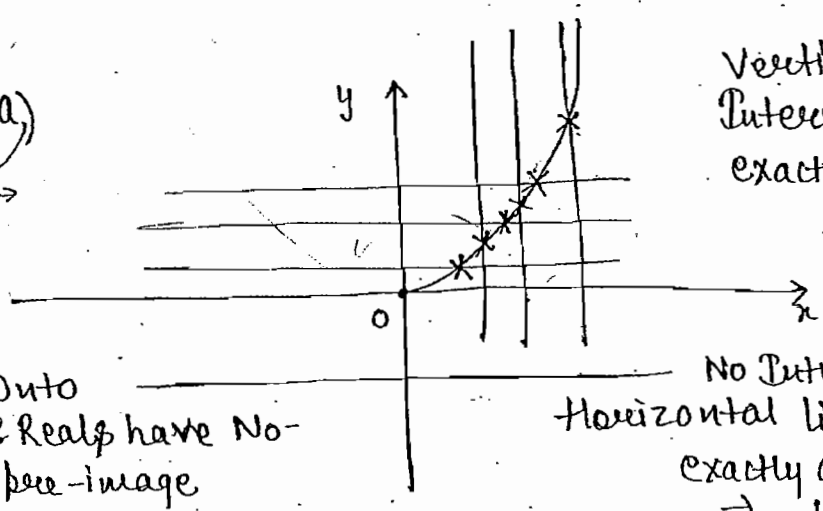
d) $f: R \to R$

0 is not included in R^+.

Sol^n →

a)

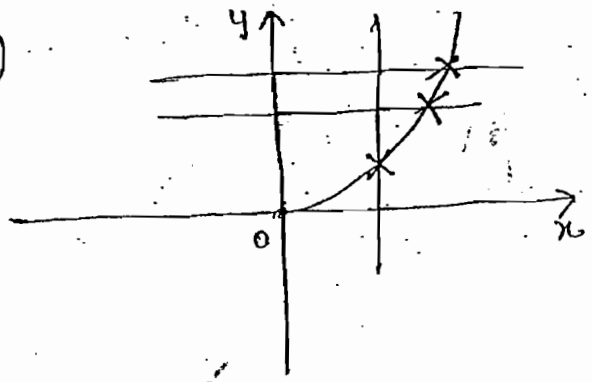




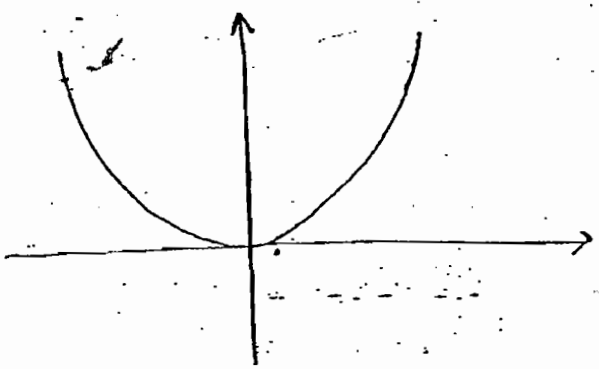
Vertical line Intersects at exactly one pt ⇒ fun^n

Not Onto
-ve Reals have No-pre-image

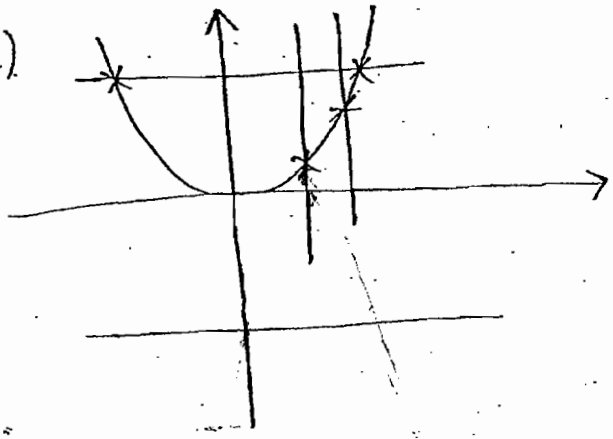
No Intersection with Curve
Horizontal line → intersects exactly at 1 pt.
⇒ 1-1

b) 

fu^n + 1-1 + Onto.

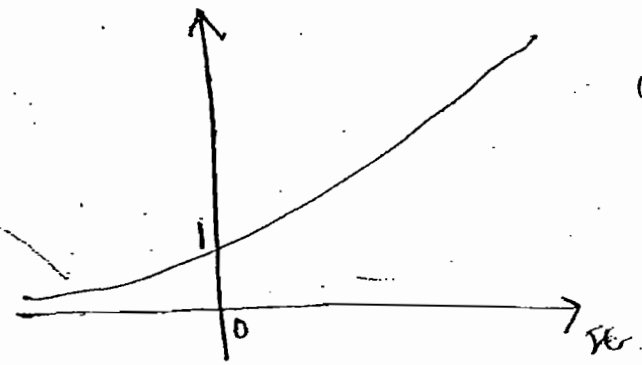
c) 

Not a fu^n
as $0 \in R$ has no image.
$(f(0)=0)$ $(\text{As } 0 \notin R^+)$

d) 

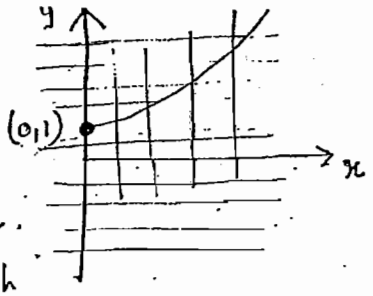
fu^n + Not 1-1 + Not Onto.

Example → let f given by $y = f(x) = e^x$

then 

a) $f: R^+ \to R$
b) $f: R \to R^+$
c) $f: R^+ \to R^+$
d) $f: R \to R$

a) $y = f(x) = e^x \quad f: R^+ \to R$

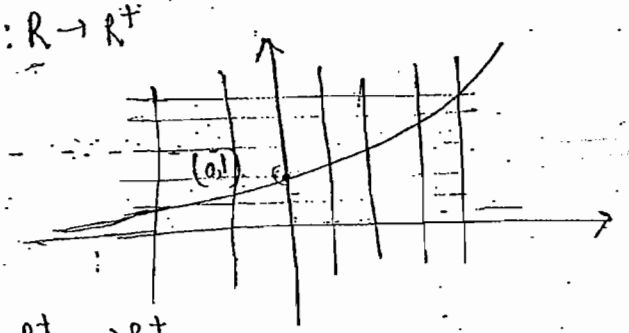


fu^n → ∵ every line passing through domain intersect must intersect the curve exactly one.

One-One → ∵ every line passing through co-domain intersect the curve atmost once.

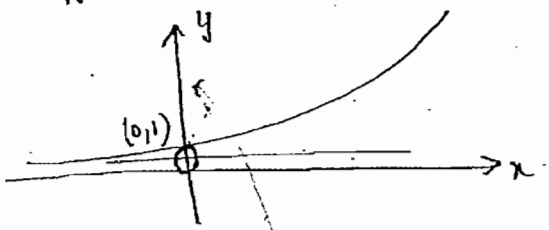
Not onto → ∵ line passing through co-domain below x-axis not intersect the curve atleast once.

b) $f: R \to R^+$



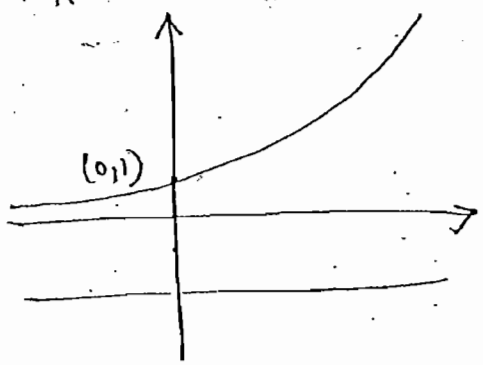
fu^n
1-1
onto
(∵ 0 ∉ R^+)

c) $f: R^+ \to R^+$



fu^n
1-1
Not onto

d) $f: R \to R$



fu^n
1-1
Not onto.

Differential Equation →

A relationship b/w the dependent variables and the derivatives of dependent variables w.r.t independent variables is called differential eqⁿ.

One Independent variable →

e.g. 1) $y = y(x)$ (One-Independent & One-Dependent variable x & y resp.)

ODE

$$\frac{d^3y}{dx^3} + \sin x \frac{dy}{dx} = e^y \quad \text{(Simple ODE)}$$

2) $y = y(x)$, $z = z(x)$ (One-Independent variable x & Two dependent variable y & z resp.)

$$\frac{dy}{dx} + x\frac{dz}{dx} = \sin x$$

$$\frac{dy}{dx} + \frac{dz}{dx} = \cos x$$

(System of ODE)

MORE THAN ONE Independent Variable →

PDE

3) $z = z(x,t)$ (One-dependent variable z & Two independent variable x,t)

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial t^2} = xt$$

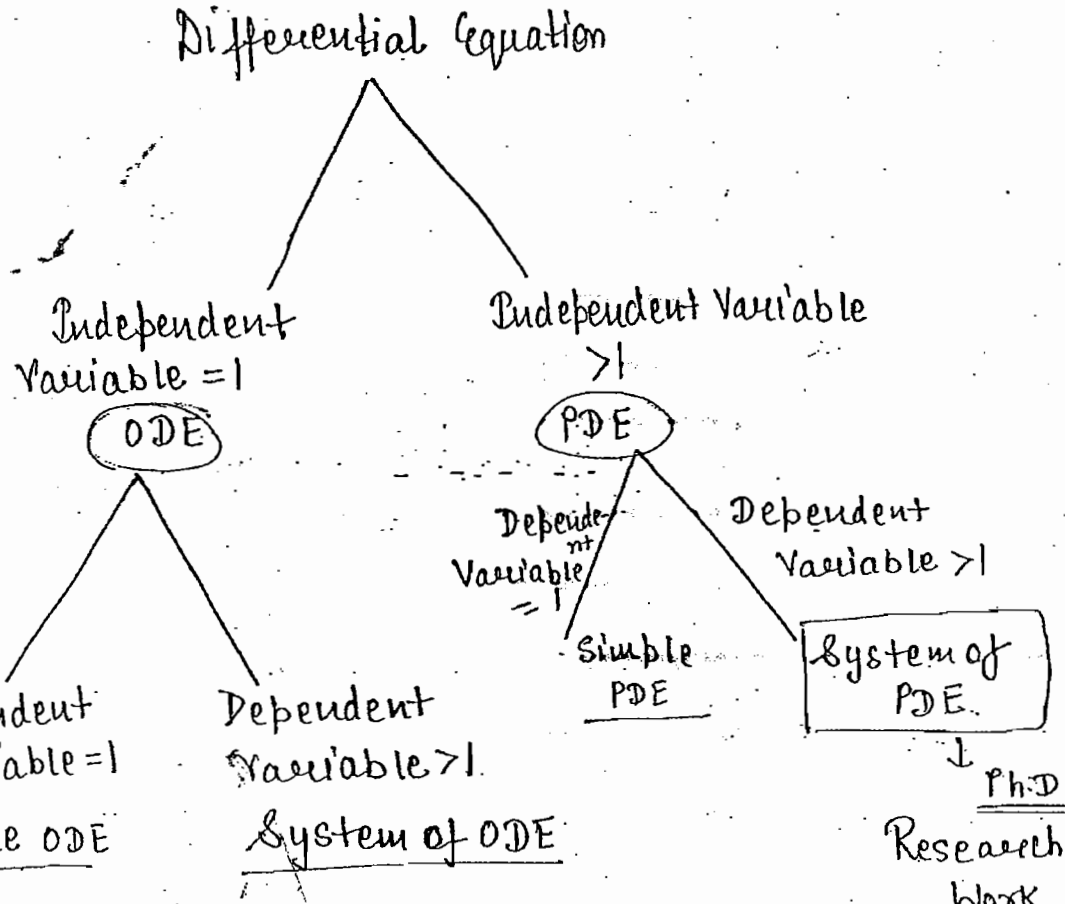
(Simple PDE)

4) $x = x(x,t)$, $u = u(x,t)$ (Two independent variable & Two dependent variable)

$$\frac{\partial z}{\partial x} + \frac{\partial u}{\partial t} = \sin x$$

$$\frac{\partial z}{\partial t} + \frac{\partial u}{\partial x} = \cos t.$$

(System of PDE)



Que: $\left|\frac{dy}{dx}\right| + |y| + a = 0$, $y = y(x)$ ← General form

Consider three values of a , $a = 0$

$a < 0$ and $a > 0$

Que 1. $\left|\frac{dy}{dx}\right| + |y| = 0$, $y = y(x)$ (case When $a=0$)

Find Number of Solutions?

Sol^n → a) let $y(x)$ be any Non-Constant Sol^n.

$\Rightarrow \exists\ x_0 \in \mathbb{R}$ s.t

$y(x_0) \neq 0$

$|y(x_0)| > 0$

$\left|\frac{dy}{dx}\right|_{at\,x_0} + |y(x)|_{at\,x_0} > 0$

+ve or zero + ve

$\Rightarrow$ $y(x)$ cannot be any, Non-Constant solution.

(If $y(x) = 0\ \forall x_0$ — denotes x-axis, $y(x) \equiv 0$ $\Rightarrow$ Constant. Assumed Non-Constant)

b) let $y(x)$ be any Constant solution (Non-Zero)

say $y(x) = a \neq 0$

$|y(x)| > 0$ (If a is -ve then $|y(x)| = |-a| = a > 0$)

$\Rightarrow \left|\frac{dy}{dx}\right| + |y(x)| > 0$ $\left(\left|\frac{dy}{dx}\right| = 0\right)$

$\Rightarrow$ $y(x)$ is not a solution.

$\therefore$ $y(x)$ Can't be any Non-Zero Constant Sol^n.

c) Only Possibility Left $y(x) \equiv 0$.

$\frac{dy}{dx} = 0$

$\Rightarrow \left|\frac{dy}{dx}\right| + |y(x)| = 0$

(Or $y(x) = b$, $y'(x) = 0$, $|y'(x)| + |y(x)| = 0 + b = 0$, $b = 0$ $\Rightarrow$ Only Trivial Sol^n)

$\Rightarrow$ There is only one solution i.e TRIVIAL SOLUTION

Ans → Only 1 Sol^n which is Trivial One.

Qus2. $\left|\frac{dy}{dx}\right| + |y(x)| + a = 0$, $a \in \mathbb{R}$, $a < 0$

Then a) diff eq^n has unique real valued sol^n.

b) diff eq^n has No real sol^n.

c) diff eq^n has infinite many real sol^n.

d) diff eq^n has exactly two real valued solutions.

Sol^n → a) Let $y(x)$ be a Non-Constant sol^n

$\Rightarrow \exists x_0$ s.t $y(x_0) \neq a$ $(\neq 0)$

$\left|\frac{dy}{dx}\right| \geq 0$ $\quad |y(x_0)| \geq 0$ $\quad a < 0$

Constant / Non-Constant → $\left|\frac{dy}{dx}\right|_{x \neq x_0} + |y(x_0)| + a \neq 0$ (1) $-a$

a = Constant (If a then $|+|$ will cancel out with a) $\quad (\neq a)$ $\quad -ve$

But $|y(x_0)| \neq a \neq 0$

Non-Constant

e.g → $y(x) = 2x$ (Const^{out} → $y' = 2 > 0$.

If $y(x) = x^2$ ← N.C

$y' = 2x$

↑ Non-Constant

a) Let $y(x)$ be any Non-Constant sol^n →

$|y'(x)| \geq 0 \quad \forall x$ $\quad (\geq 0 \because$ of Modulus value)

Then $y'(x)$ → either Constant / Non-Constant.

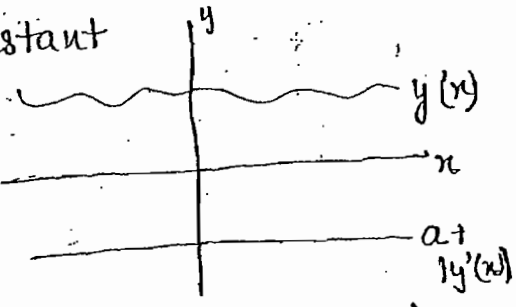
If $y'(x)$ → Constant

Then $|y'(x)| + a =$ Constant

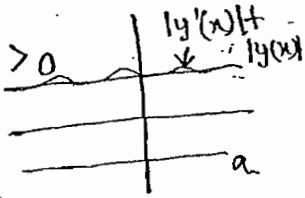
$|y'(x)| + |y(x)| + a \neq 0$

for some x

As → $y(x)$ is Non-Constant draw its curve. Then $|y(x)| + a$ will be shifted only. It can't be x-axis (i.e $|y(x)| + a \neq 0$)

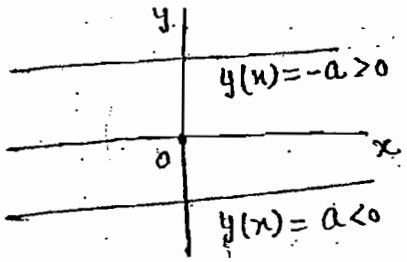


$y'(x)$ → Non-Constant Then $|y'(x)| + |y(x)| + a > 0$ (Same reason)



∴ $y(x)$ cannot be any Non-Constant sol^n.

b) Let $y(x) = b$, constant sol^n $b \in R$



Then $|y'(x)| = 0$

$$|y'(x)| + |y(x)| + a = 0$$

$$0 + |b| + a = 0$$

$$b = \pm a$$

$\therefore$ $y(x) = \pm a$ $(\because y(x) = b)$.

c) If $y(x) \equiv 0$ Not a sol^n clearly.

$\therefore$ $y(x) = a$ & $y(x) = -a$ are exactly two real valued sol

Ans → diff eq^n has exactly 2 real valued sol^{n}s.

$\left(\begin{matrix} 0 + |a| + a = 0 \\ 0 + |-a| + a = 0 \end{matrix}\right)$ (-ve, -ve)

Ques. $\left|\frac{dy}{dx}\right| + |y| + a = 0$, $a > 0$

Sol^n, Let $y(x)$ be any sol^n →

$$|y'(x)| \geq 0 \quad , \quad |y(x)| \geq 0$$

$$|y'(x)| + |y(x)| + a > 0 \qquad (\because a > 0 \therefore \neq 0)$$

$\therefore$ No Solution. ← Real valued ← Discuss ?

Ans → No Solution

May have Complex valued sol^n.

($y(x) \to$ Constant say b
$0 + |b| + a \neq 0$
So No-Sol^n)

Que.4. $\left|\frac{dy}{dx}\right| + |y| + a = 0$

The Number of sol^n's of diff. eq^n s.t $y(b) = c$, $b, c \in \mathbb{R}$

a) if $a = 0$, unique sol^n for infinite many choices of $(b, c) \in \mathbb{R}^2$

b) if $a < 0$, unique sol^n ——— ——— ———

c) if $a = 0$, No solution for infinite many choices of $(b, c) \in \mathbb{R}^2$

d) if $a < 0$, No sol^n ——— ——— ———

Sol^n → Consider $\left|\frac{dy}{dx}\right| + |y| + a = 0$ ——— ——— ———

From Previous 3 Que.'s →

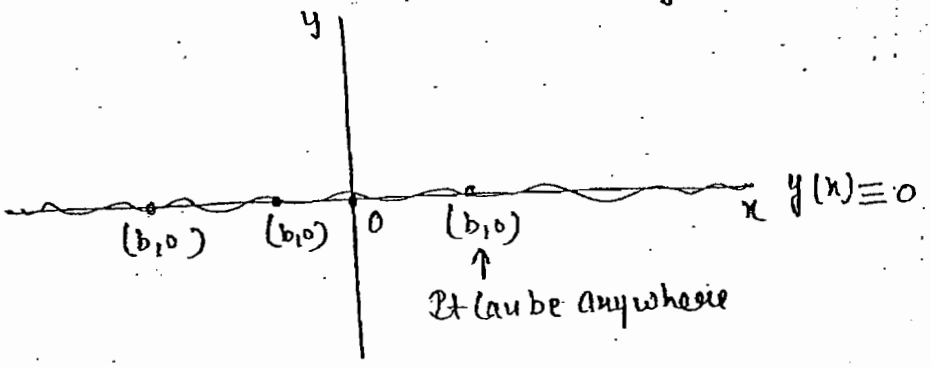
If $a = 0$ → Trivial sol^n

If $a < 0$ → Exactly Two sol^n's

If $a > 0$ → No sol^n

1st option →

If $a = 0$, We have Trivial sol^n $y(x) \equiv 0 \Rightarrow$ UNIQUE



(b, c) is any pt. in $\mathbb{R}^2$. (b, c) can be anywhere

Take $c = 0$ $(b, 0)$ lies on x-axis.

For $b \in \mathbb{R}$ We have infinite many values of b

∴ If $a=0$

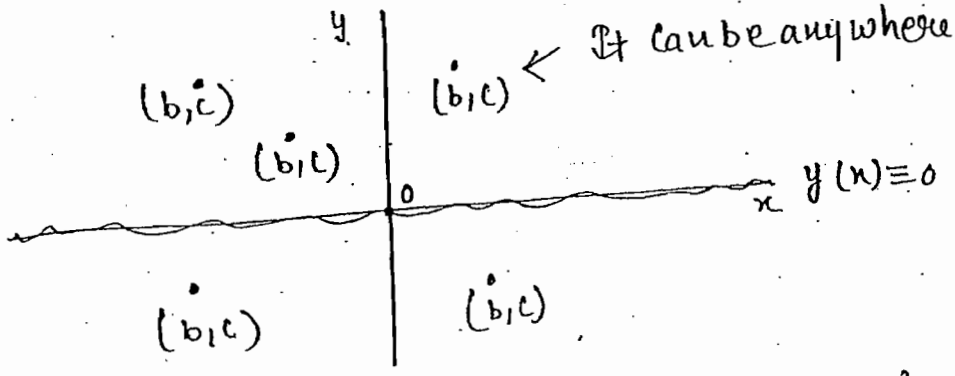
we have unique trivial sol^n, for infinite many values of $(b,c) \in \mathbb{R}^2$ [Here $c=0$ i.e $(b,0)$ infinite many values lying on x-axis]

Trivial sol^n will be satisfied by all such values.

∴ a) is True.

3rd option →

If $a=0$, we have only Trivial sol^n.



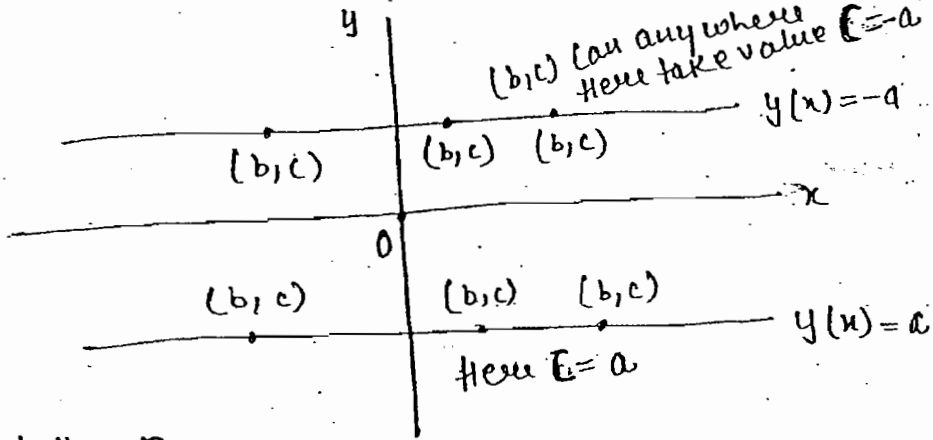
It can be anywhere.

∴ There are infinite many choices of $(b,c) \in \mathbb{R}^2$ for which ∄ no solution.

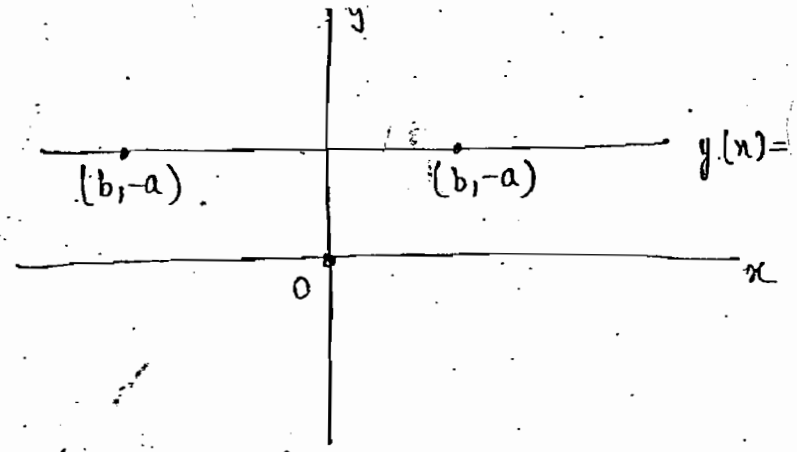
∴ c) is also True.

2nd option →

If $a<0$, In this case we have exactly two solutions.



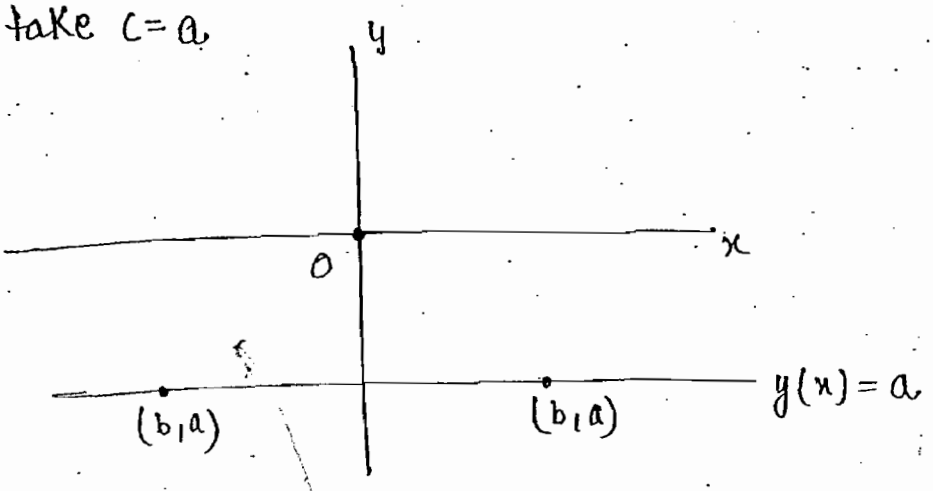
If we take $c=-a$



$\begin{pmatrix} a<0 \\ -a>0 \end{pmatrix}$

Exactly One Sol^n for infinite many choices of $(b,-a) \in \mathbb{R}^2$ (Take $c=-a$)

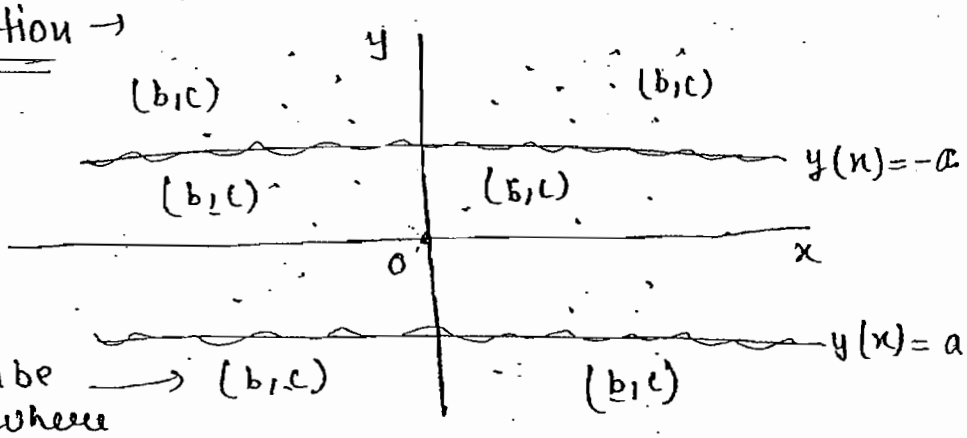
If we take $c=a$



Exactly One Sol^n for infinite many choices of $(b,a) \in \mathbb{R}^2$ (Take $c=a$)

$\therefore$ option b) is also correct.

4th option →



There are infinite many choices of $(b,c) \in \mathbb{R}^2$ for which $\exists$ No solution.

$\therefore$ option d) is also correct.

Ans. a), b), c), d) All are correct.

Note → $\int e^{x^2} dx \to ?$ Integrable or not.

For integration $\exists f(x)$ s.t

$$f'(x) = e^{x^2}$$

There does not exist such $f(x)$ s.t $f(x) = e^{x^2}$

$\therefore \int e^{x^2} dx \to$ Not Possible i.e e^{x^2} is not integrable.

Also $\frac{dy}{dx} = e^{x^2}$

Not Solvable

Integration Not Possible (To be Solved by Numerical Methods)

Note → A fu^n $f(x)$ is integrable i.e $\int f(x) \to$ When $\exists f(x)$ s.t

$$f'(x) = f(x).$$

Simple ODE →

Let $f(x, y, y', y'', \dots\dots y^{(n)}) = 0$ be nth order simple ODE.

(i.e fu^n of dependent variable y & derivatives of dependent variable w.r.t. Independent variable)

Sol^n of ODE → is a real or Complex valued fu^n ϕ

(Range $\subseteq R$) (Range $\subseteq C$)

of real variable x is a sol^n if →

($x \in I \subseteq R$) (Domain is always subset of R)

↓ subset of R

i) $\phi(x), \phi'(x), \dots\dots \phi^n(x)$ exist

ii) $\phi(x)$ satisfied the differential eq^n →

$$f(x, \phi(x) \dots\dots \phi^n(x)) = 0$$

Note →

ϕ is not the sol^n when ~~any one~~ ϕ does not satisfy i), ii) or both.

E.g → $x\,dy - y\,dx = 0$ $\quad x \in \mathbb{R}$

$\frac{dy}{y} - \frac{dx}{x} = 0$ $\quad y(x) \neq 0$, $x \neq 0$

On Integration →

$\log y - \log x = \log c$ $\quad (y(x) > 0, x > 0$ ∵ Log(-ve) → Not defined)

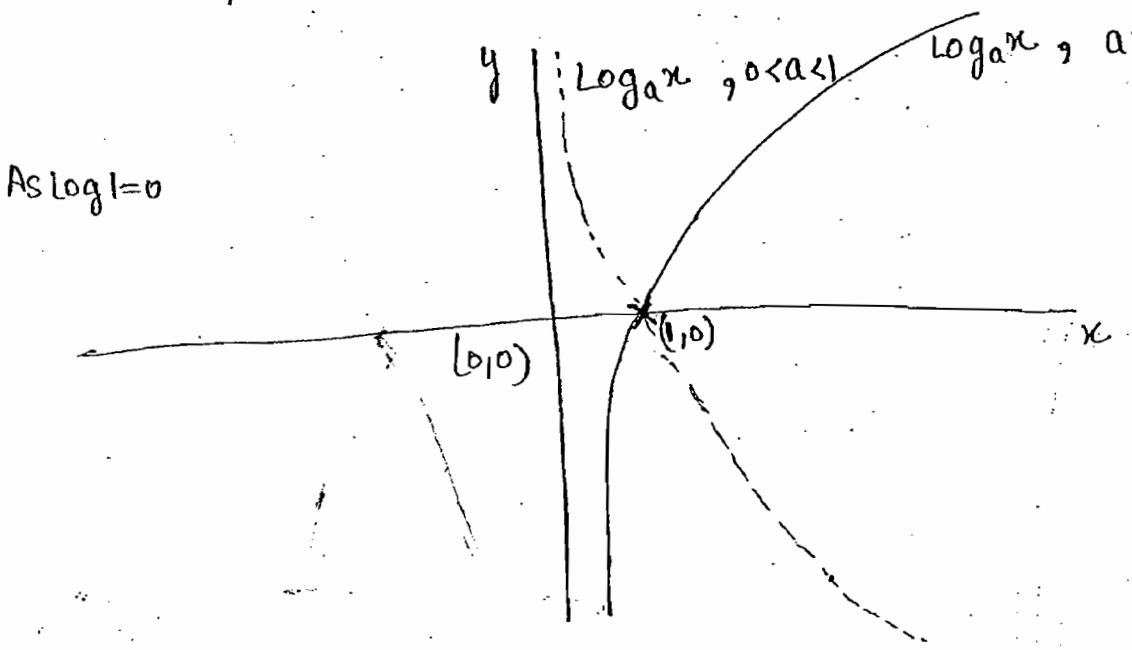
$\frac{y}{x} = c$ $\quad y(x) > 0, x > 0.$

$y(x) = cx$ $\quad y(x) > 0, x > 0.$

General Solution of given diff eq^n.

$y(x) \equiv 0$ is also the sol^n of given diff. eq^n. (In case when $c = 0$, $y(x) = 0$ But $y(x) > 0$ for any c $\therefore y(x) \equiv 0$ cannot be obtained from General sol^n).

Note $\Rightarrow$ This trivial sol^n cannot be find out from the General sol^n.



Note →

In this e.g → In starting $x \in \mathbb{R}$ (Say Domain of diff eq^n $I \subseteq \mathbb{R}$)

At Last →

We obtained $x > 0$ for $y(x) = cx$ (Domain of sol^n $\subseteq$ Domain of diff eq^n.

(Say $J \subseteq \mathbb{R}$) ← Domain of sol^n

We see that $\boxed{I \neq J}$. (I may or may not be equal to J)

(i.e Domain doesnot Remains the same)

Order of Diff. eq^n →

The highest order derivative occuring in the differential equation is called order of Diff. eq^n.

Degree of Diff. eq^n →

The exponent (power) of highest order derivative occuring in the diff. eq^n, provided power of all derivative must be natural number. (or we can say provided all the derivative are free from radical's & fractions)

E.g. → $\left(\frac{dy}{dx}\right)^{3/2} = x$

Squaring → $\left(\frac{dy}{dx}\right)^3 = x^2$

Order → 1

Degree → 3

Find Degree & Order in following e.gs ?

E.g → $(y''')^{2/3} = (y'')^{3/2}$ On Squaring & Cubing →

$(y''')^4 = (y'')^9$

Order → 3

Degree → 9 (Only when inverse does not exist)

& When inverse exist →

$1 = (y'')^5$

Order → 3, degree → 5

E.g. → $y''' - xy'' = e^{y'''}$ $\left(e^x = 1 + x + \frac{x^2}{2!} + \dots\right)$

Order → 3 degree → Does not exist

$$y''' - xy'' = 1 + y''' + \frac{(y''')^2}{2!} + \dots$$

Imp. | Order → Always exist
Degree → May | May Not exist

E.g. → $\frac{dy}{dx} + \int y\,dx = 0$

Integral term cannot be there in the differential eq^n.

∴ ~~Integrating~~ Differentiating →

$$\frac{d^2y}{dx^2} + y(x) = 0$$

Order → 2
Degree → 1

E.g. → $\frac{dy}{dx} + \int \phi(x)\,dx = 0$ (ϕ → fu^n of x alone)

As ϕ is fu^n of x only ∴ NO need of Differentiating →

$$\frac{dy}{dx} + F(x) = 0 \qquad \left(\int \phi\,dx = F(x),\ \phi(x) = F'(x)\right)$$

Order → 1
Degree → 1

e.g. → $|y''| + |y'| = 0$

Find Number of real valued sol^{n}s →

a) Unique
b) No sol^n
c) Infinite
d) Finite But Not Unique.

Sol^n → 1) Let $y(x)$ be any Non-Constant sol^n.

Then $y'(x)$ → either Constant or Non-Constant. (Shown Earlier)

When $y'(x)$ → Constant

$y'(x) \neq 0$

$\Rightarrow y'(x)$ is Non-Zero Constant fu^n.

$|y'(x)| > 0$

$\Rightarrow |y''| + |y'| > 0$

$\Rightarrow y$ is not a sol^n.

(If $y'(x) = 0$ $\Rightarrow y(x) = a$ $\Rightarrow y \rightarrow$ Constant) But y is Non-Constant Here

($y' \rightarrow$ Constant, $y'' = 0$)

When $y'(x)$ is Non-Constant.

$\exists\, x_0$ s.t $y'(x_0) \neq 0$

$\Rightarrow |y''(x_0)| + |y'(x_0)| > 0$

$\Rightarrow y(x)$ is Not a sol^n.

(If $y'(x_0) = 0$ $\forall\, x_0$ $\Rightarrow y(x) \rightarrow$ Constant But $y \rightarrow$ Non-Constant Here)

2) When $y(x) = a$ (Constant), $a \in \mathbb{R}$

$y'(x) = 0$

$y''(x) = 0$

$|y'(x)| + |y''(x)| = 0$

a is any arbitrary constant.

So, Infinitely Many real valued sol^{n}s.

Ans is c) → Infinite.

e.g → $\frac{2}{3}\frac{dy}{dx} = y^{1/3}$, $y(0) = 0$

a) Infinite many sol^{n} for any b
b) Unique sol^{n} for any b
c) No sol^{n} for any b.
d) exactly two sol^{n} for any b

Sol^{n} →

On Integration →

$$\frac{2}{3}\frac{dy}{dx} = y^{1/3}$$

$$\frac{2}{3}\frac{dy}{y^{1/3}} = dx \quad , y(x) \neq 0$$

$$\frac{2}{3} y^{-1/3}\, dy = dx$$

On Integration →

$$\frac{2}{3}\frac{y^{2/3}}{2/3} = x + C$$

$$y^{2/3} = x + C.$$

$$y(x) = (x+C)^{3/2}$$

Use $y(0) = 0$

$$0 = (0+C)^{3/2}$$

$$C = 0$$

$\therefore$ $y(x) = x^{3/2}$, ($y(x) \neq 0$) (Different value of x gives diff. y)

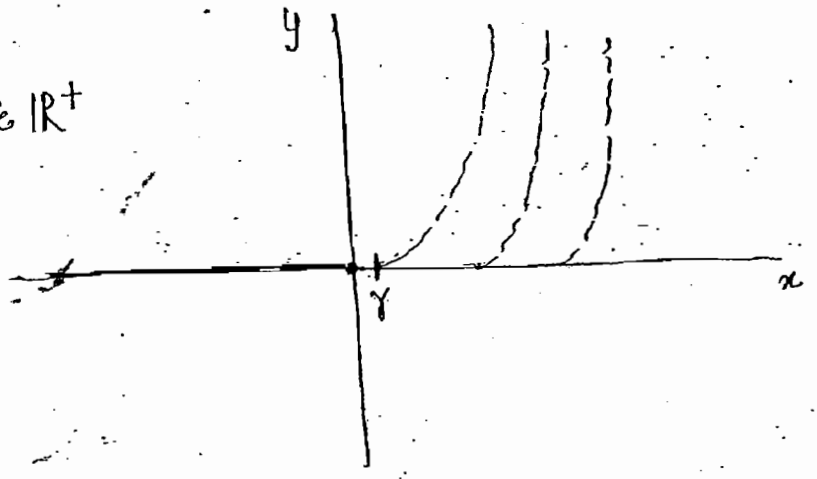
↓

One sol^{n}

$y(x) \equiv 0$ is also the sol^n of given differential Eq^n.
(As it satisfy given Diff eq^n)

↑ Another sol^n

New →
for $\gamma \in \mathbb{R}^+$
Define
$y(x)$



for $\gamma \in \mathbb{R}^+$

let $y(x) = \begin{cases} 0 & x \le \gamma \\ (x-\gamma)^{3/2} & x > \gamma \end{cases}$ ← (*)

$y'(x) = \begin{cases} 0 & x \le \gamma \\ 3/2\,(x-\gamma)^{1/2} & x > \gamma \end{cases}$ ← (1)

Put (1) in given differential eq^n →

$$\frac{2}{3} y' = y^{1/3}$$

$$\frac{2}{3} \begin{cases} 0 & x \le \gamma \\ 3/2\,(x-\gamma)^{1/2} & x > \gamma \end{cases} = \begin{cases} 0 & x \le \gamma \\ (x-\gamma)^{1/2} & x > \gamma \end{cases}$$

$$\begin{cases} 0 & x \le \gamma \\ (x-\gamma)^{1/2} & x > \gamma \end{cases} = \begin{cases} 0 & x \le \gamma \\ (x-\gamma)^{1/2} & x > \gamma \end{cases}$$

$\Rightarrow y(x)$ given by ☆ satisfy given ODE.

Hence, Infinitely many sol^{n}s as γ is arbitrary positive real

$\therefore$ option b) is correct.

General form of this Que →

$$\frac{dy}{dx} = y^\alpha \quad , y(b) = 0 \quad \boxed{\forall\, \alpha \in (0,1)},$$

$b \in \mathbb{R}$

This Diff. eq^n always has infinite many solutions.

Eg → $\frac{2}{3}\frac{dy}{dx} = y^{1/3}$, $y(1) = 0$

Then the differential eq^n has Real valued →

a) Unique sol^n
b) Infinite sol^n
c) No sol^n
d) exactly two sol^n.

Sol^n

$$\frac{2}{3} y^{-1/3}\, dy = dx \quad , y(x) \not\equiv 0$$

$$y^{2/3} = x + C$$

$$y(x) = (x+C)^{3/2}$$

Using $y(1) = 0$

$$0 = (1+C)^{3/2} \Rightarrow C = -1$$

$$y(x) = (x-1)^{3/2}$$

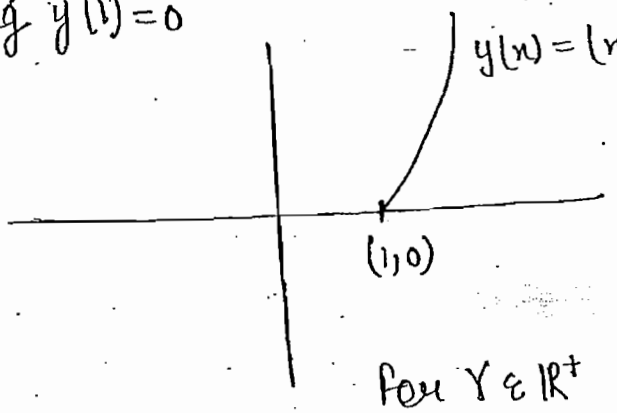
$x < 1 \rightarrow$ Complex valued
valid for $x > 1$

Domain of Sol^n → $[1,\infty)$

Domain of Differential Eq^n → $\mathbb{R}$

Domain is Restricted.

$y(x) \equiv 0$ is also the Sol^n of the differential Sol^n, satisfying $y(1)=0$



If Instead of $y(1)=0$ $y(b)=0$ is given then $\gamma \geq b$

↓ this case $y(x) = \begin{cases} 0 & x \leq \gamma \\ (x-\gamma)^{3/2} & x > \gamma \end{cases}$

If $1+\gamma=b$ only then at $x=b$ y is 0

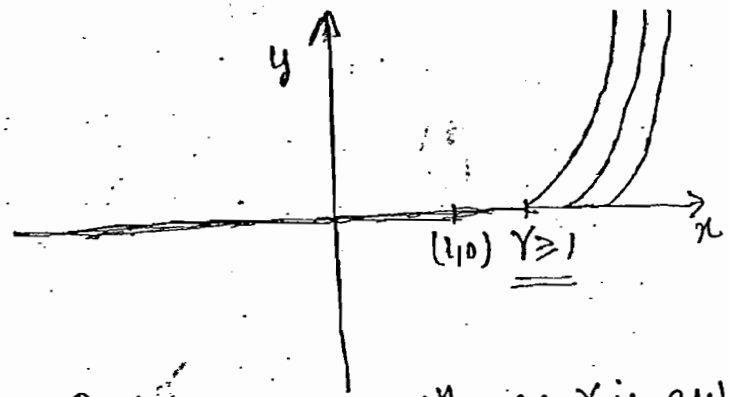
If $y(x) = \begin{cases} 0 & x \leq \gamma \\ (x-\gamma)^{3/2} & x > \gamma \end{cases}$ ——— (★)

It also satisfy the given diff. eq^n.

$$\frac{2}{3}\begin{cases} 0 & x \leq \gamma \\ \frac{3}{2}(x-\gamma)^{1/2} & x > \gamma \end{cases} = \begin{cases} 0 & x \leq \gamma \\ (x-\gamma)^{1/2} & x > \gamma \end{cases}$$

$$\begin{cases} 0 & x \leq \gamma \\ (x-\gamma)^{1/2} & x > \gamma \end{cases} = \begin{cases} 0 & x \leq \gamma \\ (x-\gamma)^{1/2} & x > \gamma \end{cases}$$

∴ $y(x)$ given by ★ satisfy the given differential eq^n.



Hence, Infinite many sol^{n}s as γ is arbitrary +ve real.
$\therefore$ Option b) is Correct → Infinitely many sol^{n}s.

Note→

$y(x)$ can't be $\begin{cases} 0 & x \leq \gamma \quad \gamma \geq 1 \\ (x-1-\gamma)^{3/2} & x > \gamma \quad \text{Take } \gamma = 1 \end{cases}$

$y(x)$ must satisfy $y(1) = 0$

At $x=1$ $\quad y(x) = 0$ $\quad$ L·H·L → i.e. $\lim_{x \to 1^-} y(x) = 0$

R·H·L $\lim_{x \to 1^+} y(x)$

$\lim_{x \to 1^+} (x-1-\gamma)^{3/2}$ $\quad \gamma = 1$

$\lim_{h \to 0} (1+h-1-\gamma)^{3/2}$

$\lim_{h \to 0} (-1+h)^{3/2} = (-1)^{3/2} \neq 0$ $\quad$ Not cts at $x=1$

$y(x)$ Can't be $\begin{cases} 0 & x \leq \gamma \Rightarrow x \leq 1 \quad \text{Take } \gamma = 1 \\ (x+\gamma)^{3/2} & x > \gamma \Rightarrow x > 1 \end{cases}$? 0

From these 2 sol^{n}s Continuity breaks.

At $x=1$ $\quad y(x) = 0$

L·H·L $\lim_{x \to 1^-} y(x) = 0$

R·H·L $\lim_{x \to 1^+} y(x) = \lim_{x \to 1^+} (x+\gamma)^{3/2} = \lim_{h \to 0} (1+h+1)^{3/2} = (2)^{3/2} \neq 0$

Not cts at $x=1$

Que: Let $y(x) = Ae^{Bx+C}$ be family of curves with A, B, C as 3 parameters (arbitrary) if $y(x)$ is the sol^n of

$$f(x, y, y', y'' \text{ ——— } y^n) = 0$$

Then order of $f(x, y, y', y'' \text{ ——— } y^n) = 0$

a) 3

b) 2

c) 1

d) None of these.

Sol^n Form the Diff eq^n with the help of sol^n.

$$y(x) = Ae^{Bx+C}$$

$$= Ae^{Bx} \cdot e^{C}$$

$$= A \cdot e^{C} \cdot e^{Bx}$$

$$= Q e^{Bx}$$

$$y'(x) = Q B e^{Bx} \quad \& \quad y'' = Q B^2 e^{Bx}$$

$$\Rightarrow \frac{y}{y'} = \frac{1}{B} \quad \& \quad \frac{y'}{y''} = \frac{1}{B}$$

$$\frac{y}{y'} = \frac{y'}{y''}$$

$$y y'' = (y')^2$$

Order → 2

Degree → 1

Here No. of arbitrary Constants (3) ≠ Order of Diff. eq^n (2)

Order depend → How many Independent arbitrary Constant? → C, & B

(*) If $\phi(x, y, c_1, c_2 \text{ ---------- } c_n) = 0$ ← (Sol. of Diff. eq.)

be n-parameter family of curves where $c_1, c_2 \text{ --- } c_n$ are Independent Constants then the order of Diff. eq. corresponding to this curve is equal to n.

(*) General Sol. → General Sol. of the Diff eq. is a Sol. in which number of arbitrary Constants is equal to the order of the Differential eq.

(*) Particular Sol. / Integral Curve → ---- ---

A Sol. of the Differential eq. obtained from General Sol. by taking some particular value of Constants.

C.g → $x\,dy + y\,dx = 0$

Sol. is →

$y = cx$, $x > 0$, $y(x) > 0$ (obtained earlier)

& $y(x) \equiv 0$

$y(x) \equiv 0$ is Not a particular Sol. (∵ this is not obtained by assigning any value to c)

Example 5 → Parabola → Sol. → $(x-a)^2 + y^2 = a^2$

On Diff → $2(x-a) + 2yy' = 0$

$(x-a) = -yy' \Rightarrow a = x + yy'$

$\Rightarrow (yy')^2 + y^2 = a^2 = (x + yy')^2$

$(yy')^2 + y^2 = x^2 + (yy')^2 + 2xyy'$

$\Rightarrow x^2 - y^2 + 2xyy' = 0$

? Envelope →

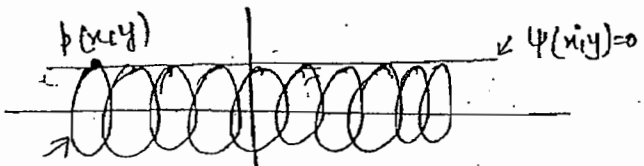
Let $f(x, y, y', y'' \text{——} y^n) = 0$ be nth order differential eqⁿ and

$$\phi(x, y, c_1, c_2 \text{——} c_n) = 0$$ be a General solⁿ.

Then a curve $\psi(x,y) = 0$ is called envelop to $\phi(x, y, c_1, c_2 \text{——} c_n) = 0$ if →

i) for each $p(x,y)$ on $\psi(x,y)=0$ ∃ a member of $\phi(x, y, c_1, c_2 \text{——} c_n) = 0$ s.t $\psi(x,y) = 0$ is tangent to $\phi(x, y, c_1, c_2 \text{——} c_n) = 0$ at $p(x,y)$.

($\phi(x, y, c_1, c_2 \text{——} c_n) = 0$ is the n-parameter family of curves.



member of ϕ s.t $\psi(x,y) = 0$ is the tangent to ϕ at $p(x,y)$)

ii) each member of $\phi(x, y, c_1, c_2 \text{——} c_n) = 0$ passes through one point $p(x,y)$ lying on $\psi(x,y)=0$

(~~It passes through more than one pt~~ i.e for every member of $\phi(x, y, c_1, c_2 \text{——} c_n)$ ∃ $p(x,y)$ such that $\psi(x,y)=0$ is tangent ϕ at $p(x,y)$.

(Envelope has 1-to-1 correspondence with the family of curves)

(i.e Unique tangent)

Que.

Consider family of circles with centre on x-axis of radius 3.

$$(x-a)^2 + y^2 = 9 \quad \text{— (1)}$$

On Diff →

$$(x-a) + yy' = 0$$

$$(x-a) = -yy'$$

$$(yy')^2 + y^2 = 9$$

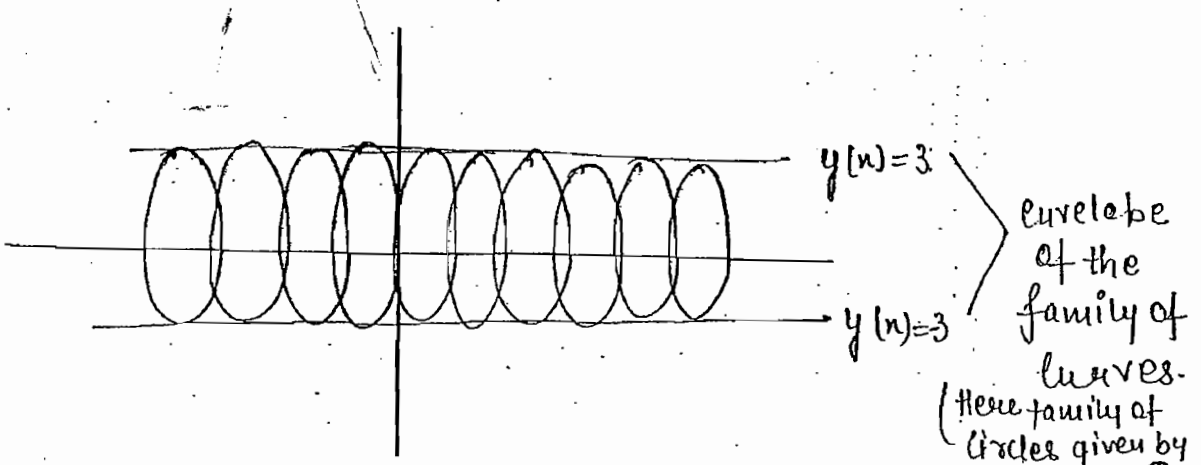
$y^2[y'^2+1] = 9$ is the Diff eq^n whose — (2) sol^n is given by (1).

$y(x) = 3$ is the sol^n of (2) →

$$9[0+1] = 9 \Rightarrow 9 = 9$$

Also $y(x) = -3$ is the sol^n of (2) → $9[0+1] = 9 \Rightarrow 9 = 9$.

for no value of 'a' these two sol^{n}s cannot be find out.



Singular Sol^n → An envelope to the General sol^n which satisfy the Differential eq^n is called Singular Sol^n.

Here $y(x) = 3$ & $y(x) = -3$ is also a Singular Solution.

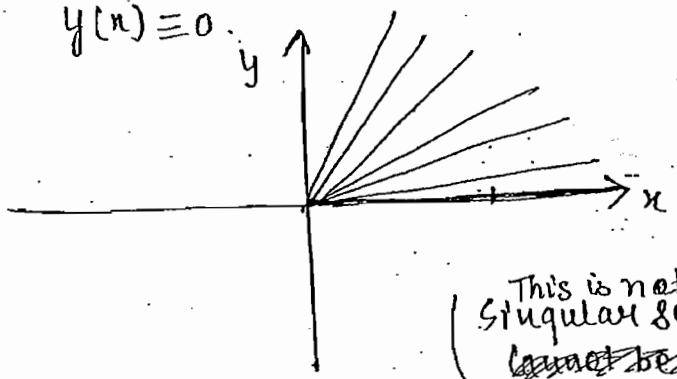
Note →

A differential eq^n can have No Singular Sol^n
Unique Singular Sol^n
Finitely many Singular Sol^n
Infinitely many Singular Sol^n.

e.g → $x\,dy - y\,dx = 0$

$y(x) = cx$, $x \geq 0$, $y(x) \geq 0$

$y(x) \equiv 0$



$y(x) \equiv 0$ Not a envelope to the family of Curves

(This is not a Singular Sol^n ~~cannot be find out from General Sol^n~~)

Linear & Non-Linear Diff. eq^n →

A diff. eq^n $f(x, y, y', y'' \text{ ---- } y^n) = 0$ is called Linear Diff eq^n if →

i) All the derivatives and the dependent variable is of degree 1.

ii) There does not exist any product terms of dependent variable and on its derivatives.

e.g → $y \cdot \frac{dy}{dx}$, $\frac{d^2y}{dx^2} \cdot \frac{dy}{dx}$

Que Let $f(x, y', y'' \text{ ——— } y^n) = 0$ be nth order Diff eq^n → Then choose the incorrect →

✓a) if deg $f = 1 \Rightarrow f = 0$ is linear diff eq^n

b) If $f = 0$ is linear $\Rightarrow$ deg $f = 1$

✓c) if deg $f > 1 \Rightarrow f = 0$ is Non-linear

✓d) If $f = 0$ is Non-linear $\Rightarrow$ deg $f > 1$

Sol^n:

a) $y' + \sqrt{y} = x$

Degree = 1

Non-linear Diff eq^n

Hence option a) is incorrect.

If →
$y'' + \sqrt{y'} = x$
$-y'' + x = \sqrt{y'}$
$y' = (x - y'')^2$

b) Correct.

c) If deg $f > 1$ Then $f = 0$ is Non-linear Hence Correct

d) $y'' + yy' = \sin x$

As $y\ y'$ are multiplied together
∴ Non-linear.
degree = 1

Hence Option d) is Incorrect.

First order & First degree diff eq^n →

General form → $\frac{dy}{dx} = f(x,y)$

or $M(x,y)\,dx + N(x,y)\,dy = 0$

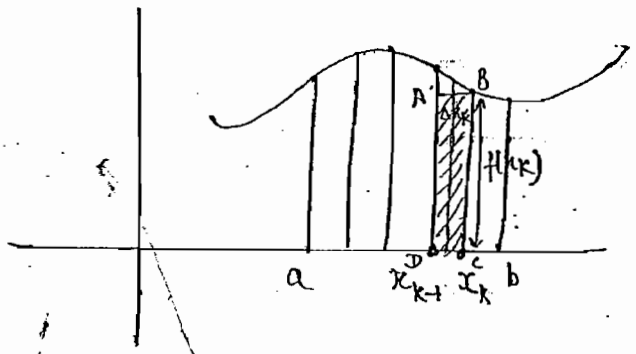
Imp → Exact diff. eq^n & Reducable into Exact diff eq^n.

→ Separable form & reducable into Separable.

→ Homog. diff eq^n & reducable into Separable form

Imp → Linear diff eq^n.

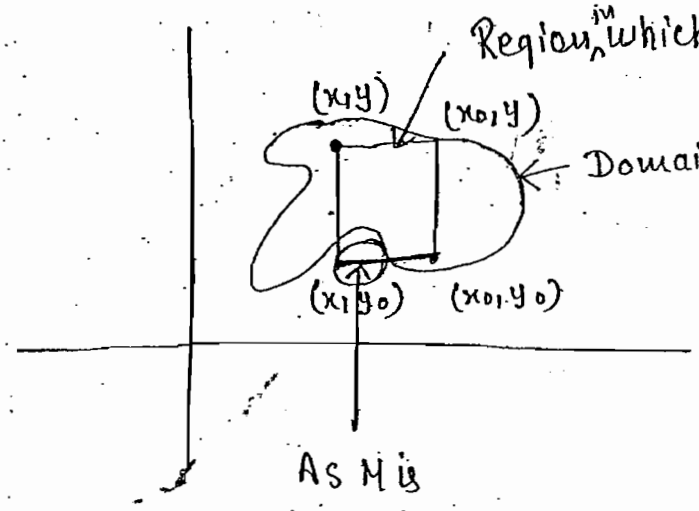
Integration → Area under Curve above x-axis.



