

Chapter 4

Derivatives and Its Application

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Derivative measures the steepness of the graph of a function at some particular point on the graph. Thus, the derivative is a slope that is a ratio of change in the value of the function to change in the independent variable. If the independent variable is time, we often think of this ratio as a rate of change. If we zoom in on the graph of the function at some point that the function looks almost like a straight line, the derivative at that point is the slope of the line. This is the same as saying that the derivative is the slope of the tangent line to the graph of the function at the given point. The slope of a secant line which is connecting two points on a graph approaches the derivative when the interval between the points shrinks down to zero. The derivative itself is also a function that varies from place to place. For example, the velocity of a car may change from moment to moment as the driver speeds up or slows down.

4.1 Definition and Properties of derivatives; basic rules; chain rules

Definition of Derivative

Definition 4.1.1 The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

■ **Example 4.1** Find an equation of the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.

Solution: Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$\begin{aligned} m &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \\ m &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ m &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \end{aligned}$$

$$\begin{aligned} m &= \lim_{x \rightarrow 1} (x + 1) \\ m &= 1 + 1 \\ m &= 2 \end{aligned}$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is $y - 1 = 2(x - 1)$ or $y = 2x - 1$. ■

Definition 4.1.2 The derivative of a function f at any variable x , denoted by $f'(x)$, is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Provided the limit exists.

Definition 4.1.3 If we write $x = a + h$, then $h = x - a$ and approaches 0 if and only if x approaches a . Therefore, an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Provided the limit exists.

The tangent line to the curve $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a .

■ **Example 4.2** Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a .

Solution: From the above definition we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{((a+h)^2 - 8(a+h) + 9) - (a^2 - 8a + 9)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a^2 + 2ah + h^2 - 8a - 8h + 9) - (a^2 - 8a + 9)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah + h^2 - 8h}{h} \\ &= \lim_{h \rightarrow 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$

■ **Example 4.3** If $f(x) = \sqrt{x-1}$, find the derivative of f . State the domain of f' .

Solution: From the definition of derivatives

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h-1) - (x-1)}{h(\sqrt{x+h-1} + \sqrt{x-1})} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h-1} + \sqrt{x-1})} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h-1} + \sqrt{x-1}} \\
 &= \frac{1}{\sqrt{x-1} + \sqrt{x-1}} \\
 &= \frac{1}{2\sqrt{x-1}}
 \end{aligned}$$

We see that $f'(x)$ exists if $x > 1$, so the domain of f' is $(1, \infty)$. ■

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y , then some common alternative notations for the derivative are as follows

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\frac{d}{dx}$ (Leibniz notation) are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating a derivative. If we want to indicate the value of a derivative $\frac{dy}{dx}$ in Leibniz notation at a specific number, we use the notation $\frac{dy}{dx}|_{x=a}$, which is a synonym for $f'(a)$.

Remark 4.1.1 A function is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$ if it is differentiable at every number in the interval.

■ **Example 4.4** Where is the function $f(x) = |x|$ differentiable?

Solution: If $x > 0$, then $|x| = x$ and we can choose small enough that $x+h > 0$ and hence $|x+h| = x+h$. Therefore, for $x > 0$ we have

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h} \\
 &= \lim_{h \rightarrow 0} 1 \\
 &= 1
 \end{aligned}$$

and so f is differentiable for any $x > 0$. Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small

enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{|x + h| - |x|}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(x + h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} \\ &= \lim_{h \rightarrow 0} -1 \\ &= -1 \end{aligned}$$

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|0 + h| - |0|}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h} \text{ if it exist} \end{aligned}$$

Let's compute the left and right limits separately:

$$\lim_{h \rightarrow 0^+} \frac{|h|}{h} = \frac{h}{h} = 1 \text{ and } \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \frac{-h}{h} = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus, f is differentiable at all x except 0. ■

If the graph of a function f has a corner, cusp or discontinuity point at a point a , then the function is not differentiable at that point.

Theorem 4.1.2 If f is differentiable at a , then f is continuous at a .

But the converse of the above statement may not be true.

Properties of Derivative

If $f(x)$ and $g(x)$ are differentiable functions and c is some constant number, then

1. Constant Function Rule

$$\frac{d}{dx} c = 0$$

2. Power Function Rule

$$\frac{d}{dx} x^n = nx^{n-1}$$

where n is any real number.

3. Constant Multiple Rule

$$\frac{d}{dx} (cf(x)) = c \frac{d}{dx} f(x)$$

4. Sum or Difference Rule

$$\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

5. Product Rule

$$\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \frac{d}{dx} g(x) + \left(\frac{d}{dx} f(x) \right) g(x)$$

6. Quotient Rule

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{\left(\frac{d}{dx} f(x) \right) g(x) - f(x) \frac{d}{dx} g(x)}{(g(x))^2}$$

provided $g(x) \neq 0$

■ **Example 4.5** If $f(x) = x^6$, then $f'(x) = 6x^5$ by using power function rule. ■

■ **Example 4.6** Find $f'(x)$ at $x = -1$, where $f(x) = 2x^7 - \sqrt{2}x^5 + \frac{3}{2}x^3 + 20x + 1$.

Solution: First we find the derivatives of the function at an arbitrary x .

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^7 - \sqrt{2}x^5 + \frac{3}{2}x^3 + 20x + 1) \\ &= 2(7x^6) - \sqrt{2}(5x^4) + \frac{3}{2}(3x^2) + 20 \\ &= 14x^6 - 5\sqrt{2}x^4 + \frac{9}{2}x^2 + 20 \end{aligned}$$

Then we evaluate the derivatives at $x = -1$

$$\begin{aligned} f'(-1) &= 14(-1)^6 - 5\sqrt{2}(-1)^4 + \frac{9}{2}(-1)^2 + 20 \\ &= 14 - 5\sqrt{2} + \frac{9}{2} + 20 \\ &= \frac{77}{2} - 5\sqrt{2} \end{aligned}$$

■ **Example 4.7** Find $f'(x)$, where $f(x) = \frac{9x^7}{x^2+1}$.

Solution: If $g(x) = 9x^7$ and $h(x) = x^2 + 1$, then $f = \frac{g}{h}$, $g'(x) = 63x^6$ and $h'(x) = 2x$. Therefore

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2} \\ &= \frac{(63x^6)(x^2+1) - (9x^7)(2x)}{(x^2+1)^2} \\ &= \frac{45x^8 + 63x^6}{(x^2+1)^2} \end{aligned}$$

Rules of Derivatives

1. Exponential Functions

The Derivative of the exponential function $f(x) = a^x$ is $f'(x) = a^x \ln a$.

Definition 4.1.4 Definition of the Number e

e is the number such that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$, where $e \approx 2.71828$.

2. Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

3. Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \text{ and } \frac{d}{dx}(\ln x) = \frac{1}{x}$$

4. Derivatives of Trigonometric Functions

- (a) $\frac{d}{dx} \sin x = \cos x$
- (b) $\frac{d}{dx} \cos x = -\sin x$
- (c) $\frac{d}{dx} \tan x = \sec^2 x$
- (d) $\frac{d}{dx} \csc x = -\csc x \cot x$
- (e) $\frac{d}{dx} \sec x = \sec x \tan x$
- (f) $\frac{d}{dx} \cot x = -\csc^2 x$

Consider $f(x) = \sin x$. From the definition of the derivative, we have

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \cos x \left(\frac{\sin h}{h} \right) \right) \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \rightarrow 0} \cos x \left(\frac{\sin h}{h} \right) \\
 &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= (\sin x)(0) + (\cos x)(1) \\
 &= \cos x
 \end{aligned}$$

In general, the derivatives of the remaining functions can also be found easily using the definition and rules of derivatives.

