MODEL ANSWERS IN ORDINARY NATIONAL CERTIFICATE MATHEMATICS FOR ENGINEERS

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PREFACE

IT WOULD not be possible in one book of moderate size to present a series of model answers that included all the topics covered by the many different syllabuses in Mathematics for the Ordinary National Certificate in Engineering. Consequently, we do not claim to cover any particular syllabus completely, but have tried instead to include that subject matter which is common to the majority of such syllabuses.

We have divided the book into a number of sections each one devoted to a particular topic, or group of related topics, and itself subdivided into Worked Examples, being complete solutions; Hinted Examples, being semi-solved problems; and Further Examples, for which only the answers are provided.

Where necessary, and space has permitted, solutions have been supplemented with explanatory notes.

We wish to thank the following Examining Boards for permission to use questions set in their examinations. The source of a particular example is acknowledged by the appropriate abbreviation.

The East Midland Educational Union (E.M.E.U.)
The Union of Educational Institutions (U.E.I.)
The Union of Lancashire and Cheshire Institutions (U.L.C.I.)
The Welsh Joint Education Committee (W.J.E.C.)

May 1965

D. W. H.
J. G. S.

MODEL ANSWERS

Ordinary National Certificate Mathematics for Engineers

LOGARITHMS

WORKED EXAMPLES

No. 1

(a) Evaluate (i) $\log_4 215$, (ii) $\log_e 0{\cdot}6$.

(b) Evaluate without using tables

$$\log_{10} 20 + 2\log_{10} \tfrac{3}{16} - \tfrac{1}{2}\log_{10} \tfrac{81}{64} + 4\log_{10} 2.$$

(a) (i) $\log_4 215 = \dfrac{\log_{10} 215}{\log_{10} 4}$

$= \dfrac{2{\cdot}3324}{0{\cdot}6021}$

$= \underline{3{\cdot}87.}$

No.	Log
2·3324	0·3678
0·6021	$\bar{1}$·7797
3·874	0·5881

(ii) $\log_e 0{\cdot}6 = \dfrac{\log_{10} 0{\cdot}6}{\log_{10} e}$

$= \dfrac{\log_{10} 0{\cdot}6}{\log_{10} 2{\cdot}718}$

$= \dfrac{\bar{1}{\cdot}7782}{0{\cdot}4343} = \dfrac{-1+0{\cdot}7782}{0{\cdot}4343}$

$= \dfrac{-0{\cdot}2218}{0{\cdot}4343} = \underline{-0{\cdot}511.}$

No.	Log
0·2218	$\bar{1}$·3459
0·4343	$\bar{1}$·6378
0·5106	$\bar{1}$·7081

(b) $\log_{10} 20+2\log_{10}\frac{3}{16}-\frac{1}{2}\log_{10}\frac{81}{64}+4\log_{10}2$

$= \log_{10} 20+2(\log_{10}3-\log_{10}16)-\frac{1}{2}(\log_{10}81-\log_{10}64)+4\log_{10}2$

$= \log_{10}(2\times10)+2\log_{10}3-2\log_{10}(2^4)$
$\quad -\frac{1}{2}\log_{10}(3^4)+\frac{1}{2}\log_{10}(2^6)+4\log_{10}2$

$= \log_{10}2+\log_{10}10+2\log_{10}3-8\log_{10}2$
$\quad -2\log_{10}3+3\log_{10}2+4\log_{10}2$

$= \log_{10}10$

$= \underline{1}.$

Note. When working with logarithms the fourth figure is unreliable, since the same calculation done in different ways may give slight variations in the fourth figure. Hence it is usual to express answers correct to three significant figures.

No. 2

(a) Evaluate (i) $(0{\cdot}36)^{-2{\cdot}1}$, (ii) $1{\cdot}76^{2{\cdot}46}\times0{\cdot}378^{1{\cdot}56}$.

(b) Solve the following equation for x

$$4{\cdot}15^{(x-3)}\times3 = 3{\cdot}09^{(x-4)}.$$

(a) (i) Let $x = (0{\cdot}36)^{-2{\cdot}1}$

then $\log_{10}x = (-2{\cdot}1)\times\log_{10}0{\cdot}36 = (-2{\cdot}1)\times(\bar{1}{\cdot}5563)$

$= (-2{\cdot}1)(-1+0{\cdot}5563) = (-2{\cdot}1)(-0{\cdot}4437)$

$= +\{0{\cdot}8874+0{\cdot}04437\} = 0{\cdot}93177 \doteq 0{\cdot}9318$

$\therefore\quad x = 8{\cdot}547. \qquad \therefore\quad \underline{(0{\cdot}36)^{-2{\cdot}1} = 8{\cdot}55.}$

(ii) Let $x = 1{\cdot}76^{2{\cdot}46}\times0{\cdot}378^{1{\cdot}56}$

No.	Log
2·46	0·3909
0·2455	$\bar{1}$·3901
0·604	$\bar{1}$·7810
1·56	0·1931
0·4225	$\bar{1}$·6258
0·659	1·8189

then $\log_{10}x = 2{\cdot}46\log_{10}1{\cdot}76+1{\cdot}56\log_{10}0{\cdot}378$

$= 2{\cdot}46\times0{\cdot}2455+1{\cdot}56\times(\bar{1}{\cdot}5775)$

$= 2{\cdot}46\times0{\cdot}2455-1{\cdot}56\times0{\cdot}4225$

$= 0{\cdot}6040-0{\cdot}6590$

$= -0{\cdot}0550 = -1+0{\cdot}9450$

$= \bar{1}{\cdot}9450$

$\therefore\quad x = 0{\cdot}8811$

$\therefore\quad \underline{1{\cdot}76^{2{\cdot}46}\times0{\cdot}378^{1{\cdot}56} = 0{\cdot}881.}$

(b) $4{\cdot}15^{(x-3)} \times 3 = 3{\cdot}09^{(x-4)}$.

Taking logarithms of both sides gives

$$(x-3)\log_{10} 4{\cdot}15 + \log_{10} 3 = (x-4)\log_{10} 3{\cdot}09$$

$$(x-3)\,0{\cdot}6180 + 0{\cdot}4771 = (x-4)\,0{\cdot}4900$$

$$\therefore\quad x(0{\cdot}6180 - 0{\cdot}4900) = (-4 \times 0{\cdot}4900) + (3 \times 0{\cdot}6180) - 0{\cdot}4771$$

$$x \times 0{\cdot}1280 = -1{\cdot}96 + 1{\cdot}854 - 0{\cdot}4771$$

$$= -0{\cdot}5831$$

$$\therefore\quad x = \frac{-0{\cdot}5831}{0{\cdot}1280}$$

$$\underline{x = -4{\cdot}56.}$$

No.	Log
0·5831	$\bar{1}{\cdot}7657$
0·1280	$\bar{1}{\cdot}1072$
4·555	0·6585

No. 3

(a) Solve the equations (i) $(2x)^{2\cdot5} = 50$, (ii) $(2{\cdot}5)^{2x} = 50$.

(b) If $pv^n = c$, $p = 120$ when $v = 100$ and $p = 760$ when $v = 25$, calculate the values of n and c.

(a) (i) $$(2x)^{2\cdot5} = 50$$

$$\therefore\quad 2{\cdot}5\log_{10} 2x = \log_{10} 50 = 1{\cdot}699$$

$$\therefore\quad \log_{10} 2x = \frac{1{\cdot}699}{2{\cdot}5} = 0{\cdot}6796$$

$$\therefore\quad 2x = 4{\cdot}782$$

$$\therefore\quad x = 2{\cdot}391. \qquad \therefore\ \underline{x = 2{\cdot}39.}$$

(ii) $$(2{\cdot}5)^{2x} = 50$$

$$\therefore\quad 2x\log_{10} 2{\cdot}5 = \log_{10} 50$$

$$\therefore\quad 2x \times 0{\cdot}398 = 1{\cdot}699$$

$$\therefore\quad 2x = \frac{1{\cdot}699}{0{\cdot}398}$$

$$= 4{\cdot}27$$

$$\therefore\quad x = 2{\cdot}135. \qquad \therefore\ \underline{x = 2{\cdot}14.}$$

(b) $$pv^n = c$$

$$\therefore \quad \log_{10} p + n \log_{10} v = \log_{10} c.$$

Since $p = 120$ when $v = 100$

$$\log_{10} 120 + n \log_{10} 100 = \log_{10} c$$

$$\therefore \quad 2{\cdot}0792 + 2n = \log_{10} c. \qquad (1)$$

Also since $p = 760$ when $v = 25$

$$\log_{10} 760 + n \log_{10} 25 = \log_{10} c$$

$$2{\cdot}8808 + 1{\cdot}398n = \log_{10} c. \qquad (2)$$

$$\therefore \quad \text{Eqn. (1)} - \text{eqn. (2)} = -0{\cdot}8016 + 0{\cdot}602n = 0$$

$$\therefore \quad n = \frac{0{\cdot}8016}{0{\cdot}602} = \underline{1{\cdot}332.}$$

Substituting in eqn. (1)

$$2{\cdot}08 + 2 \times 1{\cdot}332 = \log_{10} c$$

$$\therefore \quad \log_{10} c = 2{\cdot}08 + 2{\cdot}664$$

$$= 4{\cdot}744$$

$$\therefore \quad \underline{c = 55{,}460.}$$

Answers: $n = 1{\cdot}33$ and $c = 55{,}500.$

FURTHER EXAMPLES

No. 1. Evaluate: $\log_6 38{\cdot}4$; $\log_8 0{\cdot}47$; $\log_e 0{\cdot}25$.

No. 2. Evaluate: (i) $\sqrt[3]{0{\cdot}3541}$; (ii) $(0{\cdot}0473)^{-1{\cdot}25}$; (iii) $(0{\cdot}1516)^{1{\cdot}6}$.

No. 3. Solve for x: (a) $5^{2x+1} = 5 \log_{10} 4$, (b) $e^{5x} = 0{\cdot}5^{-2{\cdot}3}$.

No. 4. The pressure p pounds per square inch and the volume v cubic inches of a gas are connected by the relation $pv^{1{\cdot}4} = 200$. Calculate p when $v = 12{\cdot}7$. (E.M.E.U.)

No. 5. Using logarithms evaluate $(A^{1/n} + B^{1/n})^{-n}$ when $A = 15{\cdot}2$, $B = 4{\cdot}25$ and $n = 1{\cdot}5$. (U.L.C.I.)

ANSWERS

No. 1. 2·04; −0·363; −1·39.
No. 2. (i) 0·708; (ii) 45·3; (iii) 0·0489.
No. 3. (a) −0·158; (b) 0·319.
No. 4. $p = 5{\cdot}70$ lb/in^2.
No. 5. 0·0386.

FORMULAE: TRANSFORMATION AND EVALUATION

WORKED EXAMPLES

No. 1

(a) If $v = V\mathrm{e}^{-t/RC}$, calculate the value of v when $V = 500$, $t = 0{\cdot}56$, $R = 10^6$ and $C = 15\times10^{-6}$.

(b) Rearrange the following formula to make L the subject

$$K = \sqrt{\frac{L+M}{P-(L+M)/R}} \qquad \text{(U.E.I.)}$$

(a) Since $v = V\mathrm{e}^{-t/RC}$, taking logarithms to base e gives

$$\log_e v = \log_e V - \frac{t}{RC}.$$

Hence when $V = 500$, $t = 0{\cdot}56$, $R = 10^6$ and $C = 15\times10^{-6}$

$$\log_e v = \log_e 500 - \frac{0{\cdot}56}{10^6\times15\times10^{-6}}$$

$$= \log_e 500 - \frac{0{\cdot}56}{15}$$

$$= \log_e 500 - 0{\cdot}0373.$$

From tables, $\log_e 500 = \log_e 5 + \log_e 100 = 1{\cdot}6094 + 4{\cdot}6052 = 6{\cdot}2146$.

Hence $\log_e v = 6{\cdot}2146 - 0{\cdot}0373 = 6{\cdot}1773$

$= 4{\cdot}6052 + 1{\cdot}5721 = \log_e 100 + \log_e 4{\cdot}817$

$\therefore\quad v = 4{\cdot}817\times100 = 481{\cdot}7.$

Answer: $v = 482$ (3 sig. fig.).

(b)
$$K = \sqrt{\frac{L+M}{P-(L+M)/R}}$$
$$\therefore\ K^2 = \frac{L+M}{P-(L+M)/R}$$
$$\therefore\ K^2 = \frac{R(L+M)}{RP-(L+M)}$$
$$= \frac{R(L+M)}{RP-L-M}$$
$$\therefore\quad K^2(RP-L-M) = R(L+M)$$
$$\therefore\quad K^2RP-K^2L-K^2M = RL+RM$$
$$\therefore\quad K^2RP-K^2M-RM = RL+K^2L$$
$$\therefore\quad K^2RP-K^2M-RM = L(R+K^2)$$
$$\therefore\quad \frac{K^2RP-K^2M-RM}{R+K^2} = L.$$

Answer: $L = \dfrac{K^2(RP-M)-RM}{R+K^2}$.

Note. In (a), since $10^{-6} = 1/10^6$, the term RC becomes on substitution $10^6 \times 15 \times 10^{-6} = 10^6 \times 15 \times (1/10^6) = 15$.

No. 2

If $E = (K/t)(\log_e p - \log_e q)$ give an expression for p in terms of the other symbols. (U.L.C.I.)

$$E = \frac{K}{t}(\log_e p - \log_e q)$$
$$\therefore\quad E = \frac{K}{t}\log_e\left(\frac{p}{q}\right)$$
$$\therefore\quad \frac{Et}{K} = \log_e\left(\frac{p}{q}\right)$$
$$\therefore\quad \frac{p}{q} = e^{Et/K}$$
$$\therefore\quad q = qe^{Et/K}.$$

Answer: $p = qe^{Et/K}$.

No. 3

(a) If $i = (E/R)[1-e^{-Rt/L}]$, calculate i when $R = 10$, $L = 5$, $t = 0{\cdot}4$ and $E = 240$.

(b) From the equation in (a) above, find an expression for t in terms of the other symbols. (U.L.C.I.)

(a) $$i = \frac{E}{R}[1-e^{-Rt/L}].$$

Hence when $R = 10$, $L = 5$, $t = 0{\cdot}4$ and $E = 240$

$$i = \frac{240}{10}[1-e^{-10\times 0{\cdot}4/5}],$$

$$\text{i.e. } i = 24(1-e^{-0{\cdot}8}).$$

From exponential tables

$$i = 24(1-0{\cdot}4493)$$

$$i = 24\times 0{\cdot}5507$$

$$i = 13{\cdot}2168.$$

Answer: $i = 13{\cdot}2$.

(b) $$i = \frac{E}{R}[1-e^{-Rt/L}]$$

$$\therefore \quad Ri = E-Ee^{-Rt/L}$$

$$\therefore \quad Ri-E = -Ee^{-Rt/L}$$

$$\therefore \quad E-Ri = Ee^{-Rt/L}$$

$$\therefore \quad e^{-Rt/L} = \frac{E-Ri}{E}$$

$$e^{Rt/L} = \frac{E}{E-Ri}$$

$$\frac{Rt}{L} = \log_e\left(\frac{E}{E-Ri}\right)$$

$$t = \frac{L}{R}\log_e\left(\frac{E}{E-Ri}\right).$$

Answer: $t = \frac{L}{R}\log_e\left(\frac{E}{E-Ri}\right)$.

HINTED EXAMPLES

No. 1. (a) Rearrange the following formula to make R the subject

$$v = \sqrt{\frac{2gH}{1+(k/R)^2}}.$$

(b) Given that

$$A = P\left(1+\frac{v}{100}\right)^n,$$

find n when $A = 335{\cdot}8$, $P = 274{\cdot}5$ and $v = 4{\cdot}5$. (U.E.I.)

(a) Square both sides of the equation, and clear the fraction on R.H.S. to give

$$v^2\{1+(k/R)^2\} = 2gH.$$

Divide through the equation by v^2, subtract 1 from both sides and then put R.H.S. over a common denominator, v^2, to give

$$\frac{k^2}{R^2} = \frac{2gH-v^2}{v^2}.$$

Invert the fractions on both sides of the equation, multiply through by k^2, and finally take the square roots on both sides to give answer,

$$R = \frac{kv}{\sqrt{2gH-v^2}}.$$

(b) Straight substitution to give

$$\frac{335{\cdot}8}{274{\cdot}5} = (1{\cdot}045)^n$$

Take logarithms to the base 10 on both sides to give

$$n \times 0{\cdot}0191 = 2{\cdot}5260 - 2{\cdot}4386.$$

Finish calculation.

Answer: $n = 4{\cdot}58$.

No. 2. (a) Given $\theta = Ae^{-bt}\sin(2\pi t/T)$ express T in terms of the other quantities. Calculate T when $\theta = 0{\cdot}1$ radian, $t = 0{\cdot}1$, $b = 8$, $A = 2$.

(b) Given $p_1v_1^\gamma = p_0v_0^\gamma$ and $p_1v_1 = p_2v_0$ find an expression for γ in terms of p_0, p_1 and p_2. (U.E.I.)

(a) Make $\sin(2\pi t/T)$ the subject of the formula, then rewrite equation in equivalent inverse sine notation to give

$$\frac{2\pi t}{T} = \sin^{-1}\left(\frac{\theta e^{bt}}{A}\right).$$

Now multiply both sides by T and divide by the inverse sine to give the required answer

$$T = \frac{2\pi t}{\sin^{-1}(\theta e^{bt}/A)}.$$

Substitute the given values to give

$$T = \frac{0{\cdot}2 \times \pi}{\sin^{-1}(0{\cdot}1113)}.$$

From tables find the angle which has 0·1113 for its sine value, taking care to express this angle in radians, complete the calculation.

Answer: $T = 5{\cdot}58$.

(b) Let $$p_1 v_1^\gamma = p_0 v_0^\gamma \tag{1}$$

and $$p_1 v_1 = p_2 v_0 \tag{2}$$

It is required to find an expression for γ in terms of p_0, p_1 and p_2 only, hence both v_1 and v_0 must be eliminated between the two equations. Raise both sides of eqn. (2) to the power γ and *divide* the equation so formed by eqn. (1) to give

$$\frac{p_1^\gamma v_1^\gamma}{p_1 v_1^\gamma} = \frac{p_2^\gamma v_0^\gamma}{p_0 v_0^\gamma},$$

$$\text{i.e. } p_1^{(\gamma-1)} = \frac{p_2^\gamma}{p_0}.$$

Take logarithms of both sides to give

$$(\gamma - 1)\log p_1 = \gamma \log p_2 - \log p_0.$$

Now make γ the subject of the formula.

Answer: $\gamma = \log\left(\frac{p_1}{p_0}\right) \Big/ \log\left(\frac{p_1}{p_2}\right)$.

FURTHER EXAMPLES

No. 1. If

$$C = \frac{L}{2\log_e(b/a)},$$

express b in terms of e, a, L and C.

No. 2. The relation between the span, x feet, of a chain suspended at the ends of the length, s feet, is given by the equation:

$$x = k \log_e \left\{ \frac{s+\sqrt{s^2+k^2}}{k} \right\},$$

where k is a constant.

(a) Show that $s+\sqrt{s^2+k^2} = ke^{x/k}$.

(b) Find the value of s when $x = 85$ and $k = 70$. (U.E.I.)

No. 3. (a) Use the following expression for the Brinell hardness number to obtain an expression for d, the width of the impression, in terms of the other quantities.

$$B = \frac{P}{\pi . D/2 . (D-\sqrt{D^2-d^2})}.$$

(b) If

$$x = \frac{3y+5}{3-5y}$$

express

$$1+\frac{1+y}{1-y}$$

in terms of x. (E.M.E.U.)

ANSWERS

No. 1. $b = ae^{L/2C}$. No. 2. $s = 108$.

No. 3. (a) $d = \dfrac{\sqrt{4P(\pi BD^2-P)}}{\pi BD}$; (b) $\dfrac{3+5x}{4+x}$.

PROGRESSIONS

WORKED EXAMPLES

No. 1

(a) The first three terms in a progression are 1, $1\frac{3}{8}$, $1\frac{3}{4}$. Determine (i) the ninth term and (ii) the sum of the first nine terms.

(b) A lathe is to be designed to accommodate cylindrical objects having diameters from 2 to 6 in. The cutting speed, i.e. the speed of a point on the surface of the object being turned is to be 75 ft/min. Six speeds in geometrical progression are required to cover this range of work. Find these speeds correct to the nearest revolution per minute. (W.J.E.C.)

(a) Progression is an arithmetical one with first term $a = 1$ and common difference $d = \frac{3}{8}$.

Hence

(i) Ninth term $= a+8d = 1+8\times\frac{3}{8} = 1+3 = \underline{4}$.

(ii) Sum of first nine terms $S_9 = \frac{9}{2}[a+(a+8d)] = \frac{9}{2}[1+4]$

$= \frac{9}{2}+5 = \frac{45}{2} = \underline{22\frac{1}{2}}$.

Answers: (i) 4; (ii) $22\frac{1}{2}$.

(b) Let diameter of cylindrical object be d inches.

Let turning speed be V revolutions per minute.

$\therefore$ Circumference of cylinder, i.e. length of cut for one revolution is

$$\pi d \text{ in.} = \frac{\pi d}{12} \text{ ft.}$$

$\therefore$ At V rev/min cutting speed is

$$\frac{V\pi d}{12} \text{ ft/min.}$$

Hence

$$\frac{V\pi d}{12} = 75, \text{ since cutting speed must be 75 ft/min.}$$

When $d = 2$ in. $\quad V = \dfrac{75 \times 6}{\pi} = \dfrac{450}{\pi}$ rev/min.

When $d = 6$ in. $\quad V = \dfrac{150}{\pi}$ rev/min.

There are to be six speeds in geometric progression, where the first term, $a = 150/\pi$ and the sixth term $ar^5 = 450/\pi$, where r is the common ratio of the progression.

Dividing the sixth term by the first term gives

$$\frac{ar^5}{a} = r^5 = \frac{450}{\pi} \div \frac{150}{\pi} = \frac{450}{150} = 3$$

$$\underline{r = \sqrt[5]{3}.}$$

$$\text{First speed } a = \frac{150}{\pi} = 47{\cdot}74 \simeq \underline{48 \text{ rev/min.}}$$

No.	Log
150	2·1761
π	0·4972
47·74	1·6789

Term		Value	Log
	r^5	3	0·4771
	r	$\sqrt[5]{3}$	0·0954
1st	a	47·74	1·6789
2nd	ar	59·47	1·7743
3rd	ar^2	74·08	1·8697
4th	ar^3	92·28	1·9651
5th	ar^4	114·9	2·0605
6th	ar^5	143·2	2·1559

Check on working:

$$\text{sixth term} = \frac{450}{\pi} \text{ rev/min.}$$

No.	Log
450	2·6532
π	0·4972
←—CHECK———→	2·1560

Required speeds, correct to the nearest revolution per minute, are:

$$\underline{48, 59, 74, 92, 115 \text{ and } 143 \text{ rev/min.}}$$

Notes

(1) In part (b) since all six terms of the progression are required, the setting out of the problem is important if a lot of superfluous work is to be avoided. Having found the first term and the common ratio, subsequent terms can be found easily by ordered use of logarithms.

The logarithms of a and r added together give the logarithm of ar, add the logarithm of r on to this to obtain the logarithm of ar^2, add the logarithm of r on to this to obtain the logarithm of ar^3, and so on, until the logarithm of ar^5 is obtained.