Area of strip → ABCD = $f(x_k)\,\Delta x_k$

Area Under Curve $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k$

As we increase the number of partitions error in the ans reduces.

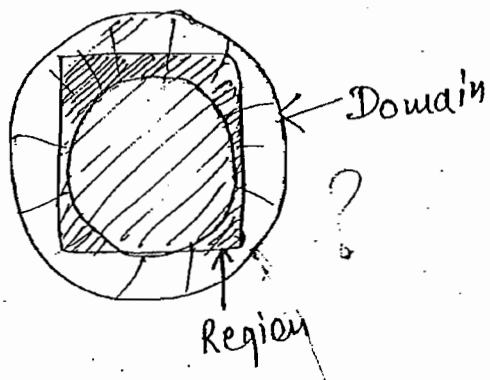


M & N are fu's of x & y defined in Domain.

$$\int_{x_0}^{x} M(x,y)\,dy + \int_{y_0}^{y} N(x,y)\,dy = C$$

As M is defined in Domain, so M can't be integrated in this region.

We shrink/expand the region so that integration can be done.



UDE → YOLLAD

Exact →

The expression $M\,dx + N\,dy$ is called exact if $\exists$ $f(x,y)$ s.t

$$d[f(x,y)] = M\,dx + N\,dy$$

($df(x,y) \Rightarrow$ f must be differentiable & hence cts)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

On comparing →

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad N = \frac{\partial f}{\partial y}$$

Now →

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

As f is cts

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Exact $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Now To show → If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Then show $M\,dx + N\,dy$ is exact.

Proof → As M & N are fu^n of (x,y)

$\therefore$ $\int_{x=\text{constant}} N\,dy = \phi(x,y)$ ($\int N(x,y)\,dy$ taking x as constant will be a fu^n of (x,y))

$$N(x,y) = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y}$

Integrating w.r.t $y \rightarrow$

$M = \frac{\partial \phi}{\partial x} + \psi(x)$ ← fu" of x

Putting value of M & N in $M dx + N dy \rightarrow$

$\left(\frac{\partial \phi}{\partial x} + \psi(x)\right) dx + \frac{\partial \phi}{\partial y} dy$

$\left[\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy\right] + \psi(x)\, dx$

$d(\phi(x,y)) + \psi(x)\, dx$

$d\left[\phi(x,y) + \int \psi(x)\, dx\right]$

$d[f(x,y)]$ Where $f(x,y) = \phi(x,y) + \int \psi(x) dx$

$\Rightarrow \int a$ fu" $f(x,y)$

$\int + \int d[f(x,y)] = M dx + N dy$

$\Rightarrow M dx + N dy$ is exact expression.

$\therefore$ $M dx + N dy$ is Exact $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Que→ $(x^2-ay)\,dx+(y^2-bx)\,dy=0$ is exact diff. eq^n

✓ a) only if $a=b$

b) for any a & $b=0$

c) for any b & $a=0$

✓ d) if $a=0$, $b=0$

Sol^n→ Here $M=x^2-ay$

& $N=y^2-bx$

As expression is given to be exact →

$$\therefore \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$-a=-b$$

$$a=b$$

Hence, option a) and d) are correct.

Que→ If $M\,dx+N\,dy$ be exact diff. eq^n then sol^n is/are true →

a) $\int M\,dx+\int N\,dy-\int\left(\frac{\partial}{\partial y}\int M\,dx\right)dy=C$

b) $\int M\,dx+\int N\,dy-\int\left(\frac{\partial}{\partial x}\int N\,dy\right)dx=C$

c) $\int M\,dx+\int N\,dy=C$, N is free from x

d) $\int M\,dx+\int N\,dy=C$, M is free from y.

Sol^{n} →

If $Mdx + Ndy = 0$ is exact diff. eq^{n} →

$\Rightarrow \exists\ f(x,y)$ s.t

$$d(f(x,y)) = Mdx + Ndy = 0$$

$$d(f(x,y)) = 0$$

$f(x,y) = c$ is the sol^{n} of diff eq^{n}. (*)

Now find $f(x,y)$ → ?

$$d(f(x,y)) = Mdx + Ndy$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

$$M = \frac{\partial f}{\partial x} \quad \text{and} \quad N = \frac{\partial f}{\partial y}$$

Take $M = \frac{\partial f}{\partial x}$

$$f(x,y) = \int M dx + \phi(y) \quad (1)$$ Find $\phi(y)$ → ?

& $N = \frac{\partial f}{\partial y}$

$$= \frac{\partial}{\partial y}\left[\int M dx + \phi(y)\right]$$

$$\frac{\partial \phi}{\partial y} = N - \frac{\partial}{\partial y}\left[\int M dx\right]$$

$$\phi = \int \left[N - \frac{\partial}{\partial y}\int M dx\right] dy.$$

Put in (1)

$$f(x,y) = \boxed{\int M dx + \int N dy - \int\left[\frac{\partial}{\partial y}\left(\int M dx\right)\right] dy = c} \quad (2)$$ (From *)

Hence option a) is correct.

Now Take

$$N = \frac{\partial f}{\partial y}$$

$$f(x,y) = \int N\,dy + \psi(x) \qquad \text{Find } \psi(x) \text{ ?}$$

$$M = \frac{\partial f}{\partial x}$$

$$M = \frac{\partial}{\partial x}\left(\int N dy + \psi(x)\right)$$

$$\psi'(x) = M - \frac{\partial}{\partial x}\int N dy$$

$$\psi(x) = \int M dx - \int\left(\frac{\partial}{\partial x}\int N dy\right)dx$$

$$\therefore f(x,y) = \boxed{\int N dy + \int M dx - \int\left(\frac{\partial}{\partial x}\int N dy\right)dx = C} \quad (3)$$

is the sol^{n}

Hence option b) is correct.

Now From (3) →

$$\int N dy + \int M dx - \int\left(\frac{\partial}{\partial x}\int N dy\right)dx = C$$

This is equal to d) option →

$$\int N dy + \int M dx = C \text{ , } M \text{ free from } y.$$

and a) option is equal to c)

E.g → $(x^2-ay)\,dx + (y^2-ax)\,dy = 0$ (H, N) Using → (3) Calculate sol^{n}

$$\int (x^2-ay)\,dx + \int (y^2-ax)\,dy - \int \left(\frac{\partial}{\partial x}\int (y^2-ax)\,dy\right)dx = 0$$

$$\left(\frac{x^3}{3}-ayx\right) + \left(\frac{y^3}{3}-axy\right) - \int \left(\frac{\partial}{\partial x}\left(\frac{y^3}{3}-axy\right)\right)dx = 0$$

$$\left(\frac{x^3}{3}-ayx\right) + \left(\frac{y^3}{3}-axy\right) - \int (0-ay)\,dx = 0$$

$$\left(\frac{x^3}{3}-ayx\right) + \left(\frac{y^3}{3}-axy\right) + axy = 0$$

Diminishing i.e Removing y terms with help of $\frac{\partial}{\partial x}$ & If there is any y term then that will cancel out with y term in $\int H\,dx$

Using d) option → Terms here with y will cancel out with this

$$\int x^2\,dx + \int (y^2-ax)\,dy = 0 \qquad \text{(Here H is free from y)}$$

$$\frac{x^3}{3} + \frac{y^3}{3} - axy = 0$$

Both gives the same ans.

Better to find answers from a) & b) option.

See How →

$$\underbrace{\cos^2 x}_{N}\,dy + \underbrace{y(-2\cos x \sin x)}_{H}\,dx = 0$$

$$\frac{\partial H}{\partial y} = -2\cos x \sin x$$

$$\frac{\partial N}{\partial x} = -2\sin x \cos x$$

$$\int H\,dx + \int N\,dy = 0$$

$N \rightarrow$ free from x

$$\therefore \int H\,dx + 0 = C$$

$$\int H\,dx = y\int(-2\cos x \sin x)\,dx = -y\int \sin 2x\,dx$$

$$= +y\frac{\cos 2x}{2} = C$$

(If $\cos^2 x$ is changed to $\frac{1+\cos 2x}{2}$ then $N \rightarrow$ free from x, $N = 1/2$.

Ans →

$$y\frac{\cos 2x}{2} + \frac{y}{2} = C$$

$$y\left(\frac{\cos 2x}{2} + 1\right) = C$$

)

Or

$$\int H\,dx + \int N\,dy - \int \frac{\partial}{\partial y}\left(\int H\,dx\right)dy = C$$

$$\frac{y\cos 2x}{2} + y\cos^2 x - \int \frac{\partial}{\partial y}\left(\frac{y\cos 2x}{2}\right)dy = C$$

$$\frac{y\cos 2x}{2} + y\cos^2 x - \int \frac{\cos 2x}{2}\,dy = C$$

$$\frac{y\cos 2x}{2} + y\cos^2 x - y\frac{\cos 2x}{2} = C$$

$$y\cos^2 x = C$$

$$y\left(\frac{1+\cos 2x}{2}\right) = C$$

$$y\left(\left(\frac{1}{2}\right) + \frac{\cos 2x}{2}\right) = C$$

↑ Missing in above Ans.

Needs No Change.

In Case Of polynomials Use any one there will be no change.

Eg2. $\cos y\,dx - (x\sin y - y^2)\,dy = 0$

$M\,dx + N\,dy = 0$

$M = \cos y$

$N = -x\sin y + y^2$

Given diff eq^n is exact only when →

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$-\sin y = -\sin y + 0$$

$$-\sin y = -\sin y.$$

⇒ Given diff eq^n is exact diff eq^n.

Eg3: $$\left(x - \frac{y}{x^2+y^2}\right)dx + \left(y + \frac{x}{x^2+y^2}\right)dy = 0$$

$$M = x - \frac{y}{x^2+y^2}$$

$$N = y + \frac{x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = 0 - \frac{(x^2+y^2)\cdot 1 - (2y)y}{(x^2+y^2)^2} = \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = 0 + \frac{(x^2+y^2)(+1) - 2x\cdot x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

∴ Given diff. eq^n is exact diff eq^n.

Eq3. $M = \frac{3}{x} - \frac{y}{x^2}$ Find $N(x,y) \to ?$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial x} = -x^{-2}$$

On Integrating →

$$N = \frac{-x^{-1}}{-1} + \phi(y)$$

$$N = \frac{1}{x} + \phi(y).$$

∴ a), b), c) and d) are all correct.

Eq4. $M = x^2 - ay$

$N = y^2 - ax$

$M_y = -a$, $N_x = -a$

⇒ Exact diff eq^n.

Find Sol^n → ?

$$\int M\,dx + \int N\,dy + \int\left(\frac{\partial}{\partial x}\int N\,dy\right)dx = c$$

$$\int (x^2 - ay)\,dx + \int (y^2 - ax)\,dy - \int \frac{\partial}{\partial x}\left(\int (y^2 - ax)\,dy\right)dx = c$$

$$\int (x^2 - ay)\,dx + \int (y^2 - ax)\,dy - \int \frac{\partial}{\partial x}\left(\frac{y^3}{3} - axy\right)dx = c$$

$$\left[\frac{x^3}{3} - axy\right] + \left[\frac{y^3}{3} - axy\right] + \int (0 + ay)\,dx = c$$

$$\frac{x^3}{3} - axy + \frac{y^3}{3} - axy + axy = c$$

$$\frac{x^3}{3} + \frac{y^3}{3} - axy = c$$

$$x^3 + y^3 - 3axy = 9 \quad \text{Ans.}$$

Que. $M dx + N dy = 0$ be exact differential eq^n. On page 4.

Sol^n From the articles of Sol^n →

$$\phi'(y) = N - \frac{\partial}{\partial y}\left[\int M dx\right]$$ Chk ?

↑ May be fu^n of y or constant

∴ Option a) is correct

and $$\psi'(x) = M - \frac{\partial}{\partial x}\left[\int N dy\right]$$

↑ May be fu^n of x or constant

∴ option b) is correct

Que. $\left\{x^3(x^2+y^2)+y\right\}dx + x\,dy = 0$

Sol^n →

$$\int M dx + \int N dy - \int \frac{\partial}{\partial y}\left(\int M dx\right) dy = C$$

$$\int \left(x^3(x^2+y^2)+y\right)dx + \int x\,dy - \int \frac{\partial}{\partial y}\left(\frac{x^6}{6} + \frac{y^2x^4}{4} + xy\right) dy = C$$

$$\frac{x^6}{6} + \frac{x^4y^2}{4} + xy - \int \left(0 + \frac{2y\,x^4}{4} + x\right) dy = C$$

$$\frac{x^6}{6} + \frac{x^4y^2}{4} + xy - \left(\frac{y^2x^4}{4} + xy\right) = C$$

$$\frac{x^6}{6} - xy = C$$ Option b) →

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$1 + x^3(2y) \neq 1$$

Que. $x^2\frac{dy}{dx} + xy\sqrt{1-x^2y^2} = 0$

$x^2 dy + xy\sqrt{1-x^2y^2}\,dx = 0$

$M = xy\sqrt{1-x^2y^2}$

$N = x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Diff. Equation is not exact.

$$\int M dx + \int N dy - \int \frac{\partial}{\partial y}\left(\int M dx\right) dy = c$$

$$\int xy\sqrt{1-x^2y^2}\,dx + \int x^2 dy - \int \frac{\partial}{\partial y}\left(\int xy\sqrt{1-x^2y^2}\,dx\right) dy = c$$

$$y\int x\sqrt{1-x^2y^2}\,dx + x^2y - \int \frac{\partial}{\partial y}\left(y\int x\sqrt{1-x^2y^2}\,dx\right) dy = c$$

$y\int$

Diff eq^n reducible into exact diff. eq^n →

Let $M(x,y)dx + N(x,y)dy = 0$ be a Non-exact diff. eq^n

Integrating factor →

A fu^n $\phi(x,y)$ is said to be I.F if

$\phi(x,y)[Mdx+Ndy]=0$ is exact diff. eq^n.

Q. $-y^2 e^{xy^2}dx + 2xy\,e^{xy^2} = 0 = g(x,y,y')$ — ①

($y(x)\equiv 0$ is a Sol^n)

$M = y^2 e^{xy^2}$

$N = 2xy\,e^{xy^2}$

$M_y = 2y\,e^{xy^2} + y^2 e^{xy^2}\cdot 2y\cdot x$

$N_x = 2y e^{xy^2} + 2xy^3 e^{xy^2}$

$M_y = N_x$

This shows ① is exact differential eq^n.

forming $f(x,y,y') = \underbrace{e^{xy^2}}_{M}dx + \underbrace{\frac{2x}{y}}_{N} e^{xy^2}dy = 0$ (By dividing y^2 in ①)

($y(x)\equiv 0$ is Not a Sol^n As y is in the denominato

$M_y \neq N_x$

⇒ f is not exact diff. eq^n.

If we multiply f by y^2, then the differential eq^n becomes exact.

Q. Whether Sol^n space of $f(x,y,y')$ = Sol^n space of $g(x,y,y')$ or ? Not-o

$f(x,y,y')=0$ Non-exact diff eq^n

$g(x,y,y')=0$ Corresponding exact diff. eq^n

$A = \{ y(x) \mid y(x) \text{ is Sol}^n \text{ of } f(x,y,y')=0 \}$

$B = \{ y(x) \mid y(x) \text{ is Sol}^n \text{ of } g(x,y,y')=0 \}$

a) $A \subseteq B$

b) $B \subseteq A$

c) $A = B$

✓d) None of these

Ans.

Exp. Non Exact → (Change) → Exact → Extra sol^n is added / Removed

$B \nsubseteq A$

In previous e.g → Extra Trivial sol^n is added to B.

$f(x,y,y') =$ $x\,dy - y\,dx = 0$ ← Non exact diff eq^n ((Having Zero sol^n) $y(x) \equiv 0$ is the sol^n)

$M = -y \quad N = x$

$M_y = -1 \quad N_x = 1$

$g(x,y,y') =$ $\dfrac{x\,dy - y\,dx}{y^2} = 0$ ← Exact sol^n. (No. Zero sol^n) $y(x) \neq 0$ Not the sol^n.

$M = \dfrac{-1}{y} \quad N = \dfrac{x}{y^2}$

$\dfrac{\partial M}{\partial y} = \dfrac{1}{y^2}, \quad N_x = \dfrac{1}{y^2}$

In this e.g → Trivial sol^n is removed $(A \nsubseteq B)$

$B \nsubseteq A$
$\because \nexists\ y(x) \equiv 0$ which belongs to B but does not belongs to A

$A \nsubseteq B$
$\because \nexists\ y(x) \equiv 0$ which belongs to A but does not belongs to B.

$\therefore$ Also $A \neq B$ $\therefore$ option d) is Correct.

Conclusion → There is no relation b/w sol^n space of Exact and Non-exact diff eq^n

Que. If $f(x,y,y')=0$ be a Non-exact Diff eq^n, then Number of Integrating factor →

a) unique

b) finite but need not be unique

c) always Infinite

d) dependency on diff. eq^n ?

Sol^n.

Let $Mdx+Ndy=0$ be Non-Exact diff eq^n.

Let $\phi(x,y)$ be an I.F

$\Rightarrow \phi M dx + \phi N dy = 0 \rightarrow$ Exact diff eq^n

$\Rightarrow \exists\ f(x,y)$ s.t

$$d(f(x,y)) = \phi M dx + \phi N dy.$$

($f(x,y)=c \Rightarrow$ Solution)

Let G_1 be any arb. fu^n of f. →

$G_1 = G_1(f)$ (← fu^n of fu^n)

$$d\psi = G_1(f)\,df = \phi G_1(f)(Mdx+Ndy)$$ (Put the value of df)

$\Rightarrow \phi G_1(f)$ is also an I.F.

(like → $g(x)dx$ integrable ? we can reverse it $\int g(x)dx$ ∵ of total diff

$\Rightarrow$ R.H.S → Integrable

$\Rightarrow$ Exact)

As G_1 is Arbitrary

∴ I.F are infinite of the form $\phi G_1(f)$

(like

$G_1(f)\,df \backsim g(x)dx$

↓

$\int g(x)dx$)

(If depends on diff eq^n →

If Non-Exact → having sol^n → Infinite I.F

Non-Exact → No sol^n → No I.F

Exact → Infinite I.F

Discussed on Next Page)

e.g. → $x\,dy - y\,dx = 0$

$\phi(x,y) = \frac{1}{y^2}$ ✓

$\frac{x\,dy - y\,dx}{y^2} = 0$ → Exact diff eq^n

($\int M\,dx + \int N\,dy = c'$ (free from x)

$\int -\frac{1}{y}\,dx = c'$

$-\frac{x}{y} = c' \Rightarrow \frac{x}{y} = c$ be the sol^n)

Sol^n → $\frac{x}{y} = c$

$f(x,y) = \frac{x}{y}$ ✓

Integrating factor is given by $\phi G_1(f)$ →

$\frac{1}{y^2} G_1\left(\frac{x}{y}\right)$ → Also Integrating factor.

G_1 → Arbitrary fu^n.

$\frac{1}{y^2} \times \frac{x}{y}$, if $G_1(u) = u$

$\frac{1}{y^2} \times \sin\frac{x}{y}$, if $G_1(u) = \sin u$

$\frac{1}{y^2} \times \tan^{-1}\frac{x}{y}$, if $G_1(u) = \tan^{-1} u$

$G_1(u) = u^2, u^3$ etc.

∴ Infinite No. of I.F

Note →

If α is any constant and ϕ is I.f → (Non-constant) then $\alpha\phi$ is also Non-Constant I.f.

No. of Non-Constant Integrating factor → Infinite

&

No. of Constant I.f → ~~Infinite~~ Does not exist

(Integrating factor → $f(x,y)$ i.e is fu^n of x & y)

No. of linearly Dependent I.f → Infinite

No. of linearly Independent I.f → Infinite

If $\phi(x,y) \to$ I.F

Of $Mdx+Ny=0$

and if α is any constant then $\alpha\phi$ is also the I.F ?

Since α is arbitrary, then we have infinite I.F also.

Also these are infinite linearly dependent I.F.

Also $\phi G(t)$ is also the I.F, where $f(x,y)=c$ is sol^n of exact diff. eq^n.

Depending on G, we get different Integrating factors and all such I.F are independent, they does not depend on each other as G is different in all I.F.

(i.e As G is arbitrary, so infinite such sol^{n}s & these sol^{n}s are L.I)

Note→

If a diff. eq^n has a sol^n (solvable) then every Non-Exact diff eq^n is reducible to Exact diff eq^n. (i.e it has I.F)

If a Non-Exact diff eq^n has a sol^n then it always has a infinite no. of I.F.

(Diff eq^n has sol^n Then
Non-Exact
↓ Reducible
Exact
↓
It has I.F & I.F are infinite)

Let $\phi(x,y)$ be an I.F of

(1) $$M\,dx + N\,dy = 0$$

Let $\boxed{\phi(x,y) = \phi(x)}$ (I.F → fu^n of x)

$\phi M\,dx + \phi N\,dy = 0$ be the exact diff eq^n.

M' N'

$$M_y' = N_x'$$

$$\phi(x)\,M_y = \phi'(x)\,N + \phi(x)\,N_x$$

$$\frac{M_y - N_x}{N} = \frac{\phi'(x)}{\phi(x)} = f(x) \rightarrow \text{fu}^n \text{ of } x \text{ only}$$

— atmost constant

$$\frac{\phi'(x)}{\phi(x)} = f(x)$$

On Integrating →

$$\log \phi(x) = \int f(x)\,dx$$

$$\Rightarrow \phi(x) = e^{\int f(x)\,dx} \rightarrow \text{I.F}$$

Que?

Let $\frac{M_y - N_x}{N} = f(x)$ (validate)

Then this is I.F → $e^{\int f(x)dx}$

($\phi(x)$ is fu^n of x only $\Rightarrow \frac{M_y - N_x}{N} = f(x)$ fu^n of x only)

$P \Rightarrow Q$

$$\frac{\partial}{\partial y}\left[M \cdot e^{\int f(x)dx}\right] = \frac{\partial}{\partial x}\left[e^{\int f(x)dx} \cdot N\right]$$

(Exactness → $M_y = N_x$ after multiplying with I.F)

$$\text{if } M_y \cdot e^{\int f(x)dx} = e^{\int f(x)dx} \cdot f(x) \cdot N + N_x \cdot e^{\int f(x)dx}$$

$$\text{if } M_y = N \cdot f(x) + N_x$$

$$\text{if } \frac{M_y - N_x}{N} = f(x), \text{ which is true}$$

$\left(\frac{\partial}{\partial x} e^{\int f dx} = e^{\int f(x)dx} \frac{\partial}{\partial x}\left(\int f(x)dx\right) = e^{\int f(x)dx} \frac{d}{dx}\left(\int f(x)dx\right) = e^{\int f dx} f(x)\right)$

⇒ ∃ an I.F which is fu^n of x only. (Existence Guarantee)

If $\frac{My - Nx}{N} = f(x,y) \mid f(y)$.

⇒ There does not exist an I.F which is fu^n of x only.

Conclusions ⇒

Let $Mdx + Ndy$ be a Non-exact diff eq^n →

(1) If $Q(x,y) = Q(x)$ is an I.F

$\Rightarrow \frac{My - Nx}{N} =$ fu^n of x only | Constant.

(2) If $\frac{My - Nx}{N}$ = Non Constant fu^n of x & y both or fu^n of y

$\Rightarrow$ ∄ any I.F which is fu^n of x-only

(3) If $\frac{My - Nx}{N} = f(x)$

$\Rightarrow$ ∃ an I.F which is fu^n of x-only given by $e^{\int f(x)\,dx}$.

← Here $\frac{\partial}{\partial x}$ changes to $\frac{d}{dx}$ ∵ $\int f(x)\,dx$ is fu^n of x-only.

e.g → $x\,dy - y\,dx = 0$

$M = -y$ $\quad M_y = -1$

$N = x$ $\quad N_x = 1$ $\quad M_y \neq N_x$

$$\frac{M_y - N_x}{N} = \frac{-1-1}{x} = \frac{-2}{x} = \text{fu}^{\text{n}} \text{ of } x \text{ only}$$

$\Rightarrow$ I.f $e^{\int -2/x^2\,dx}$

$= e^{-2\log x} = \frac{1}{x^2}$

(2) Let $\phi(x,y)$ be the I.f of $M\,dx + N\,dy = 0$

Now → $\boxed{\phi(x,y) = \phi(y) \text{ only}}$ (I.f → fu^{n} of y-only)

$\underbrace{\phi M}_{M'}\,dx + \underbrace{\phi N}_{N'}\,dy = 0$ be the exact diff. eq^{n}.

$M_y' = \phi'(y) M + M_y \phi(y)$

$N_x' = \phi(y) N_x$

$\phi'(y) M + M_y \phi(y) = \phi(y) N_x$

$(M_y - N_x)\,\phi(y) = -M \phi'(y)$

$$\frac{M_y - N_x}{M} = -\frac{\phi'(y)}{\phi(y)} = \text{fu}^{\text{n}} \text{ of } y = f(y) \quad \text{(Constant)}$$

$$\Rightarrow \frac{\phi'(y)}{\phi(y)} = -f(y)$$

On Integration →

$$\phi(y) = e^{\int -f(y)\,dy}$$

Conclusions →

(1) Let $Mdx + Ndy$ be Non-exact diff. eq^n →
If $\phi(x,y) = \phi(y)$ is an I.F
$\Rightarrow \frac{My - Nx}{M}$ is a fu^n of y only / constant

(2) If $\frac{My - Nx}{M}$ = Non-constant fu^n of x & y both or fu^n of x
$\Rightarrow$ ∄ any I.F which is fu^n of y-only.

(3) If $\frac{My - Nx}{M}$ = fu^n of y only = $f(y)$ $\Rightarrow$
∃ an I.F which is fu^n of y-only & given by $e^{\int f(y)\,dy}$

e.g →

$x\,dy - y\,dx = 0$

$M = -y$

$N = x$

$\frac{My - Nx}{M} = \frac{2}{y}$ = fu^n of y

$I.F = e^{-\int \frac{2}{y} dy}$

$= \frac{1}{y^2}$

(3) **When $\phi(x,y) = \phi(u)$, $u = x \cdot y$**

$$\frac{\partial}{\partial y}[\phi(u) \cdot M] = \frac{\partial}{\partial x}[\phi(u) \cdot N]$$

$\left(\frac{\partial \phi}{\partial y} = \frac{d\phi}{du} \cdot \frac{du}{\partial y} = \phi'(u)\, x\right)$

$$\frac{\partial \phi}{\partial y} M + \phi \cdot M_y = \frac{\partial \phi}{\partial x} N + \phi \cdot N_x$$

$$\phi'(u) x M + \phi M_y = \phi'(u)\, y N + \phi N_x$$

$$[M_y - N_x]\phi(u) = [yN - Mx]\,\phi'(u)$$

$$\frac{M_y - N_x}{yN - xM} = \frac{\phi'(u)}{\phi(u)} = f(u) = \text{fu}^{n} \text{ of } u \text{ only} \mid \text{Constant}$$

$\Rightarrow$ On Integration $\phi(u) = e^{\int f(u)\,du}$

Conclusions →

(a) If $\phi(x,y) = \phi(u)$, $u = x \cdot y$ is an I.F

$\Rightarrow \dfrac{M_y - N_x}{yN - xM} = \text{fu}^{n} \text{ of } u \mid \text{Constant}$

(b) If $\dfrac{M_y - N_x}{yN - xM}$ is not a fu^{n} of u or Constant $\Rightarrow \nexists$ any I.F which is fu^{n} of u only.

(c) If $\dfrac{M_y - N_x}{yN - xM} = f(u)$, fu^{n} of u only, then $\exists$ an I.F given by →

↓

(which is $e^{\int f(u)\,du}$ fu^{n} of u only)

[Harris Tenenbaum & Harry Pollard]

e.g → $\underset{M}{(y^3+xy^2+y)}dx + \underset{N}{(x^3+x^2y+x)}dy = 0$

$$\frac{My-Nx}{yN-xM} = \frac{(3y^2+2xy+1)-(3x^2+2xy+1)}{(yx^3+x^2y^2+xy)-(xy^3+x^2y^2+xy)}$$

$$= \frac{3(y^2-x^2)}{yx(x^2-y^2)} = \frac{-3}{yx} = \frac{-3}{u} = f^n \text{ of } u \text{ only}$$

$$I.F \rightarrow e^{-3\int \frac{1}{u}du}$$

$$= \frac{1}{u^3}$$

$$= \frac{1}{(xy)^3}$$

Validate → let $\frac{My-Nx}{yN-Mx}$ is f^n of u only, $u = x \cdot y$

Then $e^{\int f(u)du}$ is the I.F if

$$\left(\frac{\partial}{\partial y}(\phi) = \frac{d\phi}{du} \cdot \frac{du}{dx} = \phi'(u)\,y\right)$$

$$\frac{\partial}{\partial y}\left[\underbrace{e^{\int f(u)du}}_{\phi(u)} M\right] = \frac{\partial}{\partial x}\left[N e^{\int f(u)du}\right]$$

$$\text{if } \left[My\, e^{\int f(u)du} + M\frac{\partial}{\partial y}(\phi(u))\right] = \left[Nx\, e^{\int f(u)du} + N\frac{\partial}{\partial x}(\phi(u))\right]$$

$$\left[My\, e^{\int f(u)du} + M\,\phi'(u)\,x\right] = \left[Nx\, e^{\int f(u)du} + N\,\phi'(u)\,y\right]$$

$$My\, e^{\int f(u)du} + M e^{\int f(u)du} \cdot f(u)\,x = Nx\, e^{\int f(u)du} + N e^{\int f(u)du} f(u)\,y$$

$$My + M f(u)x = Nx + N f(u)y \Rightarrow \frac{My-Nx}{yN-Mx} = f(u) = f^n$$

$$\Rightarrow My - Nx = (xM + Ny) f(u)$$

which is true

(4) When $\phi(x,y) = \phi(u)$, $u = \frac{x}{y}$

$$\frac{\partial}{\partial y}[M\cdot\phi(u)] = \frac{\partial}{\partial x}[N\cdot\phi(u)]$$

$$M_y\phi + M\frac{\partial\phi}{\partial y} = N_x\cdot\phi + N\cdot\frac{\partial\phi}{\partial x}$$

$$M_y\phi + M\phi'(u)\left(\frac{-x}{y^2}\right) = N_x\cdot\phi + N\phi'(u)\frac{1}{y}$$

$$(M_y - N_x)\phi(u) = \left[\frac{N}{y} + \frac{x}{y^2}M\right]\phi'(u)$$

$$\frac{y^2(My - Nx)}{yN + xM} = \frac{\phi'(u)}{\phi(u)} = f(u)$$

On Integration →

$$\phi(u) = e^{\int f(u)\,du}$$

Conclusions →

(a) If $\phi(x,y) = \phi(u)$, $u = \frac{x}{y}$ is an I.F

$\Rightarrow \frac{y^2(My - Nx)}{yN + xM}$ is a fu^n of u only / Constant

(b) If $\frac{y^2(My - Nx)}{yN + xM}$ is not a fu^n of u only /

Constant $\Rightarrow \nexists$ any I.F which is fu^n of u only.

(c) If $\frac{y^2(My - Nx)}{yN + xM}$ is a fu^n of u only

$\Rightarrow \exists$ an I.F which is fu^n of u only

given by $e^{\int f(u)du}$

Here $\phi(u) = \phi(x/y) = \phi\left(\frac{1}{y/x}\right)$ = fu^n of y/x

(5) When $\phi(x,y) = \phi(u)$, $u = \frac{y}{x}$

$$\frac{\partial}{\partial y}\left[M \cdot \phi(u)\right] = \frac{\partial}{\partial x}\left[N \cdot \phi(u)\right]$$

$\left(\frac{\partial \phi}{\partial y} = \frac{d\phi}{du} \cdot \frac{du}{dy} = \phi'(u) \cdot \frac{1}{x}\right)$

$$My\,\phi + M\frac{\partial \phi}{\partial y} = Nx\,\phi + N\frac{\partial \phi}{\partial x}$$

$$My\,\phi(u) + \frac{M\,\phi'(u)}{x} = Nx\,\phi(u) + N\,\phi'(u)\left(\frac{-y}{x^2}\right)$$

$$(My - Nx)\,\phi(u) = \left(\frac{M}{x} - \frac{Ny}{x^2}\right)\phi'(u)$$

$$\frac{x^2(My - Nx)}{-(xM + Ny)} = \frac{\phi'(u)}{\phi(u)}$$

$$\frac{x^2(My - Nx)}{(yN + xM)} = -\frac{\phi'(u)}{\phi(u)} = f(u) \quad \text{| Constant}$$

On Integration →

$$\phi(u) = e^{-\int f(u)\,du}$$

Conclusions →

(1) If $\phi(x,y) = \phi(u)$, $u = \frac{y}{x}$ is an I.F ⇒

$\frac{(My - Nx)x^2}{yN + xM}$ is a fu^n of u only | Constant

(2) If $\frac{x^2(My - Nx)}{yN + xM}$ is not a fu^n of u only | Constant

⇒ $\nexists$ an I.F which is fu^n of u only

(3) If $\frac{x^2(My - Nx)}{yN + xM}$ is a fu^n of u only

⇒ $\exists$ an I.F which is fu^n of u only

& given by $e^{-\int f(u)\,du}$

Que. → $x\,dy - y\,dx = 0$

Which of the following is Not an I.F

a) $1/x^2$
b) $1/y^2$
c) $\frac{1}{xy}$

} These all Integrating factors.

d) None of these

(As $\frac{x}{y} = c$ is i.e $f(x,y) = c$ (i.e $f(x/y)$) the sol^n of diff. eq^n.
∴ ϕ G(f) are the Integrating factors

i) When $\phi = 1/x^2$

$G(u) = \sin^{-1} u$

∴ Integrating factor →

$\frac{1}{x^2} \sin^{-1}\left(\frac{y}{x}\right)$

ii) when $\phi = 1/y^2$

$G(u) = \log(\tan^{-1} u)$

∴ Integrating factor →

$\frac{1}{y^2} \log\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$

iii) $\frac{1}{xy}$ is the I.F)

Or

a) $\frac{1}{x^2} \sin^{-1}\left(\frac{y}{x}\right)$

b) $\frac{1}{y^2} \log\left(\tan^{-1}\frac{x}{y}\right)$

c) $\frac{1}{xy}$

d) None of these.

1) $\frac{Hy - Nx}{N} = f(x) \Rightarrow$ ∃ au I.F $e^{\int f(x)\,dx}$

2) $\frac{Hy - Nx}{H} = \pm\phi(y) \Rightarrow$ ∃ au I.F $e^{-\int \phi(y)\,dy}$

3) $\frac{Hy - Nx}{yN - xH} = \phi(u)$, $u = x \cdot y \Rightarrow$ ∃ au I.F $e^{\int \phi(u)\,du}$.

4) $\frac{y^2[Hy - Nx]}{yN + xH} = \phi(u)$, $u = \frac{x}{y} \Rightarrow$ ∃ I.F $\rightarrow e^{\int \phi(u)\,du}$

5) $\frac{x^2[Hy - Nx]}{yN + xH} = \pm\phi(u)$, $u = \frac{y}{x} \Rightarrow$ ∃ I.F $\rightarrow e^{-\int \phi(u)\,du}$.

Que. → $-x\,dy + y\,dx + e^{y/x}\,dx = 0$

a) ∄ an I.F which is fu^n of x only

b) ∄ any I.F which is fu^n of y only

c) ∄ any I.F which is fu^n of u, $u = x \cdot y$ only

d) ∄ any I.F which is fu^n of u only, $u = \frac{y}{x}$

Sol^n →

$$M = y + e^{y/x}$$

$$N = -x$$

$$M_y = 1 \qquad N_x = -1$$

∴ Non-Exact

$$M_y - N_x = 2$$

1) $\frac{M_y - N_x}{N} = \frac{2}{-x} = f(x)$ ⇒ $e^{-\int \frac{2}{x}\,dx}$ is an I.F

⇒ $\frac{1}{x^2}$ → I.F

a) is True

2) $\frac{M_y - N_x}{M} = \frac{2}{y + e^{y/x}} = f(x, y)$, Not fu^n of y.

⇒ ∄ any I.F which is fu^n of y-only

3) $\frac{M_y - N_x}{yN - xM} = \frac{2}{y(-x) - x(y + e^{y/x})} = \frac{2}{-2xy - xe^{y/x}}$

Not a fu^n of $u = x \cdot y$

⇒ ∄ any I.F which is fu^n of u-only

d) $\frac{M_y - N_x}{yN + xM} = \frac{2}{x e^{y/x}}$

$x^2\left(\frac{M_y - N_x}{yN+xM}\right) = \frac{2x}{e^{y/x}}$ = Not a fu^n of $\frac{y}{x}$

$\Rightarrow \nexists$ any I.F which is fu^n of u only, $u = \frac{y}{x}$

$\therefore$ a), b), c), d) → Are all Correct.

Que $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$

Then

a) $\frac{1}{y^3}\left(x^3e^y + x + \frac{x}{y}\right)$ is an I.F

b) $\frac{\alpha}{y^3}$ is an I.F $\forall\ \alpha \in \mathbb{R}$

c) Sol^n of diff eq^n $x^3e^y + x + \frac{x}{y} = c$

d) $y(x) \equiv 0$ is not a sol^n of Reducable Exact eq^n.

Sol^n →

$M = 3x^2y^3e^y + y^3 + y^2$, $N = x^3y^3e^y - xy$

$M_y = 9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y$

$N_x = 3x^2y^3e^y - y$

$M_y - N_x = 9x^2y^2e^y + 3y^2 + 3y$

$\frac{M_y - N_x}{M} = \frac{3(3x^2y^2e^y + y^2 + y)}{y(3y^2x^2e^y + y^2 + y)} = \frac{3}{y}$ = fu^n of y only

$\Rightarrow \exists$ an I.F which is fu^n of y-only

$I.F = e^{-\int \frac{3}{y}\,dy}$

$= e^{-3\log y} = e^{\log y^{-3}} = \frac{1}{y^3}$

$\Rightarrow \alpha \times I.F$ is also an I.F $\forall \alpha \in \mathbb{R}$

Multiply with I.F $\longrightarrow$ $(3x^2e^y + 1 + \frac{1}{y})\,dx + (x^3e^y - \frac{x}{y^2})\,dy = 0$ — (*)

Exact diff eq^n.

$M' = 3x^2e^y + 1 + \frac{1}{y}$

$N' = x^3e^y - \frac{x}{y^2}$

← Use when Polynomials are there

$\int M'\,dx + \int N'\,dy = c$

— — — N' free from x

$\int (3x^2e^y + 1 + \frac{1}{y})\,dx + 0 = c$

$x^3e^y + x + \frac{x}{y} = c$

$\Rightarrow f(x,y) = x^3e^y + x + \frac{x}{y} = c$ — (*)

We know → ϕ G(f) is an I.F Here, Take G(u) = u

$\frac{1}{y^3} \times f$

$\frac{1}{y^3}\left(x^3e^y + x + \frac{x}{y}\right)$ is also an I.F.

& $y(x) \equiv 0$ is not the Sol^n of Reducable exact diff. eq^n.

(∵ y occurs in denominaters in (*))

Que

I $(xe^{xy} + 2y)dy + ye^{xy}dx = 0$

II $(e^{2y}+1)\cos x\,dx + 2e^{2y}\sin x\,dy = 0$

1. Chk the exactness of the diff eq^n.
2. Find Sol^n of diff eq^n.
3. If Diff eq^n is non-exact find →

   I.F which fu^n of →

   x-only / x/y only if possible

   y-only / y/x only if possible.

   x.y only /

Sol^n →

I $M = ye^{xy}$ $\quad N = xe^{xy} + 2y$

$M_y = e^{xy} + xy\,e^{xy}$

$N_x = e^{xy} + xy\,e^{xy} + 0 = e^{xy} + xye^{xy}$

$M_y = N_x$

⇒ Given diff eq^n is exact diff eq^n.

Sol^n → $\int M dx + \int N dy = c$, N is free from x

$\int ye^{xy}dx + \int 2y\,dy = c$

$\frac{y\,e^{xy}}{y} + y^2 = c \Rightarrow e^{xy} + y^2 = c$ is the Sol^n.

$M = (e^{2y}+1)\cos x$

$N = 2e^{2y}\sin x$

$M_y = N_x$.

$(2e^{2y})\cos x = 2e^{2y}\cos x$

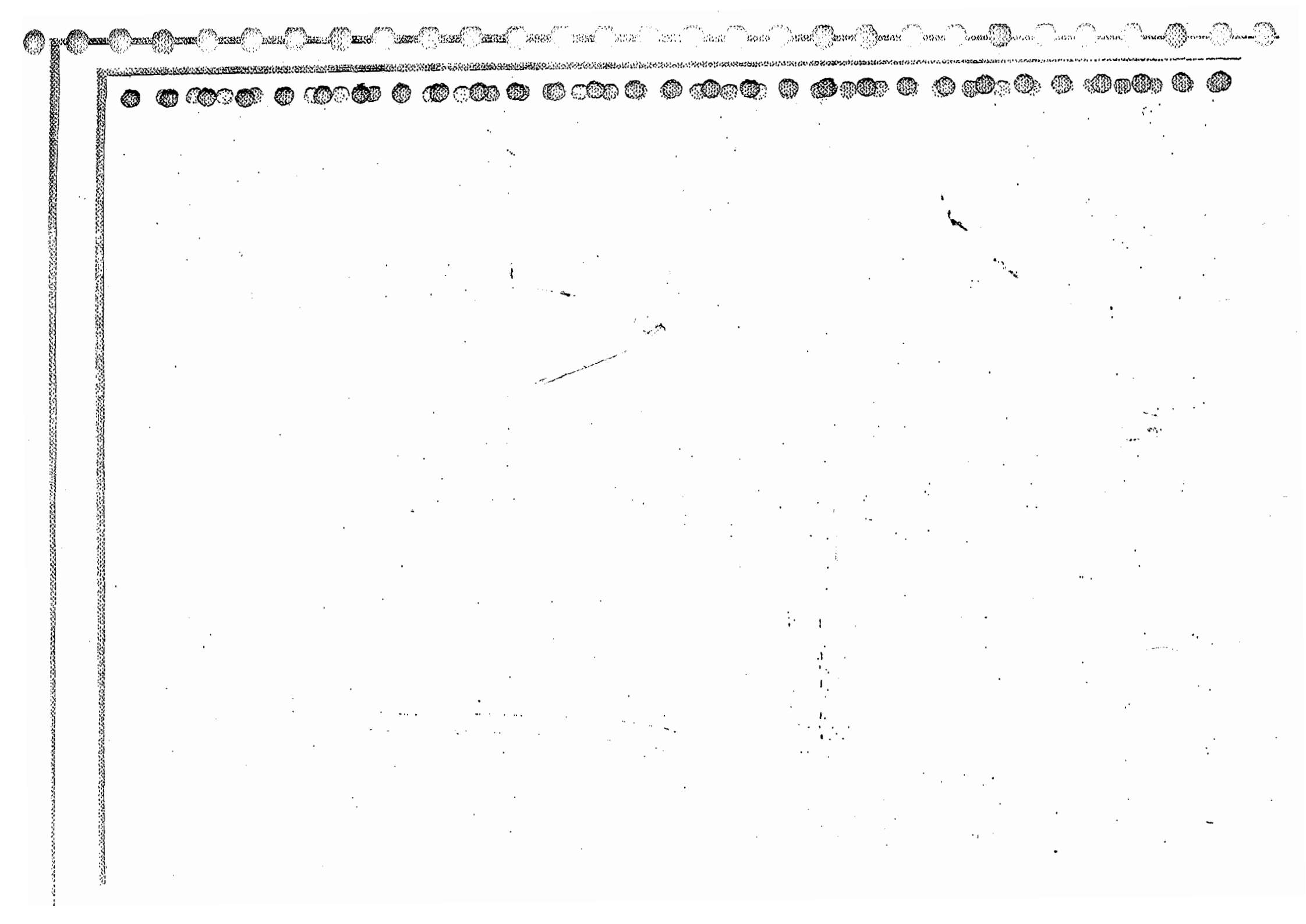
$\Rightarrow$ Given diff eq^n is exact diff. eq^n.

Sol^n is given by $\rightarrow$

$\int M\,dx + \int N\,dy = c$, N is free from x.

$(e^{2y}+1)\int \cos x\,dx + \int 0\,dy = c$

$(e^{2y}+1)\sin x = c$ is the sol^n of diff eq^n.



Homogeneous fun^n →

A fun^n $f(x,y)$ is called Homogeneous if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y), \quad \lambda > 0, \quad n \in \mathbb{R},$$

of degree n.

eg → (1) $f(x,y) = \dfrac{\sqrt{x^2+y^2}}{(x+y)}$

$$f(\lambda x, \lambda y) = \frac{\sqrt{\lambda^2 x^2 + \lambda^2 y^2}}{(\lambda x + \lambda y)}$$

$$= \frac{\lambda^0 \sqrt{x^2+y^2}}{(x+y)} = \lambda^0 f(x,y)$$

$f(x,y)$ is Homogeneous fun^n of degree 0.

(2) $f(x,y) = \dfrac{\sqrt{x^2+y^2}}{(x^2+y^2)}$

$$f(\lambda x, \lambda y) = \lambda^{-1} f(x,y)$$

$f(x,y)$ is Homog. fun^n of degree -1

(3) $f(x,y) = (x^2+y^2)^{3/2}$

$$f(\lambda x, \lambda y) = \lambda^3 f(x,y)$$

$f(x,y)$ is Homog. fun^n of degree 3

(4) $f(x,y) = (x+y)^{3/2}$

$$f(\lambda x, \lambda y) = \lambda^{3/2} f(x,y)$$

$f(x,y)$ is Homog. fun^n of degree $3/2$.

then concept of degree in Non-Homog. fun^n is meaningless.

($f(x,y) = x + \sqrt{y}$ ← Non-Homog. fun^n of deg^n

$f(x,y) = xy^2 + x^2y$.

↑

Homog. fun^n of degree 3)

$f(x,y) = (x^2+y^2)^{3/2}$ homog fun^n of deg 3.

This can be written as →

$$f(x,y) = y^3 \left(\left(\frac{x}{y}\right)^2+1\right)^{3/2} = y^3 \phi\left(\frac{x}{y}\right)$$

$$f(x,y) = x^3 \left(1+\left(\frac{y}{x}\right)^2\right)^{3/2} = x^3 \phi\left(\frac{y}{x}\right)$$] — (1)

[$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ ← Homog. fun^n of deg. 0.

$f(x,y) = e^{x+y}$

↗ Non-Homog. fun^n]

If $f(x,y) = y^n \phi\left(\frac{x}{y}\right)$

$f(x,y) = x^n \phi\left(\frac{y}{x}\right)$ then $f(x,y)$ is also the Homog. fun^n of degree 'n'. (Shown by (1))

FOFD Homog. Diff eq^n →

FOFD ← First Order First Degree

A diff. eq^n

$\frac{dy}{dx} = f(x,y)$ is Homog. diff. eq^n

if $f(x,y)$ is Hom. fu^n of deg. zero. — (*)

⇒ A diff eq^n

$\frac{dy}{dx} = f(x,y)$ is Homog. diff eq^n if

either $f(x,y) = \phi\left(\frac{y}{x}\right)$ or $f(x,y) = \phi\left(\frac{x}{y}\right)$

(i.e. of degree 0)

Note →

$M(x,y)\,dx + N(x,y)\,dy = 0$ is Homog. diff eq^n if M & N are Homog fu^n of same degree

$$\Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x,y)$$

$f(x,y)$ must be Homog. fu^n of deg. zero, so that this diff. eq^n is homogeneous. (Using (*))

$$f(dx,dy) = -\frac{M(dx,dy)}{N(dx,dy)}$$

$$= -\frac{d^n M(x,y)}{d^n N(x,y)} = -\frac{M(x,y)}{N(x,y)} = f(x,y)$$

(If M & N are homog. fu^n, not having same degree say n & n+1

$$f(dx,dy) = -\frac{M(dx,dy)}{N(dx,dy)} = -\frac{d^{n+1} M(x,y)}{d^n N(x,y)} = -\frac{d\,M(x,y)}{N(x,y)} = d\,f(x,y)$$

i.e. $f(x,y)$ is of deg 1. ∴ Diff eq^n is not homog diff eq^n)

∴ M & N are Homog. fu^n of same degree

Eg. → $\underbrace{(xy^2+x^2y)}_{M}\,dx + \underbrace{(x^3+yx^2)}_{N}\,dy = 0$

$M(x,y) = xy^2+x^2y$

$N(x,y) = x^3+yx^2$

] These are Homog. fun of degree 3. ∴ Homog. diff eqn.

$M_y = 2xy+x^2$

$N_x = 3x^2+2xy$.

$M_y \neq N_x$ ∴ Given diff. eqn is not exact

$$\frac{1}{xM+yN} = \frac{1}{x(xy^2+x^2y)+y(x^3+yx^2)}$$

$$= \frac{1}{x^2y^2+x^3y+yx^3+x^2y^2}$$

$$= \frac{1}{2x^2y^2+2x^3y} = \frac{1}{2(x^2y^2+x^3y)}$$

Multiplying this with (1) →

$$\frac{(xy^2+x^2y)}{2(x^2y^2+x^3y)}\,dx + \frac{(x^3+yx^2)}{2(x^2y^2+x^3y)}\,dx = 0$$

$M' \rightarrow$ $\qquad$ $\rightarrow N'$

$M_y' =$

$N_x' =$

Note →

Let $Mdx + Ndy = 0$ be a Homog. Non-Exact diff eq^n s.t

$$xM + yN \neq 0$$

Then $\boxed{\frac{1}{yN+xM}}$ is an Integrating factor of Diff eq^n.

Chk →

$$\frac{\partial}{\partial y}\left[\frac{M}{xM+yN}\right] = \frac{\partial}{\partial x}\left[\frac{N}{xM+yN}\right]$$

$$\frac{(xM+yN)(M_y) - M(xM_y + N + yN_y)}{(xM+yN)^2}$$

$$= \frac{(xM+yN)N_x - N(xM_x + M + yN_x)}{(xM+yN)^2}$$

As $xM + yN \neq 0$

$$M_y(xM + yN - xM) + N_y(-yM) - MN$$

$$= N_x(xM + yN - xN) - MN - xNM_x$$

$$yNM_y - yMN_y = xMN_x - xNM_x$$

$$N(yM_y + xM_x) = M(xN_x + yN_y)$$

$$N(yM_y + xM_x) - M(yN_y + xN_x) = 0$$

(Use Euler's Th^m) (Use Euler's Th^m)

Euler's Thm if $f(x,y)$ is homog. fn of deg n then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n \cdot f$$ " Imp.

Say deg $M = K$ (as M & N are Homog. fn of same degree)

deg $N = K$

$N \cdot K \cdot H - N \cdot K \cdot M = 0$. (Using Euler's Thm)

$0 = 0$

Satisfied

$\Rightarrow \frac{1}{yN+xH}$ is the I.F of Homog. Diff eqn.

Note → If $Mdx + Ndy = 0$ ← (May be Homog / Non-Homog) FOFD.

s.t $xH + yN = 0$

Then General Sol^n of $Mdx+Ndy=0$ is →

a) family of circles.

✓ b) family of straight lines.

c) family of parabola's

d) None of these.

$$\frac{dy}{dx} = -\frac{M}{N}$$

&

$$xM + yN = 0$$

$$\Rightarrow xM + yN = 0 \quad \Rightarrow \quad xM = -yN$$

$$\frac{M}{N} = -\frac{y}{x}$$

Equating two values of $\frac{M}{N}$ →

$$\frac{dy}{dx} = \frac{y}{x}$$, ($y(x) \equiv 0$ is the Sol^n of Diff eq^n)

$$\frac{dy}{y} = \frac{dx}{x}$$, $\boxed{y(x) > 0}$.

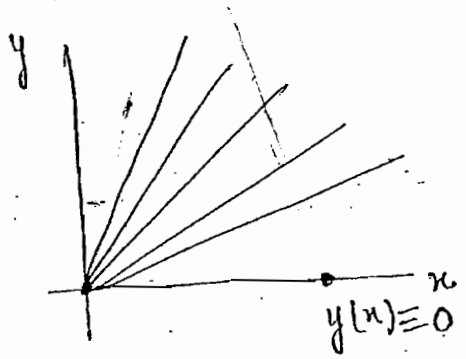
On Integration →

$$\log y = \log x + \log c$$

$$\boxed{y = cx}$$

($y(x) \equiv 0$ is not the General/Particular Sol^n as It can't be obtained from $y = cx$ for any c, $y(x) > 0$)

Represents family of straight lines.



$y(x) \equiv 0$ is not the envelope to the family of straight lines.

∴ $y(x) \equiv 0$ is not the singular Sol^n

Que. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

Find I.F of diff eq^n.

Sol^n → This is Homog. Diff eq^n.

$$xM + yN = x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$
$$= x^2y^2 \neq 0.$$

$$\frac{1}{xM+yN} = \frac{1}{(xy)^2} \text{ is an I.F}$$
$$= f(u), \quad u = x\cdot y$$

If I.F is fu^n of u, $u = x\cdot y$ then

$\frac{My - Nx}{yN - xM}$ is fu^n of u, $u = x\cdot y$ (Obtained earlier)

Chk → $$\frac{My - Nx}{yN - xM} = \frac{(x^2 - 4xy) - (-3x^2 + 6xy)}{-x^3y + 3x^2y^2 - x^3y + 2x^2y^2}$$

$$= \frac{4x^2 - 10xy}{-2x^3y + 5x^2y^2} = \frac{2x(2x - 5y)}{-xy(2x^2 - 5xy)}$$

$$= \frac{-2}{xy} = \frac{-2}{u}$$

∴ Integrating factor is $e^{\int f(u)\,du}$

$$I.F = e^{\int f(u)\,du} = e^{-\int \frac{2}{u}\,du} = e^{-2\log u} = e^{\log u^{-2}} = \frac{1}{u^2}$$
$$= \frac{1}{x^2y^2}$$

(Same integrating factor)

Multiply this with given diff eq^n →

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

$M = \frac{y}{y} - \frac{2}{x} \Rightarrow My = \frac{-1}{y^2}$

$N' = \frac{-x}{y^2} + \frac{3}{y} \Rightarrow Nx' = \frac{-1}{y^2}$

$My' = Nx' \Rightarrow$ Now the diff eq^n becomes exact.

Que → Note →

The I.f of

$$y\, f_1(xy)\, dx + x\, f_2(xy)\, dy = 0$$

is $\boxed{\frac{1}{xM - yN}}$, provided $xM - yN \neq 0$

e.g → $(xy^2 \sin xy + y\cos xy)\, dx + (x^2 y \sin xy - x\cos xy)\, dy = 0$

$y\underbrace{(xy\sin xy + \cos xy)}_{f_1(xy)}\, dx + x\underbrace{(xy\sin xy - \cos xy)}_{f_2(xy)} dy = 0$

$xM - yN = x^2y^2 \sin xy + xy\cos xy - x^2y^2 \sin xy + xy\cos xy$

$= 2xy\cos xy.$

$\therefore \frac{1}{2xy\cos xy}$ is an I.f.

Note When $xM - yN = 0$

$$y f_1(xy)\,dx + x f_2(xy)\,dy = 0 \quad \text{—(1)}$$

If $xM - yN = 0$

then General sol^n of the diff. eq^n $\rightarrow$

a) family of Circle's

b) ——— Straight line's

✓ c) ——— Rectangular Hyperbolas

d) None of these

Sol^n $\rightarrow$ $M = y f_1(xy)$, $N = x f_2(xy)$, Calculate $xM - yN$ and Put $= 0$

$$x y f_1(xy) - xy f_2(xy) = 0$$

$$xy\left[f_1(xy) - f_2(xy)\right] = 0$$

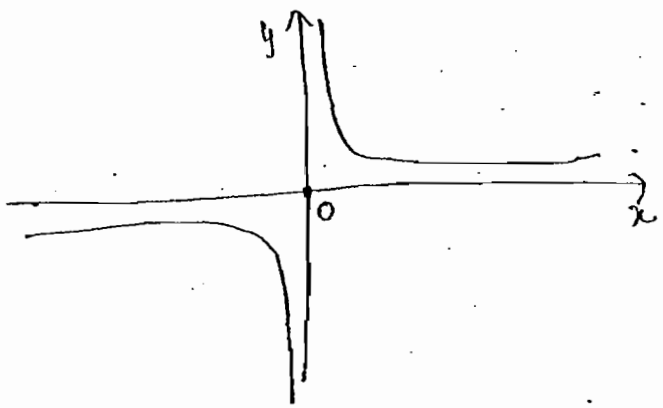
$$f_1(xy) = f_2(xy)$$

Using in (1) $\rightarrow$

$$f_1(xy)\left[y\,dx + x\,dy\right] = 0$$

$$y\,dx + x\,dy = 0 \quad , \quad \frac{dy}{y} = -\frac{dx}{x}$$

$$\Rightarrow xy = c \quad \Rightarrow y(x) = \frac{c}{x}$$



This represents family of Rectangular Hyperbolas

Example → $x^{\alpha} y^{\beta} (my\,dx + nx\,dy) = 0$ —(1)

Then an I.F of the diff. eq^{n} is →

$$\boxed{x^{Km-1-\alpha}\ y^{Kn-1-\beta}}\ , K \in \mathbb{R}.$$

Multiply with (1) →

$$\underbrace{-m\, x^{Km-1}\, y^{Kn}}_{M'}\, dx + \underbrace{n\, y^{Kn-1}\, x^{Km}}_{N'}\, dy = 0$$

$$M'_y = m\, x^{Km-1}\, Kn\, y^{(Kn-1)}$$

$$N'_x = n\, y^{(Kn-1)}\, K{\cdot}m\, x^{(Km-1)}$$

$$M_y' = N_x' \quad \Rightarrow \text{Exact}$$

Example →

$$x^{\alpha} y^{\beta} (my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'} (m'y\,dx + n'x\,dy) = 0 \quad \text{—(*)}$$

I.F will be either

$$\boxed{x^{Km-1-\alpha}\ y^{Kn-1-\beta}} \text{ or } \boxed{x^{K'm'-1-\alpha'}\ y^{K'n'-1-\beta'}}$$

such that

$$\left.\begin{aligned} Km-1-\alpha &= K'm'-1-\alpha' \Rightarrow Km-\alpha = K'm'-\alpha' \\ Kn-1-\beta &= K'n'-1-\beta' \Rightarrow Kn-\beta = K'n'-\beta' \end{aligned}\right\} \text{—(1)}$$

m, n, α, β and m', n', α', β' will be given to us with the help of two eq^{n}s given by ① and putting the values of m, n, α, β & m', n', α', β' we will find K and K'.