■ **Example 4.8** Differentiate $f(x) = \frac{\sec x}{1 - \tan x}$.
 Solution: Here $f(x) = \frac{\sec x}{1 - \tan x}$, then we have

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} \left(\frac{\sec x}{1 - \tan x} \right) \\
 &= \frac{\left(\frac{d}{dx} \sec x \right)(1 - \tan x) - \sec x \frac{d}{dx}(1 - \tan x)}{(1 - \tan x)^2} \\
 &= \frac{\sec x \tan x(1 - \tan x) - \sec x(-\sec^2 x)}{(1 - \tan x)^2} \\
 &= \frac{\sec x \tan x(1 - \tan x) + \sec^3 x}{(1 - \tan x)^2} \\
 &= \frac{\sec x(\tan x - \tan^2 x + 1 + \tan^2 x)}{1 - 2 \tan x + \tan^2 x} \\
 &= \frac{\sec x(\tan x + 1)}{\sec^2 x - 2 \tan x}
 \end{aligned}$$

■

Chain Rule

Definition 4.1.5 If h and g are both differentiable and $f = hog$ is the composition function defined by $(hog)(x) = h(g(x))$, then f is differentiable and f' is given by the product $f'(x) = h'(g(x))g'(x)$. In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

The power rule combined with the chain Rule

If n is any real number and $u = h(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$

Alternatively, $\frac{d}{dx}(h(x))^n = n(h(x))^{n-1}h'(x)$.

■ **Example 4.9** Find $f'(x)$ if $f(x) = \sqrt{x^2 + 3}$.

Solution: f can be expressed as $f(x) = (hog)(x)$, where $h(u) = \sqrt{u}$ and $g(x) = x^2 + 3$. Since

$$f'(u) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} \text{ and } g'(x) = 2x$$

We have

$$\begin{aligned} f'(x) &= h'(g(x))g'(x) \\ &= \frac{1}{2\sqrt{x^2 + 3}} 2x \\ &= \frac{x}{\sqrt{x^2 + 3}} \end{aligned}$$

■ **Example 4.10** Find the derivative of $f(x) = (x - 2x^3)^{-11}$.

Solution:

$$\begin{aligned} f'(x) &= -11(x - 2x^3)^{-12} \frac{d}{dx}(x - 2x^3) \\ &= -11(x - 2x^3)^{-12}(1 - 6x^2) \\ &= -11(1 - 6x^2)(x - 2x^3)^{-12} \\ &= (-11 + 66x^2)(x - 2x^3)^{-12} \\ &= (66x^2 - 11)(x - 2x^3)^{-12} \end{aligned}$$

■ **Example 4.11** Find the derivative of $f(x) = \sin^8 x$.

Solution: From the definition of chain rule. We let $u = \sin x$. Then $f(x) = u^8$, so it follows that

$$\begin{aligned} f'(x) &= \frac{d}{dx}f(x) \\ &= 8u^7 \cos x \\ &= 8 \sin^7 x \cos x \end{aligned}$$

■ **Example 4.12** Find the derivative of $y = x^{\sin x}$.

Solution: Since

$$y = x^{\sin x} = e^{\ln x^{\sin x}} = e^{\sin x \ln x}$$

Thus,

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{d}{dx}(e^{\sin x \ln x}) \\ &= e^{\sin x \ln x} \frac{d}{dx}(\sin x \ln x) \\ &= x^{\sin x} \frac{d}{dx}(\sin x \ln x) \\ &= x^{\sin x} \left(\frac{d}{dx}(\sin x) \ln x + \sin x \frac{d}{dx} \ln x \right) \\ &= x^{\sin x} \left(\cos x \ln x + \sin x \frac{1}{x} \right) \\ &= x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right) \end{aligned}$$

■ **Example 4.13** Find the derivative of $y = x^{x^x}$.

Solution: Since

$$y = x^{x^x} = e^{\ln x^{x^x}} = e^{x^x \ln x} = e^{e^{x \ln x} \ln x} = e^{e^{x \ln x} \ln x}$$

Thus,

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{d}{dx}(e^{e^{x \ln x} \ln x}) \\ &= e^{e^{x \ln x} \ln x} \frac{d}{dx}(e^{x \ln x} \ln x) \\ &= x^{x^x} \frac{d}{dx}(e^{x \ln x}) \ln x + e^{x \ln x} \frac{d}{dx}(\ln x) \\ &= x^{x^x} (\ln x \frac{d}{dx} e^{x \ln x} + e^{x \ln x} \frac{1}{x}) \\ &= x^{x^x} (e^{x \ln x} \ln x \frac{d}{dx}(x \ln x) \ln x + e^{x \ln x} \frac{1}{x}) \\ &= x^{x^x} (x^x (\ln x (\frac{dx}{dx} \ln x + x \frac{d}{dx} \ln x)) + x^x \frac{1}{x}) \\ &= x^{x^x} (x^x ((\ln x (\ln x + 1) + \frac{1}{x}))) \end{aligned}$$

■

4.2 Derivatives of Inverse Functions; Inverse Trigonometric Functions; Hyperbolic Functions and Inverse Hyperbolic Functions

Derivatives of Inverse Functions

Theorem 4.2.1 Suppose that f has an inverse and is continuous on an open interval I containing $\mathbf{a}$. Assume also that $f'(\mathbf{a})$ exist, $f'(\mathbf{a}) \neq 0$, and $f(\mathbf{a}) = c$. Then $(f^{-1})'(c)$ exists, and

$$(f^{-1})'(c) = \frac{1}{f'(\mathbf{a})}$$

■ **Example 4.14** Let $f(x) = 8x^3 - 1$

1. Find $(f^{-1})'(7)$ by using the fact that $f^{-1}(x) = \frac{1}{2}(x+1)^{\frac{1}{3}}$.
2. Find $(f^{-1})'(7)$ by using the fact that $f(1) = 7$.

Solution:

1. The formula for f^{-1} appeared in the above example. Differentiating f^{-1} , we find that

$$(f^{-1})'(x) = \frac{1}{6}(x+1)^{-\frac{2}{3}}$$

So that

$$(f^{-1})'(x) = \frac{1}{6}(7+1)^{-\frac{2}{3}} = \frac{1}{24}$$

2. Since $f(1) = 7$ and $f'(x) = 24x^2$,

$$(f^{-1})'(x) = \frac{1}{f'(1)} = \frac{1}{24}$$

This is the same answer that we find in the above part.

■

■ **Example 4.15** Find $(f^{-1})'(-2)$, where $f(x) = x^7 + 8x^3 + 4x - 2$.

Solution: First, we have to find the value of a for which $f(a) = -2$. But $f(a) = -2$. So, $a = 0$ and since $f'(x) = 7x^6 + 24x^2 + 4$, it follows that $f'(0) = 4$. Therefore,

$$(f^{-1})'(-2) = \frac{1}{f'(0)} = \frac{1}{4}$$

■ **Example 4.16** Let $y = x^5 + 2x$. Find $\frac{dx}{dy}$ and $\frac{dx}{dy}|_{y=-3}$.

Solution: The function y has an inverse because its derivative, $5x^4 + 2$, is a positive function. So

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{5x^4 + 2}$$

Since $y = x^5 + 2x$, it follows that $y = -3$ for $x = -1$. We conclude that

$$\frac{dx}{dy}|_{y=-3} = \frac{1}{5(-1)^4 + 2} = \frac{1}{7}$$

Derivatives of Inverse Trigonometric Functions

1. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$
4. $\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$
5. $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$
6. $\frac{d}{dx} \cot^{-1} x = -\frac{1}{x^2+1}$

■ **Example 4.17** Show that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Solution: Let $y = \sin^{-1} x$, then $\sin y = x$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Differentiating $\sin y = x$ respect to a variable y , then we have

$$\begin{aligned} \frac{dx}{dy} &= \cos y \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

Now, $\cos y \geq 0$, since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. So,

$$\begin{aligned} \cos y &= \sqrt{1 - \sin^2 y} \\ &= \sqrt{1 - x^2} \end{aligned}$$

Therefore, $\frac{dx}{dy} = \frac{1}{\sqrt{1-x^2}}$.

■ **Example 4.18** Differentiate $y = \frac{1}{\sin^{-1} x}$.

Solution: Here

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (\sin^{-1} x)^{-1} \\ &= -(\sin^{-1} x)^{-2} \frac{d}{dx} (\sin^{-1} x) \\ &= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1-x^2}} \end{aligned}$$

Hyperbolic Functions

Definition of the Hyperbolic Functions

1. $\sinh x = \frac{e^x - e^{-x}}{2}$
2. $\cosh x = \frac{e^x + e^{-x}}{2}$
3. $\tanh x = \frac{\sinh x}{\cosh x}$
4. $\operatorname{csch} x = \frac{1}{\sinh x}$
5. $\operatorname{sech} x = \frac{1}{\cosh x}$
6. $\operatorname{coth} x = \frac{\cosh x}{\sinh x}$

Remark 4.2.2 $\sinh$ has domain $\mathbb{R}$ and range $\mathbb{R}$, while $\cosh$ has domain $\mathbb{R}$ and range $[1, \infty)$.

