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Definition: An equation involving differentials or differential coefficient is called a differential equation.

For example

$$(i) \frac{dy}{dx} = x \sin x$$

$$(ii) \frac{d^3y}{dx^3} - \frac{3d^2y}{dx^2} + \frac{5dy}{dx} - 6y = \sin x$$

$$(iii) \frac{d^2s}{dt^2} - \frac{4ds}{dt} + 5s = t^2$$

$$(iv) y = \frac{xdy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$(v) \frac{x \partial z}{\partial x} + \frac{y \partial z}{\partial y} = z$$

$$(vi) \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{ad^2y}{dx^2}$$

$$(vii) \frac{d^4y}{dx^4} - y = \tan x$$

$$(viii) \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial y}$$

2 Ordinary and partial differential equations

If a differential equation contains one dependent variable (or function) and one independent variable i.e. if all the differential coefficient have reference to single independent variable, then the differential equation is called an ordinary differential equation.

The above differential equations (i), (ii), (iii), (iv), (vi) and (vii) are of this type. If there are two or more independent variables, so that derivative is partial differential equation. The equations (v) and (viii) are example of this type.

3 Degree and order of a differential equation

The order of a differential equation is the highest differential coefficient which appears in it. For example the differential equation (i), (ii) and (iii) as given above are of 1st, 3rd and 2nd respectively.

The degree of differential equation is the degree of highest order differential coefficient which appears in it, when the differential coefficients are free from radicals and fractions. For example (ii) equation is of 1st degree, (iv) is of 2nd degree and (vi) (after squaring both sides) is of 2nd degree.

General complete solution of a differential equation.

The solution of a differential equation in which the number of arbitrary constant is equal to the order of the differential equation is called the general solution (or complete integral or complete primitive).

Examples.

1. $v = \frac{A}{r} + B$ is the general solution of differential equation $\frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} = 0$, where A and B are arbitrary constants.

2. Let $y = Ce^x$ (1)

$$\Rightarrow \frac{dy}{dx} = Ce^x \quad (2)$$

Eliminating the arbitrary constant C between (1) and (2),

$$\Rightarrow \frac{dy}{dx} = y \text{ which is a differential equation of the 1st order.}$$

3. Let $y = C_1 \cos x + C_2 \sin x$ (1)

$$\Rightarrow \frac{dy}{dx} = -C_1 \sin x + C_2 \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -C_1 \cos x - C_2 \sin x = -y, \quad (\text{from (1)}) \quad (2)$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

which is a differential equation of 2nd order.

Note. No. of arbitrary constant = Order of Differential Equation

Particular solution

If particular values are given to the arbitrary constant in the general solution then the solution so obtained is called the particular solution.

For example putting $A = 2$ and $B = 3$ in ex 1, a particular solution of $\frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} = 0$ is $v = \frac{2}{r} + 3$.

independent of arbitrary constants:

The arbitrary constants appearing in the general solution must be independent and should not be reducible to a fewer number of equivalent constants

For example : The solution $y = ce^{x+k}$ apparently contains two arbitrary constants c and k , but in fact they are equivalent to only one constant, for $y = ce^{x+k} = e^x \cdot ce^k = (ce^k)e^x = \lambda e^x$.

Therefore, the two constant c and k are reduced to one constant λ and as such c and k are not independent constants.

EX 1 write down the order and degree of the following equation

$$(i) \frac{d^2x}{dy^2} + \sqrt{1 + \left(\frac{dx}{dy}\right)^3} = 0$$

$$(ii) \frac{d^2x}{dy^2} + \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = 0$$

$$(iii) \left(\frac{d^3y}{dx^3}\right)^2 - xy\left(\frac{dy}{dx}\right)^3 + y = 0$$

$$(iv) \left(\frac{dy}{dx}\right) = \sin x$$

$$(v) \left(\frac{d^2y}{dx^2}\right)^3 - xy\left(\frac{dy}{dx}\right)^4 + y = 0$$

Solution.

(1) Second order, second degree

(2) Second order, second degree

(3) Third order, Second degree

(4) First order, First degree

(5) Second order, Third degree

Ex. 2. What is the order of the differential equation whose solution is the circle $(x - \alpha)^2 + y^2 = \alpha^2$, where α is an arbitrary constant.

Solution. The given equation is

$$(x - \alpha)^2 + y^2 = \alpha^2$$

$$\Rightarrow x^2 - 2\alpha x + y^2 = 0$$

Differentiating both sides of (1) with respect to x , we get

$$2x - 2\alpha + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow x + y \frac{dy}{dx} = \alpha$$

Eliminating α from (1) and (2), we get

$$x^2 - 2\left(x + y \frac{dy}{dx}\right)x + y^2 = 0$$

$$\Rightarrow x^2 - 2x^2 - 2xy \frac{dy}{dx} + y^2 = 0$$

$$\Rightarrow 2xy \frac{dy}{dx} + x^2 - y^2 = 0 \text{ which is the differential equation of the given circle (1) and its order is first.}$$

Ex. 3. Find the differential equation of the family of curves $y = e^x (A \cos x + B \sin x)$, where A and B are arbitrary constants.

Solution. The given equation is

$$y = e^x (A \cos x + B \sin x)$$

$$\Rightarrow \frac{dy}{dx} = e^x (-A \sin x + B \cos x) + e^x (A \cos x + B \sin x)$$

$$= e^x (-A \sin x + B \cos x) + y$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x (-A \cos x - B \sin x) + e^x (-A \sin x + B \cos x) + \frac{dy}{dx}$$

$$= -e^x (A \cos x + B \sin x) + \left(\frac{dy}{dx} - y\right) + \frac{dy}{dx} \quad (\text{from (2)})$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y + \left(\frac{dy}{dx} - y\right) + \frac{dy}{dx} \quad (\text{from (1)})$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \text{ which is differential equation of 2nd order.}$$

Ex. 4. Form the differential equation of $y = A \cos(x^2) + B \sin(x^2)$

Solution. Differentiating (1) with respect to x , we get

$$\frac{dy}{dx} = A[-(\sin x^2)]2x + B[\cos(x^2)]2x$$

$$= 2x[-A \sin(x^2) + B \cos(x^2)]$$

Again differentiating both sides with respect to x, we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2[-A \sin(x^2) + B \cos(x^2)] + 2x[-A\{\cos(x^2)\}(2x) + B\{-\sin(x^2)\}(2x)] \\ \Rightarrow \frac{d^2y}{dx^2} &= 2[-A \sin(x^2) + B \cos(x^2)] - 4x^2[A \cos(x^2) + B \sin(x^2)] \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{1}{x} \frac{dy}{dx} - 4x^2 y, \text{ from (1) and (2)} \\ \Rightarrow x \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3 y &= 0 \end{aligned}$$

Ex. 5. Form the differential equation of all circles of radius r.

solⁿ: The equation of all circles of radius r is

$$(x - a)^2 + (y - b)^2 = r^2,$$

where a and b are arbitrary coordinates of the centre.

$$(x - a)^2 + (y - b)^2 = r^2 \quad (1)$$

Differentiating with respect to x, we have

$$2(x - a) + 2(y - b) \frac{dy}{dx} = 0$$

$$\Rightarrow (y - b) \frac{dy}{dx} + (x - a) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x - a)}{(y - b)}$$

(2) differentiating

$$\frac{d^2y}{dx^2} = -\left[\frac{(y - b)(1) - (x - a)\left(\frac{dy}{dx}\right)}{(y - b)^2} \right] = \frac{(x - a)\left(\frac{dy}{dx}\right) - (y - b)}{(y - b)^2}$$

$$= \frac{(x - a)\left\{-\frac{(x - a)}{(y - b)}\right\} - (y - b)}{(y - b)^2} \quad (\text{from (2)})$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(x - a)^2 + (y - b)^2}{(y - b)^3} \right] = \frac{-r^2}{(y - b)^3} \quad (\text{from (1)}) \quad (3)$$

Again from (2), we have

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x - a)^2}{(y - b)^2}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(y - b)^2 + (x - a)^2}{(y - b)^2} = \frac{r^2}{(y - b)^2} \quad (\text{from (1)})$$

$$\Rightarrow \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{3/2} = \frac{r^3}{(y - b)^3} \quad \left(\text{raising both sides to power } \frac{3}{2} \right) \quad (4)$$

From (3) and (4) we have

$$\left[1 + \left(\frac{dy}{dx}\right)^2 \right]^3 = r^2 \left(\frac{d^2y}{dx^2}\right)^2$$

Ex. 6. Show that $v = \frac{A}{r} + B$ is a solution of $\frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} = 0$.

Solⁿ: It is given that $v = \frac{A}{r} + B$ (1)

Differentiating both sides twice with respect to r, we get

$$\frac{dv}{dr} = -\frac{A}{r^2} \quad (2)$$

$$\frac{d^2v}{dr^2} = \frac{2A}{r^3} \quad (3)$$

From (2) and (3), we have

$$\frac{d^2v/dr^2}{dv/dr} = \frac{2A/r^3}{-A/r^2}$$

$$\Rightarrow \frac{d^2v/dr^2}{dv/dr} = -\frac{2}{r}$$

$$\Rightarrow \frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} = 0$$

Ex. 7. Show that $Ax^2 + By^2 = 1$ is the solution of $x \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \right] = y \frac{dy}{dx}$

Solⁿ: The given equation is $Ax^2 + By^2 = 1$

Differentiating both sides with respect to x , we get

$$2Ax + 2By \frac{dy}{dx} = 0$$

$$\Rightarrow Ax + By \frac{dy}{dx} = 0$$

Differentiating both sides of (2) with respect to x , we get

$$A + B \left[y \frac{d^2y}{dx^2} + \frac{dy}{dx} \frac{dy}{dx} \right] = 0$$

$$\Rightarrow Ax + Bx \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] = 0$$

(multiplying each term by x)

Subtracting (2), from (3), we get

$$Bx \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] - By \frac{dy}{dx} = 0$$

$$\Rightarrow x \left\{ y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right\} = y \frac{dy}{dx}$$

Ex. 8. Find the differential equation satisfied by the system of parabolas $y^2 = 4a(x + a)$

Solution. Given $y^2 = 4a(x + a)$

Differentiating both sides with respect to x , we get

$$2y \left(\frac{dy}{dx} \right) = 4a$$

Dividing (1) by (2), we get

$$\frac{y^2}{2y \left[\frac{dy}{dx} \right]} = (x + a)$$

$$\Rightarrow a = \left[\frac{y}{2 \left[\frac{dy}{dx} \right]} - x \right]$$

Substituting equation (3) in (2), we get

$$y \left(\frac{dy}{dx} \right) = 2 \left[\frac{y}{2 \left(\frac{dy}{dx} \right)} - x \right]$$

$$\Rightarrow y \left(\frac{dy}{dx} \right)^2 = y - 2x \left(\frac{dy}{dx} \right)$$

Ex. 9. By eliminating the constants a and b obtain differential equation of which $xy = ae^x + be^{-x} + x^2$ is a solution.

Solution. The given equation is $xy = ae^x + be^{-x} + x^2$

Differentiating both sides with respect to x , we get

$$y + x \left(\frac{dy}{dx} \right) = ae^x - be^{-x} + 2x$$

Differentiating both sides of (2) with respect to x , we get

$$\frac{dy}{dx} + \frac{dy}{dx} + x \frac{d^2y}{dx^2} = ae^x + be^{-x} + 2$$

$$\Rightarrow x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = (xy - x^2) + 2, \quad [\text{from (1) } xy - x^2 = ae^x + be^{-x}]$$

$$\Rightarrow x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy + x^2 - 2 = 0$$

DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

INTRODUCTION

There are two standard forms of differential equations of first order and first degree, namely

$$(1) \frac{dy}{dx} = f(x, y)$$

$$(2) M(x, y)dx + N(x, y)dy = 0$$

2(A) SEPARATION OF VARIABLES

If in an equation, it is possible to get all the functions of x and dx to one side, and all the function of y and dy to the other, the variables are said to be separable.

Rule to solve an equation in which the variables are separable.

Consider the equation $\frac{dy}{dx} = XY$, where X is a function of x only and Y is a function of y only.

1. Given differential equation is $\frac{dy}{dx} = XY$

2. $\frac{dy}{Y} = Xdx$, i. e. variables have been separated

3. Integrating both sides

$$\int \frac{dy}{Y} = \int X dx + c,$$

where c is an arbitrary constant, is the required solution.

Point to Remember

$$\begin{aligned} (1) \log x + \log y &= \log xy & (2) \log x - \log y &= \log \frac{x}{y} \\ (3) n \log x &= \log x^n & (4) \tan^{-1} x + \tan^{-1} y &= \tan^{-1} \frac{x+y}{1-xy} \\ (5) \tan^{-1} x - \tan^{-1} y &= \tan^{-1} \frac{x-y}{1+xy} \end{aligned}$$

Ex. 10. Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

Solution. The given differential equation is

$$\begin{aligned} \frac{dy}{dx} &= e^{x-y} + x^2 e^{-y} = e^x e^{-y} + x^2 e^{-y} \\ \Rightarrow \frac{dy}{dx} &= e^{-y} (e^x + x^2) \Rightarrow e^y dy = (x^2 + e^x) dx \end{aligned}$$

Integrating

$$e^y = \left(\frac{1}{3}\right) x^3 + e^x + c \text{ which is required solution.}$$

Ex. 11. Solve $\log \left(\frac{dy}{dx}\right) = ax + by$

Solution. The given differential equation is

$$\begin{aligned} \log \left(\frac{dy}{dx}\right) &= ax + by \\ \Rightarrow \frac{dy}{dx} &= e^{ax+by} = e^{ax} e^{by} \\ \Rightarrow e^{-by} dy &= e^{ax} dx \end{aligned}$$

Integrating

$$-\left(\frac{1}{b}\right) e^{-by} = \left(\frac{1}{a}\right) e^{ax} + c \text{ which is required solution.}$$

Prob. 12 $(x^2 - yx^2) \frac{dy}{dx} + (y^2 + xy^2) = 0$

$$\text{Ans. } \log \left(\frac{x}{y}\right) = \frac{1}{x} + \frac{1}{y} + c$$

Do yourself

Ex. 13. Solve $y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx}\right)$.

Solution. The given differential equation is

$$\begin{aligned} \Rightarrow y - x \frac{dy}{dx} &= a \left(y^2 + \frac{dy}{dx}\right) \\ \Rightarrow (a+x) \frac{dy}{dx} &= y - ay^2 \\ \Rightarrow \frac{dx}{x+a} &= \frac{dy}{y(1-ay)} \\ \Rightarrow \frac{dx}{x+a} &= \left[\frac{a}{1-ay} + \frac{1}{y} \right] dy \quad [\text{By partial fraction}] \end{aligned}$$

Integrating,

$$\begin{aligned} \log(x+a) &= -\log(1-ay) + \log y + \log c \\ \Rightarrow \log(x+a) &= \log \left[\frac{cy}{1-ay} \right] \\ \Rightarrow x+a &= \frac{cy}{1-ay} \\ \Rightarrow (x+a)(1-ay) &= cy, \text{ which is the required solution.} \end{aligned}$$

Prob. 14. $\cos y \log(\sec x + \tan x) dx = \cos x \log(\sec y + \tan y) dy$

$$\text{Ans. } \left[\log \left(\frac{\sec x + \tan x}{\sec y + \tan y} \right) \log((\sec x + \tan x)(\sec y + \tan y)) \right] = c$$

Do yourself

Ex. 15. Solve $3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$

Solution. The given differential equation is

$$\begin{aligned} 3e^x \tan y dx + (1 - e^x) \sec^2 y dy &= 0 \\ \Rightarrow \frac{3e^x}{1 - e^x} dx + \frac{\sec^2 y}{\tan y} dy &= 0 \end{aligned}$$

Integrating,

$$-3 \log(1 - e^x) + \log(\tan y) = \log c$$

$$\Rightarrow \tan y = c(1 - e^x)^3.$$

Prob. 16. $\frac{dy}{dx} + \tan y = \sin(x+y) + \sin(x-y)$

Ans. $(2 \cos x + \sec y = c)$

Do yourself

Ex. 17. Solve $\sqrt{(1+x^2+y^2+x^2y^2)} + xy \left(\frac{dy}{dx}\right) = 0$.

Solution The given differential equation is

$$\Rightarrow \sqrt{(1+x^2+y^2+x^2y^2)} + xy \left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \sqrt{(1+x^2)}\sqrt{(1+y^2)} + xy \left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{\sqrt{(1+x^2)}dx}{x} + \frac{ydy}{\sqrt{(1+y^2)}} = 0$$

$$\Rightarrow \frac{(1+x^2)dx}{x\sqrt{(1+x^2)}} + \frac{ydy}{\sqrt{1+y^2}} = 0$$

Integrating

$$\int \frac{dx}{x\sqrt{(1+x^2)}} + \int \frac{xdx}{\sqrt{(1+x^2)}} + \int \frac{ydy}{\sqrt{(1+y^2)}} = C \quad (1)$$

Now,

$$\begin{aligned} \int \frac{dx}{x\sqrt{(1+x^2)}} &= \int \frac{(-1/t^2)dt}{(1/t)\sqrt{1+((1/t)^2)}}, \quad \left(\text{putting } x = \frac{1}{t}\right) \\ &= - \int \frac{dt}{\sqrt{t^2+1}} = -\log\{t + \sqrt{(t^2+1)}\} \\ &= -\log\left\{\frac{1}{x} + \sqrt{\left(\frac{1}{x^2} + 1\right)}\right\} = -\log\left\{\frac{1 + \sqrt{(1+x^2)}}{x}\right\} \\ &= \log x - \log(1 + \sqrt{(1+x^2)}) \end{aligned} \quad (2)$$

Again,

$$\begin{aligned} \int \frac{xdx}{\sqrt{(1+x^2)}} &= \int \frac{1}{2} \frac{dt}{\sqrt{t}}, \quad (\text{putting } 1+x^2 = t) \\ &= \frac{1}{2} \int t^{-1/2} dt = t^{1/2} = (1+x^2)^{1/2} \end{aligned} \quad (3)$$

Similarly,

$$\int \frac{ydy}{\sqrt{(1+y^2)}} = (1+y^2)^{1/2} \quad (4)$$

Using (2), (3) and (4), (1) gives the required solution as

$$\log x - \log\{1 + \sqrt{1+x^2}\} + (1+x^2)^{1/2} + (1+y^2)^{1/2} = C \text{ which is required solution.}$$

Prob. 18. Find the function 'f' which satisfies the question $\frac{df}{dx} = 2x$ given that $f(0) = e^3$

Ans. $(f = e^{2x+3})$ Do yourself

Ex. 19. Solve $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

Solⁿ: The given differential equation is

$$\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$$

$$\Rightarrow \frac{dy}{dx} = e^y + (e^x + x^2 e^{x^3})$$

$$\Rightarrow e^{-y} dy = (e^x + x^2 e^{x^3}) dx.$$

Integrating(1),

$$\int e^{-y} dy = \int e^x dx + \int x^2 e^{x^3} dx$$

$$\Rightarrow -e^{-y} = e^x + \left(\frac{1}{3}\right) \int e^t dt + c \quad \text{putting } x^3 = t$$

$$\Rightarrow -e^{-y} = e^x + \left(\frac{1}{3}\right) e^{x^3} + c$$

$$\Rightarrow -e^{-y} = e^x + \frac{1}{3} e^{x^3} + c \quad \text{which is required solution}$$

Prob. 20. $\frac{dy}{dx} + x^2 = x^2 e^{3y}$

Ans. $(e^{3y} - 1) = ce^{(x^3+3y)}$

Do yourself

Ex. 21. If $\frac{dy}{dx} = e^{x+y}$ and $y(1) = 1$, find y when $x = -1$.

Solⁿ: The given differential equation $\frac{dy}{dx} = e^{x+y}$

$$\Rightarrow e^{-y} dy = e^x dx$$

integrating,

$$-e^{-y} = e^x + c$$

Putting, $x = 1, y = 1$ in (1),

$$-e^{-1} = e + c$$

$$\Rightarrow c = -e^{-1} - e$$

Hence (1) becomes

$$-e^{-y} = e^x - e^{-1} - e$$

Putting $x = -1$ in (2), we obtain

$$-e^{-y} = e^{-1} - e^{-1} - e$$

$\Rightarrow y = -1$ which is required solution.

Prob. 22. $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

Ans. $[\log(x/y) - (x+y)xy = c]$

Do yourself

2(B) REDUCIBLE SEPARABLE VARIABLE

Let the equation of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

Let $ax + by + c = v$

$$\Rightarrow a + b \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b} \left(\frac{dv}{dx} - a \right)$$

$\therefore$ from (1)

$$\frac{1}{b} \left(\frac{dv}{dx} - a \right) = f(v)$$

$$\Rightarrow \frac{dv}{dx} = b \cdot f(v) + a$$

$$\Rightarrow \frac{dv}{a + bf(v)} = dx$$

Integrate and find the solution.

Ex. 23. Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Solution The given differential equation is

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Let $4x + y + 1 = v$

$$\Rightarrow 4 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$$

From (1), the given equation becomes

$$\frac{dv}{dx} - 4 = v^2$$

$$\Rightarrow \frac{dv}{dx} = 4 + v^2$$

$$\Rightarrow \frac{dv}{4 + v^2} = dx$$

Integrating,

$$x + c = \frac{1}{2} \tan^{-1} \left(\frac{v}{2} \right)$$

$$\Rightarrow 2x + 2c = \tan^{-1} \left(\frac{v}{2} \right)$$

$$\Rightarrow v = 2 \tan(2x + 2c)$$

$$\Rightarrow 4x + y + 1 = 2 \tan(2x + 2c), \quad (\text{using (2)})$$

which is required solution.

Prob. 24. $\left(\frac{(x+y-a)}{(x+y-b)} \right) \frac{dy}{dx} = \frac{x+y+a}{x+y+b}$

Ans. $\log\{(x+y)^2 - ab\} = 2(x-y+c)$

Do yourself

Ex. 25. Solve $(x + y)^2 \left(\frac{dy}{dx}\right) = a^2$

Solution. The given differential equation is

$$(x + y)^2 \left(\frac{dy}{dx}\right) = a^2$$

$$\text{Let } x + y = v \quad (1)$$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1 \quad (2)$$

From (1) and (2), the given equation becomes

$$v^2 \left(\frac{dv}{dx} - 1\right) = a^2$$

$$\Rightarrow v^2 \frac{dv}{dx} = a^2 + v^2$$

$$\Rightarrow dx = \frac{v^2}{v^2 + a^2} dv$$

$$\Rightarrow dx = \left[1 - \frac{a^2}{a^2 + v^2}\right] dv$$

Integrating,

$$x + c = v - a^2 \frac{1}{a} \tan^{-1} \frac{v}{a}$$

$$\Rightarrow x + c = x + y - a \tan^{-1} \left(\frac{x + y}{a}\right)$$

$$\Rightarrow y - a \tan^{-1} \left(\frac{x + y}{a}\right) = c \text{ which is required solution.}$$

Prob. 26. $(4x + 3y + 2)dy = (8x + 6y + 12)dx$

Do yourself

Ex. 27. Solve $\frac{dy}{dx} = \sec(x + y)$

Solution The given differential equation is

$$\frac{dy}{dx} = \sec(x + y)$$

$$\text{Let } x + y = v \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dv}{dx}\right) - 1 \quad (2)$$

Using (1) & (2), the given equation becomes

$$\frac{dv}{dx} = 1 + \frac{1}{\cos v}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1 + \cos v}{\cos v}$$

$$\Rightarrow dx = \frac{\cos v}{1 + \cos v} dv = \frac{2 \cos^2 \frac{v}{2} - 1}{1 + 2 \cos^2 \frac{v}{2} - 1} dv$$

$$\Rightarrow dx = \left(1 - \frac{1}{2} \sec^2 \frac{1}{2} v\right) dv$$

Integrating,

$$x + c = v - \tan^{-1} \frac{1}{2} v \quad \Rightarrow x + c = x + y - \tan^{-1} \frac{v}{2}$$

$$\Rightarrow y - \tan^{-1} \frac{1}{2} (x + y) = c \text{ which is required solution.}$$

Ex. 28. Solve $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

Solⁿ: The given differential equation is

$$\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$$

$$\text{Let } x + y = v \quad (1)$$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1 \quad (2)$$

Using (1) and (2), the given equation becomes

$$\Rightarrow \frac{dv}{dx} = 1 + \sin v + \cos v \quad (3)$$

$$\text{But } 1 + \sin v + \cos v = 1 + 2 \sin(v/2) \cos(v/2) + 2 \cos^2(v/2) - 1 \\ = 2 \cos^2(v/2) [1 + \tan(v/2)]$$

$$\text{by (3)} \quad dx = \frac{dv}{2 \cos^2(v/2) [1 + \tan(v/2)]} = \frac{\frac{1}{2} \sec^2(v/2) dv}{1 + \tan(v/2)}$$

Integrating,

$$x + c = \log[1 + \tan(v/2)]$$

$\Rightarrow x + c = \log[1 + \tan\{(x + y)/2\}]$, (on using (1)) which is required solution.

Homogeneous Function

An equation of the form

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$$

is called homogeneous equation in x & y of degree n .

The above equation can be written as

$$f(x, y) = x^n [a_0 + a_1 (y/x) + \dots + a_n (y/x)^n]$$

$$\Rightarrow f(x, y) = x^n F(y/x)$$

3. HOMOGENEOUS DIFFERENTIAL EQUATION

Definition: A differential equation of first order and first degree is said to be homogeneous if it is of the form

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)},$$

where $f_1(x, y)$ & $f_2(x, y)$ both are homogeneous in x & y of same degree n

$$\Rightarrow \frac{dy}{dx} = \frac{x^n F_1(y/x)}{x^n F_2(y/x)} = \frac{F_1(y/x)}{F_2(y/x)}$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Note For solving homogenous differential equation

Let the given equation be homogenous. Then, by definition, the given equation can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad (1)$$

$$\text{Let } \frac{y}{x} = v \Rightarrow y = vx \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = v + x \left(\frac{dv}{dx}\right) \quad (3)$$

Using (2) and (3), (1) becomes

$$v + x \frac{dv}{dx} = f(v)$$

$$\Rightarrow x \frac{dv}{dx} = f(v) - v$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v};$$

$$\Rightarrow \log x + c = \int \frac{dv}{f(v) - v}$$

After integration, replace v by y/x and get the solution.

Ex. 29. Solve $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$.

Solution: The given differential equation is

$$(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^3 + 3xy^2}{y^3 + 3x^2y} \quad (1)$$

$$\text{Let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2),

$$v + x \frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v}$$

$$\Rightarrow x \frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v} - v = -\frac{v^4 + 6v^2 + 1}{v^3 + 3v}$$

$$\Rightarrow 4 \frac{dx}{x} = -\frac{4v^3 + 12v}{v^4 + 6v^2 + 1} dv$$

Integrating,

$$4 \log x = -\log(v^4 + 6v^2 + 1) + \log c$$

$$\Rightarrow \log x^4 = \log[c/(v^4 + 6v^2 + 1)]$$

$$\Rightarrow x^4(v^4 + 6v^2 + 1) = c$$

$$\Rightarrow x^4 + 6x^2y^2 + y^4 = c \quad \left[\because \frac{y}{x} = v \right] \text{ which is required solution.}$$

Prop. 30. $(x^2 - y^2)dx + 2xydy = 0$

Do yourself

Ans. $(y^2 + x^2) = cx$

Ex. 31. Solve $xdy - ydx = \sqrt{x^2 + y^2}dx$

Solution The given differential equation is

$$xdy - ydx = \sqrt{x^2 + y^2}dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \sqrt{\left\{1 + \left(\frac{y}{x}\right)^2\right\}} \quad (1)$$

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2),

$$v + x \frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1 + v^2}}$$

Integrating,

$$\log x + \log c = \log \left[v + \sqrt{v^2 + 1} \right]$$

$$\Rightarrow xc = v + \sqrt{v^2 + 1}$$

$$\Rightarrow x^2c = y + \sqrt{y^2 + x^2} \quad \left[\because v = \frac{y}{x} \right]$$

which is required solution.

Prob. 32. $(x^2 + y^2)dx + 2xydy = 0$

Ans. $x^{1/3}(x^2 + 3y^2)^{1/3} = 0$

Do yourself

Ex. 33. Solve $\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$

Solution The given differential equation is

$$\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$$

$$\text{Let } \frac{y}{x} = v, \quad \Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

Using (1), the given equation becomes

$$v + x \frac{dv}{dx} = v + \tan v$$

$$\Rightarrow \frac{dx}{x} = \frac{\cos v}{\sin v} dv$$

Integrating,

$$\log x = \log \sin v + \log c$$

$$\Rightarrow x = c \sin v$$

$$\Rightarrow x = c \sin \left(\frac{y}{x} \right) \text{ which is required solution.}$$

Prob. 34. $(x - y)^2 dx + 2xydy = 0$

Ans. $\log(3y^2x - 2yx + x^3) + \sqrt{2} \tan^{-1} \left(\frac{3y - x}{x\sqrt{2}} \right) = c$

Do yourself

Ex. 35. Solve $x \cos \left(\frac{y}{x} \right) (ydx + xdy) = y \sin \left(\frac{y}{x} \right) xdy - ydx$

Solution. The given differential equation is

$$x \cos \left(\frac{y}{x} \right) (ydx + xdy) = y \sin \left(\frac{y}{x} \right) xdy - ydx$$

$$\Rightarrow \frac{dy}{dx} = \frac{[x \cos(y/x) + y \sin(y/x)]y}{[y \sin(y/x) - x \cos(y/x)]x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{[\cos(y/x) + (y/x) \sin(y/x)](y/x)}{[(y/x) \sin(y/x) - \cos(y/x)]}$$

$$\text{Let } \frac{y}{x} = v \quad \Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$\therefore$ The above diff. equation becomes

$$v + x \frac{dv}{dx} = \frac{v(\cos x + v \sin v)}{v \sin v - \cos v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v(\cos v + v \sin v)}{v \sin v - \cos v} - v = \frac{2v \cos v}{v \sin v - \cos v}$$

$$\Rightarrow 2 \frac{dv}{x} = \frac{v \sin v - \cos v}{v \cos v} dv = \left[\frac{\sin v}{\cos v} - \frac{1}{v} \right] dv$$

Integrating,

$$2 \log x = -\log \cos v - \log v + \log c$$

$$\Rightarrow \log x^2 = \log \left(\frac{c}{v \cos v} \right) \Rightarrow x^2 v \cos v = c$$

$$\Rightarrow x^2 v \cos v = c \quad \left[\because v = \frac{y}{x} \right] \text{ which is required solution.}$$

Prob. 36. $(x^2 + y^2)dy = (x^2 + xy)dx$

Do yourself

Ex. 37. Solve $(4y + 3x)dy + (y - 2x)dx = 0$ **Solution** The given differential equation is

$$(4y + 3x)dy + (y - 2x)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - y}{4y + 3x} \quad (1)$$

$$\text{Let } y = xv \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (3)$$

Using (2) and (3), (1) reduces to

$$v + x \frac{dv}{dx} = \frac{2 - v}{3 + 4v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2 - v}{3 + 4v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2 - 4v - 4v^2}{3 + 4v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2 - 4v - 4v^2}{3 + 4v}$$

$$\Rightarrow \frac{2dx}{x} = \frac{3 + 4v}{1 - 2v - 2v^2} dv$$

Integrating,

$$2 \log x = \int \frac{(3 + 4v)dv}{1 - 2v - 2v^2} = \int \frac{-(-2 - 4v) + 1}{1 - 2v - 2v^2} dv$$

$$\Rightarrow \log x^2 + \log c = -\log(1 - 2v - 2v^2) + \frac{1}{2} \int \frac{dv}{(1/2) - v - v^2}$$

$$\Rightarrow \log\{cx^2(1 - 2v - 2v^2)\} = \frac{1}{2} \int \frac{dv}{(3/4) - (v^2 + v + 1/4)}$$

$$= \frac{1}{2} \int \frac{dv}{(\sqrt{3}/2)^2 - (v + 1/2)^2}$$

$$= \frac{1}{2} \times \frac{1}{2(\sqrt{3}/2)} \log \frac{(\sqrt{3}/2) + (v + 1/2)}{(\sqrt{3}/2) - (v + 1/2)}$$

$$= \frac{1}{2\sqrt{3}} \log \frac{\sqrt{3} + 2v + 1}{\sqrt{3} - 2v - 1}$$

$$\Rightarrow \log \left[cx^2 \left(1 - \frac{2y}{x} - \frac{2y^2}{x^2} \right) \right] = \frac{1}{2\sqrt{3}} \log \frac{(\sqrt{3} + 1) + 2(y/x)}{(\sqrt{3} - 1) - 2(y/x)} \quad \left[\because v = \frac{y}{x} \right]$$

$$\Rightarrow c(x^2 - 2xy - 2y^2) = \left\{ \frac{(\sqrt{3} + 1)x + 2y}{(\sqrt{3} - 1)x - 2y} \right\}^{1/2\sqrt{3}} \text{ which is required solution.}$$

Prob. 38. $x^2 y dx - (x^3 + y^3) dy = 0$ Ans. $ye^{-x^3/3y^3} = c$

Do yourself

Ex. 39. Solve $x \left(\frac{dy}{dx} \right) = y(\log y - \log x + 1)$ **Solution** The given differential equation is

$$x \left(\frac{dy}{dx} \right) = y(\log y - \log x + 1) \quad (1)$$

Let $y = xv$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (2)$$

From (1) and (2),

$$v + x \frac{dv}{dx} = v(\log v + 1)$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{v \log v}$$

Integrating,

$$\log x + \log c = \log \log v$$

$$\Rightarrow xc = \log v$$

$$\Rightarrow v = e^{xc} \Rightarrow (y/x) = e^{xc}$$

$$\Rightarrow y = xe^{xc} \text{ which is required solution}$$

$$\text{Prob. 40. } xdx + (y - \sqrt{x^2 + y^2}) dy = 0$$

$$\text{Ans, } (y + \sqrt{y^2 + x^2}) = cx^2$$

Do yourself

4. REDUCIBLE HOMOGENEOUS EQUATION (NON – HOMOGENEOUS EQUATION)

A differential Equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \quad (1)$$

is called non – homogeneous differential equation

$$\text{CASE I: When } \frac{a}{a'} \neq \frac{b}{b'}$$

$$\text{Let } x = X + h \text{ and } y = Y + k \quad (2)$$

Where X and Y are new variables and h and k constants

From (2),

$$dx = dX \text{ and } dy = dY$$

$$\Rightarrow \frac{dy}{dx} = \frac{dY}{dX} \quad (3)$$

Using (2) and (3), (1) becomes

$$\begin{aligned} \frac{dY}{dX} &= \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'} \\ &= \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \end{aligned} \quad (4)$$

In order to make (4) homogeneous,

Choose h and k so as to satisfy the following two equations

$$ah + bk + c = 0$$

$$\text{and } a'h + b'k + c' = 0 \quad (5)$$

Solving (5),

$$h = \frac{bc' - b'c}{ab' - a'b} \text{ and } k = \frac{ca' - c'a}{ab' - a'b} \quad (\text{where } ab' - a'b \neq 0) \quad (6)$$

In view of (5), (4) reduces to

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

which is surely homogeneous equation in X and Y and can be solved by putting $Y/X = v$ as usual. After getting solution in terms of X and Y, we remove X and Y by using (7) and obtain solution in terms of the original variables x and y.

$$\text{Ex, 41. Solve } \frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$$

$$\text{Sol}^n: \text{ The given differential equation is } \frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$$

$$\text{Let } x = X + h, y = Y + k \Rightarrow \frac{dy}{dx} = \frac{dY}{dX} \quad (1)$$

Given equation becomes

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)} \quad (2)$$

Choose h, k so that

$$h + 2k - 3 = 0$$

$$\text{and } 2h + k - 3 = 0 \quad (3)$$

Solving (3) we get

$$h = 1, k = 1$$

so that from (1), we have

$$X = x - 1, Y = y - 1 \quad (4)$$

Using (3) in (2), we get

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \quad (5)$$

$$\text{Let } Y = vX.$$

$$\Rightarrow \frac{dY}{dX} = v + X \frac{dv}{dX}$$

From (5) and (6),

$$v + X \frac{dv}{dX} = \frac{1+2v}{2+v}$$

$$\Rightarrow X \frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{dX}{X} = \frac{(2+v)dv}{(1-v)(1+v)} = \left[\frac{1}{2} \left(\frac{1}{1+v} \right) + \frac{3}{2} \left(\frac{1}{1-v} \right) \right] dv$$

Integrating,

$$\log X + \log c = \left(\frac{1}{2} \right) [\log(1+v) - 3 \log(1-v)]$$

$$\Rightarrow 2 \log(cX) = \log \frac{1+v}{(1-v)^3}$$

$$\Rightarrow X^2 c^2 = \frac{1+v}{(1-v)^3}$$

$$\Rightarrow X^2 c^2 \left(1 - \frac{Y}{X} \right)^3 = 1 + \frac{Y}{X} \quad \left[\because v = \frac{Y}{X} \right]$$

$$\Rightarrow c^2 (X-Y)^3 = X+Y$$

$$\Rightarrow c^2 \{x-1 - (y-1)\}^3 = x-1+y-1, \quad [\text{by (4)}]$$

$$\Rightarrow c'(x-y)^3 = x+y-2, \quad [\text{taking } c' = c^2] \text{ which is required solution.}$$

Prob. 42. $(2x + y - 3) dy = (x + 2y - 3) dx$

Ans $(x + y - 2) = c^2(x - y)^2$

Do yourself

Ex. 43. Solve $\frac{dy}{dx} + \frac{(x-y-2)}{(x-2y-3)} = 0$

Solution: The given differential equation is

$$\frac{dy}{dx} + \frac{(x-y-2)}{(x-2y-3)} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x-y-2)}{(x-2y-3)}$$

$$\text{Let } x = X + h, \quad y = Y + k \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dY}{dX} \quad (1)$$

$$\therefore \frac{dY}{dX} = -\frac{X-Y+h-k-2}{X-2Y+h-2k-3} \quad (2)$$

Choose h, k so that

$$h - k - 2 = 0$$

$$\text{and } h - 2k - 3 = 0 \quad (3)$$

Solving (3), we get

$$h = 1, \quad k = -1$$

so that from (1), we have

$$X = x - 1$$

$$\text{and } Y = y + 1 \quad (4)$$

and (2) becomes

$$\frac{dY}{dX} = -\frac{X-Y}{X-2Y} \quad (5)$$

Let $Y = vX$

$$\Rightarrow \frac{dY}{dX} = v + X \frac{dv}{dX} \quad (6)$$

From (5) and (6),

$$v + X \frac{dv}{dX} = -\frac{1-v}{1-2v}$$

$$\Rightarrow X \frac{dv}{dX} = \frac{1-2v^2}{2v-1}$$

$$\Rightarrow \frac{dX}{X} = \frac{2v-1}{1-2v^2} dv$$

$$\Rightarrow \frac{dX}{X} = \left[-\frac{1}{2} \frac{(-4v)}{1-2v^2} - \frac{1}{1-(v\sqrt{2})^2} \right] dv$$

Integrating,

$$\log X = -\frac{1}{2} \log(1-2v^2) - \frac{1}{\sqrt{2}} \log \frac{1+v\sqrt{2}}{1-v\sqrt{2}} - \frac{1}{2} \log c$$

$$\Rightarrow 2 \log X + \log(1 - 2v^2) + \log c = -\frac{1}{\sqrt{2}} \log \left(\frac{1 + v\sqrt{2}}{1 - v\sqrt{2}} \right)$$

$$\Rightarrow \log\{cX^2(1 - 2v^2)\} = \log \left(\frac{1 - v\sqrt{2}}{1 + v\sqrt{2}} \right)^{1/\sqrt{2}}$$

$$\Rightarrow c(X^2 - 2Y^2) = \left(\frac{X - Y\sqrt{2}}{X + Y\sqrt{2}} \right)^{1/\sqrt{2}} \quad [\because v = Y/X]$$

$$\Rightarrow c[(x-1)^2 - 2(y+1)^2] = \left[\frac{(x-1) - (y+1)\sqrt{2}}{(x-1) + (y+1)\sqrt{2}} \right]^{1/\sqrt{2}}$$

$$\Rightarrow c(x^2 - 2y^2 - 2x - 4y - 1) = \left(\frac{x - y\sqrt{2} - \sqrt{2} - 1}{x + y\sqrt{2} - 1 + \sqrt{2}} \right)^{1/\sqrt{2}} \quad \text{Which is required solution.}$$

Prob. 44 . $(3y - 7x - 3)dx + (7y - 3x - 7)dy = 0$

Do yourself

Ex. 45. Solve $(2x^2 + 3y^2 - 7)xdx - (3x^2 + 2y^2 - 8)ydy = 0$

Solⁿ: The given differential equation is

$$(2x^2 + 3y^2 - 7)xdx - (3x^2 + 2y^2 - 8)ydy = 0 \quad (1)$$

Let $x^2 = u$ and $y^2 = v$

$$\Rightarrow 2xdx = du \quad \text{and} \quad 2ydy = dv \quad (2)$$

From (1) and (2),

$$(2u + 3v - 7)du - (3u + 2v - 8)dv = 0$$

$$\Rightarrow \frac{dv}{du} = \frac{(2u + 3v - 7)}{(3u + 2v - 8)} \quad (3)$$

Let $u = U + h, v = V + k$

$$\Rightarrow \frac{dv}{du} = \frac{dV}{dU} \quad (4)$$

The given equation becomes

$$\frac{dV}{dU} = \frac{2U + 3V + (2h + 3k - 7)}{3U + 2V + (3h + 2k - 8)} \quad (5)$$

Choose h, k so that

$$2h + 3k - 7 = 0$$

$$\text{and} \quad 3h + 2k - 8 = 0$$

Solving, we get

$$h = 2, k = 1 \quad (6)$$

so that from (4), we have

$$U = u - 2$$

$$\text{and} \quad V = v - 1$$

$$\Rightarrow U = x^2 - 2$$

$$\text{and} \quad V = y^2 - 1 \quad [\text{by (2)}] \quad (7)$$

and (5) becomes

$$\frac{dV}{dU} = \frac{2U + 3V}{3U + 2V}$$

$$\Rightarrow V = wU. \quad (8)$$

$$\Rightarrow \frac{dV}{dU} = w + U \frac{dw}{dU} \quad (9)$$

From (8) and (9),

$$\Rightarrow U \frac{dw}{dU} = \frac{2(1 - w^2)}{3 + 2w}$$

$$\Rightarrow \frac{2dU}{U} = \frac{3 + 2w}{1 - w^2} dw = \left[\frac{3}{1 - w^2} - \frac{-2w}{1 - w^2} \right] dw$$

Integrating,

$$2 \log U = \frac{3}{2} \log \frac{1 + w}{1 - w} - \log(1 - w^2) + \frac{1}{2} \log c$$

$$\Rightarrow 4 \log U = 3 \log \left(\frac{1 + w}{1 - w} \right) - 2 \log(1 - w^2) + \log c$$

$$\Rightarrow \log \frac{U^4}{c} = \log \left(\frac{1 + w}{1 - w} \right)^3 - \log(1 - w^2)^2$$

$$\Rightarrow \log \frac{U^4}{c} = \log \left[\left(\frac{1 + w}{1 - w} \right)^3 \left(\frac{1}{(1 - w^2)^2} \right) \right]$$

$$\begin{aligned}
&\Rightarrow \frac{U^4}{c} = \frac{(1+w)^3}{(1+w)^5(1+w)^2} \\
&\Rightarrow (1-w)^5 U^4 = c(1+w) \\
&\Rightarrow \left(1 - \frac{V}{U}\right)^5 U^4 = c\left(1 + \frac{V}{U}\right) \\
&\Rightarrow (U-V)^5 = c(U+V) \\
&\Rightarrow (x^2 - y^2 - 1)^5 = c(x^2 + y^2 - 3) \text{ which is required solution.}
\end{aligned}$$

Case - II. When $\frac{a}{a'} = \frac{b}{b'}$

Let $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say)

then $a' = ma$, $b' = mb$

$\therefore$ Given equation becomes

$$\begin{aligned}
\frac{dy}{dx} &= \frac{ax + by + c}{max + mby + c'} \\
\frac{dy}{dx} &= \frac{ax + by + c}{m(ax + by) + c'}
\end{aligned}$$

(i)

Put $ax + by = z$

then $a + b \frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b} \left(\frac{dz}{dx} - a \right)$$

$\therefore$ from (i)

$$\frac{1}{b} \left(\frac{dz}{dx} - a \right) = \frac{z + c}{mz + c'}$$

$$\Rightarrow \frac{dz}{dx} - a = \frac{b(z + c)}{mz + c'}$$

$$\Rightarrow \frac{dz}{dx} = \frac{b(z + c)}{mz + c'} + a = \frac{b(z + c) + a(mz + c')}{mz + c'}$$

Separating the variables

$$\frac{mz + c'}{b(z + c) + a(mz + c')} dz = dx$$

which can be easily integrated.