Example → $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

Reducing this to term ⓚ →

$$xy(y\,dx + x\,dy) + x^2y^2(2y\,dx - x\,dy) = 0$$

On comparing we have →

$\alpha = 1 \quad \beta = 1$ $\qquad$ $\alpha' = 2 \quad m' = 2$

$m = 1 \quad n = 1$ $\qquad$ $\beta' = 2 \quad n' = -1$

I.F. → $x^{Km-1-\alpha}\, y^{Kn-1-\beta}$ → (A) Or $x^{K'm'-1-\alpha'}\, y^{K'n'-1-\beta'}$ → (B)

s.t

$$Km - 1 - \alpha = K'm' - 1 - \alpha'$$

$$K - 2 = 2K' - 3 \Rightarrow K = 2K' - 1 \quad \text{—— (1)}$$

&

$$Kn - 1 - \beta = K'n' - 1 - \beta'$$

$$Kn - \beta = K'n' - \beta'$$

$$K - 1 = K'(-1) - 2 \Rightarrow K + K' = -1 \quad \text{— (2)}$$

Solving ① & ② →

$$(2K' - 1) + K' = -1$$

$$\Rightarrow K' = 0$$

$$\Rightarrow K = -1$$

$\therefore$ I.F → $x^{-1-1-1}\, y^{-1-1-1} \Rightarrow \frac{1}{x^3y^3}$ (From Ⓐ)

Or $x^{0-1-2}\, y^{0-1-2} = \frac{1}{x^3y^3}$ (From Ⓑ)

Same Integrating Factor

Imp. → Let $Mdx + Ndy$ be any FOFD Non-exact diff. eq^n Then $\phi(x,y)$ is an I.F

$$\Leftrightarrow \left(M\frac{\partial\phi}{\partial y} - N\frac{\partial\phi}{\partial x}\right) + \phi\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0.$$

$$\left(\underbrace{\phi M}_{M'} dx + \underbrace{\phi N}_{N'} dy = 0 \right.$$

$$My' = \frac{\partial\phi}{\partial y} M + My\,\phi$$

$$Nx' = \frac{\partial\phi}{\partial x} N + Nx\,\phi$$

$$My' = Nx'$$

$$\left. \left(M\frac{\partial\phi}{\partial y} - N\frac{\partial\phi}{\partial x}\right) + (My - Nx)\,\phi = 0 \right)$$

CONCLUSIONS →

① let $Mdx+Ndy=0$ be a Homog Non.-exact Diff eq^n s.t $xM+yN \neq 0$

Then $\boxed{\frac{1}{xM+yN}}$ is the I.F

② The I.F of $y\,f_1(xy)\,dx + x\,f_2(xy)\,dy = 0$

Then $\boxed{\frac{1}{xM-yN}}$ is the I.F

③ If $x^\alpha y^\beta (my\,dx + nx\,dy) = 0$

I.F of diff eq^n is → $\boxed{x^{Km-1-\alpha}\, y^{Kn-1-\beta}}$

④ If $x^\alpha y^\beta (my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'} (m'y\,dx + n'x\,dy) = 0$

I.F of diff eq^n is $x^{Km-1-\alpha}\, y^{Kn-1-\beta}$ or $x^{K'm'-1-\alpha'}\, y^{K'n'-1-\beta'}$

s.t $\boxed{\begin{aligned} Km-1-\alpha &= K'm'-1-\alpha' \\ K'n'-1-\beta' &= Kn-1-\beta \end{aligned}}$

Separable Form →

The first order first degree diff-eq^n

$\frac{dy}{dx} = f(x,y)$ is called Separable form

if $\boxed{f(x,y) = f_1(x)\, f_2(y)}$

then $\frac{dy}{dx} = f_1(x) \cdot f_2(y)$

$$\frac{dy}{f_2(y)} = f_1(x)\,dx$$

$$\Rightarrow \int \frac{dy}{f_2(y)} = \int f_1(x)\,dx + C$$ is the sol^n of diff-eq^n.

Eg → $x \log x \frac{dy}{dx} = y$, $x > 0$

$$\frac{dy}{y} = \frac{dx}{x \log x}$$, $y(x) \neq 0$, $\log x > 0 \Rightarrow x > e^0 = 1$

$$\log y = \log(\log x) + \log C$$, $y > 0$, $x > 1$

$y(x) = C \log x$ is the sol^n of diff eq^n , $x > 1$, $y > 0$

Eg → $x \log x \frac{dy}{dx} = y$, $x > 0$ then

✓ a) domain of General sol^n is $x > 1$

✓ b) domain of General sol is Openset

c) domain of General sol^n is $x > 0$

d) None of these

Note → how to find Domain:

1) domain of $\log(\underbrace{\log(\log(x))}_{y})$ is

$\log y$

$\Rightarrow y > 0$

Put the value of y →

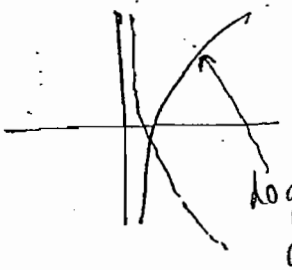
$\log(\log(x)) > 0$

Sign remains same as $\log x$ is increasing fu^n

$\log x > e^0 = 1$

$\log x > 1$

$x > e^1 = e$



$\log_a x$ → increasing fu^n

$\log_a x$ → decreasing fu^n

$0 < a < 1$

2) $\log_{1/2}(\underbrace{\log_{1/2}(\log_{1/2}(x))}_{\log_{1/2} y})$

$\Rightarrow y > 0$

Put the value of y →

$\log_{1/2}(\log_{1/2}(x)) > 0$

$\log_{1/2}(x) < (\frac{1}{2})^0 = 1$

$\log_{1/2}(x) < 1$

$x > (\frac{1}{2})^1$

$\Rightarrow x > \frac{1}{2}$

$\log_{1/2}(x)$ → decreasing fu^n

∴ Sign changes

Again sign changes.

∴ $\log_{1/2}(x)$ → decreasing fu^n

Note → Cntd →

In diff eq^n we will use open set on Real line as Domain of $y(x)$ in Diff eq^n as subset of $\mathbb{R}$.

($\log_e x = a$
$x = e^a$)

Note→

Defⁿ→ $S \subseteq \mathbb{R}$ is called **open set** if it is nbd of each of its points.

- $S \subseteq \mathbb{R}$ is called nbd of $a \in S$ if $\exists\ \delta > 0$ s.t $(a-\delta, a+\delta) \subseteq S$
- Any set becomes nbd of each of its point if it is infinite. ($(a-\delta, a+\delta) \subseteq S$; Infinite set ↑ ; Infinite ↑)

Solⁿ→

$$x \log x \frac{dy}{dx} = y \quad , \quad x > 0$$

$$\frac{dy}{y} = \frac{dx}{x \log x} \qquad y \neq 0 \ , \ \log x > 0 \ , \ x > e^0 \ , \ x > 1$$

$$\int \frac{dy}{y} = \int \frac{dx}{x \log x}$$

$$\log y = \log(\log x) + \log c \quad , \quad y > 0$$

$y = c \log x$ is general solⁿ for $x > 1$

∴ Option a) is correct

i.e Domain of General solⁿ is $x > 1$ i.e $(1, \infty)$

Also open interval is an open set.

⇒ domain of General solⁿ is an open set.

∴ a) & b) are correct

& c) is incorrect

Note→ Here, On Real line, $(1, \infty)$ is open interval ⇒ $(1, \infty)$ is an open set.

as open sets on real line are either open intervals or Union of open intervals. [∵ Arbitrary union of open interval is an open interval]

Q.q → Consider the diff eq^n →

$$x \log x \frac{dy}{dx} = y \qquad x > 0 \Rightarrow R^+ \text{ is domain of diff eq}^n$$

then

1) Every particular sol^n is defined on R^+

2) $\exists$ a particular sol^n which is defined on R^+

✓ 3) $\exists$ a sol^n which is defined on R^+

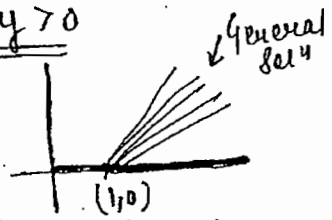
4) None of these

Sol^n. General sol^n →

$$y = c \log x \quad ; \quad x > 1 \; , \; y > 0$$

$y \equiv 0$ is Not a Particular sol^n

$y \equiv 0$ is Not a Singular sol^n



Domain of General sol^n is $(1,\infty) \subset (0,\infty) = R^+$ ($\because y(x) \equiv 0$ is Not a envelope to the family of curves)

Every particular sol^n is obtained from General sol^n. So each particular sol^n is defined on $(1,\infty)$ but not on whole R^+

$\Rightarrow$ There does not exist any particular sol^n defined on R^+

$\therefore$ a) and b) are incorrect.

($\because$ for $0 < x \leq 1$, No particular sol^n is defined)

$y(x) \equiv 0$ is also sol^n of diff eq^n.

(This is defined on whole R, but we have to take domain of diff eq^n as R^+ & $\mathbb{R}^+ \subseteq R^+$ $\therefore$ $y(x) \equiv 0$ is defined on $\mathbb{R}^+$) $\quad R^+ \subseteq R$ Bt. $R \not\subseteq R^+$

$\therefore$ c) is Correct.

$\because$ Domain of sol^n $\subseteq$ Domain of diff eq^n

(If we take → Domain as R Bt. $R \not\subseteq R^+$ $\therefore$ Take $\mathbb{R}^+$ as Domain.) $\quad R^+ \subseteq R$

Periodic function →

A fu^n f defined on $A \subseteq \mathbb{R}$ is called periodic if

$\exists\ T \in \mathbb{R}$ s.t $f(x+T) = f(x)\ \forall\ x \in A$.

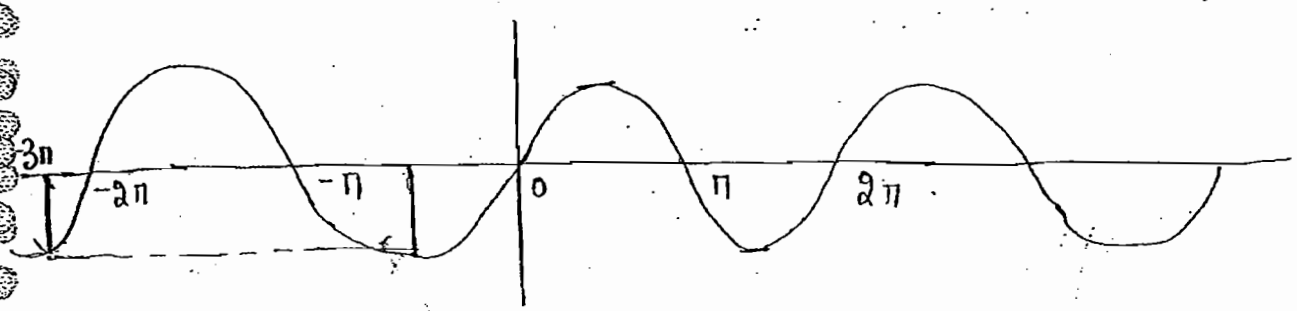
Then the smallest positive such T is called (Unique)
period of f.

e.g

$f(x) = \sin x$

$T = -2\pi$

$\sin(x-2\pi) = \sin x$ with period 2π ($\sin(x-2\pi) = -\sin(2\pi - x) = +\sin x$)



e.g Constant fu^n is periodic whose period does not exist.

Let $f(x) = c$ be a Constant fu^n

Then $\exists\ T \in \mathbb{R}$ s.t

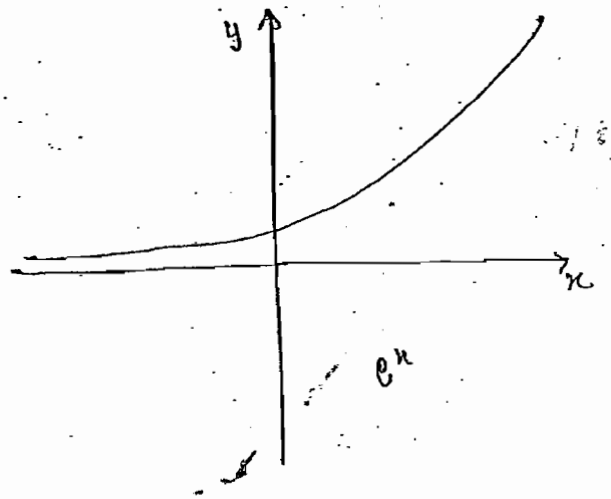
$f(x+T) = f(x) = c\ \forall\ x \in A$. (as $f(x)$ is Constant fu^n)

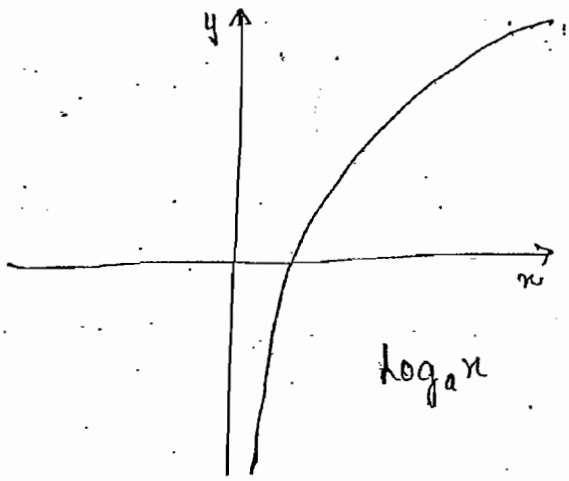
Now among these T's, the smallest such +ve T is Period

Here $f(x+\frac{1}{n}) = f(x)\ \forall\ n > 0$ (Smallest form will be of $\frac{1}{n}$)

But we can't find smallest such $\frac{1}{n} \in (0, \infty)$

Hence, Constant fu^n is Periodic, but its Period doesn't exist.





$P(x)$, $\log x$, e^x → Non-Periodic fu^n where $P(x)$ → Any Polynomial.

$\sin x$, $\cos x$, $\tan x$ → Periodic fu^n

$\sinh x$, $\cosh x$ → Non-Periodic fu^n $\left(\sinh x = \frac{e^x + e^{-x}}{2}\right)$

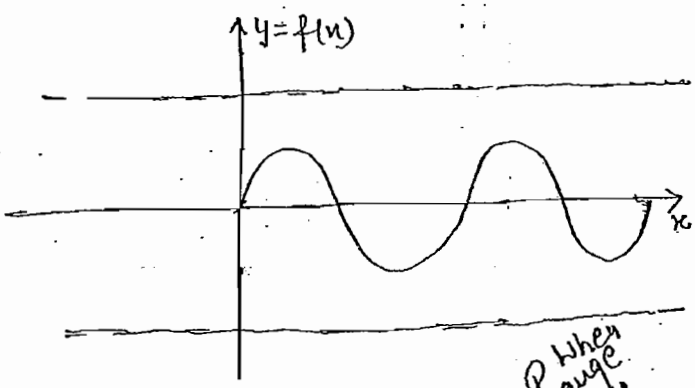
$\left(\sin x = \frac{e^{ix} - e^{-ix}}{2i}\right)$ ← By Complex we chk periodicity in another way

? **Note** → If f & g are periodic fu^n, $f+g$ → may not be periodic.

Bounded fu^n → A fu^n f defined on $A \subseteq \mathbb{R}$ is called bounded if $\exists\ M \in \mathbb{R}^+$ s.t

$$|f(x)| \leq M \quad \forall\ x \in A$$

$\left(\begin{array}{l} |f(x)| \leq M \\ -M \leq f(x) \leq M \end{array}\right)$



i.e. When Range set is bounded.

1) $\cosh x = \frac{e^x + e^{-x}}{2}$

Put $x = 0$, $\cosh x = 1$

$x \to \infty$ $e^x \to \infty$

$x \to \infty$ $\frac{e^x + e^{-x}}{2} \to \infty$

$x \to -\infty$ $e^{-x} \to \infty$

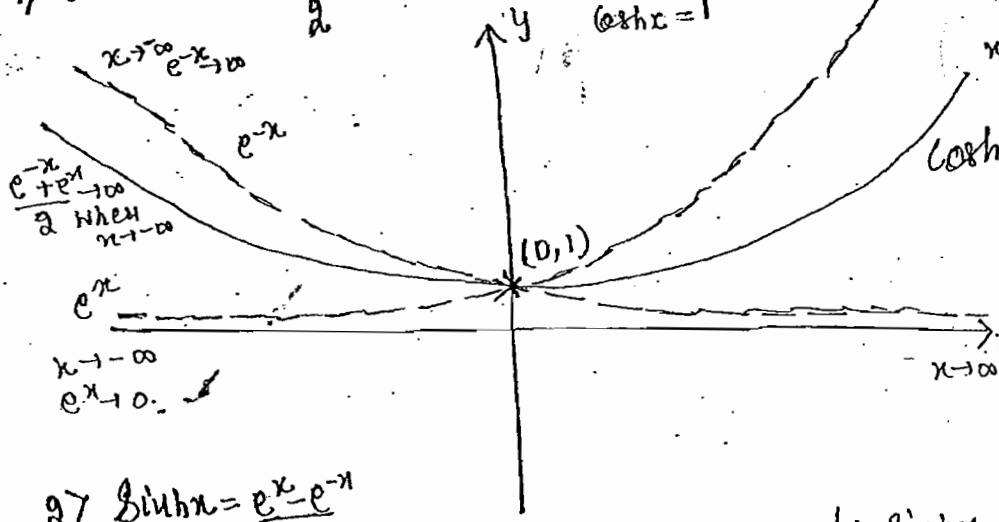
e^{-x}

$\frac{e^{-x} + e^x}{2} \to \infty$ when $x \to -\infty$

e^x

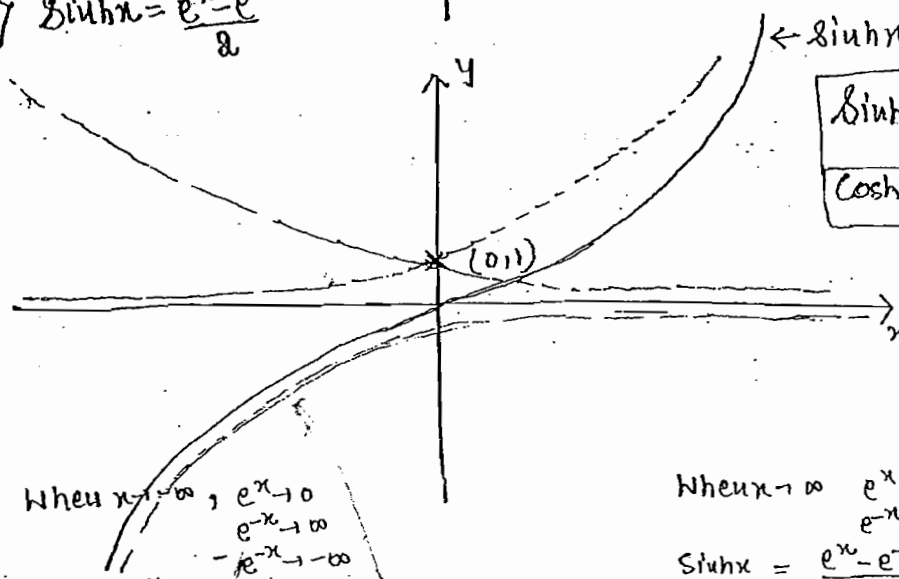
$x \to -\infty$ $e^x \to 0$

$x \to \infty$ $e^{-x} \to 0$



$\cosh x \to$ Mean of e^x and e^{-x}

2) $\sinh x = \frac{e^x - e^{-x}}{2}$



$\sinh x \to$ increasing fu^n

$\cosh x \to$ decreasing fu^n

When $x = 0$, $\sinh x = \frac{1-1}{2} = 0$

When $x \to -\infty$, $e^x \to 0$, $e^{-x} \to \infty$, $-e^{-x} \to -\infty$

$\sinh x = \frac{e^x - e^{-x}}{2} \to -\infty$ (downward)

When $x \to \infty$ $e^x \to \infty$

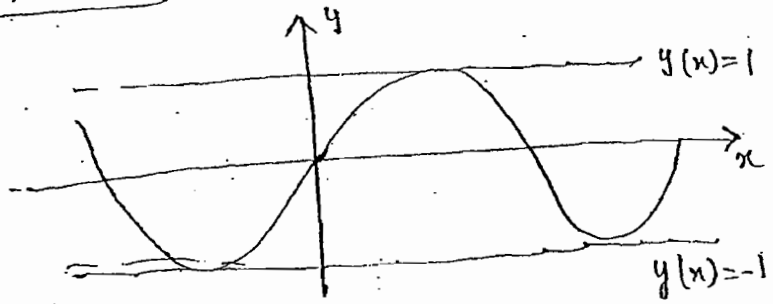
$e^{-x} \to 0 \Rightarrow -e^{-x} \to 0$

$\sinh x = \frac{e^x - e^{-x}}{2} \to \infty$ (upward)

$\sinh x$, $\cosh x$ → Both Unbounded fu^n as e^x & e^{-x} are unbounded fu^n.

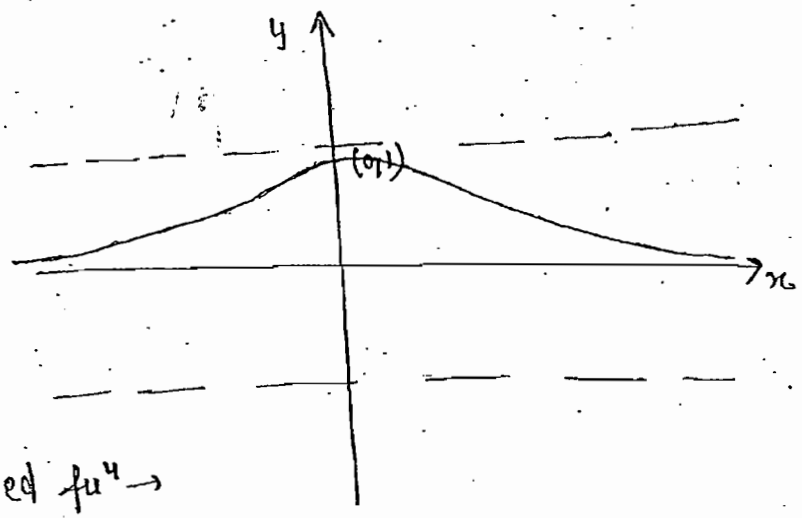
e^x, $\log x$ → Unbounded fu^n.

$\sin x$, $\cos x$ → bounded fu^n.



i) e^{-x^2}

* $x \to \infty$
  $e^{-x^2} \to 0$
* Put $x = 0$
  $e^{-x^2} = 1$
* Put $x \to -\infty$
  $e^{-x^2} \to 0$



Nearly Bounded fu^n →

$0 \leq f(x) \leq 1$

Eg → Consider the differential eq^n →

$$x \log x \frac{dy}{dx} = y \qquad x > 0$$

Then

a) Every sol^n is Bounded, Non-Periodic

b) ∃ a Bounded and Periodic sol^n

c) Every particular sol^n is Non-periodic Unbounded

d) ∃ Non-periodic, Unbounded solution.

Sol^n → sol^n of diff. eq^n is $\underbrace{y = c \log x}_{\text{General sol}^n}$, $y > 0$, $x > 1$

$\log x$ → Unbounded and Non-periodic fu^n ∴ General sol^n is Unbounded & Non-periodic.

and ∴ Every particular sol^n is Unbounded and Non-periodic (*) (∵ Every sol^n can't be bounded) as

∴ a) is wrong.

$y(x) \equiv 0$ is also the sol^n of diff. eq^n, which is bounded and periodic (being constant fu^n)

∴ b) is true

c) is true (Using *)

d) is true (∵ particular sol^n's are unbounded & Non-periodic. ∴ ∄ Non-periodic and Unbounded Sol^n)

Reducable into Separable form →

If the diff. eq^n →

$$\frac{dy}{dx} = f(ax+by+c)$$

Put →

$$ax+by+c = u$$

$$a+by' = \frac{du}{dx} \qquad \left(\frac{dy}{dx} = f(u)\right)$$

$$a+b\,f(u) = \frac{du}{dx}$$

Separable form → $dx = \dfrac{du}{a+b\,f(u)}$, $a+b\,f(u) \neq 0$

$$\int \frac{du}{a+b\,f(u)} = \int dx + c$$

$$\frac{dy}{dx} = (4x+y+1)^2$$

Put $u = 4x+y+1$

$$4 + \frac{dy}{dx} = \frac{du}{dx}$$

$$4 + u^2 = \frac{du}{dx}$$

$$\frac{du}{2^2+u^2} = dx \qquad \left(\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$\Rightarrow \quad x = \frac{1}{2}\tan^{-1}\frac{u}{2} + c$$

$$x = \frac{1}{2}\tan^{-1}\frac{(4x+y+1)}{2} + c$$

Eg. → The general sol^n of

$y'' + e^{2y}(y') = 0$ is

$$x - c = \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right)$$

$$2(x+c') = \tan^{-1}\left(\frac{4x+y+1}{2}\right) \qquad c' = -c$$

$$4x+y+1 = 2\tan\left(2(x+c')\right)$$

is the required sol^n of diff. eq^n.

Eg → The General sol^n of

$$y'' + e^{2y}(y')^3 = 0 \text{ is}$$

a) $x = \frac{1}{4}e^{2y} + Cy + C_2$

b) $x = \frac{-1}{4}e^{2y} + Cy + C_2$

c) $y = xe^{2y} + Cy + C_2$

d) None of these

Sol^n:

Put $y' = u$

(If we write → $y = y(x)$
$u' + e^{2y}u^3 = 0$
u, y & x → Three variables)

Find $\frac{d^2y}{dx^2}$ → ?

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx} = \frac{du}{dy}\cdot u$$

$$\therefore \quad u\frac{du}{dy} + e^{2y}u^3 = 0$$

$$\frac{du}{dy} + e^{2y}u^2 = 0 \qquad (u \neq 0 \text{ factor we eliminate gives General sol}^n)$$

$$\frac{du}{dy} = -e^{2y}u^2$$

$$u^{-2}\,du = -e^{2y}\,dy$$

$$\frac{-1}{u} = \frac{-e^{2y}}{2} + C$$

$$\frac{-1}{u} + \frac{e^{2y}}{2} = C \Rightarrow \frac{1}{u} = \frac{e^{2y}}{2} - C$$

$$\frac{dx}{dy} = \frac{e^{2y}}{2} - C$$

$$dx = \left(\frac{e^{2y}}{2} - C\right)dy$$

$$x = \frac{e^{2y}}{4} + Cy + C_2 \quad \text{is the required sol}^n.$$

$(C_1 = -C)$

I $yy' = \cos^2 x$: $y^2 = x + \sin 2x + C$

II $y^2 y' + x^2 = 0$: $x^3 + y^3 = C$

III $y' = e^{x+y} + x^2 e^y$: $e^{-y} + \frac{1}{3}x^3 + e^x = C$

IV $y y' = -x e^{-x} \sqrt{1-y^2}$: $\sqrt{1-y^2} = (x+1)e^{-x} + C$

I $y'' + 16y' = 0$: $y(x) = C_1 e^{-16x} + C_2$

II $xy'' + y' = 0$: $y(x) = C_1 \log x + C_2 \; ; x > 0$

III $y'' + (y')^2 = 0$: $y(x) = \log(C_1 x + C_2)$

IV $y'' + (1 + y^{-1})(y')^2 = 0$: $(y-1)e^y = C_1 x + C_2$.

First Order First Degree Homog. diff eq^n →

$\frac{dy}{dx} = f(x,y)$ is Homog diff eq^n

if $f(x,y)$ is Homog. fu^n of degree Zero.

i.e $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ or $\phi\left(\frac{x}{y}\right)$

Let $\frac{y}{x} = u$.

$y = xu$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$\phi(u) = u + x\frac{du}{dx}$

$\phi(u) - u = x\frac{du}{dx}$

$\frac{du}{\phi(u) - u} = \frac{dx}{x}$ $\quad \phi(u) \neq u$

$\int \frac{du}{\phi(u) - u} = \log x + c$ is the sol^n of given diff. eq^n.

(Note) →

Every Homog. diff. eq^n is reducible into Separable form.

$(x^2+4y^2+xy)\,dx - x^2 dy = 0$

Sol^n → Homog. diff eq^n ∴ Reducible into Separable form

$$\frac{dy}{dx} = 1 + 4\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right), \quad x \neq 0$$

$$y = ux$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = 1 + 4u^2 + u$$

$$\frac{du}{1+4u^2} = \frac{dx}{x}$$

$$\frac{1}{2}\tan^{-1}(2u) = \log x + \log c$$

$$\tan^{-1}(2u) = 2\log xc$$

$$2u = \tan(2\log(cx))$$

$$y = \frac{x}{2}\tan(2\log(cx))$$

Q.q → $(3xy+y^2)\,dx + (x^2+xy)\,dy = 0 \quad y(1)=1$

Sol^n of diff. eq^n ?

Sol^n → Homog. diff. eq^n ∴ reducible into Separable form.

$$\frac{dy}{dx} = -\frac{(3xy+y^2)}{x^2+xy}$$

$$\frac{dy}{dx} = -\left(\frac{3(y/x)+(y/x)^2}{1+(y/x)}\right)$$

Put $\frac{y}{x} = u \Rightarrow y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = -\left(\frac{3u+u^2}{1+u}\right)$$

$$x\frac{du}{dx} = \frac{-3u-u^2}{1+u} - u$$

$$= \frac{-3u-u^2-u-u^2}{1+u}$$

$$= \frac{-4u-2u^2}{1+u}$$

$$\frac{x\,du}{dx} = \frac{-2u(2+u)}{1+u}$$

$$\frac{1+u}{u(2+u)}\,du = -2\frac{dx}{x}$$

$$-2\frac{dx}{x} = \left[\frac{1}{2u} + \frac{1}{2(2+u)}\right]du$$

$$-2\log x = \frac{1}{2}\log u + \frac{1}{2}\log 2+u + \log c$$

$$\frac{1}{x^2} = u^{1/2}(2+u)^{1/2}c$$

$$\frac{1}{x^4} = u(2+u)c^2$$

$$\frac{1}{x^4} = \frac{y}{x}\left(2+\frac{y}{x}\right)c^2$$

$$x^3 y\left(2+\frac{y}{x}\right) = \frac{1}{c^2} = c' \quad \underline{\text{Ans.}}$$

Domain of fu^{n} & Domain of Def^{n} →

Domain of fu^{n}

↓

Given → Domain
and we try to find
~~Rule~~ rule / function
it means finding
'Domain of fu^{n}'.

Domain of Def^{n} →

↓

Given → function / Rule
and we try to find
domain of that fu^{n}
it means we are
finding 'domain of def^{n}'

Note → Every Separable form is reducible to exact form.

$$Mdx + Ndy = 0$$

$f_1(x)\,g_1(y)\,dx + f_2(x)\,g_2(y)\,dy = 0$ ← Variable Separable is of this form.

Multiply with $\frac{1}{g_1(y)\,f_2(x)}$ ⇒

$$\underbrace{\frac{f_1(x)}{f_2(x)}}_{M'}\,dx + \underbrace{\frac{g_2(y)}{g_1(y)}}_{N'}\,dy = 0$$

$M_y' = 0$

$N_x' = 0$ ⇒ Exact diff eq^n.

⇒ Every Separable form is reducible to Exact form.

(Homg. $\xrightarrow{\text{reducible to}}$ Separable

Separable $\xrightarrow{\text{reducible to}}$ Exact

⇒ Homg. $\xrightarrow{\text{reducible to}}$ Exact)

Note → Homog. diff eq^n is may or may not exact, it is always reducible to exact form

If diff eq^n is of the form →

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad \text{—(1)}$$

Let $A = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix}$

I $|A| \neq 0$ i.e. $ab' \neq ba'$

Then the diff eq^n (1) is reducible into Homogeneous diff eq^n

$$\frac{dy}{dX} = \frac{aX+bY}{a'X+b'Y}$$

where $X = x-h$

$Y = y-k$

And (h,k) is intersection point of

$ax+by+c=0$

$a'x+b'y+c'=0$

II $|A|=0$ and $c \neq c'$ i.e Parallel & Non-Coinciding

Then diff eq^n (1) is →

$$\frac{dy}{dx} = \frac{\lambda(a'x+b'y)+c}{a'x+b'y+c'}, \quad \frac{a}{a'} = \frac{b}{b'} = \lambda \quad \text{—(2)}$$

If $a'x+b'y = u$

Then (2) is reducible into Separable form and sol^n is →

$$\int \frac{du}{a'+b'\left(\frac{\lambda u+c}{u+c'}\right)} = x+c$$

Explanation to Above two cases →

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad \text{—— (1)}$$

Let $f(x,y) = ax+by+c$

& $g(x,y) = a'x+b'y+c$

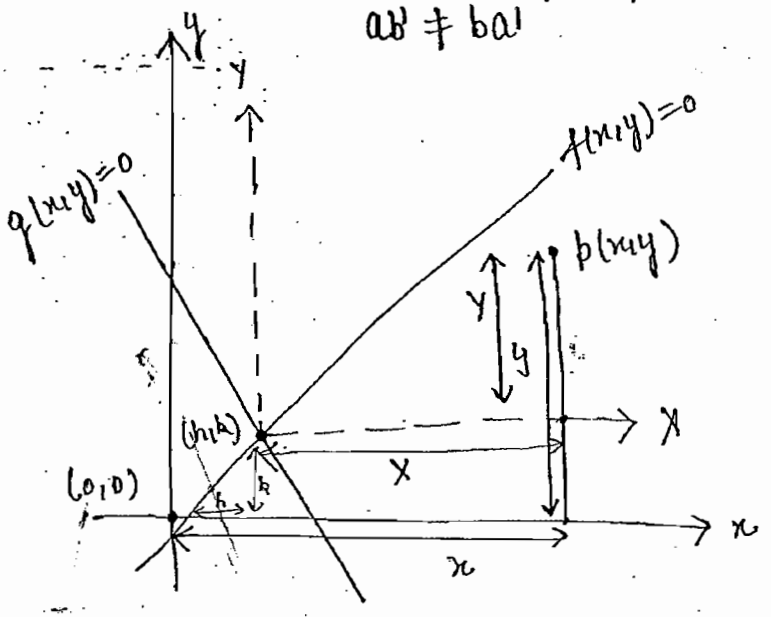
Then $f(x,y) = ax+by+c = 0$ &

$g(x,y) = a'x+b'y+c' = 0$ represents } — (*)

STRAIGHT LINES (Not passing through origin as c & c' are there)

1) Let (h,k) be the intersection pt. of these two lines.

$ab' \neq ba'$



Shifting the origin to pt. (h,k) and let (X,Y) be the new co-ordinate system.

Let P be any pt with co-ordinates (x,y) in x-y co-ordinate system & with co-ordinates (X,Y) in X-Y co-ordinate system.

$$\therefore \quad y = Y+k \quad \Rightarrow \quad Y = y-k$$
$$x = X+h \qquad\qquad X = x-h \quad \text{— (2)}$$

Now the line $g(x,y)$ in Co-ordinate System $X-Y$ passes through origin (h,k) ∴

$$\left.\begin{aligned} g(X,Y) &= a'X+b'Y \\ f(X,Y) &= aX+bY \end{aligned}\right] \text{ in New Co-ordinate System}$$

$$\therefore \frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')} \quad -(3)$$

Where $X = x-h$ &

$Y = y-k$

In order to make (3) homogeneous, choose (h,k) so as to satisfy the following two eq^{n}s →

$$ah+bk+c=0 \;\&\; a'h+b'k+c'=0$$

(i.e (h,k) is the pt of intersection of 2 lines given by *)

$$\therefore \frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} \leftarrow \text{Homog. eq}^n.$$

$$\frac{dY}{dX} = \frac{a+b\left(\frac{Y}{X}\right)}{a'+b'\left(\frac{Y}{X}\right)}$$

(Apply the Method of Separable term)

Put $\frac{Y}{X} = U \Rightarrow Y = XU$

$$\frac{dY}{dX} = X\frac{dU}{dX}+U = \frac{a+bU}{a'+b'U}$$

$$X\frac{dU}{dX} = \frac{a+bU}{a'+b'U} - U$$

Separate variables & Then integeate.

2) When the two lines are parallel & coinciding.

Trivial case, Both lines will be having the same eq^n.

$$\frac{dy}{dx} = \frac{ax+by+c}{ax+by+c} = 1$$

$$dy = dx$$

$$\boxed{y = x + c}$$

3) Parallel & Non-coinciding.

If $|A| = 0$, $\quad \frac{a}{a'} = \frac{b}{b'} = \lambda \quad , c \neq c'$

$$a = \lambda a'$$

$$b = \lambda b'$$

$$\frac{dy}{dx} = \frac{\lambda a'x + \lambda b'y + c}{a'x + b'y + c'}$$

$$= \frac{\lambda (a'x + b'y) + c}{a'x + b'y + c'} \quad \text{— (4)}$$

Let $a'x + b'y = u$

$$\Rightarrow a' + b'\frac{dy}{dx} = \frac{du}{dx}$$

From (4) →

$$a' + b'\left(\frac{\lambda u + c}{u + c'}\right) = \frac{du}{dx}$$

Sol^n is →

$$\int \frac{du}{a' + b'\left(\frac{\lambda u + c}{u + c'}\right)} = \int dx = x + c$$

Q.q → $(2x-y+1)dx - (x-y-1)dy = 0$

$$\frac{dy}{dx} = \frac{2x-y+1}{x-y-1}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -2+1 = -1 \neq 0$$

$$\frac{dy}{dx} = \frac{2x-y}{x-y} \leftarrow \text{Homog. eq}^n$$

$$\frac{dy}{dx} = \frac{2-y/x}{1-y/x}$$

Put $\frac{y}{x} = v \Rightarrow y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{2-v}{1-v}$$

$$x\frac{dv}{dx} = \frac{2-v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{2-v-v+v^2}{1-v} = \frac{2-2v+v^2}{1-v}$$

$$\frac{1-v}{v^2-2v+2}\,dv = \frac{dx}{x}$$

$$\int \frac{1-v}{(v-1)^2+1}\,dv = \int \frac{dx}{x} = \log x + \log c$$

$(v-1)^2+1 = t \Rightarrow 2(v-1)\,dv = dt$

$$-\frac{1}{2}\int \frac{dt}{t} = \log xc$$

$$-\frac{1}{2}\log t = \log xc$$

$$\log t^{-1/2} = \log xc$$

$$t^{-1/2} = xc$$

$$(1+(v-1)^2)^{-1/2} = xc$$

$$\left(1+\left(\frac{y}{x}-1\right)^2\right)^{-1/2} = xc$$

$x = X+h \quad \Rightarrow X = x+2$

$y = Y+k \quad \Rightarrow Y = y+3$

(h,k) is the pt. of intersection $\rightarrow$

$$x = y+1$$

$$2(y+1) - y + 1 = 0$$

$$y+3 = 0$$

$$y = -3 \qquad (h,k) = (-2,-3)$$

$$x = -2$$

$$\left(1+\left(\frac{y+3}{x+2}-1\right)^2\right)^{-1/2} = (x+2)c.$$

E.g → $(2x-y+1)dx - (x-y-1)dy = 0$

Another approach → (when the lines intersect)

$$(2x-y+1)dx - (x-y-1)dy = 0 \quad \text{—(1)}$$

$|A| = -1 \neq 0$

$u \quad\quad v$

$u = 2x - y + 1$

$v = -x + y + 1$

$du = 2dx - dy$

$dv = -dx + dy$

Find dx & dy →

$dx = du + dv$

$dy = 2dv + du$

∴ (1) reduces to →

$u(2u+dv) + v(2dv+du) = 0$

$\underbrace{(u+v)}_{M}du + \underbrace{(u+2v)}_{N}dv = 0$ ← Homog. diff eqn

$M_v = 1$

$N_u = 1$

$\int M\,du + \int N\,dv = 0$, N is free from u

$\frac{u^2}{2} + uv + v^2 = C$

$\frac{(2x-y+1)^2}{2} + (2x-y+1)(-x+y+1) + (-x+y+1)^2 = C$

Que. $(x+ay-1)dx + (bx-3y+2)dy = 0$

a) diff eq^n is reducible into Homog. diff eq^n for unique $a, b \in \mathbb{R}$

b) diff eq^n ~~is reducible into Homog diff eq^n~~ for unique $a \in \mathbb{R}$ & any $b \in \mathbb{R}$

✓ c) diff eq^n reducible to separable for any $a, b \in \mathbb{R}$

d) diff eq^n reducible to separable for unique $a, b \in \mathbb{R}$

Sol^n:

$$\frac{dy}{dx} = \frac{-(x+ay-1)}{bx-3y+2}$$

$$|A| = \begin{vmatrix} -1 & -a \\ b & -3 \end{vmatrix}$$

If $|A| = 0$, reducible to separable form (No need to chk)

If $|A| \neq 0$

$$3 + ab \neq 0$$

$$\Rightarrow ab \neq -3$$

- Diff eq^n is reducible into Homog. diff eq^n when $ab \neq -3$

  $\exists$ many $a, b \in \mathbb{R}$ s.t $ab \neq -3$, Not Unique

  $\therefore$ a) is wrong.

- For Unique $a \in \mathbb{R}$, 🚫 (Say $a=1$) & for any $b \in \mathbb{R}$ say $b = -3$

  $ab = -3$

  Not reducible into Homog. diff eq^n

∴ b) is wrong

option c) is correct.

∵ If $ab \neq -3$

• Reducible into Homog diff eq^n & then to separable form.

and If $ab = -3$ i.e $|A| = 0$ even then also diff eq^n is reducible to separable form

∴ for any $a, b \in \mathbb{R}$, diff eq^n is reducible to Separable form.

∴ Uniqueness of $a, b \in \mathbb{R}$ fails

& Hence

Option d) is incorrect.

(To reduce into Homog. diff eq^n $|A| \neq 0$

• diff eq^n is reducible to separable form in both cases when $|A| \neq 0$ & $|A| = 0$)

Que

$$T_1 : x + ay - 1 = u$$

$$T_2 : bx - 3y + 2 = v$$

✓ a) By using T_1 & T_2, the diff eq^n is reducible into Homog. eq^n in u, v if $ab \neq -3$

b) By using T_1 & T_2, the diff eq^n is reducible into Homog. eq^n in u, v only if $ab \neq -3$

✓ c) if $ab = -3$, then T_1 or T_2 reduced the diff eq^n into Separable form in x, u or x, v

✓ d) if $ab = -3$ then T_1 or T_2 reduced the diff eq^n into Separable form in y, u or y, v

Example in view of (i) & (ii) option

Consider diff eq^n → **When $ab=-3$** Take $a=1$ & $b=-3$

$$(x+y-1)\,dx+(-3x-3y+2)\,dy=0$$

Take any coff of dx or dy →

Say $u=x+y-1$

$$du=dx+dy$$

$$dx=du-dy \quad \text{— (1)}$$

$$\therefore u(du-dy)+(-3(u+1)+2)\,dy=0$$

$$u(du-dy)+(-3u-1)\,dy=0$$

$$u\,du-u\,dy-dy=0$$

$$u\,du=(u+1)\,dy$$

⇒ diff eq^n reduces into separable form in y, u

If we take dy in place of dx in (1) then diff eq^n reduces into separable form in x, u

(iii) If we put $v=-3x-3y+2$

By same procedure

diff eq^n reduces into separable form in v, y & v, x.

Note → Let $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a'x+b'y+c'}$ — (1)

If $A=\begin{bmatrix} a & b \\ a' & b' \end{bmatrix}$ & If $|A|=0$

Then the diff eq^n (1) can be reduced into separable form in x, u or y, u

Using $u=a'x+by+c$

& the diff eq^n (1) can be reduced into separable form in x, v or y, v

Using $v=a'x+b'y+c'$

Sol 7→

∴ In view of Note →

Options (b) and (d) are correct.

Option (a) is correct

∵ If $ab \neq -3$, then diff eq^n is reducible into Homog eq^n.

Option (b) is incorrect (∵ Of only if condition)

FOLD linear Diff eq^n →

$\frac{dy}{dx} = f(x,y)$ is linear if

$f(x,y) = y\,Q_1(x) + Q_2(x)$ (linear → y & its derivatives → degree 1)

General form →

$$\boxed{\frac{dy}{dx} + y\,P(x) = Q(x)}$$ is FOLD linear Diff eq^n.

Que. $M(x,y)dx + N(x,y)dy = 0$ is linear when
Let $y = y(x)$

a) M is free from y & N is linear in y

✓ b) M is linear in y & N is free from y.

c) Both M & N are linear in y

✓ d) Both M & N are free from y.

Sol^n →

a) N is linear in y ⇒ y is (of i.e. degree 1) there

$N(x,y)dy = y\,dy$ product term will be there a) option is Wrong

⇒ Non-linear diff eq^n

b) M is linear in y & N is free from y.

Correct. (No product term like → $y \cdot dy$ or y^2)

c) Both M & N are linear in y.

N linear in y produces Non-linear diff eq^n

∴ c) is Wrong

d) Both M & N are free from y - Also Correct.

Que. $\frac{dy}{dx} + y \cdot P(x) = Q(x)$

a) exact diff eq^n, if $P(x)=0$, $Q(x)$ any fu^n

b) exact diff eq^n, if $Q(x)=0$, $P(x)$ any fu^n

c) exact diff eq^n only if $P(x)=0$, $Q(x)$ any fu^n

d) exact diff eq^n only if $P(x)=0$, $Q(x) \neq 0$

Sol^n →

$$dy = [Q(x) - y \cdot P(x)]\,dx$$

$$[y P(x) - Q(x)]\,dx + dy = 0$$

$$M = y P(x) - Q(x)$$

$$N = 1$$

$$My = P(x)$$

$$Nx = 0$$

Exact $\Leftrightarrow$ $My = Nx$ i.e $P(x)=0$ & $Q(x)$ can be any fu^n.

$\Rightarrow$ option a) is correct. & option b) is wrong

c) is correct

∵ If $P(x)=0$, $Q(x)$ any fu^n then diff eq^n is exact.

If $P(x) \neq 0$ then diff eq^n is not exact

P only if Q

Means →

if Q is True then P is True & if Q is false then P is false

& d) option is wrong

∵ Only if Condition fails. ∵ If Q is false i.e take $P(x)=0$, $Q(x)=0$ (instead of $Q(x) \neq 0$) then P must be ~~false~~ false.

Take $P(x) = Q(x) = 0$

$\frac{dy}{dx} = 0$ i.e diff eq^n is reducible to exact form i.e P is True

∴ option d) is incorrect.

Que If $\frac{dy}{dx} + y \cdot P(x) = Q(x)$ be a Non-Exact eq^n and $\phi(x,y)$ is an Integrating factor then

a) $\phi(x,y)$ is fu^n of x only.

b) $\phi(x,y)$ cannot be fu^n of y/x

✓ c) $\exists$ $f(x)$ s.t $\phi(x,y) = f(x)$

d) None of these.

Sol^n:

$\exists$ an Integrating factor which is function of $\frac{y}{x}$

When $\frac{x^2(My - Nx)}{yN + xM} = f\left(\frac{y}{x}\right)$

$M = yP(x) - Q(x)$

$N = 1$

$My = P(x)$ & $Nx = 0$

$$\frac{x^2(My - Nx)}{yN + xM} = \frac{x^2 P(x)}{y + x(yP(x) - Q(x))}$$

Take $P(x) = 1/x$ & $Q(x) = 0$

$$= \frac{x^2/x}{y + x\left(\frac{y}{x} - 0\right)} = \frac{x}{2y} = \frac{1}{2y/x}$$

$\Rightarrow$ $\exists$ an integrating factor which is fu^n of $\frac{y}{x}$.

$\therefore$ b) is Wrong.

$\exists$ infinite number of integrating factors.

$\therefore$ $\phi(x,y)$ can't be fu^n of x-only.

It may be fu^n of x, y, xy, y/x etc.

$\therefore$ a) is Wrong

If an I.F which is fu^n of x only, when

$$\frac{My - Nx}{N} = P(x) = \text{fu}^n \text{ of } x\text{-only} = f(x)$$

I.F is given by →

$$e^{\int P(x)\,dx} = e^{\int f(x)\,dx}$$

Consider $\frac{dy}{dx} + yP(x) = Q(x)$

Then $e^{\int P(x)\,dx}$ is an I.F

$$[yP(x) - Q(x)]\,dx + dy = 0$$

∴ Multiply with I.F →

$$[yP(x)\,e^{\int P(x)\,dx} - Q(x)\,e^{\int P(x)\,dx}]\,dx + e^{\int P(x)\,dx}\,dy = 0$$

$$d\left(y\,e^{\int P(x)\,dx}\right) - Q(x)\,e^{\int P(x)\,dx} = 0.$$

$$d\left[y\,e^{\int P(x)\,dx} - \int Q(x)\,e^{\int P(x)\,dx}\right] = 0$$

$$d[f(x,y)] = 0$$

$f(x,y) = c$ is the sol^n of diff eq^n.

$$y\,e^{\int P(x)\,dx} = \int Q(x)\,e^{\int P(x)\,dx} + c$$

$$\boxed{y \cdot I.F = \int Q \cdot I.F + c}$$ is the sol^n of diff eq^n.

Que $\frac{dy}{dx} - 2xy = e^{x^2}$

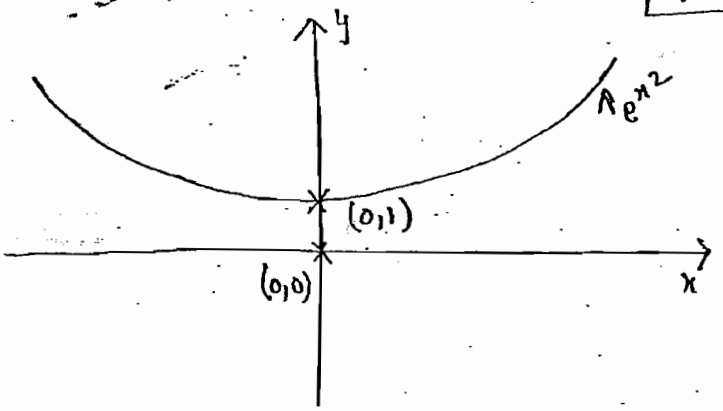
a) Bounded
b) Periodic
c) Montonically decreasing
d) Montonically increasing
e) None of these ✓

Sol^n I.F $\to e^{\int -2x\,dx} = e^{-x^2}$

Sol is →

$$y \times e^{-x^2} = \int e^{x^2} \times e^{-x^2} dx + c$$

$$y e^{-x^2} = x + c \Rightarrow \boxed{y = (x+c)e^{x^2}}$$

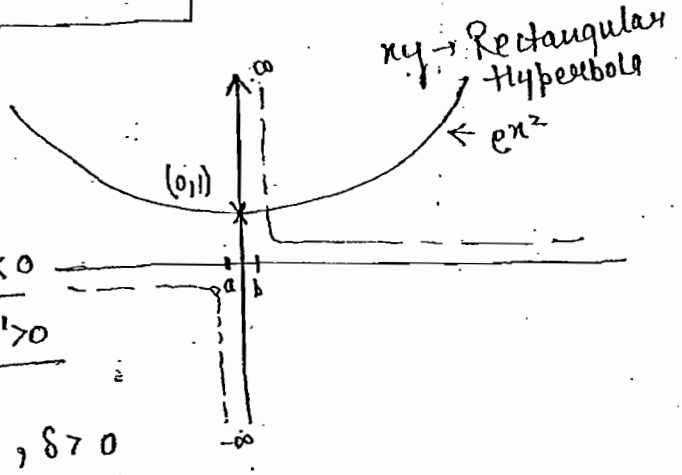


Clearly the Sol^n is unbounded and Non-Periodic

($e^x \to$ Non-Periodic)

f is ↑ if $f'(x) \geq 0 \quad \forall x$
f is ↓ if $f'(x) \leq 0 \quad \forall x$

Ans $\frac{dy}{dx} = e^{x^2} + 2xy$



When $a \cong 0^-$, $b \cong 0^+$

$e^a \cong 1$ $\quad xy \to -\infty \Rightarrow y' < 0$

$e^b \cong 1$ $\quad xy \to +\infty \Rightarrow y' > 0$

i.e Their Sum is -ve
when $x \in (0-\delta, 0)$, $\delta > 0$

& Their Sum is +ve
when $x \in (0, \delta+0)$, $\delta > 0$.

$\Rightarrow$ Neither $y(x)$ is ↑ nor ↓.

Que. $\frac{dy}{dx} + \frac{(2x+1)}{x} y = e^{-2x}$

Then

✓ a) $\exists$ a sol^n $y(x) \to 0$ as $x \to \infty$

b) $\exists$ a sol^n $y(x) \to \infty$ as $x \to \infty$

c) $\exists$ a sol^n $y(x) \to a \neq 0$, $a \in \mathbb{R}$ as $x \to \infty$

d) None of these.

Sol^n

$$I.F = e^{\int (2+\frac{1}{x})dx} = e^{2x+\log x} = e^{2x} \cdot e^{\log x}$$

$$= x e^{2x}$$

Sol^n is →

$$y \cdot x e^{2x} = \int e^{-2x} \cdot x e^{2x} + c$$

$$y \cdot x e^{2x} = \frac{1}{2} x^2 + c$$

$$y(x) = \frac{1}{2} x e^{-2x} + \frac{c}{x} e^{-2x}$$

When $x \to \infty$

$$y(x) = \infty \times 0 + \frac{c}{\infty} \times 0 \quad \text{(Indeterminate form)}$$

Using L-H rule →

$$y(x) = +\frac{1}{4} e^{-2x} - 2c e^{-2x}$$

When $x \to \infty$

$$y(x) \to 0 \qquad (e^{-\infty} = 0)$$

∴ Option a) is correct

Note 1) when $c \in \mathbb{R}$

$\Rightarrow$ c is finite

2) $c \in \mathbb{R} \cup \{\infty\}$ ← Extended Real Plane.

Indeterminate form → $\frac{0}{0}, \frac{0}{\infty}, \frac{\infty}{\infty}, \frac{\infty}{0}, -\infty \times \infty, 0 \times \infty$

3) In FOLDE, If we want to chk Montonically ↑ or ↓ (linear)

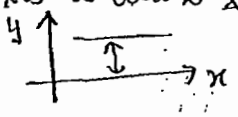
We need $\frac{dy}{dx}$.

i.e In case of $\frac{dy}{dx} + P(x)y = Q(x)$

No Need to find the sol^n & then derivative.

$\frac{dy}{dx}$ is already given.

4) Constant → Distance from x-axis is alwx same.



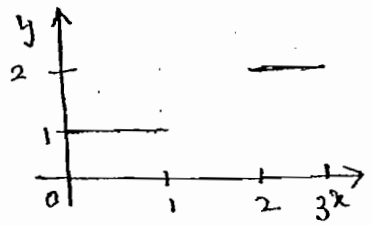
5) f is Constant $\Rightarrow f'(x) = 0$

$f'(x) = 0 \not\Rightarrow$ f is Constant

e.g →

$$f(x) = \begin{cases} 1 & x \in (0,1) \\ 2 & x \in (2,3) \end{cases}$$

$f'(x) = 0$ But $f(x)$ is not Constant



Que. f is diff. fu^n defined on

$$A = (1,2) \cup (3,4) \text{ s.t}$$

$$f'(x) = 0 \quad \forall x$$

say

$$f(x) = \begin{cases} a & x \in (1,2) \\ b & y \in (3,4) \end{cases}$$

The range of f has →

✓ a) atmost Two point

b) exactly One point

✓ c) atleast One point

d) None - of these.

Sol^n:

No. of distinct intervals = Atmost point in range of f

Here distinct Intervals = 2.

∴ atmost 2 points in range of f.

Option a) & c) both are Correct.

atleast One point.