Hyperbolic Identities

1. $\sinh(-x) = -\sinh x$
2. $\cosh(-x) = \cosh x$
3. $\cosh^2 x - \sinh^2 x = 1$
4. $1 - \tanh^2 x = \operatorname{sech}^2 x$
5. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
6. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

■ **Example 4.19** Prove that $\cosh^2 x - \sinh^2 x = 1$.

Solution: From the definition of hyperbolic function

$$\begin{aligned}
 \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
 &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
 &= \frac{4}{4} \\
 &= 1
 \end{aligned}$$

■

Inverse Hyperbolic Functions

Inverse hyperbolic functions can be expressed as

$$\begin{aligned}
 y = \sinh^{-1} x &\Leftrightarrow \sinh y = x \\
 y = \cosh^{-1} x &\Leftrightarrow \cosh y = x \text{ and } y \geq 0 \\
 y = \tanh^{-1} x &\Leftrightarrow \tanh y = x
 \end{aligned}$$

Inverse hyperbolic functions can also be expressed in terms of logarithms. In particular, we have

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), x \in \mathbb{R}$
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$
3. $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$

■ **Example 4.20** Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), x \in \mathbb{R}$.

Solution: Let $y = \sinh^{-1} x$. Then,

$$\begin{aligned}
 x &= \sinh y \\
 &= \frac{e^y - e^{-y}}{2} \\
 \Rightarrow e^y - 2x - e^{-y} &= 0
 \end{aligned}$$

multiplying by e^y , we have

$$\begin{aligned} e^{2y} - 2xe^y - 1 &= 0 \\ (e^y)^2 - 2x(e^y) - 1 &= 0 \\ e^y &= \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\ &= x \pm \sqrt{x^2 + 1} \end{aligned}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$. Thus, the minus sign will be ignored and we have $e^y = x + \sqrt{x^2 + 1}$. Therefore,

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

since $y = \sinh^{-1} x$ and the domain is the set of all real number. ■

■ **Example 4.21** Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$.

Solution: Let $y = \cosh^{-1} x$. Then,

$$\begin{aligned} x &= \cosh y \\ &= \frac{e^y + e^{-y}}{2} \\ \Rightarrow e^y - 2x + e^{-y} &= 0 \end{aligned}$$

multiplying by e^y , we have

$$\begin{aligned} e^{2y} - 2xe^y + 1 &= 0 \\ (e^y)^2 - 2x(e^y) + 1 &= 0 \\ e^y &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &= x \pm \sqrt{x^2 - 1} \\ \Rightarrow y &= \ln(x \pm \sqrt{x^2 - 1}) \end{aligned}$$

Note that $y > 0$, but $\ln(x - \sqrt{x^2 - 1}) < 0$. Thus, the minus sign will be ignored and we have $y = \ln(x + \sqrt{x^2 - 1})$. Therefore,

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

since $y = \cosh^{-1} x$ and the domain is the set of real number which is $x \geq 1$. ■

Derivatives of Hyperbolic Functions

The differentiation formula for the hyperbolic functions.

1. $\frac{d}{dx} \sinh x = \cosh x$
2. $\frac{d}{dx} \cosh x = \sinh x$
3. $\frac{d}{dx} \tanh x = \text{sech}^2 x$
4. $\frac{d}{dx} \text{csch} x = -\text{csch} x \coth x$
5. $\frac{d}{dx} \text{sech} x = -\text{sech} x \tanh x$
6. $\frac{d}{dx} \coth x = -\text{csch}^2 x$

■ **Example 4.22** Show that $\frac{d}{dx} \sinh x = \cosh x$.

Solution:

$$\begin{aligned} \frac{d}{dx} \sinh x &= \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) \\ &= \frac{e^x + e^{-x}}{2} \\ &= \cosh x \end{aligned}$$

Therefore, $\frac{d}{dx} \sinh x = \cosh x$. ■

■ **Example 4.23** Evaluate $\frac{d}{dx} \cosh \sqrt{x}$.

Solution: from derivative of hyperbolic function

$$\begin{aligned} \frac{d}{dx} \cosh \sqrt{x} &= \sinh \sqrt{x} \frac{d\sqrt{x}}{dx} \\ &= \frac{\sinh \sqrt{x}}{2\sqrt{x}} \end{aligned}$$

■

Derivatives of Inverse Hyperbolic Functions

1. $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$
2. $\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$
3. $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$
4. $\frac{d}{dx} \csc h^{-1} x = -\frac{1}{|x|\sqrt{x^2+1}}$
5. $\frac{d}{dx} \sec h^{-1} x = -\frac{1}{x\sqrt{1-x^2}}$
6. $\frac{d}{dx} \coth^{-1} x = -\frac{1}{1-x^2}$

■ **Example 4.24** Prove that $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$.

Solution: Let $y = \sinh^{-1} x \Leftrightarrow x = \sinh y \Rightarrow \frac{dx}{dy} = \cosh y$, but

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} \\ &= \frac{1}{\cosh y} \\ &= \frac{1}{\sqrt{1+\sinh^2 y}} \\ &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

Hence, $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$.

■

■ **Example 4.25** Differentiate $y = \sinh^{-1} x^3$.

Solution: Let $h(x) = \sinh^{-1} x$ and $g(x) = x^3$

But

$$\begin{aligned} y &= h(g(x)) \\ \Rightarrow \frac{dy}{dx} &= h'(g(x))g'(x) \\ &= \frac{1}{\sqrt{1+(g(x))^2}} 3x^2 \\ &= \frac{3x^2}{\sqrt{1+x^6}} \end{aligned}$$

Hence, $\frac{d}{dx} \sinh^{-1} x^3 = \frac{3x^2}{\sqrt{1+x^6}}$.

■

4.3 Implicit Differentiation; Higher Order Derivatives

4.3.1 Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable. For example, $y = \sqrt{x^3+1}$ or $y = \sin x$ or, in general, $y = f(x)$. Some functions, however,

are defined implicitly by a relation between x and y such as $x^2 + y^2 = 25$ or $x^3 + y^3 = 6xy$. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y' .

■ **Example 4.26** If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution: Differentiate both sides of the equation $x^2 + y^2 = 25$

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}25 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\ \frac{dy}{dx} &= \frac{-x}{y}\end{aligned}$$

■ **Example 4.27** Find an equation of the line that is tangent to the circle $x^2 + y^2 = 9$ at the point $(-2, \sqrt{5})$.
Solution: The point $(-2, \sqrt{5})$ lies on the top half of the circle, which is the graph of a differential function y . From the above example,

$$\left. \frac{dy}{dx} \right|_{(-2, \sqrt{5})} = \frac{2}{\sqrt{5}}$$

Therefore the slope of the tangent line we seek is $\frac{2}{\sqrt{5}}$, so an equation of the line is

$$y - \sqrt{5} = \frac{2}{\sqrt{5}}(x + 2)$$

4.3.2 Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f . Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Another notation is $f''(x) = D^2f(x)$. The third derivative f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. The alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3f(x)$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the n^{th} derivative of f is denoted by $f^{(n)}$ and is obtained from by differentiating n -times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = D^n f(x)$$

■ **Example 4.28** If $f(x) = x \cos x$, find and interpret $f^{(4)}(x)$.

Solution: Here $f(x) = x \cos x$, then we have

$$\begin{aligned}f(x) &= x \cos x \\ \Rightarrow f'(x) &= \cos x + x \sin x \\ f''(x) &= -\sin x - \sin x - x \cos x \\ &= -2 \sin x - x \cos x \\ f^{(3)}(x) &= -2 \cos x + \cos x - x \sin x \\ &= -\cos x - x \sin x \\ f^{(4)}(x) &= -x \cos x\end{aligned}$$

■ **Example 4.29** Find $f^{(55)}(x)$, where $f(x) = \cos x$.

Solution: Here $f(x) = \cos x$, then we have

$$\begin{aligned} f(x) &= \cos x \\ \Rightarrow f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

The derivatives occur in a cycle of length 4 because of this $f^{(55)}(x) = \sin x$. ■

■ **Example 4.30** Find the n^{th} derivative of f for $n \geq 1$, where $f(x) = e^x - x^2$.

