

Daniele Mundici

Advanced Łukasiewicz calculus and MV-algebras

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Advanced Łukasiewicz calculus and MV-algebras

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To Cecilia

Preface

This book is designed as a text for a second course in infinite-valued Łukasiewicz logic and its algebras, Chang’s MV-algebras. It is also intended as a source of reference for the more advanced readers, and is a continuation of the monograph by Cignoli et al., “*Algebraic Foundations of Many-Valued Reasoning*,” which may be used as a suitable text for a first course. I give complete versions of a compact body of recent results and techniques, virtually proving everything that is used throughout. So if I have accomplished my purpose, this book should be usable for individual study.

Modern Łukasiewicz logic and MV-algebra theory draw on three principal sources: polyhedral topology, functional analysis, and lattice-ordered abelian groups (ℓ -groups henceforth). This is so because

Every free MV-algebra is an algebra of $[0,1]$ -valued piecewise linear functions f over some unit cube, each linear piece of f having integer coefficients. Zerosets of these functions are, on the one hand, models of formulas in Łukasiewicz propositional logic $\mathcal{L}_\infty$, and on the other hand, they are the most general rational polyhedra contained in some cube $[0, 1]^n$.

For any MV-algebra A , regular Borel probability measures on the maximal spectral space of A correspond to de Finetti’s coherent probability assessments on the events represented by A , as an algebra of equivalence classes of formulas in Łukasiewicz logic.

There is a categorical equivalence Γ between MV-algebras A and unital ℓ -groups $(G, 1)$, those ℓ -groups having a distinguished order unit.

Just as the $\mathbb{Z}$ -module structure of $(G, 1)$ is missing in the MV-algebra $A = \Gamma(G, 1)$, several fundamental notions and constructs available in the framework of MV-algebras and Łukasiewicz logic hardly make any sense for unital ℓ -groups, despite the latter are categorically equivalent to MV-algebras. Thus, the equational definability of the class of MV-algebras gives us a way of introducing free and finitely presented objects—while the class of unital ℓ -groups is not even definable

in first-order logic. Induction on the complexity of Łukasiewicz formulas, combined with their geometric representation as McNaughton functions, is a main tool to explore syntactic and semantic consequence in $\mathcal{L}_\infty$, and the fundamental logic property of interpolation. Formulas in $\mathcal{L}_\infty$ denote continuously valued events, just as boolean formulas denote yes-no events; coherent probability assessments on these events yield Rényi conditionals, which would make no sense for unital ℓ -groups. σ -complete MV-algebras provide a natural framework for generalizations of many classical results originally proved for σ -complete boolean algebras, such as the theorem of Loomis–Sikorski and Poincaré’s recurrence theorem. Several main techniques and results of probability theory, that Carathéodory reformulated in the language of σ -complete boolean algebras, have nontrivial MV-algebraic generalizations. Bases originate as algebraically invariant counterparts of disjunctive Schauder normal forms in Łukasiewicz logic; an MV-algebra has a basis iff it is finitely presented. Classical first-order logic with identity has a generalization to a Łukasiewicz first-order logic $\mathcal{L}_{\omega\omega}$ with $[0,1]$ -valued identity. Models of $\mathcal{L}_{\omega\omega}$ are suitable sets X of unit vectors in a Hilbert space $\mathcal{H}$, and the identity degree of any two vectors $u, v \in X$ is their scalar product; functions and relations on X satisfy suitable continuity properties.

Since this book is devoted to these genuine MV-algebraic and logical topics, its overlap with books on ℓ -groups, with or without unit, is negligible.

Every chapter in this book relies on a combination of classical, as well as of recent mathematical results, well beyond the traditional domain of algebraic logic.

The first prerequisite for a profitable reading is familiarity with the main theorems of Łukasiewicz logic and MV-algebra theory, notably Chang completeness theorem, McNaughton representation of free MV-algebras, Wójcicki’s analysis of consequence in Łukasiewicz logic, and the properties of the Γ functor. Secondly, the reader is assumed to have some acquaintance with a few basic facts of polyhedral topology and functional analysis. As is often the case in the study of advanced mathematical topics, detailed knowledge of the proofs of all background results is less important than knowing a place in the literature where one can go and look—if the need ever arises to check a proof.