(2) A check on the answers is available since the sixth term can be calculated directly or "built up" from the first term.

No. 2

(a) An arithmetic progression consists of fifteen terms of which the first is 5 and the last is 12. Find the common difference and the sum of fifteen terms.

(b) The annual output of a certain gold mine decreases every year by 8% of its amount during the previous year. If the first year's output was £400,000, find

(i) the output in the tenth year,

(ii) the total output in the first 10 years. Show that the total possible output cannot exceed £5,000,000. (E.M.E.U.)

(a) Arithmetic progression with first term $a = 5$.

Number of terms $n = 15$.

Let the common difference be d.

$$\text{Fifteenth term} = a+14d = 12$$
$$\therefore \quad 5+14d = 12$$
$$\therefore \quad 14d = 7$$
$$\therefore \quad d = \tfrac{1}{2}.$$

Sum of fifteen terms,

$$S_{15} = \frac{15}{2}(2a+14d) = \frac{15}{2}(10+7) = \frac{15\times 17}{2} = 127\tfrac{1}{2}.$$

Answers: common difference is $\frac{1}{2}$; sum of fifteen terms is $127\frac{1}{2}$.

(b) Annual output decreases every year by 8% of previous year's output, i.e. annual output is 92% of previous year's output.

Hence the annual outputs for subsequent years form a geometric progression with first term a = £400,000 and common ratio $r = \frac{92}{100}$,

$$\text{i.e. } a = 4\times 10^5 \quad \text{and} \quad r = 92\times 10^{-2},$$

(i) The output in the tenth year = tenth term = ar^9

No.	Log
92	$1{\cdot}9638 \times 9$
$4{\cdot}723 \times 10^{17}$	17·6742

$$= 4 \times 10^5 \times (92)^9 \times 10^{-18}$$
$$= 4 \times 10^{-13} \times 4{\cdot}723 \times 10^{17}$$
$$= 4 \times 47230$$
$$= 188{,}920.$$

Output in the tenth year is £189,000.

(ii) The total output in the first 10 years = sum of first ten terms.

$$= \frac{a(1-r^{10})}{1-r}$$

No.	Log
92	$1{\cdot}9638 \times 10$
$4{\cdot}345 \times 10^{19}$	19·6380

$$= \frac{4 \times 10^5 \times [1-(92)^{10} \times 10^{-20}]}{8 \times 10^{-2}}$$
$$= \frac{10^7 \times (1-4{\cdot}345 \times 10^{-1})}{2}$$
$$= \frac{10^6 \times 5{\cdot}655}{2} = 10^6 \times 2{\cdot}8275.$$

$\therefore$ Total output in the first 10 years is £2,830,000.

Since the progression is a geometric progression with common ratio less than one, a sum to infinity exists, given by the formula

$$S_\infty = \frac{a}{1-r}.$$

Since the sum of any number of terms cannot exceed the sum to infinity, the total possible output cannot exceed

$$£\frac{4 \times 10^5}{1-\frac{92}{100}},$$

i.e. $$£\frac{4 \times 10^5}{8 \times 10^{-2}} = £\tfrac{1}{2} \times 10^7,$$

i.e. £5,000,000.

Note. The sum to infinity of a geometric progression only exists if the common ratio is *numerically less* than 1 (i.e. $-1 < r < +1$).

HINTED EXAMPLES

No. 1. (a) When a ball is dropped on to a horizontal surface it rebounds to 0·75 of the height from which it falls. If it was dropped from a height of 5 ft, calculate by using your knowledge of geometrical progressions, the total distance the ball has travelled when it strikes the surface for the fifth time.

(b) In an arithmetical progression the fifth term is twice the second and the ninth term is 24. Find the first term and the common difference. (U.E.I.)

(a) Subsequent heights form a geometric progression with first term 5 and common ratio $0{\cdot}75 = \frac{3}{4}$.

The total distance the ball travels depends on S_5, the sum of the first five terms of the progression. The terms of the progression give the actual height of each bounce, so the total distance travelled is $(2 \times S_5 - 5)$ ft, since for every term except the first one, when it only falls, the ball has to cover the distance twice, once up, and once down.

Answer: 25·5 ft.

(b) Let first term be a and common difference be d.

Then
$$\text{fifth term} = a+4d$$
$$\text{second term} = a+d$$
$$\text{ninth term} = a+8d$$

Putting the fifth term equal to twice the second, i.e. $a+4d = 2(a+d)$, and the ninth term equal to 24, obtain a pair of simultaneous equations in a and d. Solve the equations.

Answers: $a = 4{\cdot}8$ and $d = 2{\cdot}4$.

No. 2. (a) The fourth term of an arithmetic progression is 54·5 and the sixteenth term is 72·5. Calculate (i) the first term, (ii) the sum of the first sixteen terms.

(b) The profit of a business is expected to decrease each year by one-tenth of the profit of the previous year. If the profit during the year 1959 was £2000, determine (i) the expected profit during the year 1969, (ii) the total profit expected to be made from 1959 to 1969, inclusive.

(a) Let the first term be a and common difference d. .

Then fourth term $= a+3d = 54{\cdot}5$

and sixteenth term $= a+15d = 72{\cdot}5$.

(i) Solve the simultaneous equations.

Answer: $a = 50$.

(ii) For sum of first sixteen terms

use formula $$S^n = \frac{n}{2}(\text{first term} + \text{last term}).$$

Answer: $S_{16} = 980$.

(b) Since profit is expected to decrease each year by one-tenth of the profit of the previous year, the profits in successive years will be in geometric progression with common ratio 0·9.

(i) The first term of this progression, for year 1959, is £2000. The profit expected in 1969 will be the eleventh term of the progression.

Answer: £697.

(ii) The total profit expected to be made from 1959 to 1969, inclusive, is the sum of the first eleven terms of the progression.

Answer: £13,700.

FURTHER EXAMPLES

No. 1. The sum of the first fifteen terms of an arithmetical progression is 270 and the common difference is $1\frac{1}{2}$. Calculate the first and fifteenth terms.

No. 2. (a) A rod 85 in. long is to be cut into seventeen pieces each one $\frac{1}{4}$ in. longer than the previous one. Determine the length of the longest piece.

(b) Find the difference between the sum to infinity and the sum to the first five terms of the progression 9, 3, 1, ...

No. 3. The cutting speed of a lathe is given by the formula

$$\frac{n.\pi.d}{12} \text{ ft/min},$$

where d is the diameter of the work in inches and n revolutions per minute is the spindle speed. The lathe operates on a range of work from 1 to 10 in. diameter. Calculate the highest and lowest spindle speeds if the cutting speed is 70 ft/min. If the lathe has eight speeds ranged in geometric progression, calculate the second lowest speed. (U.L.C.I.)

No. 4. A drilling machine has a range of six speeds. The lowest speed is 450 rev/min and the fourth speed is 1050 rev/min. Calculate the six speeds for the drilling machine if they are in geometrical progression. (U.E.I.)

ANSWERS

No. 1. $7\frac{1}{2}$, $28\frac{1}{2}$. No. 2. (a) 7 in.; (b) $\frac{1}{18}$.

No. 3. 267; 26·7; 37·2. No. 4. 450, 597, 792, 1050, 1390, 1850 rev/min.

BINOMIAL EXPANSIONS

WORKED EXAMPLES

No. 1

Expand $(2x-3)^5$ by the binomial theorem reducing each term to its simplest form. (U.L.C.I.)

When n is a positive whole number the binomial theorem states that:

$$(a+b)^n = a^n + na^{n-1}.b + \frac{n(n-1)}{1.2}.a^{n-2}.b^2 + \ldots + b^n.$$

Hence $(2x-3)^5 = (2x)^5 + 5(2x)^4.(-3) + \frac{5.4}{1.2}.(2x)^3.(-3)^2$

$$+\frac{5.4.3}{1.2.3}.(2x)^2.(-3)^3 + \frac{5.4.3.2}{1.2.3.4}.(2x).(-3)^4 + (-3)^5.$$

Answer: $(2x-3)^5 = 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243.$

No. 2

Using the binomial theorem (i) expand $(x^2 - 3/x)^6$; (ii) determine the fifth term of $(2a - b/3)^7$. (U.E.I.)

(i) Using the general expansion of the binomial theorem above, and putting $a = x^2$ and $b = -3/x$:

$$\left(x^2 - \frac{3}{x}\right)^6 = (x^2)^6 + 6(x^2)^5.\left(-\frac{3}{x}\right) + \frac{6.5}{1.2}.(x^2)^4.\left(-\frac{3}{x}\right)^2$$

$$+\frac{6.5.4}{1.2.3}.(x^2)^3.\left(-\frac{3}{x}\right)^3 + \frac{6.5.4.3}{1.2.3.4}.(x^2)^2.\left(-\frac{3}{x}\right)^4$$

$$+\frac{6.5.4.3.2}{1.2.3.4.5}.(x^2).\left(-\frac{3}{x}\right)^5 + \left(-\frac{3}{x}\right)^6.$$

Answer: $\left(x^2 - \frac{3}{x}\right)^6 = x^{12} - 18x^9 + 135x^6 - 540x^3 + 1215 - \frac{1458}{x^3} + \frac{729}{x^6}.$

(ii) The $(r+1)$th term of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3.\ldots.(r-1).r}.a^{n-r}.b^r.$$

Hence fifth term of

$$\left(2a-\frac{b}{3}\right)^7 = \frac{7.6.5.4}{1.2.3.4}(2a)^3.\left(-\frac{b}{3}\right)^4 = 35\times 8a^3.\frac{b^4}{81}.$$

Answer: fifth term of $\left(2a-\frac{b}{3}\right)^7 = \frac{280.a^3.b^4}{81}$.

No. 3

Use the binomial theorem to expand $(1-x)^5$. With the series obtained evaluate accurately (to 4 places of decimals) $0{\cdot}98^5$. (U.L.C.I.)

$$(1-x)^5 = 1+5(-x)+\frac{5.4}{1.2}.(-x)^2+\frac{5.4.3}{1.2.3}.(-x)^3+\frac{5.4.3.2}{1.2.3.4}.(-x)^4+(-x)^5.$$

Answer: $(1-x)^5 = 1-5x+10x^2-10x^3+5x^4-x^5$.

$$\begin{aligned}(0{\cdot}98)^5 &= (1-0{\cdot}02)^5\\ &= 1-5(0{\cdot}02)+10(0{\cdot}02)^2-10(0{\cdot}02)^3+5(0{\cdot}02)^4-(0{\cdot}02)^5\\ &= 1-0{\cdot}1+0{\cdot}0040-0{\cdot}000080+\ldots\\ &= 1{\cdot}004-0{\cdot}10008\\ &= 0{\cdot}90392.\end{aligned}$$

Answer: $(0{\cdot}98)^5 = 0{\cdot}9039$ (correct to 4 decimal places).

Notes

(1) When raising a decimal number to a power the number of decimal places in the answer is equal to the number of places in the number multiplied by the power; e.g. $(0{\cdot}02)^3$ will have $2\times 3 = 6$ decimal places in the answer.

(2) When an answer is wanted correct to a certain number of decimal places, working must be carried far enough to obtain the figure in the next decimal place; e.g. when an answer is required correct to 4 decimal places the fifth decimal place must be shown.

No. 4

(a) If x is so small that its cube and higher powers may be neglected, find the values of A, B and C if

$$\frac{\sqrt{4-3x}}{3-x^2} = A+Bx+Cx^2+\ldots$$

(b) Using the binomial theorem, evaluate $\sqrt[3]{8{\cdot}24}$, correct to 4 decimal places. Deduce the cube root of 0·00824, correct to 4 significant figures. (U.E.I.)

(a) $$\frac{\sqrt{4-3x}}{3-x^2} = (4-3x)^{\frac{1}{2}}.(3-x^2)^{-1} = \tfrac{2}{3}(1-\tfrac{3}{4}x)^{\frac{1}{2}}.(1-x^2/3)^{-1}$$

Hence expanding by binomial theorem, and neglecting terms in x^3 and higher powers,

$$(1-\tfrac{3}{4}x)^{\frac{1}{2}} = 1+\tfrac{1}{2}(-\tfrac{3}{4}x)+\frac{\frac{1}{2}.(-\frac{1}{2})}{1.2}.(-\tfrac{3}{4}x)^2+\ldots = 1-\frac{3x}{8}-\frac{9x^2}{128}-\ldots$$

$$\left(1-\frac{x^2}{3}\right)^{-1} = 1+(-1).\left(-\frac{x^2}{3}\right)+\ldots = 1+\frac{x^2}{3}+\ldots$$

$$\therefore \quad \frac{\sqrt{4-3x}}{3-x^2} = \frac{2}{3}\left\{1-\frac{3x}{8}-\frac{9x^2}{128}-\ldots\right\}.\left\{1+\frac{x^2}{3}+\ldots\right\} \quad \text{(A)}$$

$$= \frac{2}{3}\left\{1-\frac{3x}{8}-\frac{9x^2}{128}+\frac{x^2}{3}+\ldots\right\}$$

$$= \frac{2}{3}-\frac{x}{4}+\frac{2}{3}\left(\frac{1}{3}-\frac{9}{128}\right).x^2+\ldots$$

$$\therefore \quad \frac{\sqrt{4-3x}}{3-x^2} = \frac{2}{3}-\frac{x}{4}+\frac{101}{576}x^2+\ldots$$

Hence comparing with

$$\frac{\sqrt{4-3x}}{3-x^2} = A+Bx+Cx^2.$$

Answer: $A = \frac{2}{3}$; $B = -\frac{1}{4}$; $C = \frac{101}{576}$.

(b) $\sqrt[3]{8{\cdot}24} = (8+0{\cdot}24)^{\frac{1}{3}} = [8(1+0{\cdot}03)]^{\frac{1}{3}} = 2(1+0{\cdot}03)^{\frac{1}{3}}$

$$= 2.\left\{1+\tfrac{1}{3}(0{\cdot}03)+\frac{\frac{1}{3}.(-\frac{2}{3})}{1.2}.(0{\cdot}03)^2 + \frac{\frac{1}{3}.(-\frac{2}{3})(-\frac{5}{3})}{1.2.3}(0{\cdot}03)^3+\ldots\right\}$$

$$= 2.\{1+0{\cdot}01-0{\cdot}0001+0{\cdot}00000167-\ldots\}$$

$$= 2\times 1{\cdot}00990167 = 2{\cdot}01980334.$$

Answer: $\sqrt[3]{8{\cdot}24} = 2{\cdot}0198$ (correct to 4 decimal places).

$$\sqrt[3]{0{\cdot}00824} = \sqrt[3]{\frac{8{\cdot}24}{10^3}} = \frac{\sqrt[3]{8{\cdot}24}}{10} = 0{\cdot}20198.$$

Answer: $\sqrt[3]{0{\cdot}00824} = 0{\cdot}2020$ (correct to 4 sig. fig.).

Notes

(1) The simplest form of the binomial theorem to use is the expansion of $(1+x)^n$, hence the working is easier if the bracket to be expanded is altered to this form. This is done by extracting the appropriate factor, which must be operated on by the power to which the bracket is raised.

$$[4-3x]^{\frac{1}{2}} = [4(1-\tfrac{3}{4}x)]^{\frac{1}{2}} = (4)^{\frac{1}{2}}\times\left(1-\frac{3x}{4}\right)^{\frac{1}{2}} = 2\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}.$$

(2) When terms in x of a certain order and higher can be ignored, not only are the initial expansions curtailed but when a multiplication of two expansions occurs (as at stage (A)), any product which yields a power of x equal to or higher than the order to be ignored can also be disregarded.

No. 5

(a) Show that the binomial expansion of

$$\frac{1}{(3+4x)^3}$$

may be written as

$$\frac{1}{27}.\left\{1-4x+\frac{32x^2}{3}+\ldots\right\}$$

and state for what values of x the expansion is valid.

(b) The mass of air flowing through a nozzle may be calculated from

$$m = \frac{k.h.D^2}{\sqrt{T}}.$$

Using the binomial theorem, find the appropriate percentage error in m due to errors of -2% in D and $+1{\cdot}5\%$ in T. (W.J.E.C.)

(a) The binomial theorem states that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}.x^2+\ldots$$

for all values of x if n is a positive whole number, but only for $-1<x<+1$ if n is fractional and/or negative.

$$\frac{1}{(3+4x)^3} = (3+4x)^{-3} = \left[3\left(1+\frac{4x}{3}\right)\right]^{-3} = 3^{-3}.\left(1+\frac{4x}{3}\right)^{-3}$$

$$= \frac{1}{27}\left\{1+(-3).\left(\frac{4x}{3}\right)+\frac{(-3).(-4)}{1.2}.\left(\frac{4x}{3}\right)^2+\ldots\right\}$$

$$= \frac{1}{27}.\left\{1-4x+\frac{32x}{4}+\ldots\right\}.$$

Expansion is valid in the range $-1<4x/3<+1$, i.e. valid for $-\frac{3}{4}<x<+\frac{3}{4}$.

(b) Let δm, δD and δT be the errors in m, D and T respectively.

Then $$m = \frac{k.h.D^2}{\sqrt{T}} = k.h.D^2.T^{-\frac{1}{2}}$$

and $$m+\delta m = k.h.(D+\delta D)^2.(T+\delta T)^{-\frac{1}{2}}.$$

But $$\delta D = -\frac{2}{100}D = -0{\cdot}02D \text{ and } \delta T = +\frac{1{\cdot}5}{100}T = +0{\cdot}015T.$$

Hence $$m+\delta m = kh(D-0{\cdot}02D)^2.(T+0{\cdot}015T)^{-\frac{1}{2}}$$
$$= khD^2.T^{-\frac{1}{2}}.(1-0{\cdot}02)^2.(1+0{\cdot}015)^{-\frac{1}{2}}$$
$$= m.\{1+2(-0{\cdot}02)+(-0{\cdot}02)^2\}.\{1+(-\tfrac{1}{2})(0{\cdot}015)+\ldots\}$$
$$= m.\{1-0{\cdot}04+0{\cdot}0004\}.\{1-0{\cdot}0075\}$$
$$= m(1-0{\cdot}04+0{\cdot}0004-0{\cdot}0075+0{\cdot}0003)$$
$$= m(1{\cdot}0007-0{\cdot}0475) = m(1-0{\cdot}0468)$$

$$\therefore\ \delta m = -0{\cdot}0468m = -\frac{4{\cdot}68m}{100}.$$

Answer: the error in m is $-4{\cdot}7\%$.

HINTED EXAMPLES

No. 1. (a) Write down and simplify the first four terms in the expansion of

(i) $(1+x)^7$; (ii) $\dfrac{1}{\sqrt{(1-x)^3}}$.

(b) Show that the expansion of $(1+x)^7.(1+2x+3x^2)$ in ascending powers of x begins with the terms $1+9x+38x^2$.

(c) Use the binomial theorem to determine the fourth root of 0·98 to 4 decimal places. (W.J.E.C.)

(a) (i) Straightforward expansion to give

$$(1+x)^7 = 1+7x+21x^2+35x^3+\ldots$$

(ii) $\dfrac{1}{\sqrt{(1-x)^3}} = (1-x)^{-\frac{3}{2}}$, then expand.

Answer: $\dfrac{1}{\sqrt{(1-x)^3}} = 1+\dfrac{3x}{2}+\dfrac{15x^2}{8}+\dfrac{35x^3}{16}+\ldots$

(b) Use the expansion for $(1+x)^7$ found in (a) (i), and multiply it by $(1+2x+3x^2)$ ignoring products that yield terms in x^3 or higher powers. Finally, collect up terms.

Answer: $(1+x)^7.(1+2x+3x^2) = 1+9x+38x^2+\ldots$

(c) $\sqrt[4]{0{\cdot}98} = (0{\cdot}98)^{\frac{1}{4}} = (1-0{\cdot}02)^{\frac{1}{4}}$, then expand working to 5 decimal places. Finally, collect up terms.

Answer: $\sqrt[4]{0{\cdot}98} = 0{\cdot}9950$ (correct to 4 decimal places).

No. 2. (a) Write down the first five terms in the expansion of $(1-x)^{-n}$.

(b) The quantities η, s, m, k and T are related by the expansion

$$\eta = \frac{2}{3s^2}\sqrt{\frac{mkT}{\pi^3}}.$$

Using the binomial approximations determine the percentage error in η if s is 2% too small, m is 1% too large and T is 2% too small. (U.E.I.)

(a) Answer: $(1-x)^{-n} = 1+nx+\frac{1}{2}n(n+1).x^2+n/6(n+1)(n+2).x^3$
$+n/24(n+1)(n+2)(n+3)x^4$.

(b) $$\eta = \frac{2}{3s^2}\sqrt{\frac{mkT}{\pi^3}} = \frac{2}{3}\cdot\frac{s^{-2}.m^{\frac{1}{2}}.k^{\frac{1}{2}}.T^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$$

Let $\delta\eta$, δs, δm and δT be the errors in η, s, m and T respectively, then

$$\eta+\delta\eta = \frac{2}{3}\cdot\frac{(s+\delta s)^{-2}.(m+\delta m)^{\frac{1}{2}}.k^{\frac{1}{2}}.(T+\delta T)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$$

Since

$$\delta s = \frac{-2}{100}s = -0{\cdot}02s,\ \delta m = \frac{+1}{100}m = 0{\cdot}01m \text{ and } \delta T = \frac{-2}{100}T = -0{\cdot}02T.$$

$$\eta+\delta\eta = \frac{2}{3}\cdot\frac{s^{-2}.(1-0{\cdot}02)^{-2}.m^{\frac{1}{2}}.(1+0{\cdot}01)^{\frac{1}{2}}.k^{\frac{1}{2}}.T^{\frac{1}{2}}.(1-0{\cdot}02)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}$$

$$= \eta.(1+0{\cdot}01)^{\frac{1}{2}}.(1-0{\cdot}02)^{-\frac{1}{2}}.$$

Expand both brackets and multiply out working to 4 decimal places, then subtract η from both sides of the equation to obtain $\delta\eta$ in terms of η. Finally, express as a percentage.

Answer: error in η is an increase of 3·6%.

FURTHER EXAMPLES

No. 1. Expand $[(1+x)(1-4x)]^{-\frac{1}{2}}$ as a series of ascending powers of x as far as the term in x^2.

No. 2. Using the binomial theorem, show that $(1-x)^{\frac{1}{2}}$ and $(1+\frac{1}{2}x)^{-1}$ are approximately equal when x^2 and higher powers are neglected.

No. 3. (a) Expand to four terms and simplify (i) $(1+2x)^{\frac{1}{2}}$; (ii) $(1-x)^{-2}$.

(b) Use the expansion obtained for $(1+2x)^{\frac{1}{2}}$ to find the value of $\sqrt{1{\cdot}02}$ correct to 5 significant figures.

(c) Find the coefficient of x^2 in the expansion of $(1+2x)^{\frac{1}{2}}.(1-x)^{-2}$. (W.J.E.C.)

No. 4. Using the binomial theorem:

(i) Calculate the seventh term of the expansion of $(2x-y/3)^{11}$.

(ii) Expand $(1+x/3)^6$ as far as the term containing x^3.

(iii) Evaluate $(2{\cdot}03)^5$ correct to 3 significant figures. (U.E.I.)