Solution: Using chain rule we have

$$\begin{aligned} f(x) &= e^x - x^2 \\ \Rightarrow f'(x) &= e^x - 2x \\ f''(x) &= e^x - 2 \\ f^{(3)}(x) &= e^x \\ f^{(4)}(x) &= e^x \text{ and } f^{(n)}(x) = e^x \text{ for } n \geq 3 \end{aligned}$$

■ **Example 4.31** Find the n^{th} derivative of f , where $f(x) = e^{cx}$.

Solution: By the chain rule,

$$\begin{aligned} f'(x) &= ce^{cx} \\ f''(x) &= c^2 e^{cx} \\ f^{(3)}(x) &= c^3 e^{cx} \end{aligned}$$

In general, for any positive integer n ,

$$f^{(n)}(x) = c^n e^{cx}$$

4.4 Applications of Derivative

Tangent Line Approximation

Definition 4.4.1 If $y = f(x)$ is differentiable at $x = a$, then

$$L(x) = f(a) + f'(a)(x - a)$$

is the tangent line linear approximation of the curve $t = f(x)$. The approximation $f(x) \approx L(x)$ is the standard linear approximation of f at a . The center of approximation is a .

■ **Example 4.32** Find the linear approximation to the function $f(x) = \sqrt{3+x}$ at $x = 1$.

Solution: From the above definition use $L(x) = f(a) + f'(a)(x - a)$ for $f(x) = \sqrt{3+x}$. The derivative of $f(x)$ is $f'(x) = \frac{1}{2\sqrt{3+x}}$. Substitute $a = 1$, then we have $f'(1) = \frac{1}{4}$ and $f(1) = 2$. So,

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 2 + \frac{1}{4}(x - 1) \\ &= \frac{1}{4}x + \frac{7}{4} \end{aligned}$$

Therefore, $L(x) = \frac{1}{4}x + \frac{7}{4}$ is the linear approximation of $f(x) = \sqrt{3+x}$ near $x = 1$. ■

■ **Example 4.33** Approximate $\sqrt{3.98}$ and $\sqrt{4.2}$ for a function $f(x) = \sqrt{3+x}$ at $x = 1$.

Solution: For $\sqrt{3.98}$, $f(0.98) = \sqrt{3+0.98} = \sqrt{3.98}$ and 0.98 is near $a = 1$. So,

$$f(1 + (-0.02)) \approx f(1) + f'(1)(0.98 - 1)$$

Therefore, the actual value of $\sqrt{3.98}$ is about 1.99499.

For $\sqrt{4.2}$, $f(1.2) = \sqrt{3+1.2} = \sqrt{4.2}$ and 1.2 is near $a = 1$. So,

$$f(1.2) \approx f(1) + f'(1)(1.2 - 1)$$

Therefore, the actual value of $\sqrt{4.2}$ is about 2.04939. ■

Maximum and Minimum Values

Definition 4.4.2 A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D , where D is the domain of f . The number $f(c)$ is called the maximum value of f on D . Similarly, a function f has an absolute minimum (or global minimum) at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the minimum value of f on D . The maximum and minimum values of f are called the extreme values of f .

$f(c)$ is the maximum value of f if and only if $(c, f(c))$ is as high as any point on the graph of f and $f(c)$ is the minimum value of f if and only if $(c, f(c))$ is as low as any point on the graph of f .

Definition 4.4.3 A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ when x is near c ; that is, $f(c) \geq f(x)$ that for all x in some open interval containing c . Similarly, f has a local minimum at c if $f(c) \leq f(x)$ when x is near c .

■ **Example 4.34** The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2np = 1$ for any integer N and $-1 \leq \cos x \leq 1$ for all x . Likewise, $\cos(2n+1)p = -1$ is its minimum value, where n is any integer. ■

■ **Example 4.35** If $f(x) = x^2$, then $f(x) = f(0)$ because $x^2 = 0$ for all x . Therefore, $f(0) = 0$ is the absolute (and local) minimum value of f . This corresponds to the fact that the origin is the lowest point on the parabola $y = x^2$. However, there is no highest point on the parabola and so this function has no maximum value. ■

A function f may or may not have extreme values on a set I , depending on f and on I . The following simple examples illustrate four of the possibilities

1. If $f(x) = x$, then on $[0, 1]$ the function f has the maximum value on 1 and minimum value on 0.

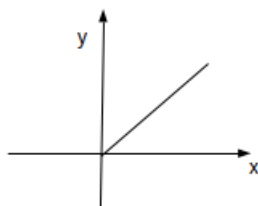
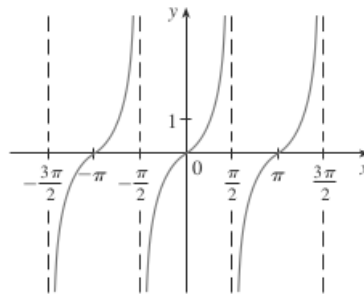
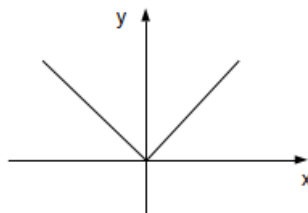
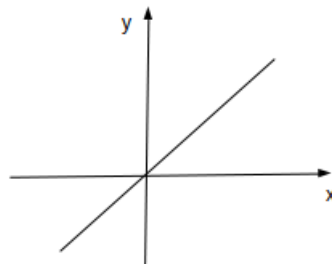


Figure 4.1: $f(x) = x$ on $[0, 1]$

2. If $f(x) = \tan x$, then on $(-\frac{\pi}{2}, \frac{\pi}{2})$ the function f has neither a maximum value nor a minimum value.
3. If $f(x) = |x|$ for $-1 \leq x < 0$ and $0 < x \leq 1$, and if $f(0) = 1$, then on $[-1, 1]$ the function has the maximum value because f does not assume the value 0.
4. If $f(x) = x$ for $-\infty < x < \infty$, then on $(-\infty, \infty)$ the function has neither a minimum value nor a maximum value.

Figure 4.2: $f(x) = \tan x$ Figure 4.3: $f(x) = |x|$ Figure 4.4: $f(x) = x, (-\infty, \infty)$ **Theorem 4.4.1** The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Theorem 4.4.2 Fermat's Theorem

Suppose c is an interior point of an interval I , and f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.

Definition 4.4.4 A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist. If f has a local maximum or minimum at c , then c is a critical number of f .

■ **Example 4.36** Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4 - x)$.

Solution: The product rule gives

$$\begin{aligned}
 f'(x) &= \frac{3}{5}x^{-\frac{2}{5}}(4-x) + x^{\frac{3}{5}}(-1) \\
 &= \frac{3(4-x)}{5x^{\frac{2}{5}}} - x^{\frac{3}{5}} \\
 &= \frac{3(4-x) - 5x}{5x^{\frac{2}{5}}} \\
 &= \frac{12-8x}{5x^{\frac{2}{5}}}
 \end{aligned}$$

Therefore, $f'(x) = 0$ if $12 - 8x = 0$, that is $x = \frac{3}{2}$ and $f'(x)$ does not exist when $x = 0$. Thus, the critical numbers are $\frac{3}{2}$ and 0. ■

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$

- Step 1: Find the critical numbers of f in (a, b) .
- Step 2: Find the values of f at the end points of the interval and at critical points.
- Step 3: The largest of the values from Step 2 is the absolute maximum value; the smallest of these values is the absolute minimum value

■ **Example 4.37** Find the absolute maximum and absolute minimum of the function $f(x) = x^2 + 4x + 5$ on the interval $[-3, 1]$.

Solution: The function $f(x) = x^2 + 4x + 5$ is continuous on the interval $[-3, 1]$

- Step 1: First we have to find the critical points of f

$$\begin{aligned}
 f'(x) &= 2x + 4 = 0 \\
 \Rightarrow 2x + 4 &= 0 \\
 2x &= -4 \\
 x &= -2 \text{ is a critical value in } (-3, 1)
 \end{aligned}$$

- Step 2: Now we have to find the values of f at the end points of the interval.

$$\begin{aligned}
 f(-3) &= (-3)^2 + 4(-3) + 5 = 2 \\
 f(-2) &= (-2)^2 + 4(-2) + 5 = 1 \\
 f(1) &= (1)^2 + 4(1) + 5 = 10
 \end{aligned}$$

- Step 3: The largest of the values from Step 2 is $f(1) = 10$ is absolute maximum value; the smallest value $f(-2) = 1$ is absolute minimum value. ■

■ **Example 4.38** Find the absolute maximum and absolute minimum of the function $f(x) = x^3 + 3x^2 - 9x - 7$ on the interval $[-4, 2]$.