In the solution, putting $z = ax + by$, we get the required solution.

Ex. 46. Solve $\frac{dy}{dx} = \frac{x + y + 7}{2x + 2y + 3}$

Solution. The given differential equation is

$$\frac{dy}{dx} = \frac{x + y + 7}{2x + 2y + 3} \quad (1)$$

Let $x + y + 7 = v$

(2)

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Therefore, from (1)

$$\left(\frac{2v - 11}{3v - 11} \right) dv = dx$$

$$\Rightarrow \left[\frac{2}{3} - \frac{11}{3(3v - 11)} \right] dv = dx$$

Integrating,

$$\frac{2}{3}v - \frac{11}{9} \log(3v - 11) = x + C$$

$$\Rightarrow 6v - 11 \log(3v - 11) = 9x + C_1 \quad \text{where } C_1 = 9C$$

$$\Rightarrow 6(x + y + 7) - 11 \log(3x + 3y + 10) = 9x + C_1, \quad \text{from (2)}$$

$$\Rightarrow 6y - 3x - 11 \log(3x + 3y + 10) = C_2, \text{ where } C_2 = C_1 - 42$$

which is required solution.

Ex. 47. Solve $\frac{dy}{dx} = \frac{(x + y + 1)}{(2x + 2y + 3)}$

Solution. The given differential equation is

$$\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3} \quad (1)$$

Let $x + y + 1 = v$

(2)

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

∴ from (1)

$$\frac{dv}{dx} - 1 = \frac{v}{2v+1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{3v+1}{2v+1}$$

$$\Rightarrow \frac{2v+1}{3v+1} dv = dx$$

$$\Rightarrow \left(\frac{2}{3} + \frac{1}{3(3v+1)} \right) dv = dx$$

Integrate and find the solution.

METHOD – II.

Let

$$x + y = v$$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\therefore \frac{dv}{dx} - 1 = \frac{v+1}{2v+3}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v+1}{2v+3} + 1 = \frac{3v+4}{2v+3} \text{ simplify and find the solution.}$$

Ex. 48. Solve $(4x + 3y + 2)dy = (8x + 6y + 12)dx$

Solution: The given Differential equation is

$$(4x + 3y + 2)dy = (8x + 6y + 12)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{8x + 6y + 12}{4x + 3y + 2} \quad (1)$$

$$\text{Let } 4x + 3y + 2 = v \quad (2)$$

$$\Rightarrow 4 + 3 \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3} \left(\frac{dv}{dx} - 4 \right)$$

Therefore, from (1)

$$\left(1 - \frac{24}{7v+24} \right) dv = 7dx$$

Integrating,

$$v - (24/7) \log(7v + 24) = 7x + c,$$

$$\Rightarrow 4x + 3y + 2 - (24/7) \log(28x + 21y + 38) = 7x + c$$

(from (2))

$$\Rightarrow 3y - 3x - (24/7) \log(28x + 21y + 38) = C_1, \text{ where } C_1 = c - 2$$

$$\Rightarrow 21(y - x) - 24 \log(28x + 21y + 38) = k, \text{ where } k = 7C_1$$

which is the required solution.

Ex. 49. Solve $(4x + 6y + 5)dy = (3y + 2x + 4)dx$

Solution. The given differential equation is

$$(4x + 6y + 5)dy = (3y + 2x + 4)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y + 2x + 4}{4x + 6y + 5} \quad (1)$$

$$\text{Let } 2x + 3y + 4 = v \quad (2)$$

$$\Rightarrow 2 + 3 \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3} \left(\frac{dv}{dx} - 2 \right)$$

Therefore, from (1)

$$\frac{1}{3} \left(\frac{dv}{dx} - 2 \right) = \frac{v}{2v-3}$$

$$\Rightarrow \frac{(2v-3)dv}{7v-6} = dx$$

$$\Rightarrow \frac{1}{7} \left[\frac{(14v-12)-9}{7v-6} \right] dv = dx$$

$$\Rightarrow \left[2 - \frac{9}{(7v-6)} \right] dv = 7dx$$

Integrating,

$$2v - \frac{9}{7} \log[7(2x + 3y + 4) - 6] = 7x + C,$$

$$\Rightarrow 2(2x + 3y + 4) - \frac{9}{7} \log[7(2x + 3y + 4) - 6] = 7x + C, \text{ from (2)}$$

$$\Rightarrow (6y - 3x) - \frac{9}{7} \log(14x + 21y + 22) = C_1, \text{ where } C_1 = C - 8$$

which is the required solution.

Ex. 50. Solve $(x + y)(dx - dy) = dx + dy$

Solution. The given differential equation is

$$(x + y)(dx - dy) = dx + dy$$

$$\Rightarrow (x + y - 1)dx = (x + y + 1)dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y - 1}{x + y + 1} \quad (1)$$

$$\text{Let } x + y = v \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dv}{dx}\right) - 1 \quad (3)$$

Using (2) and (3), (1) becomes

$$\frac{dv}{dx} - 1 = \frac{v - 1}{v + 1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{2v}{v + 1}$$

$$\Rightarrow 2dx = \left(1 + \frac{1}{v}\right)dv$$

$\therefore$ Integrating,

$$2x + c = v + \log v$$

$$\Rightarrow x - y + c = \log(x + y) \text{ which is required solution.}$$

Ex. 51. Solve $\frac{dy}{dx} = \frac{4x + 6y + 5}{3y + 2x + 4}$

Solⁿ: The given differential equation is

$$\frac{dy}{dx} = \frac{4x + 6y + 5}{3y + 2x + 4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(2x + 3y) + 5}{(2x + 3y) + 4} \quad (1)$$

$$\text{Let } 2x + 3y = v \quad (2)$$

$$\Rightarrow 2 + 3 \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3} \left(\frac{dv}{dx} - 2\right) \quad (3)$$

Using (2) and (3), (1) gives

$$\frac{dy}{dx} = \frac{1}{3} \left(\frac{dv}{dx} - 2\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{3(2v + 5)}{v + 4} + 2 = \frac{8v + 23}{v + 4}$$

$$\frac{dx}{dv} = \frac{v + 4}{8v + 23} = \frac{\frac{1}{8}(8v + 23) + \left(4 - \frac{23}{8}\right)}{8v + 23} = \left[\frac{1}{8} + \frac{9}{8(8v + 23)}\right]$$

$$\Rightarrow dx = \left[\frac{1}{8} + \frac{9}{8(8v + 23)}\right] dv$$

Integrating,

$$x + c = \left(\frac{v}{8}\right) + \left(\frac{9}{64}\right) \log(8v + 23)$$

$$\Rightarrow 8x + 8c = 2x + 3y + \left(\frac{9}{8}\right) \log(16x + 24y + 23) \text{ using (2) and multiplying by 8}$$

$$\Rightarrow 3y - 6y + \frac{9}{8} \log(16x + 24y + 23) = 8c$$

$$\Rightarrow y - 2x = \frac{3}{8} \log(16x + 24y + 23) = c', \text{ where } \frac{8c}{3} = c' \text{ is an arbitrary constant.}$$

5. EXACT DIFFERENTIAL EQUATION.

Definition If M and N are function of x and y, the equation

$$Mdx + Ndy = 0 \quad (1)$$

is called exact when there exists a function of x and y such that

$$d[f(x, y)] = Mdx + Ndy \quad (2)$$

$$\text{i. e. } \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Mdx + Ndy \quad (3)$$

Remark: The differential equation

$$y^2 dx + 2xy dy = 0 \quad (4)$$

is an exact differential equation, for there exists a function xy^2 such that

$$d(xy^2) = \frac{\partial}{\partial x}(xy^2)dx + \frac{\partial}{\partial y}(xy^2)dy$$

$$\Rightarrow d(xy^2) = y^2 dx + 2xy dy \quad (5)$$

Therefore, (4) may be written as

$$d(xy^2) = 0$$

Integrate, we get

$$xy^2 = c,$$

which is general solution of (1).

Necessary and sufficient condition for a differential equation of first order and first degree to be exact.

Statement: The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0 \quad (1)$$

to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2)$$

Proof. Necessary condition

Let (1) be exact.

Hence, by definition, there must exist a function $f(x, y)$ of x and y such that

$$d[f(x, y)] = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Mdx + Ndy \quad (3)$$

Equating coefficients of dx and dy in (3), we get

$$M = \frac{\partial f}{\partial x} \quad (4)$$

$$N = \frac{\partial f}{\partial y} \quad (5)$$

To remove the unknown function $f(x, y)$, we differentiate partially (4) and (5) with respect to y and x respectively giving

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \quad (6)$$

$$\text{and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \quad (7)$$

$$\text{Since } \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y},$$

$\therefore$ from (6) and (7)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, if (1) is exact, M and N satisfy condition (2)

Sufficient condition: We now assume that (2) holds and show that (1) is an exact differential equation. For this we must find a function $f(x, y)$ such that

$$d[f(x, y)] = Mdx + Ndy$$

$$\text{Let } g(x, y) = \int Mdx \quad (8)$$

be the partial integral of M , that is, the integral obtained by keeping y fixed.

We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is a function of x only. This is clear because

$$\begin{aligned} \frac{\partial}{\partial x} \left(N - \frac{\partial g}{\partial y} \right) &= \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \\ &= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y} \left(\frac{\partial g}{\partial x} \right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \\ &= 0 \quad [\text{using (2)}] \end{aligned}$$

$$\text{Take, } f(x, y) = g(x, y) + \int \left(N - \frac{\partial g}{\partial y} \right) dy \quad (9)$$

Hence, on total differentiation of (9), we get

$$\begin{aligned} df &= dg + \left(N - \frac{\partial g}{\partial y} \right) dy = \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) + Ndy - \frac{\partial g}{\partial y} dy \\ &= \frac{\partial g}{\partial x} dx + Ndy = Mdx + Ndy, \text{ using (8)} \end{aligned}$$

Thus, if (2) is satisfied, (1) is surely an exact equation.

NOTE: METHOD FOR SOLVING EXACT DIFFERENTIAL EQUATION.

Step – 1 Compare the given equation with $Mdx + Ndy = 0$ and find out M and N .

Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

We conclude that the given equation is exact.

Step - 2 Integrate M with respect to x treating y as a constant.

Step - 3 Integrate N with respect to y only those terms which are either constant or independent of x.

Step - 4 On adding the terms obtained in steps 2 and 3 and equating to an arbitrary constant, we get the required solution.

Ex. 52. (a) Solve $y \sin 2x dx - (y^2 + \cos^2 x) dy = 0$

(b) $y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0$

Solution: (a) The given differential equation is

$$y \sin 2x dx - (y^2 + \cos^2 x) dy = 0$$

Comparing (1) with $Mdx + Ndy = 0$

we get

$$M = y \sin 2x \text{ and } N = -(y^2 + \cos^2 x)$$

$$\therefore \frac{\partial M}{\partial y} = \sin 2x, \frac{\partial N}{\partial x} = 2 \cos x \sin x = \sin 2x.$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the given equation is exact.

$$\text{Now } \int M dx = \int y \sin 2x dx = -\frac{1}{2} y \cos 2x$$

$$\text{And } \int N dy = -\int y^2 dy = -\frac{y^3}{3} \quad [\text{Integrate only those terms which is independent of } x]$$

Hence, the required solution is

$$-\frac{1}{2} y \cos 2x - \frac{1}{3} y^3 = C_1$$

$$\Rightarrow y \cos 2x + \frac{2}{3} y^3 = C \quad [C = -2C_1]$$

which is required solution.

(b) Do yourself.

Prob. 53. $(2ax + by)ydx + (ax + 2by)x dy = 0$

Ans $ayx^2 + by^2x = c$

Do yourself

Ex. 54. Solve

$$\left\{ y \left(1 + \frac{1}{x} \right) + \cos y \right\} dx + \{ x + \log x - x \sin y \} dy = 0$$

Solution: The given differential equation is

$$\left\{ y \left(1 + \frac{1}{x} \right) + \cos y \right\} dx + \{ x + \log x - x \sin y \} dy = 0$$

$$\text{Here } M = y \left(1 + \frac{1}{x} \right) + \cos y, N = x + \log x - x \sin y$$

$$\therefore \frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y, \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ the given differential equation is exact.

$$\text{now } \int M dx = \int \left(y + \frac{y}{x} + \cos y \right) dx = yx + y \log x + x \cos y$$

$$\text{and } \int N dy = 0$$

hence the required solution is

$$yx + y \log x + x \cos y = C \text{ which is required solution.}$$

Prob. 55. $(x^3 - 2xy - y^2)dx - (x + y)^2 dy = 0$

Ans. $x^3 - y^3 - 3x^2y - 3x^2x = C$

Do yourself

Ex. 56. Solve $(ax + hy + g)dx + (hx + by + f)dy = 0$

Solution: The given differential equation is]

$$(ax + hy + g)dx + (hx + by + f)dy = 0$$

$$\text{Here, } M = ax + hy + g, N = hx + by + f$$

$$\frac{\partial M}{\partial y} = h, \frac{\partial N}{\partial x} = h$$

$$\Rightarrow \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$\therefore$ the given differential equation is exact

now $\int Mdx = \int (ax + hy + g) dx = \left(\frac{1}{2}\right) ax^2 + hxy + gx$ (1)

and $\int Ndy = \int (by + f) dy = \left(\frac{1}{2}\right) by^2 + fy$

∴ the required solution is

$$\left(\frac{1}{2}\right) ax^2 + hxy + gx + \left(\frac{1}{2}\right) by^2 + fy = C$$

$$\Rightarrow ax^2 + 2hxy + by^2 + 2gx + 2fy = C', \quad [C' = (2C)]$$

which is required solution.

Prob. 57. $(12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)dx + (2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})dy = 0$

Ans. $4x^3y + x^2y^2 + x^4 - 4xy^3 + ye^{2x} - xe^y + y^3 = C$

Do yourself

Ex. 58. The given differential equation is

$$xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$$

$$\Rightarrow \left\{x + \frac{a^2y}{x^2 + y^2}\right\} dx + \left\{y - \frac{a^2x}{x^2 + y^2}\right\} dy = 0$$

$$\therefore M = x + \frac{a^2y}{x^2 + y^2} \text{ and } N = y - \frac{a^2x}{x^2 + y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = a^2 \frac{1 \cdot (x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = a^2 \frac{(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial N}{\partial x} = -a^2 \frac{1 \cdot (x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

∴ the given equation is an exact equation.

Now,

$$\int Mdx = \int \left\{x + \frac{a^2y}{x^2 + y^2}\right\} dx = \frac{x^2}{2} + a^2y \cdot \frac{1}{y} \tan^{-1} \left(\frac{x}{y}\right) = \left(\frac{1}{2}\right) x^2 + a^2 \tan^{-1} \left(\frac{x}{y}\right)$$

$$\text{and } \int Ndy = \int ydy = \frac{y^2}{2}$$

∴ the required solution is

$$\frac{1}{2}x^2 + a^2 \tan^{-1} \left(\frac{x}{y}\right) + \frac{y^2}{2} = C$$

$$\Rightarrow x^2 + 2a^2 \tan^{-1} \left(\frac{x}{y}\right) + y^2 = C' \quad (\text{where } C' = 2C)$$

which is required solution.

59. Solve $(1 + e^{x/y})dx + e^{x/y}[1 - (x/y)]dy = 0$

Solⁿ: The given differential equation is

$$(1 + e^{x/y})dx + e^{x/y}[1 - (x/y)]dy = 0$$

Therefore,

$$M = (1 + e^{x/y}) \text{ and } N = e^{x/y}[1 - (x/y)]$$

$$\Rightarrow \frac{\partial M}{\partial y} = e^{x/y} x \left(-\frac{1}{y^2}\right) = -\frac{x}{y^2} e^{x/y}$$

$$\text{and } \frac{\partial N}{\partial x} = e^{x/y} \left(-\frac{1}{y}\right) + \left(-\frac{x}{y}\right) e^{x/y} \cdot \frac{1}{y} = -\frac{x}{y^2} e^{x/y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

hence the given equation is exact.

$$\text{Now } \int Mdx = \int (1 + e^{x/y}) dx = x + y e^{x/y}$$

$$\text{and } \int Ndy = 0$$

Hence, from (1) the required solution is

$$x + ye^{x/y} = C, \text{ which is required solution.}$$

Ex. 60. Solve $[2 \cos 2x + \sin(x + 2y)]dx + [\cos y + 2 \sin(x + 2y)]dy = 0$

Solⁿ: The given differential equation is

$$[2 \cos 2x + \sin(x + 2y)]dx + [\cos y + 2 \sin(x + 2y)]dy = 0$$

$$\therefore M = 2 \cos 2x + \sin(x + 2y), \quad N = \cos y + 2 \sin(x + 2y)$$

$$\therefore \frac{\partial M}{\partial y} = 2 \cos(x+2y) \quad \text{and} \quad \frac{\partial N}{\partial x} = 2 \cos(x+2y)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

hence the given equation is exact.

$$\begin{aligned} \text{Now} \quad \int M dx &= \int \{2 \cos 2x + \sin(x+2y)\} dx \\ &= \sin 2x - \cos(x+2y) \end{aligned} \quad (1)$$

$$\text{and} \quad \int N dy = \int \cos y dy = \sin y \quad (2)$$

$\therefore$ From (1) and (2), the required solution is $\{\sin 2x - \cos(x+2y)\} + \{\sin y\} = C$, which is required solution.

Ex. 61. Solve $(\sin x \cos y + e^{2x})dx + (\cos x \sin y + \tan y)dy = 0$

Solⁿ: The given differential equation is

$$(\sin x \cos y + e^{2x})dx + (\cos x \sin y + \tan y)dy = 0$$

$$\therefore M = \sin x \cos y + e^{2x} \text{ and } N = \cos x \sin y + \tan y$$

$$\Rightarrow \frac{\partial M}{\partial y} = -\sin x \sin y \text{ and } \frac{\partial N}{\partial x} = -\sin x \sin y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

hence the given equation is exact.

$$\begin{aligned} \text{Now} \quad \int M dx &= \int (\sin x \cos y + e^{2x}) dx \\ &= -\cos x \cos y + \frac{1}{2} e^{2x} \end{aligned} \quad (1)$$

$$\text{and} \quad \int N dy = \int \tan y dy = \log \sec y \quad (2)$$

$\therefore$ From (1), (2), the required solution is

$$\left(-\cos x \cos y + \frac{1}{2} e^{2x}\right) + \log \sec y = C,$$

$$\Rightarrow \frac{1}{2} e^{2x} - \cos x \cos y + \log(\sec y) = C \text{ which is required solution.}$$

Ex. 62. Solve $\cos x (\cos x - \sin \alpha \sin y)dx + \cos y (\cos y - \sin \alpha \sin x)dy = 0$

solⁿ: The given differential equation is

$$\cos x (\cos x - \sin \alpha \sin y)dx + \cos y (\cos y - \sin \alpha \sin x)dy = 0$$

Therefore, $M = \cos x (\cos x - \sin \alpha \sin y)$ and $N = \cos y (\cos y - \sin \alpha \sin x)$

$$\Rightarrow \frac{\partial M}{\partial y} = \cos x (-\sin \alpha \cos y) \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos y (-\sin \alpha \cos x)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

hence the given equation is exact.

$$\begin{aligned} \text{Now} \quad \int M dx &= \int \cos x (\cos x - \sin \alpha \sin y) dx \\ &= \int (\cos^2 x - \sin \alpha \sin y \cos x) dx \\ &= \frac{1}{2} \int (1 + \cos 2x - 2 \sin \alpha \sin y \cos x) dx \\ &= \frac{1}{2} \left(x + \frac{1}{2} \sin 2x - 2 \sin \alpha \sin y \sin x \right) \\ &= \frac{1}{2} x + \frac{1}{4} \sin 2x - \sin \alpha \sin y \sin x \end{aligned} \quad (1)$$

$$\begin{aligned} \text{and} \quad \int N dy &= \int \cos^2 y dy \\ &= \frac{1}{2} \int (1 + \cos 2y) dy \\ &= \frac{1}{2} \left(y + \frac{\sin 2y}{2} \right) \\ &= \frac{y}{2} + \frac{\sin 2y}{4} \end{aligned}$$

$\therefore$ From (1) and (2) the required solution is

$$\left(\frac{1}{2} x + \frac{1}{4} \sin 2x - \sin \alpha \sin y \sin x\right) + \left(\frac{1}{2} y + \frac{1}{4} \sin 2y\right) = C,$$

$$\Rightarrow 2x + \sin 2x - 4 \sin \alpha \sin y \sin x + 2y + \sin 2y = 4C \text{ which is required solution.}$$

Ex. 63. Solve $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

Solution. The given differential equation is

$$(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$$

Therefore, $M = y^2 e^{xy^2} + 4x^3$; $N = 2xye^{xy^2} - 3y^2$

$$\Rightarrow \frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 2xye^{xy^2} \text{ and } \frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xyy^2 e^{xy^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

Hence the given equation is exact.

$$\text{Now } \int M dx = \int (y^2 e^{xy^2} + 4x^3) dx = y^2 (1/y^2) e^{xy^2} + x^4 = e^{xy^2} + x^4$$

$$\text{and } \int N dy = \int -3y^2 dy = -y^3$$

$\therefore$ From (1) and (2), the required solution is

$$e^{xy^2} + x^4 - y^3 = C, \text{ which is required solution.}$$

METHOD – II

Given equation can be rewritten as

$$e^{xy^2} (y^2 dx + 2xy dy) + 4x^3 dx - 3y^2 dy = 0$$

Let $e^{xy^2} = u$, then $e^{xy^2} (y^2 dx + 2xy dy) = du$

$\therefore$ (1) reduces to $du + 4x^3 dx - 3y^2 dy = 0$

Integrating we get $u + x^4 - y^3 = C$

$$\Rightarrow e^{xy^2} + x^4 - y^3 = C,$$

Ex. 64. Solve $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$

Solution. The given differential equation is

$$(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$$

Therefore, $M = x^4 - 2xy^2 + y^4$; $N = -2x^2y + 4xy^3 - \sin y$

$$\Rightarrow \frac{\partial M}{\partial y} = -4xy + 4y^3 \quad \text{and} \quad \frac{\partial N}{\partial x} = -4xy + 4y^3$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

hence the given equation is exact.

$$\text{Now } \int M dx = \int (x^4 - 2xy^2 + y^4) dx = \frac{1}{5}x^5 - x^2y^2 + y^4x$$

$$\text{and } \int N dy = \int -\sin y dy = \cos y$$

$\therefore$ From (1) and (2), the required solution is

$$\frac{1}{5}x^5 - x^2y^2 + y^4x + \cos y = C, \text{ which is required solution.}$$

6. INTEGRATING FACTOR

Definition: Sometime the given differential equation $Mdx + Ndy = 0$ is not exact, then we multiplying some function of x and y to make it exact is called an integrating factor. We shall write I. F. for integrating factor.

Note The D. E. $Mdx + Ndy = 0$ possession infinite number of I. F.

RULE – I (INSPECTION)

By rearranging the terms of the given equation and/or by dividing by a suitable function x and y , the equation thus obtained will contain several parts integratable easily. In this connection the following list of exact differentiable should be noted carefully.

$$(1) d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$(3) d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$$

$$(5) d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy - 2xy^2dx}{x^4}$$

$$(7) d[\log(xy)] = \frac{xdy + ydx}{xy}$$

$$(9) d\left(\tan^{-1} \frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

$$(11) d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$$

$$(2) d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$(4) d\left(\frac{x^2}{y}\right) = \frac{2yx dx - x^2 dy}{y^2}$$

$$(6) d\left(\frac{x^2}{y^2}\right) = \frac{2y^2 x dx - 2yx^2 dy}{y^4}$$

$$(8) d(xy) = xdy + ydx$$

$$(10) d\left(\tan^{-1} \frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$$

$$(12) d\left[\log\left(\frac{x}{y}\right)\right] = \frac{ydx - xdy}{xy}$$

$$(13) d\left(\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$$

$$(14) d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$$

$$(15) d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$$

Ex. 65. Solve $ydx - xdy + (1 + x^2)dx + x^2 \sin ydy = 0$

Solution: The given differential equation is

$$\begin{aligned} ydx - xdy + (1 + x^2)dx + x^2 \sin ydy &= 0 && \text{(not exact, verify)} \\ \Rightarrow \frac{ydx - xdy}{x^2} + \frac{1 + x^2}{x^2}dx + \sin ydy &= 0 \\ \Rightarrow -\left(\frac{xdy - ydx}{x^2}\right) + \left(\frac{1}{x^2} + 1\right)dx + \sin ydy &= 0 \\ \Rightarrow -d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x^2}\right)dx + \sin ydy &= 0 \end{aligned}$$

Integrating,

$$\begin{aligned} -\left(\frac{y}{x}\right) + x - \left(\frac{1}{x}\right) - \cos y &= c \\ \Rightarrow -y + x^2 - 1 - x \cos y &= cx \text{ which is required solution.} \end{aligned}$$

Ex. 66. Solve $xdy - ydx + 2x^3dx = 0$

Solution: The given differential equation is

$$\begin{aligned} xdy - ydx + 2x^3dx &= 0 && \text{(not exact, verify)} \\ \Rightarrow \frac{xdy - ydx}{x^2} + 2xdx &= 0 && \left(\text{IF} = \frac{1}{x^2}\right) \\ \Rightarrow d\left(\frac{y}{x}\right) + 2xdx &= 0 \end{aligned}$$

Integrating

$$\begin{aligned} (y/x) + x^2 &= C, \\ \Rightarrow y + x^3 &= Cx \text{ which is the required solution.} \end{aligned}$$

Ex. 67. Solve $y(2xy + e^x)dx = e^x dy$

Solution: The given differential equation is

$$\begin{aligned} y(2xy + e^x)dx &= e^x dy && \text{(not exact, verify)} \\ \Rightarrow 2xy^2dx + ye^x dx - e^x dy &= 0 \\ \Rightarrow 2xdx + \frac{ye^x dx - e^x dy}{y^2} &= 0 \\ \Rightarrow 2xdx + d\left(\frac{e^x}{y}\right) &= 0 \end{aligned}$$

Integrating,

$$\begin{aligned} x^2 + e^x/y &= c \\ yx^2 + e^x &= cy \text{ which is required solution.} \end{aligned}$$

Ex. 68. Solve $ydx - xdy + (1 + x^2)dx + x^2 \sin ydy = 0$

Solution: The given differential equation is

$$\begin{aligned} ydx - xdy + (1 + x^2)dx + x^2 \sin ydy &= 0 && \text{(not exact, verify)} \\ \frac{ydx - xdy}{x^2} + \left(\frac{1 + x^2}{x^2}\right)dx + \sin ydy &= 0 \\ \Rightarrow -d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x^2}\right)dx + \sin ydy &= 0 \end{aligned}$$

Integrating

$$-\left(\frac{y}{x}\right) + \left(x - \frac{1}{x}\right) + (-\cos y) = C \text{ which is required solution.}$$

Ex. 69. Solve $y \sin 2x dx = (1 + y^2 + \cos^2 x)dy$

Solution: The given differential equation is

$$\begin{aligned} y \sin 2x dx &= (1 + y^2 + \cos^2 x)dy && \text{(not exact, verify)} \\ \Rightarrow -2y \sin x \cos x dx + \cos^2 x dy + (1 + y^2)dy &= 0 \\ \Rightarrow d(y \cos^2 x) + (1 + y^2)dy &= 0 \end{aligned}$$

Integrating,

$$y \cos^2 x + y + \frac{y^3}{3} = c \text{ which is required solution.}$$

Ex. 70. Solve $(1 + xy)ydx + (1 - xy)x dy = 0$

Solution: The given differential equation is

$$\begin{aligned} (ydx + xdy) + (xy^2 dx - x^2 y dy) &= 0 && \text{(not exact, verify)} \\ \Rightarrow d(yx) + xy^2 dx - x^2 y dy &= 0 \\ \Rightarrow \frac{d(yx)}{x^2 y^2} + \frac{1}{x} dx - \frac{1}{y} dy &= 0 && \left(\text{IF} = \frac{1}{x^2 y^2}\right) \end{aligned}$$

$$\Rightarrow \left(\frac{1}{z^2}\right) dz + \left(\frac{1}{x}\right) dx - \left(\frac{1}{y}\right) dy = 0, \quad (\text{where } z = xy)$$

Integrating

$$-\left(\frac{1}{z}\right) + \log x - \log y = \log c,$$

$$\Rightarrow \log\left(\frac{x}{cy}\right) = \frac{1}{z} = \frac{1}{xy} \Rightarrow x = cye^{1/xy} \text{ which is required solution.}$$

Ex. 71. Solve $(x^3 + xy^2 + a^2y)dx + (y^3 + yx^2 - a^2x)dy = 0$ **Solution:** The given differential equation is

$$(x^3 + xy^2 + a^2y)dx + (y^3 + yx^2 - a^2x)dy = 0 \quad (\text{not exact, verify})$$

$$\Rightarrow x(x^2 + y^2)dx + y(x^2 + y^2)dy + a^2(ydx - xdy) = 0$$

$$\Rightarrow x(x^2 + y^2)dx + y(x^2 + y^2)dy + a^2(ydx - xdy) = 0$$

$$\Rightarrow (xdx + ydy) + a^2\left(\frac{ydx - xdy}{x^2 + y^2}\right) = 0$$

$$\Rightarrow xdx + ydy + a^2d\left(\tan^{-1}\frac{x}{y}\right) = 0$$

Integrating

$$\left(\frac{1}{2}\right)x^2 + \left(\frac{1}{2}\right)y^2 + a^2 \tan^{-1}\left(\frac{x}{y}\right) = \left(\frac{1}{2}\right)c$$

$$\Rightarrow x^2 + y^2 + 2a^2 \tan^{-1}\left(\frac{x}{y}\right) = c. \text{ which is required solution.}$$

Ex. 72. Solve $(y^2e^x + 2xy)dx - x^2dy = 0$ **Solution:** The given differential equation is

$$(y^2e^x + 2xy)dx - x^2dy = 0 \quad (\text{not exact, verify})$$

$$\Rightarrow e^x dx + 2x \frac{1}{y} dx - \frac{x^2}{y^2} dy = 0$$

$$\Rightarrow e^x dx + \frac{2xydx - x^2dy}{y^2} = 0$$

Integrating

$$e^x + (x^2/y) = C,$$

$$\Rightarrow ye^x + x^2 = Cy \text{ which is the required solution.}$$

Ex. 73. Solve $y(ax + e^x)dx - e^x dy = 0$ **Solution:** The given differential equation is

$$y(ax + e^x)dx - e^x dy = 0 \quad (\text{not exact, verify})$$

$$\Rightarrow axdx + \frac{e^x}{y} dx - \frac{e^x}{y^2} dy = 0$$

$$\Rightarrow axdx + \left(\frac{ye^x dx - e^x dy}{y^2}\right) = 0$$

$$\Rightarrow axdx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating

$$a \frac{1}{2} x^2 + \left(\frac{e^x}{y}\right) = C,$$

$$\Rightarrow ax^2y + 2e^x = 2Cy$$

Ex. 74. Solve $(x^3 e^x - my^2) dx + mxydy = 0$ **Solⁿ:** The differential equation is

$$(x^3 e^x - my^2)dx + mxydy = 0 \quad (\text{not exact, verify})$$

$$\Rightarrow x^3 e^x dx + m(xydy - y^2 dx) = 0$$

$$\Rightarrow e^x dx + m \left(\frac{xydy - y^2 dx}{x^3}\right) = 0, \quad (\text{dividing by } x^3)$$

$$\Rightarrow e^x dx + \frac{m}{2} \left(\frac{x^2 2ydy - y^2 \cdot 2xdx}{x^4}\right) = 0$$

$$\Rightarrow e^x dx + \frac{1}{2} m d(y^2/x^2) = 0$$

$$e^x + \frac{1}{2} m (y^2/x^2) = C,$$

$$\Rightarrow 2x^2 e^x + my^2 = 2Cx^2 \text{ which is required solution.}$$

Ex. 75. Solve $x^2 \left(\frac{dy}{dx}\right) + xy = \sqrt{1 - x^2 y^2}$ **Solⁿ:** The differential equation is

$$x^2 \left(\frac{dy}{dx} \right) + xy = \sqrt{1 - x^2 y^2} \quad (\text{not exact, verify})$$

$$\Rightarrow \frac{xdy + ydx}{\sqrt{1 - x^2 y^2}} - \frac{dx}{x} = 0$$

Integrating,

$$\sin^{-1}(xy) - \log x = c \text{ which is required solution.}$$

Ex. 76. Solve $xdx + ydy + (x^2 + y^2) dy = 0$

solⁿ: The differential equation is

$$xdx + ydy + (x^2 + y^2) dy = 0 \quad (\text{not exact, verify})$$

$$\Rightarrow (2xdx + 2ydy) + 2(x^2 + y^2)dy = 0$$

$$\Rightarrow d(x^2 + y^2) + 2(x^2 + y^2)dy = 0$$

$$\Rightarrow \frac{d(x^2 + y^2)}{(x^2 + y^2)} + 2dy = 0$$

$$\Rightarrow \frac{dz}{z} + 2dy = 0, \quad (\text{where } z = x^2 + y^2)$$

Integrating

$$\log z + 2y = c,$$

$$\Rightarrow \log z = c - 2y \quad \Rightarrow z = e^{c-2y}$$

$$\Rightarrow x^2 + y^2 = e^{c-2y} \text{ which is required solution.}$$

Ex. 77. Solve $xdy - ydx = a(x^2 - y^2) dy$

Solⁿ: The differential equation is

$$xdy - ydx = a(x^2 - y^2) dy \quad (\text{not exact, verify})$$

$$\Rightarrow \frac{xdy - ydx}{x^2 + y^2} = a dy \quad \Rightarrow d\left(\tan^{-1} \frac{y}{x}\right) = a dy$$

Integrating,

$$\tan^{-1}(y/x) = ay + C, \text{ which is required solution.}$$

Ex. 78. Solve $(x^2 + y^2 + a^2)ydy + x(x^2 + y^2 - a^2)dx = 0$

Solⁿ: The differential equation is

$$(x^2 + y^2 + a^2)ydy + x(x^2 + y^2 - a^2)dx = 0 \quad (\text{not exact, verify})$$

$$\Rightarrow (x^2 + y^2 + a^2)ydy + (x^2 + y^2 - a^2)xdx = 0$$

$$\Rightarrow (x^2 + y^2)(2xdx + 2ydy) + 2a^2ydy - 2a^2xdx = 0$$

$$\Rightarrow (x^2 + y^2)d(x^2 + y^2) + 2a^2ydy - 2a^2xdx = 0$$

$$\Rightarrow zdz + 2a^2ydy - 2a^2xdx = 0, \quad (\text{putting } z = x^2 + y^2)$$

Integrating

$$\frac{1}{2}z^2 + a^2y^2 - a^2x^2 = C$$

$$\Rightarrow (x^2 + y^2) + 2a^2(x^2 - y^2) = 2C \text{ which is required solution.}$$

RULE II: If the given equation $Mdx + Ndy = 0$ **is homogeneous in x & y**

i.e. M, N are homogeneous function of same degree in x, y & $(Mx + Ny) \neq 0$,

$$\text{then integrating factor} = \frac{1}{(Mx + Ny)}$$

Ex. 79. Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solⁿ: The given differential equation is

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \quad (1)$$

Clearly (1) is a homogenous differential equation. Comparing (1) with $Mdx + Ndy = 0$, we have

$$M = x^2y - 2xy^2 \text{ and } N = -(x^3 - 3x^2y)$$

$$\frac{\partial M}{\partial y} = x^2 - 4y; \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given differential equation is not an exact

$$\text{Now, } Mx + Ny = x(x^2y - 2xy^2) - y(x^3 - 3x^2y) = x^2y^2 \neq 0$$

$$\therefore \text{I.F.} = \frac{1}{(Mx + Ny)} = \frac{1}{(x^2y^2)}$$

On multiplying (1) by $\frac{1}{x^2y^2}$, we have

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

The required solution

$$\int \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = 0$$

$$\Rightarrow \left(\frac{x}{y}\right) - 2 \log x + 3 \log y = c \text{ which is required solution.}$$

Prob. 80. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Ans. $3x^4y^4 + 6xy^2 + 2y^6 = C$

Do yourself.

Prob. 81. $x^2ydx - (x^3 - y^3)dy = 0$

Ans. $y = ce^{x^3/3y^3}$

Do yourself.

Rule III – If the equation $Mdx + Ndy = 0$ is of the form $f_1(xy)ydx + f_2(xy)x dy = 0$, then

integrating factor = $\frac{1}{(Mx - Ny)}$ provided $(Mx - Ny) \neq 0$.

Ex. 82. Solve $(xy \sin xy + \cos xy)ydx + (xy \sin xy - \cos xy)x dy = 0$

Solution: The given differential equation is

$$(xy \sin xy + \cos xy)ydx + (xy \sin xy - \cos xy)x dy = 0$$

Comparing (1), with $Mdx + Ndy = 0$, we have

$$M = y(xy \sin xy + \cos xy)$$

$$\text{and } N = x(xy \sin xy - \cos xy)$$

$$\text{Here } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ the given differential equation is not an exact.

$\therefore$ the given Differential Equation of the form $f_1(xy)ydx + f_2(xy)x dy = 0$

$$\therefore Mx - Ny = xy(xy \sin xy + \cos xy) - xy(xy \sin xy - \cos xy)$$

$$\Rightarrow Mx - Ny = 2xy \cos xy \neq 0$$

$$\therefore \text{I. F.} = \frac{1}{(Mx - Ny)} = \frac{1}{(2xy \cos xy)}$$

On multiplying (1) by $\frac{1}{(2xy \cos xy)}$, we have

$$\frac{1}{2} \left(y \tan xy + \frac{1}{x} \right) dx + \left(\frac{1}{2} \right) \left(x \tan xy - \frac{1}{y} \right) dy = 0 \quad (\text{which must be exact})$$

$$\begin{aligned} \text{Now, } \int Mdx &= \int \frac{1}{2} \left(y \tan xy + \frac{1}{x} \right) dx \\ &= \frac{1}{2} (\log \sec xy + \log x) \end{aligned}$$

$$\int Ndy = \int \frac{1}{2} \left(-\frac{1}{y} \right) dy = -\frac{1}{2} \log y$$

$\therefore$ the required solution is

$$\frac{1}{2} (\log \sec xy + \log x) - \frac{1}{2} \log y = \log c \text{ which is required solution is}$$

Ex. 83. Solve $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)x dy = 0$

Solⁿ: The given differential equation is

$$(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)x dy = 0 \quad (\text{not exact, verify})$$

which is of the form $f_1(xy)ydx + f_2(xy)x dy = 0$

$$\therefore \text{IF} = \frac{1}{(x^2y^2 + xy + 1)yx - (x^2y^2 - xy + 1)xy} = \frac{1}{2(x^2y^2)}$$

Multiplying the given equation by the integrating factor we get

$$\frac{1}{2} \left(1 + \frac{1}{xy} + \frac{1}{x^2y^2} \right) ydx + \frac{1}{2} \left(1 - \frac{1}{xy} + \frac{1}{x^2y^2} \right) x dy = 0$$

$$\Rightarrow (ydx + xdy) + \left(\frac{1}{x} dx - \frac{1}{y} dy \right) + \left(\frac{1}{x^2y} dx + \frac{1}{xy^2} dy \right) = 0$$

$$\Rightarrow d(xy) + \left(\frac{1}{x} dx - \frac{1}{y} dy \right) + \left(\frac{ydx + xdy}{x^2y^2} \right) = 0$$

$$\Rightarrow d(xy) + \frac{1}{x} dx - \frac{1}{y} dy + \frac{d(xy)}{x^2y^2} = 0$$

$$\Rightarrow d(xy) + \frac{1}{x} dx - \frac{1}{y} dy + \frac{dz}{z^2} = 0, \quad (\text{where } z = xy)$$

Integrating

$$xy + \log x - \log y - (1/z) = C,$$

$$\Rightarrow xy + \log x - \log y - (1/xy) = C \text{ which is the required solution.}$$

Ex. 84. Solve $(x^4y^4 + x^2y^2 + xy)ydx + (x^4y^4 - x^2y^2 + xy)x dy = 0$

Solⁿ: The given differential equation is

$$(x^4y^4 + x^2y^2 + xy)ydx + (x^4y^4 - x^2y^2 + xy)x dy = 0$$

(not exact, verify)

which is of the form $f_1(xy)ydx + f_2(xy)x dy = 0$

$$\therefore IF = \frac{1}{(x^4y^4 + x^2y^2 + xy)yx - (x^4y^4 - x^2y^2 + xy)xy} = \frac{1}{2x^3y^3}$$

Multiplying the given equation by the integrating factor, we get

$$\frac{1}{2} \left(xy + \frac{1}{xy} + \frac{1}{x^2y^2} \right) ydx + \frac{1}{2} \left(xy - \frac{1}{xy} + \frac{1}{x^2y^2} \right) xdy = 0$$

$$\Rightarrow (xy^2dx + x^2ydy) + \left(\frac{1}{x}dx - \frac{1}{y}dy \right) + \left(\frac{1}{x^2y}dx + \frac{1}{xy^2}dy \right) = 0$$

$$\Rightarrow xy(ydx + xdy) + \left(\frac{1}{x}dx - \frac{1}{y}dy \right) + \left(\frac{ydx + xdy}{x^2y^2} \right) = 0$$

$$\Rightarrow xy d(xy) + \left(\frac{1}{x}dx - \frac{1}{y}dy \right) + \frac{d(xy)}{x^2y^2} = 0$$

$$\Rightarrow zdz + \frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z^2}dz = 0, \quad (\text{putting } z = xy)$$

Integrating

$$\frac{1}{2}z^2 + \log x - \log y - \left(\frac{1}{z} \right) = C,$$

$$\Rightarrow \frac{1}{2}x^2y^2 + \log x - \log y - \left(\frac{1}{xy} \right) = C, \quad (\text{putting } z = xy).$$

which is required solution.

Ex. 85. Solve $(x^3y^3 + x^2y^2 + xy + 1)ydx + (x^3y^3 - x^2y^2 - xy + 1)x dy = 0$

Solution: The given differential equation is

$$(x^3y^3 + x^2y^2 + xy + 1)ydx + (x^3y^3 - x^2y^2 - xy + 1)x dy = 0$$

(not exact, verify)

which is of the form $f_1(xy)ydx + f_2(xy)x dy = 0$

$$\therefore IF = \frac{1}{(x^3y^3 + x^2y^2 + xy + 1)xy - (x^3y^3 - x^2y^2 - xy + 1)xy} = \frac{1}{2(x^3y^3 + x^2y^2)}$$

Multiplying the given equation by this integrating factor, we get

$$\frac{1}{2x^2y^2(xy+1)} [\{x^2y^2(xy+1) + (xy+1)\}ydx + \{ (x^3y^3+1) - xy(xy+1) \}x dy] = 0$$

$$\Rightarrow \left[\frac{(x^2y^2+1)}{x^2y^2} \right] ydx + \left[\frac{(x^2y^2-xy+1)-xy}{x^2y^2} \right] x dy = 0$$

$$\Rightarrow (ydx + xdy) + \frac{ydx + xdy}{x^2y^2} dy = 0$$

$$\Rightarrow d(xy) + \frac{d(xy)}{x^2y^2} - \frac{2}{y} dy = 0$$

$$\Rightarrow dz + \frac{dz}{z^2} - \frac{2}{y} dy = 0, \quad (\text{when } z = xy)$$

Integrating

$$z - (1/z) - 2 \log y = C$$

$$\Rightarrow xy - \left(\frac{1}{xy} \right) - 2 \log y = C, \quad (\text{putting } z = xy).$$

which is required solution.

Prob. 86. $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

$$\text{Ans. } \log(x^3y) - \frac{1}{xy} = c$$

Do yourself.

Rule IV: If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of x alone say $f(x)$, then Integrating factor = $e^{\int f(x)dx}$

Ex. 87. Solve $(x^2 + y^2 + x)dx + xy dy = 0$

Solⁿ: The given differential equation is

Comparing (1) with $Mdx + Ndy = 0$ we have

$$M = x^2 + y^2 + x \quad \text{and} \quad N = xy$$

$$\therefore \frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = y$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ The Differential Equation is not an exact.