No. 5. (i) Write down the first five terms of the expansion of $(8+3x)^{\frac{1}{2}}$ in ascending powers of x and state the range of values of x for which the expansion is valid.

(ii) In the formula $P = cd^{5\cdot5}.n^{3\cdot5}$ find the error in P due to an error of $+2\%$ in d and -1% in n. (U.E.I.)

ANSWERS

No. 1. $1+\frac{3x}{2}+\frac{43x^2}{8}$.

No. 3. (a) (i) $1+x-\frac{x^2}{2}+\frac{x^3}{2}$; (ii) $1+2x+3x^2+4x^3$.

(b) 1·0100. (c) $\frac{9}{2}$.

No. 4. (i) $\frac{4928}{243}x^5y^6$; (ii) $1+2x+\frac{5x^2}{3}+\frac{20x^3}{27}$; (iii) 34·5.

No. 5. (i) $2+\frac{x}{4}-\frac{x^2}{32}+\frac{5x^3}{768}-\frac{5x^4}{3072}$ valid in range $-\frac{8}{3}<x<+\frac{8}{3}$;

(ii) 7·5% increase.

ALGEBRAIC EQUATIONS

WORKED EXAMPLES

No. 1

(a) If $3t^2-9t = 8$, calculate the values of t correct to 1 decimal place.

(b) Solve $$\frac{2}{x-3}-\frac{3}{x-2}=\frac{2}{x}.$$

(c) Solve the following simultaneous equations

$$x^2-4y^2 = 9;\ x-6y = -7.$$ (U.E.I.)

(a) $$3t^2-9t = 8$$

$$\therefore\ 3t^2-9t-8 = 0$$

Hence using the formula for a quadratic equation

$$t = \frac{9\pm\sqrt{81+96}}{6} = \frac{9\pm\sqrt{177}}{6}$$

$$\therefore\ t = \frac{9\pm 13{\cdot}3}{6} = \frac{22{\cdot}3}{6} \text{ or } \frac{-4{\cdot}3}{6}$$

$$\therefore\ t = 3{\cdot}71 \text{ or } -0{\cdot}71.$$

Answer: $t = 3{\cdot}7$ or $-0{\cdot}7$ (correct to 1 decimal place).

(b) $$\frac{2}{x-3}-\frac{3}{x-2}=\frac{2}{x}$$

$$\therefore\ \frac{2(x-2)-3(x-3)}{(x-3)(x-2)}=\frac{2}{x}$$

$$\therefore\ \frac{-x+5}{(x-3)(x-2)}=\frac{2}{x}$$

$$\therefore\ x(5-x) = 2(x-3)(x-2)$$

$$5x-x^2 = 2x^2-10x+12$$

$$\therefore\ 3x^2-15x+12 = 0$$

$$\therefore\ x^2-5x+4 = 0$$

$$\therefore\ (x-4)(x-1) = 0.$$

Answer: $x = 4$ or $x = 1$.

(c) $$x-6y = -7 \quad (1)$$

$$x^2-4y^2 = 9 \quad (2)$$

From eqn. (1) $x = 6y-7$.

$\therefore$ Substituting in eqn. (2), $(6y-7)^2-4y^2 = 9$

$\therefore$ $36y^2-84y+49-4y^2 = 9$

$\therefore$ $32y^2-84y+40 = 0$

$\therefore$ $8y^2-21y+10 = 0$

$\therefore$ $(y-2)(8y-5) = 0.$

Answer: $y = 2$ or $y = \frac{5}{8}$.

From eqn. (1) when $y = 2$, $x-12 = -7 \quad \therefore x = 5$

and when $y = \frac{5}{8}$, $x-\frac{30}{8} = -7$

$$\therefore \quad x = -7+\frac{15}{4} = -\frac{13}{4}.$$

Answers: $x = 5$ and $y = 2$ or $x = -\frac{13}{4}$ and $y = \frac{5}{8}$.

Note. In part (b), when the product of two factors is zero, one or the other must be equal to zero. Hence since $(x-4)(x-1) = 0$, *either* $x-4 = 0$ *or* $x-1 = 0$, giving the solution:

either $x = 4$ *or* $x = 1$.

No. 2

(a) An experimental law was thought to be

$$y = A+Bx+Cx^2.$$

The readings given below were taken and assuming the law to be true find A, B and C.

y	27	2	6
x	-6	-1	1

(b) Solve the following equation for real values of x:

$$\sqrt{x+3}-\sqrt{x-6} = 1. \quad \text{(U.L.C.I.)}$$

(a) Substitution of the given pairs of values into the assumed law leads to the set of equations:

$$27 = A - 6B + 36C \quad (1)$$

$$2 = A - B + C \quad (2)$$

$$6 = A + B + C \quad (3)$$

Eqn. (3) − eqn. (2), $4 = 2B \quad \therefore B = 2.$

Substitute $B = 2$ in eqn. (1) $\quad 27 = A - 12 + 36C$

$$\therefore \quad 39 = A + 36C \quad (4)$$

Substitute $B = 2$ in eqn. (2) $\quad 2 = A - 2 + C$

$$\therefore \quad 4 = A + C \quad (5)$$

Eqn. (4) − eqn. (5) $\quad 35 = 35C. \quad \therefore C = 1.$

Substitute $C = 1$ in eqn. (5) $\quad 4 = A + 1. \quad \therefore A = 3.$

Answer: $A = 3$, $B = 2$ and $C = 1$.

(b) $$\sqrt{x+3} - \sqrt{x-6} = 1.$$

$$\sqrt{x+3} = 1 + \sqrt{x-6}$$

$\therefore$ Squaring both sides

$$x + 3 = 1 + 2.\sqrt{x-6} + (x-6)$$

$$8 = 2.\sqrt{x-6}$$

$$\therefore \quad 4 = \sqrt{x-6}$$

Squaring both sides $\quad 16 = x - 6.$

Answer: $x = 22$.

Check in original equation L.H.S. $= \sqrt{22+3} - \sqrt{22-6} = \sqrt{25} - \sqrt{16}$

$$= 5 - 4 = 1 = \text{R.H.S.}$$

Notes

(1) In part (a) since eqns. (2) and (3) were used to find the value of B, the first substitution of B must be made in eqn. (1) since all three equations must come into the calculation.

(2) In part (b) a check of the answer is required since squaring can introduce non-allowable solutions, i.e. in this case the same result would be obtained from the equation $\sqrt{x+3} + \sqrt{x-6} = 1$ but the check would disallow the result.

No. 3

Solve the equation $10^{2x}-10^{x+1}+25 = 0$.

$$10^{2x}-10^{x+1}+25 = 0$$
$$\therefore \quad (10^x)^2-10(10^x)+25 = 0.$$

Put $y = 10^x$ then equation becomes

$$y^2-10y+25 = 0$$
$$\therefore \quad (y-5)^2 = 0$$
$$\therefore \quad y = 5$$

i.e. $$10^x = 5.$$

Taking logarithms to the base 10, $x = \log_{10} 5 = 0{\cdot}6990$.

Answer: $x = 0{\cdot}699$ (correct to 3 sig. fig.).

No. 4

Solve the following equations:

$$x^2+y^2 = 27$$
$$xy = 11 \qquad \text{(U.E.I.)}$$

$$x^2+y^2 = 27 \qquad (1)$$
$$xy = 11 \qquad (2)$$

From eqn. (2) $$2xy = 22 \qquad (3)$$

Eqn. (1)+eqn. (3) $$x^2+2xy+y^2 = 49$$
$$\therefore \quad (x+y)^2 = 49$$
$$\therefore \quad x+y = \pm 7 \qquad (4)$$

Eqn. (1)−eqn. (3) $$x^2-2xy+y^2 = 5$$
$$\therefore \quad (x-y)^2 = 5$$
$$\therefore \quad x-y = \pm\sqrt{5} = \pm 2{\cdot}236 \qquad (5)$$

Since eqns. (4) and (5) are both "plus" or "minus" there are four cases to consider.

Case (1)
$$x+y = 7$$
$$x-y = 2{\cdot}236$$

Adding $2x = 9{\cdot}236$ $\quad \therefore \quad x = 4{\cdot}618$

Subtracting $2y = 4{\cdot}764$ $\quad \therefore \quad y = 2{\cdot}382$

Case (2)

$$x+y = -7$$
$$x-y = 2{\cdot}236$$

Adding $\quad 2x = -4{\cdot}764 \qquad \therefore\ x = -2{\cdot}382$

Subtracting $\quad 2y = -9{\cdot}236 \qquad \therefore\ y = -4{\cdot}618$

{*Note.* Cases (3) and (4) could be solved by exactly the same method as cases (1) and (2), but a quicker method is available.}

By inspection, eqns. (1) and (2) are symmetrical in x and y, hence another pair of possible sets of solutions must be:

$$x = 2{\cdot}382 \qquad x = -4{\cdot}618$$
and
$$y = 4{\cdot}618 \qquad y = -2{\cdot}382$$

Answers: either $x = 4{\cdot}62, y = 2{\cdot}38$ or $x = -2{\cdot}38, y = -4{\cdot}62$
or $x = 2{\cdot}38, y = 4{\cdot}62$ or $x = -4{\cdot}62, y = -2{\cdot}38$.

Note. Another check on the solutions is provided by eqn. (2); from which it can be seen that the product of x and y must always be positive. The solutions can be checked to ensure that this condition is satisfied.

HINTED EXAMPLES

No. 1. Find the values of x and y which satisfy simultaneously the equations $2x+3y = 9$ and $4x^2+2xy = 15$. (U.L.C.I.)

$$2x+3y = 9 \qquad (1)$$
$$4x^2+2xy = 15$$

Factorize $\quad 2x(2x+y) = 15 \qquad (2)$

Use eqn. (1) to substitute for $2x$ in terms of y in eqn. (2), multiply out and collect up terms to give the following quadratic in y:

$$2y^2-15y+22 = 0.$$

Factorize and solve to give $\quad y = 2$ or $y = 1\frac{1}{2}$.

Substitute these values in turn in eqn. (1).

Answer: $x = \frac{3}{2}, y = 2$ or $x = -1\frac{5}{4}, y = 1\frac{1}{2}$.

No. 2. Solve the following simultaneous equations for x, y and z.

$$x-y+3z = 12, \quad 2x+y-z = -7, \quad x+5y+2z = 2.$$

(W.J.E.C.)

Eliminate x between eqns. (1) and (3) and between eqns. (1) and (2) to give the following pair of simultaneous equations in y and z.

$$6y-z = -10$$

$$3y-7z = -31$$

Eliminate y between these two equations to obtain $z = 4$. Substitute for z in one of the last pair of equations to find $y = -1$.

Substituting for both y and z gives $x = -1$.

FURTHER EXAMPLES

No. 1. (a) Solve correct to two decimal places $x^2+5x-10 = 0$.

(b) Solve the simultaneous equations

$$2x+y = 1$$

$$3x^2+2xy+3 = 0.$$ (U.L.C.I.)

No. 2. Find the value of x which satisfies the equation

$$\sqrt{3-2x}+\sqrt{4-3x} = \sqrt{x+3}.$$

No. 3. Find the values of x, y and z which satisfy the simultaneous equations:

$$x+y+z = 6$$

$$2x-y+3z = 9$$

$$3x+2y-2z = 1.$$

No. 4. By means of the substitution $3^x = u$, $2^y = v$, or otherwise, solve the simultaneous equations

$$3^x-4.2^y = 10$$

$$9.2^y-3^x = 18.$$

ANSWERS

No. 1. (a) $x = 1{\cdot}53$ or $-6{\cdot}53$, (b) $x = 3, y = -5$ or $x = -1, y = 3$.

No. 2. $x = 1$. No. 3. $x = 1, y = 2, z = 3$.

No. 4. $x = 3{\cdot}17, y = 2{\cdot}49$.

DETERMINATION OF LAWS

HINTED EXAMPLES

No. 1. It is believed that the following values fit the law $T = te^{n\theta}$.

T cwt	19·00	34·25	45·8	59·48	105·50
θ radians	3·12	9·52	12·8	15·70	22·00

Test graphically if this is so and find the best values for n and t. (E.M.E.U.)

To express $T = te^{n\theta}$ in straight line form take natural logarithms of the equation, giving

$$\log_e T = \log_e t + n\theta \text{ (since } \log_e e^k = k).$$

Compare with $Y = c + mX$.

Put $Y = \log_e T$ and complete the following table.

$Y (= \log_e T)$	2·94				4·66
θ	3·12	9·52	12·80	15·70	22·00

Now plot Y against θ.

Suggested scales. Vertical: 1 inch to 1 unit from 2 to 5.
Horizontal: 1 inch to 4 units from 2 to 22.

Since the points are practically collinear it can be assumed that the given values fit the required law. Consequently, the plotted points lie on or close to a line which has equation of the form $Y = \log_e t + n\theta$ where n is the gradient of the line and $\log_e t$ is the value of Y when $\theta = 0$ radians. Take two points A and B on the line as far apart as is reasonable then gradient of the line $= n =$ $\dfrac{\text{value of } Y \text{ at } B - \text{value of } Y \text{ at } A}{\text{value of } \theta \text{ at } B - \text{value of } \theta \text{ at } A}$.

From the graph the value of n will be found to be 0·09 approx. Substituting for n, gives equation of line as $Y = \log_e t + 0{\cdot}09.\theta$. Take a convenient point on the line and substitute into the equation to find $\log_e t$, and hence t.

Answer: equation of law is $\log_e T = \log_e 14{\cdot}6 + 0{\cdot}09.\theta$

i.e. $T = 14{\cdot}6.e^{0{\cdot}09.\theta}$.

No. 2. Express the law $y = Ax + Bx^2$ in straight line form. Use this straight form to verify that the experimental results given below conform to this law and hence find the values of the two constants.

y	8·9	2·1	−1	5·1	13·8
x	−3	−2	−1	1	2

(U.L.C.I.)

To convert $y = Ax + Bx^2$ to straight line form, divide equation by x, i.e.

$$\frac{y}{x} = A + Bx.$$

Compare with $Y = c + mX$

Put $Y = y/x$ and complete the following table.

$Y(= y/x)$	−2·97				6·9
x	−3	−2	−1	1	2

Plot Y against x.

Suggested scales. Vertical: 1 inch to 2 units from −3 to 7.
Horizontal: 1 inch to 1 unit from −3 to 2.

Since the plotted points are practically collinear, the experimental results are consistent with the given law, $Y = A + Bx$, i.e.

$$\frac{y}{x} = A + Bx.$$

B is the gradient of the line of best fit and A is the value of Y when $x = 0$. Take two convenient points P and Q on the line, then gradient of line =

$$B = \frac{\text{value of } Y \text{ at } Q - \text{value of } Y \text{ at } P}{\text{value of } x \text{ at } Q - \text{value of } x \text{ at } P}$$

From the graph the value of B will be found to be 2 approx. A can be found from the graph, since $Y = A$ when $x = 0$.

Answer: equation of law is $Y = \frac{y}{x} = 3 + 2x$, i.e. $y = 3x + 2x^2$.

No. 3. The following tabulated values should obey one of the two laws,

$$y = \frac{b}{x} + a \quad \text{or} \quad y = \frac{c}{x^2} + d.$$

Determine which law applies by using a graphical method and find the values of the constants.

X	1	2	3	4	5
Y	11	5	3·89	3·5	3·32

(U.E.I.)

Complete the following table:

Y	11	5	3·89	3·5	3·32
$1/x$	1	0·5			0·2
$1/x^2$	1	0·25			0·04

Plot two graphs: (a) Y against $1/x$; and (b) Y against $1/x^2$.
Plot both graphs on the same axes taking Y vertically.

Suggested scales. Vertical: 1 inch to 2·0 units from 3 to 11.
Horizontal: 1 inch to 0·1 units from 0 to 1·0.

Graph (a) will show a marked curve and graph (b) will be practically a straight line, showing that the values are consistent with the second law

$$y = \frac{c}{x^2} + d.$$

Draw the most probable line and measure its gradient. Choose a convenient point *on* the line and use this to calculate d.

Answers: should be in the region (i.e. $\pm 0{\cdot}01$) of the following

$$\underline{c = 8{\cdot}0;\ d = 3{\cdot}0.}$$

FURTHER EXAMPLES

No. 1. The speed v of a vessel and the horse-power H are believed to be connected by a law of the form $H = a + bv^3$.

v	4	6	8	10	12	14
H	252	434	789	1375	2249	3468

The above values were found by experiment. Test if the law is true and find the probable values of a and b and thus the true law. Find also

(i) the value of H when $v = 9$.
(ii) the value of v when $H = 2000$. (U.E.I.)

No. 2. The current, i amperes, in an inductive circuit after t seconds is as indicated in the following table.

t	1	2	3	4	5
i	60	24·8	11·2	4·8	1·92

Show that i and t satisfy the relationship $i = Be^{kt}$. Find probable values of B and k and also the value of i when $t = 0$. (U.E.I.)

ANSWERS

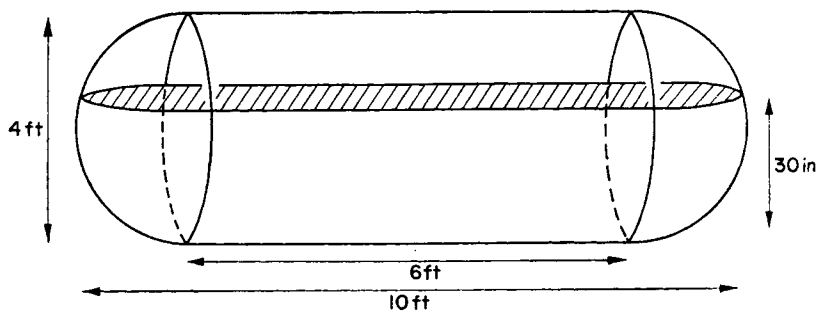
No. 1. $a = 175$, $b = 1{\cdot}2$; (i) $H = 1050$; (ii) $v = 11{\cdot}5$.
No. 2. $B = 140$, $k = -0{\cdot}85$; 140.

MENSURATION

WORKED EXAMPLES

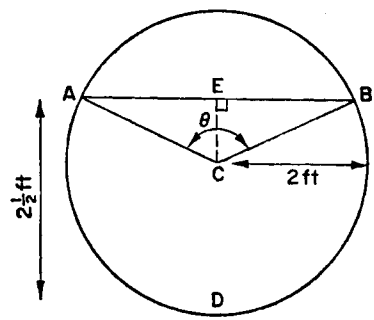
No. 1.

A fuel storage tank consists of a horizontal cylinder 6 ft long by 4 ft diameter with hemispherical ends (overall length 10 ft). Calculate the volume of liquid it contains when filled to a depth of 30 in. Express the volume in gallons. (U.L.C.I.)



The volume can be considered in two parts: (a) a horizontal cylinder 6 ft long, 4 ft diameter, filled to a depth of 30 in., and (b) the hemispherical ends combining to form a sphere of diameter 4 ft, filled to a depth of 30 in.

(a) Volume of liquid in cylinder = area of cross-section of liquid × length of cylinder.



Area of cross-section = area of sector $ACBD$ + area of $\triangle ACB$. Let angle $ACB = \theta$. Then from $\triangle CEB$

$$\cos\frac{\theta}{2} = \frac{\frac{1}{2}}{2} = \tfrac{1}{4} = 0{\cdot}25.$$

$$\therefore \quad \frac{\theta}{2} = 75\tfrac{1}{2}^\circ \quad \therefore\ \theta = 151^\circ.$$

$$\therefore \quad \text{Area sector } ACBD = \frac{(360^\circ - 151^\circ)}{360^\circ} \times \pi \times (2)^2$$

$$= \frac{209}{360} \times 4\pi = \frac{209\pi}{90}\ \text{ft}^2.$$

$$\text{Also area } \triangle ACB = \tfrac{1}{2}.2.2.\sin 151^\circ = 2.\sin 29^\circ\ \text{ft}^2.$$

$$\therefore \quad \text{Area of cross-section} = \frac{209\pi}{90} + 2.\sin 29^\circ$$

$$= \frac{20{\cdot}9 \times 3{\cdot}142}{9} + 2 \times 0{\cdot}4848\ \text{ft}^2.$$

$$\therefore \quad \text{Volume of liquid} = 6\left(\frac{20{\cdot}9 \times 3{\cdot}142}{9} + 2 \times 0{\cdot}4848\right)\ \text{ft}^3$$

$$= 43{\cdot}78 + 5{\cdot}82 = 49{\cdot}6\ \text{ft}^3.$$

(b) Volume of a cap of a sphere is given by

$$V = \frac{\pi h^2}{3}.(3r - h)$$

where r is the radius of the sphere and h is the height of the cap.

$$\therefore \quad \text{Volume of liquid} = \frac{\pi}{3}.\left(\frac{5}{2}\right)^2 \times \left(3 \times 2 - \frac{5}{2}\right) = \frac{\pi \times 25 \times 7}{24}\ \text{ft}^3$$

$$= \frac{3{\cdot}142 \times 175}{24} = 22{\cdot}9\ \text{ft}^3.$$

$\therefore$ From (a) and (b) the total volume of liquid

$$= 49{\cdot}6 + 22{\cdot}9\ \text{ft}^3$$

$$= 72{\cdot}5\ \text{ft}^3.$$

But 1 ft^3 is equivalent to $6\frac{1}{4}$ gal.

$$\therefore \quad \text{Volume of liquid} = 72{\cdot}5 \times \frac{25}{4} = 453\ \text{gal.}$$

Answer: volume of liquid in the tank is 453 gal.

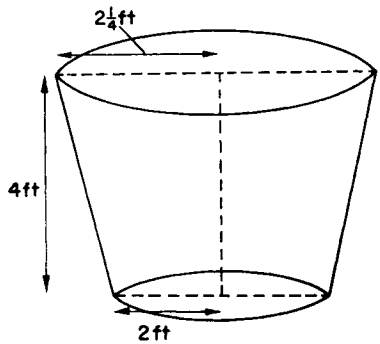
No. 2

Hollow metal hemispheres of internal and external diameter 5 and 7 in. respectively are to be cast from the molten contents of a ladle in the shape of a frustum of a cone of depth 4 ft and radii of ends 2 ft and 2 ft 3 in. Calculate how many hemispheres will be cast. (E.M.E.U.)

Volume of a single hemisphere

$$= \frac{2}{3}.\pi.\left\{\left(\frac{7}{2}\right)^3 - \left(\frac{5}{2}\right)^3\right\} = \frac{2}{3}.\pi.\left\{\frac{343}{8} - \frac{125}{8}\right\}$$

$$= \frac{2}{3}.\pi.\frac{218}{8} = \underline{\frac{109\pi}{6}\ \text{in}^3.}$$



Volume of contents of ladle

$$= \tfrac{1}{3}.\pi.h(R^2 + rR + r^2)$$

$$= \frac{1}{3}.\pi.4\left\{2^2 + 2\times\frac{9}{4} + \left(\frac{9}{4}\right)^2\right\}\ \text{ft}^3$$

$$= \frac{4}{3}\pi\left(4 + \frac{9}{2} + \frac{81}{16}\right)\ \text{ft}^3$$

$$= \frac{4}{3}\pi\times\frac{217}{16} = \frac{4\times217}{3\times16}.\pi\ \text{ft}^3$$

$$= \frac{217\times1728}{12}.\pi = \underline{217\times144\pi\ \text{in}^3.}$$

$\therefore$ Number of hemispheres that can be cast is given by

$$\frac{\text{volume of contents of ladle}}{\text{volume of a single hemisphere}},$$

i.e. number of hemispheres is

$$217\times144\pi \div \frac{109\pi}{6} = \frac{217\times144\times6\pi}{109\pi}$$

$$= \frac{217\times864}{109} = \underline{1720.}$$

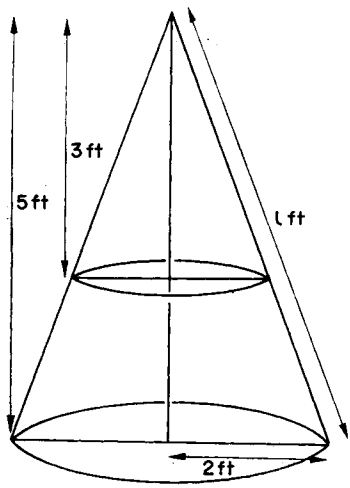
No.	Log
217	2·3365
864	2·9365
	5·2730
109	2·0374
1720	3·2356

Answer: the number of hemispheres that can be made is 1720.