Solution: The function $f(x) = x^3 + 3x^2 - 9x - 7$ is continuous on the interval $[-4, 2]$

- Step 1: First we have to find the critical points of f

$$\begin{aligned}
 f'(x) &= 3x^2 + 6x - 9 = 0 \\
 \Rightarrow 3x^2 + 6x - 9 &= 0 \\
 (x+3)(x-1) &= 0 \\
 x &= -3 \text{ or } x = 1 \text{ are a critical value in } (-4, 2)
 \end{aligned}$$

- Step 2: Now we have to find the values of f at the critical point and at the end points of the interval.

$$\begin{aligned}
 f(-4) &= (-4)^3 + 3(-4)^2 - 9(-4) - 7 = 13 \\
 f(-3) &= (-3)^3 + 3(-3)^2 - 9(-3) - 7 = 20 \\
 f(1) &= (1)^3 + 3(1)^2 - 9(1) - 7 = -12 \\
 f(2) &= (2)^3 + 3(2)^2 - 9(2) - 7 = -5
 \end{aligned}$$

- Step 3: The largest of the values from Step 2 is $f(-3) = 20$ is absolute maximum value; the smallest value $f(1) = -12$ is absolute minimum value. ■

The Mean Value Theorem

To arrive at the Mean Value Theorem we first need the following result.

Theorem 4.4.3 Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

- **Example 4.39** Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Solution: First we use the Intermediate Value Theorem to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$. Thus, the given equation has a root. To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots a and b . Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$. But $f'(x) = 3x^2 + 1 = 1$ for all x (since $x^2 = 0$) so $f'(x)$ can never be 0. This gives a contradiction. Therefore, the equation can not have two real roots. ■

Theorem 4.4.4 The Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ or, equivalently } f(b) - f(a) = f'(c)(b - a)$$

- **Example 4.40** Let $f(x) = \frac{1}{3}x^3 + 2x$. Find a number c in $(0, 3)$ such that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0}$$

Solution: Since

$$\frac{f(3) - f(0)}{3 - 0} = \frac{15 - 0}{3 - 0} = 5$$

We seek a number c in $(0, 3)$ such that $f'(c) = 5$. But

$$f'(x) = x^2 + 2$$

So that c must satisfy $c^2 + 2 = 5$.

Therefore, $c^2 = 3$ and since c must be in $(0, 3)$, we conclude that $c = \sqrt{3}$. ■

- **Example 4.41** Show that $e^x \geq 1 + x$ for $x \geq 0$.

Solution: Let $f(x) = e^x$, since f is continuous and differentiable for $x \geq 0$. By mean value theorem there is c in $(0, x)$ such that

$$f'(c) = \frac{e^x - e^0}{x - 0} \Leftrightarrow e^c = \frac{e^x - 1}{x}$$

clearly, $e^c \geq 1$ for $c \geq 0$, then

$$\begin{aligned} \frac{e^x - 1}{x} &\geq 1 \\ \Leftrightarrow e^x - 1 &\geq x \\ e^x &\geq 1 + x \end{aligned}$$

■ **Example 4.42** Show that $|\sin a - \sin b| \leq |a - b|$ for all a and b .

Solution: Let $f(x) = \sin x$, since f is continuous and differentiable for all x . Let $a > b$. By mean value theorem there is c in (a, b) such that

$$f'(c) = \frac{f(a) - f(b)}{a - b} \Leftrightarrow \cos c = \frac{\sin a - \sin b}{a - b}$$

Since, $-1 \leq \cos c \leq 1$, then

$$\begin{aligned} -1 &\leq \frac{\sin a - \sin b}{a - b} \leq 1 \\ \Leftrightarrow -(a - b) &\leq \sin a - \sin b \leq (a - b) \\ \Leftrightarrow |\sin a - \sin b| &\leq |a - b| \end{aligned}$$

Similarly, for $a < b$ we have $|\sin a - \sin b| \leq |a - b|$. ■

■ **Example 4.43** Show that $\frac{b-a}{b} \leq \ln\left(\frac{b}{a}\right) \leq \frac{b-a}{a}$ for $0 < a < b$.

Solution: Let $f(x) = \ln x$, since f is continuous on $[a, b]$ and differentiable on (a, b) . By mean value theorem there is c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow \frac{1}{c} = \frac{\ln b - \ln a}{b - a}$$

Since, $0 < a < c < b$, then

$$\begin{aligned} \frac{1}{a} &> \frac{1}{c} > \frac{1}{b} \\ \Leftrightarrow \frac{1}{a} &> \frac{\ln b - \ln a}{b - a} > \frac{1}{b} \\ \Leftrightarrow \frac{b - a}{a} &> \ln b - \ln a > \frac{b - a}{b} \\ \Leftrightarrow \frac{b - a}{b} &\leq \ln\left(\frac{b}{a}\right) \leq \frac{b - a}{a} \end{aligned}$$

■ **Example 4.44** Suppose that we know that $f(x)$ is continuous and differentiable on $[6, 15]$. Let's also suppose that $f(6) = -2$ and $f'(x) \leq 10$. What is the largest possible value for $f(15)$?

Solution: Let's start with the conclusion of the mean value theorem

$$\begin{aligned} f'(c) &= \frac{f(15) - f(6)}{15 - 6} \\ \Rightarrow f(15) - f(6) &= f'(c)(15 - 6) \end{aligned}$$

substituting the known values and rewriting this gives

$$\begin{aligned} f(15) &= f(6) + f'(c)(15 - 6) \\ &= -2 + 9f'(c) \end{aligned}$$

Now, we know that $f'(x) \leq 10$. So in particular we know that $f'(c) \leq 10$. This gives us the following

$$\begin{aligned} f(15) &= -2 + 9f'(c) \\ &\leq -2 + (9)(10) = 88 \end{aligned}$$

All we did was replace $f'(c)$ with its largest possible value. This means that the largest possible value for $f(15)$ is 88. ■

Increasing and Decreasing Test

Theorem 4.4.5 1. Let f be continuous on an interval I . If $f'(x)$ exists and equals 0 for each interior point x of I , then f is constant on I .
 2. Let f and g be continuous on an interval I . If $f'(x)$ and $g'(x)$ exist and are equal for each interior point x of I , then $f - g$ is constant on I . In other words, there is a constant c such that $f(x) = g(x) + c$ for all I .

Theorem 4.4.6 Let f be continuous on an interval I and differentiable at each interior point of I .
 1. If $f'(x) > 0$ at each interior point of I , then f is increasing on I . Moreover, f is increasing on I if $f'(x) > 0$ except for a finite number of points x in I .
 2. If $f'(x) < 0$ at each interior point of I , then f is decreasing on I . Moreover, f is decreasing on I if $f'(x) < 0$ except for a finite number of points x in I .

■ **Example 4.45** Find the interval where the function $f(x) = x^2$ is increasing and the interval where it is decreasing.

Solution: The derivative of $f(x) = x^2$ is $f'(x) = 2x > 0$ when $x > 0$, and $f'(x) = 2x < 0$ when $x < 0$, f is increasing in the interval $(0, \infty)$ and decreasing in the interval $(-\infty, 0)$. ■

■ **Example 4.46** Determine the interval where the function $f(x) = x^3 - 3x^2 - 24x + 32$ is increasing and where it is decreasing.

Solution: The derivative of f is $f'(x) = 3x^2 - 6x - 24 = 3(x+2)(x-4)$ and it is continuous everywhere. The zeros of $f'(x)$ are $x = -2$ and $x = 4$. So, $f'(x) > 0$ in the interval $(-\infty, -2)$, $f'(x) < 0$ in the interval $(-2, 4)$ and $f'(x) > 0$ in the interval $(4, \infty)$. Therefore, f is increasing on the intervals $(-\infty, -2)$ and $(4, \infty)$ and it is decreasing on the interval $(-2, 4)$. ■

The First Derivative Test

Theorem 4.4.7 Let f be differentiable on an open interval about the number c except possibly at c , where f is continuous. Suppose that c is a critical number of a continuous function f .
 1. If f' changes from positive to negative at c , then f has a local maximum at c .
 2. If f' changes from negative to positive at c , then f has a local minimum at c .
 3. If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c .