Now

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{xy} (2y - y) = \frac{1}{x} \quad \text{which is a function of } x \text{ alone.}$$

$$\text{IF, } = e^{\int (1/x)dx} = e^{\log x} = x$$

Multiplying (1) by x , we have

$$(x^3 + xy^2 + x^2)dx + x^2ydy = 0$$

$$\Rightarrow \int Mdx = \int (x^3 + xy^2 + x^2)dx$$

$$= \left(\frac{x^4}{4}\right) + \frac{(x^2y^2)}{2} + \left(\frac{x^3}{3}\right)$$

$$\text{and } \int Ndy = 0$$

$\therefore$ the required solution is

$$\left(\frac{x^4}{4}\right) + \frac{x^2y^2}{2} + \frac{x^3}{3} = c \text{ which is required solution.}$$

Ex. 88. Solve $(x^2 + y^2)dx - 2xydy = 0$

Solⁿ: The given differential equation is

$$(x^2 + y^2)dx - 2xydy = 0$$

$$M = x^2 + y^2 \text{ and } N = -2xy$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = -2y$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ The differential equation is not an exact

$$\text{Now } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{-2xy} (2y + 2y) = -\frac{2}{x}$$

(which is a function of x alone).

$$\therefore \text{IF} = e^{\int f(x)dx} = e^{\int -(2/x)dx} = e^{-2 \log x} = \frac{1}{x^2}$$

Multiplying the given equation by this integrating factor, we get

$$\frac{1}{x^2} (x^2 + y^2)dx - 2xy \frac{1}{x^2} dy = 0 \Rightarrow \left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$\Rightarrow dx + \frac{y^2}{x^2}dx - \frac{2y}{x}dy = 0$$

$$\Rightarrow dx + d\left(-\frac{y^2}{x}\right) = 0$$

Integrating

$$x - \left(\frac{y^2}{x}\right) = C,$$

$\Rightarrow x^2 - y^2 = Cx$, which is the required solution.

Ex. 89. Solve $(x^2 + y^2 + 1)dx - 2xydy = 0$

Solⁿ: The given differential equation is

$$(x^2 + y^2 + 1)dx - 2xydy = 0$$

Comparing (1) with $Mdx + Ndy = 0$

$$\therefore M = x^2 + y^2 + 1 \text{ and } N = -2xy$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = -2y$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ The differential equation is not an exact.

$$\text{Now } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{-2xy} (2y + 2y) = -\frac{2}{x},$$

(which is a function of x alone)

$$\therefore \text{IF} = e^{\int f(x)dx} = e^{\int -(2/x)dx} = e^{-2 \log x} = \frac{1}{x^2}$$

Multiplying the given equation by this integrating factor $1/x^2$

$$\frac{1}{x^2} (x^2 + y^2 + 1)dx - \frac{1}{x^2} (2xy)dy = 0$$

$$\Rightarrow \left[1 + \left(\frac{y^2}{x^2}\right) + \left(\frac{1}{x^2}\right)\right]dx - 2\left(\frac{y}{x}\right)dy = 0$$

$$\Rightarrow \left[1 + \left(\frac{1}{x^2}\right)\right]dx + \left[\left(\frac{y^2}{x^2}\right)dx - 2\left(\frac{y}{x}\right)dy\right] = 0$$

$$\Rightarrow \left[1 + \left(\frac{1}{x^2}\right)\right]dx + d\left(\frac{-y^2}{x}\right) = 0$$

Integrating

$$x - \left(\frac{1}{x}\right) + \left(\frac{-y^2}{x}\right) = C,$$

$\Rightarrow x^2 - 1 - y^2 = Cx$ which is the required solution.

Ex. 90. Solve $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(x + xy^2)dy = 0$

Solution: The given differential equation is

$$\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(x + xy^2)dy = 0 \quad (1)$$

Comparing (1) with $Mdx + Ndy = 0$

$$M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2 \text{ and } N = \frac{1}{4}(x + xy^2)$$

$$\Rightarrow \frac{\partial M}{\partial y} = 1 + y^2 \text{ and } \frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ The differential equation is not an exact.

$$\begin{aligned} \text{Now } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) &= \frac{4}{(x + xy^2)} \left\{ (1 + y^2) - \frac{1}{4}(1 + y^2) \right\} \\ &= \frac{4}{x(1 + y^2)} \times \frac{3}{4}(1 + y^2) = \frac{3}{x} \quad (\text{which is function of } x \text{ alone, } f(x)) \end{aligned}$$

$$\therefore \text{IF} = e^{\int f(x)} = e^{\int (3/x) dx} = x^3$$

Multiplying both sides of the given equation by the integrating factor x^3 , we get

$$x^3 \left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2 \right) dx + \frac{1}{4}x^3(x + xy^2)dy = 0$$

$$\Rightarrow \left(4x^3y + \frac{4}{3}x^3y^3 + 2x^5 \right) dx + (x^4 + x^4y^2)dy = 0$$

$$\Rightarrow 2x^5dx + (x^4dy + 4x^3ydx) + \frac{1}{3}(x^43y^2dy + 4x^3y^3dx) = 0$$

$$\Rightarrow 2x^5dx + d(x^4y) + \frac{1}{3}d(x^4y^3) = 0$$

Integrating

$$\frac{1}{3}x^6 + x^4y + \frac{1}{3}x^4y^3 = C,$$

$\Rightarrow x^6 + 3x^4y + x^4y^3 = 3C$ which is the required solution

Prob. 91. $(x^2 + y^2 + 2x)dx + 2y = 0$

Ans. $(x^2 + y^2)e^x = C$

Do yourself.

Rule V: If $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is a function of y alone say $f(y)$, then integrating factor $= e^{\int f(y) dy}$

Ex. 92. $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

Solution. The given differential equation is

$$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0 \quad (1)$$

Comparing (1) with $Mdx + Ndy = 0$

$$\therefore M = 2xy^4e^y + 2xy^3 + y \text{ and } N = (x^2y^4e^y - x^2y^2 - 3x) \quad (2)$$

$$\Rightarrow \frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1, \quad \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ The Differential Equation is not an exact.

$$\text{Now, } \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4(2xy^3e^y + 2xy^2 + 1) = -\frac{4}{y}(2xy^4e^y + 2xy^3 + y)$$

$$\Rightarrow \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = -\frac{4}{y} \quad (\text{which is a function of } y \text{ alone})$$

$$\therefore \text{I. F.} = e^{\int (-4/y) dy} = e^{-4 \log y} = \left(\frac{1}{y^4}\right).$$

multiplying (1) by $\left(\frac{1}{y^4}\right)$, we have

$$\left\{ 2xe^y + \left(\frac{2x}{y}\right) + \left(\frac{1}{y^3}\right) \right\} dx + \left\{ x^2e^y - \left(\frac{x^2}{y^2}\right) - 3\left(\frac{x}{y^4}\right) \right\} dy = 0 \quad (\text{which must be exact})$$

$$\text{Now } \int Mdx = \int \left\{ 2xe^y + \left(\frac{2x}{y}\right) + \left(\frac{1}{y^3}\right) \right\} dx$$

$$= x^2 e^y + \left(\frac{x^2}{y}\right) + \frac{x}{y^3}$$

$$\text{and } \int N dy = 0$$

$$\therefore \text{the required solution is } x^2 e^y + \left(\frac{x^2}{y}\right) + \frac{x}{y^3} = c.$$

which is required solution.

Ex. 93. Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

Solⁿ: The differential equation is

$$(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$$

Comparing (1) with $Mdx + Ndy = 0$

$$\therefore M = xy^3 + y \text{ and } N = 2x^2y^2 + 2x + 2y^4$$

$$\Rightarrow \frac{\partial M}{\partial y} = 3xy^2 + 1 \text{ and } \frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

$\Rightarrow$ The Differential Equation is not an exact.

$$\text{Now } \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{(xy^3 + y)} \{ (4xy^2 + 2) - (3xy^2 + 1) \}$$

$$= \frac{1}{y(xy^2 + 1)} \{ xy^2 + 1 \} = \frac{1}{y}, \quad (\text{which is a function of } y \text{ say } f(y))$$

$$\therefore \text{IF} = e^{\int f(y) dy} = e^{\int (1/y) dy} = e^{\log y} = y.$$

Multiplying the given equation by the integrating factor y , we get

$$(xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0$$

$$\Rightarrow (xy^4 dx + 2x^2y^3 dy) + (y^2 dx + 2xy dy) + 2y^5 dy = 0$$

$$\Rightarrow \frac{1}{2} d(x^2y^4) + d(y^2x) + 2y^5 dy = 0$$

Integrating

$$\frac{1}{2} (x^2y^4) + (y^2x) + \frac{1}{3} y^6 = C$$

$$\Rightarrow 3x^2y^4 + 6xy^2 + 2y^6 = 6C \text{ which is the required solution.}$$

Prob. 94. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

$$\text{Ans. } 3x^4y^4 + 2y^6 = C$$

Do yourself.

Rule VI: If the given equation $Mdx + Ndy = 0$ is of the form $x^\alpha y^\beta (mydx + nx dy) = 0$, then integrating factor $= x^{km-1-\alpha} y^{kn-1-\beta}$, where k can have any value.

Remark 1. Using rule VI, we now find the rule for finding I. F. of the equation of the form

$$x^\alpha y^\beta (mydx + nx dy) + x^{\alpha'} y^{\beta'} (m'y dx + n'x dy) = 0$$

By virtue of rule VI, we see that the factor makes the first term of (1) exact differential is $x^{km-1-\alpha} y^{kn-1-\beta}$

and that for the second term of (1) $x^{km'-1-\alpha'} y^{kn'-1-\beta'}$ is where and can have any value.

the above mentioned two factors will be identical if we choose k and k' such that

$$km - 1 - \alpha = k'm' - 1 - \alpha' \quad (2)$$

$$\text{and } kn - 1 - \beta = k'n' - 1 - \beta' \quad (3)$$

Solving (2) and (3), we evaluate the values of k and k' . Substituting these values in the factor

$$x^{km-1-\alpha} y^{kn-1-\beta}$$

or $x^{km'-1-\alpha'} y^{kn'-1-\beta'}$, we obtain the required I. F. of (1)

Ex. 95. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$

Solution. The given differential equation is

$$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$$

Rewriting (1) in the standard form

$$x^\alpha y^\beta (mydx + nx dy) + x^{\alpha'} y^{\beta'} (m'y dx + n'x dy) = 0$$

we have

$$y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$$

Comparing (2) and (3), we have

$$\alpha = 0, \beta = 1, m = 1, n = -1;$$

$$\alpha' = 2, \beta' = 0, m' = 2, n' = 2$$

Hence the I. F. for the first term on L. HS. of (3)

$$= x^{k-1} y^{-k-1-1},$$

$$= x^{k-1} y^{-k-2}$$

and the I. F. for the second term on L. H. S of (3)

$$= x^{2k'-1-2} y^{2k'-1},$$

$$= x^{2k'-3}y^{2k'-1} \quad (5)$$

For the integrating factor (4) and (5) to be identical, we have

$$k-1 = 2k'-3 \text{ and } -k-2 = 2k-1$$

$$\Rightarrow k-2k' = -2 \text{ and } k+2k' = -1$$

$$\Rightarrow k = -\frac{3}{2} \text{ and } k' = \frac{1}{4} \quad (6)$$

Substituting the value of k in (4) or in (5),

the integrating factor of (3) or (1) = $x^{-5/2}y^{-1/2}$.

Multiplying (1) by $x^{-5/2}y^{-1/2}$, we have

$$(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx + (2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2})dy = 0$$

which must be exact

$$\text{Now } \int Mdx = \int (x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx$$

$$= \frac{x^{-3/2}y^{3/2}}{(-3/2)} + \frac{2x^{1/2}y^{1/2}}{(1/2)}$$

$$\text{and } \int Ndy = 0$$

$\therefore$ the required solution is

$$\frac{x^{-3/2}y^{3/2}}{(-3/2)} + \frac{2x^{1/2}y^{1/2}}{(1/2)} = c \text{ which is required solution.}$$

Remark 2. Sometime the rule VI for finding I. F. is modified as given below.

If the given equation $Mdx + Ndy = 0$ can be put in the form $x^\alpha y^\beta (mydx + nx dy) + x^{\alpha'} y^{\beta'} (m'ydx + n'xdy) = 0$

Where $\alpha, \beta, m, n, \alpha', \beta', m', n'$ are constants, then the given equation has an I. F. $x^h y^k$, where h and k are obtained by applying the condition that the given equation must become exact after multiplying by $x^h y^k$.

Ex. 96. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$

Solⁿ: The given differential equation is

$$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$$

$$\Rightarrow (y^2dx - xydy) + (2x^2ydx + 2x^3dy) = 0$$

$$\Rightarrow y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$$

Let I. F. = $x^h y^k$

$\therefore$ Multiplying the given equation by $x^h y^k$, we have

$$(x^h y^{k+2} + 2x^{h+2} y^{k+1})dx + (2x^{h+3} y^k - x^{h+1} y^{k+1})dy = 0 \quad (1)$$

which must be exact.

$\therefore$ (1), we have

$$M = x^h y^{k+2} + 2x^{h+2} y^{k+1}, \quad N = 2x^{h+3} y^k - x^{h+1} y^{k+1}$$

Since (1) is exact.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow (k+2)x^h y^{k+1} + 2(k+1)x^{h+2} y^k = 2(h+3)x^{h+2} y^k - (h+1)x^h y^{k+1}$$

Now equating the coefficients of $x^h y^{k+1}$ and $x^{h+2} y^k$, we get

$$k+2 = -(h+1) \text{ and } 2(k+1) = 2(h+3)$$

$$\text{i.e. } h+k = -3 \text{ and } h-k = -2$$

Solving system (2), we get

$$h = -\left(\frac{5}{2}\right) \text{ and } k = -\left(\frac{1}{2}\right)$$

$$\therefore \text{I. F.} = x^{-5/2}y^{-1/2}.$$

Multiplying the given equation by I. F. $x^{-5/2}y^{-1/2}$, we get

$$(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx + (2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2})dy = 0 \quad (\text{which must be exact}).$$

The required solution is

$$-(2/3)x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c \quad (\text{verify}).$$

which is required solution.

Ex. 97 Given that the differential equation $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$ has an I. F. of the form $x^h y^k$ find its general solution.

Solⁿ: The given differential equation is

$$(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$$

$$\Rightarrow (2x^2y^2 + y)dx + (3x - x^3y)dy = 0 \quad (1)$$

Multiplying by $x^h y^k$ in (1)

$$(2x^{h+2}y^{k+2} + x^h y^{k+1})dx + (3x^{h+1}y^k - x^{h+3}y^{k+1})dy = 0 \quad (2)$$

which must be exact.

Comparing (2) with $Mdx + Ndy = 0$, we have

$$M = 2x^{h+2}y^{k+2} + x^hy^{k+1} \quad \text{and} \quad N = 3x^{h+1}y^k - x^{h+3}y^{k+1} \quad (3)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow 2(k+2)x^{h+2}y^{k+1} + (k+1)x^hy^k = 3(h+1)x^hy^k - (h+3)x^{h+2}y^{k+1}$$

$$\Rightarrow 2(k+2) = -(h+3) \text{ and } k+1 = 3(h+1) \quad (\text{verify})$$

$$\Rightarrow h+2k = -7 \text{ and } 3h-k = -2$$

$$\Rightarrow h = -\frac{11}{7} \text{ and } k = -\frac{19}{7}$$

$$\therefore \text{I.F.} = x^{-11/7}y^{-19/7}, \text{ we have}$$

Multiplying (1) by $x^{-11/7}y^{-19/7}$, we have

$$(2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7})dx + (3x^{-4/7}y^{-19/7} - x^{10/7}y^{-12/7})dy = 0 \text{ which must be exact.}$$

Hence, as usual, the required solution is

$$\frac{2x^{10/7}y^{-5/7}}{10/7} + \frac{x^{-4/7}y^{-12/7}}{(-4/7)} = \frac{7c}{20} \quad (\text{verify})$$

$$\Rightarrow 4x^{10/7}y^{-5/7} - 5x^{-4/7}y^{-12/7} = c \text{ which is required solution.}$$

Ex. 98. Solve $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$

Solⁿ: The given differential equation is

$$(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0 \quad (\text{not exact, verify})$$

$$\text{Let } \text{IF} = x^hy^k$$

Multiplying the given equation by x^hy^k , we get

$$(2x^{h+2}y^{k+1} - 3x^hy^{k+4})dx + (3x^{h+3}y^k + 2x^{h+1}y^{k+3})dy = 0 \quad (2)$$

Since, (2) is exact

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow 2(k+1)x^{h+2}y^k - 3(k+4)x^hy^{k+3} = 3(h+3)x^{h+2}y^k + 2(h+1)x^hy^{k+3}$$

Equating coefficients of $x^{h+2}y^k$ and x^hy^{k+3} on both sides, we have

$$2(k+1) = 3(h+3) \text{ and } -3(k+4) = 2(h+1)$$

$$\Rightarrow 3h - 2k + 7 = 0 \text{ and } 2h + 3k + 14 = 0$$

Solving these equations, we get $h = -49/13$ and $k = -28/13$

$$\text{Therefore, IF} = x^hy^k = x^{-49/13}y^{-28/13}$$

Multiplying the given equation (1) by the IF = $x^{-49/13}y^{-28/13}$,

$$\therefore (2x^{-23/13}y^{-15/13} - 3x^{-49/13}y^{-24/13})dx + (3x^{-10/13}y^{-28/13} + 2x^{-36/13}y^{11/13})dy = 0$$

$$\text{Now } \int Mdx = \int (2x^{-23/13}y^{-15/13} - 3x^{-49/13}y^{-24/13})dx$$

$$= -(2 \times 13/10)x^{-10/13}y^{-15/13} + (39/36)x^{-36/13}y^{24/13}$$

$$= -(13/5)x^{-10/13}y^{-15/13} + (13/12)x^{-36/13}y^{24/13}$$

$$\text{and } \int Ndy = 0$$

Hence, the required solution is

$$-(13/5)x^{-10/13}y^{-15/13} + (13/12)x^{-36/13}y^{24/13} = C,$$

$$\Rightarrow -12x^{-10/13}y^{-15/13} + 5x^{-36/13}y^{24/13} = k \quad (\text{where } k = (60C/13))$$

which is required solution.

Ex. 99. Solve $(2xydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$

Solⁿ: The given differential equation is

$$(2xydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$$

$$\Rightarrow (2y + 6xy^2)dx + (3x + 8x^2y)dy = 0 \quad (1)$$

$$\text{Let } \text{IF} = x^hy^k$$

Multiplying the given equation by x^hy^k ,

$$(2x^hy^{k+1} + 6x^{h+1}y^{k+2})dx + (3x^{h+1}y^k + 8x^{h+2}y^{k+1})dy = 0$$

Since, (2) is an exact

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow 2(k+1)y^kx^h + 6(k+2)y^{k+1}x^{h+1} = 3(h+1)x^hy^k + 8(h+2)x^{h+1}y^{k+1}$$

Equating the coefficients of x^hy^k and $x^{h+1}y^{k+1}$ on both sides we get

$$3(h+1) = 2(k+1) \text{ and } 8(h+2) = 6(k+2)$$

$$\Rightarrow 3h - 2k = -1 \text{ and } 4h - 3k = -2$$

Solving these equations we get $h = 1, k = 2$

$$\text{Therefore, IF} = x^hy^k = xy^2.$$

Multiplying the given equation (1) by the IF = xy^2 ,

$$(2xy^3 + 6x^2y^4)dx + (3x^2y^2 + 8x^3y^3)dy = 0$$

$$\text{Now } \int Mdx = \int (2xy^3 + 6x^2y^4) dx = x^2y^3 + 2x^3y^4 \quad (1)$$

$$\text{and } \int Ndy = 0$$

Hence, from (1), the required solution is $x^2y^3 + 2x^3y^4 = C$ which is required solution.

7. LINEAR DIFFERENTIAL EQUATION

Definition. A differential equation of first order is of the form

$$\frac{dy}{dx} + Py = Q \quad (1)$$

is called linear differential equation where P and Q both are function of x alone or constants.

The integral factor of equation (1) is

$$IF = e^{\int Pdx}$$

The required solution is

$$y \times (I.F.) = \int [Q \times (I.F.)] dx + c$$

Remark: Sometimes the differential equation of first order is of the form

$$\frac{dy}{dx} + P_1x = Q_1$$

where P_1 and Q_1 both are function of y or constant is also called linear differential equation of first order.

In this case $IF = e^{\int P_1 dy}$

The required solution is

$$y \times (IF) = \int Q \times (IF) dy + C$$

Ex. 100. Solve $x \cos x \left(\frac{dy}{dx} \right) - y(x \sin x + \cos x) = 1$

Solⁿ: The given differential equation is

$$x \cos x \left(\frac{dy}{dx} \right) - y(x \sin x + \cos x) = 1$$

$$\Rightarrow \frac{dy}{dx} + \frac{(x \sin x + \cos x)}{x \cos x} y = \frac{1}{x \cos x}$$

$$\Rightarrow \frac{dy}{dx} + \left(\tan x + \frac{1}{x} \right) y = \frac{\sec x}{x}$$

$$I.F. e^{\int (\tan x + 1/x) dx} = e^{\log \sec x + \log x} = e^{\log x \sec x} = x \sec x$$

Hence the required solution is

$$yx \sec x = \int \frac{\sec x}{x} x \sec x dx + c$$

$$\Rightarrow yx \sec x = \int \sec^2 x dx + c$$

$$\Rightarrow yx \sec x = \tan x + c \text{ which is required solution.}$$

Prob. 101. $\frac{dy}{dx} + \frac{y}{x} = x^2$

$$\text{Ans. } xy = \frac{x^4}{4} + C$$

Do yourself.

Ex. 102. Solve $(1-x^2) \left(\frac{dy}{dx} \right) + 2xy = x(\sqrt{1-x^2})$

Solⁿ: The given differential equation is

$$(1-x^2) \left(\frac{dy}{dx} \right) + 2xy = x(\sqrt{1-x^2})$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1-x^2} y = \frac{x}{\sqrt{1-x^2}}$$

$$\text{Here } \int Pdx = \int \frac{2x}{1-x^2} dx = -\log(1-x^2)$$

$$IF = e^{\int Pdx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

$\therefore$ the required solution is

$$\frac{y}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}} \times \frac{1}{1-x^2} dx$$

$$= -\frac{1}{2} \int t^{-3/2} dt + c$$

$$= t^{-1/2} + c = c + \frac{1}{\sqrt{t}}$$

$$[\text{Pt } 1-x^2 = t \quad \therefore -2x dx = dt]$$

$$\Rightarrow \frac{y}{1-x^2} = \frac{1}{\sqrt{(1-x^2)}} + c \quad [\because t = 1-x^2]$$

which is required solution.

Prob. 103. $(1+x) \frac{dy}{dx} - xy = (1-x)$

Ans. $y(1+x) = ce^x + x$

Do yourself.

Ex. 104. Solve $\sin\left(\frac{dy}{dx}\right) + 3y = \cos x$

Solⁿ: The given differential equation is

$$\begin{aligned} \sin\left(\frac{dy}{dx}\right) + 3y &= \cos x \\ \Rightarrow \frac{dy}{dx} + \frac{3}{\sin x} y &= \frac{\cos x}{\sin x} \\ \Rightarrow \frac{dy}{dx} + (3 \cot x) y &= \cot x. \end{aligned}$$

Here

$$IF = e^{\int P dx} = e^{\int 3 \cot x dx} = \tan^3 \frac{x}{2}$$

$\therefore$ the required solution is

$$\begin{aligned} y \tan^3 \left(\frac{x}{2}\right) &= \int \cot x \tan^3 \left(\frac{x}{2}\right) dx + c \\ \Rightarrow y \tan^3 \frac{x}{2} &= \int \frac{1 - \tan^2 x/2}{2 \tan x/2} \tan^3 \frac{x}{2} dx + c \\ \Rightarrow y \tan^3 \frac{x}{2} &= \frac{1}{2} \int (1 - \tan^2 \frac{x}{2}) \tan^2 \frac{x}{2} dx + c \\ \Rightarrow y \tan^3 \frac{x}{2} &= \int \frac{1}{2} (1 - t^2) t^2 \times \frac{2dt}{1+t^2} + c \\ &\quad \left[\text{Put } \tan \frac{x}{2} = t \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt \Rightarrow dx = \frac{2dt}{\sec^2 x/2} = \frac{2dt}{1 + \tan^2 x/2} = \frac{2dt}{1+t^2} \right] \\ \Rightarrow y \tan^3 \frac{x}{2} &= \int \frac{t^2 - t^4}{1+t^2} dt + c \\ \Rightarrow y \tan^3 \frac{x}{2} &= \int \left[-t^2 + 2 - \frac{2}{t^2+1} \right] dt + c \\ \Rightarrow y \tan^3 \left(\frac{x}{2}\right) &= -\left(\frac{1}{3}\right) t^3 + 2t - 2 \tan^{-1} t + c \\ \Rightarrow y \tan^3 \frac{x}{2} &= -\frac{1}{3} \tan^3 \frac{x}{2} + 2 \tan \frac{x}{2} - 2 \tan^{-1} \left(\tan \frac{x}{2}\right) + c \\ \Rightarrow \left(y + \frac{1}{3}\right) \tan^3 \left(\frac{x}{2}\right) &= 2 \tan \left(\frac{x}{2}\right) - x + c \text{ which is required solution.} \end{aligned}$$

Prob. 105. $\frac{dy}{dx} + \frac{2x}{(\sqrt{1-x^2})^2} y = x\sqrt{1-x^2}$

Ans. $y = c(1-x^2) + \sqrt{1-x^2}$

Do yourself.

Ex. 106. Integrate $(1+x^2) \left(\frac{dy}{dx}\right) + 2xy - 4x^2 = 0$. Obtain equation of the curve satisfying this equation and passing through the origin.

Solⁿ: The given differential equation is

$$\begin{aligned} (1+x^2) \left(\frac{dy}{dx}\right) + 2xy - 4x^2 &= 0 \\ \Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2} y &= \frac{4x^2}{1+x^2}. \end{aligned}$$

Here $\int P dx = \int \frac{2x}{1+x^2} dx = \log(1+x^2)$

$\therefore$ I.F. = $e^{\int P dx} = (1+x^2)$.

$\therefore$ the required solution is

$$\begin{aligned} y(1+x^2) &= \int \frac{4x^2}{1+x^2} (1+x^2) dx + c \\ \Rightarrow y(1+x^2) &= \left(\frac{4}{3}\right) x^3 + c \end{aligned}$$

(1)

Since the required curve passes through origin, (1) must satisfy the condition $x = 0, y = 0$.

Putting these in (1), we get $c = 0$.

$\therefore$ the required curve is

$4x^3 = 3y(1+x^2)$ which is required solution.

Prob. 107. $x(x-1) \frac{dy}{dx} - (x-2)y = x^3(2x-1)$

Ans. $\frac{y(x-1)}{x^2} = x^2 - x + c$

Do yourself.

Ex. 108. Solve $(x + 2y^3) \frac{dy}{dx} = y$

Solution. The given differential equation is

$$(x + 2y^3) \frac{dy}{dx} = y$$

$$\Rightarrow \frac{y dx}{dy} = x + 2y^3$$

$$\Rightarrow \frac{dx}{dy} = \frac{x + 2y^3}{y}$$

$$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

$$\text{IF} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

$\therefore$ the required solution is

$$\frac{x}{y} = \int 2y^2 \left(\frac{1}{y}\right) dy + c$$

$$\Rightarrow \frac{x}{y} = y^2 + c \text{ which is required solution.}$$

Prob. 109. $\frac{dy}{dx} + y \tan x = x^m \cos x$

Ans. $y \sec x = C + \left(\frac{x^{m+1}}{m+1}\right)$

Do yourself.

Ex. 110. Solve $(1 + y^2)dx = (\tan^{-1} y - x)dy$

Solution. The given differential equation is

$$(1 + y^2)dx = (\tan^{-1} y - x)dy$$

$$\Rightarrow (1 + y^2) \frac{dx}{dy} = \tan^{-1} y - x$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

Now

$$\text{IF} = e^{\int \frac{1}{(1+y^2)} dy} = e^{\tan^{-1} y}$$

Hence, the required solution is

$$xe^{\tan^{-1} y} = \int e^{\tan^{-1} y} \cdot \frac{\tan^{-1} y}{1 + y^2} dy + c$$

$$\Rightarrow xe^{\tan^{-1} y} = \int e^t \cdot t dt + c \quad \left[\text{Put } \tan^{-1} y = t \therefore \frac{dy}{1 + y^2} = dt \right]$$

$$\Rightarrow xe^{\tan^{-1} y} = te^t - e^t + c \quad [\text{Integrating by parts}]$$

$$\Rightarrow xe^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + c$$

$$\Rightarrow x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}, \text{ which is required solution.}$$

Prob. 111. $x(x^2 + 1) \frac{dy}{dx} = y(1 - x^2) + x^3 \log x$

Ans. $y(x^2 + 1) = ex + \frac{x^3}{2} \log x - \frac{x^3}{4}$

Do yourself.

Ex. 112. Solve $\frac{dy}{dx} + \frac{y}{(1 - x^2)^{3/2}} = \frac{x + \sqrt{(1 - x^2)}}{(1 - x^2)^2}$

Solution. The given differential equation is

$$\frac{dy}{dx} + \frac{y}{(1 - x^2)^{3/2}} = \frac{x + \sqrt{(1 - x^2)}}{(1 - x^2)^2}$$

$$\text{Now } \int P dx = \int \frac{1}{(1 - x^2)^{3/2}} dx = \int \frac{\cos \theta d\theta}{\cos^3 \theta} = \frac{x}{\sqrt{(1 - x^2)}} \quad (\text{putting } x = \sin \theta \Rightarrow dx = \cos \theta d\theta)$$

$$\text{I. F.} = e^{\int P dx} = e^{x/\sqrt{(1-x^2)^{1/2}}} \quad (2)$$

$\therefore$ The required solution is

$$y(\text{I. F.}) = \int Q(\text{I. F.}) dx + c \quad (3)$$

$$\text{Now, } \int Q(\text{I. F.}) dx = \int \frac{x + \sqrt{1 - x^2}}{(1 - x^2)^2} e^{x/\sqrt{(1-x^2)^{1/2}}} dx \quad (4)$$

$$\text{Let } \frac{x}{\sqrt{(1-x^2)}} = t \quad (5)$$

$$\Rightarrow \frac{\sqrt{(1-x^2)} \cdot 1 - x(1/2)(1-x^2)^{-1/2} \cdot (-2x)}{1-x^2} dx = dt$$

$$\Rightarrow \frac{\sqrt{(1-x^2)} + \left[x^2 / \sqrt{(1-x^2)} \right]}{1-x^2} dx = dt$$

$$\Rightarrow \frac{1}{(1-x^2)^{3/2}} dx = dt \quad (6)$$

∴ from (4), we have

$$\int Q(\text{I.F.}) dx = \int \frac{\left(x^2 / \sqrt{(1-x^2)} \right) + 1}{(1-x^2)^{3/2}} e^{x/(1-x^2)^{1/2}} dx$$

$$= \int (t+1)e^t \cdot dt, \quad (\text{using (5) and (6)})$$

$$\text{Thus, } = (t+1)e^t - \int e^t dt = (t+1)e^t - e^t = te^t$$

$$\int Q(\text{I.F.}) dx = \frac{x}{\sqrt{(1-x^2)}} e^{x/(1-x^2)^{1/2}} \quad (7)$$

Using (2) and (7) in (3), the required solution is

$$y e^{x/(1-x^2)^{1/2}} = \frac{x}{\sqrt{(1-x^2)}} e^{x/(1-x^2)^{1/2}} + c$$

$$\Rightarrow y = \frac{x}{\sqrt{(1-x^2)}} c e^{-x/(1-x^2)^{1/2}}$$

which is required solution.

Prob. 113. $\sqrt{a^2 + x^2} \frac{dy}{dx} + y = \sqrt{a^2 + x^2} - x$ **Ans.** $y \left[x + \sqrt{x^2 + a^2} \right] = c + a^2 \log \left[x + \sqrt{x^2 + a^2} \right]$

Do yourself.

Ex. 114. Solve $x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$

Solⁿ: The given differential equation is

$$x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$$

$$\Rightarrow x(1-x^2) \frac{dy}{dx} + y(2x^2 - 1) = ax^3$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x^2 - 1}{x(1-x^2)} y = \frac{ax^2}{1-x^2} \quad (1)$$

Compare (1) with $\left(\frac{dy}{dx}\right) + Py = Q$, we have

$$P = \frac{2x^2 - 1}{x(1-x^2)} = -\frac{1}{x} - \frac{1}{2(x+1)} - \frac{1}{2(x-1)}$$

and $Q = \frac{ax^2}{1-x^2}$. (2)

$$\text{Now } \int P dx = - \int \left[\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x-1)} \right] dx$$

$$= - \left[\log x + \frac{1}{2} \log(x+1) + \frac{1}{2} \log(x-1) \right]$$

$$= - \left[\log x + \frac{1}{2} \log(x^2 - 1) \right] = - \log \left[x \sqrt{(x^2 - 1)} \right]$$

$$= \log \left[x \sqrt{(x^2 - 1)} \right]^{-1}$$

$$\text{I.F.} = e^{\int P dx} = e^{\log \left[x \sqrt{(x^2 - 1)} \right]^{-1}} = \left[x \sqrt{(x^2 - 1)} \right]^{-1}$$

Hence, the solution of (1) is given by

$$y \cdot \text{I.F.} = \int Q(\text{I.F.}) dx + c$$

$$\Rightarrow \frac{y}{x \sqrt{(x^2 - 1)}} = \int \frac{ax^2}{1-x^2} \times \frac{1}{x \sqrt{(x^2 - 1)}} dx + c$$

$$= c - a \int \frac{xdx}{(x^2 - 1)^{3/2}}$$

$$= c - \frac{a}{2} \int \frac{dt}{t^{3/2}}, \quad (\text{putting } x^2 - 1 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{1}{2} dt)$$

$$= c - \frac{a}{2} \left[\frac{t^{-1/2}}{-1/2} \right] = c + \frac{a}{\sqrt{t}} = c + \frac{a}{\sqrt{(x^2 - 1)}}$$

$\Rightarrow y = cx\sqrt{(x^2 - 1)} + ax$ which is required solution.

Prob. 115. $(x + \log y)dx + ydx = 0$

Ans. $xy = e - y \log y + y$

Do yourself.

Ex. 116 An equation relating to stability of an aeroplane is $(dv/dt) = g \cos \alpha - kv$, where v is velocity, g, α, k being constants. Find an expression for the velocity if $v = 0$ when $t = 0$.

Solⁿ: The given equation can be rewritten as .

$$\left(\frac{dv}{dt} \right) + kv = g \cos \alpha \quad (1)$$

$\therefore$ The integrating factor $= e^{\int k dt} = e^{kt}$

$\therefore$ The required solution is

$$v \cdot e^{kt} = C + \int g e^{kt} \cos \alpha dt$$

$$\Rightarrow v e^{kt} = C + (g/k) \cos \alpha e^{kt} \quad (2)$$

Since $v = 0$ when $t = 0$, therefore from (2), we have

$$0 = C + (g/k) \cos \alpha e^0 \quad \text{or} \quad C = -(g/k) \cos \alpha$$

$$\Rightarrow v = -(g/k) \cos \alpha e^{-kt} + (g/k) \cos \alpha$$

$$\Rightarrow v = (g/k) \cos \alpha (1 - e^{-kt}) \text{ which is required solution.}$$

Ex. 117. Solve $x^2 \left(\frac{dy}{dx} \right) + 2xy = 1$

Solⁿ: The given differential equation is

$$x^2 \left(\frac{dy}{dx} \right) + 2xy = 1$$

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2} \quad (1)$$

$$\therefore \text{IF} = e^{\int (2/x) dx} = e^{2 \log x} = e^{\log x^2} = x^2$$

$\therefore$ the required solution is

$$y \cdot (\text{IF}) = c + \int Q \cdot (\text{IF}) dx$$

$$\Rightarrow y \cdot x^2 = C + \int x^2 (1/x^2) dx,$$

$$\Rightarrow y \cdot x^2 = C + \int dx = C + x.$$

$$\Rightarrow y \cdot x^2 = C + x \text{ which is required solution.}$$

Ex. 118. Solve $\cos^3 x \left(\frac{dy}{dx} \right) + y \cos x = \sin x$

Solⁿ: The given differential equation is

$$\cos^3 x \left(\frac{dy}{dx} \right) + y \cos x = \sin x$$

$$\Rightarrow \frac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x \quad (1)$$

$\therefore$ the required solution is

$$\Rightarrow y \cdot e^{\tan x} = C + \int \tan x \sec^2 x e^{\tan x} dx$$

$$= C + \int t e^t dt \text{ where } t = \tan x$$

$$= C + \left[t e^t - \int e^t dt \right]$$

$$= C + t e^t - e^t$$

$$= C + (t - 1) e^t = C + (\tan x - 1) e^{\tan x}$$

$$\Rightarrow y e^{\tan x} = C + (\tan x - 1) e^{\tan x} \text{ which is required solution.}$$

Ex. 119 Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

Solⁿ: The given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \frac{dy}{dx} + (\sec^2 x) y = \sec^2 x \tan x \quad (1)$$

Now proceed as in last problem

Ex. 120. Solve $x \left(\frac{dy}{dx} \right) - y = 2x^2 \operatorname{cosec} 2x$

Solⁿ: The given differential equation is

$$x \left(\frac{dy}{dx} \right) - y = 2x^2 \operatorname{cosec} 2x$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = 2x \operatorname{cosec} 2x \quad (1)$$

$$\therefore \text{IF} = e^{-\int (1/x) dx} = e^{-\log x} = \frac{1}{x}$$

$\therefore$ the required solution is

$$y(\text{IF}) = C + \int Q.(\text{IF}) dx$$

$$\Rightarrow y \cdot \frac{1}{x} = C + \int \left(2x \operatorname{cosec} 2x \cdot \frac{1}{x} \right) dx$$

$$\Rightarrow y \left(\frac{1}{x} \right) = C + 2 \int \csc 2x dx$$

$$\Rightarrow y \left(\frac{1}{x} \right) = C + \log(\tan x) \text{ which is required solution.}$$

Ex. 121. Solve $(1-x^2) \left(\frac{dy}{dx} \right) - xy = 1$

Solⁿ: The given differential equation is

$$(1-x^2) \left(\frac{dy}{dx} \right) - xy = 1$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{1-x^2} \right) y = \frac{1}{1-x^2} \quad (1)$$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \frac{-x}{1-x^2} dx} = e^{\frac{1}{2} \log(1-x^2)} = \sqrt{1-x^2}$$

$\therefore$ the required solution is

$$y(\text{IF}) = c + \int Q.(\text{IF}) dx$$

$$\Rightarrow y\sqrt{1-x^2} = c + \int \frac{1}{1-x^2} \sqrt{1-x^2} dx$$

$$y\sqrt{1-x^2} = c + \int \frac{1}{\sqrt{1-x^2}} dx$$

$$y\sqrt{1-x^2} = c + \sin^{-1} x \text{ which is required solution.}$$

Ex. 122. Solve $\left(\frac{dy}{dx} \right) + 2xy = x, \quad y(0) = 2$

Solⁿ: The given differential equation is

$$\left(\frac{dy}{dx} \right) + 2xy = x, \quad y(0) = 2$$

$$\therefore \text{IF} = e^{\int 2x dx} = e^{x^2}$$

$\therefore$ the required solution is

$$y(\text{IF}) = C + \int Q.(\text{IF}) dx$$

$$\Rightarrow y \cdot e^{x^2} = C + \int x e^{x^2} dx = C + \frac{1}{2} \int e^t dt$$

$$\Rightarrow y e^{x^2} = C + \frac{1}{2} e^t = C + \frac{1}{2} e^{x^2} \quad (\text{let } x^2 = t \Rightarrow 2x dx = dt)$$

$$\Rightarrow 2y = 2C e^{-x^2} + 1 \quad (1)$$

Now, when $x = 0, y = 2$

$\therefore$ from (1), we get

$$2(2) = 2C e^0 + 1 \Rightarrow 3 = 2C$$

$\therefore$ from (1) the required solution is

$$2y = 3e^{-x^2} + 1 \text{ which is required solution.}$$

Ex. 123 Obtain the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin.

Solⁿ: The differential equation of the curve is

$$\frac{dy}{dx} = y + 2x \quad \left[\because \text{slope of curve is } \frac{dy}{dx} \right]$$

$$\Rightarrow \frac{dy}{dx} - y = 2x$$

$\therefore$ the required solution is

$$y(\text{IF}) = C + \int Q.(\text{IF}) dx$$

$$\begin{aligned}
\Rightarrow y \cdot e^{-x} &= C + \int 2xe^{-x} dx \\
&= C + 2 \left[x \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1} dx \right], \quad (\text{integrating by parts}) \\
\Rightarrow ye^{-x} &= C - 2xe^{-x} + 2 \int e^{-x} dx = C - 2xe^{-x} - 2e^{-x} \\
\Rightarrow y &= Ce^x - 2x - 2 \quad (2)
\end{aligned}$$

Since, (2) passes through the origin (0,0),
 $\therefore 0 = C - 2 \Rightarrow C = 2$
The required solution is
 $y = 2e^x - 2x - 2$.

8(A). EQUATIONS REDUCIBLE TO LINEAR FORM

An equation of the form

$$f'(y) \frac{dy}{dx} = Pf(y) - Q \quad (1)$$

where P and Q are constants or function of x alone (and not of y) can be reduced to linear form as follows.

Put $f(y) = v$

$$\Rightarrow f'(y) \frac{dy}{dx} = \frac{dv}{dx}$$

Finally, replace v by f(y) to get solution in terms of x and y alone.

Another form: An equation of the form

$$f'(x) \frac{dx}{dy} + P_1 f(x) = Q_1 \quad (1)$$

where P_1 and Q_1 are constants or function of y alone can be reduced to linear form again as follows.Put $f(x) = v$

$$\Rightarrow f'(x) \left(\frac{dx}{dy} \right) = \frac{dv}{dy}$$

 $\therefore$ (1) gives

$$\frac{dv}{dy} + P_1 v = Q_1 \quad (2)$$

which is linear in variables v and y.

$$\text{I. F.} = e^{\int P_1 dy}$$

 $\therefore$ the required solution is

$$ve^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} dx + c.$$

Replace v by f(x), we obtain the required solution of (1)

8(B). BERNOULLI'S EQN.

An equation of the form

$$\left(\frac{dy}{dx} \right) + Py = Qy^n \quad (n \neq 0, \quad n \neq 1) \quad (3)$$

is called Bernoulli's equation where P and Q are constants or function of x alone and not of y. Divided by y^n , we get

$$y^{-n} \frac{dy}{dx} + Py^{1-n} = Q$$

$$\text{Let } y^{1-n} = v \quad (4)$$

$$\Rightarrow (1-n)y^{-n} \frac{dy}{dx} = \frac{dv}{dx} \quad (5)$$

$$\Rightarrow y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dv}{dx} \quad (6)$$

From (4), (5) and (6), we get

$$\frac{1}{1-n} \frac{dv}{dx} + Pv = Q$$

$$\Rightarrow \frac{dv}{dx} + P(1-n)v = Q(1-n)$$

which is linear in v and x.

$$\text{I. F.} = e^{\int P(1-n)dx} = e^{(1-n) \int P dx}$$

and hence the required solution is

$$v \cdot e^{(1-n) \int P dx} = \int Q \cdot e^{(1-n) \int P dx} dx + c$$

$$\Rightarrow y^{1-n} e^{(1-n) \int P dx} = \int Q \cdot e^{(1-n) \int P dx} dx + c$$

Remark: The differential equation

$$\frac{dx}{dy} + P_1 x = Q_1 x^n \quad (n \neq 0, \quad n \neq 1)$$

is called Bernoulli's equation, where P_1 and Q_1 both are function of y (alone) or constraint.

Ex. 124. Solve $\left(\frac{dy}{dx}\right) + x \sin 2y = x^3 \cos^2 y$

Solⁿ: The given differential equation is

$$\left(\frac{dy}{dx}\right) + x \sin 2y = x^3 \cos^2 y \quad (1)$$

$$\Rightarrow \sec^2 y \left(\frac{dy}{dx}\right) + x \left(\frac{\sin 2y}{\cos^2 y}\right) = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x(\tan y) = x^3 \quad (2)$$

Put $\tan y = v$

$$\Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dv}{dx} \quad (3)$$

Therefore, from equation (1), (2) & (3)

$$\frac{dv}{dx} + 2xv = x^3,$$

which is linear in v and x .

Now I. F. = $e^{\int 2x dx} = e^{x^2}$

The solution is given by

$$\begin{aligned} v \cdot e^{x^2} &= \int x^3 e^{x^2} dx + c \\ &= \frac{1}{2} \int t e^t dt + c \quad [\text{Put } x^2 = t \therefore 2x dx = dt] \\ &= \frac{1}{2} \int [t e^t - \int 1 \cdot e^t dt] + c \\ &= \left(\frac{1}{2}\right)(t \cdot e^t - e^t) + c \end{aligned}$$

$$\Rightarrow \tan y \cdot e^{x^2} = \left(\frac{1}{2}\right)e^{x^2}(x^2 - 1) + c \quad [\because v = \tan y \text{ and } t = x^2]$$

$$\Rightarrow \tan y = \left(\frac{1}{2}\right)(x^2 - 1) + c e^{-x^2}$$

which is required solution.

Prob. 125. $y^2 \left(\frac{dy}{dx}\right) = x + y^3$

Ans. $y^3 e^{-3x} = c - x e^{-3x} - \frac{1}{3} e^{-3x}$

Do yourself.