Note. It is often better not to substitute the numerical value of π, as in this example.

HINTED EXAMPLES

No. 1. A right circular solid cone has a perpendicular height of 5 ft and has a base radius of 2 ft. Calculate its total surface area. If a frustum is formed by removing the upper 3 ft of the cone calculate, (a) the volume and (b) the total surface area of the frustum. (U.L.C.I.)



Let total surface area be S square feet. Then S is given by $S = \pi.rl+\pi.r^2$ where r is radius of base and l is slant height.

Calculate l by Pythagoras' theorem and evaluate S.

Answer: 46·4 ft^2.

The frustum can be treated as the difference between a cone of height 5 ft and base radius of 2 ft and an exactly similar cone of height 3 ft.

(a) Find the volume V of the large cone from the formula $V = \frac{1}{3}\pi r^2 h$. Use "the volumes of similar solids are in the same ratio as the cubes of corresponding lengths".

Hence if V is volume of large cone and V_1 is volume of small cone

$$\frac{V_1}{V} = \frac{3^3}{5^3}.$$

Since V has already been found, V_1 can be calculated. Finally, volume of frustum $= V-V_1$.

Answer: 16·4 ft^3.

(b) The total surface area of the frustum is given by $S-L+A$ where S is the total surface area of the large cone, L is the area of the slant surface of the small cone and A is the area of the top of the frustum (i.e. A is the area of the base of the small cone).

Also, if S_1 is the total surface area of the small cone, $S_1 = L+A$. Hence total surface area of frustum is, eliminating L, $S-S_1+2A$. Use "the areas of similar figures are in the same ratio as the squares of corresponding lengths". Hence

$$\frac{S_1}{S} = \frac{3^2}{5^2}.$$

Since S has already been found in the first part of the problem, S_1 can be calculated.

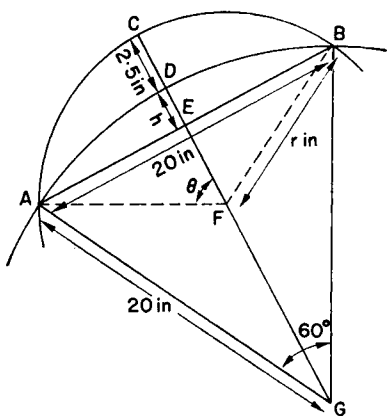
Also $A = \pi r_1^2$, where r_1 is the radius of the base of the small cone, and

$$\frac{r_1}{3} = \frac{2}{5}$$

by similar triangles. Hence A can be found and finally the area of the frustum calculated.

Answer: 38·7 ft^2.

No. 2. The inner arc of a crescent is 20 in. radius and the length of the common chord is 20 in. If the greatest width of the crescent is 2·5 in., calculate the radius of the outer arc and the area of the crescent. (U.L.C.I.)



Let the radius of the outer arc be r inches. The crescent is formed by subtracting a segment cut off from a circle of radius 20 in. (by a chord of length 20 in.) from a segment cut off from a circle of radius r inches (also by a chord of length 20 in.), i.e. crescent is formed by segment $ACBE$ − segment $ADBE$. Let the height of the segment of the larger circle be h inches, i.e. $DE = h$ inches.

Apply Pythagoras' theorem to $\triangle AEG$ to find EG (E is mid-point of AB).

Hence $$h = DG - EG = \underline{2{\cdot}68 \text{ in.}}$$

and $$CE = 2{\cdot}5 + h = \underline{5{\cdot}18 \text{ in.}}$$

If F is the centre of the smaller circle, $FA = FC = FB = r$ inches. Also

$$EF = CF - CE = r - 5{\cdot}18 \text{ in.}$$

Apply Pythagoras' theorem to $\triangle AEF$ to give equation in r. Solve to give: radius of outer arc is 12·2 in.

Area of crescent = area of segment $ACBE$ − area of segment $ADBE$.

Let angle AFE be θ, and calculate this from $\triangle AEF$. $\theta = 54{\cdot}8°$.

Area of sector

$$ACBF = \frac{2\theta}{360^\circ} \times \pi r^2$$

and since both θ and r are known this can be calculated.

Calculate area of $\triangle ABF = \frac{1}{2}AB \times EF$.

Then area of segment $ACBE$ = area of sector $ACBF$ − area of $\triangle ABF = 72{\cdot}7 \text{ in}^2$.

Note that $\triangle ABG$ is equilateral. $\therefore$ Angle $AGB = 60^\circ$.

Calculate area of sector $ADBG$ and of $\triangle ABG$ and subtract the latter from the former to give the area of the segment $ADBE$.

Area of segment $ADBE = 36{\cdot}27 \text{ in}^2$.

$\therefore$ Finally,

area of the crescent $= 72{\cdot}7 - 36{\cdot}3 = 36{\cdot}4 \text{ in}^2$.

Note. In evaluating θ from the tables it is more convenient in the subsequent calculations, if the value is taken as a decimal fraction of degrees, rather than expressing it in degrees and minutes.

FURTHER EXAMPLES

No. 1. A cube of metal weighs $17\frac{1}{2}$ lb. When it is melted down it is recast into two cubes one side 2 in. and one side 3 in. (i) What is the weight of each cube? (ii) What was the length of a side of the original cube? (U.L.C.I.)

No. 2. Calculate the thickness of a hollow sphere of inner diameter 12 in., if its weight is equal to that of a solid sphere of the same density and whose diameter is 11 in. (U.E.I.)

No. 3. (a) A swimming bath has vertical sides and all edges are straight lines. Its length, measured perpendicular to the parallel ends, is 100 ft. At the shallow end the depth of the water is 3 ft and the width of the bath is 44 ft. At the deep end the depth of water is 7 ft and the bath is 28 ft wide. Calculate the volume of water in the bath.

(b) In a constructional drawing of the bath, the plan view of the surface of the water occupies an area of 9 in^2 of the drawing. If the scale of the drawing is "1 in. represents x ft", find x. (W.J.E.C.)

ANSWERS

No. 1. (i) 4 lb, $13\frac{1}{2}$ lb; (ii) 3·27 in.

No. 2. 1·26 in. No. 3. (a) 17,500 ft^3, (b) 20 ft.

TRIGONOMETRY: EQUATIONS AND IDENTITIES

WORKED EXAMPLES

No. 1

(a) If $t = \tan x/2$, show that

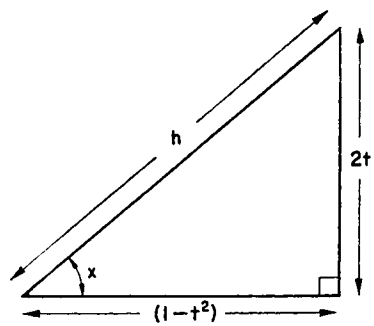
$$\sin x = \frac{2t}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}.$$

(b) Find the angles less than 360° which satisfy $2\sin x - 3\cos x = 1$. (U.L.C.I.)

(a) Since $t = \tan x/2$, $\tan x = \tan 2(x/2)$

$$\therefore \quad \tan x = \frac{2\tan(x/2)}{1-\tan^2(x/2)} = \frac{2t}{1-t^2}.$$

Consider the following triangle.



Hypotenuse h, by theorem of Pythagoras, satisfies

$$\begin{aligned} h^2 &= (1-t^2)^2 + (2t)^2 \\ &= 1 - 2t^2 + t^4 + 4t^2 \\ &= 1 + 2t^2 + t^4 \\ &= (1+t^2)^2 \qquad \therefore \quad \underline{h = 1+t^2.} \end{aligned}$$

$\therefore$ From diagram

$$\sin x = \frac{2t}{h} = \underline{\frac{2t}{1+t^2}}$$

$$\cos x = \frac{1-t^2}{h} = \underline{\frac{1-t^2}{1+t^2}}.$$

(b) To solve $2\sin x - 3\cos x = 1$. Using the substitution $t = \tan x/2$, and the relationships derived in part (a), equation becomes

$$\frac{4t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} = 1$$

$$\therefore \quad 4t - 3 + 3t^2 = 1 + t^2$$

$$2t^2 + 4t - 4 = 0$$

$$t^2 + 2t - 2 = 0$$

$$\therefore \quad t = \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm 2\sqrt{3}}{2}$$

$$t = -1 \pm \sqrt{3}$$

$$\therefore \quad t = -2{\cdot}732 \text{ or } 0{\cdot}732.$$

When $\tan x/2 = -2{\cdot}732$, $x/2$ lies in second quadrant,

$\therefore$ since $\tan 69° 54' = 2{\cdot}732$, $x/2 = 180° - 69° 54' = \underline{110° 6'}$.

When $\tan x/2 = 0{\cdot}732$, $x/2$ lies in first quadrant,

$\therefore$ since $\tan 36° 12' = 0{\cdot}732$, $x/2 = \underline{36° 12'}$ $\quad \therefore \underline{x = 72° 24' \text{ or } 220° 12'}$.

Note. In part (b) since solutions for x are required in the range 0 to 360°, $x/2$ has only to be found in the range 0 to 180°.

No. 2

Solve the following equations:

(a) $\qquad e^{-\sin x} = 0{\cdot}4066.$

(b) $\qquad \cos^2 2x - 2.\cos^2 x = 0.$

for x between 0 and 360°. (U.E.I.)

(a) $e^{-\sin x} = 0{\cdot}4066$

$\therefore \quad -\sin x = \log_e 0{\cdot}4066 = \log_e 0{\cdot}1 + \log_e 4{\cdot}066$

$= \bar{3}{\cdot}69741 + 1{\cdot}40266 = -0{\cdot}9$

$\therefore \quad \sin x = 0{\cdot}9$

$\therefore$ x lies in first or second quadrant and since $\sin 64° 10' = 0{\cdot}9$

$x = 64° 10'$ or $180° - 64° 10'$,

i.e. $\underline{x = 64° 10' \text{ or } 115° 50'.}$

(b) $\cos^2 2x - 2.\cos^2 x = 0$

$\therefore \quad (2\cos^2 x - 1)^2 - 2\cos^2 x = 0$

$\therefore \quad 4\cos^4 x - 6\cos^2 x + 1 = 0$

This is a quadratic equation in $\cos^2 x$, hence

$$\cos^2 x = \frac{6 \pm \sqrt{36-16}}{8} = \frac{6 \pm 2\sqrt{5}}{8} = \frac{3 \pm 2{\cdot}2361}{4}.$$

The positive sign gives an impossible answer, since this makes $\cos^2 x$, and consequently $\cos x$, greater than 1. Hence

$$\cos^2 x = \frac{3 - 2{\cdot}2361}{4} = 0{\cdot}191$$

$$\therefore \quad \cos x = \pm 0{\cdot}437$$

$\therefore$ There is a solution for x in each quadrant and since $\cos 64° 6' = 0{\cdot}437$

$\underline{x = 64° 6',\ 115° 54',\ 244° 6' \text{ or } 295° 54'.}$

Notes. In part (a): the tables of natural (or Naperian) logarithms only give values for numbers in the range 1 to 10, supplemented by the logarithms of powers of 10. Consequently, a number must be expressed in its standard form before taking logarithms, e.g.

$$0{\cdot}4066 = 4{\cdot}066 \times 10^{-1} \text{ giving}$$

$$\log_e 0{\cdot}4066 = \log_e 10^{-1} + \log_e 4{\cdot}066.$$

No. 3

Prove the following identities:

(a) $$\frac{\tan\theta}{(1+\tan^2\theta)^2}+\frac{\cot\theta}{(1+\cot^2\theta)^2} = \sin\theta.\cos\theta.$$

(b) $\cot\theta+\tan(180°+\theta)+\tan(90°+\theta)+\tan(360°-\theta) = 0.$

(c) $\cos 4\theta = 1-8.\cos^2\theta+8.\cos^4\theta.$ (E.M.E.U.)

(a) $$\frac{\tan\theta}{(1+\tan^2\theta)^2}+\frac{\cot\theta}{(1+\cot^2\theta)^2}$$

$$=\frac{\tan\theta}{\sec^4\theta}+\frac{\cot\theta}{\operatorname{cosec}^4\theta} = \frac{\sin\theta}{\cos\theta}.\cos^4\theta+\frac{\cos\theta}{\sin\theta}.\sin^4\theta$$

$$= \sin\theta.\cos\theta(\cos^2\theta+\sin^2\theta) = \underline{\sin\theta.\cos\theta.}$$

(b) $$\cot\theta+\tan(180°+\theta)+\tan(90°+\theta)+\tan(360°-\theta)$$

$$= \cot\theta+\tan\theta-\tan(90°-\theta)-\tan\theta$$

$$= \cot\theta-\cot\theta$$

$$= \underline{0.}$$

(c) $$\cos 4\theta = 2.\cos^2 2\theta-1$$

$$= 2\{2.\cos^2\theta-1\}^2-1$$

$$= 2.\{4.\cos^4\theta-4.\cos^2\theta+1\}-1$$

$$= \underline{8.\cos^4\theta-8.\cos^2\theta+1.}$$

Notes

(1) Part (b): use has been made of the property of complementary angles, i.e. $\tan(90°-\alpha) = \cot\alpha$.

Also: $\tan(180°+\theta) = \tan\theta$ and $\tan(360°-\theta) = -\tan\theta.$

(2) Part (c): when substituting for the cosine of a double angle, there are three identities available,

either $\cos(2A) = \cos^2A-\sin^2A$

or $\cos(2A) = 2.\cos^2A-1$

or $\cos(2A) = 1-2.\sin^2A.$

Since the expression to be derived contains only the terms in $\cos\theta$, the second of these identities was chosen.

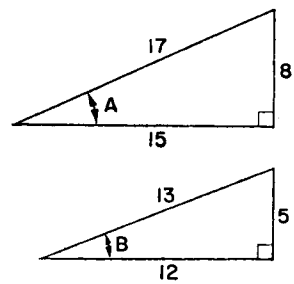
No. 4

(a) Given that A and B are both acute angles and that $\tan A = \frac{8}{15}$ and $\cos B = \frac{12}{13}$, find without the use of tables

$$\text{(i) } \sin(A+B), \quad \text{(ii) } \cos(A-B).$$

(b) Express $33\sin\theta + 56\cos\theta$ in the form $R\sin(\theta+\alpha)$. Hence, or otherwise, solve the equation $33\sin\theta + 56\cos\theta = 20$, for $0° < \theta < 360°$. (U.L.C.I.)

(a) Since A is acute and $\tan A = \frac{8}{15}$, the other ratios of A can be read off the diagram, the third side of the triangle being found by Pythagoras' theorem. Similarly for B:



$$\text{(i) } \sin(A+B) = \sin A.\cos B + \cos A.\sin B = \frac{8}{17}\times\frac{12}{13} + \frac{15}{17}\times\frac{5}{13}$$

$$= \frac{96+75}{221} \qquad \therefore \quad \sin(A+B) = \frac{171}{221}.$$

$$\text{(ii) } \cos(A-B) = \cos A\cos B + \sin A\sin B = \frac{15}{17}\times\frac{12}{13} + \frac{8}{17}\times\frac{5}{13}$$

$$= \frac{180+40}{221} \qquad \therefore \quad \cos(A-B) = \frac{220}{221}.$$

(b) $R\sin(\theta+\alpha) = R.\sin\theta\cos\alpha + R.\cos\theta\sin\alpha$

$$= (R.\cos\alpha)\sin\theta + (R.\sin\alpha)\cos\theta$$

Comparing with $33\sin\theta + 56\cos\theta$

gives
$$\left.\begin{aligned} R\sin\alpha &= 56 \\ R\cos\alpha &= 33 \end{aligned}\right\} \qquad (1)$$

From eqn. (1) dividing

$$\tan\alpha = \tfrac{56}{33}$$

No.	Log
56	1·7482
33	1·5185
tan 59° 29′	0·2297

$\therefore\ \alpha = 59°\ 29'.$

From eqn. (1) squaring and adding

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 33^2 + 56^2$$
$$\therefore\ R^2(\cos^2\alpha + \sin^2\alpha) = 1089 + 3136$$
$$\therefore\ R^2 = 4225$$
$$\therefore\ R = 65$$

$\therefore\ 33\sin\theta + 56\cos\theta = 65\sin(\theta+\alpha)$ where $\alpha = 59°\ 29'$.

$$33\sin\theta + 56\cos\theta = 20$$
$$\therefore\ 65\sin(\theta+\alpha) = 20$$
$$\therefore\ \sin(\theta+\alpha) = \frac{20}{65} = \frac{4}{13}$$

No.	Log
4	0·6021
13	1·1139
sin 17° 55′	$\bar{1}$·4882

$\therefore\ (\theta+\alpha)$ lies in first or second quadrant

$$\therefore\ \theta+\alpha = (17°\ 55') \text{ or } 162°\ 5' \text{ or } 377°\ 55'$$
$$\alpha = 59°\ 59'$$
$$\therefore\ \theta = 102°\ 36' \text{ or } 318°\ 36'.$$

Note. In part (b), one of the values of $\theta+\alpha$ turns out to be less than the value of α and hence would give rise to a negative value of θ which would in turn be outside the required range. This is overcome by adding 360° to this lowest value of $\theta+\alpha$, i.e. 17° 55′ is less than 59° 59′, hence work with $17°\ 55' + 360° = 377°\ 55'$.

HINTED EXAMPLES

No. 1. (a) Write down the expansions of $\sin(A-B)$ and $\cos(A-B)$. Prove the identity: $\sin(120°-\theta)+\sin(240°-\theta) = \sin\theta$.

(b) Prove that

$$\frac{1+\cos 2A}{\sin 2A} = \cot A. \qquad \text{(U.E.I.)}$$

(a) $\sin(A-B) = \sin A.\cos B - \cos A.\sin B$

$\cos(A-B) = \cos A.\cos B + \sin A.\sin B$

Method 1. Use the expansion for $\sin(A-B)$ on each term on the L.H.S.

Note: $\cos 120° = \cos 240° = -\frac{1}{2}$

and $\sin 120° = -\sin 240° = \frac{\sqrt{3}}{2}$.

Method 2. Use the identity

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right).\cos\left(\frac{x-y}{2}\right)$$

on the L.H.S. *Note*:

$$\sin(180° - \theta) = \sin\theta \text{ and } \cos 60° = \tfrac{1}{2}.$$

(b) Take L.H.S. and simplify using the identities

$$\sin 2A = 2 \sin A \cos A \text{ and } \cos 2A = 2\cos^2 A - 1.$$

No. 2. (a) Express $(3 \sin 7\theta + 3 \sin\theta)$ as a product, and hence find the values of θ between 0 and 90° which satisfy the equation,

$$3 \sin 7\theta + 3 \sin\theta = 2 \sin 4\theta.$$

(b) Find (i) the maximum value of $(7 \cos x - \sin x)$;
(ii) the smallest positive value of x for which this maximum occurs. (W.J.E.C.)

(a) Using the identity

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right).\cos\left(\frac{x-y}{2}\right)$$

$$3 \sin 7\theta + 3 \sin\theta = \underline{6 \sin 4\theta.\cos 3\theta.}$$

Hence the equation

$$3 \sin 7\theta + 3 \sin\theta = 2 \sin 4\theta$$

becomes $6 \sin 4\theta.\cos 3\theta = 2 \sin 4\theta$

Complete solution gives $\underline{\theta = 0°, 45°, 90° \text{ or } 23° 31'.}$

(b) Express $7\cos x - \sin x$ in the form $R\sin(\alpha - x)$.

$$\{7\cos x - \sin x = \sqrt{50}\sin(81° 50' - x)\}$$

Since $\sin(\alpha - x)$ can only vary between -1 and $+1$,

i.e. $$-1 \leqq \sin(\alpha - x) \leqq +1 \text{ for all } x.$$

Maximum value occurs when $\sin(\alpha - x) = +1$, i.e. maximum value is $+R$.

Answers: (i) 7·07; (ii) 351° 50′.

Notes

(1) In part (a) both the possibilities $\cos 3\theta = \frac{1}{3}$ and $\sin 4\theta = 0$ lead to solutions.

(2) Since θ is only required in the range 0 to 90°, $\sin 4\theta$ only needs to be solved in the range 0 to 360°.

(3) Solving $\cos 3\theta = \frac{1}{3}$ in the range 0 to 360° gives $3\theta = 70° 32'$ or $288° 28'$. Only the first of these gives a value of θ in the required range.

FURTHER EXAMPLES

No. 1. (a) Solve $3.\sin\theta - 4.\cos\theta = 2$ for values of θ between 0 and 360°.

(b) Solve $\sin\theta - \sin 3\theta + \sin 5\theta = 0$ for all values of θ between 0 and 360°. (U.E.I.)

No. 2. (a) Solve, giving all solutions for x between 0 and 360°,

$$\sin^2 x - \sin x - 3.\cos^2 x = 0.$$

(b) Express $2{\cdot}4.\sin x + \cos x$ in the form $R.\sin(x+\phi)$ determining the values of the constants R and ϕ. Hence solve the equation

$$2{\cdot}4\sin x + \cos x = 1{\cdot}3$$

for values of x between 0 and 360°. (W.J.E.C.)

No. 3. (a) If $\sin A = \frac{5}{13}$ and $\cos B = \frac{16}{65}$ where A and B are acute angles, find, without using tables, the values of (i) $\sin(A+B)$; (ii) $\sin 2B$.

(b) Prove the identities

$$\text{(i) } \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}; \quad \text{(ii) } \tan A = \frac{\sin 2A}{1+\cos 2A}.$$

No. 4. A sliding piece moves along a straight guide, its displacement D inches being given by

$$D = 4.\sin\theta + 3(1+\cos\theta).$$

Show that this may be written:

$$D = A + B.\sin(\theta + k),$$

and find the values of the constants A, B and k.

Find also (a) the minimum value of D and the smallest positive value of θ for which D takes this value, (b) the two values of θ between 0 and 360° for which $D = 7$. (E.M.E.U.)

ANSWERS

No. 1. (a) 76° 43′, 209° 33′; (b) 0°, 30°, 60°, 120°, 150°, 180°, 210°, 240°, 300°, 330°, 360°.

No. 2. (a) 90°, 228° 35′, 311° 25′; (b) 2·6, 22° 37′, 7° 23′, 127° 23′.

No. 3. (a) (i) $\frac{836}{845}$; (ii) $\frac{2016}{4225}$.

No. 4. 3, 5, 36° 52′; (a) -2, 233° 8′; (b) 90°, 16° 16′.