(imply here 2)

(No chance of having 3,4,5,6 — points) There is

Que $\frac{dy}{dx} + 2xy = e^{-x^2}$, Show → Bounded & Non-Periodic Sol^n

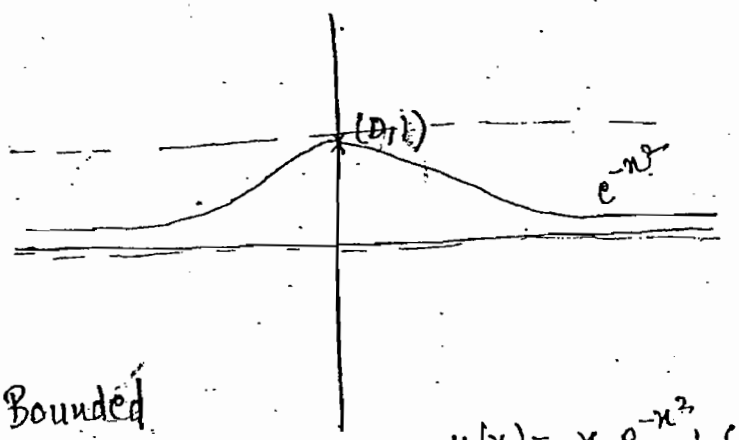
Sol^n:

$$I.F = e^{\int 2x dx} = e^{x^2}$$

$$y \cdot e^{x^2} = \int e^{-x^2} \cdot e^{x^2} dx + C$$

$$y \cdot e^{x^2} = x + C$$

$$y = (x + C) e^{-x^2}$$



Clearly, Bounded

$y(x) = x e^{-x^2} + c e^{-x^2}$

When $x \to \infty$

or

$x \to \infty$ & $e^{-x^2} \to 0$

$x \cdot e^{-x^2} \longrightarrow \infty \cdot 0$

$x \cdot e^{-x^2} \longrightarrow 0$

$\frac{x}{e^{x^2}} = \frac{1}{2x e^{x^2}}$

Note → Two fu^{n}s $f(x)$ & $g(x)$ one tending to 0 & another tending to ∞ i.e

$f(x)$ & $g(x)$

↓ ↓

0 ∞

$f(x) \cdot g(x) \longrightarrow$ 0 or ∞ depends on the fu^n having larger growth i.e having large rate of change

Hence e^{-x^2} has large rate of change always.

$\therefore x \cdot e^{-x^2} \longrightarrow 0$

Apply in case of →

$0 \cdot \infty$

$0 \cdot -\infty$

$\infty \cdot 0$

$-\infty \cdot 0$

$\therefore$ When $x \to \infty$

$y(x) \to 0 + 0 \to 0$

When $x \to -\infty$

$y(x) \to +0 + 0 \to 0$

When $x \to 0$

$y(x) = 0 \cdot \text{finite} + c \Rightarrow y(x) \to c$

From these three cases → We can also say that sol^n is bounded as bounded in each case as tending to finite value

$y(x) = x \cdot e^{-x^2} + c \cdot e^{-x^2}$ ($\because$ Polynomial X Any fu^n → Non-Periodic) is Non-Periodic

Ex. → $\frac{dy}{dx} = \frac{y^2}{1-3xy}$ ← Not linear in y

$$\frac{dx}{dy} = \frac{1-3xy}{y^2} = \frac{1}{y^2} - \frac{3x}{y}$$

$$\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$$ ← linear in x

$$I.F = e^{\int \frac{3}{y} dy} = e^{3\log y} = y^3$$

Sol^n is →

$$x \cdot y^3 = \int \frac{1}{y^2} \times y^3 \, dy + c$$

$$x \cdot y^3 = \frac{y^2}{2} + c$$

$$x = \frac{1}{2y} + \frac{c}{y^3}$$ is the sol^n of given diff eq^n.

Ex →

$$\frac{dy}{dx} + y \cdot P(x) = Q(x)\, y^n \quad \text{—— (1)}$$

for $n=0$ → linear diff eq^n

for $n=1$ → linear diff eq^n

for $n \neq 0, 1$

(1) diff eq^n is called **BERNOULLI'S Eq^n**

Writting (1) in the form →

$$y^{-n} \frac{dy}{dx} + y^{1-n} P(x) = Q(x)$$

Que $y^{-n}\frac{dy}{dx} + y^{1-n}P(x) = Q(x)$, $y(x) \neq 0$.

Put $y^{1-n} = u$ (coefficient of $P(x)$)

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx} \qquad (n \neq 1)$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)} \cdot \frac{du}{dx}$$

$$\frac{1}{(1-n)}\frac{du}{dx} + u \cdot P(x) = Q(x)$$, which is linear in u.

Most General form →

$$\boxed{f'(y)\frac{dy}{dx} + p(x)f(y) = Q(x)}$$

Put $f(y) = u$ (coefficient of $p(x)$)

$$f'(y)\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + u\,p(x) = Q(x)$$

Que $\frac{dy}{dx} + y = f(x)$, $y(0) = 0$

$$f(x) = \begin{cases} 2 & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

Then Sol^n of the diff eq^n →

a) $y(x) = \begin{cases} 2(1+e^{-x}) & 0 \le x < 1 \\ 2(e-1)e^{-x} & x \ge 1 \end{cases}$

b) $y(x) = \begin{cases} 2(1-e^{-x}) & 0 \le x < 1 \\ 2(e+1)e^{-x} & x \ge 1 \end{cases}$

✓ c) $y(x) = \begin{cases} 2(1-e^{-x}) & 0 \le x < 1 \\ 2(e-1)e^{-x} & x \ge 1 \end{cases}$

d) None of these.

Sol^n → Case I → When $0 \le x < 1$

$$f(x) = 2$$

$$\frac{dy}{dx} + y = 2$$

$$I.F = e^{\int 1 \cdot dx} = e^x$$

Sol^n is →

$$y \cdot e^x = \int 2 \cdot e^x + c$$

$$y \cdot e^x = 2e^x + c \Rightarrow y(x) = 2 + ce^{-x}, \quad 0 \le x < 1$$

Case II → When $x \geq 1$

$$f(x) = 0$$

$$\frac{dy}{dx} + y = 0$$

$$I.F = e^{x}$$

$$y \cdot e^{x} = c'$$

$$y(x) = c'e^{-x} \quad , x \geq 1$$

$$\therefore y(x) = \begin{cases} 2 + ce^{-x} & 0 \leq x < 1 \\ c'e^{-x} & x \geq 1 \end{cases}$$

Find two constants c & c' ?

Using $y(0) = 0$, find c →

$$y(0) = 0$$

$$2 + c \cdot e^{0} = 0 \Rightarrow c = -2$$

$$\therefore y(x) = 2(1 - e^{-x}) \quad 0 \leq x < 1$$

Find c'? But No condition is given.

Use → $y(x)$ → sol^{n} of FOLD Diff eq^{n} $\therefore y(x)$ is differentiable & hence cts (Use def^{n} of sol^{n} of diff eq^{n}.)

$\therefore$ Apply the concept of continuity at $x = 1$.

$$\therefore \lim_{x \to 1^-} y(x) = \lim_{x \to 1^+} y(x)$$

$$\lim_{x \to 1^-} 2(1 - e^{-x}) = \lim_{x \to 1^+} c'e^{-x}$$

$$\lim_{h \to 0} 2(1 - e^{-(1-h)}) = \lim_{h \to 0^+} c'e^{-(1+h)}$$

$$2(1 - e^{-1}) = c'e^{-1} \Rightarrow c' = 2(e - 1)$$

$$\therefore y(x) = \begin{cases} 2(1-e^{-x}) & 0 \le x < 1 \\ 2(e-1)e^{-x} & x \ge 1 \end{cases}$$ is the req. soln

∴ option (c) is correct

Que → Solve → $\frac{dy}{dx} + y = xy^3$

Soln → ← Compare with Bernoulli's eqn.

$$y^{-3}\frac{dy}{dx} + y^{-2} = x$$

Here $P(x) = 1$, $Q(x) = x$.

Put $y^{-2} = U$

$$-2y^{-3}\frac{dy}{dx} = \frac{dU}{dx}$$

$$\frac{-1}{2}\frac{dU}{dx} + U = x$$

$$\frac{dU}{dx} - 2U = -2x$$

$$I.F = e^{\int -2dx}$$

$$= e^{-2x}$$

$$U \cdot e^{-2x} = -2\int x e^{-2x} + C$$

$$U \cdot e^{-2x} = -2\left[\frac{xe^{-2x}}{-2} + \frac{e^{-2x}}{+2\times -2}\right] + C$$

$$U \cdot e^{-2x} = xe^{-2x} + \frac{e^{-2x}}{2} + C$$

$$U = \left(x + \tfrac{1}{2}\right) + Ce^{2x}$$

$y^{-2} = \left(x + \frac{1}{2}\right) + Ce^{2x}$ is the required sol^n

i.e $\frac{1}{y^2} = \left(x + \frac{1}{2}\right) + Ce^{2x}$ Ans

Que → $x\frac{dy}{dx} + y = -2x^6y^4$

Sol^n → $\frac{dy}{dx} + \frac{1}{x}\cdot y = -2x^5y^4$

$y^{-4}\frac{dy}{dx} + y^{-3}\cdot\frac{1}{x} = -2x^5$

Put $y^{-3} = u$

$-3y^{-4}\frac{dy}{dx} = \frac{du}{dx}$

$\frac{-1}{3}\frac{du}{dx} + \frac{u}{x} = -2x^5$

$\frac{du}{dx} - \frac{3u}{x} = 6x^5$

$I.F = e^{-\int \frac{3}{x}dx} = e^{-3\log x} = \frac{1}{x^3}$

$u\cdot\frac{1}{x^3} = 6\int \frac{x^5}{x^3}dx + C$

$\frac{u}{x^3} = 6\int x^2 dx + C \Rightarrow \frac{y^{-3}}{x^3} = \frac{6x^3}{3} + C$

$\Rightarrow \frac{1}{y^3x^3} = 2x^3 + C$ Ans.

Que. $\frac{dy}{dx} + ay = e^{-dx}$, $y(0)=0$

$a \neq d$, $a, d \in \mathbb{R}$

Then

1) $y(x) \to \infty$ as $x \to \infty$ then a and d are –

2) $y(x) \to -\infty$ as $x \to \infty$ then a and d are –

3) $y(x) \to 0$ as $x \to \infty$ then a and d are –

4) $y(x) \to \infty$ as $x \to -\infty$ then a and d are –

5) $y(x) \to -\infty$ as $x \to -\infty$ then a and d are –

6) $y(x) \to 0$ as $x \to -\infty$ then a and d are –

Sol^n: Integrating Factor $\to e^{\int a\,dx} = e^{ax}$

Sol^n is given by $\to$

$$y \cdot e^{ax} = \int e^{-dx} \cdot e^{ax}\,dx + c$$

$$y \cdot e^{ax} = \frac{e^{(a-d)x}}{(a-d)} + c$$

$$y(x) = \frac{1 \cdot e^{-dx}}{(a-d)} + c\,e^{-ax}$$

Use $\to$ $y(0) = 0$

$$0 = \frac{1}{a-d} + c$$

$$\Rightarrow c = \frac{-1}{a-d}$$

$$\therefore y(x) = \frac{1}{a-d}\left[e^{-dx} - e^{-ax}\right]$$

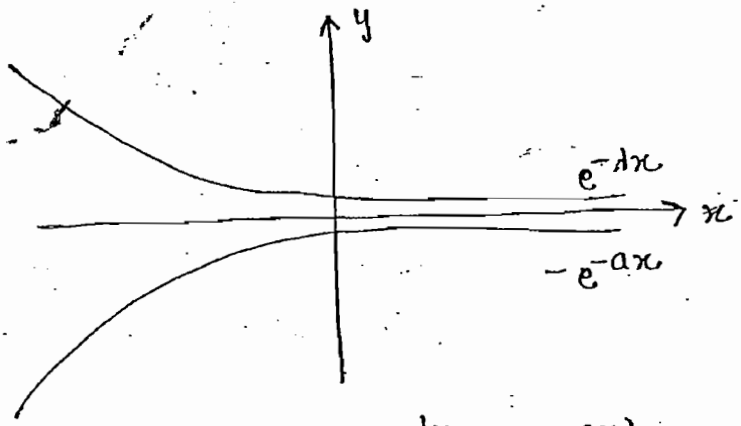
is the required Sol^n.

① When $x \to \infty$ $\quad y(x) = \frac{1}{a-d}\left[e^{-dx} - e^{-ax}\right]$

- Take $a>0$, $d>0$

$e^{-dx} \cong e^{-x}$

$e^{-ax} \cong e^{-x} \Rightarrow -e^{-ax} \cong -e^{-x}$



When $x \to \infty$ $\quad e^{-dx} + (-e^{-ax}) \to 0$

i.e $e^{-dx} - e^{-ax} \to 0$

As $a>0$, $d>0$

When $a > d$

$y(x) = (\text{finite Quantity}) \times 0$

$\Rightarrow y(x) \to 0$

When $a < d$

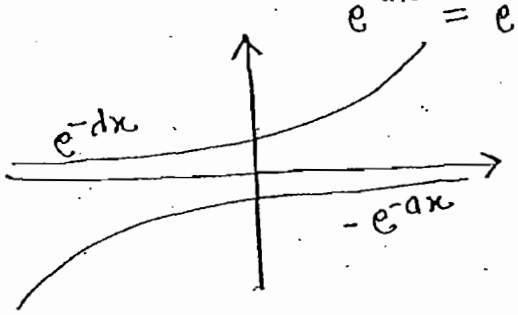
$y(x) = (\text{finite Quantity}) \times 0$

$\Rightarrow \boxed{y(x) \to 0}$

- Take $a>0$, $d<0$

$e^{-dx} \cong e^{+x}$ $\quad (\text{as } d<0)$

$e^{-ax} \cong e^{-x} \Rightarrow -e^{-ax} \cong -e^{-x}$



When $x \to \infty$ $\quad e^{-dx} \to \infty$

$-e^{-ax} \to 0$

$\Rightarrow (e^{-dx} + (e^{-ax})) \to \infty$

$\Rightarrow e^{-dx} - e^{-ax} \to \infty$

As $a>0, d<0$

$a-d>0 \Rightarrow \frac{1}{a-d} \to 0$

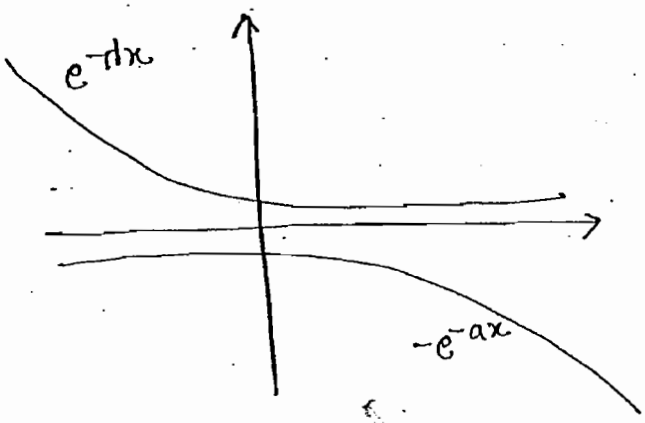
$\Rightarrow y(x) \to (+ve)(\infty)$

$\Rightarrow \boxed{y(x) \to \infty}$

- Take $a<0, d>0$

$e^{-dx} \cong e^{-x}$

$e^{-ax} \cong e^{x} \Rightarrow -e^{-ax} \cong -e^{x}$



When $x \to \infty$ $e^{-dx} \to 0$

$-e^{-ax} \to -\infty$

$e^{-dx} + (-e^{-ax}) \to -\infty$

$(e^{-dx} - e^{-ax}) \to -\infty$

As $a<0, d>0 \Rightarrow a-d \Rightarrow -ve$

$\frac{1}{a-d} \to -ve$

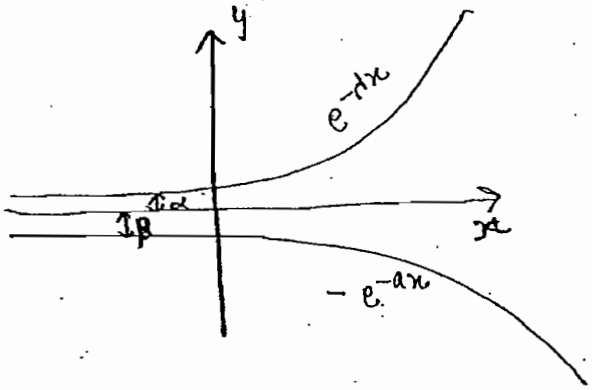
$y(x) \to (-ve)(-\infty)$

$\boxed{y(x) \to \infty}$

- Take $a<0, d<0$

$e^{-dx} \cong e^{x}$

$e^{-ax} \cong e^{x} \Rightarrow -e^{-ax} \cong -e^{x}$



When $x \to \infty$

$e^{-dx} \to \infty$

When $x \to \infty$

$-e^{-ax} \to -\infty$

Suppose → When a>λ & When a<λ

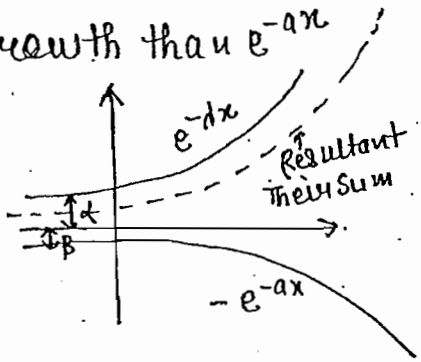
•• When $a > \lambda \Rightarrow a - \lambda > 0$

$$\Rightarrow \frac{1}{a-\lambda} > 0$$

As $a > \lambda$, $e^{-\lambda x}$ has larger growth than e^{-ax}

∴ $\alpha > \beta$.

∴ $(e^{-\lambda x} - e^{-ax}) \to \infty$



($a < 0, \lambda < 0$ & $a > \lambda$

Say $a = -1$ & $\lambda = -2$ $(\because -1 > -2)$

$a > \lambda$

$e^{-ax} = e^{x}$

$e^{-\lambda x} = e^{+2x}$

Case is When $x \to \infty$ i.e x is +ve

Say $x = 1$

$e^{-ax} = e$

& $e^{-\lambda x} = e^{2}$ $\quad e^2 > e$

i.e $e^{2x} > e^{x}$

i.e $e^{-\lambda x} > e^{-ax}$

$a > \lambda \Rightarrow e^{-\lambda x} > e^{-ax}$ When x is +ve

$\Rightarrow e^{-\lambda x}$ has larger growth

i.e $\alpha > \beta$

∴ Resultant will tend to $+\infty$)

∴ When $a > \lambda$ $\quad \boxed{y(x) \to \infty}$ $\quad ((+ve)(\infty) = \infty)$

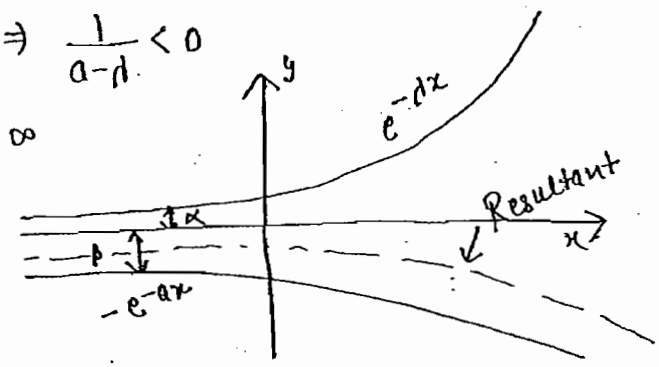
•• When $a < \lambda$

$a - \lambda < 0 \Rightarrow \frac{1}{a-\lambda} < 0$

$(e^{-\lambda x} - e^{-ax}) \to -\infty$

$\Rightarrow y(x) \to (+ve)(-\infty)$

$\boxed{y(x) \to \infty}$



$a<0$, $\lambda<0$ But $a<\lambda$

$a=-2$, $\lambda=-1$ $\quad -2<-1$

$e^{-ax} = e^{2x}$ $\quad$ ↑ ↑ $\quad a \ \lambda$

$e^{-\lambda x} = e^{x}$

When $x \to \infty$ i.e x is +ve

Say $x=1$

$e^{-ax} = e^{2}$

$e^{-\lambda x} = e$ $\quad e^2 > e$

$e^{-ax} > e^{-\lambda x}$

∴ e^{-ax} has larger growth than $e^{-\lambda x}$, When x is +ve

i.e $\beta > \alpha$

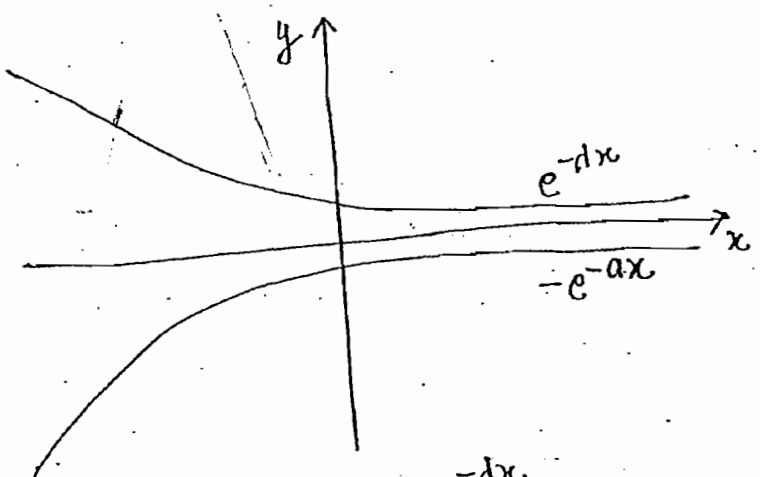
∴ Resultant will tend to $-\infty$)

(2) When $x \to -\infty$

- Take $a>0$, $\lambda>0$

$e^{-\lambda x} \cong e^{-x}$

$e^{-ax} \cong e^{-x} \Rightarrow -e^{-ax} \cong -e^{-x}$



When $x \to -\infty$ $\quad e^{-\lambda x} \to \infty$

$x \to -\infty$ $\quad -e^{-ax} \to -\infty$

Sbcases When $a>\lambda$ and When $a<\lambda$

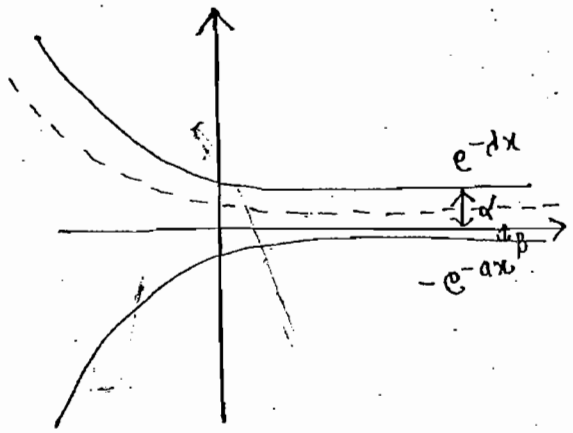
•• When $a < \alpha$

$$\frac{1}{a-\alpha} < 0$$

($a>0$, $\alpha>0$, $a<\alpha$
$a=1$ $\alpha=2$
$e^{-ax} = e^{-x}$
$e^{-\alpha x} = e^{-2x}$

When $x \to -\infty$ i.e x is -
say $x=-1$
$e^{-ax} = e^{1}$
$e^{-\alpha x} = e^{2}$

$e^{2} > e$
$e^{-\alpha x} > e^{-ax}$
i.e $a<\alpha \Rightarrow e^{-ax} < e^{-\alpha x}$
$\Rightarrow e^{-\alpha x}$ has largeer growth
i.e $\alpha > \beta$)
∴ Resultant will tend to -



$(e^{-\alpha x} - e^{-ax}) \to \infty$

∴ $y(x) \to (-ve)(\infty)$

$\boxed{y(x) \to -\infty}$

•• When $a > \alpha$

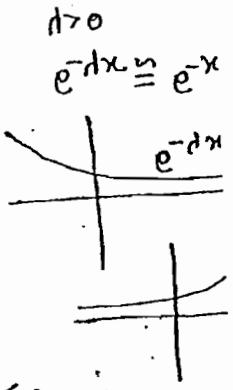
$$\frac{1}{a-\alpha} > 0$$

($a>0$, $\alpha>0$, $\boxed{a>\alpha}$
$a=2$, $\alpha=1$
$e^{-ax} = e^{-2x}$, $e^{-\alpha x} = e^{-x}$
When $x \to -\infty$ i.e x is -ve
Take $x=-1$
$e^{-ax} = e^{2}$, $e^{-\alpha x} = e$
$e^{2} > e$
$e^{-ax} > e^{-\alpha x} \Rightarrow e^{-ax}$ has largeer growth than $e^{-\alpha x} \Rightarrow \beta >$

③ When $a=0, d\neq 0$

$$y(x) = \frac{1}{a-d}\left[e^{-dx} - e^{-ax}\right]$$

$$y(x) = \frac{-1}{d}\left[e^{-dx} - 1\right]$$



Resu

	$d>0$	$d<0$
When $x\to\infty$	$e^{-dx}\to 0$ $y(x)\to\frac{-1}{d}(0-1)$ $\boxed{y(x)\to 1/d}$	$e^{-dx}\to\infty$ $y(x)\to\frac{-1}{d}(\infty-1)$ $\boxed{y(x)\to\infty}$
When $x\to-\infty$	$e^{-dx}\to\infty$ $y(x)\to\frac{-1}{d}(\infty-1)$ $\boxed{y(x)\to-\infty}$	$e^{-dx}\to 0$ $y(x)\to\frac{-1}{d}(0-1)$ $\boxed{y(x)\to 1/d}$
When $x\to 0$	$y(x)\to\frac{-1}{d}(1-1)$ $\boxed{y(x)\to 0}$	$y(x)\to\frac{-1}{d}(1-1)$ $\boxed{y(x)\to 0}$

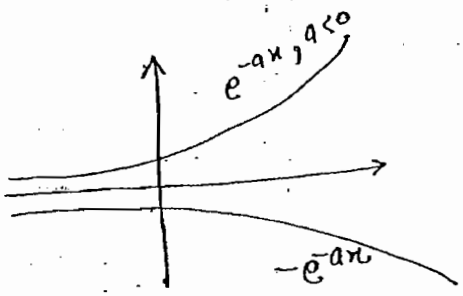
④ When $a\neq 0, d=0$

$$y(x) = \frac{1}{a}\left(1-e^{-ax}\right) = \frac{1}{a}\left(1+(-e^{-ax})\right.$$

When $a<0$

$e^{-ax} \cong e^{x}$

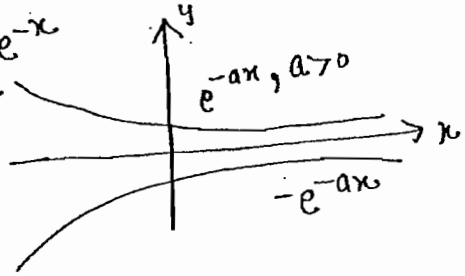
$-e^{-ax} \cong -e^{x}$



When $a>0$

$e^{-ax} \cong e^{-x}$

$-e^{-ax} \cong -e^{-x}$



	$a > 0$	$a < 0$
When $x \to \infty$	$-e^{-ax} \to 0$	$-e^{-ax} \to -\infty$
	$y(x) \to \frac{1}{a}(1+0)$	$y(x) \to \underset{-ve}{\frac{1}{a}}\underbrace{(1+(-\infty))}_{-ve}$ $\left(\frac{1}{a}<0\right)$
	$\boxed{y(x) \to \frac{1}{a}}$	$\boxed{y(x) \to \infty}$
When $x \to -\infty$	$-e^{-ax} \to -\infty$	$-e^{-ax} \to 0$
	$y(x) \to \frac{1}{a}(1+(-\infty))$	$y(x) \to \frac{1}{a}(1+0)$
	$\boxed{y(x) \to -\infty}$	$\boxed{y(x) \to \frac{1}{a}}$
When $x \to 0$	$-e^{ax} \to -1$	$-e^{-ax} \to -1$
	$y(x) \to \frac{1}{a}(1+(-1))$	$y(x) \to \frac{1}{a}(1+(-1))$
	$\boxed{y(x) \to 0}$	$\boxed{y(x) \to 0}$

Or Discuss e^{-ax} in place of $-e^{-ax}$

$$y(x) = \frac{1}{a}(1-e^{-ax})$$

When $a > 0$

When $x \to -\infty$ $\quad e^{-ax} \to \infty$

$$y(x) \to \frac{1}{a}(1-\infty)$$

$$y(x) \to -\infty$$

Same ans will be obtained

Ques1 Find a & d, $a \neq d$, $a, d \in \mathbb{R}$ s.t

$$\frac{dy}{dx} + ay = e^{-dx}$$ has a Bounded sol^n

(When $x \to \infty$, $x \to -\infty$] tends to finite value then $y(x)$ is bounded)

Ques2 Consider the diff eq^n →

$$\frac{dy}{dx} + ay = e^{-dx}, \quad a \neq d, \quad y(0) = 0$$

Then the sol^n $y(x)$ of diff eq^n →

a) Bounded if $a \neq 0$, $d = 0$

b) Bounded if $a = 0$, $d \neq 0$

c) ∄ a periodic for some a & d

d) None of these

Ques3 $\frac{dy}{dx} + by = ae^{-dx}$ $(d > 0)$

i) $y(x) \to 0$ as $x \to \infty$ ∀ b, $b \neq d$

ii) $y(x) \to 0$ as $x \to -\infty$ ∀ b, $b \neq d$

iii) $y(x) \to 0$ as $x \to \infty$ for $b = d$

iv) $y(x) \to 0$ as $x \to -\infty$ for $b = d$

Ques → $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^2 \cos^2 y$ (Form of Bernoulli's eq^n $f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)$)

Sol^n → $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^2 \cos^2 y$

$$\sec^2 y \frac{dy}{dx} + \frac{1}{x}\frac{2\sin y \cos y}{\cos^2 y} = x^2$$

$$\sec^2 y \frac{dy}{dx} + \frac{2}{x}\tan y = x^2$$

Put $\tan y = u$

$$\sec^2 y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + \frac{2}{x}u = x^2$$

$$I.F \rightarrow e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$$

Sol^n is →

$$u \cdot I.F = \int x^2 \cdot I.F\, dx + C$$

$$ux^2 = \int x^4 dx + C$$

$$ux^2 = \frac{x^5}{5} + C$$

$$\tan y\, x^2 = \frac{x^5}{5} + C$$

$$\tan y = \frac{x^3}{5} + cx^{-2}$$ **Ans**

Ques. $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y(\log y)^2}{x}$

Sol^n →

$$\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y(\log y)^2}{x}$$

$$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x}\times\frac{1}{\log y} = \frac{1}{x}$$

Put $\frac{1}{\log y} = u$

$$\frac{(-1)(\log y)^{-2}}{y}\frac{dy}{dx} = \frac{du}{dx}$$

$$-\frac{du}{dx} + \frac{u}{x} = \frac{1}{x}$$

$$\frac{du}{dx} - \frac{u}{x} = -\frac{1}{x}$$

$$I.f \rightarrow e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

Sol^n is →

$$\frac{u}{x} = -\int \frac{1}{x^2}dx + C$$

$$\frac{1}{x\log y} = \frac{+1}{x} + C$$

$$\frac{1}{x\log y} = \frac{+1+Cx}{x} \Rightarrow \log y = (Cx+1)^{-1}$$

Ans

Note →

$$\frac{dy}{dx} + y \cdot P(x) = Q(x)$$

is Homogeneous if $Q(x) = 0$

and $\frac{dy}{dx} + y \cdot P(x) = Q(x)$ is Non-Homogeneous if $Q(x) \neq 0$

Linearly Independent | Linearly Dependent fu^n →

Two fu^n's f_1 and f_2 are called Linearly Dependent on Common Domain D if $\exists$ c_1, c_2 (Not Both Zero) s.t

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \forall\, x \in D$$

↑ linear Combination of f_1 and f_2

($0 = 0 \cdot f_1(x) + 0 \cdot f_2(x)$ $\forall x$ ← Trivial L.C

Requirement → Non-Trivial L.C)

Otherwise Linearly Independent.

e.g

$f_1(x) = x$, $f_2(x) = |x|$ defined on $\mathbb{R}$

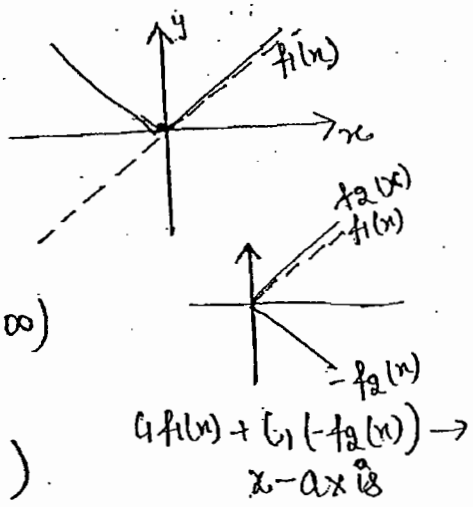
On $[0, \infty)$

$f_1(x) = x$, $f_2(x) = x$

$c_1 = -c_2 \neq 0$

$c_1 f_1(x) + c_2 f_2(x) = 0$

$\Rightarrow f_1$ & f_2 are L.D on $[0, \infty)$



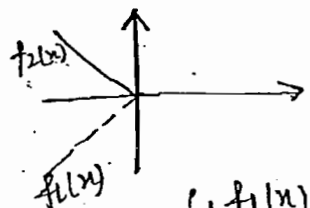
(L.D → If sum is x-axis)

$c_1 f_1(x) + c_1(-f_2(x)) \rightarrow$ x-axis

On $[-\infty, 0]$

$f_1(x) = x$

$f_2(x) = -x$



$c_1 f_1(x) + c_1(-f_1(x)) \Rightarrow$ x-axis

$\exists \; c_1 = c_2 \neq 0$

$c_1 f_1(x) + c_2 f_2(x) = 0$

$\Rightarrow f_1$ and f_2 are L.D on $[-\infty, 0]$

$\therefore$ On $\mathbb{R}^+$ & $(-\infty, 0) \rightarrow$ L.D

But On $\mathbb{R} \longrightarrow$ L.I

e.g. → $[-1, 1]$

Break → $[-1, 0]$, $[0, 1]$

Assume $c_1 f_1(x) + c_2 f_2(x) = 0 \quad \forall \; x \in [-1, 1]$

$\Rightarrow c_1 f_1(x) + c_2 f_2(x) = 0 \quad \forall \; x \in [-1, 0]$ if $c_1 = c_2$ $\Rightarrow$ L.D on $[-1, 0]$

&

$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \forall \; x \in [0, 1]$ if $c_1 = -c_2$ $\Rightarrow$ L.D on $[0, 1]$

On Whole interval Their L.C will be 0 if →

$$\left.\begin{array}{l} c_1 = c_2 \\ \& \\ c_1 = -c_2 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$$

$\therefore$ L.I on $[-1, 1]$

(In -ve interval $(-\infty, 0)$ → Two Same No's c_1 & c_2
+ve interval $(0, \infty)$ → $c_1 = -c_2$)

Que $f_1(x) = e^{mx}$ & $f_2(x) = e^{nx}$

If f_1 and f_2 are L.I then

✓ i) $m \neq n$

ii) $m = n$

✓ iii) only if $m \neq n$

iv) Can't say

Sol^n Let $c_1 e^{mx} + c_2 e^{nx} = 0$ with $c_1 \neq 0$ (let L.D)

$$c_1 e^{mx} = -c_2 e^{nx}$$

$$\frac{e^{mx}}{e^{nx}} = \frac{-c_2}{c_1}$$

$$e^{(m-n)x} = \frac{-c_2}{c_1}$$

If $m \neq n$ fu^n of x = Constant
Variable = Constant. Not Possible.

Hence $c_2 \neq 0$ Not Possible. $\Rightarrow$ If $m \neq n$ then L.I.

If $m = n$ $c_1 e^{mx} + c_2 e^{mx} = 0$

$\Rightarrow c_1 = -c_2 \neq 0 \Rightarrow$ L.D

e.g. e^x , e^{2x}

$c_1 e^x + c_2 e^{2x} = 0$

Let L.D with $c_1 \neq 0$

$$e^x = -\frac{c_2}{c_1} e^{2x}$$

$$e^{-x} = \frac{-c_2}{c_1} \Rightarrow \text{Variable} = \text{Constant}$$

Contradiction

$\therefore$ L.I

Note →

e^{mx} , e^{nx}

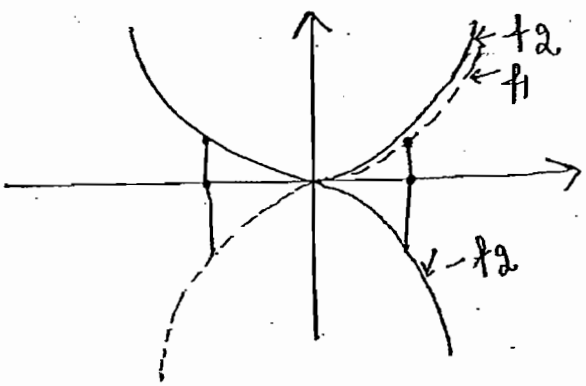
$\exists\ c_1 = -c_2 \neq 0$

s.t $c_1 e^{mx} + c_2 e^{nx} = 0$

$\Rightarrow$ Every fu^n is dependent on itself.

e.g. $x|x|$, x^2

Let $f_1(x) = x|x|$, $f_2(x) = x^2$



On $(-\infty, 0)$

$\exists\ c_1 = c_2 \neq 0$

$c_1 f_1(x) + c_2 f_2(x) = 0$

Clearly, their sum is x-axis

$\therefore$ L.D

On $(0, \infty)$

$\exists\ c_1 = -c_2$

$c_1 f_1(x) + c_2 f_2(x) = 0$

$c_1 f_1(x) + c_1 (-f_2(x)) = 0$

Clearly, again their sum is x-axis

$\therefore$ L.D

But on $\mathbb{R}$ L.I.

E.g. $f_1(x) = x f(x)$ $\quad f_2(x) = |x| f(x)$

like $f_1(x) = x \sin x$ $\quad f_2(x) = |x| \sin x$

$= x e^{3x}$ $\quad = |x| e^{3x}$

Linearly Independent → On that interval where both the x-re points are there.

a) f_1 & f_2 are L.I on $\mathbb{R}$

b) f_1 & f_2 are L.D on $(-\infty, 0]$

c) f_1 & f_2 are L.D on $[-\infty, a]$, $a > 0$

d) f_1 & f_2 are L.D on $[a, \infty)$, $a < 0$

Linear Diff eq^n of Higher Order →

Ex: → L·D·E with variable Coefficient

→ L·D·E with Constant Coefficient

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \text{---------} + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = Q(x) \quad \text{---(1)}$$

where $\boxed{a_0(x) \neq 0}$ $\forall x$ and

$a_0(x), a_1(x)$ ———— $a_n(x)$, $Q(x)$ are cts fu^{n}s

$$\left[a_0(x)D^n + a_1(x)D^{n-1} + \text{---} + a_{n-1}(x)D + a_n(x)\right]y = Q(x)$$

where $D = \frac{d}{dx}$

$$L(D)y = Q(x) \quad \text{where}$$

$$L(D) = a_0(x)D^n + a_1(x)D^{n-1} + \text{---} + a_{n-1}(x)D + a_n(x)$$

Note→ If $Q(x)=0$, then (1) is Homog. Diff eq^n.

If $Q(x)\neq 0$, then (1) is Non-Homog. Diff eq^n.

Linear Combination → Let f_1, f_2 ——— f_n are n-fu^{n}s defined on a common Domain D.

Then their linear Combination is given by →

$$\sum_{i=1}^{n} c_i f_i(x) \quad \text{where } c_i \text{ are constants}$$

And a l.c is called Convex Combination if $\sum_{i=1}^{n} c_i = 1$

Consider 2nd order L·D·E with variable Coefficient.

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = Q(x) \quad \text{—(1)}$$

Let y_1 and y_2 are 2 L·D sol^{n}s of eq^n (1) $\Rightarrow$

$$y_2(x) = c' y_1(x) \quad \text{—(2)}$$

Then $y(x) = c_1 y_1 + c_2 y_2$ (L·C of y_1 & y_2)

$y(x) = c_1 y_1(x) + c_2 c' y_1(x)$ (Using (2))

$= (c_1 + c_2 c') y_1(x)$

$= A y_1(x)$

No. of arb. Constant = 1, order = 2.

$\therefore$ $y(x)$ is not a General sol^n when y_1 and y_2 are L·D.

($y(x)$ will be a Gen. sol^n when y_1 and y_2 are L·I) then c_1 & c_2 can't be merged to 1.

Let y_1, y_2 and y_3 are 3 L·I sol^{n}s of eq^n (1)

Then $y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$

is again not a General sol^n when y_1, y_2, y_3 are L·D

($\because$ Arb. Constant $\to$ 3, Order $\to$ 2)

Note → Every linear eq^n of order n has exactly n linearly Independent solutions

And

Non-linear Diff eq^n has infinitely many L·I sol^{n}s. ($\frac{2}{3}\frac{dy}{dx} = y^{1/3}$ i.e L·I infinite sol^{n}s)

Let y_1 and y_2 are two (L.I) Sol^n of →

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)y = Q(x)$$

Then linear combination of y_1 and y_2 is sol^n of this diff eq^n i.e $y(x) = c_1 y_1(x) + c_2 y_2(x)$ if →

$$Q(x) = 0$$

or $c_1 + c_2 = 1$

$\because c_1 y_1 + (1-c_1) y_2$ ← Indep ✓ Nob

Sol^n →

Let y_1 and y_2 are two L.I sol^n of

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)y = Q(x)$$

$$\Rightarrow a_0(x)\,y_1'' + a_1(x)\,y_1' + a_2(x)\,y_1 = Q(x) \quad —(1)$$

$$\&\ a_0(x)\,y_2'' + a_1(x)\,y_2' + a_2(x)\,y_2 = Q(x) \quad —(2)$$

$y(x) = c_1 y_1(x) + c_2 y_2(x)$ is the sol^n if →

$$a_0(x)\left[c_1 y_1(x) + c_2 y_2(x)\right]'' + a_1(x)\left[c_1 y_1(x) + c_2 y_2(x)\right]' + a_2(x)\left[c_1 y_1(x) + c_2 y_2(x)\right] = Q(x)$$

$$c_1\left[a_0(x)\,y_1''(x) + a_1\,y_1'(x) + a_2(x)\,y_1(x)\right] + c_2\left[a_0(x)\,y_2''(x) + a_1(x)y_2'(x) + a_2(x)\,y_2(x)\right] = Q(x)$$

Using ① & ② →

$$c_1 Q(x) + c_2 Q(x) = Q(x)$$

$$(c_1 + c_2 - 1)\,Q(x) = 0$$

$\Rightarrow$ either $\boxed{Q(x) = 0}$

or $\boxed{c_1 + c_2 = 1}$

i.e
L.C of y_1 & y_2 is sol^n if →

$Q(x) = 0$ i.e Homog. Diff eq^n

or $c_1 + c_2 = 1$ → Convex Combination.

OR

the convex combination of any two linearly Independent sol^n of 2nd order L.D.E is also the sol^n of the Diff. eq^n.

Que → Let $f(x, y, y' \text{ ——— } y^{(n)}) = Q(x)$ be nth order L.D.E and y_1, y_2 ——— y_n are n L.I sol^n's

If $y(x) = \sum_{i=1}^{n} c_i y_i(x)$ also a sol^n. Then

a) If $Q(x) \neq 0 \Rightarrow \exists\ c_k < 1$ for some k

b) if $\forall i,\ c_i > 1 \Rightarrow Q(x) = 0$

c) if $0 < c_i$ and $Q(x) \neq 0 \Rightarrow c_i < 1\ \forall i$

d) None of these.

e) If $Q(x) \neq 0$ & $\exists\ c_k > 1 \Rightarrow \exists\ c_k < 0$.

Sol^n →

$y(x) = \sum_{i=1}^{n} c_i y_i(x)$ is also a sol^n.

$\Rightarrow$ either $Q(x) = 0$ or $\sum c_i = 1$

If $Q(x) \neq 0$

$\Rightarrow \sum c_i = 1$ (As $y(x)$ is the sol^n)

Clearly, atleast one c_k exist s.t $c_k < 1$ for some k

$\therefore$ a) is correct.

b) If $\forall i \; c_i > 1$

$\Rightarrow \sum c_i \neq 1$

$\Rightarrow Q(x) = 0$ (As $y(x)$ is the sol^n)

$\therefore$ b) is correct

c) If $0 < c_i$ and $Q(x) \neq 0$

$Q(x) \neq 0 \Rightarrow \sum c_i = 1$

and $c_i > 0$

$\Rightarrow c_i < 1 \; \forall i$

$\therefore$ c) is correct

e) If $Q(x) \neq 0$ and $\exists \; c_k > 1$

$Q(x) \neq 0 \Rightarrow \sum c_i = 1$

and $\exists \; c_k > 1$

$\Rightarrow \exists$ atleast one c_i s.t $c_i < 0$

(To make sum 1 with $c_k > 0 \Rightarrow \exists$ -ve c_i)

Que → Let $f(x, y, y', \ldots, y^n) = Q(x)$ be nth order L.D.E and $y_1, y_2, \ldots, y_n$ are n L.I Sol^{n}s.

Let $A = \{(c_1, c_2, \ldots, c_n) \mid c_i \in \mathbb{R}^+\}$

If $y(x) = \sum_{i=1}^{n} c_i y_i(x)$ is Sol^n for $(c_1, c_2, \ldots, c_n) \in A$

and $a_n = c^n$ where $c = c_k$ for some k.

a) If $Q(x) \neq 0 \Rightarrow a_n \to 0$ as $n \to \infty$

b) If $Q(x) \neq 0 \Rightarrow a_n \to 0$ or 1 as $n \to \infty$

c) If $Q(x) \neq 0 \Rightarrow a_n \to 1$ as $n \to \infty$

d) None of these

Sol^n → $y(x)$ is the Sol^n $\Rightarrow$ either $Q(x) \neq 0$ or $\sum c_i = 1$

As $Q(x) \neq 0$

$\sum c_i = 1$

$c_i \in \mathbb{R}^+ \Rightarrow c_i > 0$

$c_i > 0$ and $\sum c_i = 1 \Rightarrow 0 < c_i < 1$

$\Rightarrow c_k < 1$

$\Rightarrow c < 1 \quad (c = c_k)$

$\Rightarrow c^n \to 0$ as $n \to \infty$

$\Rightarrow a_n \to 0$ as $n \to \infty$

∴ a) is Correct

If $A = \{(c_1, c_2, \ldots, c_n) \mid c_i \in \mathbb{R}^+ \cup \{0\}\}$

$\Rightarrow c_i \geq 0$ and $\sum c_i = 1 \Rightarrow c_i \leq 1$?

Two Cases →

$c_i < 1$ for some i

Previous Case $\Rightarrow c_k < 1 \Rightarrow c_k^n \to 0$ as $n \to \infty$

$a_n \to 0$ as $n \to \infty$

$c_i = 1$ for some i (remaining zero)

$\Rightarrow c_k = 1$

$c_k^n \to 1$ as $n \to \infty$

$c^n \to 1$ as $n \to \infty$

$a_n \to 1$ as $n \to \infty$

∴ If $Q(x) \neq 0 \Rightarrow$

$a_n \to 0$ or 1 as $n \to \infty$

∴ In this case option b) is correct

Homogeneous linear Diff eq^n →

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0, \quad \text{—— (1)} \qquad a_0(x) \neq 0$$

Let y_1 and y_2 be any sol^n →

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$ is called

Wronskian of y_1 and y_2

$$= y_1 y_2' - y_1' y_2$$

Note → Wronskian is a differentiable fu^n & hence cts.

(∵ y_1, y_2 sol^n of (1) ∴ y_1', y_1'', y_2', y_2'' → exist

$\Rightarrow y_1, y_2, y_1', y_2'$ → differentiable

$\Rightarrow y_1 y_2' - y_1' y_2$ → diff. & hence cts)

As $W = y_1 y_2' - y_1' y_2$

$W' = y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2$

$= y_1 y_2'' - y_1'' y_2$ ✓

As y_1 and y_2 are 2 sol^n of ① →

$y_2 \times [a_0(x) y_1'' + a_1(x) y_1' + a_2(x) y_1] = 0$

$y_1 \times [a_0(x) y_2'' + a_1(x) y_2' + a_2(x) y_2] = 0$

On subtracting →

$a_0(x)[y_1'' y_2 - y_1 y_2''] + a_1(x)[y_2 y_1' - y_1 y_2'] = 0$

$- a_0(x) W'(x) + a_1(x)(-W) = 0$ ——— ②

$\Rightarrow \frac{W'}{W} = -\frac{a_1(x)}{a_0(x)}$, as $a_0(x) \neq 0$

On Integrating →

$\log W = -\int \frac{a_1(x)}{a_0(x)} dx + \log A$

Abel's formula → $\boxed{W = A\, e^{-\int \frac{a_1(x)}{a_0(x)} dx}}$

If $A = 0$, $W \equiv 0$ $\left(e^{-\int \frac{a_1(x)}{a_0(x)} dx} \neq 0\right)$

$A \neq 0$, W never vanishes

From (2) →

W is sol^n of

$$\boxed{a_0(x)\,y' + a_1(x)\,y = 0}$$

and

If y_1 and y_2 are two sol^n of diff eq^n (1) then their W is either identically zero or never vanishes.

Conclusions →

(1) If $W(y_1,y_2)_{x=x_0} = 0 \Rightarrow W(y_1,y_2) \equiv 0$

(2) If $W(y_1,y_2)_{x=x_0} \neq 0 \Rightarrow W(y_1,y_2) \neq 0 \ \forall x$

(3) If $W(y_1,y_2)_{x=x_0} = \alpha \neq 0$

a) If $\alpha > 0$

$\Rightarrow W(y_1,y_2) > 0 \quad \forall x$

(i.e If W at $x=x_0$ is +ve then: W is alwz +ve $\forall x$)

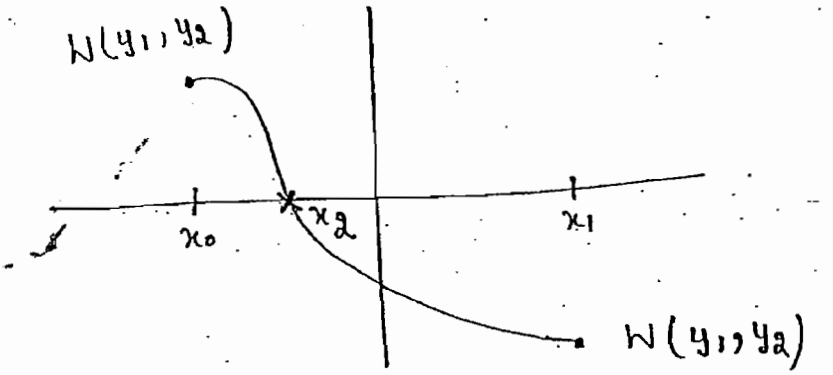
b) If $\alpha < 0$

$\Rightarrow W(y_1,y_2) < 0 \quad \forall x$

Suppose $W(y_1,y_2)_{x=x_0} = \alpha \neq 0$.

Say $\alpha > 0$ & $W(y_1,y_2)_{x=x_1} = \beta \neq 0$

Say $\beta < 0$



As Wronskian → Differentiable fu^n &

Hence cts.

There will be no break in

line joining $W(y_1,y_2)_{x=x_0}$ &

$W(y_1,y_2)_{x=x_1}$

and it will cut x-axis at pt. x_2.

i.e $W(y_1,y_2)_{x=x_2} = 0$

$\Rightarrow$ $W(y_1,y_2) \equiv 0$ (using (1))

These all results are valid when $a_0(x) \neq 0 \ \forall x$

Result →

If $\boxed{W(y_1, y_2) = 0}$

$\Rightarrow y_1 y_2' - y_2 y_1' = 0$

$y_1 y_2' = y_2 y_1'$

$\frac{y_2'}{y_2} = \frac{y_1'}{y_1}$

On Integrating →

$\log y_2 = \log y_1 + \log c$

$\Rightarrow y_2 = c y_1$

$\Rightarrow \boxed{y_1 \& y_2 \text{ are L.D}}$

Conversely, →

If $y_1 \& y_2$ are L.D

$y_2 = c y_1$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & cy_1 \\ y_1' & cy_1' \end{vmatrix}$$

$= 0$

$\therefore$

$\boxed{W(y_1, y_2) = 0 \iff y_1 \& y_2 \text{ are L.D}}$

Let y_1 and y_2 are two sol's of

$$a_0(x)\,y'' + a_1(x)\,y' + a_2\,y = 0\,,$$

$a_0(x) \neq 0$, $a_0(x), a_1(x), a_2(x)$ are cts fu's (Integrand is ∫ when it is cts.)

$\left(\Rightarrow \frac{a_1(x)}{a_0(x)} \to \text{cts} \Rightarrow \int \frac{a_1}{a_0} \to \text{possible}\right)$

Then →

(1) y_1 and y_2 are L·I $\Leftrightarrow \exists\, x_0$ s·t $W(y_1, y_2)_{x=x_0} \neq 0$

or

y_1 and y_2 are L·I $\Leftrightarrow \forall\, x$, $W(y_1, y_2) \neq 0$

(2) y_1 and y_2 are L·D $\Leftrightarrow \exists\, x_0$ s·t $W(y_1, y_2)_{x=x_0} = 0$

or

y_1 and y_2 are L·D $\Leftrightarrow \forall\, x$, $W(y_1, y_2) = 0$

Now

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Case I → $\boxed{y_1(x_0) = 0 = y_2(x_0)}$

i·e x_0 is common zero of y_1 and y_2.

$$W(y_1, y_2)_{x=x_0} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$$

$\Rightarrow \boxed{y_1 \text{ and } y_2 \text{ are L·D}}$

Conversely →

If y_1 and y_2 are L.D

$\Rightarrow y_2 = c y_1$

If $y_1(x_0) = 0$

$\Rightarrow y_2(x_0) = c\, y_1(x_0)$

$= 0$

$\Rightarrow x_0$ is the common zero of y_1 & y_2.

Case II →

$y_1(x_0) = 0 \;,\; y_1'(x_0) = 0$ or $y_2(x_0) = 0 \;,\; y_2'(x_0) = 0$

$\Downarrow$

$x_0 \rightarrow$ repeated root

$$W(y_1, y_2)_{x=x_0} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & y_2(x_0) \\ 0 & y_2'(x_0) \end{vmatrix}$$

$= 0$

$\Rightarrow$ y_1 and y_2 are L.D

Conclusion → Let y_1 & y_2 are 2 sol^n of the 2nd order linear diff eq^n.

If either y_1 or y_2 has repeated root

$\Rightarrow y_1$ and y_2 are L.D

If $y_1(x_0) = 0$ and $y_1'(x_0) = 0 \Rightarrow x_0$ is the repeated root

Proof →

$y_1(x) = (x-x_0)\, u(x)$, $u(x_0) \neq 0$

($\because y_1(x_0) = 0 \Rightarrow x_0$ is the root of y_1)

$y_1'(x) = u(x) + (x-x_0)\, u'(x)$

As $y_1'(x_0) = 0$

$\Rightarrow u(x_0) + 0 \cdot u'(x_0) = 0$

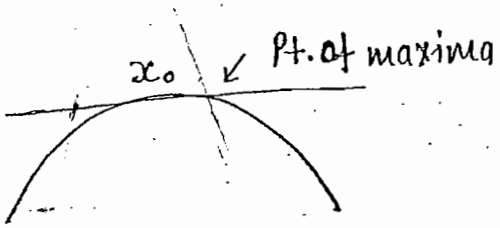
$\Rightarrow u(x_0) = 0$

$\Rightarrow u(x) = (x-x_0)\, s(x)$

$\Rightarrow y_1(x) = (x-x_0)^2 s(x)$

$\Rightarrow x_0$ is the repeated root.

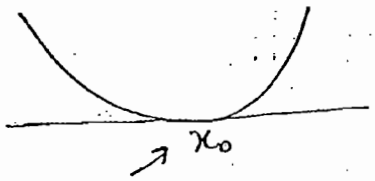
Case III → If $y_1'(x_0) = y_2'(x_0) = 0$



Derivative → slope

$\Rightarrow$ Slope = 0

$\Rightarrow$ tangent at pt x_0 is $\parallel$ to x-axis.



Pt. of minima

$\Rightarrow x_0 \rightarrow$ pt. of extrema → either maxima or minima

$$W(y_1, y_2)_{x=x_0} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ 0 & 0 \end{vmatrix}$$

$= 0 \Rightarrow y_1$ & y_2 are L.D

i.e. **If y_1 and y_2 has common point of extrema then Sol^{n}s are L.D.**

In case of Local Maxima also
As derivative at that pt. is also 0.

Que → Let y_1 and y_2 are 2 Sol^n of →

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0, \qquad a_0(x) \neq 0$$

s.t $y_1(0) = 0$, $y_2(0) = 1$

$y_1'(0) = 1$, $y_2'(0) = 1$

Then

a) y_1 and y_2 has common zero

b) y_1 & y_2 has common point of extrema.

✓c) neither y_1 nor y_2 has repeated root.

d) none of these

Sol^n →

$$W(y_1, y_2)_{at\,x=0} = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = (-1)(0) - 1 = -1 \neq 0$$

$\Rightarrow$ Sol^{n}s y_1 & y_2 are L.I.

$\Rightarrow$ y_1 & y_2 has no common zero ($p \Rightarrow q$, Not $q \Rightarrow$ Not p)

& y_1 & y_2 has no common pt of extrema ✓

∴ option c) is correct.