To help the reader, all background results used in the course of the book are collected in two final Appendices, together with references for their proofs. The notation (B21.50) will refer to entry 21.50 in Appendix B.

This book has grown out of lectures delivered at various universities and summer schools during the last ten years.

I have made much use of conversations and correspondence with many friends and colleagues. I owe a particular debt of gratitude to Ettore Casari, Roberto Cignoli, Janusz Czelakowski, Antonio Di Nola, Sergio Doplicher, Anatolij Dvurečenskij, László Fuchs, Andrew Glass, Marco Grandis, Petr Hájek, Charles Holland, Tomáš Kroupa, Ioana Leuştean, Jorge Martínez, Franco Montagna, Hiroakira Ono, Beloslav Riečan, Constantine Tsinakis, Hans Weber, and Ryszard Wójcicki. From the late Sauro Tulipani I learned that de Finetti’s coherence criterion can be applied to events described by Łukasiewicz logic.

I am also grateful to my former students Stefano Aguzzoli, Agata Ciabattoni, Brunella Gerla, and Giovanni Panti.

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Florence, November 2010

D. Mundici

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Notation and Terminology

The symbol $\Rightarrow$ is to be read “implies”. The symbol $\Leftrightarrow$ is to be read “iff”, which is short for “if, and only if”. The symbols $\exists$ and $\forall$ are to be read “there is an” and “for all”, respectively.

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively, denote the set of integer, rational and real numbers.

By a *countable* set we mean a set whose cardinality is either finite or equal to the cardinality of the set of integers.

A family $\mathcal{F}$ of subsets of a set X is said to have the *finite intersection property* if for every finite set $F_1, \dots, F_k$ of members of $\mathcal{F}$ the intersection $F_1 \cap \dots \cap F_k$ is nonempty.

For any two sets $E \subseteq F$ we let χ_E denote the *characteristic function* of E in F , i.e., the function $\chi_E: F \rightarrow \{0, 1\}$ defined by $\chi_E^{-1}(1) = E$. The ambient set F will always be clear from the context.

For every function $f: F \rightarrow G$ and $E \subseteq F$ we let $f|_E$ denote the restriction of f to E . For any two sets D and V we let V^D be the set of all functions $f: D \rightarrow V$. The notation $f: x \mapsto y$ stands for $f(x) = y$. Given functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ we denote by $gf: X \rightarrow Z$ the composite function defined by $(gf)(x) = g(f(x))$ for all $x \in X$.

For any topological space Y and subset X of Y , we denote by $\text{cl}(X)$ the closure of X in Y (the latter being always clear from the context). Similarly, $\text{int}(X)$ denotes the interior of X .

Unless otherwise specified, the adjective *linear* is understood in the affine sense.

For each $n = 1, 2, \dots$ we let $\mathbb{R}^n$ be n -dimensional euclidean space. We further let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{R}^n$, and $\pi_1, \dots, \pi_n$ the coordinate (= identity = projection) functions restricted to the unit n -cube $[0, 1]^n$.

For any subset S of $\mathbb{R}^n$ we denote by $\text{conv}(S)$ the set of all convex combinations of elements of S . Thus $x \in \text{conv}(S)$ iff there are $x_1, \dots, x_k \in S$ and real numbers $\lambda_1, \dots, \lambda_k \geq 0$ such that $\lambda_1 + \dots + \lambda_k = 1$ and $x = \lambda_1 x_1 + \dots + \lambda_k x_k$. The set S is said to be *convex* if $S = \text{conv}(S)$.

A hyperplane H is a *supporting hyperplane* of a closed convex set $T \subseteq \mathbb{R}^n$ if $H \cap T \neq \emptyset$ and $T \subseteq H^+$ or $T \subseteq H^-$, where $H^\pm$ are the two closed half-spaces bounded by H . The set $T \cap H$ is said to be a *face* of T . By convention, $\emptyset$ and T are called the *improper faces* of T . All other faces of T are said to be *proper*.