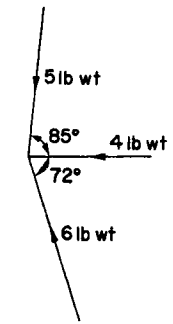
TRIGONOMETRY: SOLUTION OF TRIANGLES, ETC.

WORKED EXAMPLES

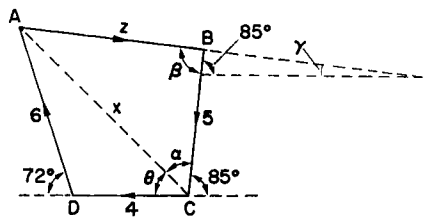
No. 1

Forces of 4, 5 and 6 lb wt act at a point such that the 5 and 6 lb forces act at angles of 85° and 72° respectively to the 4 lb force and on opposite sides of it.

Find by drawing and calculation, the resultant of these forces in magnitude and direction, giving your direction relative to the 4 lb force. (E.M.E.U.)



Force diagram



Force polygon

Let z lb wt be the fourth side of the force polygon and let γ be the angle this side makes with the direction of the force of 4 lb wt.

Let x lb wt be the diagonal indicated in the diagram and let α, β, θ be the angles marked.

By cosine rule applied to $\triangle ACD$

$$\begin{aligned} x^2 &= 6^2+4^2-2\times 4\times 6\times \cos(180°-72°) \\ &= 36+16+48.\cos 72° \\ &= 52+14{\cdot}83 \\ &= 66{\cdot}83. \end{aligned}$$

$\therefore$ $x = 8{\cdot}175$ lb wt.

By sine rule applied to $\triangle ACD$

$$\frac{\sin\theta}{6} = \frac{\sin(180°-72°)}{8{\cdot}175}$$

$$\therefore \quad \sin\theta = \frac{6.\sin 72°}{8{\cdot}175}$$

$$= 0{\cdot}6982. \qquad \therefore \quad \underline{\theta = 44°\,17'.}$$

Since $\theta+\alpha+85° = 180°$

$$\alpha = 180°-(85°+44°\,17') \qquad \therefore \quad \underline{\alpha = 50°\,43'.}$$

By cosine rule applied to $\triangle ABC$

$$z^2 = x^2+5^2-2\times 5\times x.\cos\alpha$$

$$= (8{\cdot}175)^2+5^2-10\times 8{\cdot}175.\cos 50°\,43'$$

$$= 66{\cdot}83+25-81{\cdot}75.\cos 50°\,43'$$

$$= 91{\cdot}83-51{\cdot}76$$

$$= 40{\cdot}07. \qquad \therefore \quad \underline{z = 6{\cdot}33 \text{ lb wt.}}$$

By sine rule applied to $\triangle ABC$

$$\frac{\sin\beta}{x} = \frac{\sin\alpha}{z}$$

$$\therefore \quad \frac{\sin\beta}{8{\cdot}175} = \frac{\sin 50°\,43'}{6{\cdot}33}$$

$$\therefore \quad \sin\beta = \frac{8{\cdot}175.\sin 50°\,43'}{6{\cdot}33}$$

$$= 0{\cdot}9996. \qquad \therefore \quad \underline{\beta = 88°\,23'.}$$

Since $\beta = \gamma+85°$, $\qquad \underline{\gamma = 3°\,23'.}$

Hence the resultant force is 6·33 lb wt acting at an angle of 3° 23′ to the direction of the force of 4 lb wt and on the same side of it as the force of 6 lb wt.

Note. The force z is the fourth side of the polygon and is the force required to bring the system into equilibrium, consequently the resultant of the 4, 5 and 6 lb wt forces is equal in magnitude to z but opposite in direction.

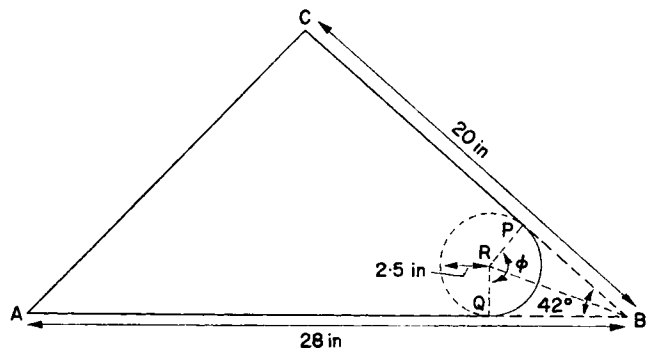
No. 2

A metal plate is in the shape of a triangle ABC where angle $ABC = 42°$. The corner at B is rounded off by an arc of a circle of radius 2·5 in., the sides BA and BC being tangential to the arc.

If $AB = 28$ in. and $BC = 20$ in. (measurements being taken to the point B before the corner was rounded off), determine the perimeter of the plate. (W.J.E.C.)

Let the centre of the circle which rounds off corner at B be R.

Let rounding start at P on CB and Q on AB. Let angle $PRQ = \phi$.



By cosine rule $AC^2 = 20^2 + 28^2 - 2.20.28.\cos 42°$

$= 400 + 784 - 832{\cdot}36$

$= 351{\cdot}64$ $\quad\therefore\quad$ $AC = 18{\cdot}75$ in.

Since BA and BC are tangential to the circle, $BP = BQ$ and RB bisects the angle at B.

Hence $BP = 2{\cdot}5.\cot 21°$

$= 2{\cdot}5 \times 2{\cdot}605$ $\quad = 6{\cdot}51$ in.

$\therefore$ $CP = 20 - 6{\cdot}51$ $\quad = 13{\cdot}49$ in.

$\therefore$ $QA = 28 - 6{\cdot}51$ $\quad = 21{\cdot}49$ in.

Since BA and BC are tangential to the circle, angle RPB = angle $RQB = 90°$.

Hence $\phi = 180° - 42°$ $\quad = 138°$.

$$\text{Arc } PQ = \frac{\phi}{360°} \times \text{circumference of circle}$$

$$= \frac{138°}{360°} \cdot 2\pi(2{\cdot}5) = \frac{23\pi}{12} \quad = 6{\cdot}02 \text{ in.}$$

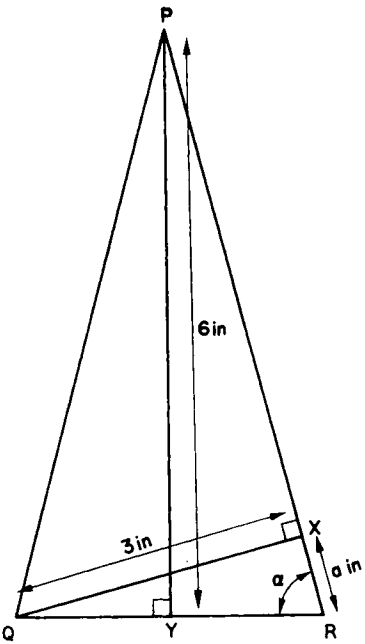
Required perimeter $= AC + CP + \text{arc } PQ + QA$

$= 18{\cdot}75 + 13{\cdot}49 + 6{\cdot}02 + 21{\cdot}49 = 59{\cdot}75.$

$\therefore$ Perimeter of plate is 59·8 in.

No. 3

PQR is an isosceles triangle in which $PQ = PR$. The perpendicular QX from Q to PR is 3 in. long and the perpendicular PY from P to QR is 6 in. long.



Find (i) the angles of the triangle.
(ii) the length of QR.
(iii) the length of PQ.
(iv) the length of PX. (U.E.I.)

Let $XR = a$ inches.

Let angle $PRQ = \alpha$.

Then since triangle is isosceles angle $PQR = \alpha$ and angle $QPR = 180° - 2\alpha$. Triangles PYR and QXR are both right-angled and have angle PRQ in common; consequently, they are equiangular, and therefore similar.

Hence

$$\frac{XR}{YR} = \frac{QR}{PR} = \frac{QX}{PY},$$

i.e. $$\frac{a}{YR} = \frac{QR}{PR} = \frac{3}{6}. \qquad (1)$$

Also since triangle PQR is isosceles $QY = YR$, i.e. $QR = 2YR$. (2)

From eqns. (1) and (2) $\underline{YR = 2a \text{ and } QR = 4a.}$

Also from eqn. (1) $PR = 2QR \quad \therefore \quad \underline{PR = 8a.}$

In $\triangle QXR$, $\cos\alpha = \dfrac{XR}{QR} = \dfrac{a}{4a} = 0{\cdot}25 \quad \therefore \quad \underline{\alpha = 75° 31'.}$

Hence (i) $\underline{\angle PQR = \angle PRQ = 75° 31' \text{ and } \angle QPR = 28° 58'.}$

In $\triangle QXR$, by Pythagoras' theorem,

$$QR^2 = XR^2 + QX^2$$

$$\therefore \quad 16a^2 = a^2 + 9$$

$$\therefore \quad a^2 = \tfrac{9}{15} = \tfrac{3}{5}. \quad \therefore \quad \underline{a = \sqrt{0{\cdot}6} = 0{\cdot}775.}$$

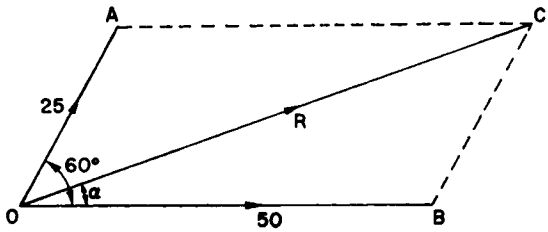
Hence (ii) $\underline{QR = 4a = 4 \times 0{\cdot}775 = 3{\cdot}1 \text{ in.}}$

Also (iii) $\underline{PQ = PR = 8a = 6{\cdot}2 \text{ in.}}$

and (iv) $\underline{PX = PR - XR = 6{\cdot}2 - 0{\cdot}775 = 5{\cdot}43 \text{ in.}}$

HINTED EXAMPLES

No. 1. Two alternating quantities are represented by vectors of lengths 25 and 50, which act at the same point. The angle between the vectors is 60°. Calculate the resultant quantity and the angle it makes with the larger of the original quantities. (U.E.I.)



Let the resultant be R, and let it make an angle α with the larger of the original quantities.

R is given by the diagonal of the parallelogram of forces.

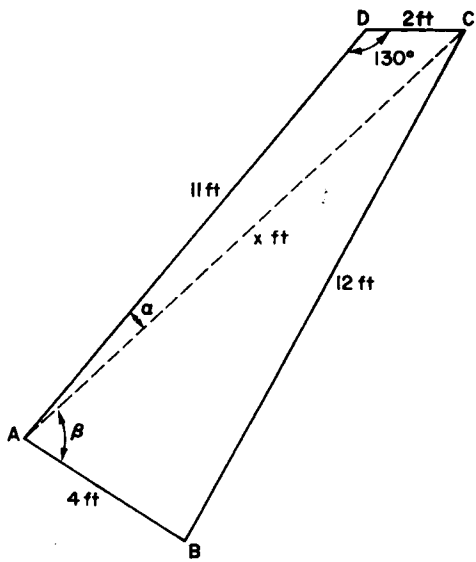
Find angle OBC, and then find R by applying the cosine rule to triangle OBC.

Answer: $R = 66{\cdot}2$.

Now find α by applying the sine rule to the same triangle.

Answer: $\alpha = 19° 6'$.

No. 2. Four straight members of a mechanism form a quadrilateral $ABCD$. $AB = 4$ ft, $BC = 12$ ft, $CD = 2$ ft, $DA = 11$ ft and the angle $ADC = 130°$. Calculate the angle BAD. (U.L.C.I.)



Let $AC = x$ ft.

Let angles DAC and BAC be α and β respectively.

Then angle $BAD = \alpha + \beta$.

In $\triangle ADC$, use the cosine rule to find x. Use the calculated value of x and the sine rule in the same triangle to find α.

Answer: $\alpha = 7° 7'$.

In $\triangle ACB$, use the cosine rule to find β, i.e.

$$\cos \beta = \frac{x^2 + 4^2 - 12^2}{8x}.$$

Since x has already been calculated, β can be found and hence angle BAD.

Answer: 82° 19′.

FURTHER EXAMPLES

No. 1. Two ladders are used to reach a point on a vertical wall which is 30 ft above ground level. The first ladder is 20 ft long and makes an angle of 60° with the horizontal ground. The base of a second ladder also 20 ft long is secured to a point on the first ladder such that its uppermost end reaches the required position of 30 ft above the ground. Calculate:

(i) The angle the second ladder makes with the horizontal.

(ii) The vertical height to the point where the two ladders are secured.

(U.E.I.)

No. 2. (a) Two forces acting at a point are inclined at an angle of 45° to each other. Calculate the magnitude and direction of the resultant force, if the magnitudes of the two forces are 20 lb and 40 lb respectively.

(b) The lengths of the sides of a triangle are 3·8 in., 4·9 in and 5·3 in. Calculate the area of the triangle. (U.E.I.)

No. 3. A circular plate has a diameter AB and O is the centre of the circle. Two hole centres C and D are marked on the plate on pitch circles of radius 5·4 and 6·8 in. respectively and on the same side of AB. Angle COA is 21° and angle DOB is 57°. Calculate the length CD and the angle of inclination of CD to AB. (U.L.C.I.)

ANSWERS

No. 1. (i) 78° 23′; (ii) 10·41 ft.

No. 2. (a) 56 lb at 30° 20′ to 20 lb force; (b) 8·94 in^2.

No. 3. 9·52 in.; 23° 23′.

GRAPHICAL SOLUTIONS

WORKED EXAMPLES

No. 1

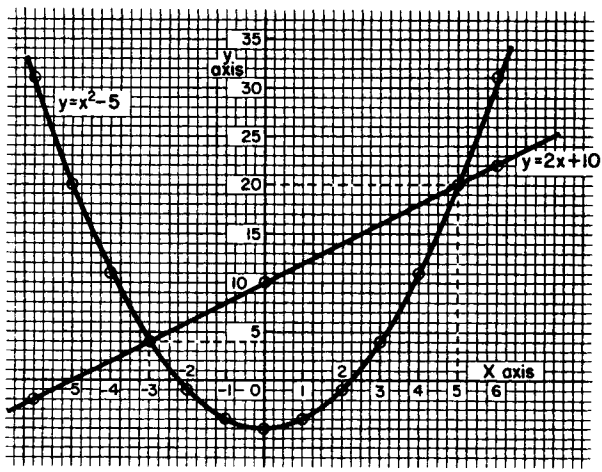
(a) Using values of x between $+6$ and -6, solve graphically the simultaneous equations: $2x-y+10 = 0$, $x^2-y-5 = 0$.

(b) Use the graph of $x^2-y-5 = 0$ to find the solutions of the quadratic equation $x^2-5 = 0$. (U.E.I.)

Equations $\begin{cases} 2x-y+10 = 0 \\ x^2-y-5 = 0 \end{cases}$ can be written in the form

$$y = 2x+10 \text{ and } y = x^2-5.$$

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$2x+10$	-2						10						22
x^2-5	31	20	11	4	-1	-4	-5	-4	-1	4	11	20	31



(a) The solutions to the simultaneous equations are given by the points of intersection of the two graphs.

Answers: $x = -3$, $y = 4$ and $x = 5$, $y = 20$.

(b) The solutions of the equation $x^2 - 5 = 0$ are given by the intersection of the curve $y = x^2 - 5$ with the line $y = 0$, i.e. with the x-axis.

Answer: $x = \pm 2{\cdot}24$.

No. 2

Find graphically the two roots of $(x-5)^{-1}.\log_e x^2 = 1$ and give the root between 1 and 10 to 3 places of decimals by again using a graphical method. (U.E.I.)

Rearranging the equation $(x-5)^{-1}.\log_e x^2 = 1$ becomes

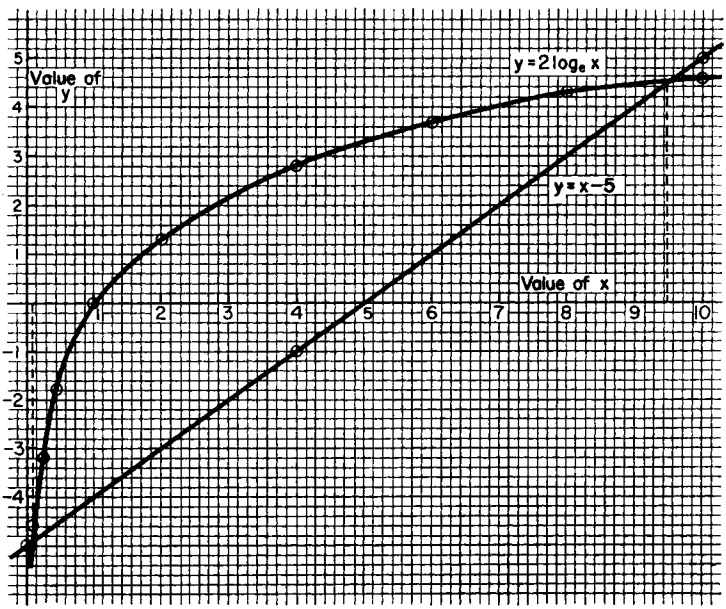
$$\log_e x^2 = (x-5),$$

i.e.

$$2\log_e x = (x-5).$$

Hence solutions to this equation will be given by the intersections of the graphs of $y = 2\log_e x$ and $y = x-5$.

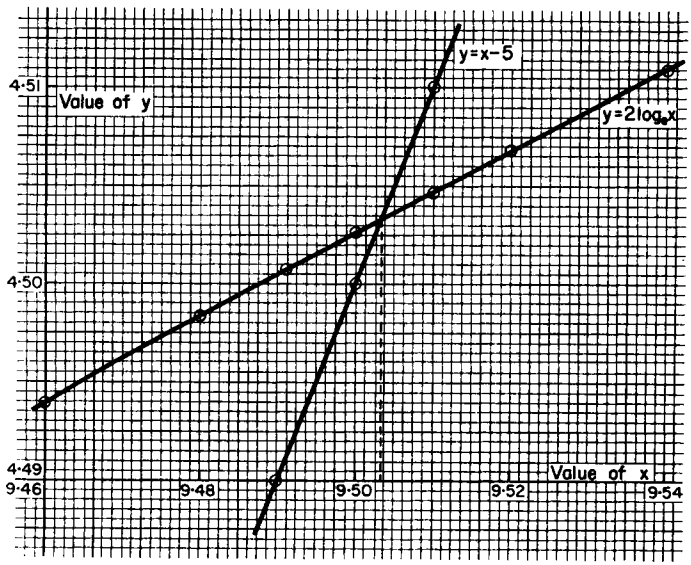
x	0	0·1	0·2	0·4	0·7	1	2	4	6	8	10
$x-5$	−5							−1			5
$\log_e x$	$-\infty$	−2·30	−1·61	−0·92	−0·36	0	0·69	1·39	1·79	2·08	2·30
$2\log_e x$	$-\infty$	−4·60	−3·22	−1·84	−0·72	0	1·38	2·78	3·58	4·16	4·60



From the graph the two roots are $x = 0{\cdot}09$ and $x = 9{\cdot}5$.

To find the root between 1 and 10 more accurately, replot the graphs on a larger scale for values of x in the range 9·46 to 9·54.

x	9·46	9·48	9·49	9·50	9·51	9·52	9·54
$x-5$			4·49	4·50	4·51		
$\log_e x$	2·2470	2·2492	2·2502	2·2513	2·2523	2·2534	2·2555
$2\log_e x$	4·4940	4·4984	4·5004	4·5026	4·5046	4·5068	4·5110



From the graph the root between 1 and 10 is $x = 9{\cdot}503$ (to 3 decimal places).

Note. In the second graph, the points for the line $y = x-5$ have been taken over the three middle points of the range of values of x. If the points had been taken at $x = 9{\cdot}46$, 9·50 and 9·54 the values of y would have had to range from 4·46 to 4·54, and this would have meant reducing the vertical scale considerably—resulting in a loss of accuracy.

No. 3

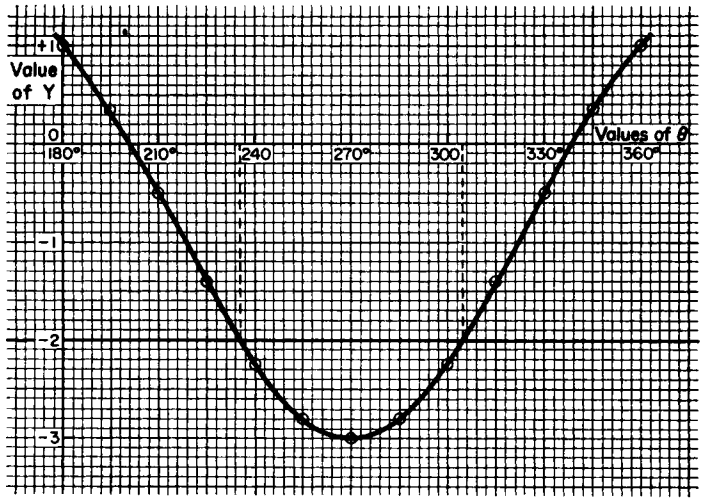
Find, graphically, the values of θ between 180 and 360° which satisfy the equation $\cos 2\theta + 2 \sin \theta = -2$. (E.M.E.U.)

Solutions will be given by the intersections of the curve

$$y = \cos 2\theta + 2 \sin \theta \text{ with the line } y = -2.$$

θ	180°	195°	210°	225°	240°	255°
$\sin \theta$	0·000	−0·269	−0·500	−0·707	−0·866	−0·966
$2 \sin \theta$ $\cos \theta$	0·000 1·000	−0·518 0·866	−1·000 0·500	−1·414 0·000	−1·732 −0·500	−1·932 −0·866
y	1·000	0·348	−0·500	−1·414	−2·232	−2·798
y (corrected to 2 dec. places)						
	1·00	0·35	−0·50	−1·41	−2·23	−2·80

θ	270°	285°	300°	315°	330°	345°	360°
$\sin \theta$	−1·000	−0·966	−0·866	−0·707	−0·500	−0·269	0·000
$2 \sin \theta$ $\cos \theta$	−2·000 −1·000	−1·932 −0·866	−1·732 −0·500	−1·414 0·000	−1·000 0·500	−0·518 0·866	0·000 1·000
y	−3·000	−2·798	−2·232	−1·414	−0·500	0·348	1·000
y (corrected to 2 dec. places)							
	−3·00	−2·80	−2·23	−1·41	−0·50	0·35	1·00



Answers: $\theta = 235° 30'$ or $304° 30'$.

Note. Unless extremely large scales are used, it is impossible to represent more than 3 significant figures graphically. This is why the values of y above have been corrected to two decimal places.

HINTED EXAMPLES

No. 1. (a) Plot a graph of $y = 2x^2 - 5x - 7$ using values of x between $+5$ and -2.

Using this graph solve the following equations:

(i) $2x^2 - 5x - 7 = 0$.

(ii) $2x^2 - 5x - 4 = 0$.

(b) On the same axes plot a graph of $y = 3x - 5$. Using the graphs solve the simultaneous equations:

$$2x^2 - 5x - y = 7;\ 3x - y = 5. \qquad \text{(U.E.I.)}$$

(a) Since $y = 2x^2 - 5x - 7 \equiv 2x^2 - (5x + 7)$.

Complete the following table, and then plot y against x.

x	-2	-1	0	1	2	3	4	5
$2x^2$	8	2						50
$5x+7$	-3	2						32
$y = 2x^2-(5x+7)$	11	0						18

Suggested scales. x: 1 inch to 1·0 units from -2 to $+5$.

y: 1 inch to 5·0 units from -10 to $+20$.