Ex. 126. Solve $\left(\frac{dy}{dx}\right) = e^{x-y}(e^x - e^y)$

Solⁿ: The given differential equation is

$$\left(\frac{dy}{dx}\right) = e^{x-y}(e^x - e^y)$$

$$\Rightarrow \frac{dy}{dx} = e^{2x} \cdot e^{-y} - e^x \Rightarrow \frac{dy}{dx} + e^x = e^{2x} \cdot e^{-y}$$

$$\Rightarrow e^y \frac{dy}{dx} + e^x \cdot e^y = e^{2x}$$

$$\Rightarrow \frac{dv}{dx} + e^x v = e^{2x}. \quad \left(\text{Let } e^y = v \Rightarrow e^y \frac{dy}{dx} = \frac{dv}{dx}\right)$$

$$\therefore \text{I. F.} = e^{\int P dx} = e^{\int e^x dx} = e^{e^x}$$

The required solution is

$$v \cdot (\text{IF}) = \int Q(\text{IF}) dx$$

$$\begin{aligned} v \cdot e^{e^x} &= \int e^{2x} \cdot e^{e^x} dx + c = \int e^x e^{e^x} \cdot e^x dx + c \\ &= \int t e^t dt + c \quad [\text{Put } e^x = t \text{ so that } e^x dx = dt] \end{aligned}$$

$$= t e^t - \int 1 \cdot e^t dt + c$$

$$= t \cdot e^t - e^t + c = e^t(t - 1) + c$$

$$\Rightarrow e^y e^{e^x} = e^{e^x}(e^x - 1) + c \quad [\because v = e^y \text{ and } t = e^x]$$

$$\Rightarrow e^y = e^x - 1 + c e^{-e^x} \text{ which is required solution.}$$

Prob. 127 $y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx}\right)$

Ans. $x + a = y(c + ax)$

Do yourself.

Ex. 128. Solve $\frac{dz}{dx} + \frac{z}{x} \log z = \frac{z}{x^2} \cdot (\log z)^2$

Solution. The given differential equation is

$$\frac{dz}{dx} + \frac{z}{x} \log z = \frac{z}{x^2} \cdot (\log z)^2$$

$$\Rightarrow \frac{1}{z(\log z)^2} \frac{dz}{dx} + \frac{1}{x(\log z)} = \frac{1}{x^2}$$

$$\text{Let } \frac{1}{\log z} = v$$

$$\Rightarrow \frac{(-1)}{(\log z)^2 z} \frac{dz}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{1}{z(\log z)^2} \frac{dz}{dx} = -\frac{dv}{dx}$$

$\therefore$ from (1)

$$-\frac{dv}{dx} + \frac{1}{x} v = \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx} - \frac{1}{x} v = -\frac{1}{x^2}$$

$$\text{now IF.} = e^{-\int (1/x) dx} = e^{-\log x} = 1/x$$

$\therefore$ the required solution is

$$v(\text{IF}) = \int Q(\text{IF}) dx$$

$$\Rightarrow \frac{v}{x} = \int \left(-\frac{1}{x^3} \right) dx + c = \frac{1}{2x^2} + c$$

$$\Rightarrow \frac{1}{x(\log z)} = \frac{1}{2x^2} + c \text{ which is required solution.}$$

Prob. 129. $(1+x^2) \frac{dy}{dx} = xy - y^2$

Ans. $\sqrt{1+x^2} = cy + y \sinh^{-1} x$

Do yourself.

Ex. 130. $x \left(\frac{dy}{dx} \right) + y \log y = xye^x$

Solution. The given differential equation is

$$x \left(\frac{dy}{dx} \right) + y \log y = xye^x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} + \frac{1}{x} \log y = e^x$$

$$\text{Let } \log y = v$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{dv}{dx}$$

Using (2), (1) gives

$$\frac{dv}{dx} + \frac{1}{x} v = e^x$$

which is linear in v & x

$$\text{Now IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

The required solution is

$$v(\text{IF}) = \int Q(\text{IF}) dx + c$$

$$\Rightarrow vx = \int xe^x dx + c$$

$$\Rightarrow vx = xe^x - \int e^x dx = xe^x - e^x + c$$

$$\Rightarrow x \log y = e^x(x-1) + c, \quad [\text{by (2)}]$$

which is required solution.

Ex. 131. Solve $(x^2 - 2x + 2y^2)dx + 2xydy = 0$

Solution. The given differential equation is

$$(x^2 - 2x + 2y^2)dx + 2xydy = 0$$

$$\Rightarrow 2xy \frac{dy}{dx} + x^2 - 2x + 2y^2 = 0$$

$$\Rightarrow 2y \frac{dy}{dx} + \frac{x^2 - 2x}{x} + \frac{2y^2}{x} = 0$$

$$\Rightarrow 2y \frac{dy}{dx} + \frac{2}{x} y^2 = \frac{2x - x^2}{x}$$

$$\text{Let } y^2 = v \Rightarrow 2y \frac{dy}{dx} = \frac{dv}{dx}$$

from equation (1)

$$\frac{dv}{dx} + \frac{2}{x}v = \frac{2x - x^2}{x}$$

which is linear in v & x

we have

$$\text{Now IF} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = x^2$$

Therefore, the required solution is

$$v(\text{IF}) = \int Q(\text{IF})dx + c$$

$$\Rightarrow vx^2 = \int \left(\frac{2x - x^2}{x} \right) x^2 dx$$

$$\Rightarrow y^2 x^2 = \int \left(\frac{2x - x^2}{x} \right) x^2 dx = \int (2x^2 - x^3) dx$$

$$\Rightarrow y^2 x^2 = \left(\frac{2}{3} \right) x^3 - \left(\frac{1}{4} \right) x^4 + c \text{ which is required solution.}$$

Prob. 132. $(x^2 y^2 + xy)dx = dy$

Do yourself.

Ex. 133. (a) Solve $2xydy - (x^2 + y^2 + 1)dx = 0$

(b) Solve $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$, given $y = 1$ when $x = 1$.

Solution. (a) The given differential equation is

$$2xydy - (x^2 + y^2 + 1)dx = 0$$

$$\Rightarrow 2xy \frac{dy}{dx} = x^2 + y^2 + 1$$

$$\Rightarrow 2xy \frac{dy}{dx} - y^2 = 1 + x^2$$

$$\Rightarrow 2y \frac{dy}{dx} - \frac{1}{x} y^2 = \left(\frac{1}{x} + x \right)$$

Putting $y^2 = v$

$$\Rightarrow 2y \left(\frac{dy}{dx} \right) = \frac{dv}{dx}$$

Therefore, from equation (1)

$$\frac{dv}{dx} - \frac{1}{x} v = \frac{1}{x} + x$$

which is linear in v & x

$$\text{Now I. F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

Therefore, the required solution is

$$v \cdot (\text{I. F.}) = \int Q \cdot (\text{I. F.}) dx + c$$

$$\Rightarrow y^2 x^{-1} = \int \left(x + \frac{1}{x} \right) x^{-1} dx = \int (1 + x^{-2}) dx$$

$$\Rightarrow y^2 x^{-1} = x - x^{-1} + c \text{ or } y^2 = (c + x)x - 1$$

which is required solution.

(b) Proceed as in part (1) and obtain

$$y^2 = (c + x)x - 1$$

Given $y = 1$, where $x = 1$

So (6) gives

$$1 = c + 1 - \text{or } c = 1$$

Hence, from (6), the required solution is

$$y^2 = x(x + 1) - 1.$$

which is required solution.

Prob. 134. $\frac{dy}{dx} = x^3 y^3 - xy$

Do yourself.

Ex. 135. Solve $x \left(\frac{dy}{dx} \right) + y = y^2 \log x$

Solution. The given differential equation is

$$x \left(\frac{dy}{dx} \right) + y = y^2 \log x$$

$$\Rightarrow y^{-2} \frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x} \log x$$

$$\text{Ans. 1} = 2y \left(1 - \frac{1}{2} x^2 \right) - c y e^{-\frac{1}{2} x^2}$$

(3)

(1)

(2)

(3)

(5)

(6)

(1)

$$\text{Let } y^{-1} = v$$

$$\Rightarrow -y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$$

Therefore, from equation (1)

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x} \log x$$

$$\Rightarrow \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x} \log x$$

which is linear in v & x

$$\text{Now IF} = e^{\int P dx} = e^{-\int (1/x) dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$$

The required solution is

$$v(\text{IF}) = \int Q(\text{IF}) dx + c$$

$$\Rightarrow vx^{-1} = -\int x^{-2} \log x dx + c$$

$$\Rightarrow y^{-1}x^{-1} = -\left[\log x \times \frac{x^{-1}}{-1} - \int \frac{1}{x} \times \frac{x^{-1}}{-1} dx\right] + c$$

$$\Rightarrow \frac{1}{y} = \log x + 1 + cx \text{ which is required solution.}$$

$$\text{Prob. 136. } \frac{dy}{dx} + 2y \tan x = y^2$$

Do yourself.

$$\text{Ex. 137. Solve } \frac{dy}{dx} - y \tan x = -y^2 \sec x$$

Solution. The given differential equation is

$$\frac{dy}{dx} - y \tan x = -y^2 \sec x$$

$$\Rightarrow y^{-2} \frac{dy}{dx} - y^{-1} \tan x = -\sec x$$

$$\text{Let } y^{-1} = v \Rightarrow -y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$$

Therefore, from equation (1)

$$-\frac{dv}{dx} - \tan x \cdot v = -\sec x$$

$$\Rightarrow \frac{dv}{dx} + \tan x \cdot v = \sec x$$

which is linear in v & x

$$\text{I. F.} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

Hence solution of (2) is

$$v \cdot \sec x = \int \sec x \cdot \sec x dx + c$$

$$\Rightarrow v \sec x = \tan x + c$$

$$\Rightarrow y^{-1} \sec x = \tan x + c \quad [\because v = y^{-1}] \text{ which is required solution.}$$

$$\text{Prob. 138. } \cos x dy = (\sin x - y) y dx$$

Do yourself.

Ex. 139. Solve the following differential equations :

$$(a) (x^3 y^2 + xy) dx = dy$$

$$(b) x^3 \left(\frac{dy}{dx} \right) - x^2 y + y^4 \cos x = 0$$

$$(c) 2x^2 \left(\frac{dy}{dx} \right) = xy = y^2$$

$$(d) x \left(\frac{dy}{dx} \right) + y^2 x = y$$

Solution (a) The given differential equation is

$$(x^3 y^2 + xy) dx = dy$$

$$\Rightarrow \frac{dy}{dx} = x^3 y^2 + xy$$

$$\Rightarrow \frac{dy}{dx} - xy = x^3 y^2$$

$$\Rightarrow y^{-2} \frac{dy}{dx} - xy^{-1} = x^3$$

$$\text{Let } y^{-1} = v \Rightarrow -y^{-2} \left(\frac{dy}{dx} \right) = \frac{dv}{dx}$$

(1) gives

$$\text{Ans. } \cos^2 x + cy + \frac{1}{2}xy + \frac{1}{4}y \sin 2x = 0$$

$$-\left(\frac{dv}{dx}\right) - xv = x^3$$

$$\Rightarrow \left(\frac{dv}{dx}\right) + xv = -x^3 \text{ which is linear in } v \text{ \& } x$$

$$\text{now I. F.} = e^{\int x dx} = e^{x^2/2}.$$

∴ The required Solution is

$$v(\text{IF}) = \int Q(\text{IF})dx + c$$

$$\Rightarrow ve^{x^2/2} = -\int x^3 e^{x^2/2} + c$$

$$\Rightarrow ve^{x^2/2} = -\int x^2 e^{x^2/2} \cdot x dx + c$$

$$\Rightarrow ve^{x^2/2} = -2 \int te^t dt = -2 \left[te^t - \int e^t dt \right] + c \quad \left(\text{Let } \frac{x^2}{2} = t \Rightarrow x dx = dt \right)$$

$$= -2(te^t - e^t) + c = -2e^t(t - 1) + c$$

$$y^{-1}e^{x^2/2} = -2e^{x^2/2} \left\{ (x^2/2) - 1 \right\} + c$$

$$\Rightarrow y^{-1} = (2 - x^2) + ce^{-x^2/2} \text{ which is required solution.}$$

(b) The given differential equation is

$$x^3 \left(\frac{dy}{dx} \right) - x^2 y = -y^4 \cos x$$

$$\Rightarrow x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$$

$$\Rightarrow y^{-4} \frac{dy}{dx} - \frac{1}{x} y^{-3} = -\frac{\cos x}{x^3}$$

$$\text{Putting } y^{-3} = v$$

$$\Rightarrow -3y^{-4} \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow y^{-4} \frac{dy}{dx} = -\frac{1}{3} \frac{dv}{dx}$$

Therefore, from equation (1)

$$-\frac{1}{3} \frac{dv}{dx} - \frac{1}{x} v = -\frac{\cos x}{x^3}$$

$$\Rightarrow \frac{dv}{dx} + \frac{3}{x} v = \frac{3 \cos x}{x^3}$$

Which is linear in v and x

$$\text{Now I. F.} = e^{\int (3/x) dx} = e^{3 \log x} = x^3$$

the required solution is

$$vx^3 = \int \left(\frac{3 \cos x}{x^3} \right) x^3 dx$$

$$\Rightarrow vx^3 = 3 \int \cos x dx$$

$$\Rightarrow vx^3 = 3 \sin x + C$$

$$\Rightarrow \frac{1}{y^3} x^3 = 3 \sin x + C$$

$$\Rightarrow x^3 = y^3 (3 \sin x + C) \text{ Which is required solution.}$$

(c) Do yourself.

(d) Do yourself

$$\text{Prob. 140. } \frac{dy}{dx} - 4xy = 16xe^{3x^2}y^2$$

Do yourself.

$$\text{Ans. } e^{2x^2} + y \left(C + \frac{8}{5} e^{5x^2} \right)$$

$$\text{Ex. 141. Solve } \left(\frac{dy}{dx} \right) (x^2 y^3 + xy) = 1$$

Solution. The given differential equation is

$$\left(\frac{dy}{dx} \right) (x^2 y^3 + xy) = 1$$

$$\Rightarrow \frac{dx}{dy} = x^2 y^3 + xy \Rightarrow \frac{dx}{dy} - yx = y^3 x^2$$

$$\Rightarrow x^{-2} \frac{dx}{dy} - x^{-1} y = y^3$$

$$\text{Let, } x^{-1} = v \Rightarrow -x^{-2} \frac{dx}{dy} = \frac{dv}{dy}$$

Therefore, from equation (1)

$$-\frac{dv}{dy} - yv = y^3, \Rightarrow \frac{dv}{dy} + yv = -y^3,$$

Which is linear in v and y

(2)

(1)

(1)

$$\text{Now IF} = e^{\int y dy} = e^{y^2/2}$$

the required solution is

$$v \cdot e^{y^2/2} = - \int y^3 e^{y^2/2} dy + c$$

$$= -2 \int t e^t dt + c \quad \left[\text{Put } \frac{y^2}{2} = t \quad \therefore y dy = dt \right]$$

$$= -2 \left[t \cdot e^t - \int 1 \cdot e^t dt + c \right] = -2(t e^t - e^t) + c$$

$$\Rightarrow x^{-1} e^{y^2/2} = -2 e^{y^2/2} \left(\frac{y^2}{2} - 1 \right) + c \quad [\because v = x^{-1}]$$

$$\frac{1}{x} = 2 - y^2 + c e^{-y^2/2} \text{ which is required solution.}$$

Ex. 142. Solve $(x^2 + y^2 + 2x)dx + 2ydy = 0$

Solution. The given differential equation is

$$(x^2 + y^2 + 2x)dx + 2ydy = 0$$

$$\Rightarrow 2y \frac{dy}{dx} + y^2 + (x^2 + 2x) = 0$$

$$\Rightarrow \frac{dv}{dx} + v + (x^2 + 2x) = 0, \quad \left(\text{let } y^2 = v \Rightarrow 2y \frac{dy}{dx} = \frac{dv}{dx} \right)$$

$$\Rightarrow \left(\frac{dv}{dx} \right) + v = -(x^2 + 2x) \quad (1)$$

which is a linear equation in v & x ,

$$\therefore \text{IF} = e^{\int dx} = e^x$$

$\therefore$ The required solution is

$$\Rightarrow v \cdot e^x = C - \int (x^2 + 2x) e^x dx,$$

$$\Rightarrow v \cdot e^x = C - \left[(x^2 + 2x) e^x - \int (2x + 2) e^x dx \right]$$

$$= C - (x^2 + 2x) e^x + \int (2x + 2) e^x dx$$

$$= C - (x^2 + 2x) e^x + \int (2x + 2) e^x dx$$

$$= C - (x^2 + 2x) e^x + \left[(2x + 2) e^x - \int 2e^x dx \right]$$

$$= C - (x^2 + 2x) e^x + (2x + 2) e^x - 2e^x$$

$$\Rightarrow y^2 e^x = C - e^x (x^2), \quad (\text{since } v = y^2)$$

$$\Rightarrow y^2 = C e^{-x} - x^2 \text{ which is required solution.}$$

Ex. 143. Solve $\tan y \left(\frac{dy}{dx} \right) + \tan x = \cos y \cos^3 x$

Solⁿ: Dividing both sides of the given equation by $\cos y$, we get

$$\tan y \sec y \left(\frac{dy}{dx} \right) + \sec y \tan x = \cos^3 x$$

$$\Rightarrow \left(\frac{dv}{dx} \right) + (\tan x) v = \cos^3 x, \quad \left(\text{Let } \sec y = v \Rightarrow \sec y \tan y \frac{dy}{dx} = \frac{dv}{dx} \right) \quad (1)$$

which is linear equation in v & x ,

$$\text{Now IF} = e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

$\therefore$ the required solution is

$$v \cdot (\text{IF}) = C + \int Q(\text{IF}) dx$$

$$\Rightarrow v \cdot \sec x = C + \int \cos^3 x \cdot \sec x dx$$

$$\Rightarrow v \cdot \sec x = C + \frac{1}{2} \int (1 + \cos 2x) dx = C + \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right)$$

$$\Rightarrow \sec y \sec x = C + \frac{1}{2} x + \frac{1}{4} \sin 2x \text{ which is required solution.}$$

Ex. 144. $\sec^2 y \left(\frac{dy}{dx} \right) + \tan y = x^3$

$$\text{Solⁿ: Let } \tan y = v \Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ the given equation becomes

$$\left(\frac{dv}{dx} \right) + v = x^3$$

(1)

which is a linear equation in v & x .

Now $IF = e^{\int dx} = e^x$

the required solution is

$$\begin{aligned} v \cdot (IF) &= C + \int Q(IF) dx \\ ve^x &= C + \int x^3 e^x dx, \\ &= C + \left[x^3 e^x - \int 3x^2 e^x dx \right], \end{aligned}$$

$$\Rightarrow ve^x = C + x^3 e^x - 3 \left[x^2 e^x - \int 2x e^x dx \right],$$

$$\Rightarrow (\tan y e^x) = C + x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x$$

$$\Rightarrow \tan y = Ce^{-x} + x^3 - 3x^2 + 6x - 6 \text{ which is required solution.}$$

Ex. 145. Solve $\sec^2 y \left(\frac{dy}{dx} \right) + 2x \tan y = x^2$. Also find the particular curve which passes through $\left(0, \frac{1}{4}\pi \right)$

Solⁿ: The given differential equation is

$$\sec^2 y \left(\frac{dy}{dx} \right) + 2x \tan y = x^2.$$

$$\text{Let } \tan y = v \Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

the given equation reduces to

$$\left(\frac{dv}{dx} \right) + 2xv = x^3$$

which is a linear equation in v and x

$$\text{Now } IF = e^{\int 2x dx} = e^{x^2}$$

the required solution is

$$\begin{aligned} v \cdot (IF) &= C + \int Q(IF) dx \\ \Rightarrow v e^{x^2} &= C + \int x^3 e^{x^2} dx, \\ \Rightarrow v e^{x^2} &= C + \frac{1}{2} \int x^2 e^{x^2} \cdot 2x dx \\ &= C + \frac{1}{2} \int t e^t dt, \quad (\text{Let } x^2 = t \Rightarrow 2x dx = dt) \\ &= C + \frac{1}{2} \left[t e^t - \int 1 \cdot e^t dt \right], \\ &= C + \frac{1}{2} [t e^t - e^t] = C + \frac{1}{2} (t - 1) e^t \\ \Rightarrow (\tan y) e^{x^2} &= C + \frac{1}{2} (x^2 - 1) e^{x^2}, \\ \Rightarrow \tan y &= C e^{-x^2} + \frac{1}{2} (x^2 - 1) \end{aligned} \quad (2)$$

which is the required solution.

Hence, the (2) passes through $\left(0, \frac{1}{4}\pi \right)$,

when putting $x = 0$ and $y = \frac{1}{4}\pi$ in (2) we get

$$\tan \left(\frac{1}{4}\pi \right) = C e^0 + \frac{1}{2} (0 - 1) \Rightarrow 1 = C - \frac{1}{2} \Rightarrow C = \frac{3}{2}$$

From (2), the required curve is

$$\tan y = \left(\frac{3}{2} \right) e^{-x^2} + \frac{1}{2} (x^2 - 1)$$

Ex. 146. Solve $x \left(\frac{dy}{dx} \right) + y \log y = x y e^x$

Solution. The given differential equation is

$$x \left(\frac{dy}{dx} \right) + y \log y = x y e^x$$

$$\Rightarrow \frac{x dy}{y dx} + (\log y) = x e^x$$

$$\Rightarrow \frac{1 dy}{y dx} + \frac{1}{x} (\log y) = e^x$$

$$\text{Let } v = \log y \Rightarrow \frac{1 dy}{y dx} = \frac{dv}{dx}$$

the above equation reduces to

$$\frac{dv}{dx} + \frac{1}{x}v = e^x, \quad (1)$$

which is a linear equation in v & x .

$$\text{Now IF} = e^{\int (1/x) dx} = e^{\log x} = x$$

$\therefore$ the required solution is

$$vx = C + \int x \cdot e^x dx,$$

$$\Rightarrow v \cdot x = C + xe^x - \int 1 \cdot e^x dx$$

$$\Rightarrow (\log y)x = C + xe^x - e^x. \text{ which is required solution.}$$

Ex. 147. Solve $y(2xy + e^x)dx - e^x dy = 0$

Solution. The given differential equation is

$$y(2xy + e^x)dx - e^x dy = 0$$

$$\Rightarrow e^x \frac{dy}{dx} = e^x y + 2xy^2$$

$$\Rightarrow \frac{dy}{dx} - y = 2xe^{-x}y^2$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} - \frac{1}{y} = 2xe^{-x}$$

$$\Rightarrow \left(\frac{dv}{dx} \right) + v = 2xe^{-x}, \quad \left(\text{where } v = -\left(\frac{1}{y} \right) \right)$$

which is linear equation in v & x ,

$$\text{Now IF} = e^{\int dx} = e^x$$

$\therefore$ the required solution is

$$v \cdot (IF) = C + \int Q(IF) dx$$

$$\Rightarrow v \cdot e^x = C + \int 2xe^{-x} e^x dx,$$

$$\Rightarrow v \cdot e^x = C + \int 2x dx$$

$$\Rightarrow v \cdot e^x = C + x^2$$

$$\Rightarrow -\left(\frac{1}{y} \right) e^x = C + x^2 \Rightarrow e^x + y(C + x^2) = 0 \text{ which is required solution.}$$

Ex. 148. Solve $(x^3 e^x - my^2)dx + mxydy = 0$

Solution. The given differential equation is

$$(x^3 e^x - my^2)dx + mxydy = 0$$

$$\Rightarrow mxy \frac{dy}{dx} - my^2 = -x^3 e^x$$

$$\Rightarrow y \frac{dy}{dx} - \frac{y^2}{x} = \frac{-x^2 e^x}{m}$$

$$\text{Let } y^2 = v \Rightarrow 2y \frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ (1) reduces to

$$\frac{1}{2} \frac{dv}{dx} - \frac{v}{x} = -\frac{x^2 e^x}{m}$$

$$\Rightarrow \frac{dv}{dx} - \frac{2}{x}v = -\frac{2x^2 e^x}{m}$$

which is a linear equation in v & x .

$$\text{Now IF} = e^{-\int (2/x) dx} = e^{-2 \log x} = x^{-2}$$

$\therefore$ the required solution is

$$v \cdot (IF) = C + \int Q(IF) dx$$

$$\frac{v}{x^2} = C - \int \frac{2e^x}{m} dx,$$

$$\Rightarrow \frac{y^2}{x^2} = C - \frac{2}{m} e^x \text{ which is required solution.}$$

Ex. 149. Solve $\left(\frac{dy}{dx} \right) + x \sin 2y = x^3 \cos^2 y$

Solution: The given differential equation is

$$\left(\frac{dy}{dx} \right) + x \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \sec^2 y \left(\frac{dy}{dx} \right) + x(2 \tan y) = x^3$$

$$\text{Let } \tan y = v \Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

The above equation reduces to

$$\frac{dv}{dx} + 2xv = x^3$$

which is linear equation in v & x ,

$$\text{Now IF} = e^{\int 2x dx} = e^{x^2}$$

$\therefore$ the required solution is

$$v \cdot (\text{IF}) = C + \int Q(\text{IF}) dx$$

$$\Rightarrow v \cdot e^{x^2} = C + \int x^3 e^{x^2} dx,$$

$$\Rightarrow v \cdot e^{x^2} = C + \frac{1}{2} \int x^2 e^{x^2} 2x dx$$

$$= C + \frac{1}{2} \int t \cdot e^t dt, \quad (\text{where } t = x^2)$$

$$= C + \frac{1}{2} \left[t \cdot e^t - \int e^t dt \right]$$

$$\Rightarrow e^{x^2} \tan y = C + \frac{1}{2} e^t (t - 1) \quad (\because v = \tan y)$$

$$\Rightarrow e^{x^2} \tan y = C + \frac{1}{2} e^{x^2} (x^2 - 1)$$

$$\Rightarrow 2 \tan y = 2C e^{-x^2} + (x^2 - 1) \text{ which is required solution.}$$

Ex. 150. Solve $y dy + by^2 dx = a \cos x dx$

Solution: The given differential equation is

$$y dy + by^2 dx = a \cos x dx$$

$$\Rightarrow 2y \frac{dy}{dx} + 2by^2 = 2a \cos x$$

$$\Rightarrow \frac{dv}{dx} + 2bv = 2a \cos x, \quad (\text{where } v = y^2)$$

which is linear equation in v & x ,

$$\text{Now IF} = e^{\int 2b dx} = e^{2bx}$$

$\therefore$ the required solution is

$$v \cdot (\text{IF}) = C + \int Q(\text{IF}) dx$$

$$v \cdot e^{2bx} = C + 2a \int e^{2bx} \cos x dx,$$

$$\Rightarrow y^2 e^{2bx} = C + 2a \int e^{2bx} \cos x dx \quad (1)$$

$$\Rightarrow y^2 e^{2bx} = C + \frac{2a}{\sqrt{(4b^2 + 1)}} e^{2bx} \cos \left(x - \tan^{-1} \frac{1}{2b} \right)$$

$$\left(\because \int e^{2bx} \cos x dx = \frac{1}{\sqrt{(4b^2 + 1)}} e^{2bx} \cos \left(x - \tan^{-1} \frac{1}{2b} \right) \right)$$

which is required solution.

Ex. 151. Solve $(1 + x^2) \frac{dy}{dx} - 4x^2 \cos^2 y + x \sin 2y = 0$

Solution: The given differential equation is

$$(1 + x^2) \frac{dy}{dx} - 4x^2 \cos^2 y + x \sin 2y = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{1 + x^2} \sin 2y = \frac{4x^2}{1 + x^2} \cos^2 y$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + \frac{2x}{1 + x^2} (\tan y) = \frac{4x^2}{1 + x^2}$$

$$\Rightarrow \frac{dv}{dx} + \frac{2xv}{1 + x^2} = \frac{4x^2}{1 + x^2}, \quad (\text{where } v = \tan y)$$

which is a linear equation in v & x ,

$$\text{Now IF} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2$$

$\therefore$ the required solution is

$$v \cdot (\text{IF}) = C + \int Q(\text{IF}) dx$$

$$\Rightarrow v(1 + x^2) = C + \int \frac{4x^2}{1 + x^2} (1 + x^2) dx,$$

$$\Rightarrow v(1+x^2) = C + \int 4x^2 dx$$

$$\Rightarrow (1+x^2) \tan y = C + \frac{4}{3}x^3, \quad (\because v = \tan y)$$

which is required solution.

Ex. 152. Solve $\frac{dy}{dx} + y \frac{d\phi}{dx} = \phi(x) \frac{d\phi}{dx}$, where ϕ is a function of x only.

Solution. The given differential equation is

$$\frac{dy}{dx} + y \frac{d\phi}{dx} = \phi(x) \frac{d\phi}{dx},$$

$$\Rightarrow \frac{dy}{d\phi} + y = \phi$$

which is a linear equation in y & x

$$\text{Now IF} = e^{\int d\phi} = e^{\phi}$$

$\therefore$ the required solution is

$$y(IF) = C + \int Q(IF)dx$$

$$\Rightarrow ye^{\phi} = C + \int \phi e^{\phi} d\phi$$

$$\Rightarrow ye^{\phi} = C + (\phi - 1)e^{\phi}$$

$$\Rightarrow y = (\phi - 1) + ce^{-\phi} \text{ which is required solution.}$$

Ex. 153. Solve $y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0$

Solution. The given differential equation is

$$y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0$$

$$\Rightarrow y \sin 2x \frac{dx}{dy} - \cos^2 x = 1 + y^2$$

$$\Rightarrow \sin 2x \frac{dx}{dy} - \frac{1}{y} \cos^2 x = \frac{1 + y^2}{y} \quad (1)$$

$$\text{Let } -\cos^2 x = v \Rightarrow -2 \cos x (-\sin x) dx = dv$$

$$\Rightarrow \sin 2x dx = dv,$$

the equations (1) reduces to

$$\frac{dv}{dy} + \frac{1}{y}v = \frac{1 + y^2}{y}$$

which is a linear equation in v & y

$$\text{Now IF} = e^{\int (1/y) dy} = e^{\log y} = y$$

$\therefore$ the required solution is

$$v \cdot (IF) = C + \int Q(IF)dx$$

$$\Rightarrow vy = C + \int (1 + y^2) dy,$$

$$\Rightarrow -y \cos^2 x = C + y + \frac{1}{3}y^3$$

$$\Rightarrow y \cos^2 x + C + y + \frac{1}{3}y^3 = 0 \text{ which is required solution.}$$

LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENT

COEFFICIENTS

Definition. We have already defined in the previous chapter that a linear differential equation is an equation in which the dependent variable y and its differential coefficients occur only in the first degree.

The general form of such an equation is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = X \quad (1)$$

where X and $P_1, P_2, P_3, \dots, P_n$ are all constants or functions of x .

If $P_1, P_2, P_3, \dots, P_n$ are all constants (X may not be constant), then the equation is said to be a linear differential equation with constant coefficients.

We shall first of all consider the differential equation in which $X = 0$

$\therefore$ Equation (1) becomes

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0 \quad (2)$$

$$\therefore (D^n + P_1 D^{n-1} + \dots + P_n)y = 0 \quad \left(\text{Let } \frac{dy}{dx} = Dy, \frac{d^2 y}{dx^2} = D^2 y \dots, \frac{d^n y}{dx^n} = D^n y \right) \quad (2)$$

which is called homogeneous linear equation with constant coefficient.

If $y_1, y_2, y_3, \dots, y_n$ are n independent linearly solution of equation (2)

Then their linear combination $c_1y_1 + c_2y_2 + \dots + c_ny_n$ is also a solution of equation (2)

which is general solution of equation (1) which contains n arbitrary constant.

The part $c_1y_1 + c_2y_2 + \dots + c_ny_n$ is known as complementary solution & the solution which does not involve particular integral (PI).

The general solution of equation (1) is

$$y = CF + PI$$

Note 1 C. F. contains n arbitrary constants, when n is the order of the given differential equation and P. I. does not contain any arbitrary constant.

Note 2 The general solution of (1) is $y = C. F. + P. I.$

THE LINEAR EQUATION WITH CONSTANT COEFFICIENTS AND SECOND MEMBER ZERO.

Let us consider the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0 \quad (1)$$

where $P_1, P_2, P_3, \dots, P_n$ are all constants.

Let the solution of the equation (1) is

$$y = e^{mx}$$

$$\therefore D^n y = m^n e^{mx}, D^{n-1} y = m^{n-1} e^{mx}, \dots, D(y) = m e^{mx}$$

$\therefore$ from (1)

$$\therefore (m^n + P_1 m^{n-1} + P_2 m^{n-2} + \dots + P_n) e^{mx} = 0,$$

$$\Rightarrow m^n + P_1 m^{n-1} + \dots + P_n = 0 \quad [e^{mx} \neq 0] \quad (2)$$

The equation (2) is called the auxiliary equation (AE).

Note. For A. E. replace D by m in the given Differential equation.

Now, we discuss the Three Cases

Case – I. Roots are Real and Distinct

If the roots of (2) are $m = m_1, m_2, \dots, m_n$

Then $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \dots, e^{m_n x}$

are all different and linearly independent. Hence the general solution of (1) is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}.$$

Note 1 If $m = \alpha \pm \sqrt{\beta}$. Then

$$CF = c_1 e^{(\alpha + \sqrt{\beta})x} + c_2 e^{(\alpha - \sqrt{\beta})x}$$

OR

$$CF = e^{\alpha x} \{c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x\}$$

2. If $y = \pm m$. Then

$$CF = c_1 e^{mx} + c_2 e^{-mx}$$

OR

$$CF = c_1 \cosh mx + c_2 \sinh mx$$

Solve $(D^3 + 6D^2 + 11D + 6)y = 0$

The given differential equation is

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

The auxiliary equation is

$$m^3 + 6m^2 + 11m + 6 = 0$$

$$\Rightarrow (m + 1)(m^2 + 5m + 6) = 0$$

$$\Rightarrow (m + 1)(m + 2)(m + 3) = 0$$

$$\Rightarrow m = -1, -2, -3$$

$\therefore$ The required solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x},$$

where C_1, C_2, C_3 are arbitrary constants.

Ex. 154. Solve $\frac{d^2 y}{dx^2} + (a + b) \frac{dy}{dx} + aby = 0$

The given differential equation is

$$\frac{d^2 y}{dx^2} + (a + b) \frac{dy}{dx} + aby = 0$$

$$\Rightarrow (D^2 + (a + b)D + ab)y = 0$$

The auxiliary equation is

$$m^2 + (a + b)m + ab = 0$$

$$\Rightarrow (m + a)(m + b) = 0$$

$$\Rightarrow m = -a, \quad m = -b$$

$\therefore$ The required solution is

$$y = C_1 e^{-ax} + C_2 e^{-bx}, \quad \text{where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

Ex. 155. Solve $\frac{d^2 x}{dt^2} - 3 \frac{dx}{dt} + 2x = 0$ given that when $t = 0, x = 0$ and $\frac{dx}{dt} = 0$.

The given differential equation is

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$$

$$\Rightarrow (D^2 - 3D + 2)x = 0 \quad \left[D = \frac{d}{dt} \right]$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m - 2)(m - 1) = 0$$

$$\Rightarrow m = 1, 2.$$

Hence the required solution is

$$x = C_1 e^t + C_2 e^{2t} \quad (1)$$

where C_1 and C_2 are arbitrary constants.

It is given that $x = 0$, when $t = 0$

$$\therefore \text{from (1), we get } 0 = C_1 + C_2 \quad (2)$$

$$\text{also from (1), } \frac{dx}{dt} = C_1 e^t + 2C_2 e^{2t}$$

and it is given that $\frac{dx}{dt} = 0$, when $t = 0$

$$\therefore 0 = C_1 + 2C_2 \quad (3)$$

Solving (2) and (3) we get $C_1 = 0$ and $C_2 = 0$

$\therefore$ From (1) the required solution is $x = 0$.

Ex. 156. Solve $(D^2 - 4D + 1)y = 0$, when $D \equiv \frac{d}{dx}$

The given differential equation is

$$(D^2 - 4D + 1)y = 0$$

The auxiliary equation is

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{(16 - 4)}}{2} = 2 \pm \sqrt{3}$$

$\therefore$ The required solution is

$$y = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x},$$

$$\Rightarrow y = e^{2x} \{ C_1 e^{x/\sqrt{3}} + C_2 e^{-x/\sqrt{3}} \}$$

$$\Rightarrow y = e^{2x} \{ c_1 \cosh \sqrt{3} + c_2 \sinh \sqrt{3} \} \quad \text{where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

Case – II Auxiliary Equation Having Equal Roots.

If the auxiliary equation (2) has two equal roots say $m = m_1, m_1$,

$$CF = y = (c_1 + c_2 x) e^{m_1 x}$$

If $m = m_1, m_1, m_1$

Then $CF = y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$

In general if $\underbrace{m = m_1, m_1, \dots, m_1}_{n \text{ times}}$

Then $CF = y = (c_1 + c_2 x + \dots + c_n x^{n-1}) e^{m_1 x}$

Note. Consider a differential equation

$$D^2 y - 2m_1 D y + m_1^2 y = 0 \quad \left(\text{where } D = \frac{d}{dx} \right) \quad (1)$$

$$\Rightarrow (D^2 - 2m_1 D + m_1^2) y = 0$$

$$\Rightarrow (D - m_1)^2 y = 0$$

$$\Rightarrow (D - m_1)(D - m_1)y = 0 \quad (2)$$

$$\text{Let } (D - m_1)y = v \quad (3)$$

Then from (2), we have

$$(D - m_1)v = 0 \Rightarrow Dv - m_1 v = 0$$

$$\Rightarrow \frac{dv}{dx} - m_1 v = 0$$

$$\Rightarrow \frac{dv}{v} = m_1 dx$$

Integrating,

$$\log v = m_1 x + \log C_1, \quad \text{where } C_1 \text{ is an arbitrary constant.}$$

$$\Rightarrow \log \left(\frac{v}{C_1} \right) = m_1 x$$

$$\Rightarrow v = C_1 e^{m_1 x}$$

$\therefore$ From (2) we get $(D - m_1)y = C_1 e^{m_1 x}$

$$\Rightarrow \frac{dy}{dx} - m_1 y = C_1 e^{m_1 x}, \quad \left[\because D \equiv \frac{d}{dx} \right]$$

which is a linear equation in y & x

$$\therefore IF = e^{\int -m_1 dx} = e^{-m_1 x}$$

$\therefore$ The solution of this equation is

$$y \cdot e^{-m_1 x} = C_2 + \int C_1 e^{m_1 x} \cdot e^{-m_1 x} dx \quad \text{where } C_2 \text{ is an arbitrary constant.}$$

$$\Rightarrow y \cdot e^{-m_1 x} = C_2 + \int C_1 dx = C_2 + C_1 x$$

$\Rightarrow y = (C_1 x + C_2) e^{m_1 x}$ is the general solution of (1). It contains two arbitrary constants.

Ex. 157. Solve $(D^3 - 3D + 2)y = 0$

The given differential equation is

$$(D^3 - 3D + 2)y = 0$$

The auxiliary equation is

$$m^3 - 3m + 2 = 0$$

$$\Rightarrow m^2(m - 1) + m(m - 1) - 2(m - 1) = 0$$

$$\Rightarrow (m - 1)(m^2 + m - 2) = 0$$

$$\Rightarrow (m - 1)\{(m - 1)(m + 2)\} = 0$$

$$\Rightarrow m = 1, 1, -2 \quad (\text{Two roots are equal})$$

$\therefore$ The general solution is

$$CF = y = (C_1 x + C_2) e^x + C_3 e^{-2x}$$

where C_1, C_2 and C_3 are arbitrary constant.

Ex. 158. Solve $(D^3 - D^2 - D + 1)y = 0$

The given differential equation is

$$(D^3 - D^2 - D + 1)y = 0$$

The auxiliary solution is

$$m^3 - m^2 - m + 1 = 0$$

$$\Rightarrow m^2(m - 1) - (m - 1) = 0$$

$$\Rightarrow (m - 1)(m^2 - 1) = 0$$

$$\Rightarrow (m - 1)(m - 1)(m + 1) = 0$$

$$\Rightarrow m = 1, 1, -1$$

$\therefore$ The required solution is

$$CF = y = (c_1 x + c_2) e^x + c_3 e^{-x}$$

where c_1, c_2 and c_3 are arbitrary constant.

Ex. 159. $(D^3 - 2D^2 - 4D + 8)y = 0$

The auxiliary equation is

$$m^3 - 2m^2 - 4m + 8 = 0$$

$$\Rightarrow m^2(m - 2) - 4(m - 2) = 0$$

$$\Rightarrow (m - 2)(m^2 - 4) = 0$$

$$\Rightarrow (m - 2)(m - 2)(m + 2) = 0$$

$$\Rightarrow m = 2, 2, -2$$

$\therefore$ The required solution is $y = (C_1 x + C_2) e^{2x} + C_3 e^{-2x}$, where C_1, C_2, C_3 are arbitrary constants.

Ex. 160. Solve $\frac{d^2 x}{dy^2} - n^2 x = 0$

The given differential equation is

$$\frac{d^2 x}{dy^2} - n^2 x = 0$$

$$\Rightarrow (D^2 - n^2)x = 0, \quad \left[\text{where } D \equiv \frac{d}{dy} \right]$$

Its auxiliary equation solution is

$$m^2 - n^2 = 0$$

$$\Rightarrow m = \pm n$$

$\therefore$ The required solution is

$$x = c_1 e^{ny} + c_2 e^{-ny}$$

Where c_1 and c_2 are arbitrary constants.

Ex. 161. Solve $(16D^2 + 24D + 9)y = 0$

The given differential equation is

$$(16D^2 + 24D + 9)y = 0$$

The auxiliary equation is

$$16m^2 + 24m + 9 = 0$$

$$\Rightarrow (4m + 3)^2 = 0$$

$$\Rightarrow m = -\frac{3}{4}, -\frac{3}{4}$$

$\therefore$ The required solution is

$$y = (C_1 x + C_2) e^{-3x/4}$$

where C_1 and C_2 are arbitrary constants

Ex. 162. Solve $(D^3 + D^2(2\sqrt{3} - 1) + D(3 - 2\sqrt{3}) - 3)y = 0$

The given differential equation is

$$(D^3 + D^2(2\sqrt{3} - 1) + D(3 - 2\sqrt{3}) - 3)y = 0$$

Here the auxiliary equation is

$$m^3 + m^2(2\sqrt{3} - 1) + m(3 - 2\sqrt{3}) - 3 = 0$$

$$\Rightarrow (m - 1)(m^2 + 2\sqrt{3}m + 3) = 0$$

$$\Rightarrow (m - 1)(m + \sqrt{3})^2 = 0$$

$$\Rightarrow m = 1, -\sqrt{3}, -\sqrt{3}$$

Hence the required solution is

$$y = C_1 e^x + (C_2 x + C_3) e^{-\sqrt{3}x} \quad \text{where } C_1, C_2 \text{ and } C_3 \text{ are arbitrary constants}$$

Ex. 163. Solve $(D^4 - 7D^3 + 18D^2 - 20D + 8)y = 0$

The given differential equation is

$$(D^4 - 7D^3 + 18D^2 - 20D + 8)y = 0$$

Here the auxiliary equation is

$$m^4 - 7m^3 + 18m^2 - 20m + 8 = 0$$

$$\Rightarrow m^3(m - 1) - 6m^2(m - 1) + 12m(m - 1) - 8(m - 1) = 0 \quad (\text{Note})$$

$$\Rightarrow (m - 1)(m^3 - 6m^2 + 12m - 8) = 0$$

$$\Rightarrow (m - 1)[m^2(m - 2) - 4m(m - 2) + 4(m - 2)] = 0$$

$$\Rightarrow (m - 1)(m - 2)(m^2 - 4m + 4) = 0$$

$$\Rightarrow (m - 1)(m - 2)(m - 2)^2 = 0$$

$$\Rightarrow m = 1, 2, 2, 2$$

$\therefore$ The required solution is

$$y = C_1 e^x + (C_2 + C_3 x + C_4 x^2) e^{2x}, \quad \text{where } C_1, C_2, C_3 \text{ and } C_4 \text{ are arbitrary constant.}$$

Prop. -III. Auxiliary equation having imaginary roots

If $m = \alpha \pm i\beta$ then

$$CF = e^{(\text{real part})x} \{c_1 \cos(\text{imaginary part})x + c_2 \sin(\text{imaginary part})x\}$$

$$\text{i.e. } CF = e^{\alpha x} \{c_1 \cos \beta x + c_2 \sin \beta x\}$$

Note

(1) If all the roots of A. E. are distinct i.e., $m_1, m_2, \dots, m_n$ then

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

(2) if any roots m_k is repeated r times & rest are different then

$$C.F. = (c_1 + c_2 x + c_3 x^2 + \dots + c_r x^{r-1}) e^{m_k x} + c_{r+1} e^{m_{k+1} x} + \dots$$

(3) If the roots of A. E. has complex root as $\alpha + i\beta$ & so other is its conjugate (complement) $\alpha - i\beta$ then

$$C.F. = c_1 e^{\alpha + i\beta} + c_2 e^{\alpha - i\beta} = e^{\alpha x} \{c_1 \cos \beta x + \dots + c_2 \sin \beta x\}$$

(4) If the complex roots be repeated two times then

$$C.F. = e^{\alpha x} \{(c_1 + c_2 x) \dots \cos \beta x + (c_3 + c_4 x) \dots \sin \beta x\}$$

(5) If a pair of roots of A. E. is irrational i.e. $\alpha \pm \sqrt{\beta}$ then

$$C.F. = c_1 e^{(\alpha + \sqrt{\beta})x} + c_2 e^{(\alpha - \sqrt{\beta})x} = e^{\alpha x} \{c_1 e^{\sqrt{\beta}x} + c_2 e^{-\sqrt{\beta}x}\} = e^{\alpha x} \{c_1 \cosh(x\sqrt{\beta}) + c_2 \sinh(x\sqrt{\beta})\}$$

Ex. 164. Solve $\left(\frac{d^2 y}{dx^2}\right) + 4y = 0$, given that $y = 2$ and $\left(\frac{dy}{dx}\right) = 0$ when $x = 0$.