Concavity Test

Theorem 4.4.8 Assume that f'' exists on an open interval I .
 1. If $f''(x) > 0$ for all x in the interval I , then the graph of f is concave upward on I .
 2. If $f''(x) < 0$ for all x in the interval I , then the graph of f is concave downward on I .

■ **Definition 4.4.5** A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P .

The Second Derivative Test

Theorem 4.4.9 Suppose f'' is continuous near c .
 1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .
 2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .
 3. If $f''(c) = 0$, then from this test alone, we can not draw any conclusion about a relative extreme value of f at c .

■ **Example 4.47** Determine the relative extreme of the function $f(x) = x^3 - 3x^2 - 24x + 32$ using the second derivative test.

Solution: We have

$$\begin{aligned} f'(x) &= 3x^2 - 6x - 24 \\ &= 3(x+2)(x-4) \end{aligned}$$

So, $f'(x) = 0$ gives $x = 2$ and $x = 4$, the critical point of f . Next we compute

$$\begin{aligned} f''(x) &= 6x - 6 \\ &= 6(x - 1) \end{aligned}$$

Since $f''(-2) = 6(-2 - 1) = -18 < 0$. The second-derivative test implies that the point $(-2, 60)$ as a relative maximum of f . Also $f''(4) = 6(4 - 1) = 18 > 0$ and the second derivative test implies that the point $(4, -48)$ is a relative minimum of f , which conforms the result obtained. ■

■ **Example 4.48** Find the absolute maximum value of the function $f(x) = 12 - x - \frac{9}{x}$ on the interval $(0, \infty)$.

Solution: First we have to find the critical values on the interval

$$\begin{aligned} f'(x) &= -1 + \frac{9}{x^2} = 0 \\ \frac{9}{x^2} &= 1 \\ \Rightarrow x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

The only critical value in the interval $(0, \infty)$ is $x = 3$. Next find the second derivative of $f(x)$

$$\begin{aligned} f''(x) &= -2\left(\frac{9}{x^3}\right) = \frac{-18}{x^3} \\ f''(3) &= \frac{-18}{3^3} = \frac{-18}{27} = \frac{-2}{3} < 0 \end{aligned}$$

Therefore, $f(3) = 12 - 3 - \frac{9}{3} = 6$ is the absolute maximum value of f on $(0, \infty)$. ■

■ **Example 4.49** Find the absolute maximum value of the function $f(x) = 24 - 2x - \frac{8}{x}$, where $x > 0$.

Solution: First we have to find the critical values by first derivative

$$\begin{aligned} f'(x) &= -2 + \frac{8}{x^2} = 0 \text{ solving for } x \text{ we get} \\ \frac{8}{x^2} &= 2 \\ \Rightarrow x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Therefore, $x = 2$ is on the interval $x > 0$. Next check whether the second order condition is satisfied or not.

$$\begin{aligned} f''(x) &= -2\left(\frac{8}{x^3}\right) = \frac{-16}{x^3} \\ f''(2) &= \frac{-16}{2^3} = \frac{-16}{8} = -2 < 0 \end{aligned}$$

Therefore, $f(2) = 24 - 2(2) - \frac{8}{2} = 16$ is the absolute maximum value of $f(x) = 24 - 2x - \frac{8}{x}$ on the interval $x > 0$. ■

■ **Example 4.50** Let $f(x) = x^3 - 3x - 2$. Using the second derivative test, find the relative extreme values of f .

Solution: By differentiation we obtain

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1) \text{ and } f''(x) = 6x$$

Therefore $f'(x) = 0$ when $x = -1$ or $x = 1$. Since

$$f''(-1) = -6 < 0 \text{ and } f''(1) = 6 > 0$$

We know from the second derivative test that $f(-1) = 0$ is a relative maximum values of f , whereas $f(1) = -4$ is a relative minimum values of f . These are the only relative extreme values of f . ■

4.5 Indeterminate Forms and L'Hopital's Rule

4.5.1 Indeterminate Forms

Indeterminate Quotient

L'Hopital Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\begin{aligned} \lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0 \text{ or} \\ \lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) = \pm\infty; \text{ that is} \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} \end{aligned}$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

L'Hopital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, $x \rightarrow a$ can be replaced by any of the symbols $x \rightarrow a^+$, $x \rightarrow a^-$, $x \rightarrow \infty$ or $x \rightarrow -\infty$.

■ **Example 4.51** Solve $\lim_{x \rightarrow 0} \frac{2^x - 1}{x}$.

Solution: Since $\lim_{x \rightarrow 0} (2^x - 1) = 0$ and $\lim_{x \rightarrow 0} x = 0$

$$\lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \frac{0}{0} \text{ is } \frac{0}{0} \text{ form}$$

By L'Hopital rule

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2^x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(2^x - 1)}{\frac{d}{dx}x} \\ &= \lim_{x \rightarrow 0} \frac{2^x \ln 2}{1} \\ &= \ln 2 \lim_{x \rightarrow 0} 2^x \\ &= \ln 2 \end{aligned}$$

Therefore, $\lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \ln 2$. ■

■ **Example 4.52** Solve $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$.

Solution: Since $\lim_{x \rightarrow 1} \ln x = 0$ and $\lim_{x \rightarrow 1} (x-1) = 0$

$$\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \frac{0}{0} \text{ is } \frac{0}{0} \text{ form}$$

By L'Hopital rule

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\ln x}{x-1} &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} (x-1)} \\ &= \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{1} \\ &= \lim_{x \rightarrow 1} \frac{1}{x} \\ &= 1\end{aligned}$$

Therefore, $\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = 1$. ■

■ **Example 4.53** Solve $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$.

Solution: Since $\lim_{x \rightarrow \infty} e^x = \infty$ and $\lim_{x \rightarrow \infty} x^2 = \infty$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \frac{\infty}{\infty} \text{ is } \frac{\infty}{\infty} \text{ form}$$

By L'Hopital rule

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^x}{x^2} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} x^2} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{2x} \\ &= \frac{\infty}{\infty} \text{ is } \frac{\infty}{\infty} \text{ form}\end{aligned}$$

Again we apply L'Hopital rule

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^x}{2x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} 2x} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{2} \\ &= \infty\end{aligned}$$

Therefore, $\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \infty$. ■

Indeterminate Products

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then it isn't clear what the value of $\lim_{x \rightarrow a} f(x)g(x)$ is called an indeterminate form of type $0 \cdot \infty$.

■ **Example 4.54** Evaluate $\lim_{x \rightarrow 0^+} x \ln x$.

Solution: Since $\lim_{x \rightarrow 0^+} x = 0$ and $\lim_{x \rightarrow 0^+} \ln x = -\infty$

$$\begin{aligned}\lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} x \lim_{x \rightarrow 0^+} \ln x \\ &= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \text{ is } \frac{\infty}{\infty} \text{ type} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \text{ by L'Hopital Rule} \\ &= \lim_{x \rightarrow 0^+} -x \\ &= 0\end{aligned}$$

Indeterminate Differences

If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$, then the limit $\lim_{x \rightarrow a} (f(x) - g(x))$ is called an indeterminate form of type $\infty - \infty$.

■ **Example 4.55** Evaluate $\lim_{x \rightarrow (\frac{-\pi}{2})^-} (\sec x - \tan x)$.

Solution: Since $\lim_{x \rightarrow (\frac{-\pi}{2})^-} \sec x = \infty$ and $\lim_{x \rightarrow (\frac{-\pi}{2})^-} \tan x = \infty$

$$\begin{aligned} \lim_{x \rightarrow (\frac{-\pi}{2})^-} (\sec x - \tan x) &= \lim_{x \rightarrow (\frac{-\pi}{2})^-} \sec x - \lim_{x \rightarrow (\frac{-\pi}{2})^-} \tan x \quad \infty - \infty \text{ type} \\ &= \lim_{x \rightarrow (\frac{-\pi}{2})^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \text{ is } \frac{\infty}{\infty} \text{ type} \\ &= \lim_{x \rightarrow (\frac{-\pi}{2})^-} \frac{1 - \sin x}{\cos x} \text{ is } \frac{0}{0} \text{ type} \\ &= \lim_{x \rightarrow (\frac{-\pi}{2})^-} \frac{-\cos x}{-\sin x} \text{ by L'Hopital Rule} \\ &= 0 \end{aligned}$$

■

Indeterminate Powers

Several indeterminate forms arise from the limit $\lim_{x \rightarrow a} (f(x))^{g(x)}$

1. $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ type 0^0
2. $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = 0$ type ∞^0
3. $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$ type $1^{\pm\infty}$

Each of these three cases can be treated either by taking the natural logarithm

Let $y = f(x)^{g(x)}$, then $\ln y = g(x) \ln f(x)$ or by writing the function as an exponential $f(x)^{g(x)} = e^{g(x) \ln f(x)}$.