For any subset S of $\mathbb{R}^n$ we denote by $\text{aff}(S)$ the *affine hull* of S , i.e., the set of all *affine combinations* in $\mathbb{R}^n$ of elements of S . Thus $x \in \text{aff}(S)$ iff there are $x_1, \dots, x_k \in S$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that $\lambda_1 + \dots + \lambda_k = 1$ and $x = \lambda_1 x_1 + \dots + \lambda_k x_k$. A set $\{y_1, \dots, y_m\}$ of points in $\mathbb{R}^n$ is said to be *affinely independent* if none of its elements is an affine combination of the remaining elements.

The *relative interior* $\text{relint}(S)$ of a convex set S in $\mathbb{R}^n$ is the interior of S in the affine hull of S .

As usual, gcd and lcm denote greatest common divisor and least common multiple.

Unless otherwise specified, in every MV-algebra considered in this book the unit and the zero element will be distinct.

We let $\text{hom}(A)$ denote the set of homomorphisms of the MV-algebra A into the MV-algebra $[0, 1]$. For every homomorphism η of A into an MV-algebra B , the *kernel* $\ker(\eta)$ of η , is defined by $\ker(\eta) = \eta^{-1}(0)$.

For each $k = 1, 2, \dots$, we denote by L_k the $(k + 1)$ -element Łukasiewicz chain $\{0, 1/k, \dots, (k - 1)/k, 1\}$. This is denoted $\mathbb{L}_{k+1}$ in [1, p. 8].

Reference

1. Cignoli, R. L. O., D'Ottaviano, I. M. L., Mundici, D. (2000). *Algebraic foundations of many-valued reasoning*. Volume 7 of Trends in Logic. Dordrecht: Kluwer.

Chapter 1

Prologue: de Finetti Coherence Criterion and Łukasiewicz Logic

In this chapter we will see that coherent probability assessments on (not necessarily yes–no) events, such as those given by the measurement of physical observables, are convex combinations of valuations in Łukasiewicz propositional logic $\mathbb{L}_\infty$. Besides familiarity with [1], the only prerequisite for this chapter is some acquaintance with the very basic properties of convex sets in euclidean space.

1.1 Events, Possible Worlds and de Finetti Coherence Criterion

Just as the measurement of an observable of a physical system in a given state outputs a real number x —and after a suitable normalization, x can be assumed to lie in the unit interval $[0,1]$ —similarly a possible world assigns a (truth-)value $x \in [0, 1]$ to any event. In particular, the value x assigned to a yes–no event X is 1 if X occurs, and 0 otherwise. If X has a continuous spectrum, our expectation “ X has a large value” is made precise by the result of the measurement/observation of X . More details will be given in Sect. 1.6 of this chapter.

Stripping away all inessentials, given an integer $n > 0$ and two sets $E = \{X_1, \dots, X_n\}$ and $W \subseteq [0, 1]^E$, let us imagine two players, Ada and Blaise, wagering money on the possible occurrence of the “events” of E in the future “possible worlds” of W . Ada, who is a mathematical bookmaker, proclaims her “betting odd” $\beta(X_i) \in [0, 1]$, and Blaise, the bettor, chooses a “stake” σ_i for each $X_i \in E$. Then Blaise pays Ada $\sigma_i \cdot \beta(X_i)$ euros ($i = 1, \dots, n$), with the stipulation that Ada will pay back $\sigma_i \cdot w(X_i)$ euros in the possible world $w \in W$ where the value $w(X_i)$ is made known. Ada is so confident in her “book” β that Blaise is allowed to put down a *negative stake* σ_i , should he rate $\beta(X_i)$ excessive. The result is a “reverse bet”: Ada now pays Blaise $|\sigma_i| \cdot \beta(X_i)$ euros, to receive $|\sigma_i| \cdot w(X_i)$ in the possible world w . The total balance of this bet on events $X_1, \dots, X_n$, with stakes $\sigma_1, \dots, \sigma_n \in \mathbb{R}$, in the possible world w is

$$\sum_{i=1}^n \sigma_i(\beta(X_i) - w(X_i)),$$

where money transfers are conventionally oriented so that “positive” means Blaise-to-Ada. Ada’s book $\beta: E \rightarrow [0, 1]$ would quickly lead her to financial disaster if in every possible world this total balance is < 0 . For, assuming the set W of possible worlds is closed in $[0, 1]^n$, by suitably rescaling his stakes, Blaise might ensure a net profit of at least one zillion euros whatever happens. Adopting the understatements which are so common in contemporary economic theory, we give the following