The given differential equation is

$$\left(\frac{d^2 y}{dx^2}\right) + 4y = 0$$

$$\Rightarrow (D^2 + 4)y = 0$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$\Rightarrow m = \pm 2i$$

$\therefore$ The solution of the given equation is

$$y = c_1 \cos 2x + c_2 \sin 2x \quad (1)$$

where c_1, c_2 are arbitrary constants

Also from (1) we get

$$\frac{dy}{dx} = -2c_1 \sin 2x + 2c_2 \cos 2x \quad (2)$$

Now given that $y = 2, \frac{dy}{dx} = 0$ when $x = 0$.

From (1) and (2), we get

$$2 = c_1 \cos 0 + c_2 \sin 0$$

$$\text{and } 0 = -2c_1 \sin 0 + 2c_2 \cos 0$$

$$\Rightarrow 2 = c_1 \text{ and } 0 = 2c_2$$

$$\Rightarrow c_1 = 2, c_2 = 0$$

$\therefore$ from (1) the required solution is

$$y = 2 \cos 2x.$$

Ex. 165. Solve $(D^2 \pm \mu^2)y = 0, \mu \neq 0$

The given differential equation is

$$(D^2 \pm \mu^2)y = 0, \mu \neq 0$$

The auxiliary equation is

$$m^2 \pm \mu^2 = 0$$

$$\Rightarrow m^2 = \mp \mu^2$$

$$\Rightarrow m^2 = -\mu^2 \quad \text{and} \quad m^2 = \mu^2$$

$$\Rightarrow m = \pm \mu i \quad \text{and} \quad \pm \mu$$

Hence the required solution is

$$y = [C_1 \cos \mu x + C_2 \sin \mu x] + C_3 e^{\mu x} + C_4 e^{-\mu x}.$$

Ex. 166. Solve $(D^4 + 4)y = 0$, where $D \equiv \frac{d}{dx}$

The given differential equation is

$$(D^4 + 4)y = 0$$

Here the auxiliary equation is

$$m^4 + 4 = 0$$

$$\Rightarrow (m^2 + 2)^2 - 4m^2 = 0$$

$$\Rightarrow (m^2 + 2)^2 - (2m)^2 = 0$$

$$\Rightarrow (m^2 - 2m + 2)(m^2 + 2m + 2) = 0$$

$$\Rightarrow m^2 - 2m + 2 = 0, m^2 + 2m + 2 = 0$$

$$\Rightarrow m = 1 \pm i \quad \text{and} \quad -1 \pm i$$

Hence the required solution is

$$y = e^x [C_1 \cos x + C_2 \sin x] + e^{-x} [C_3 \cos x + C_4 \sin x] \quad \text{where } C_1, C_2 \text{ etc. are arbitrary constants.}$$

Ex. 167. Solve $(D^3 - 8)y = 0$

The given differential equation is

$$(D^3 - 8)y = 0$$

Here the auxiliary equation is

$$m^3 - 8 = 0$$

$$\Rightarrow (m - 2)(m^2 + 2m + 4) = 0$$

which gives

$$m = 2 \quad \text{and} \quad \frac{1}{2} [-2 \pm \sqrt{(4 - 16)}]$$

$$\Rightarrow m = 2, -1 \pm i\sqrt{3}$$

 $\therefore$ The required solution is

$$y = c_1 e^{2x} + e^{-x} \{c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\} \quad \text{where } c's \text{ are arbitrary constants.}$$

Ex. 168. Solve $(D^3 - 2D^2 + 4D - 8)y = 0$

The given differential equation is

$$(D^3 - 2D^2 + 4D - 8)y = 0$$

Here auxiliary equation is

$$m^3 - 2m^2 + 4m - 8 = 0$$

$$\Rightarrow m^2(m - 2) + 4(m - 2) = 0$$

$$\Rightarrow (m - 2)(m^2 + 4) = 0$$

$$\Rightarrow m = 2, \pm 2i$$

 $\therefore$ Required solution is

$$y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x,$$

where $c's$ are arbitrary constants.**Ex. 169. Solve $(D^4 + 8D^2 + 16)y = 0$**

The given differential equation is

$$(D^4 + 8D^2 + 16)y = 0$$

Here the auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$

$$\Rightarrow (m^2 + 4)^2 = 0$$

$$\Rightarrow m^2 = -4 \quad (\text{twice})$$

$$\Rightarrow m = \pm 2i \quad (\text{twice})$$

Hence the required solution is

$$y = (C_1 x + C_2) \cos 2x + (C_3 x + C_4) \sin 2x$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.**Ex. 170. Solve $(D^4 + D^2 + 1)y = 0$**

The given differential equation is

$$(D^4 + D^2 + 1)y = 0$$

Here the auxiliary equation is

$$m^4 + m^2 + 1 = 0$$

$$\Rightarrow (m^4 + 2m^2 + 1) - m^2 = 0$$

$$\Rightarrow (m^2 + 1)^2 - m^2 = 0$$

$$\Rightarrow (m^2 + 1 + m)(m^2 + 1 - m) = 0$$

$$\text{Now } m^2 + m + 1 = 0$$

$$\Rightarrow m = \frac{1}{2} \left[-1 \pm \sqrt{(1-4)} \right] \Rightarrow m = \frac{1}{2} \left[-1 \pm i\sqrt{3} \right]$$

$$\text{Similarly, } m^2 - m + 1 = 0$$

$$\Rightarrow m = \frac{1}{2} \left[1 \pm i\sqrt{3} \right]$$

$\therefore$ The solution of the auxiliary equation (1) are

$$m = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$\therefore$ The required solution is

$$y = e^{-x/2} \left[C_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + C_2 \sin\left(\frac{x\sqrt{3}}{2}\right) \right] + e^{x/2} \left[C_3 \cos\left(\frac{x\sqrt{3}}{2}\right) + C_4 \sin\left(\frac{x\sqrt{3}}{2}\right) \right]$$

Ex. 171. Solve $(D^3 - D^2 - D - 2)y = 0$

the given differential equation is

$$(D^3 - D^2 - D - 2)y = 0$$

Here the auxiliary equation is

$$m^3 - m^2 - m - 2 = 0$$

$$\Rightarrow (m - 2)(m^2 + m + 1) = 0,$$

$$\Rightarrow (m - 2) = 0, (m^2 + m + 1) = 0$$

$$\Rightarrow m = 2, \frac{-1 \pm i\sqrt{3}}{2}$$

The required solution is

$$y = C_1 e^{2x} + e^{-x/2} \left\{ C_2 \cos\left(\frac{x\sqrt{3}}{2}\right) + C_3 \sin\left(\frac{x\sqrt{3}}{2}\right) \right\}$$

Ex. 172 Solve $(D^2 + 1)^2(D^2 + D + 1)y = 0$

The given differential equation is

$$(D^2 + 1)^2(D^2 + D + 1)y = 0$$

Here the auxiliary equation is

$$(m^2 + 1)^2(m^2 + m + 1) = 0$$

$$\Rightarrow m^2 = -1 \text{ (twice) and } m^2 + m + 1 = 0$$

$$\Rightarrow m = \pm i \text{ (twice) and } m = \frac{-1 + \sqrt{(1-4)}}{2}$$

$$\Rightarrow m = \pm i \text{ (twice) and } m = \frac{(-1 \pm \sqrt{3})}{2}$$

$\therefore$ The required solution is

$$y = (c_1 x + c_2) \cos x + (c_3 x + c_4) \sin x + \left\{ c_5 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_6 \sin\left(\frac{x\sqrt{3}}{2}\right) \right\} e^{-x/2}$$

checked upto here

Ex. 173. Solve $(D^3 - D^2 - D - 2)y = 0$

The given differential equation is

$$(D^3 - D^2 - D - 2)y = 0$$

Here the auxiliary equation is

$$(m^2 + 1)^3(m^2 + m + 1)^2 = 0$$

$$\Rightarrow m^3 = -1 \text{ (thrice) and } m^2 + m + 1 = 0 \text{ (twice)}$$

$$\Rightarrow m = \pm i \text{ (thrice) and } m = \frac{-1 \pm \sqrt{(1-4)}}{2} \text{ (twice)}$$

$$\Rightarrow m = 0 \pm i \text{ (thrice) and } \frac{(-1 \pm i\sqrt{3})}{2} \text{ (twice)}$$

$\therefore$ The required solution is

$$y = (C_1 x^2 + C_2 x + C_3) \cos x + (C_4 x^2 + C_5 x + C_6) \sin x + \left[(C_7 x + C_8) \cos\left(\frac{x\sqrt{3}}{2}\right) + (C_9 x + C_{10}) \sin\left(\frac{x\sqrt{3}}{2}\right) \right] e^{-x/2}$$

where $C_1, C_2, \dots$ etc. are arbitrary constants

Ex. 174. Find complementary function of $(D - 1)^{n-4}(D^2 + D + 1)y = 0$

Solⁿ: The given differential equation is

$$(D - 1)^{n-4}(D^2 + D + 1)y = 0$$

Here the auxiliary equation is

$$(m - 1)^{n-4}(m^2 + m + 1) = 0,$$

which gives

$$\Rightarrow m = \underbrace{1, 1, \dots, 1}_{(n-4)\text{times}} \text{ and } \frac{1}{2} \{-1 \pm i\sqrt{3}\}.$$

$\therefore$ The required solution is

$$y = [C_1 + C_2x + C_3x^2 + \dots + C_{n-4}x^{n-5}]e^x + e^{-x/2} \left[a \cos\left(\frac{1}{2}x\sqrt{3}\right) + b \sin\left(\frac{1}{2}x\sqrt{3}\right) \right]$$

where $C_1, C_2, C_3, \dots, C_{n-4}, a, b$ are arbitrary constants.

Ex. 175. Solve $(D^2 + D + 1)^2(D - 2)y = 0$

The given differential equation is

$$(D^2 + D + 1)^2(D - 2)y = 0$$

Here the auxiliary equation is

$$(m^2 + m + 1)^2(m - 2) = 0$$

$$\Rightarrow m^2 + m + 1 = 0 \text{ (twice) and } m - 2 = 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{(1-4)}}{2} \text{ (twice) and } m = 2$$

$$\Rightarrow m = 2, \frac{-1 \pm i\sqrt{3}}{2} \text{ (twice)}$$

$\therefore$ The required solution is

$$y = (c_1x + c_2) \cos(x\sqrt{3}) + (c_3x + c_4) \sin\left(\frac{x\sqrt{3}}{2}\right) + c_5e^{2x}, \quad \text{where } c' \text{ s are arbitrary constants.}$$

Ex. 176. Solve $(D^6 - 1)y = 0$

The given differential equation is

$$(D^6 - 1)y = 0$$

Here the auxiliary equation is

$$m^6 - 1 = 0$$

$$\Rightarrow (m^2 - 1)(m^4 + m^2 + 1) = 0$$

$$\Rightarrow (m - 1)(m + 1)(m^2 - m + 1)(m^2 + m + 1) = 0$$

$$\Rightarrow m = 1, -1, \frac{1}{2} \pm \frac{1}{2}i\sqrt{3} \text{ and } -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$$

$\therefore$ The required solution is

$$y = C_1e^x + C_2e^{-x} + e^{x/2} \left[C_3 \cos\left(\frac{1}{2}x\sqrt{3}\right) + C_4 \sin\left(\frac{1}{2}x\sqrt{3}\right) \right] + e^{-x/2} \left[C_5 \cos\left(\frac{1}{2}x\sqrt{3}\right) + C_6 \sin\left(\frac{1}{2}x\sqrt{3}\right) \right]$$

where $C_1, C_2, \dots$ etc. are arbitrary constants.

Ex. 177. Solve $(D^4 - m^4)y = 0$

The given differential equation is

$$(D^4 - m^4)y = 0$$

Here the auxiliary equation is

$$M^4 - m^4 = 0$$

$$\Rightarrow (M^2 - m^2, -m^2) = 0$$

$$\Rightarrow M^2 = m^2, -m^2$$

$$\Rightarrow M = \pm m, \pm mi$$

$\therefore$ The required solution is

$$y = C_1e^{mx} + C_2e^{-mx} + C_3 \cos mx + C_4 \sin mx,$$

Ex. 178. Solve $(D^4 - m^4)y = 0$

The given differential equation is

$$(D^4 - m^4)y = 0$$

Here the auxiliary equation is

$$M^4 + m^4 = 0$$

$$\Rightarrow (M^2 + m^2)^2 - 2M^2m^2 = 0$$

$$\Rightarrow (M^2 + m^2)^2 - (\sqrt{2}Mm)^2 = 0$$

$$\Rightarrow (M^2 + m^2 - \sqrt{2}Mm) = 0$$

$$\Rightarrow (M^2 + m^2 + \sqrt{2}Mm)(M^2 + m^2 - \sqrt{2}Mm)$$

Ex. 179. Solve $(D^4 + \mu)y = 0$

The given differential equation is

$$(D^4 + \mu)y = 0$$

Let $\mu = m^4$, then the above equation reduces to

$$(D^4 + m^4)y = 0,$$

Here the auxiliary equation is

$$M^4 + m^4 = 0$$

$$\Rightarrow (M^2 + m^2)^2 - 2M^2m^2 = 0$$

$$\Rightarrow (M^2 + m^2)^2 - (\sqrt{2}Mm)^2 = 0$$

$$\Rightarrow (M^2 + m^2 + \sqrt{2}Mn)(M^2 + m^2 - \sqrt{2}Mn) = 0$$

$$\Rightarrow (M^2 + m^2 + \sqrt{2}Mn)(M^2 - m^2 - \sqrt{2}Mn) = 0$$

$$\Rightarrow M - \frac{-\sqrt{(2m)} \pm \sqrt{(2m^2 - 4m^2)}}{2} : \frac{-\sqrt{(2m)} \pm \sqrt{(2m^2 - 4m^2)}}{2}$$

$$\Rightarrow M = -\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}i, \frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}i$$

∴ The required solution is

$$y + e^{-\mu x/\sqrt{2}}\{C_1 \cos(\mu x/\sqrt{2}) + C_2 \sin(\mu x/2)\} + e^{\mu x/\sqrt{2}}\{C_3 \cos(\mu x/\sqrt{2}) + C_4 \sin(\mu x/2)\}$$

where C_1, C_2 etc. are arbitrary constants.

Ex. 180. Solve $(D^2 - 2D + 5)y = 0$ given that $y = 0$ and $\frac{dy}{dx} = 4$, when $x = 0$.

The given differential equation is

$$(D^2 - 2D + 5)y = 0$$

Here the auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$\Rightarrow m = \frac{1}{2} [2 \pm \sqrt{4 - 20}] = 1 \pm 2i$$

The required solution is

$$y = e^x(C_1 \cos 2x + C_2 \sin 2x)$$

where C_1 and C_2 are arbitrary constants.

From (1),

$$\frac{dy}{dx} = e^x(C_1 \cos 2x + C_2 \sin 2x) + e^x(-2C_1 \sin 2x + 2C_2 \cos 2x)$$

Given that $y = 0$, when $x = 0$, so from (1) we get

$$0 = e^0[C_1(\cos 0) + C_2(\sin 0)]$$

$$\Rightarrow C_1 = 0$$

Similarly from (2) as $\frac{dy}{dx} = 4$ when $x = 0$, we get

$$4 = C_1 + 2C_2$$

$$\Rightarrow 4 = 0 + 2C_2$$

$$\Rightarrow C_2 = 2$$

∴ From (1), we get.

Ex. 181. Solve $1\left(\frac{d^2\theta}{dt^2}\right) + g\theta = 0$ given that $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$ when $t = 0$

The given differential equation is

$$1\left(\frac{d^2\theta}{dt^2}\right) + g\theta = 0$$

$$\Rightarrow \left[D^2 + \left(\frac{g}{1}\right)\right]\theta = 0, \quad \left(\text{where } D \equiv \frac{d}{dt}\right)$$

∴ The auxiliary equation is

$$m^2 + \left(\frac{g}{1}\right) = 0$$

$$\Rightarrow m^2 = -\left(\frac{g}{1}\right)$$

$$\Rightarrow m = \pm \sqrt{\left(\frac{g}{1}\right)}i$$

∴ The solution of the given differential equation is

$$\theta = G_1 \cos\left\{\sqrt{\left(\frac{g}{1}\right)}t + C_2\right\} \quad (1)$$

where C_1, C_2 are arbitrary constants

$$\text{and } \frac{d\theta}{dt} = -\sqrt{\left(\frac{g}{1}\right)}C_1 \sin\left\{\sqrt{\left(\frac{g}{1}\right)}t + C_2\right\} \quad (2)$$

when $t = 0, \theta = \theta_0$ and $\frac{d\theta}{dt} = 0$ (given) so from (1) and (2) we get

$$\theta_0 = C_1 \cos C_2$$

$$\text{and } 0 = -\sqrt{\left(\frac{g}{1}\right)}C_1 \sin C_2$$

From these we have $\sin C_2 = 0$, as $C_1 \neq 0$ and so $C_2 = \theta_0$

∴ from (1), the required solution is.

Ex. 182. Find the solution of $\frac{d^2i}{dt^2} + \frac{R di}{L dt} + \frac{1}{LC} = 0$, where $R^2C = 4L$ and R, C, L are constants

The given differential equation is

$$\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} = 0$$

$$\Rightarrow \frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} = 0$$

(1)

$$\Rightarrow (D^2 + aD + b)i = 0 \quad \left[\text{where } D = \frac{d}{dt}, a = \frac{R}{L}, b = \frac{1}{LC} \right]$$

The AE is

$$m^2 + am + b = 0$$

$$\Rightarrow M = \frac{1}{2} \left[-a \pm \sqrt{(a^2 - 4b)} \right] = \frac{1}{2} a \pm \frac{1}{2} \sqrt{(a^2 - 4b)}$$

$$\Rightarrow m = -\frac{1}{2} \left(\frac{R}{L} \right) \pm \frac{1}{2} \sqrt{\left[\left(\frac{R}{L} \right)^2 - 4 \left(\frac{1}{LC} \right) \right]}$$

$$= -\frac{1}{2} \left(\frac{R}{L} \right) + \frac{1}{2} \sqrt{\left(\frac{R^2 C - 4L}{L^2 C} \right)} = -\frac{R}{2L} \pm 0 \quad [\because R^2 C = 4L]$$

$$\Rightarrow m = \left(\frac{R}{2L} \right), \text{ twice}$$

$\therefore$ The required solution is

$$i = (C_1 t + C_2) e^{-(R/2L)t}, \quad \text{where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

THE PARTICULAR INTEGRAL (P.I.)

The linear DE with constant coefficient equation is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = X$$

i. e. $F(D)y = X$

The required solution is

$$y = CF + PI$$

Prop. -I When X is of the form e^{ax} , where a is any constant and $F(a) \neq 0$.

$$PI = \frac{1}{F(D)} X = \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}$$

(For PI replace D by a if $F(a) \neq 0$)

Explanation.

$$\text{We know } D(e^{ax}) = ae^{ax}$$

$$D^2(e^{ax}) = a^2 e^{ax}$$

$$D^3(e^{ax}) = a^3 e^{ax}$$

$$\text{In general } D^n(e^{ax}) = a^n e^{ax}$$

$$\therefore F(D)(e^{ax}) = F(a)e^{ax}$$

$$\Rightarrow \frac{1}{F(D)} F(D)e^{ax} = \frac{1}{F(D)} F(a)e^{ax}$$

$$\Rightarrow e^{ax} = F(a) \frac{1}{F(D)} e^{ax}, \quad [F(a) \text{ is constant.}]$$

$$\Rightarrow \frac{1}{F(a)} e^{ax} = \frac{1}{F(D)} e^{ax}$$

$$\text{Hence } \frac{1}{F(a)} e^{ax} = \frac{1}{F(D)} e^{ax}. \text{ If } F(a) \neq 0.$$

Ex. 183. Solve $(D^2 - 2D + 5)y = e^{-ax}$

The given differential equation is

$$(D^2 - 2D + 5)y = e^{-ax}$$

Here auxiliary equation is

$$m^2 - 2m + 5 = 0,$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2}$$

$$m = -1 \pm 2i$$

$$\therefore C.F. = e^{-x} [C_1 \cos 2x + C_2 \sin 2x], \quad \text{where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

$$\text{and P.I.} = \frac{1}{D^2 - 2D + 5} e^{-x} = \frac{1}{(-1)^2 - 2(-1) + 5} e^{-x} \quad [\because \text{Replace } D \text{ by } -1]$$

$$\Rightarrow \text{P.I.} = \left(\frac{1}{8} \right) e^{-x}$$

$\therefore$ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = e^{-x}[C_1 \cos 2x + C_2 \sin 2x] + \left(\frac{1}{8}\right)e^{-x}$$

Ex. 184. Solve $(D^2 - 2D + 1)y = 2e^{5x/2}$

The given differential equation is

$$(D^2 - 2D + 1)y = 2e^{5x/2}$$

here the auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m - 1)^2 = 0$$

$$\Rightarrow m = 1, 1$$

$$\begin{aligned} \text{Here C. F.} &= \frac{1}{D^2 - 2D + 1}(2e^{5x/2}) = 2 \frac{1}{(5/2)^2 - 2(5/2) + 1} e^{5x/2} \quad (\text{Replace } D \text{ by } 5/2) \\ &= 2 \frac{1}{(25/4) - 5 + 1} e^{5x/2} = \left(\frac{8}{9}\right) e^{5x/2} \end{aligned}$$

$\therefore$ The required solution is

$$y = (C_1 x + C_2)e^x + \left(\frac{8}{9}\right)e^{5x/2}$$

Ex. 185. Solve $(D^3 - 3D + 2)y = e^{5x}$

The given differential equation is

$$(D^3 - 3D + 2)y = e^{5x}$$

Here the auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m - 1)(m - 2) = 0$$

$$\Rightarrow m = 1, 2$$

Hence C. F. = $C_1 e^x + C_2 e^{2x}$, where C_1 and C_2 are arbitrary constants

$$\text{and P. I.} = \frac{1}{D^2 - 3D + 2} e^{5x} = \frac{1}{5^2 - 3(5) + 2} e^{5x} = \frac{1}{12} e^{5x} \quad (\text{Replace } D \text{ by } 5)$$

$\therefore$ Required solution is

$$y = C_1 e^x + C_2 e^{2x} + \left(\frac{1}{12}\right)e^{5x}.$$

Ex. 186. Solve $(D^3 - D^2 - 4D + 4)y = e^{3x}$

The given differential equation is

$$(D^3 - D^2 - 4D + 4)y = e^{3x}$$

Here the auxiliary equation is

$$m^3 - m^2 - 4m + 4 = 0$$

$$\Rightarrow m^2(m - 1) - 4(m - 1) = 0$$

$$\Rightarrow (m - 1)(m^2 - 4) = 0$$

$$\Rightarrow (m - 1)(m - 2)(m + 2) = 0$$

$$\Rightarrow m = 1, 2, -2$$

$\therefore$ C. F. = $C_1 e^x + C_2 e^{2x} + C_3 e^{-2x}$, where C 's are arbitrary constants

$$\text{and P. I.} = \frac{1}{D^3 - D^2 - 4D + 4} e^{3x} = \frac{1}{3^3 - 3^2 - 4(3) + 4} e^{3x} = \frac{1}{10} e^{3x}$$

$\therefore$ The required solution is

$$y = C_1 e^x + C_2 e^{2x} + C_3 e^{-2x} + \frac{1}{10} e^{3x}$$

Ex. 187. Solve $(D^2 + D + 1)y = e^{-x}$

The given differential equation is

$$(D^2 + D + 1)y = e^{-x}$$

Here the auxiliary equation is

$$m^2 + m + 1 = 0$$

$$\Rightarrow m = \frac{1}{2}[-1 \pm \sqrt{(1 - 4)}] = \frac{1}{2}[-1 \pm i\sqrt{3}]$$

Hence C. F. = $e^{-x/2} \left[c_1 \cos\left(\frac{1}{2}x\sqrt{3}\right) + c_2 \sin\left(\frac{1}{2}x\sqrt{3}\right) \right]$, where c_1 and c_2 are arbitrary constants.

$$\text{and P. I.} = \frac{1}{D^2 + D + 1} e^{-x} = \frac{1}{(-1)^2(-1) + 1} e^{-x} = e^{-x}.$$

$\therefore$ The required solution is

$$y = e^{-x/2} \left[c_1 \cos\left(\frac{1}{2}\sqrt{3}\right) + c_2 \sin\left(\frac{1}{2}x\sqrt{3}\right) \right] + e^{-x}.$$

Ex. 188. Solve $(D^2 - 3D + 2)y = e^{5x}$

The given differential equation is

$$(D^2 - 3D + 2)y = e^{5x}$$

The AE is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m - 1)(m - 2) = 0$$

$$\Rightarrow m = 1, 2$$

$$\therefore \text{C.F.} = C_1 e^{2x} + C_2 e^x, \quad \text{where } C_1 \text{ and } C_2 \text{ are arbitrary constants}$$

$$\text{and P.I.} = \frac{1}{D^2 - 3D + 2} e^{5x} = \frac{1}{(5)^2 - 3(5) + 2} e^{5x} = \frac{1}{12} e^{5x}$$

$\therefore$ The required solution is

$$y = C_1 e^{2x} + C_2 e^x + \frac{1}{12} e^{5x}.$$

Ex. 189. Solve $(D^3 + D + 1)y = e^x$

The given differential equation is

$$(D^3 + D + 1)y = e^x$$

$$\text{Here C.F.} = e^{-x/2} \left\{ C_1 \cos\left(\frac{1}{2}x\sqrt{3}\right) + C_2 \sin\left(\frac{1}{2}x\sqrt{3}\right) \right\}, \quad (\text{verify it})$$

where C_1 and C_2 are arbitrary constant,

$$\text{and P.I.} = \frac{1}{D^3 + D + 1} e^x = \frac{1}{(1)^2 + 1 + 1} e^x = \frac{1}{3} e^x$$

$\therefore$ The required solution is

$$y = e^{-x/2} \left\{ C_1 \cos\left(\frac{1}{2}x\sqrt{3}\right) + C_2 \sin\left(\frac{1}{2}x\sqrt{3}\right) \right\} + \frac{1}{2} e^x$$

Prop. -II. When X is of the e^x form and $F(a) = 0$

$$\text{P.I.} = \frac{1}{F(D)} X = \frac{1}{(D - a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$$

Ex190. Solve $(D^3 - 7D + 6)y = e^{2x}$

The given differential equation is

$$(D^3 - 7D + 6)y = e^{2x}$$

$$\text{C.F.} = C_1 e^x + C_2 e^{2x} + C_3 e^{-3x}$$

(vericy it)

$$\text{and P.I.} = \frac{1}{D^3 - 7D + 6} e^{2x}$$

$$= x \cdot \frac{1}{3D^2 - 7} e^{2x}$$

$$= x \cdot \frac{1}{3 \cdot 4 - 7} e^{2x}$$

(Replace D by 2)

$$= \frac{x}{5} e^{2x}$$

$\therefore$ The required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = C_1 e^x + C_2 e^{2x} + C_3 e^{-3x} + \frac{1}{5} e^{2x} x$$

Ex. 191. Solve $(2D + 1)^2 y = 4e^{-x/2}$

The given differential equation is

$$(2D + 1)^2 y = 4e^{-x/2}$$

$$\text{C.F.} = (C_1 x + C_2) e^{-x/2} \quad (\text{verify it})$$

$$\text{and P.I.} = \frac{1}{(2D + 1)^2} (4e^{-x/2})$$

$$= 4 \cdot \frac{1}{4 \left(D + \frac{1}{2}\right)^2} e^{-x/2}$$

$$= x \cdot \frac{1}{\left(D + \frac{1}{2}\right)^2} e^{-x/2}$$

$$= \frac{x^2}{2} e^{-x/2}$$

$$\text{OR PI} = 4 \cdot \frac{1}{4 \left(D + \frac{1}{2}\right)^2} e^{-x/2}$$

$$= \frac{x^2}{2!} e^{-x/2} = \frac{x^2}{2} e^{-x/2}$$

$\therefore$ The required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = (C_1 x + C_2) e^{-x/2} + e^{-x/2} \left(\frac{1}{2} x^2 \right)$$

Ex. 192. Solve $(D^4 + D^3 + D^2 - D - 2)y = e^x$

The given differential equation is

$$(D^4 + D^3 + D^2 - D - 2)y = e^x$$

$$C.F. = C_1 e^x + C_2 e^{-x} + e^{-x/2} \left(C_3 \cos \frac{\sqrt{7}x}{2} + C_4 \sin \frac{\sqrt{7}x}{2} \right) \quad (\text{Verify})$$

$$\text{and P.I.} = \frac{1}{(D^4 + D^3 + D^2 - D - 2)} e^x$$

$$= x \cdot \frac{1}{4D^3 + 3D^2 + 2D - 1} e^x$$

$$= x \cdot \frac{1}{4 + 3 + 2 - 1} e^x$$

(Replace D by 1)

$$= \frac{x}{8} e^x$$

∴ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = C_1 e^x + C_2 e^{-x} + e^{-x/2} \left(C_3 \cos \frac{\sqrt{7}x}{2} + C_4 \sin \frac{\sqrt{7}x}{2} \right) + \frac{1}{8} e^x (x)$$

Ex. 193. Solve $(D^2 - a^2) = \cosh ax$

The given differential equation is

$$(D^2 - a^2)y = \cosh ax$$

$$C.F. = \frac{1}{D^2 - a^2} \cosh ax$$

(Verify it)

$$\text{and P.I.} = \frac{1}{D^2 - a^2} \cosh ax$$

$$= \frac{1}{D^2 - a^2} \left(\frac{e^{ax} + e^{-ax}}{2} \right)$$

$$= \frac{1}{2} \frac{1}{D^2 - a^2} e^{ax} + \frac{1}{2} \frac{1}{D^2 - a^2} e^{-ax}$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - a^2} e^{ax} + \frac{1}{D^2 - a^2} e^{-ax} \right]$$

$$= \frac{1}{2} \left[x \cdot \frac{1}{2D} e^{ax} + x \cdot \frac{1}{2D} e^{-ax} \right]$$

$$= \frac{1}{2} \left[\frac{x}{2a} e^{ax} + \frac{x}{2(-a)} e^{-ax} \right]$$

$$= \frac{x}{2a} \left[\frac{e^{ax} - e^{-ax}}{2} \right]$$

$$= \frac{x}{2a} \sin 2ax$$

∴ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = C_1 e^{ax} + C_2 e^{-ax} + \frac{x}{2a} \sinh ax$$

I think read upto here

Ex. 194. Solve $(D + 2)(D - 1)^3 y = e^x$

The given differential equation is

$$(D + 2)(D - 1)^3 y = e^x$$

$$C.C. = c_1 e^{-2x} + (C_2 x^2 + C_3 x + C_4) e^x$$

(Verify it)

$$\text{and P.I.} = \frac{1}{(1 + 2)(D - 1)^3} e^x$$

(Replace D by 1).

$$= \frac{1}{3} \frac{x^3}{3!} e^x \quad \left\{ \because \frac{1}{(D - a)^r} e^{ax} = \frac{x^r}{r!} e^{ax} \right\}$$

∴ The required solution is

$$\Rightarrow y = C_1 e^{-2x} + (C_2 x^2 + C_3 + C_4) e^x + \frac{1}{18} x^3 e^x$$

Ex. 195. Solve $(D^2 - 4D + 4)y = 8(x^2 + e^{3x} + \sin 2x)$

The given differential equation is

$$(D^2 - 4D + 4)y = 8(x^2 + e^{3x} + \sin 2x)$$

$$C.F. = (C_1 x + C_2 +) e^{2x}$$

$$P.I. = \frac{1}{D^2 - 4D + 4} 8(x^2 + e^{3x} + \sin 2x)$$

$$= 8 \frac{1}{(D - 2)^2} x^2 + 8 \frac{1}{(D - 2)^2} e^{3x} + 8 \frac{1}{(D^2 - 4D + 4)} \sin 2x$$

$$\begin{aligned}
&= 2 \frac{1}{\left(1 - \frac{1}{2}D\right)^2} x^2 + 8e^{2x} \frac{1}{(D-2)} + 8 \frac{1}{(-4-4D+4)} \sin 2x \\
&= 2 \left(1 - \frac{1}{2}D\right)^{-2} x^2 + 8 \frac{x^2}{1} e^{2x} - 2 \frac{1}{D} \sin(2x) \\
&= 2 \left(1 + D + \frac{3}{4}D^2 + \dots\right) x^2 + 4x^2 e^{2x} - 2 \int \sin 2x dx \\
&= 2 \left(x^2 + 2x + \frac{3}{4}\right) + 4x^2 e^{2x} + \cos 2x \\
&= 2x^2 + 4x + \frac{3}{2} + 4x^2 e^{2x} + \cos 2x
\end{aligned}$$

∴ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = (C_1 x + C_2) e^{2x} + 2x^2 + 4x + \frac{3}{2} + 4x^2 e^{2x} + \cos 2x$$

Ex. 196. Solve $(D^3 - D)y = e^x + e^{-x}$

The AE is

$$\begin{aligned}
&m^3 - m = 0 \\
\Rightarrow &m(m^2 - 1) = 0 \\
\Rightarrow &m(m-1)(m+1) = 0 \\
\Rightarrow &m = 0, 1, -1
\end{aligned}$$

$$\begin{aligned}
C.F. &= C_1^{0x} + C_2 e^{-x} + C_3 e^x \\
&= C_1 + C_2 e^{-x} + C_3 e^x
\end{aligned}$$

$$\text{and } P.I. = \frac{1}{D(D^2 - 1)} + \frac{1}{D(D^2 - 1)} e^{-x}$$

$$= \frac{1}{(D^2 - 1)} e^x + \frac{1}{-1(D^2 - 1)} e^{-x}$$

$$= x \cdot \frac{1}{2D} e^x - x \cdot \frac{1}{2D} e^{-x}$$

$$= x \cdot \frac{1}{2} e^x - \frac{x}{2} \frac{1}{(-1)} e^{-x}$$

$$= \frac{x}{2} e^x + \frac{x}{2} e^{-x}$$

$$= x^2 (e^x + e^{-x})$$

∴ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = C_1 + C_2 e^{-x} + C_3 e^x + \frac{1}{2} x (e^x + e^{-x})$$

Ex. 197. Solve $(D^3 - 1)y = (e^x + 1)^2$

The given differential equation is

$$(D^3 - 1)y = (e^x + 1)^2$$

$$C.F. = C_1 e^x + e^{-x/2} [C_2 \cos(x\sqrt{3}/2) + C_3 \sin(x\sqrt{3}/2)]$$

(Verify it)

$$P.I. = \frac{1}{D^3 - 1} (e^x + 1)^2$$

$$= \frac{1}{D^3 - 1} (e^x + 2e^x + 1)^2$$

$$= \frac{1}{D^3 - 1} e^{2x} + 2 \frac{1}{D^3 - 1} e^x \frac{1}{1 - D^3} \quad (1)$$

$$= \frac{1}{2^3 - 1} e^{2x} + 2 \frac{1}{(D-1)(D^2 + D + 1)} e^x + \frac{1}{(D^3 - 1)} e$$

$$= \frac{1}{7} e^{2x} + 2 \cdot \frac{1}{(1+1+1)(D+1)} e^x + \frac{1}{(0-1)} e$$

$$= \frac{1}{7} e^{2x} + \frac{2}{3} x e^x - 1$$

∴ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = C_1 e^x + e^{-x/2} [C_2 \cos(x\sqrt{3}/2) + C_3 \sin(x\sqrt{3}/2)] + \frac{1}{7} e^{2x} + \frac{2}{3} x e^x - 1$$

Prop. -III. When X is of the form $\sin ax$ or $\cos ax$ & $F(-a^2) \neq 0$

$$P.I. = \frac{1}{F(D)} X = \frac{1}{F(D^2)} \sin ax \text{ or } \cos ax$$

(Replace D^2 by $-a^2$, D^4 by $-a^4$ and so on provided $F(-a^2) \neq 0$)

Explanation.

We know $D(\sin ax) = a \cos ax$

$$\Rightarrow D^2(\sin ax) = -a^2 \sin ax = (-a^2) \sin ax$$

$$\Rightarrow D^3(\sin ax) = -a^3 \cos ax$$

$$\Rightarrow D^4(\sin ax) = -a^4 \sin ax = (-a^2)^2 \sin ax$$

$$\Rightarrow D^5(\sin ax) = -a^5 \cos ax$$

$$\text{In general } D^{2n}(\sin ax) = (-a^2)^n \sin ax$$

$$\Rightarrow (D^2)^n (\sin ax) = (-a^2)^n (\sin ax)$$

$$\Rightarrow F(D^2) (\sin ax) = F(-a^2) (\sin ax)$$

$$\Rightarrow \frac{1}{F(D^2)} \cdot F(D^2)(\sin ax) = \frac{1}{F(D^2)} F(-a^2) \sin ax$$

$$\Rightarrow \sin ax = F(-a^2) \frac{1}{F(D^2)} \sin ax$$

$$\Rightarrow \frac{1}{F(D^2)} \sin ax = \frac{1}{F(D^2)} \sin ax, \text{ if } F(-a^2) \neq 0$$

$$\text{Similarly } \frac{1}{F(D^2)} \cos ax = \frac{1}{F(D^2)} \cos ax, \text{ if } F(-a^2) \neq 0$$

Ex. 198. Solve $(D^2 + 1)y = \sin 2x$

The given difference equation is

$$(D^2 + 1)y = \sin 2x$$

$$(D^2 + 1)y = \sin 2x$$

Here the auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

$$\therefore \text{C.F.} = C_1 \sin x + C_2 \cos x, \text{ where } C_1 \text{ and } C_2 \text{ are arbitrary constants,}$$

$$\text{and P.I.} = \frac{1}{D^2 + 1} \sin 2x = \frac{1}{(-2^2) + 1} \sin 2x, \quad (\text{Replace } D^2 \text{ by } -2^2)$$

$\therefore$ The required solution is

$$y = C_1 \cos x + C_2 \sin x - \frac{1}{3} \sin 2x$$

Ex. 199. Solve $(D^2 + 9)y = e^x - \cos 2x$

The given differential equation is

$$(D^2 + 9)y = e^x - \cos 2x$$

The AE is $m^2 + 9 = 0 \Rightarrow m = \pm 3i$

Here C.F. = $C_1 \cos 3x + C_2 \sin 3x$, where C_1 and C_2 are arbitrary constant,

$$\text{and P.I.} = \frac{1}{D^2 + 9} (e^x - \cos 2x) = \frac{1}{D^2 + 9} e^x - \frac{1}{D^2 + 9} \cos 2x$$

Ex 200. $(D^2 - 4D + 4)y = e^{-4x} + 5 \cos 3x$

The given differential equation is

$$(D^2 - 4D + 4)y = e^{-4x} + 5 \cos 3x$$

Here C.F. = $(C_1 + C_2)e^{2x}$, (Verify it)

where C's are auxiliary constants

$$\text{and P.I.} = \frac{1}{D^2 - 4D + 4} (e^{-4x} + 5 \cos 3x)$$

$$= \frac{1}{(D^2 - 4D + 4)} e^{-4x} + \frac{5}{(D^2 - 4D + 4)} \cos 3x$$

$$= \frac{1}{(-4)^2 - 4(-4) + 4} e^{-4x} + 5 \frac{1}{-4 - 4D + 4} \cos 3x$$

$$= \frac{1}{36} e^{-4x} - \frac{5}{4D} \cos 3x$$

$$= \frac{1}{36} e^{-4x} - \frac{5(4D - 5)}{(16D^2 - 25)} \cos 3x$$

$$= \frac{1}{36} e^{-4x} - \frac{5(4D^2 - 25)}{[16(-3^2) - 25]} \cos 3x$$

$$= \frac{1}{36} e^{-4x} + \frac{5(4D - 5)}{169} \cos 3x$$

$$= \frac{1}{36} e^{-4x} + \frac{5}{169} [-12 \sin 3x - 5 \cos 3x]$$

$\therefore$ Required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = \left[(C_1 x + C_2) e^{2x} + (e^{-4x}/36) - \frac{60}{169} \sin 3x - \frac{25}{169} \cos 3x \right]$$

Ex. 201. Solve $(D^2 + 9)y = \cos 4x$

The given differential equation is

$$(D^2 + 9)y = \cos 4x$$

The AE is

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$$\therefore \text{C.F.} = C_1 \cos 3x + C_2 \sin 3x,$$

$$\text{and P.I.} = \frac{1}{D^2 + 9} \cos 4x = \frac{1}{-(4^2) + 9} \cos 4x$$

$$= \frac{1}{-7} \cos 4x$$

Required solution is

$$y = C_1 \cos 3x + C_2 \sin 3x - \frac{1}{7} \cos 4x$$

Ex. 202. Solve $(D^2 - 2D + 5)y = 10 \sin x$

$$(D^2 - 2D + 5)y = 10 \sin x$$

$$\text{Here C.F.} = e^x [C_1 \cos 2x + C_2 \sin 2x],$$

(Verify if)

$$\text{and P.I.} = \frac{1}{D^2 - 2D + 5} (10 \sin x)$$

$$= 10 \frac{1}{D^2 - 2D + 5} \sin x$$

$$= 10 \frac{1}{(-1^2) - 2D + 5} \sin x$$

$$= 10 \frac{1}{4 - 2D} \sin x$$

$$= 5 \cdot \frac{1}{2 - D} \sin x$$

$$= 5 \cdot \frac{1}{(2 + D)(2 - D)} \sin x$$

$$= 5 \cdot \frac{1}{4 - D^2} \sin x$$

$$= 5 \cdot \frac{1}{4 - (-1)} (2 \sin x + D(\sin x)) = 2 \sin x + \cos x$$

Hence required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = C_1 x + C_2 + C_3 e^x + C_4 e^{-3x} + \frac{3}{20} e^{2x} + \frac{4}{5} \sin x + \frac{2}{5} \cos x$$

Ex. 204. Solve $(D^2 - 5D + 6)y = \sin 3x$

The given differential equation is

$$(D^2 - 5D + 6)y = \sin 3x$$

$$\text{Here C.F.} = C_1 e^{2x} + C_2 e^{3x},$$

(Verify it)

$$\text{and P.I.} = \frac{1}{D^2 - 5D + 6} \sin 3x$$

(Replace D^2 by -3^2)

$$= \frac{1}{-9 - 5D + 6} \sin 3x$$

$$= \frac{1}{-(5D + 3)} \sin 3x$$

$$= \frac{-1}{(5D + 3)(5D - 3)} (5D - 3) \sin 3x$$

$$= \frac{-1}{(25D^2 - 9)} (5D - 3) \sin 3x$$

$$= \frac{-1}{(25(-3^2) - 9)} (5D - 3) \sin 3x$$

$$= \frac{1}{234} [5(3 \cos 3x) - 3 \sin 3x]$$

$$= \frac{1}{78} (5 \cos 3x - \sin 3x)$$

Hence the required solution is

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{78} (5 \cos 3x - \sin 3x)$$

Ex. 205. Solve $(D^2 + 16)y = \sin 2x$, given that $y = 0$ and $\frac{dy}{dx} = \frac{5}{6}$ when $x = 0$

The given differential equation is

$$(D^2 + 16)y = \sin 2x$$

Here C. F. = $C_1 \cos 4x + C_2 \sin 4x$,

$$\text{P. I.} = \frac{1}{D^2 + 16} \sin 2x = \frac{1}{-2^2 + 16} \sin 2x = \frac{1}{12} \sin 2x$$

$\therefore$ The complete solution is

$$y = \text{C. F.} + \text{P. I.}$$

$$\Rightarrow y = C_1 \cos 4x + \frac{1}{12} \sin 2x$$

Differentiating both sides of (1) with respect to x , we get

$$\frac{dy}{dx} = -4C_1 \sin 4x + 4C_2 \cos 4x + \frac{1}{6} \cos 2x \quad (2)$$

$$\text{Given that } y = 0, \frac{dy}{dx} = \frac{5}{6} \text{ when } x = 0$$

$\therefore$ From (1) and (2) we get

$$0 = C_1 \text{ and } \frac{5}{6} = 4C_2 + \frac{1}{6}$$

These give

$$C_1 = 0, C_2 = \frac{1}{6}$$

$\therefore$ From (1) the required solution is

$$y = \frac{1}{6} \sin 4x + \frac{1}{12} \sin 2x$$

$$\Rightarrow 12y = 2 \sin 4x + \sin 2x$$

Ex. 206. Solve $(D^2 - 4)y = e^x + \sin 2x$

The given differential equation is

$$(D^2 - 4)y = e^x + \sin 2x$$

The AE is

$$m^2 - 4 = 0$$

$$\Rightarrow (m - 2)(m + 2) = 0$$

$$\Rightarrow m = 2, -2$$

$$\therefore \text{C. F.} = C_1 e^{2x} + C_2 e^{-2x},$$

$$\text{and P. I.} = \frac{1}{D^2 - 4} (e^x + \sin 2x)$$

$$= \frac{1}{D^2 - 4} e^x + \frac{1}{D^2 - 4} \sin 2x$$

$$= \frac{1}{1^2 - 4} e^x + \frac{1}{(-2^2) - 4} \sin 2x$$

$$= -\frac{1}{3} e^x - \frac{1}{8} \sin 2x$$

$\therefore$ The required solution is

$$y = \text{C. F.} + \text{P. I.}$$

$$\Rightarrow y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{8} \sin 2x$$

Ex. 207. Solve $(D^4 + 4)y = \sin 3x + e^x$

The given differential equation is

$$(D^4 + 4)y = \sin 3x + e^x$$

The AE is

$$m^4 + 4 = 0$$

$$\Rightarrow (m^2 + 2)^2 - 2m^2 = 0$$

$$\Rightarrow (m^2 + 2 - 2m)(m^2 + 2 + 2m) = 0$$

$$\Rightarrow m^2 - 2m + 2 = 0, m^2 + 2m + 2 = 0$$

$$\Rightarrow \frac{2 \pm \sqrt{4-8}}{2}, \frac{-2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow m = 1 \pm i, -1 \pm i$$

$$\therefore \text{C.F.} = e^x(C_1 \cos x + C_2 \sin x) + e^{-x}(C_3 \cos x + C_4 \sin x)$$

$$\text{P.I.} = \frac{1}{D^4 + 4}(\sin 3x + e^x)$$

$$= \frac{1}{D^4 + 4} \sin 3x + \frac{1}{D^4 + 4} e^x$$

$$= \frac{1}{(-3^2)^2 + 4} \sin x + \frac{1}{1 + 4} e^x$$

$$= \frac{1}{85} \sin 3x + \frac{1}{5} e^x$$

$\therefore$ Required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = (C_1 \cos x + C_2 \sin x)e^x + (C_3 \cos x + C_4 \sin x)e^{-x} + \frac{1}{85} \sin 3x + \frac{1}{5} e^x$$

Ex. 208. Find the integral of the equation

$$\frac{d^2 x}{dt^2} + 2n \cos \alpha \frac{dx}{dt} + a^2 x = a \cos nt,$$

which is such that when $t = 0, x = 0$ and $\frac{dx}{dt} = 0$.