If either y_1 or y_2 has repeated root ⇒ y_1 & y_2 are L.D

y_1 & y_2 are L.I ⇒ Neither y_1 nor y_2 has repeated root.

Que.→ let y_1 & y_2 are 2 Sol^n of →

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad , \underline{a_0(x) \neq 0}$$

s.t $y_1(0) = 0$, $y_2(0) = 1$

$y_1'(0) = 1$, $y_2'(0) = -1$

If $y_2(x) \neq 0 \ \forall x$ and $g(x) = \dfrac{y_1(x)}{y_2(x)}$

a) $g(x)$ is ↑ i.e strictly inc.

b) $g(x)$ is ↓ i.e strictly dec.

c) $g(x)$ is Constant Fu^n.

d) $g(x)$ has a point of extrema.

Sol^n→

Use → $f(x) \uparrow$, $f'(x) > 0$

$f(x) \downarrow$, $f'(x) < 0$

$$W(y_1, y_2)_{at\ x=0} = -1$$

Throughout

$$g'(x) = \frac{y_2 y_1' - y_1 y_2'}{y_2^2} = -\frac{W(y_1, y_2)}{y_2^2}$$

$$= \frac{1}{y_2^2} > 0$$

$$\Rightarrow g'(x) > 0 \Rightarrow g(x) \text{ is } \uparrow$$

For extrema →

$g'(x) = 0$ at any point

But $g'(x) > 0$

∴ option (d) is incorrect

As $g'(x) > 0 \Rightarrow g(x)$ is not a constant fu^n.

$\Rightarrow$ option (c) is incorrect.

Que. 2014 June

Let y_1 and y_2 form a Complete Set (Fundamental Set) → L.I Sol^{n}s of Sol^n of $y'' - 2xy' + \sin(e^{2x^2})\, y = 0 \quad \forall x \in [0,1]$

$y_1(0) = 0 \qquad y_2(0) = 1$

$y_1'(0) = 1 \qquad y_2'(0) = 0$

Then $W(x)$ of y_1 and y_2 at $x = 1$ →

a) e^2

✓ b) $-e$

c) $-e^2$

d) e

Sol^n: first chk → $a_0(x) \neq 0$ $\quad (1 \neq 0)$

$$W(y_1, y_2)_{at\, x=0} = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

By Abel's formula →

$$W(x) = A\, e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}$$

$$W(x) = A\, e^{-\int \frac{-2x}{1}\,dx}$$

$$= A\, e^{\int 2x\,dx} = \underline{A\, e^{x^2}}$$

Use → $W(0) = -1$

$-1 = A e^0 \Rightarrow A = -1$

$\Rightarrow W = -e^{x^2}$ — (1)

Find $W(y_1, y_2)_{at\ x=1} = ?$

$W(1) = -e$ (Put $x=1$ in (1))

Or:

Discard Method → $W(0) = -1$ i.e. −ve

∴ W is −ve throughout

∴ Directly discard a) and d) options.

Que MCQ 2014 June

Let y_1 and y_2 are two sol^n defined on $[0,1]$

$$y'' + y' + y = 0$$

$W(x)$ of y_1 & y_2 s.t $W\left(\frac{1}{2}\right) = 0$

Then

✓ a) $W(x) = 0 \quad \forall\ x \in [0,1]$

b) $W(x) \neq 0 \quad \forall\ x \in [0, \frac{1}{2}]$

c) $W(x) > 0 \quad \forall\ x \in [\frac{1}{2}, 1]$

d) $W(x) < 0 \quad \forall\ x \in [0, \frac{1}{2}]$

Sol^n → Chk → $a_0(x) \neq 0$ $\quad (1 \neq 0)$

$W\left(\frac{1}{2}\right) = 0$

$\Rightarrow W(x) \equiv 0 \quad \forall x \in [0,1]$

Que Let y_1 and y_2 are two L.I sol^n of

$$xy'' - 2x^2y' + e^x y = 0$$

s.t $y_1(0) = 1$, $y_1'(0) = 1$

$y_2(0) = -1$, $y_2'(0) = 1$

Find $W(2)$?

a) $2e^{-1}$
b) $2e^{-2}$
c) $2e^{4}$ ✓
d) $2e^{-4}$

Sol^n →

$a_0(x) = x$

$a_0(0) = 0$ B.t $W(0) \neq 0$

$$W(0) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2.$$

$$W(x) = A e^{-\int \frac{-2x^2}{x} dx}$$

$$W(x) = A e^{x^2}$$

$$W(0) = 2 = A \quad (x \neq 0)$$

$$W(2) = A e^4 = 2e^4.$$

Que. Let u and v are two sol^n of

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{defined on } [a,b]$$

$a_0(x) = 1 \neq 0$

✓ a) $W(u,v)_{x=x_0} = 0$

$\Rightarrow u$ & v are L.D , $x_0 \in [a,b]$

✓ b) $W(u,v) \equiv 0 \Rightarrow u$ & v are L.D

c) $W(u,v) = 0$ for some $x_0 \not\Rightarrow W(u,v) \equiv 0$

d) $W(u,v) \neq 0$ at $x = x_0$ but u & v are L.D.

In Case of Single Correct , b) is more appropriate.

Que. $y_1(x) = x$, $y_2(x) = xe^x$

are two sol^n of

$$x^2y'' - x(x+2)y' + (x+2)y = 0$$

then the largest interval on which y_1 & y_2 are L.I →

a) $x > 0$ or $x < 0$

b) $-1 < x < \infty$

c) $-1 < x < 2$

d) $-\infty < x < \infty$

Sol^{n} Let y_1, y_2 are L.D

$$c_1 x + c_2 x e^x = 0 \quad \forall x$$

$$\Rightarrow x(c_1 + c_2 e^x) = 0 \quad \forall x \qquad x \neq 0$$

$$c_1 + c_2 e^x = 0$$

Variable $\longrightarrow$ $e^{-x} = -\frac{c_2}{c_1}$ $\leftarrow$ Constant

Contradiction

$\Rightarrow c_1 = c_2 = 0$

$\Rightarrow$ Throughout L.I.

or $a_0(x) = x^2$, $a_0(0) = 0$ i.e at $x=0$, $a_0(x) = 0$

Exclude zero's of $a_0(x)$ i.e $x=0$

$$W(y_1, y_2) = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix}$$

$$= xe^x + x^2 e^x - xe^x$$

$$= x^2 e^x \neq 0 \quad \forall x \neq 0$$

$\Rightarrow$ L.I for $\forall x \neq 0$

Then same behaviour at zeroes of $a_0(x)$ i.e at $x=0$

i.e y_1 & y_2 are L.I at $x=0$

$\Rightarrow$ Throughout L.I.

$\therefore$ $-\infty < x < \infty$ is Correct.

Result →

Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$,

$a_0(x), a_1(x), a_2(x)$ are cts fn^{n} on I

If y_1 and y_2 are two sol^{n}s. Then 非 →

(1) If $W(y_1, y_2) \neq 0$ for some $x_0 \in I$ s.t $a_0(x_0) \neq 0$

$\Leftrightarrow$ y_1 and y_2 are L.I $\forall x \in I$

(i.e on the set pts also where $a_0(x)=0$)

(2) If $W(y_1, y_2) = 0$ for some $x_0 \in I$ s.t $a_0(x) \neq 0$

$\Leftrightarrow$ y_1 and y_2 are L.D.

Que → If y_1 and y_2 are L.I sol^{n} of →

$$y'' + xy' + (1-x^2)y = y \sin x$$

Then which of the following are also sol^{n} →

- ✓ a) $y_1 + y_2$
- ✓ b) $y_1 - y_2$
- ✓ c) $2y_1 - y_2$
- ✓ d) $y_1 - 2y_2$

L.I?

Sol^{n} →

$$y'' + xy' + [(1-x^2) - \sin x]y = 0$$

Here $Q(x) = 0$ i.e Homog eq^{n}.

∴ Every L.C is the sol^{n}.

(If $Q(x) \neq 0$ then c) is the correct ans ∵ $\sum c_i = 1$)

Que → Maximum no. of L.I Sol^n of →

$$\frac{d^4y}{dx^4} = 0 \quad , y(0) = 1$$

Ans. If $y(0) = 1$ Not given

Then exactly 4 L.I Sol^{n}s.

Now,

$$\frac{d^4y}{dx^4} = 0$$

$$\frac{d^3y}{dx^3} = A$$

$$\frac{d^2y}{dx^2} = Ax + C$$

$$\frac{dy}{dx} = \frac{Ax^2}{2} + Cx + D$$

$$y = \frac{Ax^3}{6} + \frac{Cx^2}{2} + Dx + E$$

$y(0) = 1$

$\Rightarrow 1 = E$

$\therefore \quad y = \frac{Ax^3}{6} + \frac{Cx^2}{2} + Dx + 1$

$y_1(x) = x^3$

$y_2(x) = x^2$

$y_3(x) = x$

$y_4(x) = 1$

$\therefore$ 4 L.I Sol^{n}s.

x^m L.I to $x^n \Leftrightarrow m \neq n$

If $y(0) = 0$

$\Rightarrow E = 0$

$$y(x) = \frac{Ax^3}{6} + \frac{Bx^2}{2} + Dx$$

$y_1(x) = x^3$

$y_2(x) = x^2$

$y_3(x) = x$

$\therefore$ 3 L.I Sol^{n}s.

Que y_1 and y_2 are L.I Sol^n of $\rightarrow$

$y'' + (\sin x)\, y = 0 \quad \forall x \in [0,1]$

$g(x) = W(y_1, y_2)$

a) $g'(x) > 0 \quad \forall x \in [0,1]$

b) $g'(x) < 0 \quad \forall x \in [0,1]$

c) $g'(x) = 0$ for exactly one $x \in [0,1]$

d) $g'(x) \equiv 0$

Sol^n $\rightarrow$ y_1 & $y_2 \rightarrow$ L.I Sol^{n}s.

$W(y_1, y_2) \neq 0$.

$$W(x) = A e^{-\int \frac{a_1}{a_0} dx} = A e^{-\int 0\, dx} = A$$

$\Rightarrow W(x) = A \quad \forall x.$

$\Rightarrow W' = 0$

$\Rightarrow W' \equiv 0.$

$\Rightarrow g'(x) \equiv 0$

> In 2nd order linear Diff eq^n $\rightarrow$
> Coff. of y' $\rightarrow$ Missing i.e a_1
> Then W = Constant

Case III

Note →

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad , a_0(x) \neq 0$$

If $a_1(x) \equiv 0$

$\Rightarrow W(y_1, y_2) = \text{constant}$.

Result proved for 2nd order linear Diff. eq^n are valid for the nth order linear Diff. eq^n.

Second order linear Diff. eq^n with Constant coefficient →

$$a_1y'' + a_2y' + a_3y = Q(x)$$

First consider Homog. case i.e $Q(x) = 0$

$$a_1y'' + a_2y' + a_3y = 0 \quad , a_1 \neq 0$$

(If $a_1 = 0$ first order)

$$\Rightarrow y'' + ay' + by = 0 \quad \text{—(1)} \quad , a = \frac{a_2}{a_1} \quad , b = \frac{a_3}{a_1}$$

Consider $y(x) = e^{mx}$

(chk on what conditions $y(x)$ is the sol^n)

$$y' = me^{mx}$$

$$y'' = m^2e^{mx}$$

$$(m^2 + am + b)e^{mx} = 0$$

$$\Rightarrow m^2 + am + b = 0 \quad \text{—(2)}$$

Let $a, b \in \mathbb{R}$

$\therefore$ eq^n ② has atmost 2 roots

Say μ_1 and μ_2

Case I → **When μ_1 and μ_2 are distinct i.e $\mu_1 \neq \mu_2$**

$\Rightarrow$ $e^{\mu_1 x}$ and $e^{\mu_2 x}$ are L.I sol^{n}s. (e^{mx} & e^{nx} are L.I when $m \neq n$)

($\because$ their L.C is also a sol^n)

$\therefore$ $y(x) = Ae^{\mu_1 x} + Be^{\mu_2 x}$ is the sol^n of eq^n ① (As $Q(x)=0$) chk?

or.

$(D^2 + aD + b)y = 0$

$(D-\mu_1)(D-\mu_2)y = 0$ (As μ_1 & μ_2 are the 2 distinct roots)

Let $(D-\mu_2)y = u$

$(D-\mu_1)u = 0$

$\Rightarrow u(x) = c_1 e^{\mu_1 x}$

($\frac{du}{dx} - \mu_1 u = 0$, I.F $= e^{-\int \mu_1 dx} = e^{-\mu_1 x}$

sol^n → $y(x)e^{-\mu_1 x} = c$, $u = c_1 e^{\mu_1 x}$)

$\therefore$ $(D-\mu_2)y = c_1 e^{\mu_1 x}$

I.F → $e^{-\int \mu_2 dx} = e^{-\mu_2 x}$

sol^n is →

$y(x) \cdot e^{-\mu_2 x} = c_1 \int e^{(\mu_1 - \mu_2)x} dx + c_2$

$y(x) \cdot e^{-\mu_2 x} = c_1 \frac{e^{(\mu_1-\mu_2)x}}{(\mu_1-\mu_2)} + c_2$

$y(x) = \frac{c_1 e^{\mu_1 x}}{(\mu_1-\mu_2)} + c_2 e^{\mu_2 x}$ $\Rightarrow$ $y(x) = Ae^{\mu_1 x} + Be^{\mu_2 x}$

sol^n when roots are distinct

Case II → When $m_1 = m_2 = m$

$$(D^2 + aD + b)y = 0$$

$$(D-m)^2 y = 0$$

Put $(D-m)y = u$

$$\therefore (D-m)u = 0$$

$$\Rightarrow u = ce^{mx}$$

$$(D-m)y = ce^{mx}$$

$$I.F = e^{-mx}$$

$\therefore$ Sol^n →

$$y \cdot e^{-mx} = \int c\,dx + c_1$$

$$y = (cx + c_1)e^{mx}$$ ← In case of repeated root this is the Sol^n

$m^2 + am + b = 0$ ← Auxillary Eq^n. Roots when are real i.e $m_1, m_2 \in \mathbb{R}$

I If roots are distinct $m_1 \neq m_2$

then $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

II If roots are same $m_1 = m_2 = m$

then $y(x) = (c_1 x + c_2)e^{mx}$

are the real valued Sol^n.

Case III → $a, b \in \mathbb{R}$ and Roots are complex

(As complex roots occur in pair when $a, b \in \mathbb{R}$)

distinct $\begin{cases} m_1 = \alpha + i\beta \\ m_2 = \alpha - i\beta \end{cases}$

$$y(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2[\cos\beta x - i\sin\beta x]\right]$$

$$= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right]$$

$$= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right] \rightarrow \text{Complex valued sol}^n$$

Que → Find real sol^n when roots are complex.

Put $B = 0$.

i.e for the real valued sol^n coefficient of $\sin x$ must be zero.

$y'' + ay' + b = 0$, $y(x_0) = y_0$, $y'(x_0) = y_1$, $a, b \in \mathbb{R}$?

Find conditions on a & b s.t General Sol^n is bounded ?

Now → Auxiliary eq^n →

$$m^2 + am + b = (m - m_1)(m - m_2)$$
$$= m^2 - (m_1 + m_2)m + m_1 m_2$$

Sum of roots $= m_1 + m_2 = -a \left[= 2\alpha\right]$

Product of roots $= m_1 m_2 = b \left[= \alpha^2 + \beta^2\right]$

(In case of complex roots → $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$)

(Discussing the case → When $m_1, m_2 \neq 0$ & +ve)

Case I When $a > 0$, $b > 0$.

As $a > 0 \Rightarrow -a < 0 \Rightarrow m_1 + m_2 < 0$

& $m_1 m_2 > 0$ (As $b > 0$)

> $b = \alpha^2 + \beta^2$ i.e Sum of 2 +ve Quantities
> $\therefore b > 0$ for complex roots
> If $b < 0$ or $b = 0$ No complex roots

$\therefore m_1 + m_2 < 0$ and $m_1 m_2 > 0$

$\Rightarrow m_1 < 0$ and $m_2 < 0$

$\Rightarrow$ Roots may be real or complex.

Complex roots when $\alpha < 0$

(i) When roots are real and distinct →

a) $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 (x)}$

As m_1 and $m_2 < 0$

$\Rightarrow y(x) \longrightarrow 0$ as $x \to \infty$

~~Sol^n is bounded~~

And $y(x) \to \infty$ when $x \to -\infty$

$\Rightarrow$ **Sol^n is Unbounded.**

b) When roots are real & same

$y(x) = (c_1 + c_2 x)\, e^{\mu_1 x}$ $\quad \mu_1 = \mu_2 = \mu < 0$

$\Rightarrow y(x) \to 0$ as $x \to \infty$

(Two fu^n $\to e^{rx} \to 0$ as $x \to \infty$ & $xe^{rx} = \frac{x}{e^{-rx}}$ $\quad -r > 0$

By L'H rule $\to$ $\frac{1}{-re^{-rx}} = \frac{e^{rx}}{-r} \to 0$ as $x \to \infty$)

And $y(x) \to \infty$ as $x \to -\infty$

$\Rightarrow$ **Sol^n is Unbounded.**

c) When roots are Complex

$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$\alpha < 0$

$y(x) \to 0$ as $x \to \infty$

$y(x) \to \infty$ as $x \to -\infty$

$\Rightarrow$ **Sol^n is Unbounded**

Case II When $a < 0$, $b > 0$

$a < 0 \Rightarrow \mu_1 + \mu_2 > 0$ & $\alpha > 0$

$b > 0 \Rightarrow \mu_1 \mu_2 > 0$

$\Rightarrow$ Complex roots When $\alpha > 0$.

$\mu_1 + \mu_2 > 0$ & $\mu_1 \mu_2 > 0 \Rightarrow \mu_1 > 0, \mu_2 > 0$

a) When roots are real & distinct $\to$

$y(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x}$

When $x \to \infty$ $\quad y(x) \to \infty$

$x \to -\infty$ $\quad y(x) \to 0$

$\Rightarrow$ **Sol^n is Unbounded**

b) When roots are real & same →

$$y(x) = (c_1 + c_2 x)e^{rx} \qquad r_1 = r_2 = r > 0$$

When $x \to \infty$ $\quad y(x) \to \infty$

$\Rightarrow$ **Sol^n is Unbounded**

(Unbounded if ∞ is obtained, No need to chk at others pt.)

c) When roots are Complex →

$$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$\alpha > 0$

When $x \to \infty$ $\quad y(x) \to \infty$

$\Rightarrow$ **Sol^n is Unbounded**

Case III When $a < 0$, $b < 0$

$b < 0 \Rightarrow$ No Complex roots

$r_1 + r_2 = -a > 0 \qquad \Rightarrow r_1 > -r_2$ (take)

$r_1 r_2 < 0$

$r_1 > -r_2$ & $r_1 r_2 < 0$

Take $r_1 > 0$ & $r_2 < 0$. $\quad r_1 > r_2$

$\Rightarrow r_2 > 0$ & $r_1 < 0$ $\quad r_2 < r_1$

$r_1 > r_2$, $r_2 < r_1$

$r_1 = 5, \ r_2 = -2 \quad 5 > 2, \ -10 < 0$

$r_1 = -2, \ r_2 = -3 \quad -6 < 0$

$-2 > -3$

$-8 > -5$

~~a) When~~ roots are real & distinct

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$y(x) \to \infty$ as $x \to \infty$

$\Rightarrow$ **Sol^n is Unbounded**

(I) When roots are real & same →

~~$y(x) = (c_1 + c_2 x)$~~

Not Possible as $r_1 > 0$ and $r_2 < 0$.

Case IV → When $a > 0$, $b < 0$

$b < 0 \Rightarrow$ No Complex roots

$r_1 + r_2 = -a < 0$

$r_1 r_2 = b < 0$

Take $r_1 > 0$ and $r_2 < 0$

Clearly roots are real & distinct

$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

$y(x) \to \infty$ as $x \to \infty$

$\Rightarrow$ Sol^n is Unbounded

Conclusion →

When $a, b \neq 0$

Sol^n is Unbounded.

When roots are real & equal → Sol^n is Never Bounded

" & distinct → "

Bounded Sol^n is possible only when →

$$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

When $\alpha \neq 0$ $\begin{cases} \alpha < 0 & \text{As } x \to -\infty \quad y(x) \to \infty \\ \alpha > 0 & \text{As } x \to \infty \quad y(x) \to \infty \end{cases}$

$\Rightarrow$ Sol^n is Unbounded.

When $\alpha = 0$ $\quad (m_1 = i\beta, m_2 = -i\beta \quad m_1 m_2 = \beta^2 = b > 0)$

$a = 2\alpha \Rightarrow \boxed{a = 0}$.

& $b > 0$ (for complex roots as $y(x)$ is complex valued Sol^n.)

$y'' + ay' + b = 0$ i.e. $\boxed{y' \text{ term must be missing for the Bounded Sol}^n}$.

$\therefore$ Sol^n is bounded only when $a = 0$, $b > 0$

$$y'' + ay' + by = 0 \quad , a, b \in \mathbb{R}$$

i) Diff eq^n has bounded sol^n if $a = 0, b > 0$

ii) Diff eq^n has Unbounded Sol^n if

$a = 0$, $b \leq 0$

or $a \neq 0$, $b \leq 0$.

Find conditions on a & b when the diff eq^n has periodic sol^n?

As $e^{ix} \rightarrow$ Not Periodic

1) When $a>0, b>0$.

In all three possible cases $e^{\alpha x}$ term is there so not periodic

2) Similarly, for $a<0, b>0$

$a>0, b<0$

$a<0, b<0$

Periodic sol^n is possible when →

$$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

and $\alpha = 0$ i.e $\boxed{a=0}$

$$y(x) = A\cos\beta x + B\sin\beta x$$

and as $y(x)$ is complex valued sol^n $\therefore \boxed{b>0}$

$\therefore$ Diff eq^n has Periodic sol^n if

$$\boxed{\begin{array}{c} a=0 \\ \& \ b>0 \end{array}}$$

Que. Let y_1 and y_2 are two sol^n of →

$$\alpha y'' + ay' + by = 0 \quad , \quad \alpha \neq 0$$

s.t $y_1(\pi/2) = 0 \quad , \quad y_2(\pi/2) = 1$

$y_1'(\pi/2) = 1 \quad , \quad y_2'(\pi/2) = 0$

if $y(x) = c_1 y_1(x) + c_2 y_2(x)$ be a periodic sol^n then

a) $W(y_1, y_2) = 1$

b) $a = 0 \; , \; b = 1$

c) $a = 0 \; , \; b = 2$

d) $a = 0 \; , \; b = 1/2$

Sol^n As $y(x)$ is the periodic sol^n ∴ roots of A.E are imag. (Need Complex valued sol^n.)

$m_1 = \alpha' + i\beta$ with $a = 0$ & $b > 0$

$m_2 = \alpha' - i\beta$

$m_1 + m_2 = 2\alpha' = \frac{-a}{\alpha} \Rightarrow \alpha' = 0$

$m_1 = i\beta \; , \; m_2 = -i\beta$

$y(x) = A \cos\beta x + B \sin\beta x.$

∴ $y_1(x) = \cos\beta x \; , \; y_2(x) = \sin\beta x$ are 2 L.I sol^{n}s.

$$W(y_1, y_2)_{x=\pi/2} = \begin{vmatrix} y_1(\pi/2) & y_2(\pi/2) \\ y_1'(\pi/2) & y_2'(\pi/2) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$W = A e^{-\int \frac{a_1}{a_0} dx}$$

$W = A$ = Constant ($\because a_1 = a = 0$) Throughout.

$\Rightarrow W = 1$

$$W(y_1, y_2) = \begin{vmatrix} \cos\beta x & \sin\beta x \\ -\beta\sin\beta x & \beta\cos\beta x \end{vmatrix}$$

$$= \beta$$

$$\Rightarrow \beta = 1$$

$$\therefore \text{product of roots} = \beta^2 = \frac{b}{a}$$

$$\Rightarrow \frac{b}{a} = 1$$

$$\Rightarrow b = a > 0.$$

$$\Rightarrow W(y_1, y_2) = 1 \;,\; a = 0 \;,\; b = a > 0$$

$\therefore$ all 4 options are correct.

Que. Let $L(y) = y'' + ay' + by = 0$, $a, b \in \mathbb{R}$

If the Auxillary eq^n $L(m) = 0$ has a root $\alpha - i\beta$ then →

a) $\alpha = -a/2$, $\beta^2 = b - \frac{a^2}{4}$

b) If $\alpha < 0$, every sol^n $y(x) \to 0$ as $x \to \infty$

c) If $\alpha > 0$ then magnitude of every sol^n, $y(x) \to \infty$ as $x \to \infty$

d) None of these.

Sol^n A.E →

$$m^2 + am + b = 0$$

$$m_1 = \alpha - i\beta$$

As $a, b \in \mathbb{R}$

∴ Complex roots occur in pair

∴ $m_2 = \alpha + i\beta$

$m_1 + m_2 = -a$ & $m_1 m_2 = \alpha^2 + \beta^2$

$2\alpha = -a$ $\quad b = \frac{a^2}{4} + \beta^2$

$\alpha = -a/2$ $\quad \beta^2 = b - \frac{a^2}{4}$

Sol^n is given by →

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

If α is less than 0 i.e $\alpha < 0$

$e^{\alpha x} \to 0$ as $x \to \infty$

$y(x) \to 0 \times$ Bounded Value

$y(x) \to 0$

If $\alpha > 0$

$e^{\alpha x} \to \infty$ as $x \to \infty$

$c_1 \cos\beta x + c_2 \sin\beta x$ may be +ve or -ve

But we have to consider magnitude. ∴

$y(x) \to \infty \times$ +ve value

$y(x) \to \infty$

∴ Options a), b), c) are correct.

Fundamental Th^m of Linear Diff. eq^n

$a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \cdots\cdots + a_n(x)\,y = 0$ defined on I

where $a_0(x) \neq 0 \quad \forall x$

$a_0(x), a_1(x), \cdots\cdots a_n(x)$ are cts (If not cts we will not exist)

Then for $x_0 \in I$ and $\alpha_0, \alpha_1 \cdots\cdots \alpha_{n-1}$ (constants)

I $\exists$ unique sol^n $y(x)$ s.t

$$y(x_0) = \alpha_0$$
$$y'(x_0) = \alpha_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = \alpha_{n-1}$$

← Initial Value problem

If $\exists\ \alpha_i \neq 0 \Rightarrow$ Non-Trivial Unique sol^n

II If $\alpha_0 = \alpha_1 = \cdots\cdots = \alpha_{n-1} = 0$

$\Rightarrow$ diff eq^n has unique trivial sol^n

$y(x) \equiv 0$.

(Also true for nth order I.V.P)

Consider 2nd order Linear Diff eq^n →

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0 \quad , \text{ defined on } I$$

$$a_0(x) \neq 0$$

& $a_0(x)$, $a_1(x)$, $a_2(x)$ are cts.

$x_0 \in I$, s.t $y(x_0) = \alpha_0$

$$y'(x_0) = \alpha_1$$

Let y_1 and y_2 are two L.I ∴ remaining sol^{n}s are L.C of y_1 & y_2

$$\therefore y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(x_0) = \alpha_0 = c_1 y_1(x_0) + c_2 y_2(x_0)$$

$$y'(x_0) = \alpha_1 = c_1 y_1'(x_0) + c_2 y_2'(x_0)$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \quad — (1)$$

Non-Homogeneous System of eq^n.

Use → Consider System of Linear eq^{n}s

$$AX = B \qquad (X = A^{-1}B)$$

Non-Homog. → it has unique sol^n $\Leftrightarrow$ $|A| \neq 0$

and

$$AX = 0 \leftarrow \text{Homog.} \qquad (X = 0)$$

it has Unique $\Leftrightarrow$ $|A| \neq 0$

Trivial sol^n

(C) ∴ ① has Unique sol^n

As $\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1y_2' - y_2y_1' = W(y_1,y_2) \neq 0$ at $x=x_0$

(As the sol^{n}s are L.I)

$\Rightarrow$ System has Unique sol^n. (N-H case)

(Non-Trivial)

If $d_0 = d_1 = 0$ (Homog. case)

$\Rightarrow$ System has Unique Trivial sol^n.

Que: $y'' + \sin x^2 y' + \cos x^3 y = 0$ be second order linear Diff eq^n. Then diff eq^n has → $I = \mathbb{R}$

a) Unique trivial sol^n if $y(1)=0$, $y'(1)=0$

b) Unique Non-trivial sol^n if $y(-1)=0$, $y'(-1)=1$

c) Unique Non-Trivial sol^n if $y(1)=1$, $y'(1)=-1$

d) Two sol^n if $y(0)=y'(0)=1$

Sol^n a) $y(1)=0$, $y'(1)=0$ $\quad d_0=d_1=0$

$\Rightarrow$ Unique Trivial sol^n

b) $y(-1)=0$, $y'(-1)=1$

$d_0=0$, $d_1=1$

$\Rightarrow$ Unique Non-Trivial sol^n.

c) $y(1)=1$, $y'(1)=-1$

$d_0=1$, $d_1=-1$

$\Rightarrow$ Unique Non-Trivial sol^n

d) $y(0)=y'(0)=1$ $\Rightarrow$ $d_0=d_1=1$

$\Rightarrow$ Unique Non-Trivial sol^n.

Que Consider the diff eq^n →

$$y''' + 4xy'' + 2x^2y' + \sin xy = 0$$

s.t $y(0) = \alpha$

$y'(0) = \beta$

$y''(0) = \gamma$

Then

a) $\exists\ (\alpha, \beta, \gamma) \in \mathbb{R}^3$ s.t diff. eq^n has bounded sol^n.

b) $\exists\ (\alpha, \beta, \gamma) \in \mathbb{R}^3$ s.t diff eq^n has periodic sol^n

c) Every sol^n is bounded if $\alpha = 0 = \beta = \gamma$

d) Every sol^n is periodic if $\alpha = 0 = \beta = \gamma$

Sol^n →

Think for trivial sol^n always.

$y(x) \equiv 0$ is the sol^n of given Diff. eq^n.

$\exists\ (\alpha, \beta, \gamma) \in \mathbb{R}^3$ i.e $(0,0,0) \in \mathbb{R}^3$

s.t diff eq^n has bounded sol^n

& for $(0,0,0) \in \mathbb{R}^3$

$y(x) \equiv 0$ is the periodic sol^n. (Being constant)

for $\alpha = 0 = \beta = \gamma$

$y(x) \equiv 0$ is the only trivial sol^n of given diff eq^n which is bounded & periodic

Que. If $y'' - (x^2+1)y = 0 \quad \forall x \in \mathbb{R}$, $(q(x) < 0)$

If $y(x)$ be a Non-Trivial Sol^n of diff. eq^n s.t $y'(x_0) = \alpha$, $x_0 \in \mathbb{R}$, $\alpha \in \mathbb{R}$. Then

a) if $\alpha > 0 \Rightarrow y(x) \uparrow$ $(\because y'(x_0) > 0)$

b) if $\alpha < 0 \Rightarrow y(x) \downarrow$ $(\because y'(x_0) < 0)$

c) for any $x_0 \in \mathbb{R}$ either $\alpha > 0$ or $\alpha < 0$

d) $y(x)$ has atmost 1 zero. (i.e there does not exist any maxima or minima)

Case II → $y''(x) + q(x)y = 0$, $q(x) > 0 \quad \forall x > 0$

if $\int_{a=1}^{\infty} q(x)\,dx = \infty$, then

Conclusions →

i) Every Non-Trivial Sol^n has Infinite number of Zero's on the +ve x-axis

ii) Every Non-Trivial Sol^n is oscillatory (i.e Non-monotonic)

iii) Every Non-Trivial Sol^n has infinite number of point of extrema

Proof →

Let $y(x)$ be a Non-Trivial sol^n s.t $y(x)$ has finite numbers of zeroes.

$\Rightarrow \exists\ x_0 > 1\ (=a)$ s.t $y(x) \neq 0 \quad \forall\ x \geq x_0$? (Assumption)

Either $y(x) > 0$ or $y(x) < 0$

Let $y(x) > 0\ \forall\ x \geq x_0$

Define $q(x) = \dfrac{-y'(x)}{y(x)}$, $x \geq x_0$

$$q'(x) = -\frac{y'(x)\,y'' - (y'(x))^2}{[y(x)]^2}$$

$$= -\left[\left(\frac{y''}{y}\right) - \left(\frac{y'}{y}\right)^2\right]$$

$$= -\frac{y''}{y} + \left(\frac{y'}{y}\right)^2$$

As $y'' + Q(x)\,y = 0$

$$\frac{y''}{y} = -Q(x)$$

$$\Rightarrow q'(x) = Q(x) + [q(x)]^2$$

Integrating →

$$\int_{x_0}^{x} q'(x)\,dx = \int_{x_0}^{x} Q(x)\,dx + \int_{x_0}^{x} q^2(x)\,dx$$

$$q(x) = q(x_0) + \int_{x_0}^{x} Q(x)\,dx + \int_{x_0}^{x} q^2(x)\,dx$$

$q(x_0)$ ↓ finite value; $\int_{x_0}^{x} Q(x)\,dx$ ↑ ∞ (as $x \to \infty$); $\int_{x_0}^{x} q^2(x)\,dx$ ↑ +ve

and hence A is finite.)

A is Bounded $\Leftrightarrow$ B is bounded

d) A is Bounded
$\Rightarrow$ B is bounded
& hence finite

$\therefore$ All 4 options are correct.

(Isolated pt. $\rightarrow$ x_0 is Isolated pt. of if $(x_0-\delta, x_0+\delta) \cap A = \{x_0\}$, for $\delta > 0$

Limit pt. $\rightarrow$ $(x_0-\delta, x_0+\delta) \cap A - \{x_0\} = \phi$ $\forall \delta > 0$
$\Rightarrow$ x_0 is not a limit pt of A)

Isolated Pt. & Limit Pt. are opposite.

Conclusion of option c) $\rightarrow$

Let $y'' + Py' + Qy = 0$ be linear Differential equation. If y_1 and y_2 are two linearly Independent sol^n, then the set of zeros of y_1 and y_2 cannot be infinite bounded set.

i.e the set of zeros of y_1 & y_2 is bounded $\Leftrightarrow$ it is finite.

Que. Let $y'' + P(x)y' + Q(x)y = 0$ be linear Diff-eq^n. If y_1 & y_2 are 2 L.I sol^{n}s. Let

$$A = \{\alpha \in \mathbb{R} \mid y_1(\alpha) = 0\}$$
$$B = \{\beta \in \mathbb{R} \mid y_2(\beta) = 0\}$$

Let $\langle a_n \rangle$ be a sequence defined on A. $\rightarrow$

a) iff $\langle a_n \rangle \uparrow \Leftrightarrow \exists \langle b_n \rangle$ seq. on B s.t $\langle b_n \rangle \uparrow$

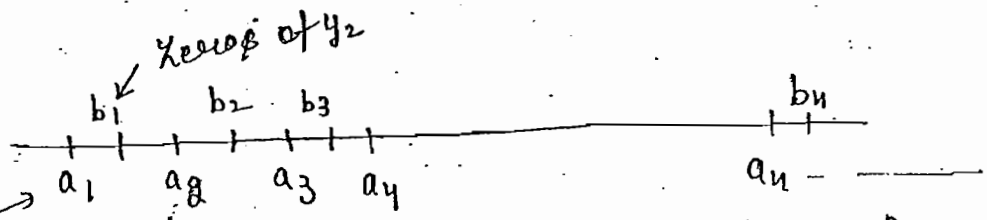
b) iff $\langle a_n \rangle \downarrow \Leftrightarrow \exists \langle b_n \rangle$ seq. on B s.t $\langle b_n \rangle \downarrow$

c) If Range $\langle a_n \rangle$ is infinite $\Rightarrow$ B is unbounded

d) If B is unbounded $\Rightarrow$ Range Set $\langle a_n \rangle$ is infinite.

Sol^n $\rightarrow$

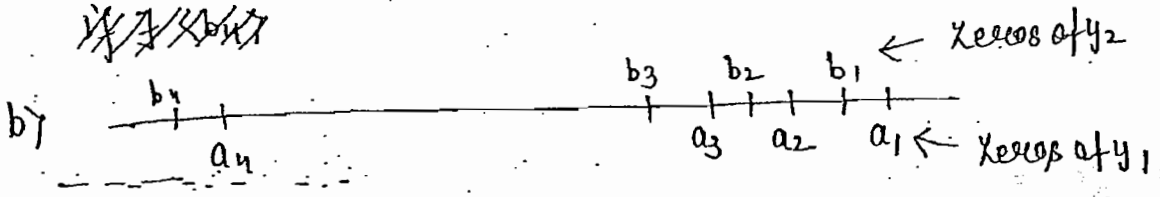
a)



Zeros of y_1

If $f\langle a_n \rangle \uparrow \Leftrightarrow f \langle b_n \rangle$ seq. on B s.t $\langle b_n \rangle \uparrow$

b)



If $f\langle a_n \rangle \downarrow \Leftrightarrow f\langle b_n \rangle$ seq on B s.t $\langle b_n \rangle \downarrow$

c) If Range $\langle a_n \rangle$ is infinite

$\Rightarrow$ A is infinite $\Rightarrow$ B is infinite

$\Rightarrow$ B is Unbounded ($\because$ Set of zeros cannot be infinite bounded)

d) If B is unbounded

$\Rightarrow$ B is infinite $\Rightarrow$ A is infinite

Choose Constant seq. from A.

i.e $a_n = \alpha \; \forall \; n$

Range set $= \{\alpha\}$ $\leftarrow$ finite

$\therefore$ d) option is wrong.

(e.g $\rightarrow$
$y'' + y = 0$
$y_1(x) = \sin x$
$y_2(x) = \cos x$
$B = \{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}$
$A = \{n\pi \mid n \in \mathbb{Z}\}$
$\langle a_n \rangle$ on A
s.t $a_n = n\pi \; \forall n$)

If d) option is ~~$\rightarrow$~~ If B is unbounded $\Rightarrow \exists$ Range Set $\langle a_n \rangle$ is infinite

If B is Unbounded

$\Rightarrow$ B is infinite $\Rightarrow$ A is infinite

$\exists$ range set $\langle a_n \rangle$ which is infinite

This is correct

Que: Let $y'' + P(x)y' + Q(x)y = 0$ be L.D.E
s.t that y_1 and y_2 are 2 L.I sol^{n}s then →

a) $P(x) = -\dfrac{y_2y_1'' - y_1y_2''}{W(y_1,y_2)}$, $Q(x) = \dfrac{y_2'y_1'' - y_1'y_2''}{W(y_1,y_2)}$

b) $P(x) = \dfrac{y_1y_2'' - y_2y_1''}{W(y_1,y_2)}$, $Q(x) = \dfrac{y_2'y_1'' - y_1'y_2''}{W(y_1,y_2)}$

c) $P(x) = \dfrac{y_1y_2'' - y_2'y_1''}{W(y_1,y_2)}$, $Q(x) = \dfrac{y_1y_2'' - y_2y_1''}{W(y_1,y_2)}$

✓ d) None of these.

Sol^n: As y_1 & y_2 are 2 L.I sol^{n}s ⇒ $W(y_1,y_2) \neq 0$

As y_1 & y_2 are the sol^{n}s ⇒

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \quad \text{— 1}$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0 \quad \text{— 2}$$

Multiply ① by y_2 & ② by y_1 →

$$P(x)\left[y_1'y_2 - y_2'y_1\right] = -\left(y_1''y_2 - y_1y_2''\right)$$

$$P(x) = \frac{y_2''y_1 - y_1''y_2}{y_1'y_2 - y_2'y_1} \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= -\frac{\left(y_2''y_1 - y_1''y_2\right)}{W(y_1,y_2)}$$

∴ Ans is d) None of these

$y_2'\left(y_1'' + P(x)y_1' + Q(x)y_1 = 0\right)$

$y_1'\left(y_2'' + P(x)y_2' + Q(x)y_2 = 0\right)$

$$Q(x)\left[y_1y_2' - y_1'y_2\right] = -\left(y_2'y_1'' - y_1'y_2''\right)$$

$$Q(x) = \frac{-\left(y_2'y_1'' - y_1'y_2''\right)}{y_1y_2' - y_1'y_2}$$

$$= \frac{-\left(y_2'y_1'' - y_1'y_2''\right)}{W(y_1, y_2)}$$

Note →

With the help of y_1 & y_2, we can find $P(x)$ & $Q(x)$ i.e. We can form linear Differential eqn.

Results → Let $y'' + P(x)y' + Q(x)y = 0$ be linear Diff. eqn s.t y_1 and y_2 are $\boxed{2\ L.I}$ Sol's then →

(1) There does not exist Common zero of y_1 & y_2

(2) The Zero's of y_1 and y_2 are in Alternate form.
i.e B/w any two Successive zeros of $y_1(x)$ $\exists$ $\boxed{\text{exactly one}}$ zero of $y_2(x)$. and Conversely.

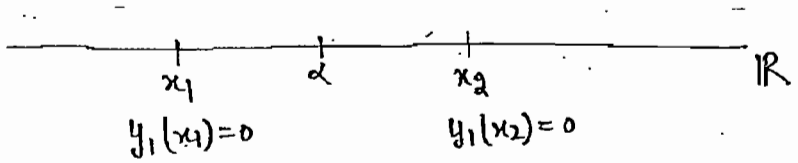
(3) the set of zeros of $y_1(x)$ and $y_2(x)$ are isolated sets.

(4) The set of zeros of y_1 and y_2 are c'able sets.

Proof:

① If ∄ any common zero of y_1 and y_2
then y_1 and y_2 are L.D
But y_1 & y_2 are 2 L.I Sol"s
∴ ∄ any common zero of y_1 and y_2.

②



$\exists\ \alpha \in (x_1, x_2)$
s.t $y_2(\alpha)=0$

Let x_1 and $x_2 \in \mathbb{R}$ are 2 successive zeros of $y_1(x)$ i.e $y_1(x_1)=0$ and $y_1(x_2)=0$

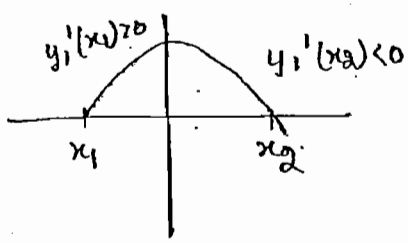
Clearly $y_1'(x_1) \neq 0$ and $y_1'(x_2) \neq 0$

(If any one of them is zero
say $y_1'(x_1)=0$
$\Rightarrow$ x_1 is the repeated root
$\Rightarrow$ y_1 & y_2 are L.D
But y_1 & y_2 are L.I)

As $y_1'(x_1) \neq 0 \Rightarrow$ either >0 or <0

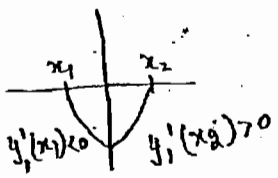
I case When $y_1'(x_1) > 0$

$\Rightarrow y_1'(x_2) < 0$



II case When $y_2'(x_1) < 0$

$\Rightarrow$ $y_1'(x_2) > 0$



$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

$$W(y_1, y_2)_{at\ x=x_1} = -y_2 y_1'$$

$$W(y_1, y_2)_{at\ x=x_2} = -y_2 y_1'$$

As y_1 & y_2 are two L.I Sol^{n}s $\Rightarrow W(y_1, y_2) \neq 0$.
either >0 or <0

Case A → When $W(y_1, y_2) > 0$

$$-y_2(x_1)\, y_1'(x_1) > 0 \quad \& \quad -y_2(x_2)\, y_1'(x_2) > 0$$

$$\boxed{y_2(x_1)\, y_1'(x_1) < 0} \qquad \boxed{y_2(x_2)\, y_1'(x_2) < 0}$$

I Case → $y_1'(x_1) > 0$ & $y_1'(x_2) < 0$

$$y_2(x_1)\,(+ve) < 0 \quad \& \quad y_2(x_2)\,(-ve) < 0$$

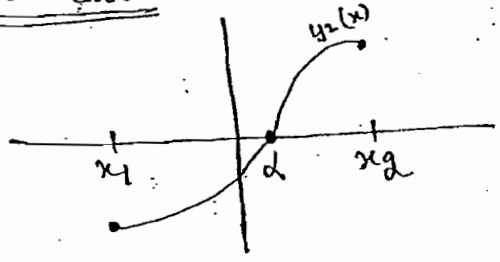
$$\boxed{y_2(x_1) < 0} \quad \& \quad \boxed{y_2(x_2) > 0}$$

II Case → $y_1'(x_1) < 0$ & $y_1'(x_2) > 0$.

$$y_2(x_1)\,(-ve) < 0 \quad \& \quad y_2(x_2)\,(+ve) < 0$$

$$\boxed{y_2(x_1) > 0} \quad \& \quad \boxed{y_2(x_2) < 0}$$

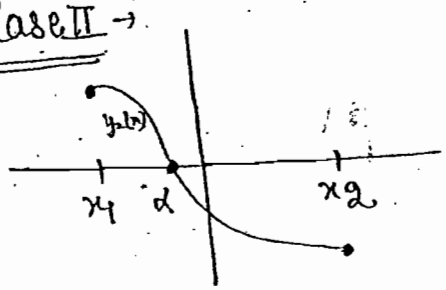
From Case I →



As y_2 → cts No Break

$\Rightarrow y_2(\alpha) = 0$

From Case II →



$\Rightarrow y_2(\alpha) = 0$.

This shows b/w any two successive zeros of $y_1(x)$ i.e x_1 & x_2 ∃ exactly one zero of $y_2(x)$ i.e α

(∵ When we consider two successive zeros of y_2, there is one zero of y_1 & b/w two successive zeros of y_1, there is 1 zero of y_2)

Case B →

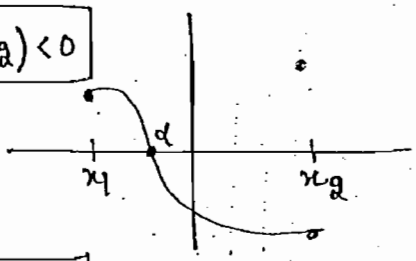
When $W(y_1, y_2) < 0$

$\Rightarrow \boxed{y_2(x_1)\, y_1'(x_1) > 0}$

and $\boxed{y_2(x_2)\, y_1'(x_2) > 0}$

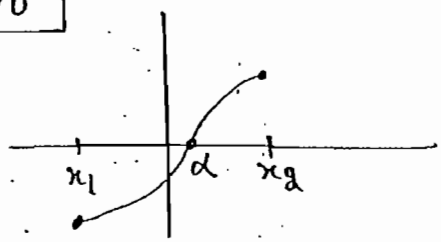
I Case → $y_1'(x_1) > 0$ & $y_1'(x_2) < 0$

$\Rightarrow \boxed{y_2(x_1) > 0}$ & $\boxed{y_2(x_2) < 0}$



II Case → $y_1'(x_1) < 0$ & $y_1'(x_2) > 0$

$\Rightarrow \boxed{y_2(x_1) < 0}$ & $\boxed{y_2(x_2) > 0}$



This shows b/w any 2 successive zeros of $y_1(x)$ i.e x_1 & x_2 ∃ exactly one zero of $y_2(x)$.

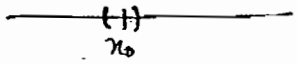
Hence the result

(3)

Isolated Point →

Let A be any set

$A = \{ x \in \mathbb{R} \mid \text{———} \}$



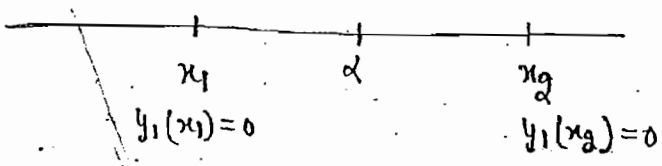
Any pt x_0 is said to be isolated pt of A if

$\exists\ \delta > 0$

$$(x_0 - \delta, x_0 + \delta) \cap A = \{x_0\}$$

(Isolated → Separate / Alone
x_0 is isolated when
it can be separated
from all the elements of A.
i.e Baki Sabse Alag kiya
ja sake.)

Set of zeros of y_1 & y_2 are isolated sets as i.e zeros are isolated points

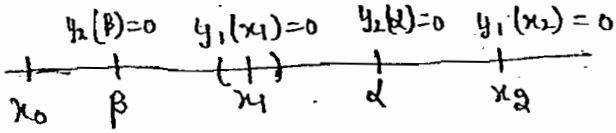


$\exists\ \alpha \in (x_1, x_2)$
s.t $y_2(\alpha) = 0$

$A = \{ \alpha \in \mathbb{R} \mid y_1(\alpha) = 0 \}$

$B = \{ \beta \in \mathbb{R} \mid y_2(\beta) = 0 \}$

& b/w any two successive zeros of y_1 ∃ exactly one zero of y_2



$y_1(x_0) = 0$

$\exists\ \delta > 0$

$$(x_1 - \delta, x_1 + \delta) \cap A = \{x_1\}$$

(∵ there is no zero of y_1)

Que Let $y'' + P(x)y' + Q(x)y = 0$

If y_1 & y_2 are 2 L.I. Sol^{n}s defined on $[a,b]$

$$A = \{\alpha \in \mathbb{R} \mid y_1(\alpha) = 0\}$$

$$B = \{\beta \in \mathbb{R} \mid y_2(\beta) = 0\}$$

a) if $a, b \in A \Rightarrow |A| \neq |B|$

b) if $a \in A, b \in B \Rightarrow |A| = |B|$

c) if $a \in B, b \in A \Rightarrow |A| = |B|$

d) if $|A| =$ infinite $\Rightarrow I = [a,b]$ unbounded interval.

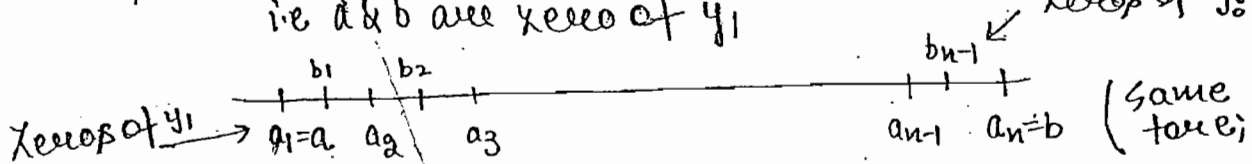
e) if $a, b \in B \Rightarrow |A| \neq |B|$

Sol^n Assume $[a,b]$ is of finite length $\Rightarrow$ A and B are finite



i) if $a, b \in A$

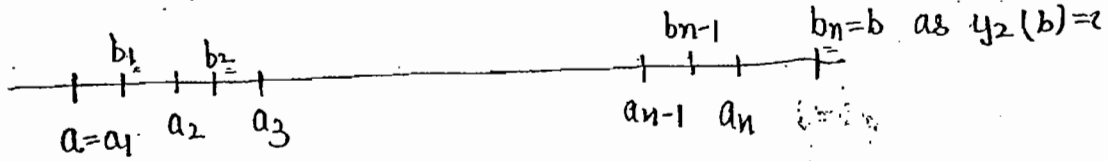
i.e a & b are zeros of y_1



No. of zeros of $y_1 = n$ i.e $|A| = n$

No. of zeros of $y_2 = n-1$ i.e $|B| = n-1$

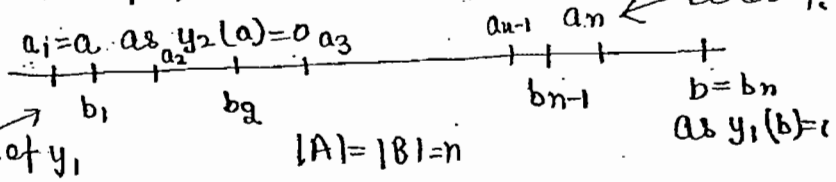
ii)



as $y_1(a) = 0$

$|A| = n \quad |B| = n$

iii) same as ii)



iv) $|A| =$ infinite $\Rightarrow$ A is Unbounded

$\Rightarrow I = [a,b]$ is the Unbounded interval

($A \subseteq [a,b] \Rightarrow [a,b]$ cannot be bounded)

Conclusion →

(1) If $a, b \in A$ & y_1 has n zeros $\Rightarrow$ y_2 has $n-1$ zeros.

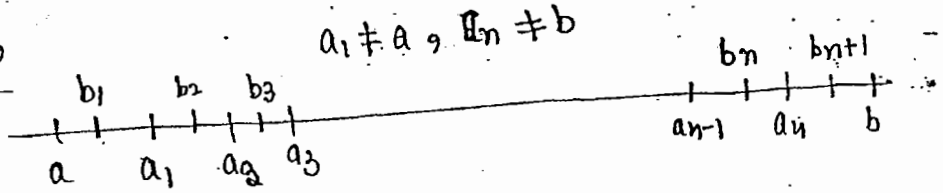
(2) If $a, b \in B$ & y_2 has n zeros $\Rightarrow$ y_1 has $n-1$ zeros

(3) If y_1 has zero's in zeros in (a, b) then y_2 has

- first & last zero is of y_2 → $n+1$ zeros
- or [either first zero of y_1 & last zero of y_2 → n zeros
- or [first zero of y_2 & last zero of y_1
- first and last zero is of y_1 → $n-1$ zeros

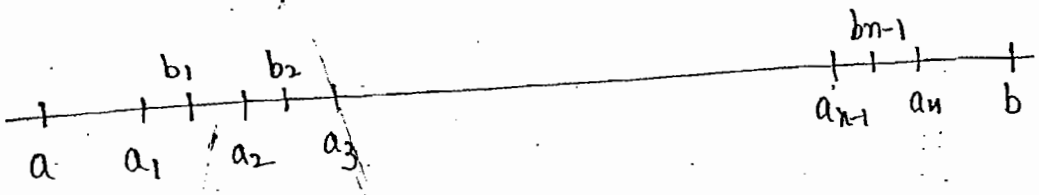
Proof of (3) →

$a_1 \neq a$, $b_n \neq b$



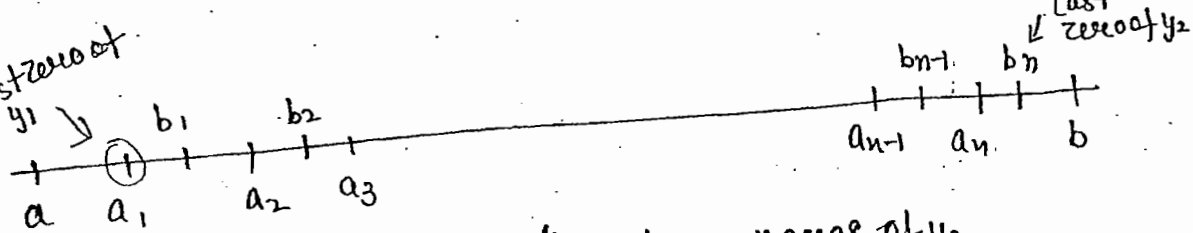
n-zeros of y_1

Clearly $n+1$ zeros of y_2



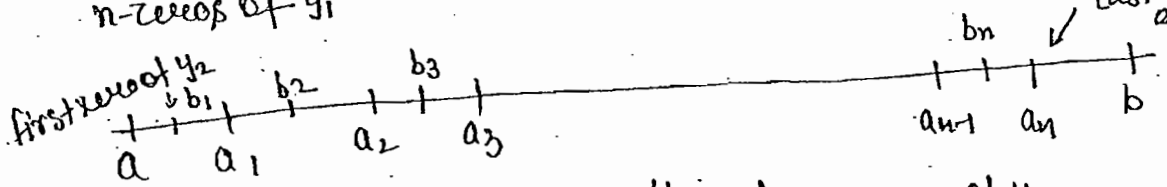
n-zeros of y_1

Clearly $n-1$ zeros of y_2



n-zeros of y_1

Clearly n zeros of y_2



Clearly n zeros of y_2

Result: Let $y(x)$ be a non-trivial sol^n of

$$y'' + P(x)y' + Q(x)y = 0$$

The ~~Number~~ Set of zeros of $y(x)$ cannot be Infinite Bounded

Proof: Following Results will be used →

* Every infinite Bounded set has a limit point
* If a is a limit point of a set A. Then $\exists$ a seq. $\langle a_n \rangle$ on A s.t

$$a_n \neq a \ \forall n \ \& \ a_n \to a$$

* if f is cts at $x = a \in A \Rightarrow$

a seq. $\langle a_n \rangle$ on A s.t if $a_n \to a$

$$\Rightarrow f(a_n) \to f(a)$$

Sol^n:

Let $A = \{ x \in \mathbb{R} \mid y(x) = 0 \}$ is a infinite Bounded set

$\Rightarrow$ A has a limit point, $a \in \mathbb{R}$ (Limit pt may or may not be a member of that set)

$\Rightarrow \exists \langle a_n \rangle$ seq on A s.t

$a_n \neq a$ and $a_n \to a$

$\Rightarrow$ As $y(x)$ is a sol^n $\Rightarrow$ $y(x)$ is differentiable & hence cts fu^n (throughout $\mathbb{R}$)

as $a_n \to a$

$\Rightarrow y(a_n) \to y(a)$

as $a_n \in A \Rightarrow y(a_n) = 0 \ \forall n$

$\Rightarrow y(a) = 0$ (Sequence of zeros will converge to)

$$y'(a) = \lim_{n\to\infty} \frac{y(a_n) - y(a)}{a_n - a} = 0$$

$\Rightarrow y(x) \equiv 0$ (By Fundamental Th^m $c_0 = c_1 = 0$)

Which is contradiction, as the sol^n is non-trivial.

∴ The set of zero's of $y(x)$ cannot be infinite Bounded.

Hence the result.