Definition 1.1 Fix an integer $n > 0$. For any two sets $E = \{X_1, \dots, X_n\}$ and $W \subseteq [0, 1]^E$, we say that a map $\beta: E \rightarrow [0, 1]$ is *W-incoherent* if for some $\sigma_1, \dots, \sigma_n \in \mathbb{R}$ the inequality $\sum_{i=1}^n \sigma_i(\beta(X_i) - w(X_i)) < 0$ holds for all $w \in W$. Otherwise, β is *W-coherent*.

1.2 Coherence and Valuations in Łukasiewicz Logic

Theorem 1.4 will establish a first connection between coherent assessments and valuations in Łukasiewicz logic. The theorem will be continued in Theorem 10.7.

In preparation for these results, for each $n = 1, 2, \dots$, we let FORM_n denote the set of formulas $\psi(X_1, \dots, X_n)$ whose variables are contained in the set $\{X_1, \dots, X_n\}$; ψ is the same as a formula in boolean logic, except that conjunction and disjunction are written as $\odot$ and $\oplus$ instead of $\wedge$ and $\vee$. As is well known, the lattice connectives $\wedge$ and $\vee$ in $\mathbb{L}_\infty$ are different from the basic connectives $\odot$ and $\oplus$.

We will write $\alpha \rightarrow \beta$ as an abbreviation of $\neg\alpha \oplus \beta$. As usual, $\alpha \leftrightarrow \beta$ stands for $(\alpha \rightarrow \beta) \odot (\beta \rightarrow \alpha)$.

More generally, for any set $\mathcal{X}$ of variables, we denote by $\text{FORM}_\mathcal{X}$ the set of formulas whose variables are among those of $\mathcal{X}$. For each formula ϕ we let $\text{var}(\phi)$ be the set of variables occurring in ϕ . For any set $\Phi \subseteq \text{FORM}_\mathcal{X}$ we also use the notation $\text{var}(\Phi)$ for $\bigcup\{\text{var}(\psi) \mid \psi \in \Phi\}$.

As usual, when writing formulas we assume that $\neg$ is more binding than $\odot$, and the latter is more binding than $\oplus$.

Definition 1.2 A *valuation* (of FORM_n in $\mathbb{L}_\infty$) is a function $V: \text{FORM}_n \rightarrow [0, 1]$ such that

$$\begin{aligned} V(\neg\phi) &= 1 - V(\phi) \\ V(\phi \oplus \psi) &= V(\phi) \oplus V(\psi) = \min(1, V(\phi) + V(\psi)) \\ V(\phi \odot \psi) &= V(\phi) \odot V(\psi) = \max(0, V(\phi) + V(\psi) - 1). \end{aligned}$$

We let VAL_n denote the set of valuations of FORM_n . For each $w = (w_1, \dots, w_n) \in [0, 1]^n = [0, 1]^{\{X_1, \dots, X_n\}}$ we let V_w be the only valuation of VAL_n such that $V_w(X_i) = w_i$ for all $i = 1, \dots, n$. Thus, $w = V_w \upharpoonright \{X_1, \dots, X_n\}$.

More generally, for any set $\mathcal{X}$ of variables, $\mathbf{VAL}_{\mathcal{X}}$ denotes the set of valuations $V : \mathbf{FORM}_{\mathcal{X}} \rightarrow [0, 1]$. A formula ϕ is a *tautology* if $V(\phi) = 1$ for all valuations $V \in \mathbf{VAL}_{\text{var}(\phi)}$. To signify that ψ is a tautology we write $\vdash \psi$.