Solution The given differential equation is

$$\frac{d^2 x}{dt^2} + 2n \cos \alpha \frac{dx}{dt} + a^2 x = a \cos nt$$

$$\Rightarrow (D^2 + 2n \cos \alpha D + n^2)x = a \cos nt, \quad \left(\text{where } D = \frac{d}{dt}\right)$$

Its auxiliary equation is $m^2 + (2n \cos \alpha)m + n^2 = 0$

$$\text{or } m = \frac{-(2n \cos \alpha) \pm \sqrt{(4n^2 \cos^2 \alpha - 4n^2)}}{2}$$

$$= -n \cos \alpha \pm \sqrt{(-n^2 \sin^2 \alpha)}$$

$$= (-n \cos \alpha) \pm i(n \sin \alpha)$$

$$\therefore \text{C.F.} = e^{-n \cos \alpha t} \{C_1 \cos(n \sin \alpha)t + C_2 \sin(n \sin \alpha)t\}$$

$$\text{and P.I.} = \frac{1}{(D^2 + 2n \cos \alpha D + n^2)} a \cos nt$$

$$= a \cdot \frac{1}{(-n^2 + 2n \cos \alpha D + n^2)} \cos nt,$$

$$= a \cdot \frac{1}{2n \cos \alpha} \cos nt$$

$$= \frac{a}{2n \cos \alpha} \int \cos nt \, dt = \left[\frac{a}{(2n \cos \alpha)} \right] \left\{ \frac{(\sin nt)}{n} \right\} = \frac{a \sin nt}{2n^2 \cos \alpha}$$

$\therefore$ The complete solution is $x = \text{C. F.} + \text{P. I.}$

$$\Rightarrow x = e^{-nt \cos \alpha} \{C_1 \cos(n \sin \alpha) t + C_2 \sin(n \sin \alpha) t\} + \frac{(a \sin nt)}{(2n^2 \cos \alpha)}$$

Differentiating both sides of (1) with respect to t , we get

$$\frac{dx}{dt} = e^{-nt \cos \alpha} n \sin \alpha \{-C_1 \sin(n \sin \alpha) t + C_2 \cos(n \sin \alpha) t\}$$

$$+ n \cos \alpha e^{-nt \cos \alpha} (C_1 \cos(n \sin \alpha) t + C_2 \sin(n \sin \alpha) t) + \frac{(a \cos nt)}{(2n \cos \alpha)}$$

It is given that when $t = 0, x = 0$ and $\frac{dx}{dt} = 0$

$\therefore$ From (1) and (2), we have

$$0\{C_1(1) + C_2(0) + \alpha(0)\} \Rightarrow C_1 = 0$$

$$\text{and } 0 = n \sin \alpha \{-C_1(0) + C_2(1)\} - n \cos \alpha \{C_1(1) + C_2(0)\} + \frac{a}{2n \cos \alpha}$$

$$\Rightarrow 0 = (n \sin \alpha)\{C_2\} - (n \cos \alpha)(C_1) + \left\{ \frac{a}{2n \cos \alpha} \right\} \quad [C_1 = 0]$$

$$\Rightarrow C_2 = -\frac{a}{2n^2 \cos \alpha \sin \alpha} = -\frac{a}{n^2 \sin 2\alpha}$$

Substituting these values of C_1 and C_2 in (1), we get the required solution as

$$x = e^{-nt \cos \alpha} \left\{ -\frac{a}{n^2 \sin 2\alpha} \sin(n \sin \alpha) t \right\} + \frac{a \sin nt}{2n^2 \cos \alpha}$$

Ex. 209. Solve $(D^3 + 1)y = \cos 2x$

The given differential equation is

$$(D^3 + 1)y = \cos 2x$$

The AE is

$$m^3 + 1 = 0$$

$$\Rightarrow (m + 1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1, \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$\Rightarrow m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$\text{C. F.} = C_1 e^{-x} + e^{x/2} \left\{ C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x \right\},$$

$$\text{and P. I.} = \frac{1}{D^3 + 1} \cos 2x$$

$$\begin{aligned}
&= \frac{1}{D^2 D + 1} \cos 2x \\
&= \frac{1}{(-2^2)D + 1} \cos 2x \quad (\text{Replace } D^2 \text{ by } -2^2) \\
&= \frac{1}{1 - 4D} \cos 2x \\
&= \frac{1}{(1 - 41)(1 + 4D)} (1 + 4D) \cos 2x \\
&= \frac{1}{(1 - 16D^2)} (1 + 4D) \cos 2x \\
&= \frac{1}{1 - 16(-2^2)} (1 + 4D) \cos 2x \\
&= \frac{1}{1 + 64} (\cos 2x + 4D(\cos 2x)) \\
&= \frac{1}{65} \{\cos 2x + 4D(\cos 2x)\} \\
&= \frac{1}{65} (\cos 2x - 8 \sin 2x)
\end{aligned}$$

∴ The required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = C_1 e^{-x} + e^{x/2} \{C_2 \cos(x\sqrt{3}/2) + C_2 \sin(x\sqrt{3}/2)\} + \frac{1}{65} (\cos 2x - 8 \sin 2x)$$

Ex. 210. Solve $(D^2 - 3D + 2)y = 6e^{3x} + \sin 2x$

The given differential equation is

$$(D^2 - 3D + 2)y = 6e^{3x} + \sin 2x$$

$$\text{C.F.} = C_1 e^x + C_2 e^{2x},$$

(Verify it)

where C's are arbitrary constants

$$\begin{aligned}
\text{and P.I.} &= \frac{1}{D^2 - 3D + 2} 6e^{3x} + \frac{1}{D^2 - 3D + 2} \sin 2x \\
&= 6 \cdot \frac{1}{3^2 - 3(3) + 2} e^{3x} + \frac{1}{-2^2 - 3D + 2} \sin 2x \\
&= 3e^x - \frac{1}{(3D + 2)} \sin 2x \\
&= 3e^{3x} - \frac{(3D - 2)}{(3D + 2)(3D - 2)} \sin 2x \\
&= 3e^{3x} - \frac{(3D - 2)}{(9D^2 - 4)} \sin 2x \\
&= 3e^{3x} - \frac{(3D - 2)}{9(-2^2) - 4} \sin 2x \\
&= 3e^{3x} + \frac{1}{40} \{3D(\sin 2x) - 2 \sin 2x\} \\
&= 3e^{3x} + \frac{1}{40} [6 \cos 2x - 2 \sin 2x] = 3e^{3x} + \frac{3 \cos 2x - \sin 2x}{20}
\end{aligned}$$

∴ Required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = C_1 e^x + C_2 e^{2x} + 3e^{3x} + \frac{(3 \cos 2x - \sin 2x)}{20}.$$

Prop. -IV. When X is of the form x^m

$$\text{In this case PI} = \frac{1}{F(D)} x^m = \frac{1}{[1 \pm F(D)]} x^m = [1 \pm F(D)]^{-1} x^m$$

(Expand binomially & simplify)

Ex. 211. Solve $(D^3 - D^2 - 6D)y = x^2 + 1$ **Solution.** The AE is

$$m^3 - m^2 - 6m = 0$$

$$\Rightarrow m(m^2 - m - 6) = 0$$

$$\Rightarrow m(m - 3)(m + 2) = 0$$

$$\Rightarrow m = 0, 3, -2$$

$$\therefore CF = C_1 e^{0x} + C_2 e^{3x} + C_3 e^{-2x},$$

where C_1, C_2 and C_3 are arbitrary constants

$$\text{and PI} = \frac{1}{D^3 - D^2 - 6D} (x^2 + 1)$$

$$= \frac{1}{-6D + \left(1 + \frac{1}{6}D - \frac{1}{6}D^2\right)(x^2 + 1)}$$

$$= -\frac{1}{6D} \left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1} (x^2 + 1)$$

$$= -\frac{1}{6D} \left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 - \dots\right] (x^2 + 1)$$

$$= -\frac{1}{6D} \left[1 - \frac{D}{6} + \frac{7D^2}{36} \dots\right] (x^2 + 1)$$

$$= -\frac{1}{6D} \left[1 + x^2 - \frac{1}{6}D(x^2) + \frac{7}{36}D^2(x^2)\right] \quad (\because D(1) = 0)$$

$$= -\frac{1}{6D} \left(1 + x^2 - \frac{1}{3}x + \frac{7}{18}\right)$$

$$= -\frac{1}{6} \int \left[1 + x^2 - \frac{1}{3}x + \frac{7}{18}\right] dx, \quad \left(\text{since } D = \frac{d}{dx}\right)$$

$$\Rightarrow PI = -\frac{1}{6} \left(x + \frac{1}{3}x^3 - \frac{1}{6}x^2 + \frac{7}{18}x\right)$$

$$= -\frac{1}{6} \left[\frac{25}{18}x + \frac{1}{3}x^3 - \frac{1}{6}x^2\right]$$

 $\therefore$ the required solution is

$$y = CF + PI$$

$$\Rightarrow y = C_1 + C_2 e^{3x} + C_3 e^{-2x} - \frac{24}{108}x - \frac{1}{18}x^3 + \frac{1}{36}x^2$$

Ex. 212. Solve $(D^3 - 4)y = x^2$ **Solution** $CF = C_1 e^{2x} + C_2 e^{-2x}$

(Verify it)

where C_1 and C_2 are arbitrary constants

$$\text{and PI} = \frac{1}{D^2 - 4} x^2$$

$$= \frac{1}{-4 \left(1 - \frac{1}{4}D^2\right)} x^2$$

$$= -\frac{1}{4} \left(1 - \frac{1}{4}D^2\right)^{-1} x^2$$

$$= -\frac{1}{4} \left[1 + \frac{1}{4}D^2 + \dots\right] x^2$$

$$= -\frac{1}{4} \left[x^2 + \frac{1}{4}(2)\right] = -\frac{1}{4} \left(x^2 + \frac{1}{2}\right)$$

The required solution is $y = CF + PI$

$$\Rightarrow y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{4} \left(x^2 + \frac{1}{2}\right)$$

Ex. 213. Solve $(D^2 + 2D + 2)y = x^2$ **Solution** The AE is

$$m^2 + 2m + 2 = 0$$

$$\Rightarrow m = \frac{1}{2} \{-2 \pm \sqrt{(4 - 8)}\} = -1 \pm i$$

$$\therefore CF = e^{-x} [C_1 \cos x + C_2 \sin x], \quad \text{where } C_1 \text{ and } C_2 \text{ are arbitrary constants}$$

$$\text{and PI} = \frac{1}{D^2 + 2D + 2} x^2$$

$$= \frac{1}{2 \left(1 + D + \frac{1}{2}D^2\right)} x^2$$

$$= \frac{1}{2} \left[1 + \left(D + \frac{D^2}{2}\right)\right]^{-1} x^2$$

$$= \frac{1}{2} \left[1 - \left(D + \frac{1}{2} D^2 \right) + D^2 + \dots \right] x^2$$

$$= \frac{1}{2} \left[1 - D + \frac{1}{2} D^2 + \dots \right] x^2 = \frac{1}{2} (x^2 - 2x + 1)$$

∴ the required solution is

$$y = CF + PI$$

$$\Rightarrow y = e^{-x} (C_1 \cos x + C_2 \sin x) + \frac{1}{2} (x^2 - 2x + 1)$$

Ex. 214. Solve $(D^2 - 5D + 6)y = x$

Solution Its CF = $C_1 e^{2x} + C_2 e^{3x}$, (Verify it)

where C_1 and C_2 are arbitrary constants

$$\text{and PI} = \frac{1}{D^2 - 5D + 6} x$$

$$= \frac{1}{6 \left[1 - \frac{5}{6} D + \frac{1}{6} D^2 \right]} x$$

$$= \frac{1}{6} \left[1 - \frac{1}{6} (5D - D^2) \right]^{-1} x$$

$$= \frac{1}{6} \left[1 + \frac{1}{6} (5D - D^2) + \dots \right] x$$

$$= \frac{1}{6} \left[x + \frac{5}{6} \right] = \frac{1}{6} x + \frac{5}{36}$$

∴ The required solution is

$$y = CF + PI$$

$$\Rightarrow y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{6} x + \frac{5}{36}$$

Ex. 215. Solve $(D^2 - 5D + 6)y = x + \sin 3x$

Solution The AE is

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow m = 3, 2$$

∴ CF = $C_1 e^{3x} + C_2 e^{2x}$, where C_1 & C_2 are arbitrary constants

$$\text{and PI} = \frac{1}{D^2 - 5D + 6} (x + \sin 3x)$$

$$= \frac{1}{D^2 - 5D + 6} x + \frac{1}{D^2 - 5D + 6} \sin 3x$$

$$= \frac{1}{6 \left[1 - \left(\frac{5}{6} \right) D + \frac{1}{6} D^2 \right]} x + \frac{1}{-3^2 - 5D + 6} \sin 3x$$

$$= \frac{1}{6} \left[1 - \left(\frac{5}{6} \right) D + \frac{1}{6} D^2 \right]^{-1} x + \frac{1}{(-3 - 5D)} \sin 3x$$

$$= \frac{1}{6} \left[1 + \left(\frac{5}{6} \right) D + \dots \right] x - \frac{(5D - 3)}{(5D + 3)(5D - 3)} \sin 3x$$

$$= \frac{1}{6} \left[x + \left(\frac{5}{6} \right) \right] - \frac{1}{25D^2 - 9} (5D - 3) \sin 3x$$

$$= \frac{1}{6} \left[x + \left(\frac{5}{6} \right) \right] + \left(\frac{1}{234} \right) [5D(\sin 3x) - 3(\sin 3x)]$$

$$= \frac{1}{6} \left[x + \left(\frac{5}{6} \right) \right] + \left(\frac{1}{234} \right) (15 \cos 3x - 3 \sin 3x)$$

∴ The required solution is $y = CF + PI$

$$\Rightarrow y = C_1 e^{3x} + C_2 e^{2x} + \frac{1}{6} \left(x + \frac{5}{6} \right) + \left(\frac{1}{234} \right) (15 \cos 3x - 3 \sin 3x).$$

Ex. 216. Solve $(D^2 + D - 2)y = x + \sin x$

Solution The AE is

$$m^2 + m - 2 = 0$$

$$\Rightarrow (m + 2)(m - 1) = 0$$

$$\Rightarrow m = 1, -2$$

∴ CF = $C_1 e^x + C_2 e^{-2x}$, where C_1 and C_2 are arbitrary constants,

$$\text{and PI} = \frac{1}{D^2 + D - 2} (x + \sin x)$$

$$= \frac{1}{D^2 + D - 2} x + \frac{1}{D^2 + D - 2} \sin x$$

$$\begin{aligned}
&= \frac{1}{-2\left(1 - \frac{1}{2}D + \frac{1}{2}D^2\right)}x + \frac{1}{-1 + D - 2}\sin x \\
&= -\frac{1}{2}\left(1 - \frac{D}{2} + \frac{D^2}{2}\right)^{-1}x + \frac{1}{D-3}\sin x \\
&= -\frac{1}{2}\left(1 - \frac{D}{2} + \dots\right)x + \frac{1}{D^2-9}(D+3)\sin x \\
&= -\frac{1}{2}\left(x + \frac{1}{2} \cdot 1\right) + \frac{1}{-1-9}(D+3)\sin x \\
&= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3\sin x) \\
&= -\frac{x}{2} - \frac{1}{4} - \frac{1}{10}(\cos x) - \frac{3}{10}\sin x
\end{aligned}$$

∴ The required solution is

$$y = CF + PI$$

$$\Rightarrow y = C_1 e^x + C_2 e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \left(\frac{1}{10}\right)\cos x - \left(\frac{3}{10}\right)\sin x$$

Ex. 217. Solve $(D^3 + 8)y = x^4 + 2x + 1$

Solution The AE is

$$m^3 + 8 = 0$$

$$\Rightarrow (m+2)(m^2 - 2m + 4) = 0$$

$$\Rightarrow m = -2, 1 \pm i\sqrt{3}$$

$$\therefore CF = C_1 e^{-2x} + e^x \left\{ C_2 \cos \frac{x}{\sqrt{3}} + C_3 \sin \frac{x}{\sqrt{3}} \right\}, \text{ where } C_1, C_2 \text{ and } C_3 \text{ are arbitrary constant}$$

$$\text{Also PI} = \frac{1}{D^3 + 8}(x^4 + 2x + 1)$$

$$= \frac{1}{8\left(1 + \frac{1}{8}D^3\right)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{1}{8}D^3\right)^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{1}{8}D^3 + \dots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}[(x^4 + 2x + 1) - 3x] = \frac{1}{8}(x^4 - x + 1)$$

∴ the required solution is

$$y = CF + PI$$

$$\Rightarrow y = C_1 e^{-2x} + e^x \left\{ C_2 \cos \frac{x}{\sqrt{3}} + C_3 \sin \frac{x}{\sqrt{3}} \right\} + \frac{1}{8}(x^4 - x + 1)$$

Ex. 218. Solve $(D^3 + 2D^2 + D)y = e^{2x} + x^2$

Solution The AE is

$$m^3 + 2m^2 + m = 0$$

$$\Rightarrow m(m^2 + 2m + 1) = 0$$

$$\Rightarrow m = 0, -1, 1$$

$$\therefore CF = C_1 e^{0x} + (C_2 x + C_3) e^{-x}, \quad \text{where } C' \text{ sare arbitrary constant}$$

$$\text{and PI} = \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} (x^2)$$

$$= \frac{1}{2^3 + 2(2^2) + 2} e^{2x} + \frac{1}{D(D+1)^2} (x^2)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} (1+D)^{-2} (x^2)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} (1 - 2D + 3D^2 - \dots) (x^2)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} [(x^2) - 2(2x) + 3(2)]$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} [x^2 - 4x + 6]$$

$$= \frac{1}{18} e^{2x} + \int (x^2 - 4x + 6) dx$$

$$= \frac{1}{18} e^{2x} + \frac{x^3}{3} - 4\left(\frac{x^2}{2}\right) + 6x$$

∴ the required solution is

$$y = CF + PI$$

$$\Rightarrow y = C_1 + (C_2x + C_3)e^{-x} + \frac{e^{2x}}{18} + \frac{x^3}{3} - 2x^2 + 6x$$

Ex. 219. Solve $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$

Solution CF = $C_1 + (C_2x + C_3)e^{-x}$

$$\begin{aligned} \text{and PI} &= \frac{1}{D^3 + 2D^2 + D} (e^{2x} + x^2 + x) \\ &= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} (x^2 + x) \\ &= \frac{1}{(2^3 + 2 \cdot 2 + 2)} e^{2x} + \frac{1}{D(1 + 2D + D^2)} (x^2 + x) \\ &= \frac{1}{18} e^{2x} + \frac{1}{D} (1 + 2D + D^2)^{-1} (x^2 + x) \\ &= \frac{1}{18} e^{2x} + \frac{1}{D} [1 - (2D + D^2) + (2D + D^2)^2 + \dots] (x^2 + x) \\ &= \frac{1}{18} e^{2x} + \frac{1}{D} [1 - 2D + 3D^2 + \dots] (x^2 + x) \\ &= \frac{1}{18} e^{2x} + \frac{1}{D} [(x^2 + x) - 2D(x^2 + x) + 3D^2(x^2 + x)] \\ &= \frac{1}{18} e^{2x} + \frac{1}{D} [(x^2 + x) - 2(2x + 1) + 3(2)] \\ &= \frac{1}{18} e^{2x} + \int (x^2 - 3x + 4) dx = \frac{1}{18} e^{2x} + \frac{1}{3} x^3 - \frac{3}{2} x + 4x \end{aligned}$$

The required solution is

$y = CF + PI$

$$\Rightarrow y = C_1 + (C_2x + C_3)e^{-x} + \frac{1}{18} e^{2x} + \frac{1}{3} x^3 - \frac{3}{2} x^2 + 4x$$

Ex. 220. Find the solution of the equation $(D^2 - 1)y = 1$ which vanishes when $x = 0$ and tends to a finite limit as $x \rightarrow \infty$ and D stands for $\frac{d}{dx}$.

Solution The AE is

$$m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

$\therefore CF = C_1 e^x + C_2 e^{-x}$ where C_1 are arbitrary constants

$$\text{and PI} = \frac{1}{D^2 - 1} 1$$

$$= \frac{-1}{(1 - D^2)} 1$$

$$= -(1 - D^2)^{-1} 1$$

$$= -(1 + D^2 + D^4 + \dots) 1 = -1$$

$\therefore$ the complete solution of the given equation is

y

$$= CF + PI$$

$$\Rightarrow y = C_1 e^x + C_2 e^{-x} - 1$$

Now, it is given that $y = 0$ when $x = 0$

$\therefore$ From (1),

$$0 = C_1 e^0 + C_2 e^0 - 1$$

$$\text{i.e. } 0 = C_1$$

$$= C_2 - 1$$

Also $y \rightarrow$ a finite limit as $x \rightarrow -\infty$ (given)

This is possible only if $C_1 = 0$

$$\therefore C_2 = 1$$

Hence, from (1) the required solution is

$$y = e^{-x} - 1.$$

Ex. 221. If $\frac{d^2x}{dt^2} (x - a) = 0$, (a, b and g being positive numbers) and $\frac{dx}{dt} = 0$, when $t = 0$, show that $x = a + (a' - a) \cos \left[\sqrt{(g/b)t} \right]$.

Solution The given equation can be written as

$$\left(D^2 + \frac{g}{b} \right) x = \frac{ga}{b}, \quad \text{where } D = \frac{d}{dt}$$

$\therefore$ The AE is

$$m^2 + \frac{g}{b} = 0 \Rightarrow m = \pm i \left(\sqrt{\frac{g}{b}} \right)$$

$\therefore$ CF = $C_1 \cos \sqrt{\left(\frac{g}{b}\right)} t + C_2 \sin \sqrt{\left(\frac{g}{b}\right)} t$, where C_1 and C_2 are arbitrary constants and

$$\begin{aligned} \text{PI} &= \frac{1}{(D^2 + g/b)} \frac{ga}{b} \\ &= \frac{ga}{b} \cdot \frac{1}{(D^2 + g/b)} e^0 \\ &= \frac{ga}{b} \cdot \frac{b}{g} = a \end{aligned}$$

$\therefore$ the required solution is

$$x = \text{CF} + \text{PI}$$

$$\Rightarrow x = C_1 \cos \left\{ \sqrt{\left(\frac{g}{b}\right)} t \right\} + C_2 \sin \left\{ \sqrt{\left(\frac{g}{b}\right)} t \right\} + a$$

Differentiating (1) w. r. t. t, we have

$$\frac{dx}{dt} = -C_1 \sqrt{\left(\frac{g}{b}\right)} \sin \left\{ \sqrt{\left(\frac{g}{b}\right)} t \right\} + C_2 \sin \left\{ \sqrt{\left(\frac{g}{b}\right)} t \right\} \cos \left\{ \sqrt{\left(\frac{g}{b}\right)} t \right\}$$

Now, it is given that $x = a'$, when $t = 0$

$\therefore$ From (1),

$$a' = C_1 + a \Rightarrow C_1 = a' - a$$

$$\text{Also } \frac{dx}{dt} = 0 \text{ when } t = 0$$

$\therefore$ From (2)

$$0 = C_2 \sqrt{\left(\frac{g}{b}\right)} \Rightarrow C_2 = 0$$

Substituting these values of C_1 and C_2 in (1), the required solution as

$$x = (a' - a) \cos \left\{ \sqrt{\left(\frac{g}{b}\right)} t \right\} + a.$$

Prop. -V. When X is of the form $e^{ax}v$, where v is the function of x

$$\text{P.I.} = \frac{1}{F(D)} X = \frac{1}{F(D)} e^{ax} v = e^{ax} \frac{1}{F(D+a)} v \quad (\text{Replace } D \text{ by } D+a)$$

Ex. 222. Solve $(D^2 - 2D + 1)y = e^x \cdot x^2$

The given differential equation is

$$(D^2 - 2D + 1)y = e^x \cdot x^2,$$

$$\text{C.F.} = (C_1x + C_2)e^x, \quad (\text{Verify it})$$

$$\text{and P.I.} = \frac{1}{D^2 - 2D + 1} e^x \cdot x^2$$

$$= e^x \frac{1}{\{(D+1)^2 - 2(D+1) + 1\}} x^2$$

$$= e^x \frac{1}{(D^2 + 2D + 1 - 2D - 2 + 1)} x^2$$

$$= e^x \cdot \frac{1}{D^2} x^2$$

$$= e^x \cdot \frac{1}{D} \int x^2 dx$$

$$= e^x \cdot \frac{1}{D} \left(\frac{x^3}{3} \right) = e^x \int \frac{x^3}{3} dx = \frac{e^x x^4}{12}$$

$\therefore$ The required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = (C_1x + C_2)e^x + \frac{1}{12} e^x \cdot x^4.$$

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Ex. 224. Solve $(D^2 - 2D + 5)y = e^{2x} \sin x$

The given differential equation is

$$(D^2 - 2D + 5)y = e^{2x} \sin x$$

$$\text{C.F.} = e^x (C_1 \cos 2x + C_2 \sin 2x) \quad (\text{Verify it})$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{[(D+2)^2 - 2(D+2) + 5]} \sin x$$

$$\begin{aligned}
&= e^{2x} \frac{1}{D^2 2D + 5} \sin x \\
&= e^{2x} \frac{1}{(-1 + 2D + 5)} \sin x \\
&= e^{2x} \frac{1}{2D + 4} \sin x \\
&= e^{2x} \frac{1}{(2D + 4)(2D - 4)} (2D - 4) \sin x \\
&= e^{2x} \frac{1}{4D^2 - 16} (2D - 4) \sin x \\
&= e^{2x} \frac{1}{\{4(-1^2) - 16\}} (2D - 4) \sin x \\
&= e^{-2x} \frac{1}{(-20)} \{2 \cos x - 4 \sin x\} \\
&= -\frac{1}{10} e^{2x} (\cos x - 2 \sin x) \\
\therefore \text{The required solution is} \\
y &= \text{C.F.} + \text{P.I.}
\end{aligned}$$

$$\Rightarrow y = e^x (C_1 \cos 2x + C_2 \sin 2x) - \frac{1}{10} e^{2x} (\cos x - 2 \sin x)$$

Ex. 225 Solve $(D^2 - 2D + 4)y = e^x \cos x$

The given differential equation is

$$(D^2 - 2D + 4)y = e^x \cos x$$

$$\text{C.F.} = e^x [C_1 \cos(x\sqrt{3}) + C_2 \sin(x\sqrt{3})] \quad (\text{Verify it})$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x = e^x \frac{1}{-1 + 3} \cos x = e^x \frac{1}{2} \cos x$$

$\therefore$ The required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = e^x [C_1 \cos(x\sqrt{3}) + C_2 \sin(x\sqrt{3})] + \frac{1}{2} e^x \cos x$$

Ex. 226. Solve $(D^4 - 1)y = e^x \cos x$

The given differential equation is

$$(D^4 - 1)y = e^x \cos x$$

The A.E. is

$$m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow m = \pm 1, \pm i$$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x$$

$$\text{and P.I.} = \frac{1}{D^4 - 1} e^x \cos x$$

$$= e^x \frac{1}{\{(D+1)^4 - 1\}} \cos x$$

$$= e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D} \cos x$$

$$= e^x \frac{1}{(-1^2)^2 + 4(-1^2)D + 6(-1)^2 + 4D} \cos x$$

$$= e^x \frac{1}{1 - 4D - 6 + 4D} \cos x = \frac{1}{5} e^x \cos x$$

$\therefore$ The required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow y = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x - \frac{1}{5} e^x \cos x$$

Ex. 227 Solve $(D^2 - 2D + 1)y = x^2 e^{3x}$

The given differential equation is

$$(D^2 - 2D + 1)y = x^2 e^{3x}$$

$$\text{C.F.} = (C_1 x + C_2) e^x \quad (\text{Verify it})$$

$$\text{and P.I.} = \frac{1}{D^2 - 2D + 1} x^2$$

$$= e^{3x} \frac{1}{(D+3)^2 - 2(D+3) + 1} x^2$$

$$\begin{aligned}
&= e^{3x} \frac{1}{D^2 + 4D + 4} x^2 \\
&= e^{3x} \frac{1}{(2 + D)^2} x^2 \\
&= e^{3x} \frac{1}{4 \left(1 + \frac{1}{2}D\right)^2} x^2 \\
&= \frac{e^x}{4} \left(1 + \frac{1}{2}D\right)^{-2} x^2 \\
&= \frac{e^x}{4} \left[1 - 2 \left(\frac{1}{2}D\right) + 3 \left(\frac{1}{2}D\right)^2 + \dots\right] x^2 \\
&= \frac{e^x}{4} \left(1 - D + \frac{3}{4}D^2 + \dots\right) x^2 \\
&= \frac{1}{4} e^{3x} \left(x^2 - D(x^2) + \frac{3}{4} D^2(x^2)\right) \\
&= \frac{1}{4} e^{3x} \left\{x^2 - 2x + \left(\frac{3}{2}\right)\right\} \\
&= \frac{1}{8} e^{3x} (2x^2 - 4x + 3)
\end{aligned}$$

∴ The required solution is

$y = C.F. + P.I.$

$$\Rightarrow y = (C_1 x + C_2) e^x + \frac{1}{8} e^{3x} (2x^2 - 4x + 3)$$

Prop. -VI. When X is of the form $\sin ax$ or $\cos ax$ and $F(-a^2) = 0$

Ex. 228. Solve $(D^2 + a^2)y = \cos ax$

The given differential equation is

$$(D^2 + a^2)y = \cos ax$$

$$C.F. = C_1 \cos ax + C_2 \sin ax$$

(Verify it)

$$P.I. = \frac{1}{D^2 + a^2} \cos ax$$

$$= \frac{1}{D^2 + a^2} (\text{Real part } e^{iax})$$

$$\because e^{iax} = \cos ax + i \sin ax$$

$$= R.P. \text{ of } e^{mx} \frac{1}{(D + ia)^2 + a^2} 1$$

$$= R.P. \text{ of } e^{iax} \frac{1}{D^2 + 2aiD} 1$$

$$= R.P. \text{ of } e^{iax} \frac{1}{2aiD[1 + (D/2ai)]} 1$$

$$= R.P. \text{ of } \frac{e^{iax}}{2ai} \cdot \frac{1}{D} \left(1 + \frac{D}{2ai}\right)^{-1} 1$$

$$= R.P. \text{ of } \frac{e^{iax}}{2ai} \cdot \frac{1}{D} \left(1 - \frac{D}{2ai}\right)^{-1} + \dots$$

$$= R.P. \text{ of } \frac{e^{iax}}{2ai} \cdot \frac{1}{D} 1$$

$$= R.P. \text{ of } \frac{e^{iax}}{2ai} x$$

$$= R.P. \text{ of } \left[-\frac{ix}{2a} (\cos ax + i \sin ax) \right] = \frac{x}{2a} \sin ax$$

∴ The required solution is

$y = C.F. + P.I.$

$$\Rightarrow y = C_1 \cos ax + C_2 \sin ax + \frac{x}{2a} \sin ax$$

$$\text{Note. } \frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$$

Ex. 229. Solve $(D^2 + a^2)y = \sin ax$

The given differential equation is

$$(D^2 + a^2)y = \sin ax$$

$$C.F. = C_1 \cos ax + C_2 \sin ax$$

(Verify it)

$$P.I. = \frac{1}{D^2 + a^2} \sin ax$$

$$= \frac{1}{D^2 + a^2} (\text{Imaginary part of } e^{iax})$$

$$\begin{aligned}
&= \text{I. P. of } e^{iax} \frac{1}{(D+ia)^2 + a^2} 1 \\
&= \text{I. P. of } e^{iax} \frac{1}{D^2 + 2iaD} 1 \quad [\because i^2 a^2 = -a^2] \\
&= \text{I. P. of } e^{iax} \frac{1}{2iaD} \left(1 + \frac{D}{2ia}\right)^{-1} 1 \\
&= \text{I. P. of } e^{iax} \frac{1}{2iaD} \left(1 - \frac{D}{2ia} + \dots\right) 1 \\
&= \text{I. P. of } e^{iax} \frac{1}{2iaD} \frac{1}{D} (1) \\
&= \text{I. P. of } \frac{e^{iax}}{-2a} x \\
&= \text{I. P. of } \frac{-ix(\cos ax + i \sin ax)}{2a} = \frac{-\{x \cos(ax)\}}{2a} = -\frac{x}{2a} \cos ax
\end{aligned}$$

$\therefore$ The required solution is

$y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = C_1 \cos ax + C_2 \sin ax = \frac{x}{2a} \cos(ax)$$

$$\text{Note. } \frac{1}{D^2 + a^2} \sin ax = \frac{-x}{2a} \cos ax$$

Ex. 230. Solve $(D^2 + a^2)y = \sin ax$

The given differential equation is

$$(D^2 + a^2)y = \sin ax$$

The AE is

$$m^2 + a^2 = 0$$

$$\Rightarrow m(m^2 + a^2) = 0$$

$$\Rightarrow m = 0, m = \pm ai$$

$$\text{C.F.} = C_1 + C_2 \cos ax + C_3 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sin ax$$

$$= \frac{1}{D^2 + a^2} (\text{Imaginary part of } e^{iax})$$

$$= \text{I. P. of } e^{iax} \frac{1}{(D+ia)^2 + a^2} (1)$$

$$= \text{I. P. of } e^{iax} \frac{1}{D^2 + 3iaD - 2a^2} (1)$$

$$= \text{I. P. of } e^{iax} \frac{1}{-2a^2 D \left[1 - \frac{3i}{2a} D - \frac{1}{2a} D^2\right]} (1)$$

$$= \text{I. P. of } \frac{e^{iax}}{-2a^2} \frac{1}{D} \left(1 - \frac{3i}{2a} D - \frac{1}{2a} D^2\right)^{-1} (1)$$

$$= \text{I. P. of } \frac{e^{iax}}{-2a^2} \frac{1}{D} \left(1 + \frac{3i}{2a} D + \dots\right) (1)$$

$$= \text{I. P. of } \frac{e^{iax}}{-2a^2} \frac{1}{D} (1)$$

$$= \text{I. P. of } \frac{e^{iax}}{-2a^2} \frac{1}{D} (x)$$

$$= \text{IP of } \left[\frac{x}{-2a^2} (\cos ax + i \sin ax) \right] = -\frac{1}{2a^2} x \sin ax$$

$\therefore$ The required solution is

$y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = C_1 + C_2 \cos ax + C_3 \sin ax - \frac{1}{2a^2} x \sin ax$$

Ex. 231. Solve $(D^2 + 4)y = e^x + \sin 2x$

The given differential equation is

$$(D^2 + 4)y = e^x + \sin 2x$$

The AE is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \text{C.F.} = C_1 \cos 2x + C_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2 + 4} e^x + \frac{1}{D^2 + 4} \sin 2x$$

$$= \frac{1}{1+4} e^x + \left(-\frac{x}{2 \times 2} \cos 2x\right) = \frac{1}{5} e^x - \frac{1}{4} x \cos 2x \quad \left[\because \frac{1}{D^2 + ax} \sin ax = -\frac{x}{2a} \cos ax\right]$$

∴ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{5}e^x - \frac{1}{4}x \cos 2x$$

Ex. 232. Solve $(D^2 + 4)y = \sin x$

The given differential equation is

$$(D^2 + 4)y = \sin x$$

$$C.F. = C_1 \cos 2x + C_2 \sin 2x \quad (\text{Verify it})$$

$$P.I. = \frac{1}{D^2 + 4} \sin^2 x$$

$$= \frac{1}{D^2 + 4} \left[\frac{1}{2} (2 \sin^2 x) \right]$$

$$= \frac{1}{D^2 + 4} \left[\frac{1}{2} (1 - \cos 2x) \right]$$

$$= \frac{1}{2} \frac{1}{D^2 + 4} e^{0 \cdot x} - \frac{1}{2} \cdot \frac{1}{D^2 + 4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{0 + 4} e^{0 \cdot x} - \frac{1}{2} \left\{ \frac{x}{2 \cdot 2} \right\} \sin 2x$$

$$= \frac{1}{8} - \frac{1}{8} x \sin 2x$$

∴ required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8} - \frac{1}{8} x \sin 2x$$

Case – VI. When X is of the form $x \cdot V$ where V is the function of x

$$P.I. = \frac{1}{F(D)} X = \frac{1}{F(D)} x \cdot V$$

$$= x \frac{1}{F(D)} V - \frac{F'(D)}{[F(D)]^2} \cdot V$$

Ex. 233. Solve $(D^2 - 2D + 1)y = x \sin x$

The given differential equation is

$$(D^2 - 2D + 1)y = x \sin x$$

$$C.F. = (C_1 x + C_2)e^x \quad (\text{Verify it})$$

$$P.I. = \frac{1}{D^2 - 2D + 1} x \sin x$$

$$= \frac{1}{D^2 - 2D + 1} \sin x = \frac{(2D - 2)}{(D^2 - 2D + 1)^2} \sin x$$

$$= \frac{1}{-1 - 2D + 1} \sin x = \frac{(2D - 2)}{(-1 - 2D + 1)^2} \sin x$$

$$= -\frac{1}{2} x \frac{1}{D} \sin x - \frac{1}{4} \frac{1}{D^2} (2D - 2) \sin x$$

$$= -\frac{1}{2} x \int \sin x - \frac{1}{2} \frac{1}{D^2} (\cos x - \sin x)$$

$$= -\frac{1}{2} x \cos x - \frac{1}{2} \frac{1}{D} \int (\cos x - \sin x) dx$$

$$= -\frac{1}{2} x \cos x - \frac{1}{2} \int (\sin x + \cos x) dx$$

$$= \frac{1}{2} \cos x - \frac{1}{2} (-\cos x + \sin x) = \frac{1}{2} (x \cos x + \cos x - \sin x)$$

∴ The required solution is

$$y = C.F. + P.I.$$

$$\Rightarrow y = (C_1 x + C_2)e^x + \frac{1}{2} (x \cos x + \cos x - \sin x)$$

Ex. 234..