■ **Example 4.56** Calculate $\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x}$.

Solution: Since $\lim_{x \rightarrow 0^+} (1 + \sin 4x) = 1$ and $\lim_{x \rightarrow 0^+} \cot x = \infty$

$$\begin{aligned} \lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} &= \lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\lim_{x \rightarrow 0^+} \cot x} \text{ is } 1^\infty \text{ type} \\ &= e^{\lim_{x \rightarrow 0^+} \ln(1 + \sin 4x)^{\cot x}} \\ &= e^{\lim_{x \rightarrow 0^+} \cot x \ln(1 + \sin 4x)} \\ &= e^{\lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}} \text{ is } \frac{0}{0} \text{ type} \\ &= e^{\lim_{x \rightarrow 0^+} \frac{\frac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x}} \text{ by L'Hospital Rule} \\ &= e^4 \end{aligned}$$

■

■ **Example 4.57** Evaluate $\lim_{x \rightarrow 0^+} x^x$.

Solution: Notice that this is indeterminate form of 0^0 . So by indeterminate form of power, we have

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} x^x &= \lim_{x \rightarrow 0^+} e^{\ln x^x} \\
 &= \lim_{x \rightarrow 0^+} e^{x \ln x} \\
 &= e^{\lim_{x \rightarrow 0^+} x \ln x} \\
 &= e^0, \text{ since } \lim_{x \rightarrow 0^+} x \ln x = 0 \\
 &= 1
 \end{aligned}$$

■

Curve Sketching

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. For instance, a given curve might not have an asymptote or possess symmetry. But the guidelines provide all the information that need to make a sketch that displays the most important aspects of the function.

1. Domain
It's often useful to start by determining the domain D of f , that is, the set of values of x for which $f(x)$ is defined.
2. Intercepts
Find x and y intercept.
3. Symmetry
 - (a) If $f(-x) = f(x)$ for all x in D , the function is an even function and the curve is symmetric about the y -axis.
 - (b) If $f(-x) = -f(x)$ for all x in D , then f is an odd function and the curve is symmetric about the origin.
 - (c) If $f(x+p) = f(x)$ for all x in D , where p is a positive constant, then f is called a periodic function and the smallest such number is called the period. For instance, $y = \sin x$ has period 2π .
4. Asymptotes
 - (a) Horizontal Asymptotes.
If either $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x \rightarrow \infty} f(x) = \pm\infty$, then we do not have an asymptote to the right.
 - (b) Vertical Asymptotes.
The line $x = a$ is a vertical asymptote if at least one of $\lim_{x \rightarrow a^-} f(x) = \infty$, $\lim_{x \rightarrow a^+} f(x) = \infty$, $\lim_{x \rightarrow a^-} f(x) = -\infty$ or $\lim_{x \rightarrow a^+} f(x) = -\infty$ is true. For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.
 - (c) Oblique Asymptotes.
Some curves have asymptotes that are oblique, that is, neither horizontal nor vertical. If $\lim_{x \rightarrow \infty} (f(x) - (mx + b)) = 0$, then the line $y = mx + b$ is called an oblique/slant asymptote. For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the oblique asymptote can be found by long division.
5. Intervals of Increase or Decrease
Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).
6. Local Maximum and Minimum Values
7. Concavity and Points of Inflection
Compute $f''(x)$ and use the Concavity Test.

8. Sketch the Curve Using the information in items 1 – 7, draw the graph. Sketch the asymptotes as dashed lines.

■ **Example 4.58** Use the guidelines to sketch the curve $y = \frac{2x^2}{x^2-1}$.

Solution:

1. The domain is

$$\{x|x^2 - 1 \neq 0\} = \{x|x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

2. The x-and y-intercept are both 0.
 3. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.
 4. The line $y = 2$ is a horizontal asymptote since

$$\lim_{x \rightarrow \pm\infty} \frac{2x^2}{x^2-1} = \lim_{x \rightarrow \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = 2$$

since the denominator is 0 when $x = \pm 1$, the limit of y as x approaches to 1^+ and -1^- is ∞ , respectively and the limit of y as x approaches to 1^- and -1^+ is $-\infty$, respectively.

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes.

- 5.

$$f'(x) = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Since $f'(x) > 0$ when $x < 0$ ($x \neq -1$) and $f'(x) < 0$ when $x > 0$ ($x \neq 1$), f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

6. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0)$ is a local maximum by first derivative test.

- 7.

$$f''(x) = \frac{-4(x^2-1)^2 + 2x \cdot 2(x^2-1)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

Since $12x^2 + 4 > 0$ for all x , we have

$$f''(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow |x| > 1$$

and $f''(x) < 0 \Leftrightarrow |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f .

8. Sketch the Curve Using the information in items 1 – 7, draw the graph. Sketch the asymptotes as dashed lines.

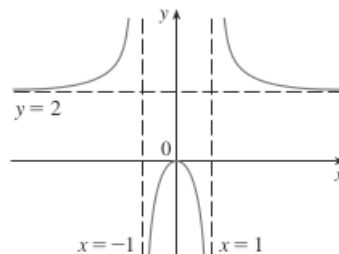


Figure 4.5: $f(x) = \frac{2x^2}{x^2-1}$

Application Problems

1. Rate of Change

The first interpretation/application of a derivative is rate of change. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

■ **Example 4.59** Suppose that the amount of water in holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 20$. Determine each of the following

- Whether the volume of water in the tank increasing or decreasing at $t = 2$ minute?
- Whether the volume of water in the tank increasing or decreasing at $t = 6$ minute?
- Is the volume of the water in the tank changing faster at $t = 2$ or $t = 6$ minute?
- Is the volume of the water in the tank ever changed? If so, when?

Solution: First compute the rate of change of the volume by finding the derivative of this function since that give a formula for the rate of change at any time t . So,

$$\frac{dV}{dt} = 4t - 16$$

By first derivative test if the rate of change is positive, then the quantity will be increasing and if the rate of change is negative, then the quantity will be decreasing.

- at $t = 2$ minute, the rate of change of the volume $V'(2) = -8$. So, at $t = 2$ the rate of change is negative and the volume must be decreased at this time.
- at $t = 6$ minute, the rate of change of the volume $V'(6) = 8$. So, at $t = 6$ the rate of change is positive and the volume must be increased at this time.
- To answer this question all that we look at is the size of the rate of change and we do not worry about the sign of the rate of change. All that we need to know here is the larger the number the faster the rate of change.
- The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$\frac{dV}{dt} = 4t - 16 = 0 \Rightarrow t = 4$$

So, at $t = 4$ the volume is not changing.

■

2. Velocity

■ **Definition 4.5.1** If $f(t)$ is the position of a particle moving on a coordinate line, then the instantaneous velocity of the particle at time t is defined by $v(t) = v'(t) = \frac{dv}{dt}$.

■ **Example 4.60** Suppose that the position of an object after t hours is given by,

$$f(t) = \frac{t}{t+1}$$

- Is the object moving to the right or the left at $t = 9$?
- Does the object ever stop moving?

Solution: The derivative is $f'(t) = \frac{1}{(t+1)^2}$.

- To determine if the object is moving to the right (Velocity is positive) or left (Velocity is negative) we need the derivative at $t = 9$.

$$f'(9) = \frac{1}{(10)^2} = \frac{1}{100}$$

So, the velocity at $t = 9$ is positive and so the object is moving to the right at $t = 9$.

- The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative and hence the speed is a constant it can not be zero. Therefore, the velocity will never stop moving. In fact, the object will always be moving to the right since velocity is always positive.

■

3. Acceleration

Definition 4.5.2 If $f(t)$ is the position function of a particle moving on a coordinate line, then the instantaneous acceleration of the particle at time t is defined by $a(t) = v'(t) = \frac{dv}{dt}$ or alternatively, since $v(t) = f'(t)$, $a(t) = f''(t) = \frac{d^2f}{dt^2}$

■ **Example 4.61** Let $f(t) = t^3 - 6t^2$ be the position function of a particle moving along a y-axis, where f is in meters and t is in seconds. Find the instantaneous acceleration $a(t)$.