Definition 1.3 For every $n = 1, 2, \dots$ and nonempty set $Y \subseteq [0, 1]^n$ we define

$$\text{Th } Y = \{\psi \in \mathbf{FORM}_n \mid V_w(\psi) = 1 \text{ for all } w \in Y\}. \quad (1.1)$$

For any set $\Phi \subseteq \mathbf{FORM}_{\mathcal{X}}$ and $V \in \mathbf{VAL}_{\mathcal{X}}$ we say that V *satisfies* Φ if $V(\psi) = 1$ for all $\psi \in \Phi$. If there is a valuation V satisfying Φ we say that Φ is *satisfiable*. Otherwise Φ is *unsatisfiable*. When Φ is a singleton $\{\phi\}$ we define the (un)satisfiability of formula ϕ in the obvious way.

As usual, by a *convex combination* C of valuations $V_1, \dots, V_r \in \mathbf{VAL}_n$ we mean a function $C \in \mathbb{R}^{\mathbf{FORM}_n}$ of the form

$$C(\psi) = \lambda_1 V_1(\psi) + \dots + \lambda_r V_r(\psi) \text{ for all } \psi \in \mathbf{FORM}_n,$$

where $\lambda_1, \dots, \lambda_r$ are real coefficients ≥ 0 whose sum is 1. In general, C is not a valuation.

For any $n = 1, 2, \dots$ and set $E = \{X_1, \dots, X_n\}$ we will freely identify

$$[0, 1]^E = [0, 1]^{\{1, \dots, n\}} = [0, 1]^n.$$

Theorem 1.4 For any set $E = \{X_1, \dots, X_n\}$, closed nonempty set $W \subseteq [0, 1]^E$, and map $\beta : E \rightarrow [0, 1]$ the following conditions are equivalent:

- (i) β is W -coherent.
- (ii) There do not exist $\sigma_1, \dots, \sigma_n \in \mathbb{R}$ such that $\sum_{i=1}^n \sigma_i (\beta(X_i) - v(X_i)) < -1$ for all $v \in W$.
- (iii) β is a convex combination of points in W , in symbols, $\beta \in \text{conv}(W)$, (equivalently, β is a convex combination of at most $n + 1$ points in W).
- (iv) $\beta = C \upharpoonright \{X_1, \dots, X_n\}$ for some convex combination C in $\mathbb{R}^{\mathbf{FORM}_n}$ of valuations, all satisfying $\text{Th } W$.
- (v) $\beta = D \upharpoonright \{X_1, \dots, X_n\}$ for some convex combination D of at most $n + 1$ valuations $V_0, \dots, V_n \in \mathbf{VAL}_n$, each V_i satisfying $\text{Th } W$.

Proof of (i) $\Leftrightarrow$ ii $\Leftrightarrow$ iii) The implication (i $\Rightarrow$ ii) is trivial. For the converse, let us assume condition (i) fails for $\beta = (\beta(X_1), \dots, \beta(X_n)) \in \mathbb{R}^n = \mathbb{R}^E$. Using the notation $\circ$ for scalar product in $\mathbb{R}^n$, for some $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ we have $c \circ (\beta - v) < 0$ for all $v = (v(X_1), \dots, v(X_n)) \in W$. Since W is closed, for some $\epsilon > 0$ the continuous function $x \mapsto c \circ (\beta - x)$ attains its maximum value $-\epsilon$ at some point in W . Then the n -tuple $(\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$ given by $\sigma_i = 2c_i/\epsilon$ is a counterexample to (ii).

(iii $\Rightarrow$ i) Evidently, each $v \in W$ is a W -coherent map. We *claim* that W -coherence is preserved by convex combinations in $\mathbb{R}^E$ of elements $v_1, \dots, v_m \in W$.