Solve $(D^2 + m^2)y = x \cos mx$

The given differential equation is

$$(D^2 + m^2)y = x \cos mx$$

$$C.F. = C_1 \cos mx + C_2 \sin mx \quad (\text{Verify it})$$

$$P.I. = \frac{1}{D^2 + m^2} x \cos mx$$

$$\begin{aligned}
&= x \frac{1}{D^2 + m^2} x \cos mx \\
&= x \left[\frac{x}{2m} \sin mx \right] - 2 \frac{1}{(D^2 + m^2)^2} \cos mx \\
&= \frac{x^2}{2m} \sin mx + 2m \frac{1}{D^2 + m^2} \cdot \frac{1}{D^2 + m^2} (\sin mx) \\
&= \frac{x^2}{2m} \sin mx + 2m \frac{1}{D^2 + m^2} \left(-\frac{x}{2m} \cos mx \right) \\
\therefore \frac{1}{D^2 + m^2} x \cos cx &= \frac{x^2}{2m} \sin mx - \frac{1}{D^2 + m^2} (x \cos mx) \\
\Rightarrow \frac{2}{D^2 + m^2} (x \cos mx) &= \frac{x^2}{2m} \sin mx \\
\Rightarrow \frac{1}{D^2 + m^2} (x \cos mx) &= \frac{x^2}{4m} \sin mx \\
\therefore \text{PI} &= \frac{x^2}{4m} \sin mx
\end{aligned}$$

$\therefore$ The required solution is

$y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = C_1 \cos mx + C_2 \sin mx + \frac{x^2}{4m} \sin mx$$

$$\text{Note. } \frac{1}{(D^2 + m^2)} x \cos mx = \frac{x^2}{4m} \sin mx$$

Ex. 235. Solve $(D^2 - 2D + 1)y = xe^x \sin x$

The given differential equation is

$$(D^2 - 2D + 1)y = xe^x \sin x$$

$$\text{C.F.} = (C_1 x + C_2)e^x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} xe^x \sin x$$

$$= e^x \frac{1}{\{(D+1)^2 - 2(D+1) + 1\}} x \sin x$$

$$= e^x \left(\frac{1}{D^2} \right) x \sin x$$

$$= e^x \left(\frac{1}{D} \right) \int x \sin x dx$$

$$= e^x \left(\frac{1}{D} \right) \left[-x \cos x + \int 1 \cdot \cos x dx \right]$$

$$= e^x \int (-x \cos x + \sin x) dx$$

$$= e^x [-x \sin x - 2 \cos x]$$

$$= -e^x (x \sin x + 2 \cos x)$$

$\therefore$ The required solution is

$y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = (C_1 x + C_2)e^x - e^x (x \sin x + 2 \cos x)$$

Ex. 236. Solve $(D^4 + 2D^2 + 1)y = x^2 \cos x$

The given differential equation is

$$(D^4 + 2D^2 + 1)y = x^2 \cos x$$

$$\text{C.F.} = (C_1 x + C_2) \cos x + (C_3 x + C_4) \sin x$$

(Verify it)

$$\text{P.I.} = \frac{1}{(D^2 + 1)} x^2 \cos x$$

$$= \frac{1}{(D^2 + 1)} x^2 (\text{R.P. of } e^{ix})$$

$$= (\text{R.P. of } e^{ix}) \frac{1}{[(D+1)^2 + 1]^2} x^2$$

$$= (\text{R.P. of } e^{ix}) \frac{1}{(D^2 + 2iD - 1 + 1)^2}$$

$$= (\text{R.P. of } e^{ix}) \frac{1}{(D^2 + 2iD)^2} x^2$$

$$= (\text{R.P. of } e^{ix}) \frac{1}{(2iD)^2 \left[1 + \frac{D}{2i} \right]^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D^2 [1 + (D/2i)]^2} x^2$$

$$\begin{aligned}
&= \text{R. P. of } \frac{e^{ix}}{-4} \frac{1}{D^2} \left(1 + \frac{D}{2i}\right)^{-2} x^2 \\
&= \text{R. P. of } \frac{e^{ix}}{-4} \frac{1}{D^2} \left(1 + iD - \frac{3}{4}D^2 + \dots\right) x^2 \quad [\because 1^2 = -1] \\
&= \text{R. P. of } \frac{e^{ix}}{-4} \frac{1}{D^2} \left\{x^2 + 2ix - \left(\frac{3}{2}\right)\right\} \quad [\because D(x^2) = 2x \text{ and } D^2(x^2) = 2] \\
&= \text{R. P. of } \left\{-\frac{1}{4} e^{ix} \left\{\left(\frac{1}{12}\right)x^4 + 2i\left(\frac{1}{6}x^3\right) - \frac{3}{2}\left(\frac{1}{2}x^2\right)\right\}\right\} \\
&= \text{R. P. of } \left\{-\frac{1}{4} (\cos x + i \sin x)\right\} \left\{\frac{1}{12}x^4 + \frac{1}{3}ix^3 - \frac{3}{4}x^2\right\} \\
&= -\frac{1}{4} \left\{\left(\frac{1}{12}\right)x^4 - \frac{3}{4}x^2\right\} \cos x + \frac{1}{4} \left\{\left(\frac{1}{3}\right)x^3\right\} \sin x \\
&= -\frac{1}{48}(x^4 - 9x^2) \cos x + \frac{1}{12}x^3 \sin x \\
\therefore \text{ The required solution is } \\
&y = \text{C. F.} + \text{P. I.}
\end{aligned}$$

$$\Rightarrow y = (C_1x + C_2) \cos x + (C_3x + C_4) \sin x - \frac{1}{48}(x^4 - 9x^2) \cos x + \frac{1}{12}x^3 \sin x$$

Ex. 237 . Solve $(D^2 + 1)y = x^2 \sin 2x$

The given differential equation is

$$(D^2 + 1)y = x^2 \sin 2x$$

C. F. = $C_1 \cos x + C_2 \sin x$ (Verify it)

$$\text{P. I.} = \frac{1}{D^2 + 1} x^2 \sin 2x .$$

$$\begin{aligned}
&= \frac{1}{D^2 + 1} x^2 \sin 2x \quad (\text{Verify it}) \\
&\text{P. I.} = \frac{1}{(D^2 + 1)} x^2 (\text{P. I. of } e^{2ix}) \\
&= \text{P. I. of } e^{2ix} \frac{1}{\{(D + 2i)^2 + 1\}} x^2 \\
&= \text{P. I. of } e^{2ix} \frac{1}{(D^2 + 4iD - 3)} x^2 \\
&= \text{P. I. of } e^{2ix} \frac{1}{-3 \left\{1 - \left(\frac{4}{3}\right)iD - \frac{1}{3}D^2\right\}} x^2 \\
&= \text{P. I. of } \frac{e^{2ix}}{-3} \left[1 - \left(\frac{4iD + D^2}{3}\right)\right]^{-1} \\
&= \text{P. I. of } \frac{e^{2ix}}{-3} \left[1 + \left(\frac{4iD + D^2}{3}\right) + \left(\frac{16i^2D^2}{9}\right)\right] x^2 \\
&= \text{P. I. of } -\frac{1}{3} e^{2ix} \left\{1 + \left(\frac{4}{3}\right)iD - \left(\frac{13}{9}\right)(2)\right\} \\
&= \text{P. I. of } -\frac{1}{3} (\cos 2x + i \sin 2x) \left[\left\{x^2 - \left(\frac{26}{9}\right)\right\} + \left(\frac{8}{3}\right)ix\right] \\
&= -\frac{1}{3} \left(\frac{8}{3}\right)x \cos 2x - \frac{1}{3} \left\{x^2 - \left(\frac{26}{9}\right)\right\} \sin 2x \\
&= -\frac{1}{27} [24x \cos 2x + (9x^2 - 26) \sin 2x] \\
\therefore \text{ The required solution is } \\
&y = \text{C. F.} + \text{P. I.}
\end{aligned}$$

$$\Rightarrow y = C_1 \cos x + C_2 \sin x - \frac{1}{27} [24x \cos 2x + (9x^2 - 26) \sin 2x]$$

Ex. 238. Solve $(D^2 + 4)y = x \sin x$

The given differential equation is

$$(D^2 + 4)y = x \sin x$$

C. F. = $C_1 \cos 2x + C_2 \sin 2x$ (Verify it)

$$\begin{aligned}
&\text{P. I.} = \frac{1}{D^2 + 4} (x \sin x) \\
&= x \frac{1}{D^2 + 4} \sin x - \frac{2D}{(D^2 + 4)^2} \sin x
\end{aligned}$$

$$= x \frac{1}{-1^2 + 4} \sin x - \frac{2D}{(-1^2 + 4)^2} \sin x$$

$$= \frac{1}{3} x \sin x - \frac{2}{9} \cos x$$

∴ The required solution is

y = C.F. + P.I.

$$\Rightarrow y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3} x \sin x - \frac{2}{9} \cos x$$

Prop. -VII. $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$, where X is a function of x.

Ex. 239. Solve $(D^2 + a^2)y = \sec ax$

The given differential equation is

$$(D^2 + a^2)y = \sec ax$$

C.F. = $C_1 \cos ax + C_2 \sin ax$ (Verify it)

$$P.I. = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D - ia)(D + ia)} \sec ax$$

$$= \frac{1}{2ia} \left[\frac{1}{D - ia} - \frac{1}{D + ia} \right] \sec ax$$

$$\text{Now } \frac{1}{D - ia} \sec ax = e^{iax} \int \sec ax \cdot e^{-iax} dx$$

$$= e^{iax} \int \sec ax (\cos ax - a \sin ax) dx$$

$$[\because e^{iax} = \cos ax - i \sin ax]$$

$$= e^{iax} \int (1 - i \tan ax) dx$$

$$= e^{iax} \left[x + \left(\frac{i}{a} \right) \log \cos ax \right]$$

$$\text{Similarly } \frac{1}{D + ia} \sec ax = e^{-iax} \int \sec ax e^{iax} dx$$

$$= e^{-iax} \int \sec ax (\cos ax + i \sin ax) dx$$

$$= e^{-iax} \int (1 + i \tan ax) dx$$

$$= e^{-iax} \left[x - \left(\frac{i}{a} \right) \log \cos ax \right]$$

∴ form (1)

$$P.I. = \frac{1}{2ia} \left[e^{iax} \left\{ x + \frac{i}{a} \log \cos ax \right\} - e^{-iax} \left\{ x - \frac{i}{a} \log \cos ax \right\} \right]$$

$$= \left[x \left(\frac{e^{iax} - e^{-iax}}{2ia} \right) + \frac{i}{a} (\log \cos ax) \left(\frac{e^{iax} + e^{-iax}}{2ia} \right) \right]$$

$$= \left[\frac{x}{a} \left(\frac{e^{iax} - e^{-iax}}{2i} \right) + \frac{i}{a^2} \left(\frac{e^{iax} + e^{-iax}}{2} \right) (\log \cos ax) \right]$$

$$= \left(\frac{x}{a} \right) \sin ax + \left(\frac{1}{a^2} \right) \cos ax (\log \cos ax)$$

∴ The required solution is

y = C.F. + P.I.

$$\Rightarrow y = C_1 \cos ax + C_2 \sin ax + \left(\frac{x}{a} \right) \sin ax + \left(\frac{1}{a^2} \right) \cos ax (\log \cos ax)$$

ADDITIONAL PROBLEMS

Ex. 240. Solve $(D^2 - 3D + 2)y = \cos^2 y$

The given differential equation is

$$(D^2 - 3D + 2)y = \cos^2 y$$

C.F. = $c_1 e^x + c_2 e^{2x}$

$$P.I. = \frac{1}{D^2 - 3D + 2} \cos^2 x$$

$$= \frac{1}{(D^2 - 3D + 2)2} (2 \cos^2 x)$$

$$= \frac{1}{2} \frac{1}{D^2 - 3D + 2} (1 + \cos 2x)$$

$$= \frac{1}{2} \frac{1}{D^2 - 3D + 2} e^{0x} + \frac{1}{2} \cdot \frac{1}{D^2 - 3D + 2} \cos 2x$$

Pg 59

$$y(0) = 9$$

Que 54

$$t = 0, y = 9 \dots (*)$$

$$y(1) = 4 \quad \left. \begin{array}{l} t = 1 \\ y = 4 \end{array} \right\} \dots \dots \dots (**)$$

$$A(y) \frac{dy}{dt} = -k \sqrt{y}.$$

In case of cylinder A(y) is constant

$$\frac{dy}{dx} = -\frac{k}{A(y)} \sqrt{y}$$

$$B = \frac{k}{A(y)}$$

$$\frac{dy}{dt} = -B\sqrt{y}$$

$$\frac{dy}{dt} = -B\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}} = -Bdt$$

$$2\sqrt{y} = -Bt + c$$

Now from (*) and (**), we get

$$2\sqrt{9} = -B \times 0 + c \Rightarrow c = 6$$

$$\text{and } 2\sqrt{y} = -2t + 6$$

y = 0 when water is finished

$$\text{ie } 2\sqrt{0} = -2t + 6$$

$$6 = 2t \Rightarrow t = 3$$

55. a = ?

$$r = 1 \text{ inch} = \frac{1}{12} \text{ ft}$$

$$a = \pi r^2$$

$$= \pi \left(\frac{1}{12} \right)^2, \quad g = 32 \text{ ft/sec}^2$$

$$y(0) = 9$$

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy}$$

$$\pi(3)^2 \frac{dy}{dt} = -\pi \left(\frac{1}{12} \right)^2 \sqrt{2g} \sqrt{y}$$

$$\frac{dy}{dt} = -8 \left(\frac{1}{12} \right)^2 \frac{1}{9} \sqrt{y}$$

$$\frac{1}{\sqrt{y}} dy = -\frac{8}{144 \times 9} dt$$

$$2\sqrt{y} = -\frac{8}{144 \times 9} t + c \Rightarrow 2\sqrt{y} = \frac{-t}{\sqrt{159}} + c$$

$$2 \times \sqrt{9} = 0 + c \Rightarrow c = 6$$

$$2\sqrt{y} = \frac{-t}{154} + 6$$

$$-\sqrt{y} = \frac{-t}{308} + 3$$

$$y = 0 \\ \text{then } t = ?$$

$$56 \quad \frac{R}{y} = \tan \alpha \Rightarrow R = y \tan \alpha_{\text{cons.}}$$

$$R \propto y \Rightarrow R^2 \propto y^2$$

$$\Rightarrow R^2 = c^2 y^2$$

$$\pi R^2 = \pi c y^2$$

$$A(y) = \pi c y^2$$

$$\frac{2}{5} y^{5/2} = -kt + C$$

$$y(0) = 16$$

$$\frac{2}{5} \times 2^{10} = C \Rightarrow C = \frac{2^{11}}{5}$$

$$y(1) = 9$$

$$\frac{2}{5} \times 3^5 = -k + \frac{2^{11}}{5} \Rightarrow k = \frac{2^{11}}{5} - \frac{2 \times 3^5}{5} \Rightarrow k = \frac{1562}{5}$$

$$(57) \quad V_0 = \text{intial vol.}$$

$$2\sqrt{4} = -B + 6 \\ B = 2$$

y → dis b/w
hole & the water level.

h = initial height.

$$V_o = \pi r^2 h \quad r \rightarrow \text{radius}$$

$$t = T \quad | \quad t = 0$$

$$y = 0 \quad | \quad y = h$$

$$A(y) \frac{dy}{dt} = -k\sqrt{y} \quad A(y) = \text{constant.}$$

$$\frac{dy}{dt} = -k\sqrt{y} \Rightarrow 2\sqrt{y} = -kt + c$$

$$t = 0, y = h$$

$$2\sqrt{h} = c$$

$$t = , y = 0$$

$$0 = -kT + c$$

$$c = KT$$

$$K = \frac{c}{T} = \frac{2\sqrt{h}}{T}$$

$$2\sqrt{y} = -\frac{2\sqrt{h}}{T} + 2\sqrt{h}$$

$$\sqrt{y} = \sqrt{h} \left[1 - \frac{1}{T} \right] \Rightarrow y = h \left[1 - \frac{1}{T} \right]^2$$

$$V = \pi r^2 y = \pi r^2 h \left[1 - \frac{1}{T} \right]^2$$

$$V = V_o \left[1 - \frac{1}{T} \right]^2$$

$$(58) \quad y = x^{4/3}$$

$$t = 0 \quad | \quad t = 1$$

$$y = 12 \quad | \quad y = 6$$

$$A(y) \frac{dy}{dt} = -k\sqrt{y}$$

area of cross section

$$A(y) = \pi x^2$$

$$= \pi y^{3/2}$$

$$\pi y^{3/2} \frac{dy}{dt} = -K\sqrt{y}$$

$$\frac{y^{3/2}}{\sqrt{y}} dy = -K dt$$

$$\frac{y^2}{2} = -Kt + C$$

$$\text{Sol}^n: \frac{y^2}{2} = -54t + 72 \Rightarrow t = \frac{4}{3} = 1.33 \text{ hr} = 80 \text{ minutes}$$

$$\therefore \text{time} = 1:20 \text{ P:M}$$

$$(59) \quad x^2 = by$$

$$A(y) = \pi x^2 \pi by$$

$$A(y) \frac{dy}{dt} = -K\sqrt{y} \Rightarrow \pi by \frac{dy}{dt} = -K\sqrt{y}$$

$$y^{1/2} dy = -\frac{K}{\pi b} dt, A = \frac{K}{\pi b}$$

$$\sqrt{y} dy = -\frac{K}{\pi b} dt, A = \frac{K}{\pi b}$$

$$\sqrt{y} dy = -A dt$$

$$\sqrt{y} dy = -A dt$$

$$t = 0, y = 4$$

$$t = 1, y = 1$$

$$\frac{2}{3} y^{3/2} = -At + C$$

$$\left(\frac{2}{3}\right) (4)^{3/2} = C$$

$$\frac{16}{3} = C \Rightarrow \frac{2}{3} (1)^{3/2} = -A + C$$

$$A = \frac{14}{3}$$

$$\frac{2}{3} y^{3/2} = \frac{-14t}{3} + \frac{16}{3}$$

$$A(y) \frac{dy}{dt} = -a \sqrt{2} g \sqrt{y} \quad k = a\sqrt{2} g$$

$$x^2 = by$$

$$\left. \begin{array}{l} x = 2\text{ft} \\ x = 4t \end{array} \right\} \text{initially}$$

$$4 = 4b \quad (\text{Put in } x^2 = by)$$

$$b = 1$$

$$\text{Now } A = \frac{K}{\pi b} \Rightarrow \frac{14}{3} = \frac{K}{\pi b} \quad (b = 1)$$

$$\Rightarrow K = \frac{14}{3} \pi$$

$$\text{Now } K = a \sqrt{2} g$$

$$a = \frac{K}{\sqrt{2} g} \quad ; \Rightarrow \frac{14}{3} \frac{\pi}{\sqrt{64}}$$

$$(a = \text{area of the hole}) \Rightarrow \frac{14\pi}{24} = \frac{7}{12} \pi$$

$$a = \frac{7}{12} \pi \Rightarrow \pi r^2 = \frac{7}{12} \pi \quad (r = \text{radius of hole})$$

$$r = \sqrt{7/12} \text{ ft.}$$

(60) FIGURE

$$R = 1 \text{ inch} = \frac{1}{12} \text{ ft}$$

$$t = 0, y = 3$$

Surface of cross section is a rectangular rectangle

$$MN = ?$$

$$MN = 2 ON$$

$$ON = \sqrt{6y - y^2}$$

$$MN = 2\sqrt{6y - y^2}$$

$$A(y) = 5 \times 2 \sqrt{6y - y^2}$$

$$= 10 \sqrt{6y - y^2}$$

$$A(y) \frac{dy}{dx} = -a \sqrt{2} g \sqrt{y}$$

$$10 \sqrt{6y^2 - y^2} \frac{dy}{dx} = -\frac{22}{7} \times \frac{1}{144} dt$$

$$\sqrt{6-y} dy = -\frac{22}{7} \times \frac{1}{10} \times \frac{8}{144}$$

$$-\frac{2}{3} (6-y)^{3/2} = -\frac{11}{630} t + A$$

$$A = -\frac{2}{3} 3^{3/2} \quad t = 0, y = 3$$

$$\therefore 2(6)^{3/2} = \frac{11}{20} + 2.3^{3/2}$$

$$\Rightarrow t = 362.75 \text{ sec}$$

$$t = 6.0458 \text{ min.}$$

$$\text{ie. } 6 \text{ min } 3 \text{ sec}$$

(61) $t = 0, y = 8$

$$t = ? y = 0.$$

$$r = 1 \text{ inch} = \frac{1}{12} \text{ ft.}$$

$$R = \sqrt{16 - (4-y)^2}$$

$$= \sqrt{16 - 16 - y^2 + 8y}$$

$$R = \sqrt{8y - y^2}$$

$$A(y) \frac{dy}{dt} = -a \sqrt{2} g \sqrt{y}$$

$$\pi R^2 \frac{dy}{dt} = -\pi \left(\frac{1}{12}\right)^2 \times 8 \sqrt{y}$$

$$R^2 \frac{dy}{dt} = \frac{-1}{144} \times 8 \times \sqrt{y}$$

$$\frac{(8y - y^2)}{\sqrt{y}} dy = \frac{-8}{144} dt$$

$$t = \frac{16}{14} = 1.14 \text{ hr} = 68.57 \text{ min}$$

$$68 \text{ min} + 0.57 \text{ min}$$

$$68 \text{ min} + 34 \text{ sec}$$

$$\therefore \text{time was } 1:08 : 34 \text{ sec}$$

pg 32/Example 2.5 Solve the equation $(3x^2 + 4xy)dx + (2x^2 + 2y) dy = 0$

$$(3x^2 + 4xy)dx + (2x^2 + 2y) dy = 0$$

$$Mdx + Ndy = 0$$

$$M(x, y) = 3x^2 + 4xy$$

$$N(x, y) = 2x^2 + 2y$$

$$\frac{\partial M}{\partial y} = 4x \quad \& \quad \frac{\partial N}{\partial x} = 4x$$

∴ It is an exact differential equation

$$\therefore \text{Sol}^n: \rightarrow \int (3x^2 + 4xy) dx + \int 2y dy = c$$

$$x^3 + \frac{4yx^2}{2} + y^2 = C$$

$$\text{ie } x^3 + 2x^2y + y^2 = C$$

pg 33/Example 2.6 Solve the initial value problem $(2x \cos y + 3x^2y) dx + (x^3 - x^2 \sin y - y) dy = 0$
 $y(0) = ?$

$$(2x \cos y + 3x^2y) dx + (x^3 - x^2 \sin y - y) dy = 0$$

$$\frac{\partial M}{\partial y} = -2x \sin y + 3x^2 = 3x^2 - 2x \sin y$$

$$\frac{\partial N}{\partial x} = 3x^2 - 2x \sin y$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}, \Rightarrow \text{diff equation is exact}$$

$$\Rightarrow \text{sol}^n: \int (2x \cos y + 3x^2y) dx - \int y dy = C$$

$$\text{ie. } x^2 \cos y + x^3y - \frac{y^2}{2} = C$$

$$y(0) = 2 \Rightarrow y = 2 \text{ when } x = 0.$$

$$\text{ie. } -2 = C$$

$$\therefore x^2 \cos y + x^3y - \frac{y^2}{2} = -2$$

Integrating Factor

Pg 36/Example 2.7 Consider the differential equation $(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= 3 + 8xy \\ \frac{\partial N}{\partial x} &= 2 + 6xy \end{aligned} \right\} \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \therefore \text{not exact}$$

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$$

$$y(3 + 4xy)dx + x(2 + 3xy)dy = 0$$

$$\therefore \text{If } = \frac{1}{Mx - Ny} = \frac{1}{3xy + 4x^2y^2 - 2xy - 3x^2y^2} = \frac{1}{xy + x^2y^2}$$

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$$

Multiplying $x^\alpha \cdot y^\beta$ in this eqⁿ

$$(3x^\alpha y^{1+\beta} + 4x^{1+\alpha} y^{2+\beta}) dx + (2x^{\alpha+1} y^\beta + 3x^{2+\alpha} y^{1+\beta}) dy = 0$$

$$\frac{\partial M}{\partial y} = (1 + \beta) 3x^\alpha y^\beta + 4x^{1+\alpha} (2 + \beta) y^{1+\beta}$$

$$\frac{\partial N}{\partial x} = y^\beta (\alpha + 1) 2x^\alpha + 3y^{1+\beta} (2 + \alpha) x^{1+\alpha}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow (1 + \beta) 3x^\alpha y^\beta + 4x^{1+\alpha} (2 + \beta) y^{1+\beta} = y^\beta (\alpha + 1) 2x^\alpha + y^{1+\beta} (2 + \beta) 3x$$

$$\Rightarrow 3(1 + \beta)x^\alpha y^\beta + 4(2 + \beta)x^{1+\alpha} y^{1+\beta} = 2(\alpha + 1)x^\alpha y^\beta + 3(2 + \beta)xy^{1+\beta}$$

$$\Rightarrow 3(\beta + 1) = 2(\alpha + 1) \quad \& \quad 4(2 + \beta) = 3(2 + \alpha)$$

$$\Rightarrow 2\alpha - 3\beta = 1 \quad \quad \quad 3\alpha - 4\beta = 2$$

$$\Rightarrow 2 = \alpha, \quad \beta = 1$$

$$\text{I.F.} = x^2y$$

Now multiplying x^2y

$$\text{ie. } (3x^2y^2 + 4x^3y^3)dx + (2x^3y + 3x^4y^2)dy = 0$$

$$\text{Now } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \int (3x^2y^2 + 4x^3y^3)dx + 0 = C$$

$$\Rightarrow \int (3x^2y^2 + 4x^3y^3)dx = C \Rightarrow x^3y^2 + y^3x^4 = C$$

In Exercise 1-10 determine whether or not each of the given equation is exact; solve those that are exact.

Pg.37/Q1 $(3x + 2y)dx + (2x + y)dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ exact diff}$$

$$\text{Sol}^n: \int (3x + 2y)dx + \int ydy = C$$

$$\frac{3x^2}{2} + 2yx + \frac{y^2}{2} = C$$

$$(3) (2xy + 1)dx + (x^2 + 4y)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$$

$$\therefore \text{Sol}^n: \int (2xy + 1)dx + \int 4ydy = C$$

$$\Rightarrow x^2y + x + \frac{4y^2}{2} = C \Rightarrow x^2y + x + 2y^2 = C$$

$$(5) (6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0$$

$$\frac{\partial M}{\partial y} = 6x + 4y, \quad \frac{\partial N}{\partial x} = 6x + 4y$$

$$\therefore \text{Sol}^n \int (6xy + 2y^2 - 5)dx + \int -6dy = c$$

$$\Rightarrow 3x^2y + 2xy^2 - 5x - 6y = c$$

$$(7) (y \sec^2 x + \sec x \cdot \tan x)dx + (\tan x + 2y)dy = 0$$

$$\frac{\partial M}{\partial y} = \sec^2 x, \quad \frac{\partial N}{\partial x} = \sec^2 x$$

$$\therefore \text{Sol}^n: \int (y \sec^2 x + \sec x \cdot \tan x)dx + \int 2y dy = c$$

$$\Rightarrow y \tan x + \sec x + y^2 = c$$

$$(9) \left(\frac{2s-1}{t} \right) ds + \left(\frac{s-s^2}{t^2} \right) dt = 0$$

$$\frac{\partial M}{\partial t} = \frac{1-2s}{t^2} \& \frac{\partial N}{\partial s} = \frac{(1-2s)}{t^2}$$

D. E exact.

$$\therefore \text{Sol}^n = \int \left(\frac{2s-1}{t} \right) ds + 0 = c \quad (\text{b/c not contain term free of } s)$$

$$\Rightarrow \frac{s^2}{t} - \frac{s}{t} = c \Rightarrow s^2 - s = ct$$

$$(11) (2xy - 3)dx + (x^2 + 4y)dy = 0, \quad y(1) = 2$$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x \Rightarrow \text{exact D. E}$$

$$\text{Sol}^n \int (2xy - 3)dx + \int 4ydy = c$$

$$\Rightarrow x^2y - 3x + 2y^2 = c$$

$$\text{when } x = 1, \quad y = 2$$

$$1 \times 2 - 3 + 8 = c \Rightarrow c = 7$$

$$\text{Particular Sol}^n \equiv x^2y - 3x + 2y^2 = 7$$

$$(13) (2y \sin x \cos x + y^2 \sin x)dx + (\sin^2 x - 2y \cos x)dy = 0$$

$$y(0) = 3$$

$$\frac{\partial M}{\partial y} = 2 \sin x \cos x + 2y \sin x, \quad \frac{\partial N}{\partial x} = 2 \sin x \cos x + 2y \sin x$$

$\Rightarrow$ Diff. eqⁿ in exact form.

$$\Rightarrow \text{Sol}^n \equiv \int (2y \sin x \cdot \cos x + y^2 \sin x) dx + 0 = c$$

$$\Rightarrow \int (y \sin 2x + y^2 \sin x) dx = c$$

$$\Rightarrow \frac{-y}{2} \cos 2x - y^2 \cos x = c$$

$$\text{when } x = 0, y = 3$$

$$\frac{-3}{2} \times 1 - 9 \times 1 = c \Rightarrow \frac{-3}{2} - 9 = c$$

$$\text{Particular Sol}^n \equiv \frac{-y}{2} \cos 2x - y^2 \cos + \frac{21}{2} = 0$$

$$(15) \left(\frac{3-y}{x^2} \right) dx + \left(\frac{y^2-2x}{xy^2} \right) dy = 0, \quad y(-1) = 2, \dots$$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2}, \quad \frac{\partial N}{\partial x} = \frac{-1}{x^2} - 0$$

$\therefore$ given D.E is exact.

$$\Rightarrow \text{Sol}^n \equiv \int \left(\frac{3-y}{x^2} \right) dx + \int \frac{-2}{y^2} dy = c$$

$$\frac{-(3-y)}{x} - 2 \frac{(-1)}{y} = c$$

for particular solⁿ put $x = (-1)$ & $y = 2$

$$\Rightarrow \frac{-(3-2)}{-1} - \frac{2(-1)}{2} = c \Rightarrow \frac{-3+2}{-1} + \frac{2}{2} = c$$

$$\Rightarrow 1 + 1 = c \Rightarrow c = 2$$

$\therefore$ particular solⁿ

$$-(3-y)y + 2x = 2xy$$

$$\Rightarrow -3y + y^2 + 2x = 2xy \Rightarrow 2x - 3y + y^2 - 2xy = 0$$

Separable equation & reducible to this form

$$\text{Ex: 2.8} \quad \frac{40}{(x-4)y^4} dx - x^3(y^2-3)dy = 0$$

$$\left(\frac{x-4}{x^3} \right) dx - \frac{(y^2-3)}{y^4} dy = 0$$

$$\text{ie.} \quad \int \left(\frac{1}{x^2} - \frac{4}{x^3} \right) dx - \int \left(\frac{1}{y^2} - \frac{3}{y^4} \right) dy = c$$

$$\Rightarrow -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$$

$$\text{Ex. 2.9} \quad \frac{40}{x \sin y} dx + (x^2 + 1) \cos y dy = 0$$

$$\left(\frac{x}{x^2+1} \right) dx + \left(\frac{\cos y}{\sin y} \right) dy = 0$$

Now integrating

$$\frac{1}{2} \log(x^2+1) + \log|\sin y| = \log c$$

$$\Rightarrow \log(x^2+1) + 2\log|\sin y| = 2\log c$$

$$\Rightarrow \log(x^2+1) + \log(\sin y)^2 = \log c_1$$

$$\Rightarrow \log(x^2+1)(\sin^2 y) = \log c_1$$

$$\Rightarrow c_1 = (x^2+1)(\sin^2 y)$$

$$\Rightarrow c_1 = (1+1) \sin^2 \frac{\pi}{2}$$

$$\Rightarrow c_1 = 2$$

$$\Rightarrow (x^2+1) \sin^2 y = 2$$

$$\text{Ex 2.10} \quad \frac{42}{\frac{dy}{dx} = \frac{3y^2-x^2}{2xy}}$$

$$\frac{dy}{dx} = \frac{3y}{2x} - \frac{1}{2} \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{3}{2} \left(\frac{y}{x} \right) - \frac{1}{2 \left(\frac{y}{x} \right)}$$

homogeneous, $\therefore$ Put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + \frac{xdv}{dx} = \frac{3}{2} v - \frac{1}{2v} \Rightarrow \frac{xdv}{dx} = \frac{3}{2} v - \frac{1}{2v} - v$$

$$x \frac{dv}{dx} = \frac{3v^2-1-2v^2}{2v} = \frac{v^2-1}{2v}$$

$$\frac{xdv}{dx} = \frac{v^2-1}{2v} \Rightarrow \frac{2v}{(v^2-1)} dv = \frac{1}{x} dx$$

$$\log(v^2-1) = \log x + \log c$$

$$v^2 - 1 = x c$$

$$\frac{y^2}{x^2} - 1 = x c$$

Ex. 2.11
42 $(y + \sqrt{x^2 + y^2}) dx - x dy = 0$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2 + y^2}{x^2}} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

y/x form makes homognious DE

Ex 1.12
45 $(x^2 - 3y^2)dx + 2xy dy = 0$

$$\frac{dy}{dx} = \frac{x}{-2y} + \frac{3y}{2x}$$

put $y = vx$

$$v + x \frac{dv}{dx} = -\frac{1}{2v} + \frac{3}{2} v \Rightarrow x \frac{dv}{dx} = -\frac{1}{2v} + \frac{3v}{2} - v$$

$$x dv \frac{-1 + 3v^2 - 2v^2}{2v} = \frac{-1 + v^2}{2v} = \frac{v^2 - 1}{2v}$$

$$\frac{dx}{2vdv} \text{ incomplete}$$

Q1
47 $4xy dx + (x^2 + 1)dy = 0$

$$\frac{4x}{x^2 + 1} dx + \frac{1}{y} dy = 0$$

$$2 \log (x^2 + 1) + \log y = \log c$$

$$(x^2 + 1)^2 y = c$$

ie. $y = c(x^2 + 1)^{-2}$

(3) $2r(s^2 + 1)dr + (r^4 + 1)ds = 0$

$$\frac{2r}{r^4 + 1} dr + \frac{1}{(s^2 + 1)} ds = 0$$

$$\tan^{-1}(r^2) + \tan^{-1} s = \tan^{-1} c$$

ie. $\tan^{-1} \frac{(r^2 + s)}{(1 - r^2 s)} = \tan^{-1} c \Rightarrow \frac{r^2 + s}{1 - r^2 s} = c$

(5) $\tan \theta dr + 2r d\theta = 0$

$$\frac{dr}{2r} + \frac{1}{\tan \theta} d\theta = 0$$

$$\frac{1}{2} \log r + \log (\sin \theta) = \log c$$

$$\sqrt{r} \sin \theta = c$$

(7) $(x + 4)(y^2 + 1)dx + y(x^2 + 3x + 2)dy = 0$

$$\left(\frac{x + 4}{x^2 + 3x + 2}\right) dx + \frac{y}{y^2 + 1} dy = 0$$

$$\left(\frac{x + 4}{(x + 1)(x + 2)}\right) + \frac{1}{2} \left(\frac{2y}{y^2 + 1}\right) dy = 0$$

$$\left(\frac{3}{x + 1} - \frac{1}{2(x + 2)}\right) dx + \frac{1}{2} \left(\frac{2y}{y^2 + 1}\right) dy = 0$$

$$\frac{A}{(x + 1)} + \frac{B}{(x + 2)} = \frac{x + 4}{(x + 1)(x + 2)} \quad \text{by partial fraction}$$

$$3 \log (x + 1) - \frac{1}{2} \log (x + 2) + \frac{1}{2} \log (y^2 + 1) = \log c$$

ie. $\frac{(x + 1)^3 (y^2 + 1)^{1/2}}{(x + 2)^{1/2}} = c$

(9) $(2xy + 3y^2)dx - (2xy + x^2)dy = 0$

$$\frac{dy}{dx} = \left(\frac{2xy + 3y^2}{2xy + x^2}\right)$$

Put $y = vx \Rightarrow v + x \frac{dv}{dx} = \frac{dy}{dx}$

$$v + x \frac{dv}{dx} = \frac{2v + 3v^2}{2v + 1} \Rightarrow \frac{xdv}{dx} = \frac{2v + 3v^2 - 2v^2 - v}{2v + 1}$$

$$= \frac{v^2 + v}{2v + 1}$$

$$\text{ie. } \left(\frac{2v+1}{v^2+v} \right) dv = \frac{1}{x} dx$$

$$\log(v^2+v) = \log x + \log c$$

$$v^2+v = xc$$

$$\left(\frac{y}{x} \right)^2 + \frac{y}{x} = x c$$

$$y^2 + yx = x^3 c$$

$$(11) \left(x \tan \frac{y}{x} + y \right) dx - dy = 0$$

$$\left(\tan \frac{y}{x} + \frac{y}{x} \right) dx - dy = 0$$

$$\frac{dy}{dx} = \tan \frac{y}{x} + \frac{y}{x}$$

$$v + x \frac{dv}{dx} = \tan v + v$$

$$\cot v dv = \frac{1}{x} dx$$

$$\log(\sin v) = \log x + \log c \Rightarrow \sin v = x c$$

$$\Rightarrow \sin \left(\frac{y}{x} \right) = x c$$

$$(13) (x^3 + y^2 \sqrt{x^2 + y^2}) dx - xy \sqrt{x^2 + y^2} dy = 0$$

$$\text{Put } y = vx$$

$$x^3 + v^2 x^3 \sqrt{1+v^2} dx - x v x^2 \sqrt{1+v^2} dy = 0$$

$$1 + v^2 \sqrt{1+v^2} dx - v \sqrt{1+v^2} dy = 0$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} = \frac{1 + v^2 \sqrt{1+v^2}}{v \sqrt{1+v^2}}$$

$$\frac{xdv}{dx} = \frac{1 + v^2 \sqrt{1+v^2} - v^2 \sqrt{1+v^2}}{v \sqrt{1+v^2}} = \frac{1}{v \sqrt{1+v^2}}$$

$$v \sqrt{1+v^2} dv = \frac{1}{x} dx$$

$$t^2 dt = \frac{1}{x} dx$$

$$\text{put } 1 + v^2 = t^2$$

$$2v dv = 2t dt$$

$$v dv = t dt$$

$$\Rightarrow \frac{t^3}{3} = \log x + \log c \Rightarrow \frac{(\sqrt{1+v^2})^3}{3} = \log xc \Rightarrow \left(1 + \frac{y^2}{x^2} \right)^{\frac{3}{2}} = 3 \log(xc)^3$$

$$= \log(xc)^3$$

$$\Rightarrow (x^2 + y^2)^{3/2} = x^3 \log(xc)^3$$

$$(14) (\sqrt{x+y} + \sqrt{x-y}) dx + (\sqrt{x-y} - \sqrt{x+y}) dy = 0$$

$$\frac{dy}{dx} = \frac{(\sqrt{x+y} + \sqrt{x-y})}{(\sqrt{x+y} - \sqrt{x-y})} = \frac{(\sqrt{x+y} + \sqrt{x-y})^2}{x+y-x+y} = \frac{x+y+(x-y)+2\sqrt{(x+y)(x-y)}}{2y}$$

$$= \frac{2x+2\sqrt{x^2-y^2}}{2y} = \frac{x+\sqrt{x^2-y^2}}{y}$$

$$y = vx$$

$$v + \frac{xdv}{dx} = \frac{1 + \sqrt{1-v^2}}{v}$$

$$\frac{xdv}{dx} = \frac{1 + \sqrt{1-v^2} - v^2}{v} \text{ ie. } \frac{(1-v^2) + \sqrt{1-v^2}}{v}$$

$$\frac{1}{x} dx = \frac{1}{(1-v^2) + \sqrt{1-v^2}} dv$$

$$\text{Put } 1 - v^2 = t^2$$

$$-2v dv = 2t dt$$

$$v dv = -t dt$$

$$\frac{1}{x} dx = \frac{-t dt}{t^2 + t} = \frac{-dt}{t+1}$$

$$\log x = -\log(t+1) + \log c = \log \frac{1}{(t+1)} c$$

$$\log x = \log \frac{c}{(t+1)}$$

$$\Rightarrow x = \frac{c}{(\sqrt{1-v^2}+1)} = \frac{c}{\sqrt{1-\frac{y^2}{x^2}}+1}$$

$$\Rightarrow x = \frac{c}{\frac{\sqrt{x^2-y^2}}{x}+1} \Rightarrow c = x \frac{(\sqrt{x^2-y^2}+x)}{x}$$

$$\Rightarrow c = \sqrt{x^2-y^2}+x$$

$$\Rightarrow c-x = \sqrt{x^2-y^2}$$

$$\Rightarrow c^2+x^2-2cx = x^2-y^2$$

$$\Rightarrow c^2-2cx = -y^2 \Rightarrow 2cx - c^2 = y^2 \Rightarrow y = (2cx - c^2)^{1/2}$$

$$(15) (y+2) dx + y(x+4) dy = 0, \quad y(-3) = -1$$

$$\frac{dx}{x+4} + y \frac{dy}{y+2} = 0$$

$$\frac{dx}{x+4} + \left(\frac{y+2}{y+2} - \frac{2}{y+2} \right) dy = 0$$

$$\log(x+4) + y - 2\log(y+2) = \log c$$

$$\log(x+4) + \log \frac{1}{(y+2)^2} + y = \log c$$

$$\Rightarrow \log \frac{x+4}{(y+2)^2} + y = \log c$$

$$\Rightarrow \log \frac{c}{(x+4)} (y+2)^2 = -y$$

$$\Rightarrow \log C - \log \frac{(x+4)}{(y+2)^2} = y$$

$$\log \frac{C(y+2)^2}{(x+4)} = y$$

$$\Rightarrow C \frac{(y+2)^2}{x+4} = e^y$$

$$\Rightarrow C = e^y \frac{(x+4)}{(y+2)^2}$$

$$\text{put } x = -3 \text{ \& } y = -1$$

$$\Rightarrow C = \frac{e^{-1}(1)}{1} \Rightarrow C = \frac{1}{e}$$

$$\Rightarrow \frac{1}{e} = e^y \frac{(x+4)}{(y+2)^2}$$

$$(17) (3x+8)(y^2+4)dx - 4y(x^2+5x+6)dy = 0$$

$$\frac{(3x+8)}{(x^2+5x+6)} dx - \frac{4y}{(y^2+y)} dy$$

$$\frac{(3x+8)}{(x+2)(x+3)} dx - \frac{4y}{y^2+4} dy$$

$$\frac{1}{(x+3)} dx + \frac{2}{(x+2)} dx - \frac{2.2y}{y^2+4} dy = 0$$

$$\log(x+3) + 2\log(x+2) - 2\log(y^2+4) = \log c$$

$$\Rightarrow \frac{(x+3)(x+2)}{(y^2+4)^2} = c \quad y(1) = 2$$

$$\Rightarrow \frac{(1+3)(1+2)^2}{(4+4)^2} = c \Rightarrow \frac{4 \times 3 \times 3}{64} = c$$

$$\Rightarrow c = \frac{9}{16}$$

$$\Rightarrow \frac{(x+3)(x+2)^2}{(y^2+4)^2} = \frac{9}{16}$$

$$\Rightarrow 16(x+3)(x+2)^2 = 9(y^2+4)^2$$

$$(19) (2x-5y) dx + (4x-y) dy = 0, \quad y(1) = 4$$

$$\text{put } y = xv$$

$$(2x-5xv) dx + (4x-xv) dy = 0$$

$$(2-5v) dx + (4-v) dy = 0$$

$$\frac{dy}{dx} = \frac{2-5v}{-4+v} = v + \frac{x dv}{dx}$$

$$\Rightarrow \frac{x dv}{dx} = \frac{2-5v+4v-v^2}{(v-4)}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2-v^2-v}{(v-4)}$$

$$\frac{(v-4)}{(v+2)(v-1)} dv = \frac{1}{x} dx$$

$$\frac{-2}{(v+2)} + \frac{1}{(v-1)} = \log x + \log c$$

$$\log(v+2)^{-2} + \log(v-1) = \log x + \log c$$

$$\frac{\left(\frac{y}{x} - 1\right)}{\left(\frac{y}{x} + 2\right)^2} = x c \quad \text{Put } x = 1, y = 4$$

$$\frac{(4-1)}{(6)^2} = c \Rightarrow \frac{3}{36} = c \Rightarrow c = \frac{1}{12}$$

$$\text{ie. } \frac{\left(\frac{y}{x} - 1\right)}{\left(\frac{y}{x} + 2\right)^2} = \frac{x}{12} \Rightarrow \frac{\frac{y-x}{x}}{\left(\frac{y+2x}{x}\right)^2} = \frac{x}{12}$$

$$\Rightarrow x \frac{(y-x)}{(y+2x)^2} = \frac{x}{12} \Rightarrow 12(y-x) = (y+2x)^2$$

$$(21) (Ax + By)dx + (Cx + Dy)dy = 0 \text{ exact iff } B = C$$

homogenous

Forward part: assume that the given homogenous eqⁿ is exact.

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow B = C$$

Conversely assume that $B = C$

$$(Ax + By)dx + (Bx + Dy)dy = 0$$

$$\frac{\partial M}{\partial y} = B, \quad \frac{\partial N}{\partial x} = B$$

$$\Rightarrow \text{exact}$$

$$(b) (Ax^2 + Bxy + Cy^2)dx + (Dx^2 + Exy + Fy^2)dy = 0$$

$$M dx + N dy = 0$$

$$\frac{\partial M}{\partial y} = Bx + 2Cy, \quad \frac{\partial N}{\partial x} = 2Dx + Ey$$

$$Bx + 2Cy = 2Dx + Ey$$

$$\Rightarrow B = 2D, E = 2C$$

$$\text{Linear eq}^n: \frac{dy}{dx} + p(x)y = Q(x)$$

$$\text{Ex. 2.14 } \frac{dy}{dx} + \frac{(2x+1)}{x}y = e^{-2x}$$

$$P(x) = \frac{(2x+1)}{x}, \quad Q = e^{-2x}$$

$$\text{I.F.} = e^{\int p(x)dx}$$

$$= e^{\int (2 + \frac{1}{x})dx} = e^{2x \log x} = e^{2x}$$

$$\text{Sol}^n \equiv y(xe^{2x}) = \int (e^{-2x} e^{2x} x)dx + c$$

$$xe^{2x} y = \frac{x^2}{2} + c$$

$$\text{Ex: 2.15 } (x^2 + 1) \frac{dy}{dx} + 4xy = x$$

$$\Rightarrow \frac{dy}{dx} + \frac{4xy}{(x^2+1)} = \frac{x}{x^2+1}$$

$$P(x) = \frac{4x}{(x^2+1)}, \quad Q(x) = \frac{x}{x^2+1}$$

$$\text{I.F.} = e^{\int \frac{4x}{x^2+1}dx} = e^{2\log(x^2+1)} = (x^2+1)^2$$

$$\text{Sol}^n: y \cdot (x^2+1)^2 = \int \frac{x}{(x^2+1)} (x^2+1)^2 dx + c$$

$$y(x^2+1)^2 = \int x^3 + x dx + c$$

$$y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + c$$

$$\text{Ex 2.16 } y^2 dx + (3xy - 1)dy = 0$$

$$\frac{dy}{dx} = \frac{y^2}{1-3xy} \Rightarrow \frac{dx}{dy} = \frac{1-3xy}{y^2}$$

$$\frac{dx}{dy} = \frac{1}{y^2} - \frac{3xy}{y^2} = \frac{1}{y^2} - \frac{3x}{y}$$

$$\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$$

$$\frac{dx}{dy} + p(y)x = Q(y)$$

$$P(y) = \frac{3}{y}, \quad Q(y) = \frac{1}{y^2}$$

$$\therefore \text{IF} = e^{\int p(y)dy} = e^{\int \frac{3}{y} dy} = e^{3 \log y} = y^3$$

$$\text{Sol}^n: x \text{ I.F.} = \int Q(y) \text{ I.F. } dy + C$$

$$xy^3 = \int \frac{1}{y^2} y^3 dy + C = \int y dy + C$$

$$xy^3 = \frac{y^2}{2} + C$$

II Type

$\frac{dy}{dx} + p(x)y = Q(x)y^n$ is called a Bernoulli differential eqⁿ

Ex. 2.17 $\frac{dy}{dx} + y = xy^3$
55

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{y^2} = x$$

$$-\frac{1}{2} \frac{dv}{dx} + v = x$$

$$\frac{dv}{dx} - 2v = -2x$$

Now it is linear eqⁿ in v

$$P(x) = -2$$

$$Q(x) = -2x$$

$$\text{IF} \equiv e^{\int P(x)dx} = e^{\int -2 dx} = e^{-2x}$$

$$\text{Sol}^n: v \cdot \text{I.F.} = \int Q(x) \text{IF} dx + c$$

$$v \cdot e^{-2x} = \int -2xe^{-2x} dx + c$$

$$\begin{aligned} v \cdot e^{-2x} &= -2x \int e^{-2x} dx - \int \frac{d}{dx}(-2x) \int e^{-2x} dx + c \\ &= -2x \cdot \frac{e^{-2x}}{-2} - \int \frac{-2e^{-2x}}{-2} dx + c \\ &= xe^{-2x} - \int e^{-2x} dx + c \end{aligned}$$

$$v \cdot e^{-2x} - xe^{-2x} = -\frac{e^{-2x}}{-2} + c$$

$$ve^{-2x} - xe^{-2x} = \frac{e^{-2x}}{2} + c \quad \text{put } \frac{1}{y^2} = v$$

$$\frac{1}{y^2} e^{-2x} - xe^{-2x} = \frac{e^{-2x}}{2} + c$$

$$\frac{1}{y^2} - x = \frac{1}{2} + ce^{2x}$$

$$\text{i.e. } \frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$$

Excercise.