Solution: $a(t) = \frac{dv}{dt} = 6t - 12$. ■

Remark 4.5.1 Interpreting the sign of acceleration. A particle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.

4. Optimization Problem

■ **Example 4.62** An open-top box is to be made by cutting small congruent squares from the corners of a 12 by 12 meters sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution: In the figure below, the corner squares are x meter on a side. The volume of the box is a function of this variable

$$\begin{aligned} v &= lwh \\ v(x) &= x(12-x)(12-x) \text{ for } 0 \leq x \leq 6 \\ \frac{dv}{dx} &= 144 - 96x + 12x^2 \\ \frac{dv}{dx} &= 0 \text{ if and only if } x = 2 \text{ or } x = 6 \end{aligned}$$

Thus, $v(2) = 128$, $v(0) = v(6) = 0$. Therefore, $v(2) = 128m^3$ is the maximum volume obtained by

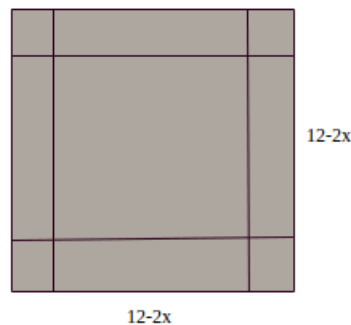


Figure 4.6: 12 m by 12 m square sheet

cut out squares 2 m on each side. ■

■ **Example 4.63** Four meter of wire is to be used to form a square and a circle. How much of the wire should be used and how much should be used for the circle to enclose the maximum total area?

Solution: The total area is given by let x be the side of the square and r are radius of the circle

$$A = \text{Area of Square} + \text{Area of circle} = x^2 + \pi r^2 \quad (4.1)$$

and the sum perimeter of square and circumference of circle is 4, Since the total amount of wire is 4 m, $4x + 2\pi r = 4$, thus

$$r = \frac{2(1-x)}{\pi} \text{ for } 0 \leq x \leq 1 \quad (4.2)$$

By substituting equation(4.2) into equation (4.1), we have

$$\begin{aligned} A(x) &= x^2 + \pi\left(\frac{2(1-x)}{\pi}\right)^2 \\ \Rightarrow \frac{dA}{dx} &= \frac{1}{2}(2(\pi+4)x - 8) \end{aligned}$$

Thus, the only critical number is $x = \frac{4}{\pi+4} \approx 0.56$, since $A(0) \approx 1.273$, $A(0.56) \approx 0.56$ and $A(1) = 1$. Therefore, the maximum area occurs when $x = 0$; that is, all the wire is used for the circle. ■

■ **Example 4.64** You have been asked to design a 1 liter(1000cm^3) oil-can shaped like a right circular cylinder. What dimensions will use the least material?

Solution: Let the right circular cylinder can have height h and diameter $2r$ as shown in the figure below. Here h and r are related by the equation

$$\pi r^2 h = 1000 \quad (4.3)$$

and total surface area is given by



Figure 4.7: Cylinder having radius r and height h

$$A = 2\pi r^2 + 2\pi r h \quad (4.4)$$

solving equation (4.3) for h , we have

$$h = \frac{1000}{\pi r^2} \quad (4.5)$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= \pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2}\right) \\ &= \pi r^2 + \frac{2000}{r} \\ A(r) &= \pi r^2 + \frac{2000}{r} \\ \frac{dA}{dr} &= 4\pi r - \frac{2000}{r^2} \\ 0 &= 4\pi r - \frac{2000}{r^2} \text{ by setting } \frac{dA}{dr} = 0 \\ 4\pi r^3 &= 2000 \\ \Rightarrow r &= \sqrt[3]{\frac{500}{\pi}} \end{aligned}$$

Therefore a critical point is $r = \sqrt[3]{\frac{500}{\pi}}$. If the domain of A were a closed interval, we could find out by evaluating A at this critical point and the endpoints and comparing the results. But the domain is

not a closed interval. So, we have to check at $r = \sqrt[3]{\frac{500}{\pi}}$ by second derivative test.

$$\begin{aligned}\frac{dA}{dr} &= 4\pi r - \frac{2000}{r^2} \\ \frac{d^2A}{dr^2} &= 4\pi + \frac{4000}{r^3}\end{aligned}$$

The second derivative is positive throughout the domain of A . The value of a at $r = \sqrt[3]{\frac{500}{\pi}}$ is therefore an absolute minimum because the graph of A is concave up and

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(\sqrt[3]{\frac{500}{\pi}})^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r \quad (4.6)$$

The above equation (4.6) tells us that the most efficient can has its height equal to its diameter; that is, $r \approx 5.42\text{cm}$ and $h \approx 10.84\text{cm}$. ■

4.6 Exercise

1. Find the derivative of $f(x) = 5x - 9x^2$ using the definition of derivatives. State the domain of the function and the domain of its derivative.

Ans. $f'(x) = 5 - 18x$, R , R

2. Let ℓ be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The angle of inclination of ℓ is the angle ϕ that ℓ makes with the positive direction of the x -axis. Calculate ϕ correct to the nearest degree.

Ans. 63°

3. Find the derivative of $y = (x^2 + 1)(x^3 + 1)$.

Ans. $5x^4 + 3x^2 + 2x$

4. Differentiate $y = \frac{x^2}{1+\sqrt{x}}$.

Ans. $y' = \frac{x(4+\sqrt{x})}{2(1+\sqrt{x})^2}$

5. Find an equation of the tangent line to the curve $y = \frac{2x}{x+1}$ at the point $(1, 1)$.

Ans. $y = \frac{1}{2}x + \frac{1}{2}$

6. Find an equation of the tangent line and normal line to the curve $y = \frac{3x+1}{x^2+1}$ at the point $(1, 2)$.

Ans. $y = -\frac{1}{2}x + \frac{5}{2}$, $y = 2x$

7. Find the first and second derivative of the function $f(x) = \frac{x^2}{1+2x}$.

Ans. $y = \frac{2x^2-2x}{(1+2x)^2}$, $f'(x) = \frac{2}{(1+2x)^3}$

8. If $f(x) = \sqrt{x}g(x)$, where $g(4) = 8$ and $g'(4) = 7$, find $f'(4)$.

Ans. 16

9. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

10. Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y = 5$.

Ans. $y = \frac{1}{3}x - \frac{1}{3}$

11. For what value of x is the function $f(x) = |x^2 - 9|$ differentiable? find a formula for f' .

Ans. No differentiation at $x = -3$ and $x = 3$, $f'(x) = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$

12. For what value of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

Ans. $a = -\frac{1}{2}$, $b = 2$

13. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

14. Differentiate $f(\theta) = \frac{\sec \theta}{1 + \sec \theta}$.

Ans. $f'(\theta) = \frac{\sec \theta \tan \theta}{(1 + \sec \theta)^2}$

15. Find the limit $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$.

Ans. 3

16. If $f(x) = g(h(x))$, where $g(-2) = 8$, $g'(-2) = 4$, $g'(5) = 3$, $h(5) = -2$, and $h'(5) = 6$, find $f'(5)$.

Ans. 24

17. Find the implicit differentiation $\frac{dy}{dx}$ for a function $\sqrt{xy} = 1 + x^2y$.

$$\text{Ans. } y' = \frac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$$

18. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

$$\text{Ans. } -\frac{16}{13}$$

19. Find the local maximum and minimum values of $f(x) = x^5 - 5x + 3$ using both the first and second derivative test

$$\text{Ans. Local maximum } f(-1) = 7 \text{ and local minimum } f(1) = -1$$

20. Let $f(x) = \frac{2x^2}{x^2-1}$, then find

- domain
- intercept
- symmetry
- all the existing asymptotes
- interval of increase and decrease
- local maximum and minimum values
- interval of concavity and inflection point
- sketch the graph of f

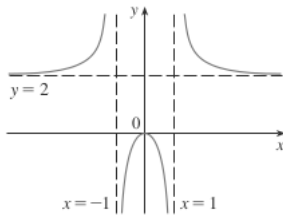


Figure 4.8: $f(x) = \frac{2x^2}{x^2-1}$