(i) The solutions to $2x^2-5x-7 = 0$ are given by the intersections of the graph with the x-axis.

Answers: $x = -1$ or $x = 3{\cdot}5$.

(ii) The solutions to $2x^2-5x-4 = 0$ are given by the intersections of the graph with the line $y = -3$; since $2x^2-5x-4 = 0$ can be rewritten in the form $2x^2-5x-7 = -3$.

Answers: $x = -0{\cdot}64$ or $x = 3{\cdot}14$.

(b) Since $y = 3x-5$ is a straight line only three points will be needed for plotting the graph. The solutions to the simultaneous equations are given by the points of intersection of the graphs.

Answers: $x = -0{\cdot}24$, $y = -5{\cdot}7$ or $x = 4{\cdot}24$, $y = 7{\cdot}7$.

No. 2. Plot the graph of $y = \sin\theta+\sin 2\theta$ for values of θ from 0 to 180°. Use the graph to solve the equation $\sin\theta+\sin 2\theta-1 = 0$. (U.L.C.I.)

Complete the following table and then plot y against θ.

θ	0°	10°	20°	30°—and hence at intervals of 10° to:	170°	180°
$\sin\theta$	0	0·174	0·342		0·174	0
$\sin 2\theta$	0	0·342	0·643		−0·342	0
y	0	0·516	0·985		−0·168	0

Suggested scales. θ: 1 inch to 20°.
y: 1 inch to 0·5 units.

The equation $\sin\theta+\sin 2\theta-1 = 0$ can be rewritten in the form $\sin\theta+\sin 2\theta = 1$ and hence the solutions are given by the intersections of the graph with the line $y = 1$.

Answers: $\theta = 20°\,30'$ or $\theta = 90°$.

FURTHER EXAMPLES

No. 1. Draw, using the same scales and axes, the graphs of $y = 5/x$ and $y = x^2/5$ between the values of $x = 1$ and $x = 5$. Explain how the approximate value of $\sqrt[3]{25}$ can be obtained from the graphs and give its value. (U.L.C.I.)

No. 2. Plot the curve $y = \sin\theta - \cos 2\theta$ for values of θ at 15° intervals between $\theta = 0°$ and $\theta = 180°$. Use the graph to find the values of θ between 0 and 180° which satisfy the equation $2(\sin\theta - \cos 2\theta) = 1$. (U.L.C.I.)

No. 3. Find, graphically, the root of

$$\tan x = \frac{180}{\pi x}$$

between 0 and 90°, to the nearest 0·1 of a degree.

ANSWERS

No. 1. 2·92. No. 2. 40·7°; 139·3°. No. 3. 49·3°.

DIFFERENTIATION

WORKED EXAMPLES

No. 1

(a) Find from first principles the differential coefficient of x^4 with respect to x.

(b) Differentiate by rule (i) $4x^2+2x-1$; (ii) $x^4(x^3+1)(x-1)$. (E.M.E.U.)

(a) Let $y=x^4$.

Let δx be a small alteration in x and δy be the corresponding alteration in y, then

$$y+\delta y = (x+\delta x)^4$$

$$= x^4+4x^3.\delta x+6x^2.(\delta x)^2+4x(\delta x)^3+(\delta x)^4.$$

But $$y = x^4$$

$$\therefore \quad \delta y = 4x^3.\delta x+6x^2.(\delta x)^2+4x(\delta x)^3+(\delta x)^4$$

$$\therefore \quad \frac{\delta y}{\delta x} = 4x^3+6x^2.\delta x+4x(\delta x)^2+(\delta x)^3.$$

The differential coefficient

$$\frac{dy}{dx} = \operatorname*{Lim}_{\delta x \to 0} \frac{\delta y}{\delta x} = \operatorname*{Lim}_{\delta x \to 0} [4x^3+6x^2.\delta x+4x(\delta x)^2+(\delta x)^3]$$

$$= 4x^3.$$

The differential coefficient of x^4 with respect to x is $4x^3$.

(b) (i) If $y = 4x^2+2x-1$, $\underline{\dfrac{dy}{dx} = 8x+2.}$

(ii) If $y = x^4.(x^3+1).(x-1)$

then $y = x^4.(x^4-x^3+x-1)$

$\therefore \quad y = x^8-x^7+x^5-x^4$

$\therefore \quad \dfrac{dy}{dx} = 8x^7-7x^6+5x^4-4x^3$

$\therefore \quad \underline{\dfrac{dy}{dx} = x^3(8x^4-7x^3+5x-4).}$

Note. Part (b) (ii) could have been done by extending the rule for differentiating a product, but since none of the terms are very complicated it is easier and quicker to multiply out before differentiation.

No. 2

(a) Differentiate with respect to x

(i) $(4x^2+3)^{10}$; (ii) $\dfrac{e^{2x}}{4+3x}$

(b) If $y = x \sin \pi x$, evaluate

$\dfrac{dy}{dx}$ when $x = \frac{1}{3}$. (W.J.E.C.)

(a) (i) Let $y = (4x^2+3)^{10}$

$$\frac{dy}{dx} = 10(4x^2+3)^9.(8x)$$

$$= \underline{80x(4x^2+3)^9}$$

(ii) Let $y = \dfrac{e^{2x}}{4+3x}$

$$\frac{dy}{dx} = \frac{(4+3x).2e^{2x}-e^{2x}.3}{(4+3x)^2}$$

$$= \frac{e^{2x}.(8+6x-3)}{(4+3x)^2}$$

$$= \underline{\frac{e^{2x}.(6x+5)}{(4+3x)^2}}$$

(b) $y = x.\sin \pi x$

$$\therefore \quad \frac{dy}{dx} = x(\pi.\cos \pi x) + \sin \pi x$$

$$\therefore \quad \text{when } x = \tfrac{1}{3}, \frac{dy}{dx} = \frac{\pi}{3}\cos\frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{\pi}{3}.\frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{3{\cdot}142}{6} + \frac{1{\cdot}732}{2} = 0{\cdot}5237 + 0{\cdot}866$$

$$= 1{\cdot}3897$$

$$\therefore \quad \text{when } x = \tfrac{1}{3}, \frac{dy}{dx} = 1{\cdot}39.$$

No. 3

(a) Differentiate the following with respect to x:

(i) $(x^2-1).\log_e(2x+2)$

(ii) $4.\cos 2x.\cos 3x$

(iii) $\dfrac{\cos x}{x^2+2x+1}$

(b) If $y = e^{3x}.\sin 4x$, show that

$$\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 25y = 0. \qquad \text{(U.E.I.)}$$

(a) (i) Let $y = (x^2-1).\log_e(2x+2)$

$$\therefore \quad \frac{dy}{dx} = (x^2-1).\frac{2}{(2x+2)} + 2x.\log_e(2x+2)$$

$$= \frac{2(x+1)(x-1)}{2(x+1)} + 2x.\log_e(2x+2)$$

$$\therefore \quad \frac{dy}{dx} = (x-1) + 2x.\log_e(2x+2).$$

(ii) $y = 4.\cos 2x.\cos 3x$

$$\therefore \quad \frac{dy}{dx} = 4\{\cos 2x(-3.\sin 3x)+(-2\sin 2x).\cos 3x\}$$

$$= -4(3.\sin 3x.\cos 2x+2.\sin 2x.\cos 3x)$$

$$= -4\{\sin 3x.\cos 2x+2(\sin 3x.\cos 2x+\sin 2x.\cos 3x)\}$$

$$\therefore \quad \underline{\frac{dy}{dx} = -4(\sin 3x.\cos 2x+2\sin 5x).}$$

(iii) Let $y = \dfrac{\cos x}{x^2+2x+1} = \dfrac{\cos x}{(x+1)^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+1)^2.(-\sin x)-2(x+1).\cos x}{(x+1)^4}$$

$$= \frac{-(x+1).\{(x+1).\sin x+2\cos x\}}{(x+1)^4}$$

$$\underline{\frac{dy}{dx} = \frac{-\{(x+1).\sin x+2\cos x\}}{(x+1)^3}.}$$

(b) $y = e^{3x}.\sin 4x$

$$\therefore \quad \frac{dy}{dx} = 3.e^{3x}.\sin 4x+e^{3x}.4.\cos 4x$$

$$\therefore \quad \frac{dy}{dx} = 3y+4.e^{3x}.\cos 4x \qquad (1)$$

$$\therefore \quad \frac{d^2y}{dx^2} = 3\frac{dy}{dx}+4.\{3.e^{3x}.\cos 4x-e^{3x}.4.\sin 4x\}$$

$$\therefore \quad \frac{d^2y}{dx^2}-3\frac{dy}{dx} = (4\times 3).e^{3x}.\cos 4x-16.e^{3x}.\sin 4x.$$

Hence, since $y = e^{3x}.\sin 4x$, and as from eqn. (1)

$$4.e^{3x}.\cos 4x = \frac{dy}{dx}-3y,$$

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx} = 3\left\{\frac{dy}{dx}-3y\right\}-16y$$

$$= 3\frac{dy}{dx}-9y-16y$$

$$\therefore \quad \frac{d^2y}{dx^2}-6\frac{dy}{dx}+25y = 0.$$

Answer: $\underline{\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+25y = 0.}$

Note. When deriving a differential equation, as in part (b), the main technique is, after each differentiation, to substitute back; i.e. replace terms in x by the corresponding terms in y. Since the terms in y are much less complicated, further differentiation is thus made a lot easier.

After the first differentiation substitute back to reintroduce y. After the second differentiation substitution can reintroduce y and $\mathrm{d}y/\mathrm{d}x$; after the third differentiation substitution can reintroduce all or any of y, $\mathrm{d}y/\mathrm{d}x$, $\mathrm{d}^2y/\mathrm{d}x^2$; and so on until the required equation is formed.

HINTED EXAMPLES

(a) Differentiate the following functions with respect to x:

(i) $\dfrac{x^3-1}{\sqrt{x}}$; (ii) $\log_e(x^2+3x+1)$;

(iii) $x.\cos^2 x$; (iv) $\dfrac{3-\sin x}{\cos x}$.

(b) If $y = Ae^{2x}+Be^{-3x}$ where A and B are constants, prove that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-6y = 0.$$

(a) (i) Let $y = \dfrac{x^3-1}{\sqrt{x}}$, i.e. $y = \dfrac{x^3-1}{x^{\frac{1}{2}}}$.

Now divide out and differentiate term by term.

Answer: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{5x^3+1}{2\sqrt{x^3}}$

(ii) Straightforward differentiation of a logarithm.

Answer: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x+3}{x^2+3x+1}$.

(iii) Straightforward differentiation of a product.

Answer: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x\,(\cos x-2x.\sin x)$.

(iv) Differentiation of a quotient. After differentiation simplify numerator by use of the identity $\cos^2 x+\sin^2 x = 1$.

Answer: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3.\sin x-1}{\cos^2 x}$.

(b) $$y = Ae^{2x} + Be^{-3x}.$$

Differentiate, and then use equation above to substitute for Be^{-3x} in terms of y and Ae^{2x}. This will give

$$\frac{dy}{dx} = -3y + 5Ae^{2x}. \tag{1}$$

Differentiate again, and then use eqn. (1) to eliminate Ae^{2x}. Collect up terms to obtain the required equation.

FURTHER EXAMPLES

No. 1. (a) Differentiate from first principles $y = 2x^2 - 2x$.

(b) Differentiate by inspection

(i) $y = 3x^4 + 3x + 2$,

(ii) $s = 8t^3 - 4t^2 + 2t + 3$. (U.E.I.)

No. 2. Differentiate with respect to x:

(i) $(4-3x^2)^{\frac{1}{2}}$; (ii) $\log_e \sqrt{1+x}$; (iii) $\log_e (\cos x)$;

(iv) $e^{2x}.\cos x$; (v) $x^3.\sin 2x$; (vi) $\dfrac{e^{2x}+1}{2e^x}$;

(vii) $\dfrac{\sin 2x}{x^2}$.

No. 3. (a) Differentiate with respect to x:

(i) $\sin x.e^{\sin x}$; (ii) $\dfrac{4x^2}{x-1}$; (iii) $\frac{1}{2}\log_e (x^2+1)$.

(b) If $y = e^x.\sin x$, show that

$$\frac{d^2y}{dx^2} = 2.\left\{\frac{dy}{dx} - y\right\}. \tag{U.L.C.I.}$$

No. 4. Write down the differential coefficient (dy/dx) when

(a) $y = \dfrac{x^2+3}{\sin x}$; (b) $y = x.\log_e x$;

(c) $y = \sin 3x.\cos 2x$; (d) $y = 5e^{3x}.\cos(2x+\alpha)$. (U.E.I.)

ANSWERS

No. 1. (a) $4x-2$; (b) (i) $12x^3+3$; (ii) $24t^2-8t+2$.

No. 2. (i) $-9x(4-3x^2)^{\frac{1}{2}}$; (ii) $\frac{1}{2(1+x)}$; (iii) $-\tan x$;

(iv) $e^{2x}.(2\cos x-\sin x)$; (v) $x^2.(3.\sin 2x+2x.\cos 2x)$;

(vi) $\frac{e^{2x}-1}{2e^x}$; (vii) $\frac{2(x.\cos 2x-\sin 2x)}{x^3}$.

No. 3. (a) (i) $\cos x.e^{\sin x}.(1+\sin x)$; (ii) $\frac{4x(x-2)}{(x-1)^2}$;

(iii) $\frac{x}{x^2+1}$.

No. 4. (a) $\frac{2x.\sin x-(x^2+3).\cos x}{\sin^2 x}$; (b) $1+\log_e x$;

(c) $\cos 3x.\cos 2x+2.\cos 5x$; (d) $5e^{3x}.\{3.\cos(2x+\alpha)-2.\sin(2x+\alpha)\}$.

RATES OF CHANGE
(INCLUDING TANGENTS TO CURVES)

WORKED EXAMPLES

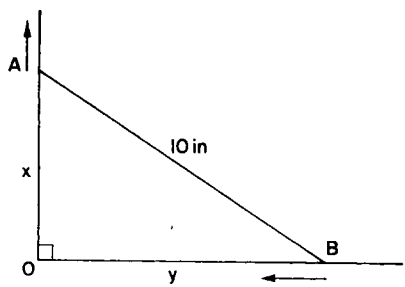
No. 1

(a) An arm AB 10 in. long is arranged so that its ends can slide freely in two slots OA, OB which intersect at O and are at 90° to each other. Find the velocity of point A when point B is 6 in. away from O and approaching it at a velocity of 3 ft/sec.

(b) The displacement s feet of a particle after time t seconds is given by

$$s = t^4 - 3t^3 + 2t^2 + 5.$$

Find: (i) an expression for the velocity at time t seconds;
(ii) an expression for the acceleration at time t seconds;
(iii) the minimum value of this acceleration;
(iv) the distance travelled from rest to the instant of minimum acceleration. (U.E.I.)



(a) Let the distances OA, OB be x, y respectively at any instant. Then since $\angle AOB = 90°$ and $AB = 10$ in. are constant,

$$x^2 + y^2 = 100 \qquad (1)$$

at all times.

Differentiating eqn. (1) with respect to time, t seconds,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \therefore \quad x\frac{dx}{dt} = -y\frac{dy}{dt} \qquad (2)$$

The velocity of the point B at any instant is the rate of change of y with respect to time at that instant, i.e. velocity of B is $\mathrm{d}y/\mathrm{d}t$. Similarly, velocity of A is $\mathrm{d}x/\mathrm{d}t$.

When $y = 6$ in., $\mathrm{d}y/\mathrm{d}t = -3$ ft/sec and hence from eqn. (2)

$$x\frac{\mathrm{d}x}{\mathrm{d}t} = +\frac{6}{12}.3 = \frac{18}{12} = \frac{3}{2}.$$

But from eqn. (1) when $y = 6$ in., $x^2+36 = 100$; i.e. $x^2 = 64$, i.e. $x = 8$ in.

$$\therefore \quad \frac{8}{12}.\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{2}, \quad \text{i.e. } \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{9}{4} = \underline{+2\tfrac{1}{4} \text{ ft/sec.}}$$

$\mathrm{d}y/\mathrm{d}t$ was taken as negative since y is decreasing with time (i.e. B is approaching O). Since $\mathrm{d}x/\mathrm{d}t$ is positive, x is increasing with time.

$\therefore$ When B is 6 in. away from O and approaching at the rate of 3 ft/sec, A is going away from O at the rate of $2\frac{1}{4}$ ft/sec.

(b) Distance moved, s feet, after time t seconds is given by

$$s = t^4-3t^3+2t^2+5. \tag{1}$$

(i) Velocity, v feet per second, is the rate of change of distance with time;

i.e. $$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \underline{4t^3-9t^2+4t.}$$

(ii) Acceleration, f ft/sec^2, is the rate of change of velocity with time;

i.e. $$f = \frac{\mathrm{d}v}{\mathrm{d}t} = \underline{12t^2-18t+4.} \tag{2}$$

(iii) Values of t leading to extreme values of the acceleration f are given by $\mathrm{d}f/\mathrm{d}t = 0$. Hence from eqn. (2):

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 24t-18. \tag{3}$$

Also $$\frac{\mathrm{d}^2f}{\mathrm{d}t^2} = 24.$$

Since $\mathrm{d}^2f/\mathrm{d}t^2$ is always positive any extreme value of f must be a minimum value.

From eqn. (3), when $\dfrac{\mathrm{d}f}{\mathrm{d}t} = 0$, $24t-18 = 0$,

i.e. $24t = 18$, $\underline{\text{i.e. } t = \tfrac{3}{4} \text{ sec.}}$

Substituting $t = \frac{3}{4}$ in eqn. (2), $f = 12\times\frac{9}{16}-18\times\frac{3}{4}+4$

$$\therefore \quad f = \frac{27}{4}-\frac{27}{2}+4 = \frac{27-54+16}{4} = -\frac{11}{4}$$

$$= \underline{-2\tfrac{3}{4} \text{ ft/sec}^2.}$$

(iv) Body is initially at rest when $t = 0$.

From eqn. (1), when $t = 0$, $s = 5$ ft.

Also from eqn. (1),

$$\text{when } t = \frac{3}{4}, \quad s = \frac{81}{256} - \frac{81}{64} + \frac{18}{16} + 5 \text{ ft.}$$

$\therefore$ The distance travelled from rest to the position of minimum acceleration is

$$\frac{81 - 324 + 288}{256} = \frac{45}{256} = 0{\cdot}176 \text{ ft}$$

$$= 0{\cdot}176 \times 12 \text{ in.} = 2{\cdot}112 = \underline{2{\cdot}11 \text{ in.}}$$

No. 2

The velocity in feet per second of a point moving in a straight line at any time t seconds is given by $2\pi \sin(4\pi t)$. After a time of 1·5 sec, the point has moved 5 ft. Find expressions for the distance moved and the acceleration after t seconds. (U.L.C.I.)

After t seconds, let the distance moved be s feet, the velocity be v feet per second, and the acceleration f feet per second

$$\therefore \quad v = 2\pi \sin(4\pi t). \tag{1}$$

But velocity is rate of change of distance with time, i.e. $v = \mathrm{d}s/\mathrm{d}t$.

$$\therefore \quad \frac{\mathrm{d}s}{\mathrm{d}t} = 2\pi \sin(4\pi t).$$

$\therefore$ Integrating $s = -\frac{1}{2}\cos(4\pi t) + C$ where C is a constant.
But $s = 5$ ft when $t = 1{\cdot}5$ sec.

$$\therefore \quad 5 = -\tfrac{1}{2}\cos(6\pi) + C$$

$$\therefore \quad 5 = 0 + C \quad \therefore \quad C = 5$$

$$\therefore \quad \underline{s = 5 - \tfrac{1}{2}\cos(4\pi t).}$$

Acceleration is rate of change of velocity with time, i.e. $f = \mathrm{d}v/\mathrm{d}t$.

$\therefore$ From eqn. (1) $\quad f = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \underline{8\pi^2 \cos(4\pi t).}$

No. 3

A line which has a slope of 0·5 is a tangent to the curve $y = x^2+x$. Find (a) the coordinates of the point where the line touches the curve, (b) the coordinates of the points where the line cuts the y-axis and the x-axis.

(U.L.C.I.)

(a) The slope of a tangent is given by the value of dy/dx at the point of contact of the tangent with the curve.

$$y = x^2+x \tag{1}$$

$$\therefore \quad \frac{dy}{dx} = 2x+1.$$

Let the tangent with slope 0·5 touch the curve at the point (x_1, y_1) then

$$0{\cdot}5 = 2x_1+1, \text{ i.e. } \tfrac{1}{2} = 2x_1+1 \quad \therefore \underline{x_1 = -\tfrac{1}{4}.}$$

Since the point (x_1, y_1) lies on the curve its coordinates satisfy the equation of the curve.

$\therefore$ From eqn. (1) $y_1 = (-\frac{1}{4})^2+(-\frac{1}{4}) = \frac{1}{16}-\frac{1}{4} = \underline{-\frac{3}{16}.}$

$\therefore$ The coordinates of the point of contact are $\underline{(-\frac{1}{4}, -\frac{3}{16}).}$

(b) The tangent has slope $\frac{1}{2}$ and passes through the point $(-\frac{1}{4}, -\frac{3}{16})$. The equation of the tangent can be put in the form $y = mx+c$, where m is the gradient and c is a constant. But $m = \frac{1}{2}$, therefore equation of tangent is

$$y = \tfrac{1}{2}x+c.$$

But line passes through $(-\frac{1}{4}, -\frac{3}{16})$

$$\therefore \quad -\tfrac{3}{16} = \tfrac{1}{2}(-\tfrac{1}{4})+c$$

$$\therefore \quad -\tfrac{3}{16} = -\tfrac{1}{8}+c \qquad \therefore \quad c = -\tfrac{1}{16}.$$

$\therefore$ Equation of tangent is

$$\underline{y = \tfrac{1}{2}x-\tfrac{1}{16}.} \tag{2}$$

The equation of the y-axis is $x = 0$.

$\therefore$ Intersection of tangent with the y-axis is found by putting $x = 0$ in eqn. (2), whence $y = -\frac{1}{16}$.

$\therefore$ Tangent cuts the y-axis in the point $\underline{(0, -\frac{1}{16}).}$

The equation of the x-axis is $y = 0$.

Intersection of tangent with the x-axis is found by putting $y = 0$ in eqn. (2), whence $0 = \frac{1}{2}x-\frac{1}{16}$, i.e. $x = \frac{1}{8}$.

$\therefore$ Tangent cuts the x-axis in the point $\underline{(\frac{1}{8}, 0).}$

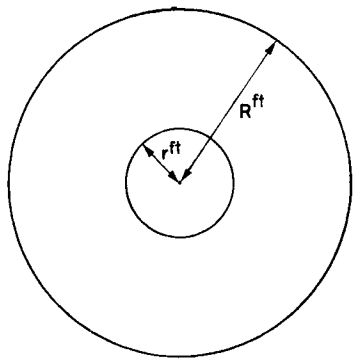
HINTED EXAMPLES

No. 1. At a certain instant the radii of two concentric spheres are 8 ft and 2 ft; the radius of the outer sphere is increasing at the rate of 2 in/sec, and the radius of the inner one is increasing at the rate of 10 in/sec. Find the rate (in cubic feet per second) at which the volume between the spheres is changing, and state whether it is increasing or decreasing.