(1) $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$

$$p(x) = \frac{3}{x}, \quad Q(x) = 6x^2$$

$$\text{I.f.} = e^{\int \frac{3}{x} dx} = e^{\log x^3} = x^3$$

$$yx^3 = \int 6x^2 x^3 dx + c$$

$$yx^3 = x^6 + c$$

$$(3) \frac{dy}{dx} + 3y = 3x^2 e^{-3x}$$

$$\text{If } = e^{\int 3dx} = e^{3x}$$

$$ye^{3x} = \int 3x^2 e^{3x-3x} dx + C$$

$$ye^{3x} = \int 3x^2 dx + C$$

$$ye^{3x} = x^3 + C$$

$$(5) \frac{dx}{dt} + \frac{x}{t^2} = \frac{1}{t^2}$$

$$p(t) = \frac{1}{t^2}, \quad Q(t) = \frac{1}{t^2}$$

$$\text{I.F. } e^{\int \frac{1}{t^2} dt} = e^{-\frac{1}{t}}$$

$$\therefore \text{Sol}^n : x e^{-1/t} = \int \frac{1}{t^2} e^{-1/t} dt + C$$

$$x e^{-1/t} = \int dp + c$$

$$x e^{-1/t} = p + C$$

$$x e^{-1/t} = e^{-1/t} = C$$

$$(x-1) e^{-1/t}$$

$$\begin{aligned} \text{let } e^{-1/t^2} &= p \\ e^{-1/t^2-2} &= dp/dt \\ pt^{-2} dt &= dp \Rightarrow \frac{1}{t^2} dt = 1/p dp \end{aligned}$$

Miscellaneous[विबिध, फुटकर] Review Pg. 60

$$(1) 6x^2 y dx - (x^3 + 1) dy = 0$$

$$\frac{dy}{dx} = \frac{6x^2 y}{(x^3 + 1)}$$

$$\frac{dy}{dx} - \frac{6x^2}{(x^3 + 1)} y = 0$$

$$\text{IF} = e^{\int \frac{-6x^2}{x^3+1} dx} = e^{-2 \int \frac{1}{t} dt}$$

$$\begin{aligned} &= e^{\log t^{-2}} = t^{-2} \\ &= (x^3 + 1)^{-2} \end{aligned}$$

$$y (x^3 + 1)^{-2} = c$$

$$y = c(x^3 + 1)^2$$

also apply variable of separation method

$$(3) (y-1)dx + x(x+1)dy = 0$$

$$\frac{dx}{x(x+1)} + \frac{dy}{(y-1)} = 0$$

$$\left(\frac{1}{x} - \frac{1}{x+1}\right) dx + \frac{dy}{y-1} = 0$$

$$\log x - \log(x+1) + \log(y-1) = \log c$$

$$\log \frac{x(y-1)}{(x+1)} = \log c \Rightarrow x \frac{(y-1)}{(x+1)} = c$$

$$x(y-1) = c(x+1)$$

$$1 + xy = c(x+1) + (x+1)$$

$$1 + xy = (c+1)(x+1)$$

$$1 + xy = C(x+1)$$

$$(5) (3x-5y)dx + (x+y)dy = 0$$

$$\frac{dy}{dx} = \frac{(5y-3x)}{x+y}$$

Put $y = vx$

$$\frac{dy}{dx} = \frac{5vx-3x}{x+vx} = \frac{5v-3}{1+v}$$

$$v + x \frac{dv}{dx} = \frac{5v-3}{1+v} \Rightarrow x \frac{dv}{dx} = \frac{5v-3-v-v^2}{1+v}$$

$$x \frac{dv}{dx} = \frac{4v-3-v^2}{1+v}$$

$$\frac{(1+v)}{4v-3-v^2} dv = \frac{1}{x} dx$$

$$\Rightarrow \frac{(1+v)dv}{(v-3)(v-1)} = \frac{1}{x} dx$$

$$\Rightarrow \left[\frac{-2}{(v-3)} + \frac{1}{v-1} \right] dv = \frac{1}{x} dx \quad \text{by partial fraction}$$

$$\Rightarrow \log(v-3)^{-2} + \log(v-1) = \log x + \log c$$

$$\Rightarrow (v-1) = c x (v-3)^2$$

$$y-x = \frac{cx}{x} (y-3x)^2$$

$$y-x = \frac{cx}{x} (y-3x)^2$$

$$y = x + c (y-3x)^2$$

$$(7) (8x^3y - 12x^3) dx + (x^4 + 1)dy = 0$$

$$\frac{\partial M}{\partial y} = 8x^3, \quad \frac{\partial N}{\partial x} = 4x^3 \quad \text{not exact}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{8x^3 - 4x^3}{x^4 + 1} = \frac{4x^3}{x^4 + 1} = f(x)$$

$$\text{IF } e^{\int \frac{4x^3}{x^4+1} dx} = (x^4 + 1)$$

$$(x^4 + 1)(8x^3y - 12x^3) dx + (x^4 + 1)(x^4 + 1)dy = 0$$

$$(x^4 + 1)(8x^3y - 12x^3)dx + (x^4 + 1)^2 dy = 0$$

$$\text{Now } \frac{\partial M}{\partial y} = 8x^3(x^4 + 1), \quad \frac{\partial N}{\partial x} = 2(x^4 + 1)4x^3$$

$$= 8x^3(x^4 + 1)$$

∴ in exact form

$$\therefore \text{Sol}^n: \int (8x^7y - 12x^7 + 8x^3y - 12x^3)dx + \int dy = c$$

$$\Rightarrow 8y \frac{x^8}{8} - \frac{12x^8}{8} + 8y \frac{x^4}{4} - \frac{12x^4}{4} + y = c$$

$$\Rightarrow x^8y - \frac{3}{2}x^8 + 2yx^4 - 3x^4 + y = c$$

$$\Rightarrow y(x^8 + 2x^4 + 1) = c + \frac{3}{2}x^8 + 3x^4$$

$$\Rightarrow y(x^4 + 1)^2 = c + \frac{3}{2}x^8 + 3x^4$$

$$\Rightarrow y = c(x^4 + 1)^{-2} + \frac{3}{2}[x^8 + 2x^4 + 1](x^4 + 1)^{-2} - \frac{3}{2}(x^4 + 1)^{-2}$$

$$\Rightarrow y = c(x^4 + 1)^{-2} - \frac{3}{2}(x^4 + 1)^{-2} + \frac{3}{2}(x^4 + 1)^{-2}(x^4 + 1)^2y = \frac{3}{2} + (x^4 + 1)^{-2}.c$$

$$(9) \frac{dy}{dx} = \frac{4x^3y^2 - 3x^2y}{x^3 - 2x^4y} \Rightarrow (4x^3y^2 - 3x^2y)dx + (2x^4y - x^3)dy = 0$$

$$\frac{\partial M}{\partial y} = 8x^3y - 3x^2, \quad \frac{\partial N}{\partial x} = 2y4x^3 - 3x^2$$

$$= 8x^3y - 3x^2$$

exact.

$$\text{Sol}^n: \int (4x^3y^2 - 3x^2y) dx + 0 = c$$

$$\Rightarrow 4y^2 \frac{x^4}{4} - 3y \frac{x^3}{3} = c$$

$$\Rightarrow x^4y^2 - x^3y = c$$

$$(11) \frac{dy}{dx} = \frac{2x - 7y}{3y - 8x} = \frac{2 - 7(y/x)}{3(y/x) - 8}$$

put $y = vx$

$$v + x \frac{dv}{dx} = \frac{2 - 7v}{3v - 8}$$

$$x \frac{dv}{dx} = \frac{2 - 7v - 3v^2 + 8v}{3v - 8} = \frac{2 - 3v^2 + v}{3v - 8}$$

$$\frac{2 - 3v^2 + v}{8 - 3v} dv = \frac{1}{x} dx \Rightarrow \frac{3v^2 - v - 2}{3v^2 - v - 2} = \frac{1}{x} dx$$

$$\frac{3v^2 - 3v + 2v - 2}{8 - 3v} = \frac{1}{x} dx$$

$$\Rightarrow \frac{3v(v-1) + 2(v-1)}{8-3v} dv = \frac{1}{x} dx$$

$$\Rightarrow \frac{(3v+2)(v-1)}{8-3v} dv = \frac{1}{x} dx$$

$$\frac{1}{(3v+2)(v-1)} = \frac{A}{(3v+2)} + \frac{B}{v-1}$$

$$8-3v = A(v-1) + B(3v+2)$$

$$8-3v = Av - A + 3Bv + 2B = (A+3B)v + (2B-A)$$

$$2B - A = 8 \quad \& \quad A + 3B = -3$$

$$6B - 3A = 24 \quad \& \quad 3A + 9B = -9$$

$$3A + 6B = 24$$

$$15B = 15 \Rightarrow B = 1$$

$$A = -6$$

$$\Rightarrow \frac{8-3v}{(3v+2)(v-1)} dv = \left(\frac{-6}{3v+2} \right) dv + \frac{1}{v-1} dv = \frac{1}{x} dx$$

$$-2 \log(3v+2) + \log(v-1) = \log x + \log c$$

$$(v-1)(3v+2)^{-2} = xc$$

$$\frac{\frac{y-x}{x}}{\frac{3y+2x}{x^2}} = xc \Rightarrow \frac{y-x}{(3y+2x)^2} = c$$

Start 13

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0 \dots \dots \dots (A)$$

Case I $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$ then the transformation is

$$x = X + h$$

$$y = Y + K$$

Where (h, k) is the solⁿ of the system

$$a_1h + b_1k + c_1 = 0$$

$$a_2h + b_2k + c_2 = 0$$

then we reduced it in the homogenous diff. eqⁿ

Case 2 If $\frac{a_2}{a_1} = \frac{b_2}{b_1} = k$ then the transformation is

$Z = a_1x + b_1y$ reduces the eqⁿ (A) to a seprable equation in the variables x & z

$$\text{Ex. 2.19 } (x - 2y + 1) dx + (4x - 3y - 6) dy = 0$$

$$\frac{a_2}{a_1} = 4, \quad \frac{b_2}{b_1} = \frac{-3}{-2} = \frac{3}{2} \Rightarrow \frac{a_2}{a_1} \neq \frac{b_2}{b_1}$$

$$\therefore \text{ put } x = X + h, \quad y = Y + K$$

$$\left. \begin{aligned} h - 2K + 1 &= 0 \\ 4h - 3K - 6 &= 0 \\ -4h - 8K + 4 &= 0 \end{aligned} \right\} \begin{aligned} x &= X + 3 \\ y &= Y + 2 \end{aligned}$$

$$5K - 10 = 0 \Rightarrow K = 2$$

$$h - 4 + 1 = 0 \Rightarrow h = 3$$

$$(X - 2Y)dX + (4X - 3Y)dY = 0$$

$$\frac{dY}{dX} = \frac{1 - 2(Y/X)}{3(Y/X) - 4}$$

$$Y = VX$$

$$V + X \frac{dV}{dX} = \frac{1 - 2V}{3V - 4}$$

$$\Rightarrow \log \left| \frac{(3V+1)^5}{V-1} \right| = \log \left| \frac{c^4}{X^4} \right|$$

$$\Rightarrow X^4 |(3V+1)^5| = c|V-1|$$

$$|(3Y+X)^5| = c|Y-X|$$

$$|3(y-2) + (x-5)|^5 = c|y-2-x+3|$$

$$|x+3y-9|^5 = c|y-x+1| \text{ Ans.}$$

$$\text{Ex. 2.20 } (x + 2y + 3) dx + (2x + 4y - 1) dy = 0$$

$$a_1 = 1, \quad b_1 = 2, \quad C_1 = 3$$

$$a_2 = 2, \quad b_2 = 4, \quad C_2 = -1$$

$$\frac{a_2}{a_1} = 2, \quad \frac{b_2}{b_1} = 2 \Rightarrow \frac{a_2}{a_1} = \frac{b_2}{b_1} = K$$

therefore, transformation

$$z = x + 2y \quad \text{by} \quad z = a_1x + b_1y$$

$$(z + 3)dx + (2z - 1) \left(\frac{dz - dx}{2} \right) \quad \left\{ \begin{aligned} dz &= dx + 2xy \\ \text{ie. } dy &= \left(\frac{dz - dx}{2} \right) \end{aligned} \right.$$

$$\left[(z + 3) - \left(z - \frac{1}{2} \right) \right] dx + \left(\frac{2z - 1}{2} \right) dz = 0$$

$$\frac{7}{2} dx + \left(\frac{2z - 1}{2} \right) dz = 0$$

$$7 dx + (2z - 1)dz = 0$$

$$7x + 2\frac{z^2}{2} - z = c$$

$$\Rightarrow 7x + (x + 2y)^2 - (x + 2y) = c$$

$$\Rightarrow x^2 + 4xy + 4y^2 + 6x - 2y = c$$

$$(1) (5xy + 4y^2 + 1) dx + (x^2 + 2xy)dy = 0 \dots \dots \dots (1)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{5x + 8y - 2x - 2y}{x^2 + 2xy} = \frac{3x + 6y}{x^2 + 2xy}$$

$$= \frac{3(x + 2y)}{x(x + 2y)} = \frac{3}{x} = f(x)$$

$$IF = e^{\int \frac{3}{x} dx}$$

$$= e^{3 \log x} = x^3$$

$\therefore$ Now multiplying this I.F by eqⁿ (i)

$$x^3 (5xy + 4y^2 + 1)dx + x^3(x^2 + 2xy)dy = 0$$

$$(5x^4y + 4x^3y^2 + x^3)dx + (x^5 + 2x^4y) dy = 0$$

$$\text{Now } \frac{\partial M}{\partial y} = 5x^4 + 8x^3y \quad \frac{dN}{dx} = 5x^4 + 8x^3y$$

$$\text{Sol}^n: \int (5x^4y + 4x^3y^2 + x^3) dx = c$$

$$\Rightarrow 5y \frac{x^5}{5} + 4y^2 \frac{x^4}{4} + \frac{x^4}{4} = c$$

$$\Rightarrow yx^5 + x^4y^2 + \frac{x^4}{4} = c$$

$$\Rightarrow 4x^5y + 4x^4y^2 + x^4 = C$$

$$(3) [y^2(x + 1) + y]dx + (2xy + 1)dy = 0$$

$$[y^2x + y^2 + y]dx + (2xy + 1)dy = 0$$

$$\frac{\partial M}{\partial y} = 2xy + 2y + 1, \quad \frac{\partial N}{\partial x} = 2y + 0$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy + 2y + 1 - 2y}{2xy + 1} = \frac{2xy + 1}{2xy + 1} = 1$$

$$\therefore \text{If} = e^{\int dx} = e^x$$

$$\Rightarrow e^x [y^2(x + 1) + y]dx + e^x(2xy + 1)dy = 0$$

$$[e^xy^2x + e^xy^2 + e^xy]dx + (2xye^x + e^x) dy = 0$$

exact.

$$\text{Sol}^n: \int (e^xy^2x + e^xy^2 + e^xy)dx + 0 = c$$

$$\Rightarrow y^2 \int e^x x dx + y^2 \int e^x dx + y \int e^x dx = c$$

$$\Rightarrow y^2xe^x - y^2e^x + y^2e^x + ye^x = c$$

$$y^2xe^x + ye^x = c$$

$$(5) (4xy^2 + 6y)dx + (5x^2y + 8x)dy = 0 \dots \dots \dots (1)$$

$$M = 4xy^2 + 6y$$

$$N = 5x^2y + 8x$$

$$\frac{\partial M}{\partial y} = 8x + 6, \quad \frac{\partial N}{\partial x} = 2 \times 5yx + 8$$

Now multiplying by $x^\alpha y^\beta$ in the eqⁿ (1)

$$(4x^{\alpha+1}y^{2+\beta} + 6x^\alpha y^{\beta+1})dx + (5x^{2+\alpha}y^{\beta+1} + 8x^{\alpha+1}y^\beta) dy = 0$$

$$\frac{\partial M}{\partial y} = 4x^{\alpha+1}(2 + \beta)y^{\beta+1} + 6x^\alpha(\beta + 1)y^\beta$$

$$\frac{\partial N}{\partial x} = 5y^{\beta+1}(2 + \alpha)x^{\alpha+1} + 8y^\beta(\alpha + 1)x^\alpha$$

comparing

$$4(2 + \beta) = 5(2 + \alpha) \quad \& \quad 6(\beta + 1) + 8(\alpha + 1)$$

$$4\beta - 5\alpha = 2 \dots \dots \dots (A)$$

$$\beta = 3$$

$$\begin{cases} 6\beta - 8\alpha = 2 \\ 4\beta - 5\alpha = 2 \end{cases}$$

$$12\beta - 16\alpha = 4$$

$$12\beta - 15\alpha = 6$$

$$-\alpha = -2 \Rightarrow \alpha = 2$$

$\therefore$ integrating factor ie. x^2y^3

$$\text{Now } x^2y^3(4xy^2 + 6y)dx + x^2y^3(5x^2y + 8x)dy = 0$$

$$(4x^3y^5 + 6x^2y^4)dx + (5x^4y^4 + 8x^3y^3)dy = 0$$

$$\text{Now } \frac{\partial M}{\partial y} = 4x^3 \times 5y^4 + 6x^2 \times 4y^3 = 20x^3y^4 + 24x^2y^3$$

$$\frac{\partial N}{\partial x} = 20x^3y^4 + 24x^2y^3 \quad \text{exact}$$

$$\text{Sol}^n : \int (4x^3y^5 + 6x^2y^4) dx = c$$

$$4y^5 \frac{x^4}{4} + 6y^4 \frac{x^3}{3} = c$$

$$\Rightarrow y^5x^4 + 2y^4x^3 = c$$

$$\text{ie. } x^3y^4(xy + 2) = c \quad \text{Ans.}$$

$$(7) (5x + 2y + 1)dx + (2x + y + 1)dy = 0$$

$$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2 \quad \text{exact.}$$

$$\text{Sol}^n: \int (5x + 2y + 1)dx + \int (y + 1)dy = c$$

$$\Rightarrow \frac{5x^2}{2} + 2yx + x + \frac{y^2}{2} + y = c$$

$$\Rightarrow 5x^2 + 4xy + 2x + y^2 + 2y = 2c$$

$$(9) (x - 2y - 3)dx + (2x + y - 1)dy = 0$$

$$\frac{a_2}{a_1} = \frac{2}{1}, \quad \frac{b_2}{b_1} = \frac{1}{-2}$$

$$\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$$

$$\therefore \text{transformation } x = X + h$$

$$y = Y + k$$

$$\frac{dY}{dX} = \frac{2Y + 2k - X - h + 3}{2X + 2h + Y + k - 1}$$

$$\frac{dY}{dX} = \frac{(2Y - X) + (2k - h + 3)}{(2X + Y) + (2h + k - 1)}$$

$$2k - h + 3 = 0$$

$$2h + k - 1 = 0$$

$$\begin{array}{r} 2k - h = -3 \\ 2k + 4h = 2 \\ \hline -5h = -5 \Rightarrow h = 1 \end{array}$$

$$k = -1$$

$$\text{ie } x = X + 1 \Rightarrow X = (x - 1)$$

$$y = Y - 1 \Rightarrow Y = (y + 1)$$

$$\frac{dY}{dX} = \frac{2Y - X}{2X + Y}$$

$$\text{Now put } Y = XV$$

$$V + X \frac{dV}{dX} = \frac{2XV - X}{2X + XV}$$

$$X \frac{dV}{dX} = \left(\frac{2V - 1}{2 + V} \right) - V$$

$$X \frac{dV}{dX} = \frac{2V - 1 - 2V - V^2}{2 + V} = \frac{-1 - V^2}{2 + V} = -\frac{(V^2 + 1)}{V + 2}$$

$$\frac{-(V + 2)}{(V^2 + 1)} dV = \frac{1}{X} dX$$

$$\Rightarrow \left(\frac{-V}{V^2 + 1} - \frac{2}{V^2 + 1} \right) dV = \frac{1}{X} dX$$

$$\Rightarrow \frac{-1}{2} \log(V^2 + 1) - 2 \tan^{-1} V = \log X + \log c$$

$$\Rightarrow \log \left(\frac{Y^2}{X^2} + 1 \right) + 4 \tan^{-1} \left(\frac{Y}{X} \right) + \log X^2 = C$$

$$\Rightarrow \log \left(\frac{(y + 1)^2}{(x - 1)^2} + 1 \right) + 4 \tan^{-1} \left(\frac{(y + 1)}{(x - 1)} \right) + \log(x - 1)^2 = C$$

$$\Rightarrow \log[(y + 1)^2 + (x - 1)^2] + 4 \tan^{-1} \frac{(y + 1)}{(x - 1)} = C$$

$$\Rightarrow \log[y^2 + 1 + 2y + x^2 + 1 - 2x] + 4 \tan^{-1} \frac{(y+1)}{(x-1)} = C$$

$$\text{OR } \log[C(x^2 + y^2 - 2x + 2y + 2)] + 4 \tan^{-1} \frac{(y+1)}{x+1} = 0$$

$$(11) (6x + 4y + 1) dx + (4x + 2y + 2) dy = 0$$

$$\frac{\partial M}{\partial y} = 4 = \frac{\partial N}{\partial x} \quad \text{exact}$$

$$\int (6x + 4y + 1) dx + \int 2y + 2 dy = 0$$

$$\frac{6x^2}{2} + 4xy + x + \frac{2y^2}{2} + 2y = c$$

$$3x^2 + 4xy + x + y^2 + 2y = c$$

$$(13) (2x + 3y + 1) dx + (4x + 6y + 1) dy = 0 \quad y(-2) = 2$$

$$\frac{a_2}{a_1} = \frac{4}{2}, \quad \frac{b_2}{b_1} = \frac{6}{3}$$

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = K$$

$$\therefore z = a_1 x + b_1 y$$

$$\text{ie. put } z = 2x + 3y \Rightarrow 2z = 4x + 6y$$

$$6y = (2z - 4x)$$

$$dz = 2dx + 3dy$$

$$dy = \left(\frac{dz - 2dx}{3} \right)$$

$$(z + 1) dx + (2z + 1) dy = 0$$

$$(z + 1) dx + (2z + 1) \left(dz - \frac{2dx}{3} \right) = 0$$

$$3(z + 1) dx + (2z + 1) dz - 2(2z + 1) dx = 0$$

$$(3z + 3 - 4z - 2) dx + (2z + 1) dz = 0$$

$$(1 - z) dx - (2z + 1) dz = 0$$

$$(z - 1) dx - (2z + 1) dz = 0$$

$$dx = \left(\frac{2z + 1}{z - 1} \right) dz = 0$$

$$dx - \left[\frac{2z}{z - 1} + \frac{1}{z - 1} \right] dz = 0$$

$$dx - \left(\frac{2(z - 1)}{z - 1} + \frac{2}{z - 1} \right) dz - \frac{1}{(z - 1)} dz = 0$$

$$x - 2z - 2 \log(z - 1) - \log(z - 1) = c$$

$$x - 2z - 3 \log(z - 1) = c$$

$$x - 2(2x + 3y) - 3 \log(2x + 3y - 1) = c$$

Proceed yourself

$$(1) M dx + N dy = 0 \text{ is st. } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

(not exact)

$$\text{homogenous \& exact but } M_x - N_y \neq 0 \text{ If } = \left(\frac{1}{M_x - N_y} \right)$$

$$(2) \text{ If } M dx + N dy = 0 \text{ is homogenous D. E. \& } M_x + N_y \neq 0$$

$$\text{then IF} = \frac{1}{(M_x + N_y)} \quad \text{homogenous but Not exact}$$

$$(3) \text{ If } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \text{const or fun or } x \text{ alone ie } f(x) \text{ then If} = e^{\int f(x) dx}$$

$$(4) \text{ If } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \text{const or fun or } y \text{ alone ie } f(y) \text{ then If} = e^{\int f(y) dy}$$

Reducible 2nd order D. E.

$$\text{ie. } F(x, y, y', y'') = 0$$

(i) when y is missing

(ii) when x is missing

$$(i) F(x, y', y'') = 0 \text{ then put } y' = \frac{dy}{dx} = p$$

$$y'' = \frac{dp}{dx} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$F(x, p, p') = 0$$

$$(ii) F(y, y', y'') = 0$$

$$F\left(y, p, \frac{dp}{dy}\right) = 0$$

$$\begin{aligned} \text{put } y' &= \frac{dy}{dx} = p \\ y'' &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dp}{dx} \\ \frac{dp}{dx} &= \frac{dp}{dy} \frac{dy}{dx} = p \frac{dp}{dy} \end{aligned}$$

Que 33/pg 58. A certain city had a population of 25000 in 1960 and a population of 30000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

$$\text{Sol}^n: x_0 = 25000$$

$$x = x_0 e^{kt}$$

$$\frac{30000}{25000} = e^{k \times 10} \quad \text{i.e. } e^{k \times 10} = \left(\frac{6}{5}\right)$$

$$x = x_0 e^{kt}$$

$$x = 30000 \times (e^{10k})^3 = 30000 \left(\frac{6}{5}\right)^3 = 30000 \times \frac{36 \times 6}{25 \times 5}$$

$$= 51840$$

$$\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = rx \quad \text{rate of interest}$$

$$x = x_0 e^{rt}$$

Q. 47. A certain piece of dubious[not sure] information about phenyl ethylamine in the drinking water began to spread one day in a city with a population of 100,000. Within a week 10,000 people had heard rumor. Assume that the rate of increase of the number who have heard the rumor[rumour] is proportional to the number who have not heard it. How long will it be until half the population of the city has heard the rumor[अफवाह, ऐसी खबर जिसकी सच्चाई संदिग्ध हो] ?

Solⁿ: Let N(t) be the no. of people who have heard the rumour (in thousands)

total population = 100,000

(100 - N) not have heard rumour.

$$\text{given } \frac{dN}{dt} \propto (100 - N)$$

$$\frac{dN}{dt} = k(100 - N) \Rightarrow \frac{dN}{(100 - N)} = k dt$$

$$-\log(100 - N) = kt + \log c$$

$$100 - N = Ae^{-kt}$$

$$t = 0, N_0 = 0$$

$$100 = Ae^0 \Rightarrow A = 100$$

$$100 - N = 100e^{-kt}$$

$$t = 7, N = 10$$

$$100 - 10 = 100e^{-7k}$$

$$e^{7k} = \frac{10}{9} \Rightarrow k = \frac{1}{7} \log \frac{10}{9}$$

$$t = ? \text{ When } N = 50.$$

$$50 = 100e^{-kt}$$

$$\frac{1}{2} = e^{-kt} \Rightarrow e^{kt} = 2$$

$$t = \frac{1}{k} \log_e 2 \Rightarrow t = \frac{7 \log 2}{\log \frac{10}{9}}$$

Rate of Decay: Let x(t) be the amount of Radioactive Substance at any time t.

$$\frac{dx}{dt} \propto x(t) \Rightarrow \frac{dx}{dt} = -kx$$

$$\frac{dx}{x} = -k dt$$

$$\log x = -kt + \log c$$

$$x = Ce^{-kt} \text{ or } x = x_0 e^{-kt} \quad C = x_0$$

$$\text{i.e. } x = x_0 e^{-kt} \quad c = x_0$$

carbon dating fixed k = 0.0001216

Decay Problem में ज्यादा वाला initial

(1) If the half life of radium is 1600 yrs. What is the % remaning after 800 yrs. ?

$$x = x_0 e^{-kt}$$

$$t = 1600 \text{ yrs}, x = \frac{x_0}{2}$$

$$\frac{x_0}{2} = x_0 e^{-1600k} \Rightarrow e^{1600k} = 2$$

$$k = \frac{1}{1600} \log 2$$

$$x = x_0 e^{-800k}, \quad t = 800 \text{ yrs}, \quad x = ?$$

$$x = x_0 e^{-800 \frac{1}{1600} \log 2}$$

$$\therefore x = \frac{x_0}{e^{\log 2}}$$

$$x = x_0 e^{-\frac{1}{2} \log 2} \Rightarrow x = x_0 e^{\log \left(\frac{1}{2}\right)^{\frac{1}{2}}}$$

$$\therefore x = \frac{x_0}{(e^{\log 2})^{\frac{1}{2}}}$$

$$x = x_0 \times \frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{x_0}{\sqrt{2}}$$

$$x = \left(\frac{x_0}{\sqrt{2}}\right)$$

Que(35)(36) (Radiocarbon dating) Carbon extracted [किसी वस्तु को बाहर निकालना (विशेषतः कठिनाई से)] from an ancient skull contained only one – sixth as much ^{19}C as carbon extracted from present day bone. How old is the skull?

[skull : मनुष्य या पशु की अस्थि संरचना, खोपड़ी, कपाल]

$$k = 0.0001216$$

$$x = x_0 e^{-kt}$$

$$x = \frac{1}{6} x_0, \quad \frac{x_0}{6} = x_0 e^{-0.0001216 t}$$

$$e^{0.0001216 t} = 6$$

$$t = \frac{\log 6}{0.0001216}$$

(44,45) When sugar is dissolved in water, the amount A that remains undissolved after t minutes satisfies the differential eqⁿ $\frac{dA}{dt} = -kA$ ($k > 0$). If 25% of the sugar dissolves after 1 min, how long does it take for half of the sugar to dissolve?

Solⁿ: $t = 1 \text{ min}$

$$A = A_0 e^{-kt}$$

25% sugar dissolve
 $\Rightarrow$ 75% sugar undissolved

$$A = 0.75A_0, \quad t = 1$$

$$0.75 = e^{-k}$$

$$k = \log \frac{1}{0.75}$$

$$\frac{1}{2} A_0 = A_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-\log \frac{1}{0.75} t}$$

45/58 The intensity I of light at a depth of x meters below the surface of a lake satisfies the differential eqⁿ

$$\frac{dI}{dx} = (-1.4)I.$$

(a) At what depth is the intensity half the intensity I_0 at the surface (where $x = 0$)?

(b) What is the intensity at a depth of 10 m (as a fraction of I_0)?

(c) At what depth will the intensity be 1% of that at the Surface.

$$\text{Sol}^n: \text{eq}^n \frac{dI}{dx} = (-1.4)I \Rightarrow \frac{dI}{I} = -1.4 dx$$

$$\Rightarrow \log I = -1.4x + c$$

$$\Rightarrow I = e^{-1.4x} e^c = C_1 e^{-1.4x}$$

$$I = C_1 e^{-1.4x}$$

$$I = I_0 e^{-1.4x}$$

$$C = I_0, \quad x = 0$$

(a) $x = ?$

$$\frac{I_0}{2} = I_0 e^{-1.4x}$$

$$e^{1.4x} = 2 \Rightarrow x = \frac{1}{1.4} \log 2$$

(b) $x = 10 \text{ m}$

$$I = I_0 e^{-1.4 \times 10}$$

$$I = I_0 e^{-14}$$

$$\text{fraction intensity} = \frac{I_0 e^{-14}}{I_0} = e^{-14}$$

(c) on surface I_0

$$I = 1\% \text{ of } I_0$$

$$x = ?$$

Newton Law of Cooling – Heating3

Let T = temperature (of cooling body)

T_0 = Temp of surrounding

$$T > T_0$$

Newton law of cooling

$$i. e \frac{dT}{dt} = -k(T - T_0)$$

$$\frac{dT}{dt} = k(T_0 - T)$$

$$T_0 > T$$

temp $\uparrow$

$$\frac{dT}{T - T_0} = -k dt$$

$$\log(T - T_0) = -kt + \log c$$

$$T - T_0 = Ce^{-kt}$$

$$T = T_0 + Ce^{-kt}$$

$$\text{heating } \frac{dT}{dt} = k(T_0 - T)$$

$$\frac{dT}{T_0 - T} = k dt$$

$$\Rightarrow -\log(T_0 - T) = kt + \log c$$

$$T_0 - T = Ce^{-kt}$$

A body whose Temp is 100°C is allowed to cool in a room 20° of constant temp. after 10 minutes the body has cooled to 40°C . Find the temp. of the body after half an hour.

Solⁿ: Initial temp. = 100°C

$$T_0 = 20$$

$$t = 10, T = 40^\circ\text{C}$$

$$T = T_0 + Ce^{-kt}$$

$$T = 20 + Ce^{-kt}$$

$$t = 0, T = 100$$

$$100 = 20 + C \Rightarrow C = 80$$

$$T = 20 + 80e^{-kt}$$

$$40 = 20 + 80e^{-10k} \Rightarrow 20 = 80e^{-10k}$$

$$k = \frac{1}{10} \log 4$$

$$t = 30, T = ?$$

50: The amount $A(t)$ of atmospheric pollutant in a certain mountain valley grows naturally and is tripling every 7.5 yrs.

(a) If the initial amount is 10 μm (pollutant units), write a formula for $A(t)$ giving the amount (in μm) present after 1 yr.

(b) What will be the amount (in μm) of pollutants present in the valley atmosphere after 5 yrs.

(c) If it will be dangerous to stay in the valley when the amount of pollutants reaches 100 μm , How long will it take?

$$(a) x = x_0 e^{kt}$$

$$x_0 = 10, x = 10e^{kt}$$

$$t = 7.5$$

$$x = 3x_0 = 30$$

$$30 = 10e^{k(7.5)}$$

$$k = ?$$

$$(c) x = 100, x_0 = 10$$

$$t = ?$$

Q. $\frac{40}{58}$. The half life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the level acceptable for human habitation. How long will it be until the region is again habitable? (Ignore the probable presence of other radioactive isotopes.)

$$A = A_0 e^{-kt}$$

$$\text{half life} = 5.27 \text{ yrs.}$$

$$A = \frac{A_0}{2}$$

$$\frac{A_0}{2} = A_0 e^{-kt} = A_0 e^{-k \times 5.27}$$

$$\frac{1}{2} = e^{-5.27k} \Rightarrow k = \frac{1}{5.27} \log 2$$

$$100 A = A_0$$

$$A_0 e^{-kt} = 0.01 A_0$$

$$t = ?$$

$$t = \frac{1}{k} \log \left[\frac{1}{.01} \right]$$

$$t = \frac{1}{k} \log 100$$

Que. 41. Suppose that a mineral body formed in an ancient cataclysm – perhaps the formation of the earth itself – originally contained the uranium isotopes ^{238}U (which has a half – life of 4.51×10^9 yrs) but no lead, the end product of the radioactive decay of ^{238}U . If today the ratio of ^{238}U atoms to lead atoms in the mineral body is 0.9, when did the catalysms occur?

$$A = A_0 e^{-kt}$$

$A_0 \rightarrow$ initial amount of U_{238}

$\text{U}_{238} \rightarrow$ lead

$$t = 4.51 \times 10^9 \text{ yrs}$$

$$\frac{A_0}{2} = A_0 e^{-4.51 \times 10^9 \times k}$$

$$k = \frac{1}{4.51 \times 10^9} \log 2$$

$A(t) =$ amount of U_{238} at time t .

$A_0 \rightarrow$ initial

$A_0 - A(t) =$ amount of lead atoms

$$\text{Ratio} = \frac{A(t)}{A_0 - A(t)} = 0.9$$

$$A_0 = .9A_0 - 0.9A(t)$$

$$\frac{19}{9} = \frac{A_0}{A(t)} = \frac{A_0}{A e^{-kt}}$$

$$\Rightarrow kt = \log \frac{19}{9} \Rightarrow t = \frac{1}{k} \log \left(\frac{19}{9} \right)$$

Que. 42: A certain moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half life is about 1.28×10^9 years) and that one of every nine potassium atom disintegration fields an argon atom. What is the age of the rock, measured from the time it contained only potassium.

Solⁿ: Let $P(t)$ be the amount of potassium

Let $A(t)$ be the amount of argon atom

$$P(t) = A(t)$$

$$P(t) = P_0 e^{-kt}$$

$$t = 1.28 \times 10^9$$

$$\frac{P_0}{2} = P_0 e^{-1.28 \times 10^9 k}$$

$$k = \frac{1}{1.28 \times 10^9} \log(2)$$

$$A(t) = \frac{1}{9} [P_0 - P(t)]$$

$$9P(t) = P_0 - P(t)$$

$$10 P(t) = P_0$$

$$10 P_0 e^{-kt} = P_0$$

$$e^{kt} = 10$$

$$t = \frac{1}{k} \log 10$$

Que. 48. According to one cosmological theory, there were equal amounts of the two uranium isotopes ^{235}U and ^{238}U at the creation of the universe in the 'big bang'. At present there are 137.7 atoms of ^{238}U for each atom of ^{235}U using the half – lives 4.51×10^9 years for ^{238}U and 7.10×10^8 yrs. for ^{235}U , calculate the age of the universe

$N_8 =$ amount of U_{238}

$N_5 =$ amount of U_{235}

$$N_8 = N_0 e^{-k_1 t}$$

$$N_5 = N_0 e^{-k_2 t}$$

$$\frac{N_0}{2} = N_0 e^{-4.51 \times 10^9 k_1}$$

$$\frac{N_0}{2} = N_0 e^{-7.10 \times 10^8 k_2}$$

$$k_1 = \frac{1}{4.51 \times 10^9} \log 2$$

$$k_2 = \frac{1}{7.1 \times 10^8} \log 2$$

$$\text{given } N_8 = 137.7 N_5$$

$$N_0 e^{-k_1 t} = 137.7 N_0 e^{-k_2 t}$$

$N_5 \rightarrow$ amount of U_{235}
at time t

$$N_8 = 137.7 N_5$$

$$N_5 = 1$$

$$N_8 = 137.7$$

$$N_5 = 10$$

$$N_8 = 137.7 \times 10$$

$$t = \frac{1}{k_2 - k_1} \log(137.7)$$

$$\frac{Q86}{88} \cdot (1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

$$\frac{\partial M}{\partial y} = e^{xy} + ye^{xy}x$$

$$\frac{\partial N}{\partial x} = e^{xy} + xe^{xy}y$$

eqⁿ is exact.

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

$$\Rightarrow \frac{\partial F}{\partial x} = (1 + ye^{xy})$$

$$F = x + \frac{ye^{xy}}{y} + g(y)$$

$$F = x + e^{xy} + g(y)$$

$$\frac{\partial F}{\partial y} = xe^{xy} + g'(y)$$

$$N = xe^{xy} + g'(y)$$

$$2y + xe^{xy} = xe^{xy} + g'(y)$$

$$g'(y) = 2y$$

$$\frac{dg}{dy} = 2y \Rightarrow g = y^2 + C_1$$

$$\Rightarrow F = x + e^{xy} + y^2 + C_1$$

$$\text{Que. 41 } \left(\frac{2x}{y} - \frac{3y^2}{x^4} \right) dx + \left(\frac{2y}{x^3} - \frac{x^2}{y^2} + \frac{1}{\sqrt{y}} \right) dy = 0$$

$$M = \left(\frac{2x}{y} - \frac{3y^2}{x^4} \right), N = \frac{2y}{x^3} - \frac{x^2}{y^2} + \frac{1}{\sqrt{y}}$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= \frac{-2x}{y^2} - \frac{6y}{x^4} \\ \frac{\partial N}{\partial x} &= \frac{-6y}{x^4} - \frac{2x}{y^2} \end{aligned} \right\} \Rightarrow \text{exact D.E.}$$

$$F_x = M \Rightarrow \frac{\partial F}{\partial x} = M$$

$$\partial F = M dx$$

$$F = \int \left(\frac{2x}{y} - \frac{3y^2}{x^4} \right) dx + g(y)$$

$$F = \frac{x^2}{y} + \frac{3y^2}{3x^3} + g(y) = \frac{x^2}{y} + \frac{y^2}{x^3} + g(y)$$

$$\frac{\partial F}{\partial y} = \frac{-x^2}{y^2} + \frac{2y}{x^3} + g'(y)$$

$$\frac{2y}{x^3} - \frac{x^2}{y^2} + \frac{1}{\sqrt{y}} = \frac{-x^2}{y^2} + \frac{2y}{x^3} + g'(y)$$

$$\frac{1}{\sqrt{y}} = g'(y) = \frac{dg}{dy}$$

$$\frac{1}{\sqrt{y}} dy = dg$$

$$y^{-1/2} dy = dg$$

$$g = 2\sqrt{y} + C$$

$$F = \frac{x^2}{y} + \frac{y^2}{x^3} + 2\sqrt{y} + C$$

$$\text{Ex } \frac{9}{84} (6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2, \quad \frac{\partial N}{\partial x} = 6x - 3y^2 \quad \text{exact}$$

$$\frac{\partial F}{\partial x} = M \Rightarrow \partial F = M dx$$

$$\partial F = (6xy - y^3) \partial x$$

$$F = 3x^2y - y^3x + g(y)$$

$$\frac{\partial F}{\partial y} = 3x^2 - 3y^2x + g'(y)$$

$$4y + 3x^2 - 3xy^2 = 3x^2 - 3xy^2 + g'(y)$$

$$g'(y) = 4y$$

$$\frac{dg}{dy} = 4y \Rightarrow dg = 4ydy$$

$$g = 2y^2 + C$$

$$\Rightarrow F = 3x^2y - xy^3 + 2y^2 + C$$

Ex 7/55 A 4-lb roast, initially at $50^\circ F$, is placed in a $375^\circ F$ oven at 5:00 pm. After 75 minutes it is found that the temperature $T(t)$ of the roast is $125^\circ F$. When will the roast be $150^\circ F$ (medium rare) ?

rare अधपका मांस

Roast आग या oven में भूना, भूनना

$$\begin{cases} T = 50^\circ F \\ T_0 = 375^\circ \end{cases} \quad \begin{cases} T = T_0 + Ce^{-kt} \text{ cooling} \\ T_0 = T + Ce^{-kt} \text{ Heating} \end{cases}$$

$$T_0 = T + Ce^{-kt} \text{ (heating)}$$

$$375 - 50 = Ce^{-kt}$$

$$325 = C \times 1$$

$$C = 325 \quad (1)$$

$$t = 75, \quad T = 125, \quad T_0 = 375$$

$$T_0 = T + Ce^{-kt}$$

$$375 - 125 = 325e^{-75k}$$

$$\frac{250}{325} = e^{-75k} \Rightarrow -75k = \log\left(\frac{250}{325}\right)$$

$$k = \frac{1}{75} \log\left(\frac{325}{250}\right) \quad k = \frac{-1}{75} \log\left(\frac{250}{325}\right)$$

$$k = 0.0035$$

$$\text{Now, } T = 150, \quad c = 325$$

$$T_0 = 325, \quad K = 0.0035, \quad t = ?$$

$$T_0 = T + Ce^{-kt}$$

$$325 = 150 + 325e^{-0.0035t}$$

$$\frac{225}{325} = e^{-0.0035t}$$

$$\log\left(\frac{225}{325}\right) = -0.0035t$$

$$\text{i.e. } t = \frac{-1}{0.0035} \log\left(\frac{225}{325}\right)$$

$$t = 10 \text{ (min)}$$

$$\text{i.e. 1 hr \& 45 min}$$

$\therefore$ the roast was placed in the oven at 5:00 pm. It should be removed at about 6:45 pm.

Homogeneous Linear D. E with Constant Coefficient

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = F(x) \text{ Non Homo.}$$

If $F(x) = 0$ Homogeneous

auxillary eqⁿ or characteristic eqⁿ (complementary fun^c)

$$a_0 D^n + a_1 D^{n-1} + \dots + a_n = 0$$

$$\text{Put } D = \frac{d}{dx}$$

This eqⁿ has n roots

(i) If all the roots are distinct and real then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

(ii) If some ($r < n$) roots are equal (to m means $m = m_1 = m_2 = \dots = m_r$) and all other are distincts

$$y = (c_1 + c_2 x + \dots + c_r x^{r-1}) e^{mx} + c_{r+1} x^{m_{r+1}} + \dots + c_n x^{m_n}$$

particularly If $m_1 = m_2 = m_3$, all other are distinct

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

(iii) If $m_1 = m_2, m_3 = m_4$ and other are distinct

$$y = (c_1 + c_2 x) e^{m_1 x} + (c_3 + c_4 x) e^{m_3 x} + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$$