Otherwise, (absurdum hypothesis) for some $0 \leq \lambda_1, \dots, \lambda_m$ with $\sum_{j=1}^m \lambda_j = 1$ the convex combination $c = \sum_{j=1}^m \lambda_j v_j$ is W -incoherent. There are real numbers

$\sigma_1, \dots, \sigma_n$ such that $\sum_{i=1}^n \sigma_i(c(X_i) - v(X_i)) < 0$, for all $v \in W$. In particular, for each $j = 1, \dots, m$, $\sum_{i=1}^n \sigma_i(c(X_i) - v_j(X_i)) < 0$, i.e., $\sum_{i=1}^n \sigma_i c(X_i) < \sum_{i=1}^n \sigma_i v_j(X_i)$. It follows that

$$\sum_{j=1}^m \lambda_j \left(\sum_{i=1}^n \sigma_i c(X_i) \right) < \sum_{j=1}^m \lambda_j \left(\sum_{i=1}^n \sigma_i v_j(X_i) \right)$$

i.e.,

$$\sum_{i=1}^n \sigma_i c(X_i) < \sum_{i=1}^n \sigma_i \sum_{j=1}^m \lambda_j v_j(X_i) = \sum_{i=1}^n \sigma_i c(X_i),$$

which is impossible. Having thus settled our claim, we have proved (iii $\Rightarrow$ i).

(i $\Rightarrow$ iii) Let us suppose $\beta \notin \text{conv}(W)$. Since W is compact, by (B21.50) so is its convex hull $\text{conv}(W)$. Then the classical separation argument (B21.51) yields a real number $\xi > 0$, together with vectors $a, b \in \mathbb{R}^n$ and a hyperplane $H = a^\perp + b \subseteq \mathbb{R}^n$ such that $a \circ \beta - a \circ w < -\xi$ for all $w \in \text{conv}(W)$. Here $a \neq 0$ and $a^\perp$ denotes the hyperplane $\{x \in \mathbb{R}^n \mid a \circ x = 0\}$, i.e., the orthogonal complement of a . The stakes $\sigma_i = a_i/\xi$, ($i = 1, \dots, n$) show that (ii) is false, whence a fortiori, β is not W -coherent. The parenthetical remark in (iii) follows from Carathéodory theorem (B21.55).

Also the equivalence (iv $\Leftrightarrow$ v) follows from Carathéodory theorem.

The proof that (iv $\Leftrightarrow$ iii) requires the introduction of additional material on Łukasiewicz logic $\mathbb{L}_\infty$ and free MV-algebras, to be used throughout the book.

1.3 McNaughton Functions and Free MV-Algebras

For $n = 1, 2, \dots$, a *McNaughton function* $f: [0, 1]^n \rightarrow [0, 1]$ is a continuous piecewise linear function all of whose linear pieces have integer coefficients. In other words, f is continuous and there are linear (affine) polynomials $l_1, \dots, l_k$ with integer coefficients such that for each $x \in [0, 1]^n$ there is $i = 1, \dots, k$ with $f(x) = l_i(x)$. We denote by $\mathcal{M}([0, 1]^n)$ the MV-algebra of all McNaughton functions over $[0, 1]^n$. More generally, for any nonempty $Y \subseteq [0, 1]^n$ we will denote by $\mathcal{M}(Y)$ the MV-algebra of restrictions to Y of the functions in $\mathcal{M}([0, 1]^n)$.

We say that formulas $\phi, \psi \in \text{FORM}_n$ are *equivalent*, in symbols, $\phi \equiv \psi$, if $V(\phi) = V(\psi)$ for all valuations $V \in \text{VAL}_n$. We denote by $\psi/\equiv$ the equivalence class of ψ . The dependence on $n = 1, 2, \dots$ will always be clear from the context.

Theorem 1.5 *For each $i = 1, \dots, n$, let $\pi_i: [0, 1]^n \rightarrow [0, 1]$ be the i th coordinate function.*

- (i) *The set $\text{FORM}_n/\equiv$ of equivalence classes of formulas of FORM_n equipped with the MV-algebraic operations inherited from the connectives $\neg, \odot$ and*

$\oplus$, coincides with the free MV-algebra $\mathbf{FREE}_n$ over the free generating set $\{X_1/\equiv, \dots, X_n/\equiv\}$.