Que. Imp. $y'' + q(x)y = 0$

Case I → When $q(x) < 0 \ \forall x$ i.e $q(x) = -x^2$, $q(x) < 0$

Conclusions →

i) Every Non-Trivial sol^n has atmost one zero.

ii) Every Non-Trivial sol^n is strictly monotonic (increasing or decreasing)

Proof →

Let $y(x)$ be a Non-Trivial sol^n.

If it has no zeros, then proved.

Let $x_0 \in \mathbb{R}$ s.t $y(x_0) = 0$ (Suppose ∃ only 1 zero)

$\Rightarrow y'(x_0) \neq 0$

(If $y'(x_0) = 0$ By Fundamental Th^m, $y(x) \equiv 0$ i.e Unique Trivial sol^n But $y(x)$ is Non-Trivial sol^n)

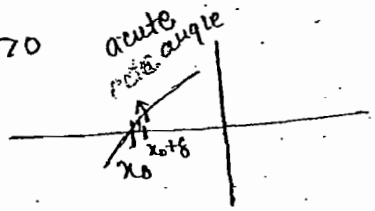
Either $y'(x_0) > 0$ or $y'(x_0) < 0$

i) Let $y'(x_0) > 0$

$$y''(x) = -q(x)y(x)$$

As $y'(x_0) > 0$ acute positive angle

$\Rightarrow$ Strictly increasing

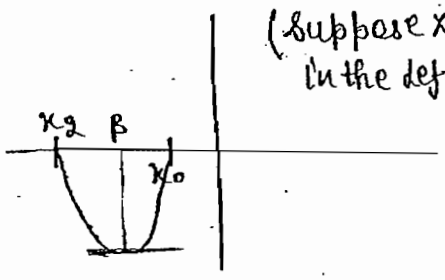


Let $\exists\, x_2 < x_0$

s.t $y(x_2) = 0$

$y'(x_2) < 0$

(Suppose zero in the left of x_0)



Now →

$y'(x_0) > 0$, $y'(x_2) < 0$

$\Rightarrow \exists\, \beta \in (x_2, x_0)$ s.t

$y'(\beta) = 0$ and $y(\beta) < 0$ ($\because$ of pt of minima)

$$y''(\beta) = \underbrace{-q(\beta)}_{-ve}\,\underbrace{y(\beta)}_{-ve}$$

$y''(\beta) < 0$

$y'(\beta)$ is $\downarrow$ at $x = \beta$

Contradiction. (as $y'(\beta) = 0$)

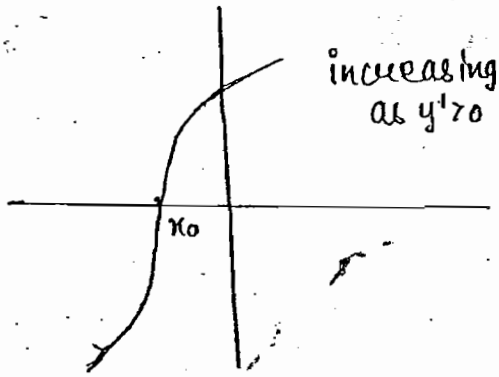
$\therefore$ No zero in the left of x_0. $\Rightarrow$ y is strictly increasing

$\Rightarrow$ Only One zero.

$\therefore$ Every Non-Trivial Sol^n has atmost one zero.

and

Every Non-Trivial Sol^n is either Strictly increasing (if $y' > 0$)



or

Strictly decreasing (if $y' < 0$) (II case when $y'(x_0) < 0$)

Note → If we are given I.V.P then exactly one Unique Sol^n will exists having these properties.

for $\delta > 0$

$y(x_0+\delta) > 0$ (in the nbd of x_0)

$y''(x_0+\delta) = -\underbrace{q(x_0+\delta)}_{-ve}\ \underbrace{y(x_0+\delta)}_{+ve}$ (As $q(x) < 0\ \forall x$)

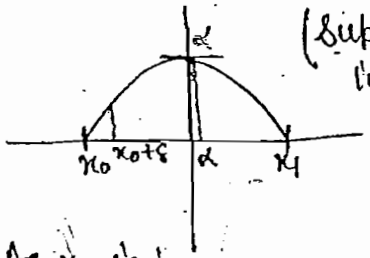
$y''(x_0+\delta) > 0$

$\Rightarrow y'(x)$ is $\uparrow$ in the nbd of x_0

$\therefore y'(x_0+\delta) > 0$

Let $\exists\ x_1 > x_0$

s.t $y(x_1) = 0$



(Suppose $\exists$ zero in the right of x_0)

Clearly, from graph →

$y'(x_1) < 0$ [As x_1 is in the right of x_0]

Now →

$y'(x_0) > 0$, $y'(x_1) < 0$

$\Rightarrow \exists\ \alpha \in (x_0, x_1)$ s.t

$y'(\alpha) = 0$ and $y(\alpha) > 0$? (∵ α pt. of maxima)

$y''(\alpha) = -\underbrace{q(\alpha)}_{+ve}\ \underbrace{y(\alpha)}_{+ve}$

$\Rightarrow y''(\alpha) > 0$

$\Rightarrow y'(\alpha)$ $\uparrow$ at $x = \alpha$

Contradiction (as $y'(\alpha) = 0$)

$\Rightarrow$ No zero in the right of x_0.

Set of Zeros of linear Diff eq^n → Isolated set provided that the Sol^n is Non-Trivial.

(4) Isolated set / Set of isolated pt → is always c'able

∴ the set of zeros of y_1 & y_2 are c'able sets.

Que → Let $y'' + P(x)y' + Q(x)y = 0$ be linear Diff eq^n s.t y_1 and y_2 are 2 L.I Sol^{n}s

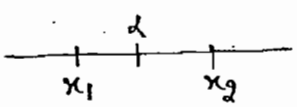
Let $A = \{\alpha \in \mathbb{R} \mid y_1(\alpha) = 0\}$

$B = \{\beta \in \mathbb{R} \mid y_2(\beta) = 0\}$

Then
- a) If $|A| \geq 2 \Rightarrow |B| \geq 1$
- b) A is infinite $\Leftrightarrow$ B is infinite
- c) A is bounded $\Leftrightarrow$ B is bounded
- d) A is bounded $\Leftrightarrow$ B is finite

Sol^n →

a) If $|A| \geq 2$



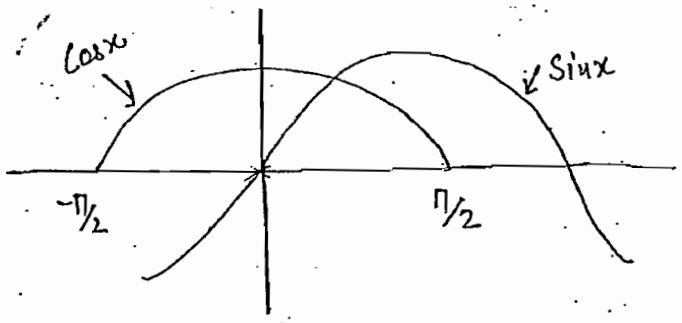
∵ b/w any 2 successive zeros of y_1 ∃ exactly one zero of y_2.

$\Rightarrow |B| \geq 1$

Converse is not true →

If $y'' + y = 0$ $\quad I = (-\pi/2, \pi/2)$

$y_1 = \sin x$
$y_2 = \cos x$ } are 2 L.I sol^{n}s.



$B = \{\alpha \in I \mid y_1(\alpha) = 0\}$

$A = \{\beta \in I \mid y_2(\beta) = 0\}$

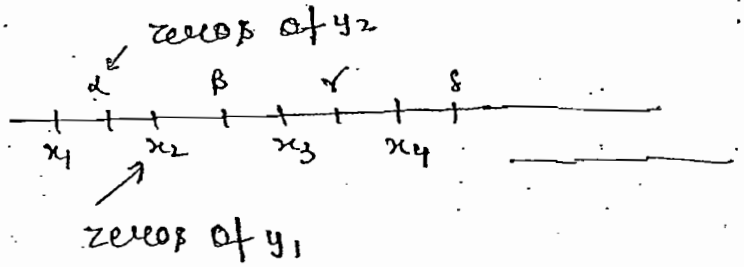
$|B| = 1$

$|A| = 0$

Here $|B| = 1$ & $|A| \neq 2$

(There may be a possibility that both y_1 & y_2 has only 1 zero i.e $|B| = 1$ & $|A| = 1$)

b) True



c) A is bounded ⇒ A is finite

(∵ If A is infinite By BWT → Infinite bounded set has a limit point.

But we considering the set of zeros which is an isolated set ∴ No limit pt.

$\Rightarrow g(x) > 0$ for some large value of x $(x > x_0, x \to \infty)$.

i.e $\exists\, x_1 > x_0$ s.t

$$g(x_1) > 0$$

$$\Rightarrow -\frac{y'(x_1)}{y(x_1)} > 0$$

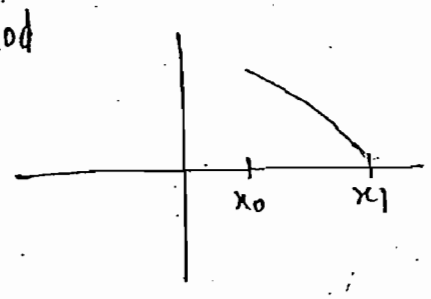
$$\Rightarrow \frac{y'(x_1)}{y(x_1)} < 0$$

$(y(x) > 0 \ \forall\ x \geq x_0$
As $x_1 > x_0$
$\therefore y(x_1) > 0)$

$$\Rightarrow y'(x_1) < 0$$

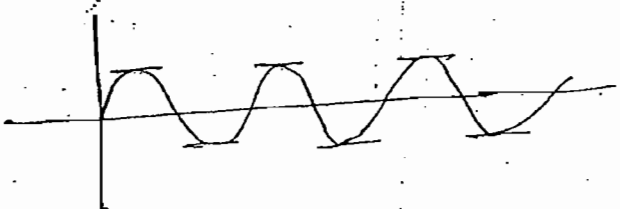
$\Rightarrow$ fn^n is in decreasing mod

$\Rightarrow$ Zero exists after x_0



Contraction to our assumption that there is no after x_0.

Hence, Every Non-Trivial Sol^n has <u>infinite number of Zeros on the x-axis</u>.



$\Rightarrow$ Every non-trivial sol^n is Oscillatory

And so has infinite number of pts of extrema.

Note →

(1) If the diff. eqⁿ is →

$$y'' - Q(x)y = 0 \quad , Q(x) > 0 \ \forall x$$

(i.e $Q(x) > 0 \ \forall x > 0$ also)

thus consider eqⁿ as →

$$y'' + [-Q(x)]y = 0 \quad , \ -Q(x) < 0 \ \forall x$$

and this refers to Case I When $Q(x) < 0$

(2) If $Q(x) > 0 \ \forall x \in \mathbb{R}$,

then we can't say anything about Zeros on the left side of origin (i.e for $x \leq 0$)

(As in 2nd case → $Q(x) > 0 \ \forall x > 0$)

(3) $Q(x)$ can be any fun ↓

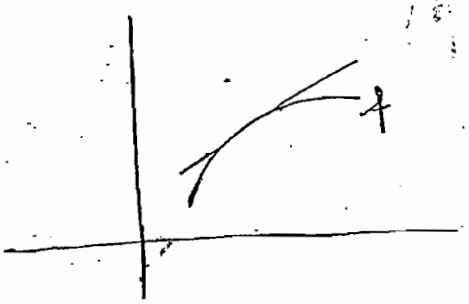
$Q(x) = x^2$

$Q(x) = e^{2x}$

$q(x) = \sin^2 x$

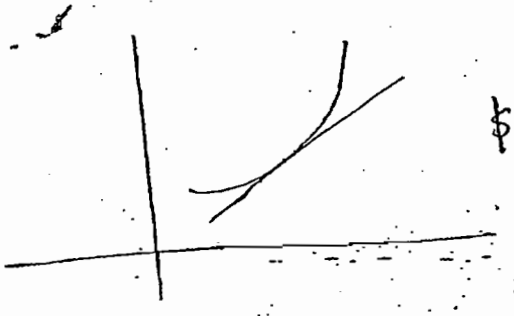
$\Big\} \ q(x) > 0$

Convex & Concave fu^n's →



A fu^n is concave if $f'' < 0$

∴ Tangent is above the curve at every pt.



A fu^n is convex if $f'' > 0$

∴ Tangent is below the curve at every pt.

Que $y'' + P(x)y' + Q(x)y = 0$

If $y_1(x) = x$ and $y_2(x) = \frac{1}{x}$ are two sol^n's. $(x \neq 0)$

a) $P(x)$ is concave and $Q(x)$ is convex

b) $P(x)$ is convex and $Q(x)$ is concave

c) Both $P(x)$ and $Q(x)$ are convex

d) Both $P(x)$ and $Q(x)$ are concave

✓ e) None-of-these.

Sol^n:

$$P(x) = -\frac{y_1 y_2'' - y_2 y_1''}{y_1 y_2' - y_2 y_1'} \quad \text{(obtained earlier)}$$

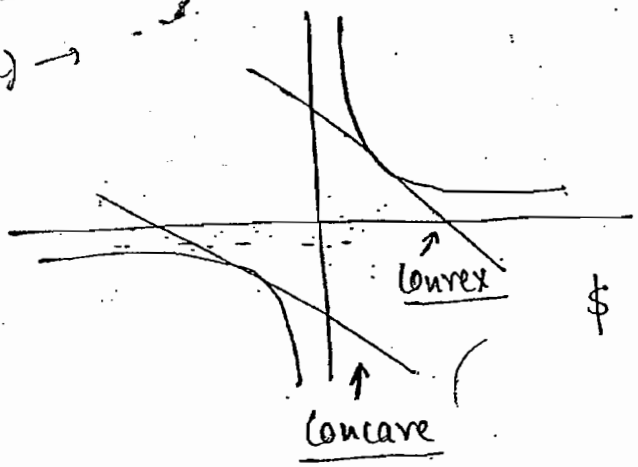
$$= -\frac{x \times \frac{2}{x^3} - \frac{1}{x} \times 0}{x \times \frac{-1}{x^2} - \frac{1}{x} \times 1} = \frac{-2/x^2}{-2/x} = \frac{1}{x}$$

$$Q(x) = -\frac{y_2' y_1'' - y_1' y_2''}{y_1 y_2' - y_2 y_1'}$$

$$= -\frac{\frac{-1}{x^2}\times 0 - 1\times\frac{2}{x^3}}{x\times\frac{-1}{x^2} - \frac{1}{x}\times 1} = \frac{2/x^3}{-2/x} = \frac{-1}{x^2}$$

Graphically →

$P(x)$ →



Tangent is below the curve on $\mathbb{R}^+$

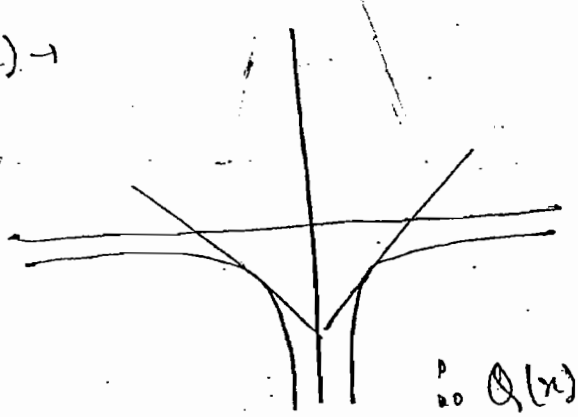
∴ $P(x)$ is convex on $\mathbb{R}^+$

$ Tangent is above the curve on $\mathbb{R}^-$

∴ $P(x)$ is concave on $\mathbb{R}^-$

∴ $P(x)$ → Neither concave nor convex on $\mathbb{R}-\{0\}$

$Q(x)$ →



Every tangent is above the curve on $\mathbb{R}-\{0\}$

∴ $Q(x)$ is concave

Or $P'(x) = \frac{-1}{x^2} \Rightarrow P''(x) = \frac{2}{x^3} > 0$ for $x > 0$

∴ Convex on $\mathbb{R}^+$

& $\frac{2}{x^3} < 0$ for $x < 0$

∴ Concave on $\mathbb{R}^-$

$\Rightarrow$ Neither Concave nor Convex on $\mathbb{R}-\{0\}$

And

$$Q'(x) = \frac{2}{x^3} \Rightarrow Q''(x) = \frac{-6}{x^4} < 0 \quad \forall x \neq 0$$

$\therefore$ $Q(x)$ is Concave on $\mathbb{R}-\{0\}$

$\therefore$ Ans is (c) None of these.

When $x \to \infty$ $\quad P \to 0$, $Q \to 0$

When $x \to -\infty$ $\quad P \to 0$, $Q \to 0$

When $x \to 0$ $\quad P(x) \to \infty$, $Q(x) \to -\infty$

$\therefore$ P & Q are Unbounded

Que Consider the differential eqn →

$$y'' + P(x)y' + Q(x)y = 0 \qquad \forall x \in \mathbb{R} - \{0\}$$

If $y_1(x) = \frac{1}{x}$ and $y_2(x) = x$ are two sol^{n}s

Then

✓ a) $P(x) \to 0$ as $x \to \infty$

b) $P(x) \to \infty$ as $x \to \infty$

✓ c) $Q(x) \to 0$ as $x \to \infty$

d) None of these.

Sol^n $P(x) = \frac{1}{x}$, $Q(x) = \frac{-1}{x^2}$ (obtained in previous e.g)

$P(x) \to 0$ as $x \to \infty$

$Q(x) \to 0$ as $x \to \infty$

Que Consider the diff eq^n $y'' + P(x)y' + Q(x)y = 0$

If $y_1(x) = \frac{1}{x}$, $y_2(x) = e^x$ are the sol^{n}s

Then

a) $P(x) \to \infty$ as $Q(x \to \infty$

✓ b) $Q(x) \to 0$ as $x \to \infty$

c) $P(x) \to 0$ and $Q(x) \to 0$ as $x \to \infty$

d) $P(x) \to \infty$ and $Q(x) \to \infty$ as $x \to \infty$

Sol^n $y_1(x) = \frac{+1}{x^2}$

$\Rightarrow y'(x) = \frac{-1}{x^2}$ and $y''(x) = \frac{2}{x^3}$

$y_2(x) = e^x$

$\Rightarrow y_2'(x) = e^x$ and $y_2''(x) = e^x$

$\frac{2}{x^3} - P(x)\frac{1}{x^2} + Q(x)\cdot\frac{1}{x} = 0 \Rightarrow 2 - xP(x) + x^2Q(x) = 0$ —(1)

$e^x + P(x)e^x + Q(x)e^x = 0 \Rightarrow 1 + P(x) + Q(x) = 0$ —(2)

Multiply (2) by $x^2 \rightarrow$ and sub. $\rightarrow$

$2 - xP(x) - x^2 - x^2P(x) = 0$

$-x(x+1)P(x) = x^2 - 2$

$P(x) = -\frac{x^2-2}{x(x+1)}$

$= -\frac{1 - 2/x^2}{(1+1/x)}$

$P(x) \rightarrow -1$ as $x \rightarrow \infty$

$Q(x) = -1 - P(x)$

$= -1 + \frac{1 - 2/x^2}{1 + 1/x}$

$Q(x) \rightarrow 0$ as $x \rightarrow \infty$

or $Q(x) = \frac{-2-x}{x(x+1)}$

$= \frac{-2/x - 1}{x(1+\frac{1}{x})}$

$\rightarrow 0$ as $x \rightarrow \infty$

Que Let e^{mx} be a sol^n of →

$$y'' + P(x)y' + Q(x)y = 0$$

Then

a) $m^2 - mP(x) + Q(x) = 0$

✓ b) $m^2 + mP(x) + Q(x) = 0$

✗ c) $m^2 + mP(x) - Q(x) = 0$

d) None of these.

Sol^n:

$$y(x) = e^{mx}$$

$$y'' = m^2 e^{mx}$$

$$y' = m e^{mx}$$

$$m^2 e^{mx} + m e^{mx} P(x) + Q(x) e^{mx} = 0$$

$$e^{mx}\left[m^2 + mP(x) + Q(x)\right] = 0$$

$$\Rightarrow \boxed{m^2 + mP(x) + Q(x) = 0} \quad (e^{mx} \neq 0)$$

Ans.

Que Let x^m be a sol^n of

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

Then

a) $m(m-1) + mx\,P(x) - x^2 Q(x) = 0$

b) $m(m-1) - mx\,P(x) + x^2 Q(x) = 0$

✓c) $m(m-1) + mx\,P(x) + x^2 Q(x) = 0$

d) None of these.

Sol^n

$$y(x) = x^m$$

$$y' = m x^{m-1}$$

$$y'' = m(m-1)\,x^{m-2}$$

$$\therefore\quad m(m-1)\,x^{m-2} + P(x)\,m\,x^{m-1} + Q(x)\,x^m = 0$$

$$x^{m-2}\left[m(m-1) + xm\,P(x) + x^2 Q(x)\right] = 0$$

$$x^{m-2} \neq 0$$

$$\Rightarrow \boxed{m(m-1) + mx\,P(x) + x^2 Q(x) = 0}$$

Que. If $\frac{1}{x}$ is the sol^n of $y'' + P(x)\,y' + Q(x)\,y = 0$

Put $m = -1$

$$2 - x\,P(x) + x^2 Q(x) = 0$$

Que. If x is the sol^n →

$$x\,P(x) + x^2 Q(x) = 0 \;\Rightarrow\; P(x) + x\,Q(x) = 0$$

Result → If one sol^n of $y'' + Py' + Qy = 0$ is known, Other L.I sol^n can be easily find out.

Procedure → P & Q → fu^n's of x

Let $\phi(x)$ be the known sol^n of

$$y'' + Py' + Qy = 0 \quad \text{— (1)}$$

Let $\boxed{y(x) = \phi \cdot u}$, u is fu^n of x

$$y' = \phi' u + u' \phi \quad \text{— (2)}$$

$$y'' = \phi'' u + \phi' u' + u' \phi' + u'' \phi$$

$$= \phi'' u + 2\phi' u' + u'' \phi \quad \text{— (3)}$$

Using (2) and (3) in (1) →

$$(\phi'' u + 2\phi' u' + \phi u'') + P(\phi' u + u' \phi) + Q\phi \cdot u = 0$$

$$\phi u'' + [2\phi' + P\phi] u' + [\phi'' + P\phi' + Q\phi] u = 0$$

As ϕ is the sol^n

$$\therefore \phi'' + P\phi' + Q\phi = 0$$

$$\therefore \phi u'' + (2\phi' + P\phi) u' = 0$$

$$u'' + \left(2\frac{\phi'}{\phi} + P(x)\right) u' = 0$$

Linear in u'.

$$Pu + u' = v$$

$$v' + \left(2\frac{\phi'}{\phi} + P(x)\right) v = 0$$

Linear in v, can be be easily solved.

$$\frac{dv}{dx} + \left(P(x) + \frac{2\phi'}{\phi}\right)v = 0$$

$$\frac{dv}{v} = -\left(P(x) + \frac{2\phi'}{\phi}\right)dx$$

Integrating →

$$\log v = -\int P(x)\,dx - 2\log\phi \quad \text{— (4)}$$

$$\log \frac{v}{[\phi(x)]^2} = -\int P(x)\,dx$$

$$\Rightarrow v(x) = \frac{e^{-\int P(x)dx}}{[\phi(x)]^2}$$

$$\Rightarrow u'(x) = \frac{e^{-\int P(x)dx}}{[\phi(x)]^2}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int P(x)dx}}{[\phi(x)]^2}\,dx$$

(No effect of adding constant in this & 4 Ap it will Merged in c_2)

∴ ϕ, ϕu are the two L.I Sol^{n}s.

∴ their L.C $y(x) = c_1\phi + c_2\phi u$ is also a Sol^n.

→ ϕ and ϕu are two L.I Sol^{n}s Ap →

$$W(\phi, \phi u) = \begin{vmatrix} \phi & \phi u \\ \phi' & \phi' u + \phi u' \end{vmatrix}$$

$$= \phi\phi' u + \phi^2 u' - \phi\phi' u$$

$$= \phi^2 u'$$

$$W(\phi, \phi u) = \phi^2 \times \frac{e^{-\int P(x)\,dx}}{\phi^2}$$

Wronskian can be directly find out from this of ϕ & ϕu.

$$\boxed{W(\phi, \phi u) = e^{-\int P(x)\,dx}} \neq 0 \Rightarrow \text{L.I}$$

$\therefore$ ϕ and ϕu are two L.I Sol^n

$\therefore$ their L.C $y(x) = c_1\phi + c_2\phi u$ is also a Sol^n.

~~Que.~~ ~~Ket~~ Conclusion →

Let $y'' + P(x)y' + Q(x)y = 0$

be second order L.D.E

If $\phi(x)$ is a Sol^n then General Sol^n is →

$$y(x) = c_1\phi + c_2\phi u$$

where $u = \int \frac{e^{-\int P\,dx}}{[\phi(x)]^2}\,dx \quad (P = P(x))$

Que $(x^2+1)y'' - 2xy' + 2y = 0$. Find General Sol^n?

Sol^n → First chk whether

x , $\frac{1}{x}$, e^x , e^{-x} , x^m etc

is a Sol^n or not. If yes Take as ϕ.

And find u. & $\therefore$ General Sol^n is →

$$y(x) = c_1\phi + c_2\phi u$$

$$(x^2+1)y'' - 2xy' + 2y = 0$$

$$y'' - \frac{2x}{(x^2+1)}y' + \frac{2}{(x^2+1)}y = 0$$

$$P(x) = \frac{-2x}{x^2+1}, \quad Q(x) = \frac{2}{x^2+1}$$

First chk $y=x$ is a sol^n or not →

$$m=1$$

$$m(m-1) + mxP(x) + x^2Q(x) = 0$$

$$1(1-1) + x\cdot x\frac{-2x}{x^2+1} + x^2\cdot\frac{2}{x^2+1} = 0$$

$$\frac{-2x^2}{x^2+1} + \frac{2x^2}{x^2+1} = 0$$

$$\Rightarrow 0 = 0$$

$\therefore$ $y(x) = x$ is the sol^n.

Take $\phi(x) = x$

Now

$$u = \int \frac{e^{+\int \frac{2x}{1+x^2}dx}}{x^2}dx$$

$$e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$$

$$\therefore u = \int \frac{1+x^2}{x^2}dx = \int\left(\frac{1}{x^2}+1\right)dx$$

$$= \frac{-1}{x} + x = x - \frac{1}{x}$$

∴ General sol^n is →

$$y(x) = c_1\phi + c_2\phi u$$

$$= c_1 x + c_2 x\left(x - \frac{1}{x}\right)$$

$= c_1 x + c_2(x^2 - 1)$ is the required General sol^n.

Special Eq^n → Cauchy Eq^n →

$$x^2y'' + axy' + by = 0 \quad \text{--- (1)}$$

$a, b \in \mathbb{R}$

(i.e. If nth order derivative is there, its coefficient is x^n) like $x^6 \frac{d^6y}{dx^6}$

I Find condition on a, b s.t sol^n is bounded

II The condition on a, b s.t sol^n is periodic

III Find the condition on a, b s.t every sol^n is power of x.

Changing the independent variable from x to z

Put $x = e^z$

$\Rightarrow z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \quad \text{--- (2)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dx} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dx} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dx}$$

$$= \frac{-1}{x^2}\frac{dy}{dx} + \frac{1}{x}\frac{d^2y}{dx^2}\cdot\frac{1}{x}$$

$$= \frac{-1}{x^2}\frac{dy}{dx} + \frac{1}{x^2}\frac{d^2y}{dx^2} \quad \text{— (3)}$$

Put $D = \frac{d}{dx}$, $D_1 = \frac{d}{dx}$

$x^2y'' + axy' + by = 0$ (from ①)

~~$x^2\left(\frac{-1}{x^2}\frac{dy}{dx} + \frac{1}{x^2}\frac{d^2y}{dx^2}\right)$~~ from ② and ③ ⟶

$D = \frac{1}{x}D_1 \Rightarrow \boxed{xD = D_1}$

and $x^2D^2 = -D_1 + D_1^2$

$= D_1^2 - D_1 = \boxed{D_1(D_1-1)}$

(Illy $\boxed{x^3D^3 = D_1(D_1-1)(D_1-2)}$ and so on)

From ① ⟶

$$(D_1(D_1-1) + aD_1 + b)\,y(z) = 0$$

$$(D_1^2 + (a-1)D_1 + b)\,y(z) = 0$$

↑

Constant Coefficient

Case I → If the roots of Auxillary eq^n are real and distinct say m_1 and m_2.

$\therefore$ $y(x) = c_1 e^{m_1 z} + c_2 e^{m_2 z}$ (As x is New I.V Not in x) Rem

$x = e^z$

$\Rightarrow$ $\boxed{y(x) = c_1 x^{m_1} + c_2 x^{m_2}}$

i.e. Solution is of polynomial type when roots are real and distinct Also $\boxed{m_1, m_2 > 0}$

(if $m_1 < 0$ say $m_1 = -2$ then $\frac{1}{x^2}$ term is there ↑ Not a polynomial)

Case II → If the roots of Auxillary eq^n are real and repeated. say $m_1 = m_2 = m$

$\therefore$ $y(x) = (c_1 + c_2 x) e^{mz}$ (when $B^2 - 4AC = 0$)

$\Rightarrow$ $\boxed{y(x) = (c_1 + c_2 \log x) x^m}$

↑ Solution is not of polynomial type as $\log x$ is there

Case III → If the roots of Auxillary eq^n are complex say $m_1 = \alpha + i\beta$

$m_2 = \alpha - i\beta$

(when $B^2 - 4AC < 0$)

$\therefore y(z) = e^{\alpha z}(c_1 \cos\beta z + c_2 \sin\beta z)$

$\Rightarrow \boxed{y(x) = x^{\alpha}(c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x))}$

Auxiliary eq^n →

$m^2 + (a-1)m + b = 0$

$m_1 + m_2 = (1-a)$

$m_1 m_2 = b$

Sol^n will be in polynomial form when roots are real and distinct and $m_1, m_2 > 0$

$\Rightarrow m_1 + m_2 > 0$

$1 - a > 0$

$\Rightarrow \boxed{a < 1}$

\$ Roots are real & distinct when

$B^2 - 4AC > 0$

$\boxed{(a-1)^2 - 4b > 0}$

and $m_1 m_2 > 0$

$\boxed{b > 0}$

i.e Sol^n will be in polynomial form when →

$(a-1)^2 - 4b > 0$, $a < 1$ \$ $b > 0$

Bounded Sol^n →

Solution is not bounded when roots are real & distinct and when roots are repeated

($\because y(x) \to \infty$ as $-\infty$ When $x \to \infty$ as $-\infty$)

Bounded sol^n is possible if roots of Auxillary eq^n are Complex roots with $\boxed{\alpha = 0}$

($e^{\alpha x} \to \infty$ as $x \to \infty$ for $\alpha > 0$
$e^{\alpha x} \to 0$ as $x \to -\infty$
$\alpha < 0$
$e^{\alpha x} \to 0$ as $x \to \infty$
$e^{\alpha x} \to \infty$ as $x \to -\infty$)

i.e $B^2 - 4AC < 0$ (For Complex roots)

$$\boxed{(a-1)^2 - 4b < 0}$$

$m_1 = +i\beta \quad m_2 = -i\beta \quad (\text{As } \alpha = 0)$

($r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$
$r_1 + r_2 = 2\alpha = 1 - a$
$r_1 r_2 = \beta^2 + \alpha^2 = b$)

$m_1 + m_2 = 1 - a = 0 \Rightarrow \boxed{a = 1}$

$m_1 m_2 = \beta^2 = b > 0 \Rightarrow \boxed{b > 0}$

∴ Condition for bounded sol^n →

$(a-1)^2 - 4b < 0$,

$a = 1$,

$b > 0$

Use Unbounded

Periodic Sol^n →

(In Ist and 2nd case Periodic Sol^n is NOT POSSIBLE ∵ Polynomial fu^n & Log. fu^n → Not Periodic)

fu^n in fix interval may | may not be periodic

for $x>0$, $a=1$, $b>0$ and $\alpha=0$

then Sol^n is →

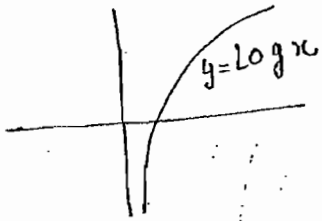
$$y(x) = C_1 \cos\beta(\log x) + C_2 \sin\beta(\log x)$$

$$y_1 = \cos\beta(\log x) \quad , \quad y_2 = \sin\beta(\log x)$$

Consider $\cos\beta(\log x)$ →

$$y = \log x$$

$$u = \cos\beta y$$



for $x>0$, $y \in (-\infty, \infty)$

Now for $u = \cos\beta y$

Domain is $(-\infty,\infty)$ i.e $\mathbb{R}$

& Cosine fu^n is periodic on $\mathbb{R}$ (Not on $(-\pi/2, \pi/2)$ i.e on $\mathbb{R}$)

$\Rightarrow \cos\beta y$ is periodic on $\mathbb{R}$ (Cosy is periodic but on $(-\pi/2, \pi/2)$ → Not periodic)

∴ y_1 is periodic fu^n for $x>0$

Illy y_2 is periodic fu^n for $x>0$

∴ $y(x) = C_1 \cos\beta(\log x) + C_2 \sin\beta(\log x)$

is the Periodic fu^n for $x>0$

If any interval is given →

Validate (as discussed above)

Periodicity → Domain based property.

(1) If $B \subseteq A$

If f is periodic on $A \not\Rightarrow f$ is periodic on B

e.g $(-\pi/2, \pi/2) \subseteq \mathbb{R}$

$\cos x$ is not periodic on $(-\pi/2, \pi/2)$

$\cos x$ is periodic on $\mathbb{R}$

(2) If $A \subseteq B$

If f is periodic on A, then f may or may not periodic on B.

Que $x^2y'' + xy' + 4y = 0$, $x > 0$ $\qquad x^2y'' + axy' + by = 0$

Then Non-Trivial Sol^n is →

a) Bounded and Non-Periodic

b) Unbounded and periodic

c) Unbounded and Non-periodic

✓ d) Bounded and periodic.

Sol^n → Auxiliary eq^n → $m^2 + (a-1)m + b = 0$

Here $a = 1$, $b = 4 > 0$ and $x > 0$

⇒ Bounded & periodic

Que $x^2y'' - 5xy' + 9y = 0$

Then General Sol^n is →

a) $(c_1 + c_2 x)e^{3x}$

✓ b) $(c_1 + c_2 \log x)x^3$

c) $(c_1 + c_2 x)x^3$

d) $(c_1 + c_2 \log x)e^{x^3}$

Sol^n

$a = -5$, $b = 9$

Discriminant $= B^2 - 4AC$

$= (a-1)^2 - 4b$

$= (-5-1)^2 - 36$

$= 36 - 36 = 0$

$\Rightarrow$ Roots of auxiliary eq^n are real and repeated

$\therefore$ form of $y(x) = (c_1 + c_2 \log x)x^4$ ← Sol^n is of this form

Ans is b) clearly

Rem (Alwz find the → Discriminant first $\because$ that will give the idea about the Sol^{n}s i.e the Sol^n is of which form)

Linear Diff eqⁿ with Constant Coefficient →

(Homogeneous)

$$y^{(n)} + a_1 y^{(n-1)} + \text{——————} + a_{n-1} y' + a_n y = 0, \quad a_i \in \mathbb{R}$$

$$F(D)\, y = 0$$

where

$$F(D) = D^n + a_1 D^{n-1} + \text{——————} + a_{n-1} D + a_n$$

Auxillary eqⁿ →

$F(m) = 0$ is nth degree eqⁿ

i.e $m^n + a_1 m^{n-1} + \text{——————} + a_{n-1} m + a_n = 0$

Case I → If all roots are real and distinct.

Say $m_1, m_2 \text{ ——————} m_n$

$$y(x) = \sum_{i=1}^{n} c_i e^{m_i x}$$

Case II → If the roots are real and repeated

$$m_1 = m_2 = \text{——————} = m_k = m, m_{k+1}, \text{——}, m_n$$

$$y(x) = (c_1 + c_2 x + \text{——————} + c_k x^{k-1}) e^{mx} + \sum_{i=k+1}^{n} c_i e^{m_i x}$$

Case III → If $m_1 = \alpha + \sqrt{\beta}$

→ $m_2 = \alpha - \sqrt{\beta}$ (Irrational roots occur in pair if $a_i \in \mathbb{R}$), $m_3, m_4 \text{ ——} m_n$ are real and distinct

$$y(x) = e^{\alpha x} (c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x) + \sum_{i=3}^{n} c_i e^{m_i x}$$

As $\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

In this Case, Sol^n is alwx Unbounded even if $\alpha = 0$

$\because$ $\sinh x$ and $\cosh x$ are Unbounded.

Case IV → Some roots are Complex and Non-Repeated.

$m_1 = \alpha + i\beta$

$m_2 = \alpha - i\beta$; $\underbrace{m_3, m_4 \text{ ---------- } m_n}$

Non-repeated real & distinct.

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + \sum_{i=3}^{n} c_i e^{m_i x}$$

Case V → Some roots are Complex and Repeated.

$m_1 = m_2 = \alpha + i\beta$

$m_3 = m_4 = \alpha - i\beta$; $\underbrace{m_5, m_6 \text{ ---------- } m_n}$

real & distinct

$$y(x) = e^{\alpha x}\left((c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right) + \sum_{i=5}^{n} c_i e^{m_i x}$$

are called the Complementary fu^n | Solution

Denoted by $y_c(x)$

Linear Diff eqⁿ with Constant Coefficient → (Non-Homogeneous)

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \text{---------} + a_{n-1} y' + a_n y = Q(x)$$

$$f(D)\, y = Q(x)$$

where

$$f(D) = D^n + a_1 D^{n-1} + \text{---------} + a_{n-1} D + a_n$$

Then P.I / Particular sol^n is

$$y_p(x) = \frac{Q(x)}{f(D)}$$

$\left[\frac{1}{f(D)} \rightarrow \text{inverse } fu^n \text{ of } f(D)\right]$

General finding Method of particular sol^n →

Que $(D+1)(D+2)\, y = \sin(e^x)$

u

$$(D+1)\, u = \sin(e^x)$$

$$\frac{du}{dx} + u = \sin(e^x) \leftarrow \text{linear in } u$$

$$\text{I.F} = e^{\int 1 \cdot dx} = e^x$$

$$u \cdot e^x = \int \sin(e^x) \cdot e^x \, dx$$

$$e^x = t \Rightarrow e^x dx = dt$$

$$= \int \sin t \, dt$$

$$u = -\cos(e^x)\, e^{-x$$

(No need to add constant As they will merge in C.F)

Note → Particular sol^n contains no constant

Ap $u = (D+2)y$

$(D+2)y = -\cos(e^x)\cdot e^{-x}$

Linear in y

$I.F = \int e^{\int 2\,dx} = e^{2x}$

$e^{2x}y = -\int \cos(e^x)\cdot e^{-x}\cdot e^{2x}\,dx$

$e^{2x}y = -\int \cos(e^x)\,e^x\,dx$

$e^x = t \Rightarrow e^x dx = dt$

$e^{2x}y = -\sin e^x$

$y_p = -e^{-2x}\sin e^x$

Note →

$(D+P(x))(D+Q(x))y = \sin e^x$

is not same as $D = \frac{d}{dx}$

$(D+Q(x))(D+P(x))y = \sin e^x$

$(D+P(x))(D+Q(x))y = D^2y + D(Q(x)y) + P(x)Dy + P(x)Q(x)y$

$= D^2y + Q'(x) + P(x)Dy + P(x)Q(x)y$

$+Q(x)Dy$

$(D+Q(x))(D+P(x))y = D^2y + D(P(x)y) + Q(x)Dy + P(x)Q(x)y$

$= D^2y + P'(x) + Q(x)Dy + P(x)Q(x)y$

$+P(x)Dy$

But in case when $P(x) \& Q(x) \Rightarrow$ Are Constants then $P'(x) = Q'(x) = 0$

$(D+1)(D+2)y$ is same as $(D+2)(D+1)y$

$\downarrow$ $\downarrow$

$D^2+3Dy+2y$ $\quad$ $D^2+3Dy+2y$

In Prev. Eq →

If we interchange $(D+2)$ & $(D+1)$

Integration becomes Difficult.

(We can change accordingly → To make ∫ simple)

Case I → When $Q(x) = e^{ax}$

$$P.I = \frac{e^{ax}}{f(a)}, \text{ provided } f(a) \neq 0$$

If $f(a) = 0$

$\Rightarrow f(D) = (D-a)^n G_1(D)$, $G_1(a) \neq 0$ ($D=a$ is the root of multiplicity n)

Then $P.I = y_p(x) = \frac{x^n}{n!} \frac{e^{ax}}{G_1(a)} = \frac{x^n e^{ax}}{f^n(a)}$

Eg. $(D-1)(D-2)y = e^{3x}$

$$y_p(x) = \frac{e^{3x}}{(3-1)(3-2)} = \frac{e^{3x}}{2}$$ ($f(D) = (D-1)(D-2)$, $f(3) \rightarrow ?$)

— (1)

By General Method →

$$(D-1)\underbrace{(D-2)y}_{u} = e^{3x}$$

$$(D-1)u = e^{3x} \leftarrow \text{linear in } u.$$

$$I.F = e^{\int -1\,dx} = e^{-x}$$

$$u \cdot e^{-x} = \int e^{3x} \cdot e^{-x}\,dx + c$$

$$u\,e^{-x} = \frac{e^{2x}}{2}$$

$$u(x) = \frac{e^{3x}}{2}$$

Linear in y → $(D-2)y = \frac{e^{3x}}{2}$

$$I.F = e^{\int -2\,dx} = e^{-2x}$$

$$y \cdot e^{-2x} = \int \frac{e^{3x}}{2} \cdot e^{-2x}$$

$$y \cdot e^{-2x} = \frac{e^{x}}{2}$$

$$y = \frac{e^{3x}}{2}$$ (same as obtained in ①)

6.9 $(D-1)(D-3)^2 y = e^{3x}$

$F(D) = (D-1)(D-3)^2$

$F(3) = 0$

$y_p(x) = \frac{x^2}{2!} \frac{e^{3x}}{G_1(3)}$ $\quad (G_1(D) = D-1)$

$= \frac{x^2 e^{3x}}{2 \cdot 2} = \frac{x^2 e^{3x}}{4}$ — (1)

By General Method →

$(D-1)\underbrace{(D-3)^2 y}_{u} = e^{3x}$

Linear in u → $(D-1)u = e^{3x}$

I.F → e^{-x}

$u e^{-x} = \int e^{3x} \cdot e^{-x}$

$u e^{-x} = \frac{e^{2x}}{2}$

$u = \frac{e^{3x}}{2}$

$(D-3)^2 y = \frac{e^{3x}}{2}$

$(D-3)\underbrace{(D-3) y}_{v} = \frac{e^{3x}}{2}$

Linear in v → $(D-3)v = \frac{e^{3x}}{2}$

C)

$$I.F \rightarrow e^{-3x}$$

$$v \cdot e^{-3x} = \int \frac{e^{3x}}{2} \cdot e^{-3x}$$

$$\therefore \quad v e^{-3x} = \frac{x}{2} \Rightarrow v = \frac{x e^{3x}}{2}$$

Put the value of $v \rightarrow$

$$(D-3)y = \frac{x e^{3x}}{2}$$

$$I.F = e^{-3x}$$

$$y \cdot e^{-3x} = \int \frac{x}{2} e^{3x} \cdot e^{-3x}$$

$$y \cdot e^{-3x} = \frac{x^2}{4}$$

$$\Rightarrow y_p(x) = \frac{x^2 e^{3x}}{4}$$ (Same as obtained in ①)

Que → Let $y'' + py' + qy = e^{ax}$, $a \neq 0$

If Particular sol^n is of the form $x^2 e^{qx}$

a) $p > 0$, $q < 0$

b) $p < 0$, $q > 0$

c) $p \neq 0$, $q > 0$

d) $p < 0$, $q < 0$

Sol^n →

A Particular Sol^n is of the form → $x^2 e^{ax}$

(⇒ $D=a$ is the root of multiplicity 2. i.e a is the repeated root)

$F(D) = F'(D) = 0$ at $D=a$

$f(D) = D^2 + pD + q$

$f'(D) = 2D + p$

⇒ $f(a) = f'(a) = 0$

$a^2 + pa + q = 0$ — (1)

$2a + p = 0 \Rightarrow p = -2a$

Since $a \neq 0$

$\Rightarrow \boxed{p \neq 0}$

Putting the value of p in (1) →

$a^2 - 2a^2 + q = 0$

$q = a^2$

$\Rightarrow \boxed{q > 0}$

∴ option (c) is correct.

Que Let $y'' + py' + qy = e^{ax}$, $a \neq 0$

If P.I is of the form $x^2 e^{ax}$, then

a) $|p| > a$

b) $|p| < a$

c) $q > a$

d) $|q| < a$

Sol^n $p = -2a$ and $q = a^2$ (obtained in previous Que)

$|p| = |-2a|$

$|p| = 2|a|$

$\Rightarrow |p| > |a|$

$\Rightarrow \boxed{|p| > a}$

and $q = a^2 \not\Rightarrow q > a$

For $a \in (-1,1)$

Say $a = 1/2 \Rightarrow a^2 = 1/4 = 0.25$

$= 0.5$

$q = a^2 = 0.25 \not> 0.5$

i.e $q = a^2 < a$ for $a \in (-1,1)$

\$ $q = a^2 > a$ for $a \in (-\infty,1) \cup (1,\infty)$

$\therefore$ option c) is Wrong.

Que. Let $y'' + py' + qy = e^{ax}$, $a \neq 0$

if the Particular solution is of the form xe^{ax}

Then

a) $p > 0$, $q < 0$

b) $p < 0$, $q > 0$

c) $p > 0$, $q > 0$

d) None of these

Sol^n As the P.S is of the form $xe^{ax} \Rightarrow D = a$ is the root with multiplicity 1

$\Rightarrow f(a) = 0$ But $f'(a) \neq 0$

$F(D) = D^2 + pD + q$

$F'(D) = 2D + p$

$F(a) = a^2 + pa + q = 0$

$F'(a) = 2a + p \neq 0$

$\Rightarrow p \neq -2a$

$q = -(a^2 + pa) = -a(a+p)$

Since $a \neq 0$

$q \neq 0$, any real no.

$\therefore$ d) option is correct.

Que Let $y'' + py' + qy = e^{ax}$, $a \neq 0$

If the P.S is of the form xe^{ax} Then

a) $p \neq -2a$, $q \neq 0$

b) $p \neq -2a$, $q > 0$

c) $p \neq -2a$, $q < 0$

d) None of these

e) $p \neq -2a$, $q \in \mathbb{R}$? $(\mathbb{R} - \{0\})$

Soln $p \neq -2a$

& $q \neq 0$, any real no. (obtained in previous Que)

Result → Find condition on p, q, a s.t the differential eqn → $y'' + py' + qy = e^{ax}$ has bounded particular sol^n.

Sol^n →

Particular sol^n is given by → $\frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$ — (1)

When $f(a) = 0$ and $f'(a) \neq 0$ ⇒ a is not the repeated root

Then P.S is given by → $\frac{x e^{ax}}{G_1(a)}$, $\left(f(D) = (D-a)G_1(D)\right)$

$G_1(a) \neq 0$ — (2)

If $f(a) = 0$ and $f'(a) = 0$ & $f''(a) \neq 0$ ⇒ a is the rep. root

Then P.S $= \frac{x^2 e^{ax}}{2 G_1(a)}$, $G_1(a) \neq 0$ $\left(f(D) = (D-a)^2 G_1(D)\right)$ — (3)

From (2) & (3) → Sol^n cannot be bounded ($\because$ P.I.S $\to \infty$ as $x \to \infty$)

i.e Sol^n is Unbounded

From (1) →

P.I.S is of the form $\frac{e^{ax}}{f(a)}$, when $f(a) \neq 0$

For bounded sol^n 'a' must be 0

i.e $a = 0$

Bt $f(0) \neq 0$ ($\because f(a) \neq 0$)

$$P.S = \frac{1}{f(a)} = \frac{1}{f(0)}$$

In this case, Bounded P.S is obtained

($e^{ax} \xrightarrow{a>0} \infty$ as $x \to \infty$ ⇒ Sol^n is Unbounded for $a<0$ → Unbounded when $x \to -\infty$)

$$f(D) = D^2 + pD + q$$

As $f(0) \neq 0$

$$\Rightarrow 0 + 0 + q \neq 0$$

$$\Rightarrow q \neq 0$$

There is no condition on p.

Conclusion →

for Bounded particular sol^n →

$a = 0$

$q \neq 0$

and there is no condition on p.

i.e p is arbitrary

Que Find the condition on p, q and a s.t diff. eq^n has bounded General sol^n →

Diff eq^n → $y'' + py' + qy = e^{ax}$, $a \neq 0$

Sol^n → General sol^n = Complementary Fu^n + Particular Integral.

i.e $y(x) = y_c(x) + y_p(x)$

$y_c(x)$: $p = 0$, $q > 0$ (obtained in case of Homogeneous diff eq^n)

$y_p(x)$: $p \rightarrow$ arb., $q \neq 0$, $a = 0$

($y'' + ay' + b = 0$ For Bounded sol^n → $a = 0, b > 0$)

∴ For Bounded General Sol^{n} →

$p = 0$
and $q > 0$
and $a = 0$

Case II →

$$f(D)y = \sin ax \,/\, \cos ax$$

P.I is given by →

$$y_p(x) = \frac{\sin ax \,/\, \cos ax}{f(-a^2)}, \quad f(-a^2) \neq 0$$

(D^2 is replaced by $-a^2$ & $f(-a^2) \neq 0$)

Eg →

$$(D^2+4D+4)y = \sin x$$

$$y_p(x) = \frac{\sin x}{D^2+4D+4}$$ Here $a = 1$, $D^2 = -a^2 = -1$

$$= \frac{\sin x}{-1+4D+4} = \frac{\sin x}{4D+3}$$

$$= \frac{(4D-3)\sin x}{(4D+3)(4D-3)} = \frac{(4\cos x - 3\sin x)}{16D^2-9}$$

$$= \frac{4\cos x - 3\sin x}{-25}$$ Ans.

When $f(-a^2) = 0$ then

$$y_p(x) = \frac{x \sin ax \;\big|\; x \cos ax}{f'(-a^2)}, \quad f'(-a^2) \neq 0$$

When $f'(-a^2) = 0$ then

$$y_p(x) = \frac{x^2 \sin ax \;\big|\; x^2 \cos ax}{f''(-a^2)}, \quad f''(-a^2) \neq 0$$

and so on.

Example → $(D^2+4)y = \sin 2x$

$$y_p(x) = \frac{\sin 2x}{D^2+4}$$

$a = 2$, $D^2 = -a^2 = -4$

$$f(D) = D^2+4$$

$$f(-a^2) = -4+4 = 0$$

$$f'(D) = 2D$$

$$y_p(x) = \frac{x \sin 2x}{f'(-a^2)}$$

$$= \frac{x \sin 2x}{2D}$$

$$= \frac{1}{2} \frac{D(x \sin 2x)}{D^2}$$

$$= \frac{1}{2} \cdot \frac{\left[\sin 2x + 2x(\cos 2x)\right]}{-4}$$

$$= \frac{-1}{8} \left(\sin 2x + 2x \cos 2x\right) \quad \text{Ans.}$$

Que. Let $y'' + \alpha y = \cos 2x$

If P.I is of the form $x \sin x \rightarrow$

a) $\alpha = 0$

b) $\alpha = 4$

c) $\alpha = -4$

d) None of these.

Soln. As P.I is of the form $x \sin x$

$\Rightarrow f(-a^2) = 0$

and $f'(-a^2) \neq 0$.

$a = 2$; $D^2 = -a^2 = -4$

$f(D) = D^2 + \alpha$

$f(-a^2) = -a^2 + \alpha$ ($\because D^2$ is replaced by $-a^2$)

$0 = -4 + \alpha$

$\Rightarrow \alpha = 4$ Ans.

Result → $y'' + py' + qy = \sin ax \mid \cos ax$

Find condition on p, q and a s.t Particular Sol^n is bounded.

Sol^n →

Particular is of the form $= \dfrac{\sin ax \mid \cos ax}{f(-a^2)}$, $f(-a^2) \neq 0$ — (I)

When $f(-a^2) = 0$

P.S is of the form $= \dfrac{x \sin ax \mid x \cos ax}{f'(-a^2)}$, $f'(-a^2) \neq 0$ — (2)

When $f'(-a^2) = 0$

P.S is of the form $= \dfrac{x^2 \sin ax \mid x^2 \cos ax}{f''(-a^2)}$, $f''(-a^2) \neq 0$ — (3)

and so on.

In (2) and (3) Case → As x & x^2 term is there in particular Sol^n ∴ Bounded particular Sol^n is not possible (As $x \to \infty$, P.S $\to \infty$)

i.e Unbounded particular Sol^n.

∴ Bounded particular Sol^n is obtained in Case I → when $f(-a^2) \neq 0$ (As 1st Case is possible only when $f(-a^2) \neq 0$)

$f(D) = D^2 + pD + q$

$f(-a^2) = -a^2 + pD + q \neq 0$

Similarly →

Periodic particular sol^{n} is obtained when $f(-a^2) \neq 0$

(Ap x & x^2 are Non-Periodic)

Conclusion →

For Bounded Particular sol^{n} →

$f(-a^2) \neq 0$

If $f(-a^2) = 0$, alwx gives Unbounded Particular sol^{n}

Special Case → When $p=0$ in $y'' + py' + qy = \sin ax \,|\, \cos ax$

i.e $y'' + qy = \sin ax \,|\, \cos ax$

$$F(D) = D^2 + q$$

For Bounded Particular sol^n →

$$F(-a^2) \neq 0 \qquad D^2 = -a^2$$

i.e $-a^2 + q \neq 0$

$$\Rightarrow q \neq a^2$$

∴ For Bounded P.S →

$$\boxed{p=0 \;,\; q \neq a^2}$$

∴ For Periodic P.S → $\boxed{p=0 \;,\; q \neq a^2}$

Que Find the condition on p, q and a s.t diff eq^n has Bounded General sol^n →

$$y'' + py' + qy = \sin ax \,|\, \cos ax$$

Sol^n:

$$y(x) = y_c(x) + y_p(x)$$

$y_c(x)$: $p=0$, $q>0$ (obtained in case of Homog. Diff eq^n)

$y_p(x)$: $p=0$, $q \neq a^2$

∴ for Bounded General Sol^n →

$$p = 0 \; ; \; q > 0 \; , \; q \neq a^2$$

Same Condition for Periodic General Sol^n.

Que. General Sol^n of $y'' + py' + qy = \sin ax$ is periodic if →

a) $p = 0 \; , \; q = 4$

b) $p = 0 \; , \; q = 2$

c) $p = 0 \; , \; q = 4 \; , \; a = 2$

d) $p = 0 \; , \; q = 9 \; , \; a = -3$

e) None of these

Sol^n → Periodic General Sol^n is obtained When →

$$p = 0 \; , \; q > 0 \; , \; q \neq a^2$$

a) $p = 0 \; , \; q = 4$

If $a = 2$

$q = a^2 = 4$ But $q \neq a^2$

So, Not possible

b) $p = 0 \; , \; q = 2$

If $a = \sqrt{2}$

$q = a^2 = 2$ But $q \neq a^2$

So, Not possible

c) $p = 0 \; , \; q = 4 \; , \; a = 2$

$\Rightarrow a^2 = 4$ But $q \neq a^2$ ∴ Not Possible

If $q=9$, $a=-3$, $b=0$

$q = a^2 = 9$

But $q \neq a^2$

∴ Not Possible.

∴ None of these

Case III → If $f(D)y = x^m$ (Polynomial Form)

$$P.I = \frac{x^m}{f(D)}$$

$$= \frac{x^m}{D^r(1+\psi(D))}$$

$$= \frac{1}{D^r}(1+\psi(D))^{-1}x^m$$

i.e lowest degree of D is taken out as common.

E.g → $(D^2+4D+4)y = x^2$

$$P.I = \frac{x^2}{D^2+4D+4}$$

$$= \frac{x^2}{D^0(D^2+4D+4)}$$

$$= \frac{x^2}{(D+2)^2}$$

$$= (D+2)^{-2} x^2$$

$$= \frac{1}{4}\left(\frac{D}{2}+1\right)^{-2} x^2$$

$$= \frac{1}{4}\left[1 - 2\frac{D}{2} + \frac{(-2)(-3)}{2}\frac{D^2}{4} + \frac{(-2)(-3)(-4)}{3}\frac{D^3}{8} + \dots\right] x^2$$

$$= \frac{1}{4}\left[1 - \frac{2D}{2} + \frac{3D^2}{4} + \dots\right] x^2$$

$$= \frac{1}{4}\left[1 - D + \frac{3D^2}{4} - \dots\right] x^2$$

$$= \frac{1}{4}\left[x^2 - 2x + \frac{3\cdot 2}{4} + 0 - \dots\right]$$

$$= \frac{1}{4}\left(x^2 - 2x + \frac{3}{2}\right)$$ Ans.

Case IV → If $f(D)y = \underbrace{e^{ax}\phi(x)}$ $\phi \rightarrow$ Trignometric fu^n / Polynomial

i.e $Q(x)$ is product of 2 fu^{n}s

$$y_p(x) = e^{ax}\frac{\phi(x)}{f(D+a)}$$

Note → If $Q(x) = \sec x, \tan x$ then use the General Method to find the Particular Integral.