Let the radius of the outer sphere be R feet, and the radius of the inner sphere be r feet, at any instant. Let the volume between the two spheres be V cubic feet. Let t denote time in seconds.

Using the formula for the volume of a sphere, obtain an equation for V in terms of R and r, i.e.

$$V = \frac{4}{3}\pi(R^3 - r^3).$$



The rates of change of R, r and V with respect to time are

$$\frac{dR}{dt}, \frac{dr}{dt} \text{ and } \frac{dV}{dt}$$

respectively.

Differentiate the equation connecting V, R and r with respect to t, to obtain the relationship

$$\frac{dV}{dt} = 4\pi\left\{R^2.\frac{dR}{dt} - r^2.\frac{dr}{dt}\right\}. \qquad (1)$$

At a certain instant $R = 8$ ft, $r = 2$ ft. $dR/dt = 2$ in/sec, $dr/dt = 10$ in/sec. Substitute these values into eqn. (1), having first converted dR/dt and dr/dt into feet per second.

Evaluate to obtain $dV/dt = 92{\cdot}2$ ft^3/sec. Since this is positive volume is increasing.

Answer: volume is increasing at the rate of 92·2 ft^3/sec.

No. 2. (a) A particle travels so that its displacement S inches from a fixed point O after t seconds, is given by

$$S = 2t^3 - 13t^2 - 4t + 5.$$

At what subsequent time is the particle at rest and what is its acceleration at this time in ft/sec^2 units?

(b) The equation of a curve is given by $y = x^2 + 5x - 2$. Find the co-ordinates of the point on the curve at which the tangent to the curve is parallel to the line $2y + x - 15 = 0$.

(a) Let v and f represent the velocity and acceleration respectively. Then

$$v = \frac{ds}{dt} \text{ and } f = \frac{dv}{dt}.$$

By differentiation find the expression for v in terms of t. To find at what time the particle is at rest put $v = 0$ in the expression and solve the resulting quadratic equation in t.

The negative root is not a possible solution.

Answer: $t = 4{\cdot}48$ sec.

From the expression for v in terms of t, obtain an expression for f in terms of t.

To find the required acceleration substitute $t = 4{\cdot}48$ sec into this expression for f, and convert to the required units.

Answer: $2{\cdot}32$ ft/sec^2.

(b) The expression for dy/dx obtained from the equation $y = x^2+5x-2$, will give the gradient of the tangent to any point of the curve.

Show that the gradient of the line $2y+x-15 = 0$ is $-\frac{1}{2}$. Let (x_1, y_1) be the coordinates of the point on the curve at which the tangent to the curve is parallel to the line $2y+x-15 = 0$; i.e. the point on the curve at which the tangent has a gradient of $-\frac{1}{2}$.

Substitute $x = x_1$ into the expression for dy/dx, put $dy/dx = -\frac{1}{2}$, and solve for x_1. $x_1 = -\frac{1}{4}\frac{1}{4}$.

Since (x_1, y_1) is a point on the curve its coordinates satisfy the equation of the curve, i.e. $y_1 = x_1^2+5x_1-2$.

Substitute $x_1 = -\frac{1}{4}\frac{1}{4}$ and solve for y_1.

Answer: point is $(-\frac{1}{4}\frac{1}{4}, -8\frac{3}{1}\frac{3}{6})$.

Note. Part (b). The gradient of the line $2y+x-15 = 0$ can either be found by comparing with the standard form for a straight line, $y = mx+c$; or by differentiation.

FURTHER EXAMPLES

No. 1. A body moves s (ft) in t (sec) where $s = 6+4t+10t^2-t^3$. By means of the calculus, find: (i) its speed at the end of 2 sec; (ii) its acceleration at the end of 3 sec; (iii) when its speed is zero; (iv) when its acceleration is zero. (U.E.I.)

No. 2. A curve is represented by $y = 2x^3 - 18x^2 + 32x - 4$. Find an expression for the gradient of the curve and hence find the coordinates of the points where the gradient is $+2$. (U.L.C.I.)

No. 3. The angular acceleration of a wheel is given by $d^2\theta/dt^2 = \pi/6$ (radians/sec^2), where θ radians is the angle turned through after t seconds. If $\theta = 0$ and $d\theta/dt = 4\pi$ when $t = 0$, obtain a formula for θ in terms of t and find the number of revolutions which the wheel makes in the first minute. (E.M.E.U.)

No. 4. The torque in the crankshaft of a certain engine in pound-feet is $T = 1200 + 500.\sin\theta - 200.\sin 2\theta$, where θ is the angle turned through by the crank. If the crankshaft is rotating at a speed of 200 rad/sec find in lb ft/min the rate at which the torque is changing when $\theta = \pi/3$.

No. 5. The equation of a curve is given by $y = x^3 - 3x^2 - 6x + 10$. Find the equations of the two tangents to this curve that both have a gradient of $+3$.

ANSWERS

No. 1. (i) 32 ft/sec; (ii) 2 ft/sec^2; (iii) 6·9 sec; (iv) $3\frac{1}{3}$ sec.

No. 2. (1, 12); (5, -44).

No. 3. $\theta = \pi/12.t^2 + 4\pi t$; 270.

No. 4. 5,400,000 lb ft/min.

No. 5. $y - 3x - 15 = 0$;
$y - 3x + 17 = 0$.

MAXIMA AND MINIMA

WORKED EXAMPLES

No. 1

Find the maximum and minimum values of the expression x^3+4x^2-3x+1, stating the corresponding values of x, and distinguish between them.

Let $$y = x^3+4x^2-3x+1$$

$$\therefore \quad \frac{dy}{dx} = 3x^2+8x-3.$$

At turning points of the function $dy/dx = 0$, hence turning points are given by

$$3x^2+8x-3 = 0,$$

i.e. $$(3x-1)(x+3) = 0.$$

$\therefore$ Turning points are given by $x = \frac{1}{3}$ and $x = -3$.

$$\frac{d^2y}{dx^2} = 6x+8.$$

When $x = \frac{1}{3}$, $\dfrac{d^2y}{dx^2}$ is positive.

$\therefore$ $x = \frac{1}{3}$ gives a minimum value of y.

When $x = -3$, $\dfrac{d^2y}{dx^2}$ is negative.

$\therefore$ $x = -3$ gives a maximum value of y.

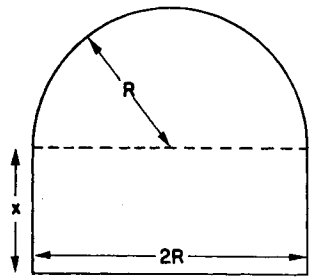
When $x = \frac{1}{3}$: $y = \frac{1}{27}+\frac{4}{9}-1+1 = \frac{13}{27}$.

When $x = -3$: $y = -27+36+9+1 = 19$.

Answer: maximum value is 19, when $x = -3$: minimum value is $\frac{13}{27}$, when $x = \frac{1}{3}$.

No. 2

A tunnel has a cross-section in the shape of a rectangle surmounted by a semicircle. The total cross-sectional area is 120 ft^2. Find an expression for the perimeter in terms of the radius R feet and hence determine the value of R which makes the perimeter a minimum. Calculate this minimum value. (U.E.I.)



Let height of rectangle be x feet.
Radius of semicircle is R feet.
$\therefore$ Width of rectangle is $2R$ feet.
Area of rectangle is $2Rx$ square feet.
Area of semicircle is $\frac{1}{2}\pi R^2$ square feet.
Total cross-sectional area is 120 ft^2.

$$\therefore \quad 2Rx+\tfrac{1}{2}\pi R^2 = 120$$

$$\therefore \quad x = \frac{1}{2R}(120-\tfrac{1}{2}\pi R^2). \qquad (1)$$

Let P feet be the perimeter of the section, then

$$P = \pi R+2R+2x.$$

Hence substituting for x from eqn. (1)

$$P = \pi R+2R+\frac{1}{R}(120-\tfrac{1}{2}\pi R^2)$$

$$\therefore \quad P = \tfrac{1}{2}\pi R+2R+\frac{120}{R}. \qquad (2)$$

A maximum or minimum value of P is given by a value of R satisfying $dP/dR = 0$,

$$\frac{dP}{dR} = \tfrac{1}{2}\pi+2-\frac{120}{R^2}.$$

When $dP/dR = 0$,

No.	Log
120	2·0792
3·571	0·5528
R^2	1·5264
5·797	0·7632

$$\frac{120}{R^2} = \tfrac{1}{2}\pi+2 = \frac{3\cdot142}{2}+2$$

$$= 1\cdot571+2 = 3\cdot571$$

$$\therefore \quad R^2 = \frac{120}{3\cdot571}. \qquad \underline{R = \pm 5\cdot8 \text{ ft.}}$$

(The negative root may be disregarded.)

$$\frac{d^2P}{dR^2} = \frac{240}{R^3}.$$

$\therefore$ When $R = +5\cdot8$ feet $\dfrac{dP}{dR} = 0$ and $\dfrac{d^2P}{dR^2}$ is positive.

Hence $R = 5\cdot8$ ft makes the perimeter a minimum.

The actual minimum value is obtained by substituting $R = 5{\cdot}8$ ft in eqn. (2).

No.	Log
3·571	0·5528
5·8	0·7634
20·71	1·3162
120	2·0792
5·8	0·7634
20·69	1·3158

$\therefore$ Minimum value of perimeter

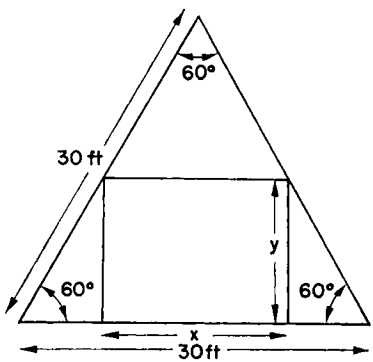
$$= 3{\cdot}571 \times 5{\cdot}8 + \frac{120}{5{\cdot}8}$$

$$= 20{\cdot}71 + 20{\cdot}69 = \underline{41{\cdot}4 \text{ ft.}}$$

Note. The extreme value *must be shown* to be a minimum value by considering d^2P/dR^2 or otherwise.

HINTED EXAMPLE

Find the area of the ground plan of the greatest rectangular building which can be erected on a plot of ground in the shape of an equilateral triangle whose side is 30 ft, assuming that one side of the building lies along the side of the triangle. (E.M.E.U.)



Let the side of the rectangle lying along the side of the triangle be x feet, and the adjacent sides be y feet each. If A square feet is the area of the rectangle then

$$A = xy. \qquad (1)$$

By trigonometry establish the relationship

$$x + \frac{2y}{\sqrt{3}} = 30. \qquad (2)$$

Eliminate x between eqns. (1) and (2) to give

$$A = 30y - \frac{2y^2}{\sqrt{3}}. \tag{3}$$

Differentiate, put $dA/dy = 0$ and solve:

$$y = \frac{15\sqrt{3}}{2} = 13{\cdot}0 \text{ (correct to 3 sig. fig.).}$$

Obtain d^2A/dy^2 and use this to show that

$$y = \frac{15\sqrt{3}}{2}$$

does give a maximum value of A.

To find this maximum value, substitute

$$y = \frac{15\sqrt{3}}{2}$$

in eqn. (3).

Answer: 195 ft^2.

FURTHER EXAMPLES

No. 1. A two-stage compressor has an interstage pressure of x. The work done (W) by the compressor can be represented by the expression

$$W = 10\left\{\frac{x^2}{225} + 18225x^{-2} - 2\right\}.$$

Show that the work done is a minimum when $x = 45$. (U.L.C.I.)

No. 2. A sheet metal rectangular box, without a lid, has square ends of side x feet. If its volume is $20\frac{5}{6}$ ft^3, show that the area of sheet metal is given by

$$A = 2x^2 + \frac{125}{2x}.$$

Find the least area of sheet metal required to construct the box. (Overlapping of metal may be ignored.) (W.J.E.C.)

No. 3. Show that for a conical hopper without a lid, the greatest volume for a given surface area occurs when the radius r is given by $r = h/\sqrt{2}$, where h is the vertical height.

No. 4. $ABCD$ is a trapezium in which $AB = 2$ in., $BC = 4$ in., angle BCD is acute and $= \theta$, angle BAD = angle $CDA = 90°$. Prove that the area A of the trapezium $= 8 \sin\theta + 4 \sin 2\theta$ square inches. Show that $dA/d\theta = 16 \cos^2\theta + 8 \cos\theta - 8$ and determine the value of θ which makes A a maximum, verifying that your result does give a true maximum. Find also the maximum value of A. (W.J.E.C.)

ANSWERS

No. 2. $37\frac{1}{2}$ ft^2. No. 4. $\theta = 60°$; $A = 10{\cdot}4$ in^2.

INTEGRATION

WORKED EXAMPLES

No. 1

Evaluate the following integrals:

$$\int_0^{\pi/3} (\sin x + 2x + 1)\,dx; \quad \int_1^3 \frac{1}{x+1}\,dx; \quad \int_0^2 e^{2x}dx \qquad \text{(U.L.C.I.)}$$

$$\int_0^{\pi/3} (\sin x + 2x + 1)\,dx = \left[-\cos x + \frac{2x^2}{2} + x \right]_0^{\pi/3}$$

$$= \left[-\tfrac{1}{2} + \left(\frac{\pi}{3}\right)^2 + \frac{\pi}{3} \right] - [-1+0+0]$$

$$= \frac{1}{2} + \frac{\pi}{3}\left(\frac{\pi}{3}+1\right)$$

$$= \underline{2{\cdot}64.}$$

$$\int_1^3 \frac{1}{x+1}\,dx = [\log_e (x+1)]_1^3 = \log_e 4 - \log_e 2$$

$$= \log_e (\tfrac{4}{2})$$

$$= \log_e 2$$

$$= \underline{0{\cdot}693.}$$

$$\int_0^2 e^{2x}\,dx = [\tfrac{1}{2}e^{2x}]_0^2 = \tfrac{1}{2}(e^4 - e^0) = \tfrac{1}{2}(e^4 - 1)$$

$$= \underline{26{\cdot}8.}$$

Notes

(1) The integral of "sin x" is "minus cos x" $(-\cos x)$.

(2) When a function integrates to a logarithm form, the logarithm is to the *base e*.

(3) Any quantity raised to the power zero is unity.

No. 2

(a) Obtain the following integrals:

$$\text{(i) } \int e^{x/2}dx; \text{ (ii) } \int_1^{10}\left(3x+\frac{1}{x}\right)dx; \text{ (iii) } \int \sin^2 x\,dx; \text{ (iv) } \int_0^8 \sqrt{3x+1}\,dx.$$

(b) If the slope of a curve at any point is given by $(3x+2)^2$ and the curve passes through the origin, find its equation. (W.J.E.C.)

(a) (i) $\int e^{x/2}dx = \underline{2e^{x/2}+C.}$

$$\text{(ii) } \int_1^{10}\left(3x+\frac{1}{x}\right)dx = \left[\frac{3x^2}{2}+\log_e x\right]_1^{10} = \left(\frac{300}{2}+\log_e 10\right)-\left(\frac{3}{2}+\log_e 1\right)$$

$$= \frac{297}{2}+\log_e 10 = 148{\cdot}5+2{\cdot}303.$$

Answer: 150·803.

(iii) $\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}(x-\frac{1}{2}\sin 2x)+C.$

Answer: $\frac{1}{4}(2x-\sin 2x)+C.$

$$\text{(iv) } \int_0^8 \sqrt{3x+1}\,dx = \int_0^8 (3x+1)^{\frac{1}{2}}.dx.$$

Put $3x+1 = u$, then $3dx = du$, and when $x = 0$, $u = 1$ and when $x = 8$, $u = 25$, hence integral becomes

$$\int_1^{25} u^{\frac{1}{2}}.\tfrac{1}{3}.du = \tfrac{1}{3}\int_1^{25} u^{\frac{1}{2}}.du = \tfrac{1}{3}\left[\tfrac{2}{3}.u^{\frac{3}{2}}\right]_1^{25}$$

$$= \tfrac{2}{9}\{(25)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\} = \tfrac{2}{9}(5^3-1) = \tfrac{2}{9}(125-1).$$

Answer: $27\frac{5}{9}$.

(b) Slope of a curve is given by dy/dx, hence

$$\frac{dy}{dx} = (3x+2)^2.$$

$\therefore$ Integrating $\quad y = \frac{1}{3}(3x+2)^3 . \frac{1}{3} + C = \frac{1}{9}(3x+2)^3 + C.$

But curve passes through the origin, i.e. the point where $x = 0$ and $y = 0$, hence

$$0 = \tfrac{1}{9}(2)^3 + C = \tfrac{8}{9} + C. \qquad \therefore \quad C = -\tfrac{8}{9}.$$

$\therefore$ Equation is $\quad y = \frac{1}{9}(3x+2)^3 - \frac{8}{9}.$

Answer: $9y = (3x+2)^3 - 8.$

No. 3

Evaluate the following:

(a) $\displaystyle\int_0^{\pi/3} (\cos 2\theta - \sin\theta)\, d\theta$ (b) $\displaystyle\int_4^5 \frac{2x+9}{x^2-x-6}.dx$

(c) $\displaystyle\int_2^3 \frac{x^3+3x^2+2x+8}{x}.dx$ (U.E.I.)

(a) $\displaystyle\int_0^{\pi/3} (\cos 2\theta - \sin\theta)\, d\theta = [\tfrac{1}{2}.\sin 2\theta + \cos\theta]_0^{\pi/3}$

$$= \left\{\frac{1}{2}.\sin\frac{2\pi}{3} + \cos\frac{\pi}{3}\right\} - \{\tfrac{1}{2}.\sin 0 + \cos 0\}$$

$$= \left\{\frac{1}{2}\times\frac{\sqrt{3}}{2} + \frac{1}{2}\right\} - \{0+1\} = \tfrac{1}{4}(\sqrt{3}-2)$$

$$= \underline{-0{\cdot}067.}$$

(b) $$\int_4^5 \frac{2x+9}{x^2-x-6}.dx = \int_4^5 \frac{2x+9}{(x+2)(x-3)}.dx$$

Splitting into partial fractions, let

$$\frac{2x+9}{(x+2)(x-3)} \equiv \frac{A}{x+2} + \frac{B}{x-3} \equiv \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$$

$$\therefore \quad 2x+9 \equiv A(x-3)+B(x+2).$$

This is an identity and thus true for all values of x.

Put $x = 3$ $\quad 15 = 5B$ $\quad \therefore \quad B = 3.$

Put $x = -2$ $\quad 5 = A(-5)$ $\quad \therefore \quad A = -1.$

Hence

$$\int_4^5 \frac{2x+9}{x^2-x-6}.\mathrm{d}x = \int_4^5 \left(\frac{3}{x-3}-\frac{1}{x+2}\right).\mathrm{d}x$$

$$= [3.\log_e(x-3)-\log_e(x+2)]_4^5$$
$$= \{3.\log_e 2-\log_e 7\}-\{3.\log_e 1-\log_e 6\}$$
$$= 3.\log_e 2-\log_e 7-0+\log_e 6$$
$$= 3.\log_e 2-\log_e 7+\log_e 2+\log_e 3$$
$$= 4.\log_e 2+\log_e 3-\log_e 7.$$

Answer: 1·93.

(c) $$\int_2^3 \frac{x^3+3x^2+2x+8}{x}.\mathrm{d}x = \int_2^3 \left(x^2+3x+2+\frac{8}{x}\right).\mathrm{d}x$$

$$= \left[\frac{x^3}{3}+\frac{3x^2}{2}+2x+8.\log_e x\right]_2^3 = \left\{\frac{(3)^3}{3}+\frac{3.(3)^2}{2}+2.3+8.\log_e 3\right\}$$
$$-\left\{\frac{(2)^3}{3}+\frac{3.(2)^2}{2}+2.2+8.\log_e 2\right\}$$

$$= 9+\tfrac{27}{2}+6+8.\log_e 3-\tfrac{8}{3}-6-4-8.\log_e 2 = \tfrac{95}{6}+8.\log_e(\tfrac{3}{2}).$$

Answer: 19·1.

Note. In part (a) the lower limit in the definite integral is zero, but care must be taken not to ignore any term, especially in trigonometric functions. The cosine of a zero angle is unity.

HINTED EXAMPLES

No. 1. Evaluate:

(a) $\displaystyle\int_1^3 \frac{x^3+3}{x}.\mathrm{d}x$; (b) $\displaystyle\int_0^{\pi/6} (\sin 2\theta-\sin\theta)\,\mathrm{d}\theta$; (c) $\displaystyle\int_3^4 \frac{x+1}{x^2+3x-10}.\mathrm{d}x$.

(U.E.I.)

(a) Divide out and integrate.

Answer: $8\frac{2}{3}+3.\log_e 3 = 11{\cdot}963.$

(b) Straightforward integration, take care not to ignore lower limit (as $\cos 0 = 1$).

Answer: $\frac{1}{4}(2\sqrt{3}-3) = 0{\cdot}116.$

(c) Split into partial fractions

$$\frac{x+1}{x^2+3x-10} = \frac{3}{7(x-2)}+\frac{4}{7(x+5)}.$$

Integrate and evaluate.

Answer: $\frac{1}{7}(-9.\log_e 2+8.\log_e 3) = 0{\cdot}364.$

No. 2. If $d^2y/dx^2 = 5x^2-7x+2$ and $y = 2$ when $x = 1$, and $y = 1$ when $x = 0$, find y in terms of x. (W.J.E.C.)

Two integrations are required to obtain y from d^2y/dx^2 and an arbitary constant must be introduced at each integration. Integrate twice to obtain $y = \frac{5}{12}x^4-\frac{7}{6}x^3+x^2+Bx+C$.

Since $y = 1$, when $x = 0$, substitution of these values in the above equation will give $C = 1$.

Since $y = 2$ when $x = 1$, substitution of these values in the above equation will give $B+C = \frac{7}{4}$. Evaluate B and C.

Answer: $y = \frac{1}{12}(5x^4-14x^3+12x^2+9x+12).$

FURTHER EXAMPLES

No. 1. Integrate the following with respect to x:

(i) ae^{bx}; (ii) $-3.\cos(-3x)$; (iii) $-1/(2x)$.

No. 2. Evaluate:

(i) $\int_1^2 (2x-3)^4\,dx$; (ii) $\int_0^1 (e^{-x}+\cos x)\,dx$; (iii) $\int_2^4 \frac{dx}{(5x-3)^2}$;

(iv) $\int_0^{\pi/2} \left(\cos\frac{3\theta}{2}-\sin\frac{\theta}{2}\right) d\theta$; (v) $\int_1^4 \frac{dx}{4x+2}$; (vi) $\int_2^3 \left(x+\frac{1}{x}\right) dx$.

No. 3. Integrate using partial fractions:

(a) $\int_2^3 \frac{2x-8}{2x^2-x-1}\,.\,dx$; (b) $\int_0^1 \frac{(13-3x)\,dx}{2+5x-3x^2}$.

No. 4. (a) Obtain the following integrals:

(i) $\int \frac{dx}{\sqrt{1-2x}}$; (ii) $\int 2.e^{2x}\,dx$; (iii) $\int_2^{20} \frac{3}{x}\,dx$; (iv) $\int_0^{\pi/2} \cos^2 x\,dx$.