- (ii) The map $X_i/\equiv \mapsto \pi_i$ uniquely extends to a homomorphism ι of $\mathbf{FREE}_n$ into $\mathcal{M}([0, 1]^n)$.
- (iii) In fact, ι is an isomorphism of $\mathbf{FREE}_n$ onto $\mathcal{M}([0, 1]^n)$.
- (iv) For each $\psi \in \mathbf{FORM}_n$, let

$$\hat{\psi} = \iota(\psi/\equiv) \quad (1.2)$$

denote the McNaughton function represented by ψ . Then

$$\hat{\psi}(w) = V_w(\psi) \text{ for all } w \in [0, 1]^n. \quad (1.3)$$

- (v) For every nonempty set $Y \subseteq [0, 1]^n$ and formula $\psi \in \mathbf{FORM}_n$,

$$\psi \in \text{Th } Y \Leftrightarrow \hat{\psi}(y) = 1 \text{ for all } y \in Y. \quad (1.4)$$

Proof A proof of (i) is obtainable from [1, 4.4.4, 4.5.5]. The universal property of free MV-algebras immediately yields (ii). By Chang completeness theorem (A21.17), ι is one-one. By McNaughton theorem (A21.48), ι is onto $\mathcal{M}([0, 1]^n)$. This proves (iii). Arguing by induction on the number of connectives in ψ , one routinely verifies (1.3) and settles (iv). Finally, (v) is a direct consequence of (iv). $\square$

Definition 1.6 For any set of formulas $\Phi \subseteq \mathbf{FORM}_n$, the set $\text{Mod}(\Phi) \subseteq [0, 1]^n$ is defined by $\text{Mod}(\Phi) = \{w \in [0, 1]^n \mid V_w(\phi) = 1 \text{ for all } \phi \in \Phi\}$. For $\theta \in \mathbf{FORM}_n$, instead of $\text{Mod}(\{\theta\})$ we write $\text{Mod}(\theta)$, or even $\text{Mod}_{\{X_1, \dots, X_n\}}(\theta)$, if clarity so demands.

Thus a formula $\phi \in \mathbf{FORM}_n$ is satisfiable iff $\text{Mod}_{\{X_1, \dots, X_n\}}(\phi)$ is nonempty; ϕ is a tautology iff $\text{Mod}_{\{X_1, \dots, X_n\}}(\phi) = [0, 1]^n$. Further, from (1.3) we have

$$\text{Mod}(\theta) = \hat{\theta}^{-1}(1). \quad (1.5)$$

1.4 $\Phi \vdash \psi$, ψ is a Consequence of Φ

The following fundamental result will find repeated use throughout this book:

Theorem 1.7 For all $n = 1, 2, \dots$ and $\theta, \phi \in \mathbf{FORM}_n$ the following conditions are equivalent:

- (i) Every valuation $V \in \mathbf{VAL}_n$ satisfying θ also satisfies ϕ ;
- (ii) $\text{Mod}(\theta) \subseteq \text{Mod}(\phi)$;
- (iii) For some integer $k > 0$ the formula $\theta^k \rightarrow \phi$ is a tautology, where θ^k is short for $\underbrace{\theta \odot \dots \odot \theta}_k$.
 $k \text{ occurrences of } \theta$

(iv) For some integer $k > 0$ the formula

$$\underbrace{\theta \rightarrow (\theta \rightarrow (\theta \rightarrow \dots \rightarrow (\theta \rightarrow (\theta \rightarrow \phi)) \dots))}_{k \text{ occurrences of } \theta} \quad (1.6)$$

is a tautology.

- (v) For some integer $k > 0$ there is a sequence of formulas $\alpha_0, \dots, \alpha_{k+1}$ such that $\alpha_0 = \theta$, $\alpha_{k+1} = \phi$, and for each $i = 1, \dots, k+1$ either α_i is a tautology, or there are $p, q \in \{0, \dots, i-1\}$ such that α_q is the formula $\alpha_p \rightarrow \alpha_i$.
- (vi) For some integer $k > 0$ there is a sequence of formulas $\alpha_0, \dots, \alpha_{k+1}$ such that $\alpha_0 = \theta$, $\alpha_{k+1} = \phi$, and for each $i = 1, \dots, k+1$ either α_i is a tautology in FORM_n , or there are $p, q \in \{0, \dots, i-1\}$ such that α_q is the formula $\alpha_p \rightarrow \alpha_i$.

Proof (i $\Leftrightarrow$ ii) By Definition 1.6.