RESULT → (fix → $f(D)$, vary → $Q(x)$)

If $f(D)y = Q(x)$ is the linear Differential eqⁿ with constant coefficient

If y_1 is the solⁿ of $\rightarrow$ $f(D)y = Q_1(x)$

If y_2 is the solⁿ of $\rightarrow$ $f(D)y = Q_2(x)$

⋮

If y_n is the solⁿ of $\rightarrow$ $f(D)y = Q_n(x)$

then

$y_1 + y_2 + \text{---} + y_n$ will be a solⁿ of

$$f(D)y = Q_1(x) + Q_2(x) + \text{---} + Q_n(x)$$

OR

If $y_1, y_2 \text{---} y_n$ are solⁿs of

$f(D)y = Q_1(x)$, $f(D)y = Q_2(x)$ ---

--- , $f(D)y = Q_n(x)$

then

$$f(D)y = Q_1(x) + Q_2(x) + \text{---} + Q_n(x)$$

has solⁿ $y_1 + y_2 + \text{---} + y_n$

Cor

If $y'' + py' + qy = Q(x)$ has sol^n y_1

& $y'' + py' + qy = R(x)$ has sol^n y_2

then

$$y'' + py' + qy = Q(x) + R(x) \text{ has sol}^n \; y_1 + y_2.$$

(Here L.H.S remains same R.H.S vary)

Another Method To find P.I →

Que. $(D^2 + 2D + 4)y = x^2$ ← (*) If x^m is there Diff. m times.

Diff 2 times →

$$(D^3 + 2D^2 + 4D)y = 2x \quad — (1)$$

$$(D^4 + 2D^3 + 4D^2)y = 2 \quad — (2)$$

$$P.I = \frac{2e^{0x}}{D^4 + 2D^3 + 4D^2} = \frac{2e^{0x}}{D^2(D^2 + 2D + 4)} = \frac{2}{4D^2}$$

P.I of last term and last eq^n are same.

$$P.I \text{ of last term} = \frac{2}{4D^2}$$

∴ Take $4D^2 y = 2$ — (3) (instead of $(D^4 + 2D^3 + 4D^2)y = 2$)

Using in (1) →

$$0 + 2 \cdot \frac{1}{2} + 4Dy = 2x$$

$$Dy = \frac{2x - 1}{4}$$

$$\frac{dy}{dx} = \frac{2x - 1}{4} \quad — (4)$$

Using ③ and ④ in ⋇ ⟶

$$\frac{1}{2} + 2\left(\frac{2x-1}{4}\right) + 4y = x^2$$

$$4y(x) = x^2 - \frac{1}{2} - \frac{1}{2}(2x-1)$$

$$= x^2 - \frac{1}{2} - x + \frac{1}{2}$$

$$= x^2 - x$$

$$y(x) = \frac{1}{4}(x^2 - x) = \frac{x}{4}(x-1)$$

Case I → If $f(D)y = x \cdot Q(x)$

Then

$$P.I = \left\{ x - \frac{f'(D)}{f(D)} \right\} \frac{1}{f(D)} Q(x)$$

Que $y'' + 4y = x \sin 2x$ find P.I?

Sol^n → $f(D) = D^2 + 4$

$$P.I = \left\{ x - \frac{(D^2+4)'}{(D^2+4)} \right\} \frac{1}{D^2+4} \sin 2x$$

$$= \left\{ x - \frac{2D}{D^2+4} \right\} \frac{1}{D^2+4} \sin 2x$$

$$= \frac{x}{D^2+4} \sin 2x - \frac{2D}{D^2+4} \left\{ \frac{x \sin 2x}{2D} \right\}$$

$$= x \cdot \frac{x^2 \sin 2x}{2D} - \frac{1}{D^2+4} \{ x \sin 2x \}$$

$$= x \cdot \frac{x^2 \sin 2x}{2D} - P.I$$

P.T.O.

$$2P.I = x \cdot \frac{x^2 \sin 2x}{2D}$$

$$P.I = \frac{1}{2} \cdot \frac{x^3 \sin 2x}{2D}$$

$$= \frac{1}{4} \int x \sin 2x \, dx$$

$$f(D) = D^2+4$$

Then

$$P.I = \left\{x - \frac{2D}{D^2+4}\right\}\frac{1}{D^2+4}\sin 2x$$

$$= x\left\{\frac{\sin 2x}{D^2+4}\right\} - \frac{2D}{D^2+4}\left\{\frac{1}{D^2+4}\sin 2x\right\}$$

$$= x^2\left\{\frac{\sin 2x}{2D}\right\} - \frac{2D}{D^2+4}\frac{x}{2D}\sin 2x$$

$$= \frac{x^2}{2}\int \sin 2x\,dx - \frac{2D}{D^2+4}\cdot\frac{x}{2}\int \sin 2x\,dx$$

$$= -\frac{x^2}{4}\cos 2x + \frac{Dx}{D^2+4}\left(\frac{\cos 2x}{2}\right)$$

$$= -\frac{x^2}{4}\cos 2x + \frac{1}{2(D^2+4)}\left(\cos 2x - 2x\sin 2x\right)$$

$$= -\frac{x^2}{4}\cos 2x + \frac{1}{2x}\left\{\left[\frac{1}{D^2+4}\cos 2x\right] - \left[\frac{x\sin 2x}{D^2+4}\right]\right\}$$

$$= -\frac{x^2}{4}\cos 2x + \frac{1}{2x}\left(\frac{x(\cos 2x)}{2D}\right) - P.I$$

$$= -\frac{x^2}{4}\cos 2x + \frac{x}{2\cdot 2\cdot 2}\sin 2x - P.I$$

$$2P.I = -\frac{x^2}{4}\cos 2x + \frac{x\sin 2x}{8}$$

$$P.I = -\frac{x^2}{8}\cos 2x + \frac{x\sin 2x}{16}$$

Rule

$$f(D)y = e^{ax}Q(x)$$

$$P.I = e^{ax}\frac{1}{D+a}Q(x)$$

$f(D) = D^n + a_1 D^{n-1} + \ldots + a_{n-1}D + a_n$

a_{n-1} X $\quad D(e^{ax}Q(x)) = ae^{ax}Q(x) + e^{ax}D(Q(x))$

$$= e^{ax}(D+a)Q(x)$$

a_{n-2} X $\quad D^2(e^{ax}Q(x)) = e^{ax}(D+a)^2Q(x)$

⋮

Multiplying with coefficients & adding $\rightarrow$

$$f(D)e^{ax}Q(x) = e^{ax}\left(f(D+a)\right)Q(x)$$

Second order linear Diff eq^n → Method of variation of parameters

Pair of complementary Fu^n

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{—(1)}$$

Let u and v are two L.I solutions of

$$y'' + P(x)y' + Q(x)y = 0$$ (Homog. Second order linear Diff eq^n)

C.F → $y(x) = c_1 u + c_2 v$

Assume

Sol^n of $y''+Py'+Qy=R$ ← $y_p(x) = A(x)u + B(x)v$

Converting c_1 and c_2 into variables $A(x)$ & $B(x)$ for this Rsn → Variation of parameters.

$$y_p'(x) = A'u + Au' + B'v + Bv' \quad \text{—(2)}$$

Let $A'u + B'v = 0$ —(*)

Otherwise u & v are L.D

$$y_p'(x) = Au' + Bv' \quad \text{—(3)}$$

$$y_p''(x) = A'u' + Au'' + B'v' + Bv'' \quad \text{—(4)}$$

Use (3) and (4) in (1) →

$$(A'u' + Au'' + B'v' + Bv'') + P(x)(Au' + Bv') + Q(x)(Au + Bv) = R(x)$$

$$A(u'' + P(x)u' + Q(x)u) + B(v'' + P(x)v' + Q(x)v) + A'u' + B'v' = R(x)$$

$$\Rightarrow A'u' + B'v' = R(x) \quad \text{—(5)}$$

From (*) and (5) →

$$A'u + B'v = 0$$

$$A'u' + B'v' - R(x) = 0$$

$$\frac{A'}{-vR} = \frac{B'}{uR} = \frac{1}{W[u,v]}$$

$$\Rightarrow A' = \frac{-vR}{W[u,v]}, \quad B' = \frac{uR}{W[u,v]}$$

$$A(x) = -\int \frac{vR}{W} dx + C_1$$

$$B(x) = \int \frac{uR}{W} dx + C_2$$

If only P.S is asked Don't add Constants.

$$\therefore y = -u\int \frac{vR}{W} dx + v\int \frac{uR}{W} dx + C_1 u + C_2 v$$

G.S (as it Contains arb. Constants)

Que can be asked in 2 ways →

Find A(x) and B(x)?

or

When A & B are given Find R(x)?

Que: The P.I

$$y_p(x) = x v_1(x) + \frac{1}{x} v_2(x)$$

of $x^2y'' + xy' - y = \frac{1}{x+1}$, $x > 0$

Then v_1 and v_2 are given by $\longrightarrow$

a) $x \cdot v_1' - \frac{1}{x} v_2'(x) = 0$, $v_1'(x) - \frac{1}{x^2} v_2' = \frac{1}{x^2(x+1)}$

b) $x \cdot v_1' + \frac{1}{x} v_2'(x) = 0$, $v_1'(x) - \frac{1}{x^2} v_2' = \frac{1}{x^2(x+1)}$

c) $x \cdot v_1' - \frac{1}{x} v_2'(x) = 0$, $v_1'(x) + \frac{1}{x} v_2' = \frac{1}{x+1}$

d) None of these.

Sol^n: $y_p(x) = x\underbrace{v_1(x)}_{A(x)} + \frac{1}{x}\underbrace{v_2(x)}_{B(x)}$

Find $A(x)$ & $B(x)$? As $A'(x)$ & $B'(x)$ are in options so use eq^n * and (5)

$u = x$, $v = \frac{1}{x}$

$$W[u,v] = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -1/x - 1/x = -\frac{2}{x}$$

$A'u + B'v = 0$

& $A'u' + B'v' = R(x)$

$A'x + \frac{B'}{x} = 0 \Rightarrow x v_1' + \frac{1}{x} v_2' = 0$

$A' - \frac{1}{x^2} B' = \frac{1}{x^2(x+1)} \Rightarrow v_1' - \frac{1}{x^2} v_2' = \frac{1}{x^2(x+1)}$

Ans

Que. If $f(x)$ & $x f(x)$ are Particular Sol^n of $y'' + R(x) y' + S(x) y = 0$ Homog?

Then Gr.S of $y'' + R(x) y' + S(x) y = f(x)$.

a) $\left(\frac{-x^2}{2} + \alpha x + \beta\right) f(x)$ c) $(-x^2 + \alpha x + \beta) f(x)$

b) $\left(\frac{x^2}{2} + \alpha x + \beta\right) f(x)$ d) $(x^3 + \alpha x + \beta) f(x)$

Sol^n:

$$y_p(x) = A(x)\, u + B(x)\, v$$

$u = f(x)$

$v = x f(x)$

$$W[u,v] = \begin{vmatrix} f(x) & x f(x) \\ f'(x) & f(x) + x f'(x) \end{vmatrix}$$

$$= f^2(x) + x f(x) f'(x) - x f(x) f'(x)$$

$$= f^2(x)$$

$R(x) = f(x)$

$$A(x) = -\int \frac{x f^2(x)}{f^2(x)}\, dx + c_1 = -\frac{x^2}{2} + c_1$$

$$B(x) = \int \frac{f(x) f(x)}{f^2(x)}\, dx + c_2 = x + c_2$$

As Gr.S is asked so add c_1 & c_2.

$$\text{Gr.S} = \left(\frac{-x^2}{2} + c_1\right) f(x) + (x + c_2)\, x f(x)$$

$$= \frac{x^2}{2} f(x) + c_1 f(x) + c_2 x f(x)$$

$$= \left(\frac{x^2}{2} + c_2 x + c_1\right) f(x)$$ Ans.

Reduction of order

Let $u(x)$ be a sol^n of

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{---(1)}$$

Let Particular sol^n is

$$\rightarrow y(x) = u\,y_1(x) \text{ of } y'' + P(x)y' + Q(x)y = R(x) \quad \text{---(2)}$$

$$y' = y_1'u + y_1u'$$

$$y'' = y_1''u + 2y_1'u' + y_1u''$$

Using in (2) $\rightarrow$

$$(y_1''u + 2y_1'u' + y_1u'') + P(x)(y_1'u + y_1u') + Q(x)y_1u = R(x)$$

$$y_1''u + y_1'(2u' + P(x)u) + y_1(u'' + P(x)u' + Q(x)u) = R(x)$$

As u is the sol^n of eq^n (1)

$$\therefore \quad y_1''u + y_1'(2u' + P(x)u) = R(x)$$

$$\boxed{y_1'' + y_1'\left(P(x) + \frac{2u'}{u}\right) = \frac{R(x)}{u}} \quad \text{---(3)}$$

Reduction of order

Let $\boxed{y_1' = v}$

$$v' + v\left(P(x) + \frac{2u'}{u}\right) = \frac{R(x)}{u} \leftarrow \text{linear in } v$$

$$\text{I.F} = e^{\int P(x)dx} \cdot e^{\int \frac{2u'}{u}dx}$$

$$= e^{\int P(x)dx}\, e^{2\log u}$$

$$= u^2 e^{\int P dx}$$

$$v(x)\cdot u^2 = \int \frac{R(x)}{u}\cdot u^2 e^{\int P dx} + q$$

(If q is added then G.S Without $q \rightarrow$ P.S)

$$\therefore \quad y_1'u^2 = \int R(x)\,u\,e^{\int P(x)dx} + q$$

So add constant accordingly what we need G.S/P.S

gives the value of $y_1(x)$

2nd to 1st order $\quad y(x) = u\,y_1(x) \rightarrow$ G.S if constants are there P.S if without constants

Que $xy'' - y' + (1-x)y = x e^{-x}$

Find P.I ?

Sol^n → First find u?

(In options → One sol^n will be either coff. of q1 or q2)

$P(x) = -1/x$

$Q(x) = \frac{1-x}{x} = \frac{1}{x} - 1$

$R(x) = e^{-x}$

$u = e^x$ is the sol^n of $xy'' - y' + (1-x)y = 0$.

P.I is obtained from eq^n (3) →

$$y_1'' + y_1'\left(-\frac{1}{x} + \frac{2e^x}{e^x}\right) = \frac{x^2 e^{-x}}{e^x}$$

$$y_1'' + y_1'\left(-\frac{1}{x} + 2\right) = x^2 e^{-2x}$$

Linear in y_1' → $y_1' = v$

$$I.F = e^{\int -\frac{1}{x} + 2\,dx} = e^{-\log x + 2x} = \frac{1}{x} e^{2x}$$

$$y_1' \cdot \frac{1}{x} e^{2x} = \int x^2 e^{-2x} \cdot \frac{1}{x} e^{2x}\,dx$$ (No need to add Constant)

$$y_1' \frac{e^{2x}}{x} = x$$

$$y_1' = x^2 e^{-2x}$$

$$y_1 = \int x^2 e^{-2x}\,dx$$

$$= -\frac{x^2 e^{-2x}}{2} + \int x e^{-2x}\,dx$$

$$= \frac{-x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{1}{4} e^{-2x}$$

$$y_p(x) = u y_1(x)$$

$$= e^x\left(-\frac{1}{2}x^2e^{-2x} - \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right)$$

$$= -\frac{1}{2}\left(x^2e^{-x} + xe^{-x} + \frac{1}{2}e^{-x}\right)$$

$\to 0 \quad \to 0 \quad \to 0$

$\to \infty \quad \to \infty \quad \to \infty$

L·H·rule $\left(\frac{x^2}{e^x} = \frac{2x}{e^x} = \frac{2}{e^x}\right)$

When $x \to \infty$ $\quad y(x) \to 0$

When $x \to -\infty$ $\quad y(x) \to -\infty$

When $x \to 0$ $\quad y(x) \to -1/4$

E.g $\quad xy'' - (2x+1)y' + (x+1)y = 0$ $\quad$ Counter E.g. →

Let $\alpha_0, \alpha_1 \in \mathbb{R}$ s.t

$y(0) = \alpha_0$

$y'(0) = \alpha_1$

} ⇒ Find curve passing through $(0, \alpha_0)$ & whose derivative y' curve passing through $(0, \alpha_1)$

Then

a) $\forall (\alpha_0, \alpha_1) \in \mathbb{R}^2$ diff eq^n has unique sol^n

b) $\forall \alpha_0 = \alpha_1 \in \mathbb{R}^2$ diff eq^n has unique sol^n

c) $\forall \alpha_0 \neq \alpha_1 \in \mathbb{R}^2$ diff eq^n has unique sol^n

d) $\exists\ \alpha_0, \alpha_1$ s.t diff eq^n has No sol^n

e) $\exists\ \alpha_0, \alpha_1$ s.t diff eq^n has Infinite sol^n

Sol^n →

$$P(x) = -2 + \frac{1}{x}$$

$$Q(x) = 1 - \frac{1}{x}$$

$$R(x) = 0$$

Let $u = e^x$ is the sol^n of $xy'' - (2x-1)y' + (x-1)y = 0$

Let $y(x) = u \cdot y_1$ is a Gr.S (∵ Initial Values are given at 0)

from ③ →

$$\therefore y_1'' + y_1'\left[\left(-2 + \frac{1}{x}\right) + 2\right] = 0$$

$$y_1'' + \frac{1}{x} y_1' = 0$$

Let $y_1' = v_1$

linear in v_1 →

$$v_1' + \frac{1}{x} v_1 = 0$$

$$I.F = e^{\int 1/x\, dx} = e^{\log x} = x$$

$$v_1 \cdot x = c_1$$

$$y_1' \cdot x = c_1$$

$$y_1' = \frac{c_1}{x}$$

$$y_1 = c_1 \log x + c_2$$

$$\therefore y(x) = (c_1 \log x + c_2) e^x$$

$$y(x) = c_1 e^x \log x + c_2 e^x \quad \text{—— (1)}$$

4 Cases arise →

1) When $c_1 = c_2 = 0$

$\Rightarrow y(x) \equiv 0$

2) When $c_1 \neq 0$, $c_2 = 0$

$\Rightarrow y(x) = c_1 e^x \log x$

3) When $c_1 = 0$, $c_2 \neq 0$

$\Rightarrow y(x) = c_2 e^x$

4) When $c_1 \neq 0$, $c_2 \neq 0$

$\Rightarrow y(x) = c_1 e^x \log x + c_2 e^x$

Note →
Never put Initial Conditions in (1) directly.
If we put $y(0) = a_0$, $y'(0) = a_1$ in (1) then it seems that for any choice of a_0 and a_1 c_1 has no solution)
So discuss all the 4 cases in Que.

At $x = 0$ $\log 0$ → does not exist $(-\infty)$

$\therefore$ 2) & 4) → Not Possible.

In 1st Case →

$y(0) = 0 \Rightarrow a_0 = 0$

$y'(0) = 0 \Rightarrow a_1 = 0$

$\Rightarrow a_0 = a_1 = 0$

In 3rd case → $y'(x) = c_2 e^x$

$y(0) = c_2 \Rightarrow a_0 = c_2$

$y'(0) = c_2 \Rightarrow a_1 = c_2$

$\Rightarrow a_0 = a_1 = c_1 \neq 0$

$\Rightarrow a_0 = a_1 \longrightarrow$ Unique sol^n.

(In both cases → $a_0 = a_1 = 0$, $a_0 = a_1 = c_1 \neq 0$)

Sol^{n} exists only when $a_0 = a_1 = 0$

or $a_0 = a_1 = q \neq 0$

For $a_0 \neq a_1$ → No Sol^{n}.

Infinite Sol^{n} does not exist

Method

Change of I.V →

$$y'' + P(x)y' + Q(x)y = R(x) \quad -①$$

Let transformation is $z = f(x)$ ← Changing Independent Variable from x to z

Using this Reducing into Constant Coff. diff eq^{n}.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= z' \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(z' \frac{dy}{dz}\right)$$

$$= z'' \frac{dy}{dz} + z' \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= z'' \frac{dy}{dz} + z' \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$$

$$= z'' \frac{dy}{dz} + (z')^2 \frac{d^2y}{dz^2}$$

Now y is fu^{n} of z.

$$\left(z'' \frac{dy}{dz} + (z')^2 \frac{d^2y}{dz^2}\right) + P(x)z'y' + Q(x)y = R(x)$$

$$(z')^2 y'' + \left(z'' + z'P(x)\right)y' + Q(x)y = R(x)$$

$$y'' + \left(\frac{z'' + z' P(x)}{(z')^2}\right) y' + \frac{Q(x)\,y}{(z')^2} = \frac{R(x)}{(z')^2}$$

Let $\boxed{\frac{Q(x)}{(z')^2} = c}$ ← Coefficient of $Q(x)y$.

(When we use the transf. c will cancel out)

$$(z')^2 = Q(x)$$

$$z' = \sqrt{Q(x)}$$

$$\boxed{z(x) = \int \sqrt{Q(x)}\, dx}$$

When $\frac{Q(x)}{(z')^2}$ is Constant Only then Variable Coefficient diff eq^n is reducible into Constant Coff diff. eq^n.

$$\frac{Q(x)}{(z')^2} = c \iff \frac{z'' + z' P(x)}{(z')^2} = \text{Constant}$$

Ap $(z')^2 = Q(x)$

$$2z'z'' = Q'(x)$$

$$z'' = \frac{Q'(x)}{2\sqrt{Q(x)}}$$

$$\frac{z'' + z' P(x)}{(z')^2} = \frac{\frac{Q'(x)}{2\sqrt{Q(x)}} + \sqrt{Q(x)} \cdot P(x)}{Q(x)}$$

$$= \frac{Q'(x) + 2Q(x)P(x)}{2\,[Q(x)]^{3/2}}$$

∴ When $\boxed{\frac{Q' + 2PQ}{2(Q)^{3/2}}}$ is Constant

Then the transformation $z(x) = \int \sqrt{Q(x)}\, dx$ is used by which diff eq^n is with Constant Coefficient.

As in Cauchy eq^n → Consider particular case of above article →

$$x^2y'' + xy' + y = 0$$

$$P(x) = 1/x$$

$$Q(x) = 1/x^2$$

$$Q' = \frac{-2}{x^3}$$

$$\frac{Q' + 2PQ}{2(Q)^{3/2}} = \frac{-2/x^3 + 2 \cdot 1/x \cdot 1/x^2}{2 \cdot 1/x^3}$$

$$= \frac{-2/x^3 + 2/x^3}{2/x^3} = 0 = \text{Constant}$$

∴ Transformation is given by →

$$z = \int \sqrt{Q(x)}\, dx$$

$$= \int \frac{1}{x}\, dx$$

$$= \log x$$

$$\boxed{x = e^z}$$, same we have applied

Apply this Method →
Only when $Q(x)$ is of the square form.

Qu. $y'' + \tan x y' + 2\cos^2 y = 2\cos^4 x$ find P.S?

a) Bd & Periodic
b) UnBounded & Periodic
c) Bd & Non-Periodic
d) Unbounded & Non-Periodic

Sol^n,

$P(x) = \tan x$

$Q(x) = 2\cos^2 x$

$Q'(x) = 4\cos x \sin x$

$R(x) = 2\cos^4 x$

$$\frac{Q' + 2PQ}{2(Q)^{3/2}} = \frac{4\cos x \sin x - 4\tan x \cos^2 x}{2(-2\cos^2 x)^{3/2}}$$

$= 0$ = Constant.

Transformation →

$$z = \int \sqrt{Q(x)}\, dx$$

Choose $\frac{Q(x)}{(z')^2} = C = -2$ (Coefficient of $Q(x)y$)

$\Rightarrow Q(x) = -2(z')^2$

$-2\cos^2 x = (z')^2 \times -2$

$(z')^2 = \cos^2 x$

$z' = \cos x$

$z = \sin x$ is the required transformation.

then diff eq^n reduces to Constant Coefficient.

$$y'' + \left(\frac{z'' + z'P(x)}{z'^2}\right)y' + \frac{Q(x)}{z'^2}y = \frac{R(x)}{z'^2}$$

$$y'' - 2y' = \frac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x \longrightarrow$$

$y'' - 2y' = 2(1 - \sin^2 x) = 2(1 - z^2)$

I.f $= e^{-2z}$

$y'' + py' + qy = \underset{I}{e^{ax}} \mid \underbrace{\sin ax \mid \cos ax}_{II} \mid \underset{III}{x^m} \mid x^m e^{ax} \mid x^m \sin ax \mid e^{ax} \sin bx$, $\cos bx$

Then the P.I $\longrightarrow$

I Case $\rightarrow$ form of P.I $\longrightarrow$

$$y_p(x) = A e^{ax}$$

$$= Ax e^{ax}$$

$$= Ax^2 e^{ax}$$

II Case $\rightarrow$

$$y_p(x) = A \sin ax + B \cos ax$$

P.T.O

Sol^n $\rightarrow$ P.I $\rightarrow \frac{2(1-z^2)}{D^2-2}$

$f(x) \rightarrow$ Period α

$f(\beta x) \rightarrow$ Period $\frac{\alpha}{|\beta|}$

$\cos x \rightarrow$ Period 2π

$\cos 2x \rightarrow$ Period π

$= z^2$

$= \sin^2 x$

$$y \cdot e^{-2z} = 2\int (1-z^2) e^{-2z} dz$$

$$= 2\left[\frac{(1-z^2) e^{-2z}}{-2} - \int \frac{-2z \cdot e^{-2z}}{-2} dz \right]$$

$$= 2\left(\frac{(z^2-1) e^{-2z}}{2} - \frac{z e^{-2z}}{-2} + \frac{e^{-2z}}{4} \right)$$

$$= (z^2-1) e^{-2z} + z e^{-2z} + \frac{e^{-2z}}{2}$$

$$y = z^2 - 1 + z + 1/2$$

$$= z^2 + z - \frac{3}{2}$$

$$= \sin^2 x + \sin x - \frac{3}{2}$$

$a + f(x) \rightarrow$ has same period as that of $f(x)$

$y_p(x) = \sin^2 x \rightarrow$ Product of 2 bdd fu^{n}s so bounded

$= \frac{1 - \cos 2x}{2} = 1/2 - \cos 2x \rightarrow$ Periodic with Period π

III case $y'' + py' + qy = x^m$ —(1)

Write possible derivatives of x^m →

$\{x^m, x^{m-1}, x^{m-2}$ _____ $x, 1\}$

$y_p(x) = A_1 x^m + A_2 x^{m-1} + A_3 x^{m-2} +$ _____ $+ A_n x + A_{n+1}$

Put this in (1) and Compare Coefficient to get the particular Sol^n.

Set of derivatives → finite (otherwise Linear Combination is not possible)

This is the method of

Undetermined Coefficients.

In case I → e^{ax} $\quad y'' + py' + qy = e^{ax}$

Possible → $\{e^{ax}\}$

Derivative

$y_p(x) = Ae^{ax}$

$A(a^2 + ap + q)e^{ax} = e^{ax}$

$A = \frac{1}{a^2 + ap + q} = \frac{1}{F(a)}$

$$y'' + py' + qy = Q_1(x) + Q_2(x) + Q_3(x) + \text{_____}$$

where $Q_i(x)$ is U.C functions.

Method → To Find Particular Sol^n is of Which form.

(1) Find U.C set →

i.e $S_i = U_{Q_i(x)} = \{Q_i(x), Q_i'(x), \text{____}\}$

(2) If $S_i \subseteq S_j$ then remove S_i

(3) If Complementary Sol^n → y_1, y_2 (nth order → n C.F.)

(4) If any $y_i \in S_j$ (y_i → C.S.)

Then multiply S_j with x^k s.t

$y_i \notin S_j \; \forall j$

(5) If any $S_i \subseteq S_j$ then remove S_i (Validate step 2 again)

(6) Take the Linear combination of fu^n's of US_i.

This method Gives only Form of Particular Sol^n

(Donot Find Particular Sol^n by this Method)

E.g. $y'' - 4y' + 4y = x^2 \sin 2x$

① Find U.C set. $\{1, x, x^2\}$ $\{\sin 2x, \cos 2x\}$

↑ No need to write 2, as it will merge in L.C.

$\{1, x, x^2\} \times \{\sin 2x, \cos 2x\}$

$\{\sin 2x, \cos 2x, x\sin 2x, x\cos 2x, x^2 \sin 2x, x^2 \cos 2x\}$ — (1)

$y_p(x)$ will be L.C of these.

Find C.F → ?

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = 2$$

C.F includes 2 factors → e^{2x}, xe^{2x}

P.S form will be given by (1).

E.g → $y'' - 4y' + 4y = x^2 e^{2x}$

$\{1, x, x^2\} \times \{e^{2x}\}$

$\{e^{2x}, xe^{2x}, x^2e^{2x}\}$

$y_p(x) = Ae^{2x} + Bxe^{2x} + Cx^2e^{2x}$ — (2)

This is not P.I

(As 1st two terms will merge with C.F terms)

C.F includes 2 factors → e^{2x}, xe^{2x}

Multiply (2) with $x^{\text{least power}}$ s.t if any of the term of P.I is not merged with C.F.

$$y_p(x) = x^2\left(Ae^{2x} + Bxe^{2x} + Cx^2e^{2x}\right)$$
$$= Ax^2e^{2x} + Bx^3e^{2x} + Cx^4e^{2x}$$

Eg → $y'' - 4y' + 4y = x^2e^{2x} + xe^{2x}$

$\{1, x, x^2\} \times \{e^{2x}\}$

$\{e^{2x}, xe^{2x}, x^2e^{2x}\}$

I

$\{1, x\} \times \{e^{2x}\}$

$\{e^{2x}, xe^{2x}\}$

J

$J \subseteq I$

Directly Remove J

Eg → $y'' - 4y' + 4y = x\sin 2x$

$\{1, x\} \times \{\sin 2x, \cos 2x\}$

$\{\sin 2x, \cos 2x, x\sin 2x, x\cos 2x\}$

System of eq →

Let $y_1(t)$ and $y_2(t)$ are two dependent variable of t.

$$\times G_1(D) \quad f_1(D)\,y_1 + f_2(D)\,y_2 = Q(t) \quad \text{—(1)}$$

$$\times f_1(D) \quad G_1(D)\,y_1 + G_2(D)\,y_2 = R(t) \quad \text{—(2)}$$

After Multiplying & then Subtracting →

$$(G_1 f_2 - G_2 f_1)\,y_2 = G_1 Q(t) - f_1(D)\,R(t)$$

Which is simple ODE in y_2.

From here we can easily find $y_2 = C.F + P.I$

and then using in (2) we can easily find y_1.

E.g →

$$D \times (D+1)\,y_1 + (D+2)\,y_2 = e^t \quad \text{—(1)}$$

$$(D+2) \times (2D+1)\,y_1 - D\,y_2 = \sin t \quad \text{—(2)}$$

$$[D(D+1) + (D+2)(2D+1)]\,y_1 = e^t + (D+2)\sin t$$

$$[D^2 + D + 2D^2 + 4D + D + 2]\,y_1 = e^t + \cos t + 2\sin t$$

$$[3D^2 + 6D + 2]\,y_1 = e^t + \cos t + 2\sin t$$

A.E → $3m^2 + 6m + 2 = 0$

$$m_1, m_2 = -1 \pm \frac{1}{\sqrt{3}} \qquad \left(\frac{-6 \pm \sqrt{36-24}}{6}, \frac{-6 \pm \sqrt{12}}{6}\right)$$

$$C.F = C_1 e^{(-1+1/\sqrt{3})t} + C_2 e^{(-1-1/\sqrt{3})t}$$

$$P.I = \frac{1}{3D^2+6D+2}(e^t + \cos t + 2\sin t)$$

$$= \frac{1}{11}e^t + \frac{1}{6D-1}\cos t + \frac{2}{6D-1}\sin t$$

$$\frac{6D+1}{(D+1)(6D-1)} = \frac{6D+1}{6D^2-1} = \frac{6D+1}{-37}$$

$$= \frac{1}{11}e^t - \frac{1}{37}(\cos t + 6(\sin t) + 12\cos t + 2\sin t$$

$$= \frac{1}{11}e^t - \frac{1}{37}(13\cos t - 4\sin t)$$

Using in (2) →

$$Dy_2 = (2D+1)y_1 - \sin t$$

$$= 2Dy_1 + y_1 - \sin t$$

$$y_2 = 2y_1 + \int y_1 - \sin t$$

from here we will get y_2

$$y_2 = 2\left\{\left[c_1 e^{(-1+1/\sqrt{3})t} + c_2 e^{(-1-1/\sqrt{3})t}\right] + \frac{1}{11}e^t - \frac{1}{37}(13\cos t - 4\sin t)\right\}$$

$$+ \frac{c_1 e^{(-1+1/\sqrt{3})t}}{(-1+1/\sqrt{3})} + \frac{c_2 e^{(-1-1/\sqrt{3})t}}{(-1-1/\sqrt{3})} + \frac{1}{11}e^t$$

$$- \frac{1}{37}(13\sin t + 4\cos t) - \sin t$$

This method is for Non-Homogeneous.

Eg →

$$(D+1)y_1 + (D+2)y_2 = e^t \quad \text{— (1)}$$

$$(2D+1)y_1 - Dy_2 = 0 \quad \text{— (2)}$$

$$y_1(t) = c_1 e^{(-1+1/\sqrt{3})t} + c_2 e^{(-1-1/\sqrt{3})t} + \frac{1}{11}e^t$$

Using in (2) →

$$Dy_2 = (2D+1)y_1$$

$$= 2Dy_1 + y_1 \qquad (\text{Diff \& } \int \text{ will cancel out})$$

$$y_2 = 2y_1 + \int y_1$$

$$= 2\left(c_1 e^{(-1+1/\sqrt{3})t} + c_2 e^{(-1-1/\sqrt{3})t} + \frac{1}{11}e^t\right)$$

$$+ \frac{c_1 e^{(-1+\frac{1}{\sqrt{3}})t}}{-1+\frac{1}{\sqrt{3}}} + \frac{c_2 e^{(-1-1/\sqrt{3})t}}{-1-1/\sqrt{3}} + \frac{1}{11}e^t$$

We can find y_2 simply also as we find y_1 →

$$[(2D+1)(D+2) + (D+1)D]\, y_2 \overset{(2D+1)t}{=} e^t$$

$$[2D^2+5D+2+D^2+D]\, y_2 = 2e^t + e^t$$

$$[3D^2+6D+2]\, y_2 = 3e^t$$

$$y_2 = C.F + P.I$$

$$C.F = c_1 e^{(-1+1/\sqrt{3})t} + c_2 e^{(-1-1/\sqrt{3})t} + \frac{1}{(3D^2+6D+2)} 3e^t$$

$$= c_1 e^{(-1+1/\sqrt{3})t} + c_2 e^{(-1-1/\sqrt{3})t} + \frac{3e^t}{11}$$

Only change will be in constants.

Que will be asked as → Sol^n tends to $y_1 \longrightarrow ?$, $y_2 \longrightarrow ?$

Find y_1 & y_2 directly.

In this process →

We find y_1 and y_2, which are the common sol^n to the system of ODE. by converting into simple ODE (i.e of 1 dependent variable)

Reverse Process is also possible →

Given simple ODE can be converted into the system of ODE.

$$y^{(n)} + a_0(t)\, y^{(n-1)} + a_1(t)\, y^{(n-2)} + \text{---------} + a_{n-1}(t)\, y' + a_n(t)\, y = Q(t)$$

Define n Dependent variables →

$$y_1 = y$$
$$y_2 = y'$$
$$y_3 = y''$$
$$\vdots$$
$$y_{n-1} = y^{n-2}$$
$$y_n = y^{n-1}$$

(No. of New(n) Dependent variable defined = Order of given ODE which is n)

$$\Rightarrow y_1' = y' = y_2$$
$$y_2' = y'' = y_3$$
$$y_3' = y''' = y_4$$
$$\vdots$$
$$y_{n-2}' = y^{n-2} = y_{n-1}$$
$$y_{n-1}' = y^{n-1} = y_n$$

System of Eq^n

As $y_n = y^{n-1}$

$$y_n' = y^n$$
$$= Q(t) - \left[a_0(t)\, y^{(n-1)} + a_1(t)\, y^{(n-2)} + \text{---------} + a_n(t)\, y\right]$$

$$y_n' = Q(t) - \left[a_0(t)\, y_n + a_1\, y_{n-1} + \text{---------} + a_n y_1\right]$$

6.9)

$$(3D^2+6D+2)\,y = 3e^t$$

$$3y'' + 6y' + 2y = 3e^t \quad \Rightarrow \quad y'' = \frac{1}{3}\left(3e^t - 6y' - 2y\right)$$

$$y_1 = y \quad \Rightarrow \quad y_1' = y' = y_2$$

$$y_2 = y' \qquad y_2' = y''$$

$$= \frac{1}{3}\left(3e^t - 6y_2 - 2y_1\right)$$

$$\therefore \quad y_1' = y_2$$

$$y_2' = \frac{1}{3}\left(3e^t - 6y_2 - 2y_1\right) = e^t - 2y_2 - \frac{2}{3}y_1$$

In Matrix form →

$$Y' = AY + f(t)$$

where $Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_1' = y_2$$

$$y_2' = e^t - 2y_2 - \frac{2}{3}y_1$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{3} & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

Que $y'' + P(t)y' + Q(t)y = 0$

2nd order Homog. L.D.E & corresponding System of eq^n is

$$\frac{dY}{dt} = A(t)Y$$

$$Y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \frac{dY}{dt} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

$A(t)_{2\times 2}$ matrix then A is →

a) $\begin{pmatrix} 1 & 0 \\ -p & -q \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$

d) $\begin{pmatrix} -p & -q \\ 1 & 0 \end{pmatrix}$

Sol^n Matrix obtained has →

Trace = − (Coff of y')

Det = y Coff.

(nth order → Trace = −(Coff of y^{n-1}), Det = ± Coff of y)

Trace = $-p$

Det = q

a) Trace → $1-q$ ✗

b) Trace → $-p$ ✓

Det → q ✓

c) Trace → p ✗

d) Trace → $-p$ ✓

Det → q ✓

Ans is either b) or d)

$y_1 = y$

$y_2 = y'$

$\Rightarrow$

$y_1' = y_2$

$y_2' = y''$

$= -P(t)y' - q(t)y$

$= -py_2 - qy_1$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

∴ Ans is b) (Discussed further)

In Que.

$Y(t)$ can be given as $\begin{bmatrix} y_2 \\ y_1 \end{bmatrix}$

& $\frac{dY}{dt} = \begin{bmatrix} y_2' \\ y_1' \end{bmatrix}$ → so chk accordingly.

Now →

$$y_1' = a_{11} y_1 + a_{12} y_2 + \text{———} + a_{1n} y_n$$

$$y_2' = a_{21} y_1 + a_{22} y_2 + \text{———} + a_{2n} y_n$$

|

$$y_n' = a_{n1} y_1 + a_{n2} y_2 + \text{———} + a_{nn} y_n$$

$$Y' = AY \quad \text{—(1)}$$

(For $y' = my$, $y' - my = 0$, $y = ce^{mx}$)

$A = [a_{ij}]_{n \times n}$

$ \boxed{Y(t) = Ce^{\lambda t}} \quad \text{—(2)}$

$Y(t) \rightarrow n \times 1$ Column vector

∴ $C \rightarrow n \times 1$ Column vector

$C = [c_1 \text{———} c_n]'$

Using (2) in (1) →

$$\lambda C e^{\lambda t} = A C e^{\lambda t}$$

$$\Rightarrow AC = \lambda C$$

Where
$\lambda \rightarrow$ Eigen Value
$C \rightarrow$ Eigen Vector

($A_{n\times n}$? $X \neq 0$
$AX = \lambda X$
$\lambda \rightarrow$ eigen Value of A
$X \rightarrow$ eigen vector)

Imp → For a given eigen value →

No. of eigen vectors → **Infinite**
(Taking Constant multiples)

No. of L.I eigen vectors → **finite**

All the remaining eigen vectors will be linear combination of these L.I eigen vectors.

For 3x3 matrix →

In Case 3 distinct eigen Value

In Sol^n → $e^{\lambda_1 x}$, $e^{\lambda_2 x}$, $e^{\lambda_3 x}$

2 Same eigen value & 1 diff.

In Sol^n → $e^{\lambda_1 x}$, $xe^{\lambda_1 x}$, $e^{\lambda_2 x}$

All Same

In Sol^n → $e^{\lambda x}$, $xe^{\lambda x}$, $x^2 e^{\lambda x}$

For 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Char eq^n →

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0$$

Sum of roots → $\lambda_1 + \lambda_2 = a_{11} + a_{22}$ = Trace of Matrix

Product of roots → $\lambda_1 \lambda_2 = a_{11}a_{22} - a_{21}a_{12}$

$= |A|$

Que will be asked as →

Trace > 0 i.e $\lambda_1 + \lambda_2 > 0$

$|A| < 0$ i.e $\lambda_1 \lambda_2 < 0$

$\Rightarrow \lambda_1 < 0 \quad \lambda_2 > 0$ (Suppose $\lambda_2 > \lambda_1$)

for $t \to \infty$

$y_1 \longrightarrow ?$

$y_2 \longrightarrow ?$

$\lambda_1 \neq \lambda_2$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda_1 t} + B \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} e^{\lambda_2 t}$$

for $t \to \infty$

$y_1 \to \infty$

for $t \to \infty$ $y_2 \to \infty$

$\lambda_1 < 0 \quad e^{\lambda_1 t} \to 0$

$\lambda_2 > 0 \quad e^{\lambda_2 t} \to \infty$

$\lambda_1 < 0 \quad e^{\lambda_1 t} \to \infty$

$\lambda_2 > 0 \quad e^{\lambda_2 t} \to 0$

Eigen Value → Let $A_{n\times n}$ matrix defined on a field F

Then def is called **Eigen Value** or **Characteristic value** if

$\exists\ X \neq 0$ s.t

$$AX = \lambda X$$

and Non-Zero vector is called characteristic vector / Eigen Vector.

(Matrix → Linear Transf. defined over field F ↓ ∵ λ → atmost n values)

NBHM →

Let $y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be a unique soln of $y' = Ay$

s.t $y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ — (1)

where A is Symmetric matrix s.t

$$Tr(A) < 0 \quad \& \quad |A| > 0$$

a) $y_1(t) \to \infty$ as $t \to \infty$

b) $y_1(t) \to 0$ & $y_2(t) \to 0$ as $t \to \infty$

c) $y_1(t) \to \infty$ as $t \to \infty$

d) $y_2(t) \to \infty$ as $t \to \infty$

Soln

$$\lambda_1 + \lambda_2 < 0$$

$$\lambda_1 \lambda_2 > 0$$

$$\Rightarrow \lambda_1 < 0 \ , \lambda_2 < 0$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = AC e^{\lambda_1 t} + BC' e^{\lambda_2 t}$$

$$= A \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{\lambda_1 t} + B \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} e^{\lambda_2 t}$$

$$y_1(t) = A c_1 e^{d_1 t} + B c_1' e^{d_2 t}$$

$$y_2(t) = A c_2 e^{d_1 t} + B c_2' e^{d_2 t}$$

As $d_1, d_2 < 0$

$\therefore y_1(t) \to 0 \quad$ as $t \to \infty$

$y_2(t) \to 0 \quad$ as $t \to \infty$

(Here, No use of ①.

When $d_1 > 0$ & $d_2 < 0$ or $d_1 < 0$ & $d_2 > 0$ } Condition ① will be used in finding A & B)

$$Y' = AY \quad ; \quad A_{2\times 2}$$

let $d_1 \neq d_2$ are 2 Eigen-values

$d_1, d_2 \in \mathbb{R}$

$|Y(t)| \to ?$

Constant → fix

As we are considering the modulus value

∴ Constant → +ve.

C → Eigen Vector Corresponding to d_1

C' → ——— to d_2

$$\boxed{\text{G.S}} \to Y(t) = A C e^{d_1 t} + B C' e^{d_2 t}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{d_1 t} + B \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} e^{d_2 t}$$

		$t \to \infty$	$t \to -\infty$	$t \to 0$
① $d_1 > 0$, $d_2 > 0$	$\lvert Y(t)\rvert$	∞	0	Finite value
② $d_1 < 0$, $d_2 > 0$	$\lvert Y(t)\rvert$	∞	∞	Finite Value
③ $d_1 > 0$, $d_2 < 0$	$\lvert Y(t)\rvert$	∞	∞	Finite Value
④ $d_1 < 0$, $d_2 < 0$	$\lvert Y(t)\rvert$	0	∞	Finite Value

Qu

$$\frac{dy_1}{dt} = 2y_1 + 5y_2$$

$$\frac{dy_2}{dt} = y_1 - 2y_2$$

Find Eigen values & Eigen vectors.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$d_1 + d_2 = 0$$

$$d_1 d_2 = -9$$

$$d_2^2 = 9$$

$$d_2 = \pm 3$$

$$d_1 = 3 \quad , d_2 = -3$$

Two distinct Eigen Values

Eigen Vector corresponding to $d_1 = 3$ →

$$AX = 3X$$

$$(A - 3I)X = 0$$

$$\begin{bmatrix} 2-3 & 5 \\ 1 & -2-3 \end{bmatrix} X = 0$$

$$\begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$-c_1 + 5c_2 = 0$$
$$c_1 - 5c_2 = 0$$

Take $c_2 = 1$

$\Rightarrow c_1 = 5$

Eigen vector $= \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda_2 = -3$

$$[A + 3I]X = 0$$

$$\begin{bmatrix} 2+3 & 5 \\ 1 & -2+3 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$c_1 + c_2 = 0$$

Take $c_2 = -1$

$\Rightarrow c_1 = 1$

(Choose acc. to options given)

Eigen vector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{3t} + B \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$$

(For finding A & B we use $Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ As given in Last e.g)

If $Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$y_1 = 5A e^{3t} + B e^{-3t}$$

$$1 = 5A + B$$

$$y_2 = A e^{3t} - B e^{-3t}$$

$$1 = A - B$$

$$\Rightarrow 6A = 2$$

$$A = 1/3$$

$$\Rightarrow B = A - 1 = 1/3 - 1 = -2/3$$

$\Rightarrow$

$$y_1 = \frac{5}{3} e^{3t} - \frac{2}{3} e^{-3t}$$

$$y_2 = \frac{1}{3} e^{3t} + \frac{2}{3} e^{-3t}$$

When $t \to \infty$ $\quad y_1(t) \to \infty$, $y_2(t) \to \infty$

When $t \to -\infty$ $\quad y_1(t) \to -\infty$, $y_2(t) \to \infty$

Note →

→ Algebraic Multiplicity $\geq$ Geometric Multiplicity

No. of times a root is repeated

Geometric Multiplicity = $N(A - \lambda I)$ ← Nullity

$= 2 - \rho(A - \lambda I)$

↓ Rank

No. of L.I eigenvectors corresponding to repeated eigen values.

System of Eq^n →

$$Y' = AY$$

(1) $A_{2\times 2}$ → matrix $\quad d_1 \neq d_2$

Then Sol^n is →

G.S → $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{d_1 t} + B\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} e^{d_2 t}$$

(2) When $d_1 = d_2 = d$ repeated Eigen value

Algebraic Multiplicity $= 2$

Geometric multiplicity ≤ 2

Case I →

Chk → $(AX_1 + BX_2)e^{dt}$ is G.S ← (G.M $(d) = N(A - dI)$

$= 2 - \rho(A - dI)$

When G.M $= 2$

If G.M $(d) = 2$

$A - dI = 0$

$(A - dI)X = 0$

$0X = 0$

$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

X_1, X_2 → Two L.I Eigen vector)

$\begin{cases} y_1 = c_1 e^{dt} \\ y_2 = c_2 e^{dt} \end{cases}$

or not BX_2 ...

$$\begin{vmatrix} AX_1 e^{dt} & BX_2 e^{dt} \\ dAX_1 e^{dt} & dBX_2 e^{dt} \end{vmatrix} = 0$$

$\Rightarrow X_1 e^{dt}$ & $X_2 e^{dt}$ are L.D ...

$\therefore (AX_1 + BX_2)e^{dt}$ is not the General Sol^n.

$$Y(t) = (AX_1 + BX_2)e^{dt}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{dt} + B\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{dt}$$

$y_1 = Ae^{dt}$

$y_2 = Be^{dt}$ ... Not Possible

Suppose $V_1 \rightarrow$ Eigen Vector corresponding to $\lambda_1 = \lambda$

When $\rho(A-\lambda I) = 1$
G.M = 1
$\Rightarrow$ Only 1 L.I Eigen Vector.

Let $X_1 = V_1 e^{\lambda t}$

Let $X_2 = (V_1 t + W) e^{\lambda t}$ Sol^n of $Y' = AY$.

$\therefore X_2' = AX_2$

$$[V_1 + \lambda(V_1 t + W)] e^{\lambda t} = A[V_1 t + W] e^{\lambda t}$$

$$V_1 + \lambda W + \lambda V_1 t = AV_1 t + AW$$

Comparing $\rightarrow$

$$AV_1 = \lambda V_1$$

& $V_1 + \lambda W = AW$

$$(A - \lambda I) W = V_1$$

If X_1 & X_2 are L.I

then $AX_1 + BX_2$ will be a Sol^n of $Y' = AY$.

$$\begin{vmatrix} X_1 & X_2 \\ X_1' & X_2' \end{vmatrix} = \begin{vmatrix} V_1 e^{\lambda t} & (V_1 t + W) e^{\lambda t} \\ \lambda V_1 e^{\lambda t} & \lambda (V_1 t + W) e^{\lambda t} \end{vmatrix}$$

$$= V_1^2 e^{2\lambda t} \neq 0$$

$\neq 0 \quad \neq 0$

$\Rightarrow X_1$ & X_2 are L.I.

$\therefore AX_1 + BX_2$ will be Sol^n of $Y' = AY$

No. of Eigen Vectors → Infinite

L.I Eigen Vectors → Finite

Remaining L.C of these.

Conclusion →

System of eq^n

$$Y' = AY$$

$A_{2\times 2}$ $\quad d_1 = d_2 = d$

then if d has Unique L.I Eigen Vector

Then One Sol^n is $X_1 = v_1 e^{dt}$ where v_1 is eigen Vector of d

And another L.I Sol^n is

$$X_2 = (v_1 t + w) e^{dt}$$

Where W is Sol^n of $(A - dI) W = v_1$

$\$$ Gen Sol is $Y(t) = A X_1 + B X_2$

E.g →

$$Y' = AY$$

$$A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$$

$$d_1 + d_2 = 3 - 9 = -6$$

$$d_1 d_2 = 9$$

$$\Rightarrow d_1 = d_2 = -3$$

$$(A + 3I) v_1 = 0$$

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rank of $(A+3I) = 1$. (2 L.I rows)

G.M = 1

∴ Only 1 L.I Eigen Vector

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$6v_1 - 18v_2 = 0$$

$$v_1 = 3v_2$$

Choose $v_2 = 1$ ∴ $V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

∴ $X_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t}$

∴ Now find X_2.

$$X_2 = (V_1 t + W) e^{-3t}$$

$$(A+3I)W = V_1$$

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$6w_1 - 18w_2 = 3$$

$$2w_1 - 6w_2 = 1$$

$$\Rightarrow \quad w_1 - 3w_2 = 1$$

$$2w_1 - 6w_2 = 1$$

$\Rightarrow$ Choose $w_2 = 1$.

$\Rightarrow w_1 = 7/2$ ∴ $W = \begin{bmatrix} 7/2 \\ 1 \end{bmatrix}$

$$\therefore \quad x_2 = \left[t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 7/2 \\ 1 \end{bmatrix} \right] e^{-3t}$$

$$= \begin{bmatrix} 3t + 7/2 \\ t+1 \end{bmatrix} e^{-3t}$$

$$\therefore \quad Y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + B \begin{bmatrix} 3t+7/2 \\ t+1 \end{bmatrix} e^{-3t}$$

Que $$xy'' + y' + (1-x)y = 0 \quad \text{—} (1)$$

find the system of eqn equivalent to (1).

Sol^n

Let $Y' = AY$ be a system of eqn.

Then

a) A is unique matrix

b) A has exactly two choices

c) A has Infinite choices

d) Can't say.

First Consider →

$$y'' + Py' + Qy = 0$$

$y_1 = y$ $\qquad$ $y_1' = y_2$

$y_2 = y'$ $\qquad$ $y_2' = -Py_2 - Qy_1$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -Q & -P \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Assume $y_2 = y$

$y_1 = y'$

$y_2' = y' = y_1$

$y_1' = y'' = -p y_1 - q y_2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -p & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Corresponding to →

$y_1 \rightarrow$ n choices

$y_2 \Rightarrow n-1$

$y_3 \Rightarrow n-2$

⋮

$y_n \rightarrow 1$

n! Choices for A.

No. of choices for A = n!

$y_1 = y$ $y_2 = y'$ $y_3 = y''$ $\$$ $y_1 = y$ $y_3 = y'$ $y_2 = y''$ $\$$ $y_2 = y$ $y_1 = y'$ $y_3 = y''$

(123)
(132)
(213)
(231)
(312)
(321)

Sol^{n} →

$y_1 = y$ $\quad y_1' = y_2$

$y_2 = y'$ $\quad y_2' = y'' + \frac{1}{x} y_2 - \frac{(1-x)}{x} y_1$

$$Y' = \begin{bmatrix} 0 & 1 \\ -\frac{(1-x)}{x} & +\frac{1}{x} \end{bmatrix} Y$$

Note →

$$Y' = A(x)\,Y$$

Then System has →

- Unique Sol^{n},
- $\forall\, \alpha_0, \alpha_1 \in \mathbb{R}$ s.t

$$Y(x_0) = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

Q. System has Unique sol^{n} →

1) $Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → $y(0) = 1 \ (= \alpha_0)$, $y'(0) = 1 \ (= \alpha_1)$ Infinite Sol^{n}

2) $Y(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → Unique (By Fundamental Th^{m})

3) $Y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → No Sol^{n}

4) $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → No Sol^{n}

$$P(x) = 1/x \quad , \quad Q(x) = 1/x - 1$$

$1 + P(x) + Q(x) = 0 \Rightarrow e^x$ is one sol^n.

$$y_1(x) = y_1 e^x$$

$$y_1'' + \left(P(x) + \frac{2u'}{u}\right) y_1' = \frac{R(x)}{u}$$

$$y_1'' + \left(\frac{-1}{x} + 2\right) y_1' = 0$$

I.F $\rightarrow$ $e^{-\int 1/x\, dx} \cdot e^{2x}$

$= e^{-\log x}\, e^{2x}$

$= \frac{1}{x} e^{2x}$

$y_1' \cdot \frac{1}{x} e^{2x} = q$

$y_1' = q x e^{-2x}$

$y_1(x) = q \int x e^{-2x} dx + c_2$

$y_1(x) = q \left(\frac{-2x}{2} e^{-2x} - \frac{1}{4} e^{-2x}\right) + c_2$

$y(x) = q \left(\frac{-x}{2} e^{-x} - \frac{1}{4} e^{-x}\right) + c_2 e^x$

$= q \left(\frac{-x}{2} - \frac{1}{4}\right) e^{-x} + c_2 e^x$ — (1)

17 $Y\begin{bmatrix}0\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

$y_1(0) = 1 \quad \Rightarrow \quad y_1 = y$

$y_2(0) = 1 \qquad y_2 = y'$

$$\left.\begin{aligned} y_1(0) &= y(0) = 1 \\ y_2(0) &= y'(0) = 1 \end{aligned}\right\}$$

Using this in (1) $\longrightarrow$

$$1 = \frac{-c_1}{4} + c_2 \quad \Rightarrow 4c_2 - c_1 = 4$$

$$y'(x) = c_1\left(\tfrac{-1}{2}\right)e^{-x} - c_1\left(\tfrac{-x}{2} - \tfrac{1}{4}\right)e^{-x} + c_2 e^{x}$$

$$\Rightarrow \quad 1 = -\frac{c_2}{2} + \frac{c_1}{4} + c_2$$

$$\Rightarrow \quad 4c_2 - c_1 = 4$$

Two constants c_1 & c_2.

Only 1 eq^n

$\Rightarrow$ Infinite many sol^{n}s.