(b) A curve passes through the point (0, 1) and its gradient, dy/dx, is $(2x-\frac{1}{2}x^2)$ for all values of x. Find the value of y when $x = 3$. (W.J.E.C.)

ANSWERS (C is an arbitary constant)

No. 1. (i) $(a/b)\,e^{bx}+C$; (ii) $\sin(-3x)+C$; (iii) $-\frac{1}{2}\log_e x+C$.

No. 2. (i) $\frac{1}{5}$; (ii) 1·47; (iii) $\frac{2}{119}$; (iv) $\frac{2}{3}.(2\sqrt{2}-3) = -0{\cdot}115$;
(v) $\frac{1}{4}\log_e 3 = 0{\cdot}275$; (vi) $\frac{5}{2}+\log_e(\frac{3}{2}) = 2{\cdot}905$.

No. 3. (a) $3.\log_e 1{\cdot}4-2.\log_e 2 = -0{\cdot}337$; (b) $\log_e 32 = 3{\cdot}47$.

No. 4. (a) (i) $-(1-2x)^{\frac{1}{2}}+C$; (ii) $e^{2x}+C$; (iii) $3.\log_e 10 = 6{\cdot}91$; (iv) $\pi/4$.
(b) $y = 5\frac{1}{2}$.

AREAS, VOLUMES, CENTROIDS AND SECOND MOMENTS OF AREA

WORKED EXAMPLES

No. 1

(a) Prove that the ratio of the mean value to the maximum value of the function $x(10-x)$, in the range $x = 0$ to $x = 10$, is $\frac{2}{3}$.

(b) Sketch the graphs of $y = x^2$ and $y = 5x-4$. Calculate the coordinates of their points of intersection and the area of the segment of the curve cut off by the line. (W.J.E.C.)

(a) $$y = x(10-x). \tag{1}$$

Mean value of the function $= \dfrac{1}{10-0}\cdot\int_0^{10} .x(10-x)\,dx$

$$= \frac{1}{10}\cdot\int_0^{10}(10x-x^2)\,dx$$

$$= \frac{1}{10}\cdot\left[5x^2-\frac{x^3}{3}\right]_0^{10}$$

$$= \frac{1}{10}\left(500-\frac{1000}{3}\right) = 50-\frac{100}{3}$$

$$= \frac{150-100}{3} = \underline{\frac{50}{3}}. \tag{2}$$

From eqn. (1) $$\frac{dy}{dx} = 10-2x \tag{3}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -2.$$

Since d^2y/dx^2 is negative any extreme value of y must be a maximum. The value of x which gives this maximum value of y is found by putting $dy/dx = 0$.

From eqn. (3) $$\frac{dy}{dx} = 10-2x = 0,$$

i.e. $$2x = 10 \quad \therefore \quad \underline{x = 5.}$$

Substituting in eqn. (1), the maximum value of y in the range is

$$y = 5(10-5) = \underline{25.} \tag{4}$$

From eqns. (2) and (4) the ratio of the mean value to the maximum value is

$$\frac{50}{3} \div 25 = \frac{50}{3 \times 25} = \underline{\frac{2}{3}}.$$

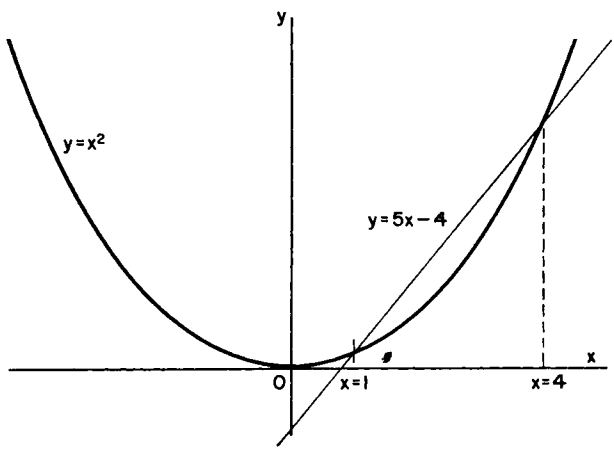
(b) The values of x at the points of intersection of $y = 5x-4$ and $y = x^2$ are given by the equation

$$x^2 = 5x-4,$$

i.e. $$x^2-5x+4 = 0,$$

i.e. $$(x-1)(x-4) = 0.$$

$\therefore$ $$\underline{x = 1 \text{ or } x = 4.}$$



From the equation $y = 5x-4$, when $x = 1$, $y = 5-4 = 1$
and when $x = 4$, $y = 20-4 = 16$.

$\therefore$ The coordinates of the points of intersection are (1, 1) and (4, 16).

Area of the segment of the curve cut off by the line is A, where

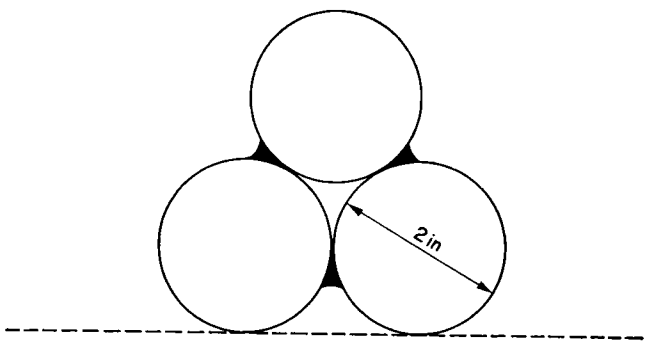
$$A = \int_1^4 (5x-4)\,dx - \int_1^4 x^2\,dx$$
$$= \int_1^4 (5x-4-x^2)\,dx$$
$$= \left[\tfrac{5}{2}x^2-4x-\tfrac{1}{3}x^3\right]_1^4$$
$$= (40-16-\tfrac{64}{3})-(\tfrac{5}{2}-4-\tfrac{1}{3})$$
$$= 25\tfrac{1}{2}-21 = 4\tfrac{1}{2} \text{ square units.}$$

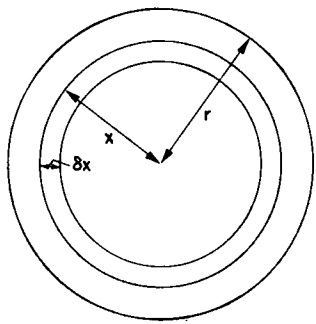
Notes. Part (b). The area of the segment of the curve cut off by the line is found by subtracting the area between the curve $y = x^2$, the x-axis and the limits $x = 1$, $x = 4$, from the area between the line $y = 5x-4$, the x-axis and the same limits $x = 1$, $x = 4$. (The limits are determined by the x-coordinates of the points of intersection of the two graphs.)

No. 2

Show that the second moment of area of a circular lamina about a diameter is $\frac{1}{4}\pi r^4$.

A girder is made of three steel bars each 2 in. diameter welded together in such a way that each bar touches the other two. Neglecting the area of the welds, find the second moment of area of the cross-section of the girder about an axis which is a common tangent to two of the bars. (U.L.C.I.)





Consider an axis through the centre of the circular lamina perpendicular to the lamina. Consider an elemental ring of the lamina of width δx and radius x. The area of the ring is $\simeq 2\pi x.\delta x$.

The second moment of the ring about the axis is $x^2.2\pi x.\delta x$, i.e. $2\pi x^3.\delta x$.

$\therefore$ The second moment of area of the circular lamina about this axis is

$$\int_0^r 2\pi x^3.\mathrm{d}x$$

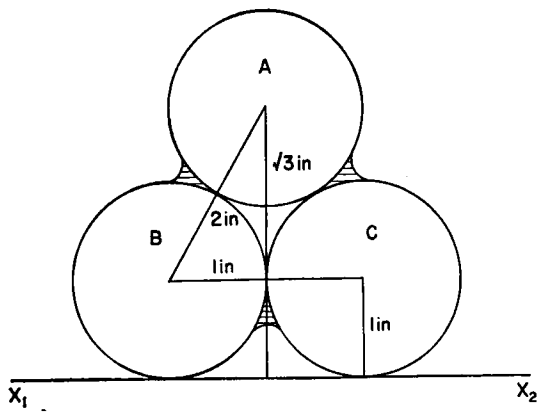
$$= 2\pi.\left[\frac{x^4}{4}\right]_0^r = \underline{\frac{\pi r^4}{2}}.$$

By the perpendicular axes theorem the sum of the second moments of area about two perpendicular axes in the plane of the lamina is equal to the second moment of area about a third axis perpendicular to the plane of the lamina and passing through the point of intersection of the first two axes.

Hence, as any axis in the plane of the circular lamina that passes through the centre of the lamina must be a diameter, and if the second moment of area about a diameter is I,

$$2I = \frac{\pi r^4}{2},$$

$$\underline{I = \frac{\pi r^4}{4}}.$$



Radius of each circle is 1 in.

$\therefore$ Second moment of area of each circle about a diameter is $\pi/4$ in^4.

Area of each circle is π in^2.

Centres of circles B and C are each 1 in. from the axis X_1X_2.

$\therefore$ By parallel axis theorem, the second moment of area of each of circles B and C about X_1X_2 is

$$\frac{\pi}{4}+\pi\times(1)^2 = \underline{\frac{5\pi}{4}\text{ in}^4.}$$

Centre of circle A is $(1+\sqrt{3})$ in. from the axis X_1X_2.

$\therefore$ By parallel axis theorem, the second moment of area of circle A about X_1X_2 is

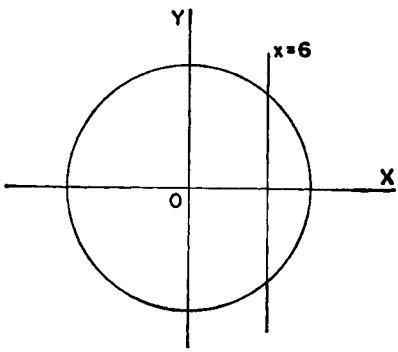
$$\underline{\frac{\pi}{4}+\pi(1+\sqrt{3})^2\text{ in}^4.}$$

$\therefore$ The second moment of area of the complete cross-section about axis X_1X_2 is

$$\begin{aligned}
&2\times\left(\frac{5\pi}{4}\right)+\left\{\frac{\pi}{4}+\pi(1+\sqrt{3})^2\right\}\\
&=\frac{11\pi}{4}+\pi(1+2\sqrt{3}+3)\\
&=2{\cdot}75\pi+4\pi+2\sqrt{3}.\pi\\
&=\pi.(6{\cdot}75+3{\cdot}464)\\
&=\pi(10{\cdot}214)\\
&=\underline{32{\cdot}1\text{ in}^4.}
\end{aligned}$$

No. 3

A steel hemisphere of radius 10 in. is cut into two parts by a plane parallel to its flat face and 6 in. from it. Find the volume of the larger of the two parts and also the position of its centre of gravity. (U.L.C.I.)



Consider the circle radius 10 in. with centre at the origin. This has equation

$$x^2+y^2 = 100. \qquad (1)$$

If the portion of this circle in the first quadrant is rotated about the x-axis, a hemisphere of radius 10 in. will be obtained.

Hence the volume of the required portion of a hemisphere can be represented by V, where

$$V = \pi \int_0^6 y^2 \, dx$$

$\therefore$ From eqn. (1)

$$V = \pi \int_0^6 (100 - x^2) \, dx$$

$$= \pi[100x - \tfrac{1}{3}x^3]_0^6$$

$$= \pi.(600 - 72) \qquad = \underline{528\pi \text{ in}^3}.$$

By symmetry the centre of gravity will lie on the axis of the hemisphere, i.e. on the x-axis. Hence the centre of gravity has coordinates $(\bar{x}, 0)$, where $\bar{x}$ is given by

$$\bar{x} = \frac{\pi \int_0^6 xy^2 \, dx}{V}$$

$$\therefore \quad \bar{x} = \frac{\pi \int_0^6 x(100 - x^2) \, dx}{528\pi}$$

No.	Log
369	2·5670
132	2·1206
2·795	0·4464

$$= \frac{1}{528}[50x^2 - \tfrac{1}{4}x^4]_0^6$$

$$= \frac{1}{528}(1800 - 324) = \frac{1476}{528}$$

$$= \frac{369}{132} = \underline{2{\cdot}8 \text{ in.}}$$

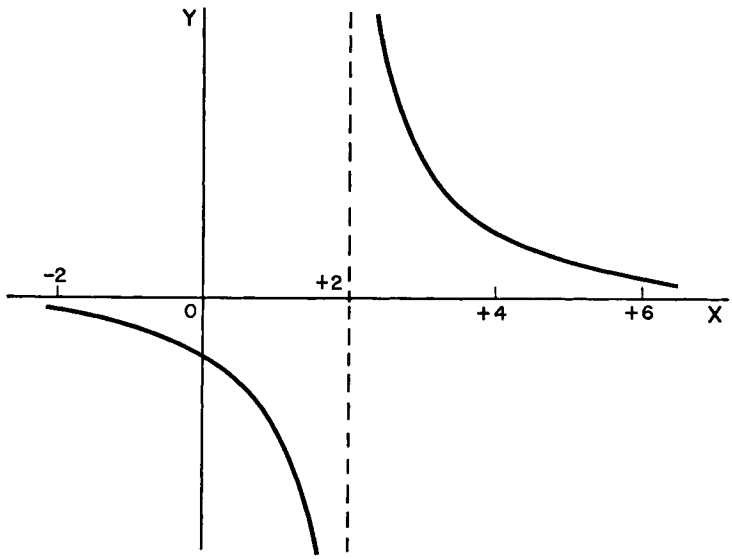
Answers: volume of section of steel hemisphere is 528π in^3 and its centre of gravity lies on the perpendicular to the faces through their centres at a distance of 2·8 in. from the larger face.

HINTED EXAMPLES

No. 1. Sketch the curve $y = 1/(x-2)$ between the limits $x = -2$ and $x = +6$.

Find the area bounded by the ordinates at $x = 3$ and $x = 6$, the curve and the x-axis.

Find also the distances of the centroid of this area from the axes of coordinates. (U.E.I.)



To find the area use the formula, area $= \int_3^6 y\,dx$. Substitute for y and evaluate the integral. Area = 1·386 square units.

Let the centroid of the area have coordinates $(\bar{x}, \bar{y})$, then these are given by the formulae:

$$\bar{x} = \frac{\int_3^6 xy\,dx}{\text{area}} \quad \text{and} \quad \bar{y} = \frac{\int_3^6 (y^2/2)\,dx}{\text{area}}.$$

Substitute for y in terms of x and use the value already calculated for the area as denominator in each case.

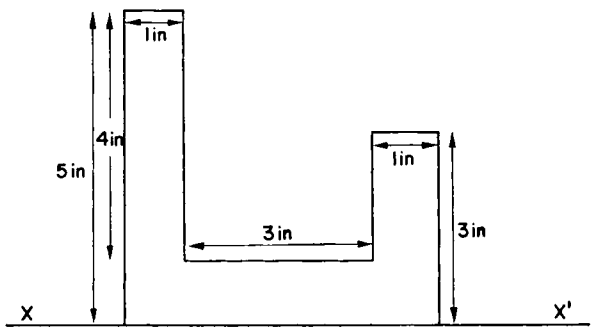
When evaluating $\bar{x}$, divide out the fraction under the integral sign to obtain

$$\bar{x} = \frac{1}{1{\cdot}386}\int_3^6 \left(1+\frac{2}{x-2}\right) dx.$$

Answers: $\bar{x} = 4{\cdot}16$, $\bar{y} = 0{\cdot}270$ units.

No. 2. (a) Show, by integration, that the second moment of a rectangle, whose sides are of length b inches, d inches, about an axis parallel to the side of length b inches and passing through the centroid is $bd^3/12$ in^4. Use the parallel axis theorem to find the second moment of area about a side of length b inches.

(b) Find the second moment of area and the radius of gyration of the figure shown about the axis XX'.



(W.J.E.C.)

(a) Theory.

(b) Divide the area into three rectangles,

(1) 5 in. by 1 in. with the 1 in. side on axis XX',

(2) 3 in. by 1 in. with the 3 in. side on axis XX',

(3) 3 in. by 1 in. with the 1 in. side on axis XX'.

Calculate the second moments of these areas about the axis XX' separately, using the formula found in part (a). Then add the results together to obtain the second moment of area of the complete figure.

Answer: $51\frac{2}{3}$ in^4.

To find radius of gyration use $I = Ak^2$, where I is the second moment of area, A is the area and k is the radius of gyration.

Thus $k^2 = 4{\cdot}70$ and $k = 2{\cdot}17$ in.

Answer: radius of gyration is 2·17 in.

FURTHER EXAMPLES

No. 1. Calculate the mean value of $2x(x^2-1)$ over the range $x = 0$ to $x = 4$. (U.E.I.)

No. 2. (a) Prove that the radius of gyration of a circular disc, radius r, about an axis perpendicular to the disc through its centre is $r/\sqrt{2}$.

Deduce, or find otherwise, the radius of gyration of a circular disc, radius r, about a diameter.

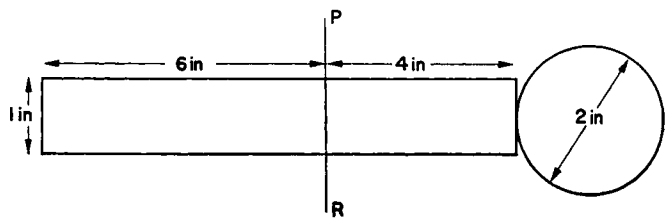
(b) Find, by integration, the radius of gyration of a right cone, radius of base R, about its geometric axis. (U.E.I.)

No. 3. Find the volume generated when the area A, which is bounded by the curve $y = \sin x$, the x-axis and the ordinates $x = \pi/3$ and $x = 2\pi/3$ is rotated about the x-axis. (W.J.E.C.)

No. 4. Find the second moment of area of a uniform rectangle, with sides of lengths L and b, about a side of length b.

Find the second moment of area of the vane shown in the diagram about the axis PR.

(The figure may be regarded as a rectangle and a circle tangential to the rectangle.)



(U.E.I.)

No. 5. Sketch the area above the line $y = 6$, bounded by the circle whose equation is $x^2+y^2 = 100$. Calculate the area of this segment and find by integration the volume generated when the segment is revolved through 360° about the x-axis.

Deduce the coordinates of the centroid of the area. (E.M.E.U.)

ANSWERS

No. 1. 28. No. 2. (a) $r/2$; (b) $\sqrt{0{\cdot}3}.R = 0{\cdot}548R$.

No. 3. 3·01 cubic units. No. 4. $L^3b/3$; 175 in^4.

No. 5. 44·8 square units; 2150 cubic units; (0, 7·6).

SIMPSON'S RULE

WORKED EXAMPLES

No. 1

The cross-sectional areas of masonry in a structure 36 ft high are given below at intervals of 3 ft vertically.

Height (ft)	0	3	6	9	12	15	18
Area (ft^2)	162	162	161	155	143	129	106
Height (ft)	21	24	27	30	33	36	
Area (ft^2)	68	45	35	26	26	26	

Using Simpson's rule find the volume of masonry in the structure. If the material weighs 140 lb/ft^3, calculate the weight of the structure to the nearest ton. (U.L.C.I.)

By Simpson's rule,

volume $= (w/3)\{y_1 + y_n + 4$ (even ordinates) $+ 2$ (remaining odd ordinates)$\}$,

where "ordinate" denotes a value of the cross-sectional area, y_1 is the first "ordinate", y_n is the last "ordinate" and w is the distance between successive measurements of cross-sectional area. In this case $w = 3$.

No. of ordinate	Value	Multiplication factor	
1	162	$\times 1$	$= 162$
2	162	$\times 4$	$= 648$
3	161	$\times 2$	$= 322$
4	155	$\times 4$	$= 620$
5	143	$\times 2$	$= 286$
6	129	$\times 4$	$= 516$
7	106	$\times 2$	$= 212$
8	68	$\times 4$	$= 272$
9	45	$\times 2$	$= 90$
10	35	$\times 4$	$= 140$
11	26	$\times 2$	$= 52$
12	26	$\times 4$	$= 104$
13	26	$\times 1$	$= 26$
		$\therefore$ Total	$= 3450$

$\therefore$ Volume $= \frac{1}{3} \times 3450\ ft^3$.

Answer: volume $= 3450\ ft^3$.

Material weighs 140 lb/ft^3, hence weight of structure is

$$\frac{3450 \times 140}{2240} \text{ tons} = \frac{1725}{8} \text{ tons} = 215\tfrac{5}{8} \text{ tons.}$$

Answer: weight of structure is 216 tons (to the nearest ton).

Note. In any purely numerical work, care in the setting out will help avoid many mistakes. In the above method the left-hand column is not absolutely necessary.

No. 2

Evaluate, using Simpson's rule with nine ordinates, the area given by

$$\int_0^4 \frac{1}{\sqrt{2+x^2}} . dx. \qquad \text{(U.E.I.)}$$

Interval from 0 to 4 with nine ordinates gives width of strip as 0·5, i.e. $w = 0{\cdot}5$.

x	$2+x^2$	$\sqrt{2+x^2}$	$y = \frac{1}{\sqrt{2+x^2}}$	
0	2	1·41	0·709	×1 = 0·709
0·5	2·25	1·5	0·667	×4 = 2·668
1·0	3·0	1·73	0·578	×2 = 1·156
1·5	4·25	2·06	0·486	×4 = 1·944
2·0	6·0	2·45	0·408	×2 = 0·816
2·5	8·25	2·87	0·348	×4 = 1·392
3·0	11·0	3·32	0·301	×2 = 0·602
3·5	14·25	3·78	0·265	×4 = 1·060
4·0	18·0	4·24	0·236	×1 = 0·236
				Total = 10·583

Hence value of integral $= \frac{0{\cdot}5}{3} \times 10{\cdot}583$

$$= \frac{10{\cdot}583}{6} = 1{\cdot}764.$$

Answer: value of integral = 1·76.

Notes

(1) Evaluation of a definite integral implies the calculation of an area under a curve. In this case the area between the curve

$$y = \frac{1}{\sqrt{2+x^2}}$$

and the x-axis and the ordinates $x = 0$, $x = 4$.

(2) Since the area is to be calculated using *nine ordinates*, the area has to be divided into *eight strips*. The width of the strip is therefore the width of the total area divided by eight, i.e. $w = \frac{4}{8} = 0{\cdot}5$.

FURTHER EXAMPLES

Note. When using Simpson's rule an *even* number of strips must be taken.

No. 1. The table below gives the relationship between x, the distance in feet from one end of a beam, and A, the corresponding cross-sectional area in square feet.

x	0	4	8	12	16	20	24
A	1·4	1·8	1·95	2·15	2·4	2·8	3·0

Estimate its volume using Simpson's rule. If the beam weighs 1·15 tons and is of uniform density, express the density in lb/in^3. (U.L.C.I.)

No. 2. Using Simpson's rule evaluate $\int_0^4 \sqrt{16-x^2}\,dx$.

No. 3. Draw the graph of $y = 2.\sin(\frac{1}{2}x+0{\cdot}2)$ from $x = 0$ to $x = 2$ radians. From this graph obtain the value of $\int_0^2 2.\sin(\frac{1}{2}x+0{\cdot}2)\,dx$ as accurately as possible and check your result by integration. (U.E.I.)

ANSWERS

No. 1. Volume = 53·5 ft^3. Density = 0·028 lb/in^3.
No. 2. Integral = 12·6.
No. 3. Integral = 2·47.