$\therefore$ System has infinite many sol^{n}s

3) $y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$4c_2 - c_1 = 4$

$4c_2 - c_1 = 0$

$c_1 = 4c_2$

$4c_2 - 4c_2 = 4$

$0 = 4$ X

∴ No Sol^n

4) $y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$4c_2 - c_1 = 0$

$4c_2 - c_1 = 4$

Gives $0 = 4$

∴ No Sol^n

$$Y' = AY$$

$$Y(x_0) = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$ Condition

$$A = [a_{ij}], \quad a_{ij} \in F$$

By Fundamental Thm → Unique Sol^n

Q.E.D. H.W

$$\left.\begin{aligned} y'' + p(x)y' &= 0 \\ y'' + q(x)y' &= 0 \end{aligned}\right] \begin{matrix} -(1) \\ -(2) \end{matrix}$$

$p(x)$ & $q(x)$ are cts $\forall x$

$$p(x) > q(x)$$

If y_1 is Non-Trivial sol^n of (1)

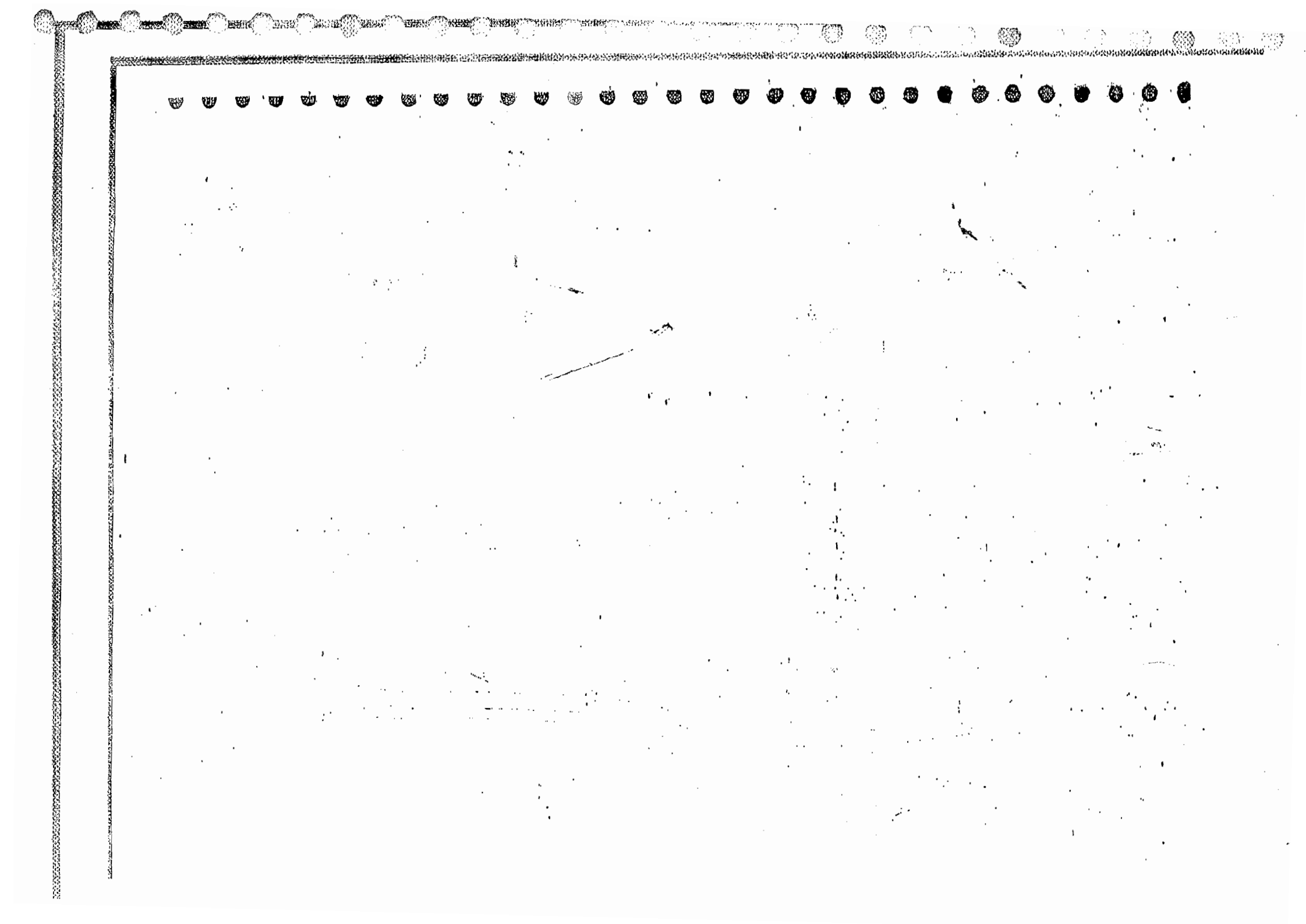
y_2 is Non-Trivial sol^n of (2)

Then

If x_0, x_1 are two successive zeros of y_1

$\Rightarrow \exists$ exactly One zero α of y_2

s.t $x_0 < \alpha < x_1$



Sturm Liouville Problem → (SLP)

Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad \forall x \in [a,b]$ — (1)

where $a_0(x) \neq 0$, $a_0(x), a_1(x), a_2(x)$ are cts on $[a,b]$

Adjoint of (1) →

$$\frac{d^2}{dx^2}[a_0(x)y] - \frac{d}{dx}[a_1(x)y] + a_2(x)y = 0 \quad \text{— (2)}$$

and (2) is adjoint of (1)

If eq^n 1 and 2 are same then Self-adjoint

Expanding (2) →

$$\frac{d}{dx}[a_0'y + a_0y'] - [a_1'y + a_1y'] + a_2y = 0$$

$$\Rightarrow a_0''y + a_0'y' + a_0'y' + a_0y'' - a_1'y - a_1y' + a_2y = 0$$

$$\Rightarrow a_0y'' + (2a_0' - a_1)y' + [a_0'' + a_2 - a_1']y = 0 \quad \text{— (3)}$$

i.e (1) = (3) then (1) is called Self-adjoint

Necessary & Sufficient for 2nd order Diff eq^n to be Self-adjoint

If $\boxed{a_0'(x) = a_1(x)} \Rightarrow a_0'' = a_1'$

Using in (3) →

$$a_0y'' + a_1(x)y' + a_2(x)y = 0$$

which is same as (1)

Every Non-self adjoint differential eq^n reducable into self-adjoint

(All SLP are self-adjoint)

Define $\mu(x) = e^{\int \frac{a_1(x)}{a_0(x)} dx}$ (Similar to Wronskian)

Multiply this with (1) →

$$\frac{\mu(x)}{a_0(x)} \left[a_0(x) y'' + a_1(x) y' + a_2(x) y \right] = 0 \quad - (4)$$

4th is alwx self adjoint

e.g. $x^2 y'' + xy' + y = 0$ — (1)

$a_0'(x) \neq a_1(x)$ $(2x \neq x)$

$\Rightarrow$ (1) is not-self adjoint.

$\mu(x) = e^{\int \frac{1}{x} dx} = x$

$$\frac{\mu(x)}{a_0(x)} \left[x^2 y'' + xy' + y \right] = 0$$

$$\left[x y'' + y' + \frac{1}{x} \right] = 0 \quad - (2)$$

$$a_0'(x) = a_1(x)$$

$\Rightarrow$ eq^n (2) is self adjoint i.e (1) is reducible to self-adjoint

$$\frac{d}{dx}[p(x)y'] + [Q(x) + \lambda r(x)]y = 0 \quad \text{---(1)}$$
$$\forall x \in [a,b]$$

$$py'' + p'y' + [Q(x) + \lambda r(x)]y = 0 \leftarrow \text{always Self-adjoint}$$

Conditions
$$\$ \begin{cases} a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0 \end{cases} \text{---(*)}, \quad \begin{matrix} a_1^2 + a_2^2 \neq 0 \\ b_1^2 + b_2^2 \neq 0 \end{matrix}$$
$$\Downarrow$$
atleast one is Non-zero

Eq^n (1) with Boundary Conditions called SLP.

Also $p(x) > 0$, $r(x) > 0$ $\forall x \in [a,b]$

(Eg → $y'' + \lambda y = 0$
$y(0) = 1$, $y'(0) = 1$ } Not SLP As I.V.P
With $y(0) = 1$, $y'(1) = 1$ } SLP As B.V.P)

(* shows relation b/w $y(a)$ & $y'(a)$ and $y(b)$ & $y'(b)$)

II form →
$y'' + \lambda y = 0$

$y(0) = y(\pi)$ → Not SLP
$y'(0) = y'(\pi)$ Periodic Boundary Value Problem

(Relation b/w $y(0)$ & $y(\pi)$ Not SLP)

I form $\quad y'' + \lambda y = 0 \quad \forall x \in [0, \pi]$

$$\left.\begin{aligned} y(0) &= 0 \\ y'(\pi) &= 0 \end{aligned}\right\} \text{ SLP}$$

Classification of B.V.P →

Regular SLP →

$$\frac{d}{dx}\left[P(x)y'\right] + \left[Q(x) + \lambda r(x)\right] y = 0 \quad \forall x \in [a,b]$$

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

if $r(x) > 0,\ P(x) > 0 \quad \forall x \in [a,b]$

Example ↓ ?

Singular SLP →

$$\frac{d}{dx}\left[P(x)y'\right] + \left[Q(x) + \lambda r(x)\right] y = 0 \quad \forall x \in [a,b]$$

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

$P(x) > 0 \quad \forall x \in (a,b)$

either $P(a) = 0$ and $P(b) = 0$

$r(x) > 0 \quad \forall x \in [a,b]$

Periodic B.V.P →

If $\frac{d}{dx}[P(x)y'] + [Q(x) + \lambda R(x)] = 0$

if $P(x) > 0$, & $R(x) > 0$ $\forall x \in [a,b]$

& $P(a) = P(b)$ (At end pts value same But +ve)

$$\left.\begin{aligned} y(a) &= y(b) \\ y'(a) &= y'(b) \end{aligned}\right]$$

Que $(x(x-1)+1)y'' + dy = 0$ $\forall x \in [0,1]$

$y(0) = y(1)$

$y'(0) = y'(1)$

Periodic B.V.P

$P(x) = x(x-1)+1$

$P(0) = 0+1 = 1$ } value at end pts is same

$P(1) = 1$

Eigen Values of B.V.P →

Let $\frac{d}{dx}\left(p(x)y'\right) + \left(q(x) + \lambda r(x)\right)y = 0$

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

Then the value of λ for which $\exists$ a Non-Trivial Sol^n $y_\lambda(x)$ is called Eigen value / characteristic value and corresponding Sol^n $y_\lambda(x)$ is called Eigen fu^n

(0 → eigen value exists in case of SLP)

Que $y'' + \lambda y = 0 \quad \forall x \in [0, \pi]$

$$y(0) = 0$$
$$y'(\pi) = 0$$

← Regular SLP

Let λ is the eigen value.

① $\lambda > 0$

$y'' + \lambda y = 0$

$\lambda = \mu^2 \quad \mu \neq 0$

$y'' + \mu^2 y = 0$

$m = \pm i\mu$

$\therefore y(x) = c_1 \cos \mu x + c_2 \sin \mu x$

$y(0) = 0$

$\Rightarrow 0 = c_1$

$y'(\pi) = 0 \Rightarrow c_2 = 0$

$\Rightarrow$ Only Trivial Sol^n

$\Rightarrow$ λ any λ re eigen value.

$y'(n) \neq 0$

$y'(x) = -\lambda c_1 \sin\lambda x + \lambda c_2 \cos\lambda x$

$\sin\lambda\pi = 0 \quad \cos\lambda\pi = 0$

$\lambda\pi = n\pi \quad \lambda\pi = (2n+1)\frac{\pi}{2}$

$\lambda = n \quad \lambda = \frac{(2n+1)}{2}$

Depends on λ)

(2) When $\lambda = 0$

$y(x) = c_1 + c_2 x$

$y(0) = 0$

$c_1 = 0$

$y'(x) = c_2$

$\Rightarrow y'(\pi) = 0$

$c_2 = 0$

$\Rightarrow$ Only Trivial Sol^n

$\Rightarrow \lambda = 0$ is not an eigen value

(3) When $\lambda < 0$

$\lambda = -\mu^2 \quad \mu \neq 0$, $\mu \rightarrow$ Non-Zero real no.

$y'' = \mu^2 y$

$m = \pm\mu$

$\therefore y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$y'(x) = \mu c_1 e^{\mu x} - c_2 \mu e^{-\mu x}$

$y(0) = 0$

$\Rightarrow c_1 + c_2 = 0 \quad \longrightarrow (1)$

$y(\pi) = 0$

$\Rightarrow \mu c_1 e^{\mu\pi} - \mu c_2 e^{-\mu\pi} = 0$

$\Rightarrow c_1 e^{\mu\pi} - c_2 e^{-\mu\pi} = 0 \quad (\mu \neq 0)$

$\longrightarrow (2)$

$c_1 = -c_2$

$c_1 e^{2\mu\pi} = c_2 \quad \Rightarrow \frac{c_1}{c_2} = e^{-2\mu\pi}$

$\frac{c_1}{c_2} = -1$

$\Rightarrow \boxed{e^{-2\mu\pi} = -1}$ ✗ Not Possible

$e^{-2\mu\pi} = e^{-i\pi}$ ∴ e^{x} & e^{-x} Always +ve

$-2\mu = -i$

$\mu = i/2$

$\Rightarrow c_1 = c_2 = 0.$

$\Rightarrow$ Trivial Sol^{n}.

(i) When $d > 0$

$d = p^2$, $p \neq 0$, $p \in \mathbb{R}$

$y(x) = c_1 \cos px + c_2 \sin px$

$y(0) = 0$

$\Rightarrow c_1 = 0$

$y(x) = c_2 \sin px$

$y'(x) = p c_2 \cos px$

$y'(\pi) = 0$

$\Rightarrow p c_2 \cos p\pi = 0$

$p \neq 0$

If $c_2 = 0$ Only Trivial soln

$\Rightarrow \cos p\pi = 0$

$p\pi = (2n+1)\frac{\pi}{2}$

$p = \frac{(2n+1)}{2}$, $n \in \mathbb{Z}$

(p can be +ve or -ve)

In Ques → Discard those value of n for which $p = 0$

As $p \neq 0$

Here $\forall n$

$p \neq 0$

$$\lambda = \mu^2 = \frac{1}{4}(2n+1)^2$$

Q. Total No. of Eigen Value

Q. Repeated Eigen Value

Q. Set of Eigen Values → Table

Q. Non-Repeated Eigen Value

Q. Eigen Value bounded below | bounded above.

Que will be asked as →

(for $n=1$, $\lambda = 9/4$; $n=-2$, $\lambda = 9/4$ ⇒ Repeated E.V
$n=0 \Rightarrow \lambda = 1/4$, $n=-1$, $\lambda = 1/4$

$$y(x) = c_2 \sin \mu x.$$

For $\lambda = \frac{1}{4}$ with multiplicity 2. For $\lambda = 1/4$, $\mu = 1/2$

$$y(x) = \sin \mu x$$

$$= \sin \frac{x}{2} \quad \text{(1)} \rightarrow \text{Only 1 L.I Eigen vector}$$

$\Rightarrow G.M \leq 2$

$G.M = 1$

(If $\mu = -1/2$?

$y(x) = -\sin x/2$ — (2)

(1) & (2) are L.D

$\therefore G.M = 1$)

Que $Y'(t) = AY + BY$

$Y(0) = Y_0$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Then $Y(t)$ is given by →

a) $e^{tA} e^{tB} Y_0$

b) $e^{tB} e^{tA} Y_0$

c) $e^{t(A+B)} Y_0$

d) $e^{-t(A+B)} Y_0$

Sol^n:

Consider it as $Y'(t) = (A+B)Y(t)$

with $Y(0) = Y_0$

Sol^n is →

$$Y(t) = C e^{(A+B)t}$$

$$Y(0) = Y_0 \Rightarrow C = Y_0$$

$\therefore$ Sol^n is → $Y(t) = Y_0 e^{t(A+B)}$

Prev. Eg →

$\lambda = \frac{1}{4}(2n+1)$ ← eigen value

$y_\lambda(x) = \sin\left(\frac{1}{2}(2n+1)x\right)$ ← eigen vector

No. of L.I eigen vectors → ?

Que

Let $S_\lambda = \{ y_\lambda(x) \mid$ where $y_\lambda(x)$ is an eigen fu^n corresponding to $\lambda \}$

Then →

a) S_λ is L.I

b) The L.I fu^n in S_λ is exactly one $\forall \lambda$

c) If $y_1(x) \in S_{\lambda_1}$ & $y_2(x) \in S_{\lambda_2}$ $\lambda_1 \neq \lambda_2 \Rightarrow$ y_1 and y_2 are L.I

d) $\exists \lambda$ s.t No. of L.I fu^n in S_λ are more than one

Find $|S_\lambda| = ?$

Soln

For $\lambda = 1/4$ $\quad |S_{1/4}| =$ ~~Finite~~ Infinite

Choose $c_2 = 1$ $\quad$ # No. of L.I Eigen fu^n corresponding to $\lambda = 1/4$ $= 1 \rightarrow$ ($\because \sin x/2$ is the only L.I eigen vector)

$y_{1/4}(x) = \sin x/2$

$y_{1/4}(x) = c \sin(x/2)$

$c \neq 0$

For every c, there are infinite eigen fu^n.

No. of eigen fu^n corresponding to any E.V → Infini

$\therefore |S_\lambda| =$ ~~finite~~ Infinite $\forall \lambda$

~~finite~~ No. of L.I eigen vectors = finite
corresponding to
eigen value λ

[If $S_\lambda = \{ y_\lambda(x) \mid$ where $y_\lambda(x)$ is the L.I eigen fu^n corresponding to $\lambda \}$

Then $|S_\lambda| = 1 \ \forall \lambda$]

for $\lambda = 1/4$ $\quad y_{1/4} = \sin x/2$ is the

$\lambda = 9/2$ $\quad y_{9/2} = \sin 3x/2$

$$N = \begin{vmatrix} \sin x/2 & \sin 3x/2 \\ \frac{1}{2}\cos x/2 & 3/2 \cos 3x/2 \end{vmatrix}$$

$$= 3/2 \sin x/2 \cos 3x/2 - 1/2 \sin 3x/2 \cos x/2$$

$$\neq 0$$

$\Rightarrow y_{1/4}$ and $y_{9/2}$ are L.I

Note → (1) $V_\lambda(F) = \langle S_\lambda \rangle$

↑

Eigen space of λ generated by L.I Eigen Vectors corresponding to

Dimension → No. of L.I Eigen Vector of $V_\lambda(F)$ Corresponding to λ

In the previous e.g →

Dimension of $V_\lambda(F) = 1$ (As No. of L.I Eigen Vector Corresponding to λ is 1)

∴ All the Eigen space of d Vector space are of dimension 1

(2) $S = \cup S_\lambda$

S = Infinite No. of L.I Eigen Vectors. (∵ λ's are alwx infinite in S.L.P)

$V(F) = S$

∴ $V(F)$ is alwx of infinite dimension

(3) In SLP →

No. of Eigen Vector → Alwx infinite

and G.M ≤ 2

⇒ Atmost Atleast 2 L.I Vectors in SLP

Alwx 1

In regular SLP → L.I Vector Alwx One.

Now →

$\lambda_1 = 1/4$ $\quad \sin x/2$

$\lambda_2 = 9/4$ $\quad \sin 3x/2$

$\lambda_3 = 25/4$ $\quad \sin 5x/2$

y_{λ_1} $\quad y_{\lambda_2}$ $\quad\quad\quad\quad\quad\quad\quad\quad$ y_{λ_n}

$\lambda_1 < \lambda_2 < \lambda_3 < \cdots\cdots < \lambda_n < \cdots\cdots$

We can arrange E.V in increasing order alwz

$\langle \lambda_n \rangle \longrightarrow \infty$, $n \to \infty$ & $\langle \lambda_n \rangle$ is Obtained.

Which is Bounded Below

& Unbounded above.

Imp → In (a,b) →

$y_{\lambda_n}(x)$ → exactly $n-1$ zeroes.

In $[a,b]$ →

$y_{\lambda_n}(x)$ → exactly n zeroes

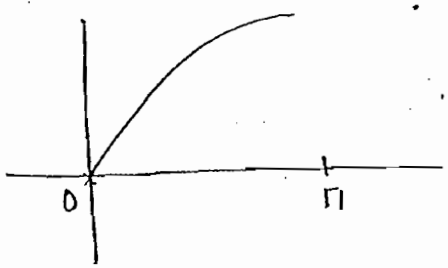
$y_{\lambda_1} = \sin x/2$

In $[0,\pi]$

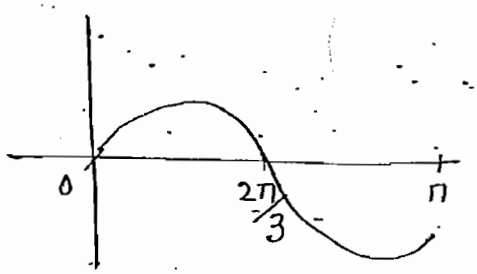
exactly 1 zero

In $(0,\pi)$

No zero.



$y_{d2} = \sin\frac{3x}{2}$



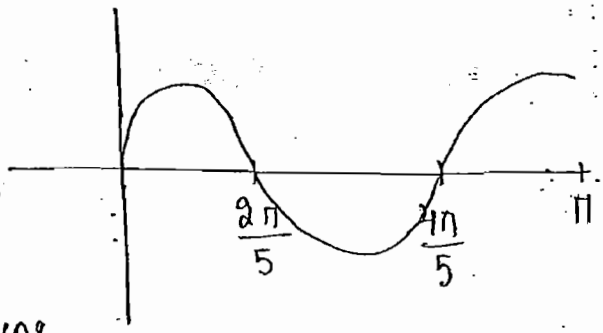
In $[0, \pi]$

exactly 2 zeroes

In $(0, \pi)$

exactly 1 zero.

$y_{d3} = \sin\frac{5x}{2}$



In $[0, \pi]$

exactly 3 zeros

In $(0, \pi)$

exactly 2 zeros

All these results are true for Regular SLP only

Que $y'' + \lambda y = 0$,

$y'(1) = 0$, $y'(10) = 0$.

Que $(x^2+1)\, y'' + 2xy' + \lambda x^2 y = 0$

$y'(1) = 0$, $y'(10) = 0$

Then which of the following possible values of λ →

X a) $\lambda \leq 0$ → Bounded above

✓ b) $\lambda \geq 0$

X c) $\lambda \neq 0$ → λ can approach to ∞ or $-\infty$

X d) $\lambda < 0$ → Bounded above

Sol^n →

Regular SLP ($P(x) = x^2+1 > 0$, $a_1 = 0, a_2 = 1$; $q(x) = x^2 > 0$, $b_1 = 0, b_2 = 1$)

In Regular SLP → Set of Eigen values → Bounded below & Unbounded above

Summary →

Regular SLP

weight fu^n

$$\frac{d}{dx}[P(x)y'] + [q(x) + \lambda r(x)]y = 0 \quad \forall x \in [a,b]$$

$$P(x) > 0, \; r(x) > 0$$

$$\left.\begin{aligned} a_1 y(a) + a_2 y'(a) &= 0 \\ b_1 y(b) + b_2 y'(b) &= 0 \end{aligned}\right\} (*), \quad \begin{aligned} a_1^2 + a_2^2 &\neq 0 \\ b_1^2 + b_2^2 &\neq 0 \end{aligned}$$

(1) All the Eigen values are real

(2) If λ_1 and λ_2 are 2 distinct Eigen values and y_{λ_1} and y_{λ_2} are Corresponding Eigen fu^n's then $y_{\lambda_1} \perp y_{\lambda_2}$ (orthogonal)

i.e $\int_a^b r(x)\, y_{\lambda_1}(x)\, y_{\lambda_2}(x)\, dx = 0$

(If Integration of $y_{\lambda_1}(x)$ and $y_{\lambda_2}(x)$ w.r.t weight fu^n is 0 then y_{λ_1} & y_{λ_2} are orthogonal)

(1) and (2) are true for $\forall$ B.V.P

i.e for Regular SLP, Periodic, Singular SLP

orthonormal

If $\int_a^b r(x)\, y_{\lambda_i}^2 = 1 \quad \forall i$

then Orthonormal Set.

Proof: Let λ_1 and λ_2 are a distinct eigen values and y_1 and y_2 are corresponding eigen fu^{n}s.

(2)

$$y_2 \times \frac{d}{dx}\{P(x)y_1'\} + q(x)y_1 + \lambda_1 r(x) y_2 = 0$$

$$y_1 \times \frac{d}{dx}\{P(x)y_2'\} + q(x)y_2 + \lambda_2 r(x) y_2 = 0$$

On Sub. $\rightarrow$

$$y_2\left(P(x)y_1'\right)' - y_1\left(P(x)y_2'\right)' + (\lambda_1-\lambda_2) r(x) y_1 y_2 = 0$$

Integrate both sides from a to b $\rightarrow$

$$\left[y_2 P(x) y_1'\right]_a^b - \int_a^b y_2' P(x) y_1' dx - \left[y_1 P(x) y_2'\right]_a^b - \int_a^b y_1' P(x) y_2' dx + \int_a^b (\lambda_1-\lambda_2) r(x) y_1(x) y_2(x) dx = 0$$

$$P(b)\left[y_2(b)y_1'(b) - y_1(b)y_2'(b)\right] + P(a)\left[y_1(a)y_2'(a) - y_2(a)y_1'(a)\right] = (\lambda_2-\lambda_1)\int_a^b r(x) y_1(x) y_2(x) dx \quad \text{---(1)}$$

Using (4) $\rightarrow$

$$a_1 y_1(a) + a_2 y_1'(a) = 0$$

$$a_1 y_2(a) + a_2 y_2'(a) = 0$$

(As y_1 & y_2 are the eigen fu^{n}s)

$$\begin{bmatrix} y_1 & y_1' \\ y_2 & y_2' \end{bmatrix}_{x=a} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a_1^2 + a_2^2 \neq 0$$

$\therefore$ Case of $AX = 0$

$\therefore |A| = 0$

$$\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}_{x=a} = 0 \quad \text{——} \quad **$$

Also $$\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}_{x=b} = 0$$

$\Rightarrow$ From (1) $\rightarrow$

$$(d_2 - d_1) \int_a^b r(x)\, y_1(x)\, y_2(x)\, dx = 0$$

$$d_2 \neq d_1$$

$$\Rightarrow \int_a^b r(x)\, y_1(x)\, y_2(x)\, dx = 0$$

$\Rightarrow$ y_1 and y_2 are orthogonal.

(1 (3) In Case of Regular SLP →

If λ be any eigen value

Then any two eigen fu's $y_1(x)$, $y_2(x)$ Corresponding to λ are Constant multiple Of each other i.e Linearly Dependent

$$W(y_1, y_2) = A e^{-\int \frac{P'(x)}{P(x)} dx} = \frac{A}{P(x)}$$

~~As y_1 and y_2 are ~~

? $W(y_1, y_2)_{x=a} = 0$ $\quad \begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} =$

(From ** of (2)

$\Rightarrow \frac{A}{P(x)} = 0$

$\Rightarrow A = 0 \quad (\because P(x) > 0)$

$\Rightarrow W(y_1, y_2) \equiv 0$

$\Rightarrow y_1$ & y_2 are L.D

In Case of Singular SLP → $P(a) = P(b) = 0$

So we can't apply this.

($W(y_1, y_2)_{x=a} = 0$

$\frac{A}{P(a)} = 0 \Rightarrow \frac{A}{0} = 0$

So $A = 0$ Not Possible)

In Case of Periodic SLP →

?

N

(4) The eigen value of regular can be arranged into a ↑ sequence →

i.e $d_1 < d_2 < d_3 < \cdots\cdots < d_n < \cdots\cdots$

i.e The set of eigen value is alwx infinite.

- Bounded below and unbounded above

and $\langle d_n \rangle \to \infty$ as $n \to \infty$

(In case of Singular SLP We can't say this)

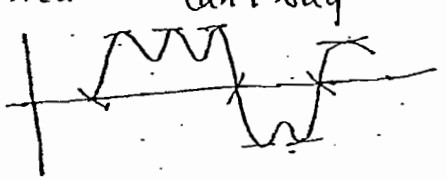
(5) If $y_n(x)$ be an eigen fu^n Corresponding to d_n then y_{d_n} has exactly $(n-1)$ Zero's in (a,b)

(6) For $n \geq 3$

If y_{d_n} is an eigen fu^n Corresponding to d_n then y_{d_n} has atleast $(n-2)$ point of extrema.

(7) Every eigen fu^n corresponding d_n, $n \geq 3$ are oscillatory

(Atmost How many pts of extrema → Can't say



If there is a twice differentiable fu^n y → having more than Zero.

Repeated root $\begin{cases} y_1(x_0) = 0 \\ y_2(x_0) = 0 \end{cases}$ Then fu^n is Monotone or not ?

Function is Not Monotonic $y'(x_0) = 0$

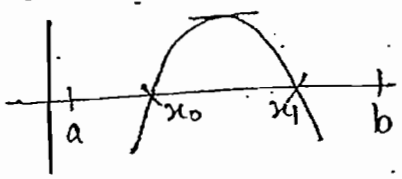
$\Rightarrow x_0$ is the pt. of extrema

y_{d_1} y_{d_2} y_{d_3} y_{d_4}

$d_1 < d_2 < d_3 < d_4 < \cdots\cdots < d_n < \cdots\cdots$

for $n \geq 3$

y_{d_3} has 2 Zeros in (a, b)

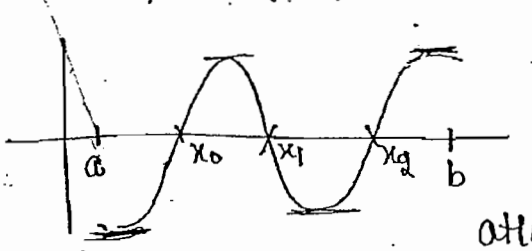


Not Monotonic

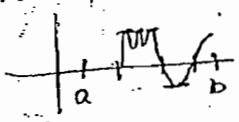
Point of extrema $\Rightarrow$ atleast $3-2=1$

y_{d_4} has 3 Zeros in (a, b)

Maximum pts of extrema → Can't say



Point of extrema → atleast $4-2=2$



Clearly oscillatory for $n \geq 3$

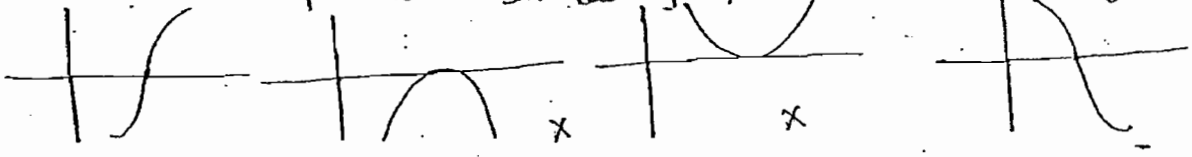
$y(x_0) = 0$ & $y'(x_0) = 0$ } Repeated Root Not possible $\Rightarrow$ No point of extrema $\Rightarrow$ Function is Monotonic

I.V.P

By Fundamental Th^m → Only Trivial Sol^n

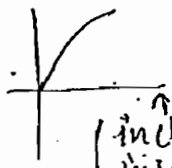
But y is eigen fu^n $(\neq 0)$

∴ In Case of $n=2$ → In Case of open interval



for $n=1 \longrightarrow d_1$ in open - interval

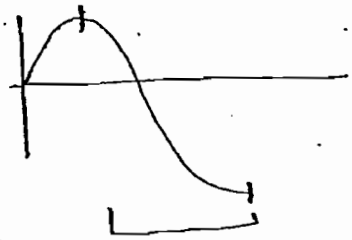
y_{d_1} has No Zero → Case of Monotonic



(in closed interval Zero may exist)

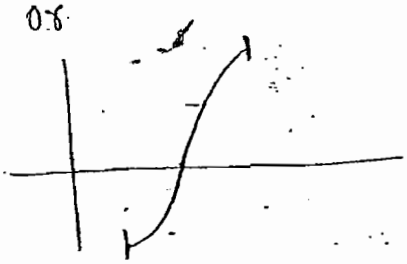
for $n=2 \longrightarrow d_2$

y_{d_2} has 1 Zero in open interval.



Case of Monotonic

or



with in this interval

Q.9 → $y'' + \lambda y = 0$ Periodic SLP

$y(0) = y(\pi)$; $y'(0) = y'(\pi)$

Sol^n → 1st Case →

When $\lambda < 0$ $\lambda = -\mu^2$, $\mu \neq 0$

$$y'' - \mu^2 y = 0$$

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$y'(x) = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}$$

$$c_1 + c_2 = c_1 e^{\mu\pi} + c_2 e^{-\mu\pi}$$

$$c_1 \mu - c_2 \mu = c_1 \mu e^{\mu\pi} - c_2 \mu e^{-\mu\pi}$$

$$c_1 (1 - e^{\mu\pi}) + c_2 (1 - e^{-\mu\pi}) = 0$$

$$c_1 (1 - e^{\mu\pi}) - c_2 (1 - e^{-\mu\pi}) = 0$$

$$2c_1 (1 - e^{\mu\pi}) = 0$$

$e^{\mu\pi} \neq 1$ (Only at $\mu = 0$ But $\mu \neq 0$)

$\Rightarrow c_1 = 0$

& $\Rightarrow c_2 = 0$

Only Trivial Sol^n in this case

2nd case → When $\lambda = 0$

$y(x) = c_1 x + c_2$ $\quad y'(x) = c_1$

$y(0) = y(\pi)$ $\quad y'(0) = y'(\pi)$

$c_2 = c_1 \pi + c_2$ $\quad c_1 = c_1$

$\Rightarrow c_1 = 0$

$y(x) = c_2$

$\Rightarrow y_\lambda(x) = 1$ $\quad$ (Case when $c_2 = 1$)

3rd case → When $\lambda > 0$

$\lambda = \mu^2$, $\mu \neq 0$

$y(x) = c_1 \cos \mu x + c_2 \sin \mu x$

$y'(x) = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x$

$y(0) = y(\pi)$

$\Rightarrow c_1 = c_1 \cos \mu \pi + c_2 \sin \mu \pi$

$y'(0) = y'(\pi)$

$c_2 \mu = -c_1 \mu \sin \mu \pi + c_2 \mu \cos \mu \pi$

$$c_1(1-\cos p\pi) - c_2 \sin p\pi = 0$$
$$+c_1 \sin p\pi + c_2(1-\cos p\pi) = 0$$

$$\begin{bmatrix} 1-\cos p\pi & -\sin p\pi \\ \sin p\pi & 1-\cos p\pi \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$c_1 = c_2 = 0 \Rightarrow$ Trivial Sol^n

for Non-Trivial Sol^n →

$$\begin{vmatrix} 1-\cos p\pi & -\sin p\pi \\ \sin p\pi & 1-\cos p\pi \end{vmatrix} = 0$$

$$(1-\cos p\pi)^2 + \sin^2 p\pi = 0$$

$$\Rightarrow \cos p\pi = 1$$

$$\therefore p\pi = 2n\pi$$

$$\Rightarrow p = 2n$$

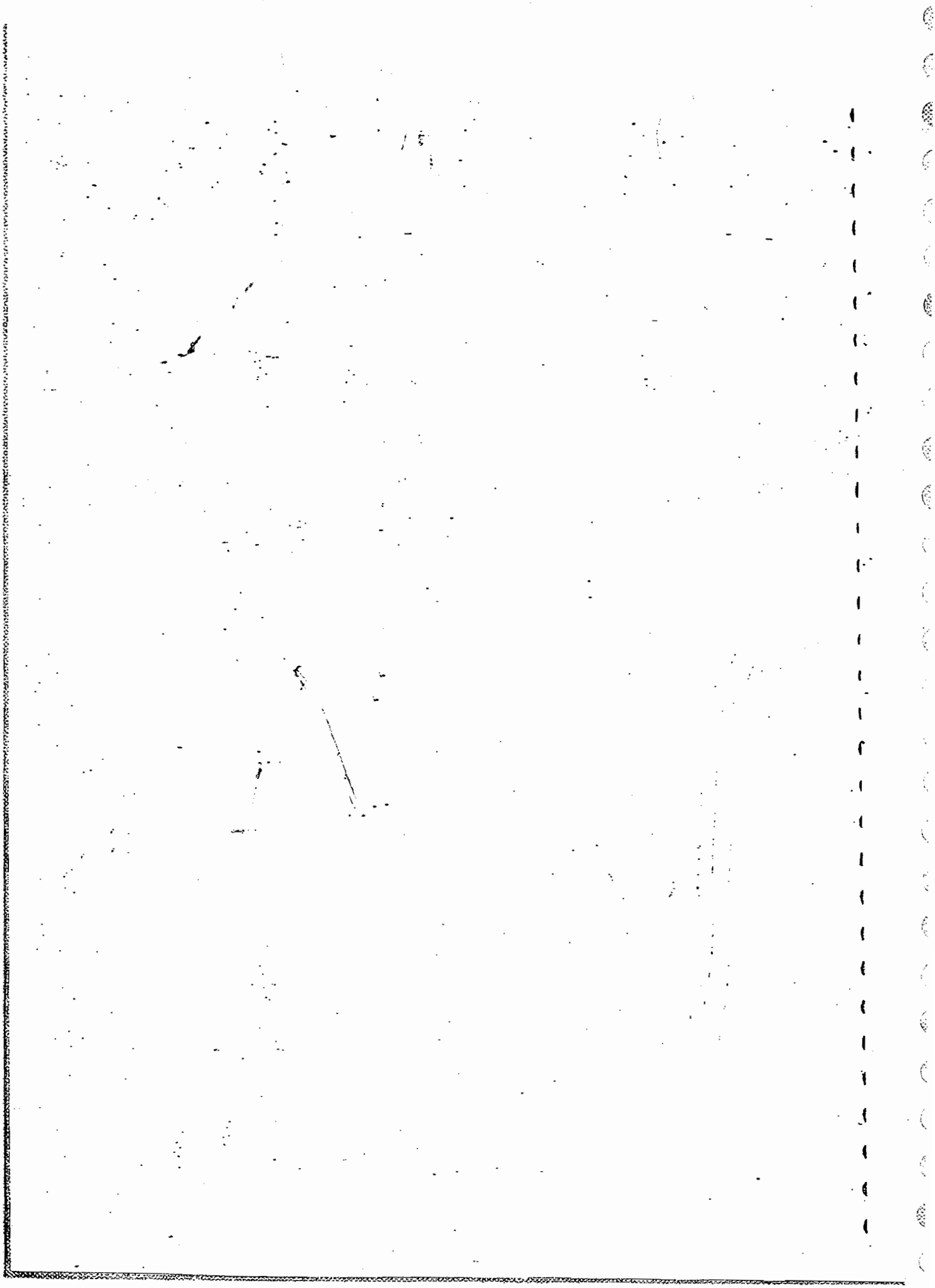
$$\therefore \lambda = p^2 = 4n^2$$

In case of Periodic SLP →

Corresponding to Least Eigen Value → Only 1 Eigen fu^n

for Rest → Two Eigen fu^{n}s

Eigen Values.



$\lambda = \lambda_0$ Only 1 Eigen fu^n → $\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots\cdots\cdots < \lambda_n < \cdots\cdots$

λ_0 ↓ 0 — λ_1 ↓ 4 — λ_2 ↓ 16 — λ_3 ↓ 36

Two L.I eigen values fu^{n}s corresponding to $\lambda = \lambda_1$ Also $\neq n$. $\lambda \neq \lambda_0$ — $\cos 2x$, $\sin 2x$ — $\cos 4x$, $\sin 4x$ — $\cos 6x$, $\sin 6x$

Bounded below

Unbounded above

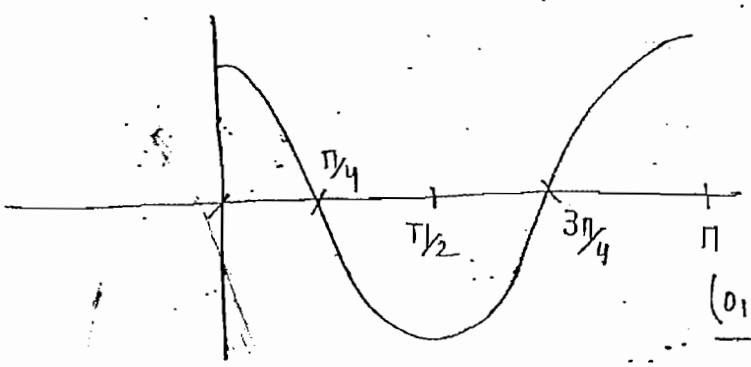
All the eigen values are +ve + 0 is the eigen value

So → Non-negative set of eigen values

For Non-Zero Eigen Value →

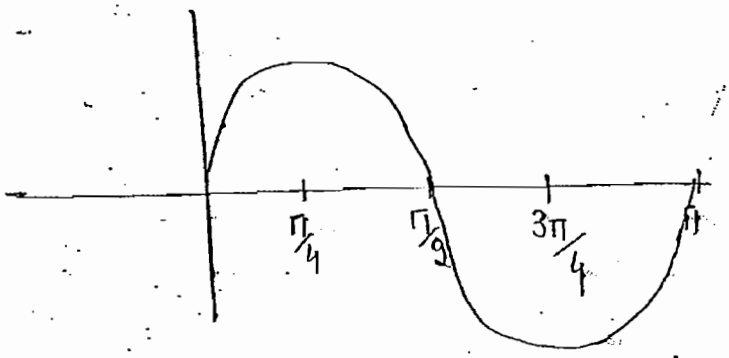
Eigen fu^{n}s are → $\cos(2\pi x)$ & $\sin(2\pi x)$

When $\lambda_1 = 4$ In $(0, \pi)$



$\cos 2x$

$(0, \pi) \rightarrow$ 2 zeroes

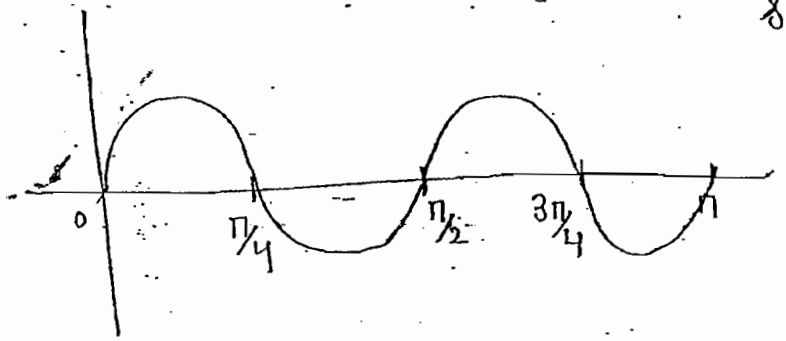


$\sin 2x$

$(0, \pi) \rightarrow$ 1 zero.

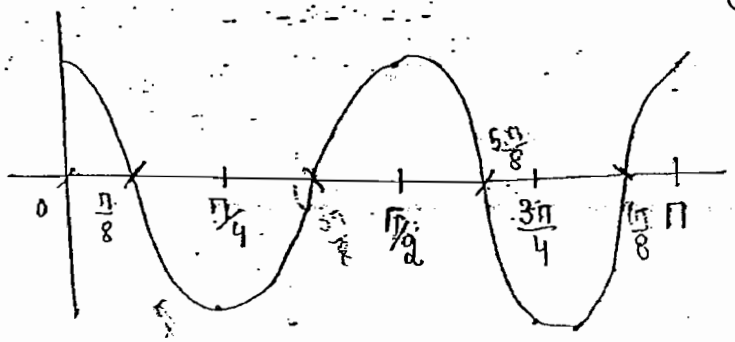
When $\lambda = 16$

In $[0, \pi]$



$\sin 4x$

3 - Zeros in $(0, \pi)$



$\cos 4x$

4 - Zeros in $(0, \pi)$

$\sin nx \longrightarrow 2n-1$ Zeros

$\cos nx \longrightarrow 2n$ Zeros

} exactly

Que $y'' + \lambda y = 0$

$y(-\pi) = y(\pi)$, $y'(-\pi) = y'(\pi)$

Then

~~X~~ $\exists$ λ the eigen space corresponding to λ is of dimension 1.

~~X~~ All the eigen values are +ve (Non-ve exists)

~~X~~ $\exists$ Uncountable value of λ for which diff. eq^n has Unique sol^n

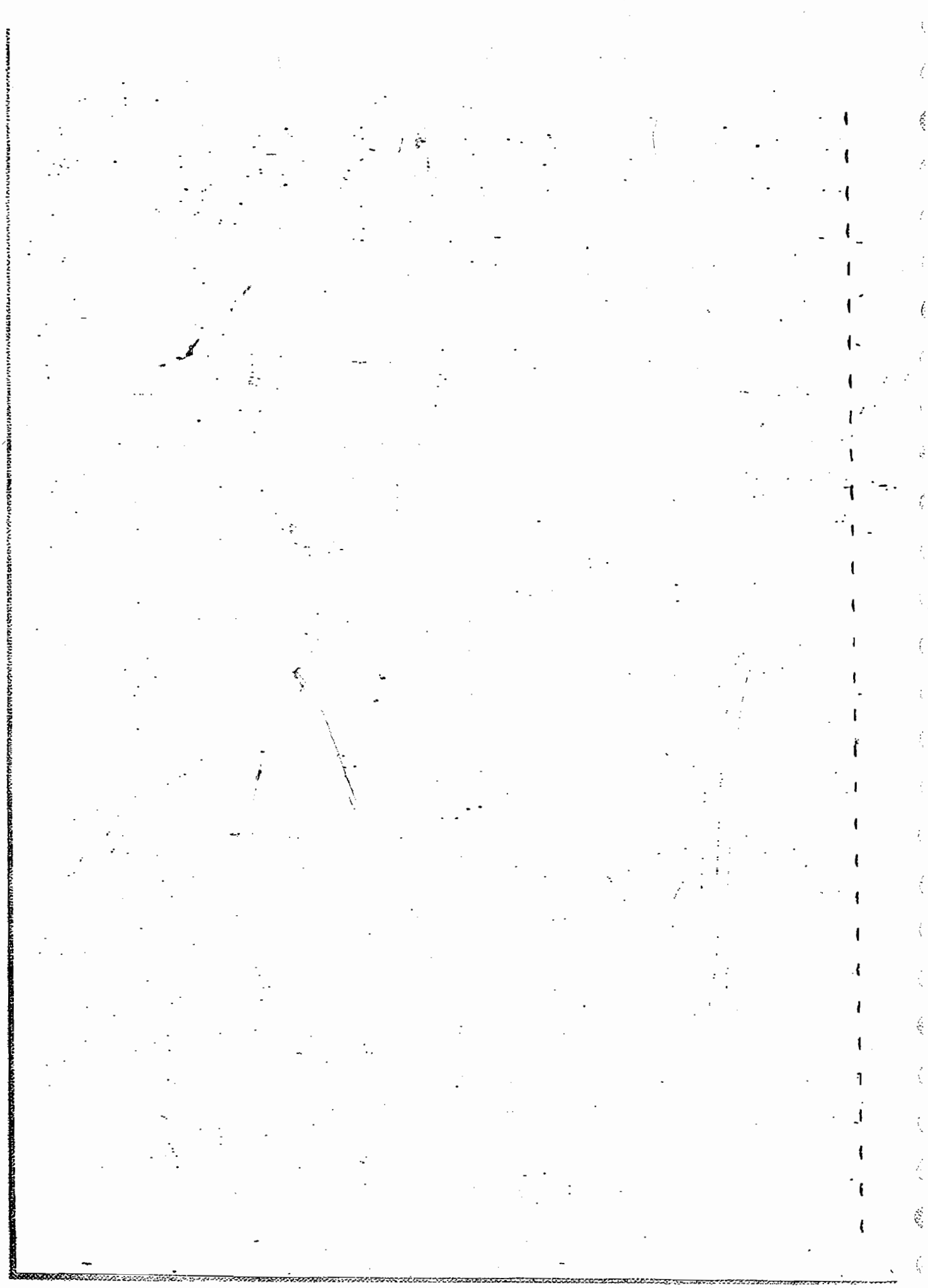
d) ∃ λ such that the eigen vector corresponding to λ is zero ∀ x ∈ [−π, π]

c) ~~∃ λ such that eigen vector corresponding to λ is zero ∀ x ∈ [−π, π]~~

f) ∃ λ such that
changes its sign in [−π, π]

g) ∃ countably Infinite λ s.t the diff eqⁿ has Infinite solⁿ

h) ∀ λ diff eqⁿ has either unique solⁿ or Infinite solⁿ



(1) All the eigen values are real

(2) The Eigen Vector Corresponding to distinct eigen value are orthogonal

(3) The Eigen Vector Corresponding to distinct are Linearly Independent

(4) The Eigen Space Corresponding to least Eigen Value is of 1 dimension

(5) The Eigen Space Corresponding to any Eigen Value except least One is of 2 dimension.

(6) If $d=0$ is an eigen value and Corresponding Eigen space is of dimension 1, then all the eigen values are non-negative

(7) If $d=0$ is an eigen value & the Eigen space Corresponding to 0 is of dimension 2, then there must exist atleast One negative Eigen Values s.t Eigen space of this eigen value is of dimension 1

(8) The set of -ve eigen values is alwx finite

(9) The set of +ve eigen values is alwx ctably infinite

(10) ~~If $\lambda_1 \neq \lambda_2$ are 2 eigen value's (Not least One)~~

If λ is an eigen value (except least One) and $y_{\lambda}, y_{\lambda'}$ are two L.I eigen functions.

- No. of Zero's of y_λ and $y_{\lambda'}$ is finite in $[a,b]$.
- B/w any two successive Zero's of y_λ ∃ exactly one Zero of $y_{\lambda'}$ and conversely.
- If Smallest and Largest Zero in (a,b) is of y_λ then no. of Zeros of y_λ = No. of Zero's of $y_{\lambda'}$ + 1
- If Smallest Zero is of y_λ and Largest Zero is of $y_{\lambda'}$ in (a,b) $\Rightarrow$ # of Zero's of y_λ = # of Zeros of $y_{\lambda'}$

(11) If $\lambda_i < \lambda_j$ are two eigen values and $y_{\lambda i}$ & $y_{\lambda j}$ are corresponding eigen fu's then

- # of Zeros of $y_{\lambda i}$ < # of Zeros of $y_{\lambda j}$.

Que $\frac{d}{dx}\left\{(x^2+1)\frac{dy}{dx}\right\} + \frac{1}{1+x^2}y = 0$

$y(-1) = 0 \quad , \quad y(1) = 0$

Variable Coefficients

Apply Method of Change of Invariable

Put $x = \tan t$

$t = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{1+x^2}\frac{dy}{dt}$$

$$(1+x^2)\frac{dy}{dx} = \frac{dy}{dt}$$

$$\therefore \frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{1}{1+x^2}y = 0$$

$$\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}+\frac{\lambda}{x^2+1}y=0$$

$$\frac{1}{1+x^2}\left(\frac{d^2y}{dt^2}+\lambda y\right)=0$$

$$\frac{d^2y}{dt^2}+\lambda y=0.$$

$x=-1 \quad y(-1)=0$

$y\left(-\frac{\pi}{4}\right)=0$ ✓

$x=1 \quad y(1)=0$

$y\left(\frac{\pi}{4}\right)=0$ ✓

$$\frac{d^2y}{dt^2}+\lambda y=0 \quad , \quad y\left(-\frac{\pi}{4}\right)=0 \quad , \quad y\left(\frac{\pi}{4}\right)=0$$

Regular SLP

When $\lambda<0$;

$$y(t)=c_1e^{\mu t}+c_2e^{-\mu t}$$

$$y\left(-\frac{\pi}{4}\right)=0$$

$$c_1e^{-\frac{\pi}{4}\mu}+c_2e^{\frac{\pi}{4}\mu}=0$$

& $y\left(\frac{\pi}{4}\right)=0 \Rightarrow c_1e^{\frac{\mu\pi}{4}}+c_2e^{-\frac{\mu\pi}{4}}=0$

$\Rightarrow c_1=0 \; , \; c_2=0$

Only Trivial sol^n

$\lambda=0$:

$$y(t)=c_1t+c_2$$

$c_1=0 \; , \; c_2=0$ Only Trivial sol^n

$d > 0$

$$y(x) = C_1 \cos px + C_2 \sin px$$

$$y\left(-\tfrac{\pi}{4}\right) = 0$$

$$C_1 \cos\frac{p\pi}{4} - C_2 \sin\frac{p\pi}{4} = 0 \quad \text{—①}$$

$$y\left(\tfrac{\pi}{4}\right) = 0 \Rightarrow C_1 \cos\frac{p\pi}{4} + C_2 \sin\frac{p\pi}{4} = 0 \quad \text{—②}$$

$$2C_1 \cos\frac{p\pi}{4} = 0$$

$$\|C_1 = 2C_2 \sin\frac{p\pi}{4} = 0$$

for Non-Trivial sol^n of C_1 & C_2 →

$$\begin{bmatrix} \cos\frac{p\pi}{4} & -\sin\frac{p\pi}{4} \\ \cos\frac{p\pi}{4} & +\sin\frac{p\pi}{4} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0$$

$$\begin{vmatrix} \cos\frac{p\pi}{4} & -\sin\frac{p\pi}{4} \\ \cos\frac{p\pi}{4} & \sin\frac{p\pi}{4} \end{vmatrix} = 0$$

$$2 \sin\frac{p\pi}{4} \cos\frac{p\pi}{4} = 0$$

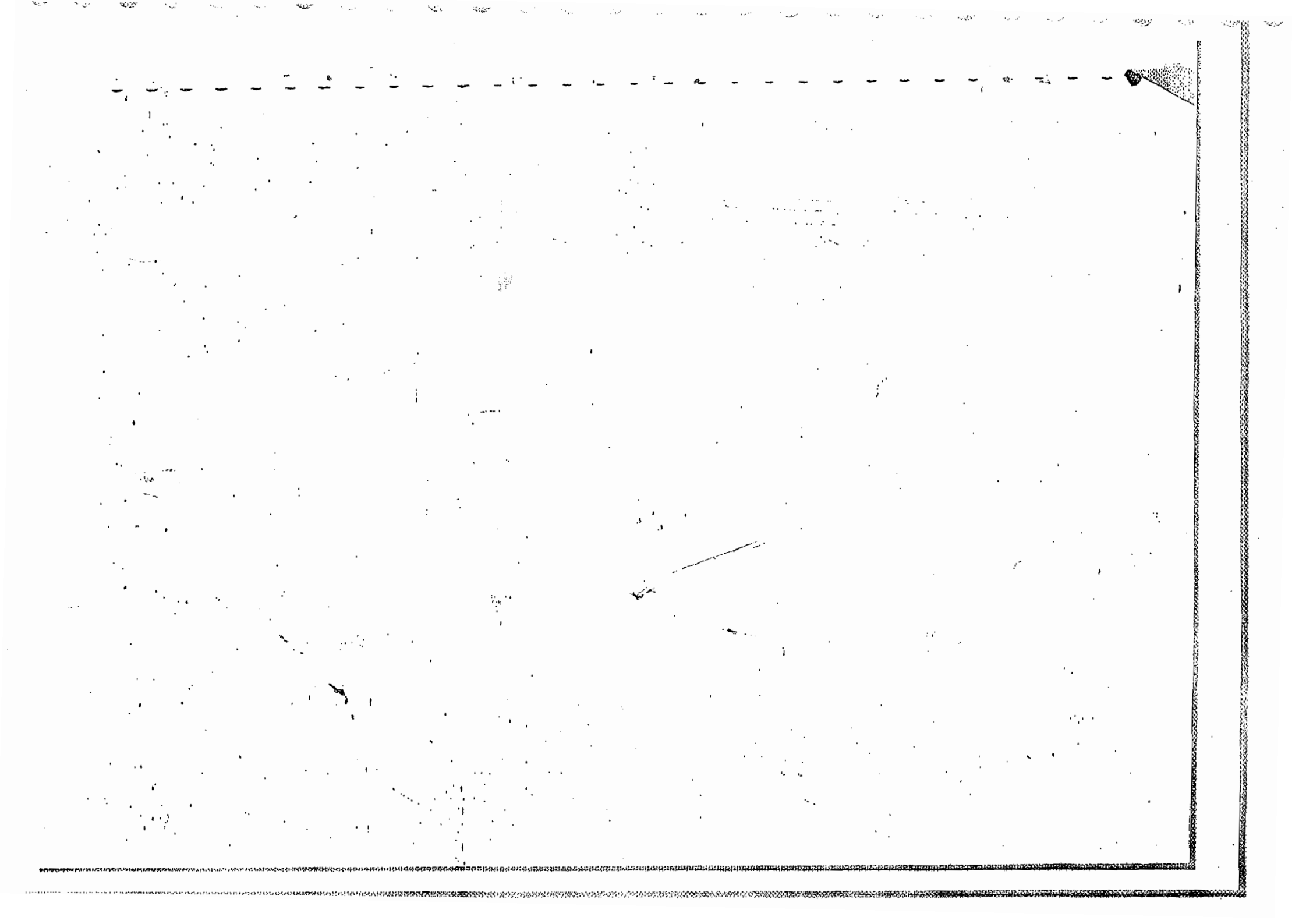
$$\sin\frac{p\pi}{2} = 0$$

$$\sin\frac{p\pi}{2} = 0$$

$$\frac{p\pi}{2} = n\pi \Rightarrow \underline{p = 2n}$$

$2 \cos \frac{p\pi}{4} = 0$

$p = 4n+2$ $\qquad$ $p = 4n$



Que $\frac{d}{dx}\{xy'\} + \frac{dy}{x} = 0$; $y(1)=0$, $y(e^{\pi})=0$

Que $y'' + \lambda y = 0$

Let $\alpha \in \mathbb{R}$ $\alpha y(0) + y'(0) = 0$, $y(1) = 0$

Then

i) $\forall\, \alpha \in \mathbb{R}$ SLP has infinite eigen values

ii) if $\alpha < 1$ all the eigen values are +ve and the smallest eigen value $\lambda \to 0$ as $\alpha \to 1$

iii) if $\alpha > 1$ $\exists$ exactly 1 -ve eigen value λ and $\lambda \downarrow$ as $\alpha \uparrow$

iv) $\lambda = 0$ is an eigen value only if $\alpha = 1$

If given SLP is in the form →

$$-(p(x)y')' + q(x)y - \lambda w(x) y = 0 \quad \text{in } [a,b]$$

$$p(x) > 0 \ , \ w(x) > 0$$

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

Then if $-\frac{a_1}{a_2} > 0$, and $\frac{b_1}{b_2} > 0$, $q(x) > 0 \ \forall x$

$\Rightarrow$ All the eigen values are +ve.

If $a_1 = 0$, $b_1 = 0$, $q(x) \equiv 0$

$\Rightarrow \lambda = 0$ is an eigen value

($y'' + \lambda y = 0$, $y'(0) = 0$, $y'(b) = 0$)