(iii $\Leftrightarrow$ iv) One promptly verifies that the two formulas (1.6) and $\theta^k \rightarrow \phi$ are equivalent.

(v $\Leftrightarrow$ i) Follows from [1, 4.5.2, 4.6.7].

(v $\Leftrightarrow$ iv) Follows from [1, 4.6.4].

(vi $\Rightarrow$ v) Trivial.

(iv $\Rightarrow$ vi) By induction on k , one verifies that ϕ can be obtained as the final formula α_{k+1} of a sequence $\alpha_0, \dots, \alpha_{k+1}$ as in (v), only needing the assumed tautology (1.6). $\square$

Definition 1.8 For $\mathcal{X}$ a set of variables, $\emptyset \neq \Phi \subseteq \text{FORM}_{\mathcal{X}}$, and ψ a formula, we write $\Phi \vdash \psi$ [read: “ ψ is a (syntactic) consequence of Φ ”] if there is an integer $k > 0$ and a set $\{\phi_1, \dots, \phi_t\} \subseteq \Phi$ such that $(\phi_1 \odot \dots \odot \phi_t)^k \rightarrow \psi$ is a tautology.

By Chang completeness theorem together with Theorem 1.7(iii $\Leftrightarrow$ v), this definition is promptly seen to agree with the definition of syntactic consequence given in [1, 4.3.2].

Corollary 1.9 For every set $\mathcal{X}$ of variables, nonempty set $\Phi \subseteq \text{FORM}_{\mathcal{X}}$, and arbitrary formula ψ , the following conditions are equivalent:

- (a) $\Phi \vdash \psi$.
- (b) There is an integer $k > 0$ and a sequence $\phi_1, \dots, \phi_k \in \Phi$ such that the formula $\phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow \dots \rightarrow (\phi_{k-1} \rightarrow (\phi_k \rightarrow \psi)) \dots))$ is a tautology.
- (c) For some integer $t > 0$ there is a sequence of formulas $\beta_1, \dots, \beta_t$ such that $\beta_t = \psi$ and for each $i = 1, \dots, t$ either $\beta_i \in \Phi$, or β_i is a tautology, or there are $p, q \in \{1, \dots, i-1\}$ such that β_q is the formula $\beta_p \rightarrow \beta_i$.
- (d) For some integer $t > 0$ there is a sequence of formulas $\beta_1, \dots, \beta_t$ such that $\beta_t = \psi$ and for each $i = 1, \dots, t$ either $\beta_i \in \Phi$, or β_i is a tautology in $\text{FORM}_{\mathcal{X} \cup \text{var}(\psi)}$, or there are $p, q \in \{1, \dots, i-1\}$ such that β_q is the formula $\beta_p \rightarrow \beta_i$.

Proof This easily follows from Theorem 1.7(iii $\Leftrightarrow \dots \Leftrightarrow$ vi). $\square$

In the particular case when $\Phi = \{\phi\}$ we write $\phi \vdash \psi$ instead of $\{\phi\} \vdash \psi$. If $\Phi = \{\phi_1, \dots, \phi_r\}$, then $\Phi \vdash \psi$ iff $\phi_1 \odot \dots \odot \phi_r \vdash \psi$ iff $\phi_1 \wedge \dots \wedge \phi_r \vdash \psi$.

Corollary 1.10 For all $n = 1, 2, \dots$ and $\phi, \psi, \theta \in \text{FORM}_n$,

$$\theta \vdash \phi \leftrightarrow \psi \text{ iff } \hat{\phi} \upharpoonright \text{Mod}(\theta) = \hat{\psi} \upharpoonright \text{Mod}(\theta).$$

Proof Combine (1.2–1.5) with Theorem 1.7, and note that, by definition of $\rightarrow$, $\theta \vdash \phi \rightarrow \psi$ iff $\hat{\phi} \upharpoonright \text{Mod}(\theta) \leq \hat{\psi} \upharpoonright \text{Mod}(\theta)$. $\square$

1.5 Lindenbaum Algebras, End of Proof of Theorem 1.4

Definition 1.11 Fix $n = 1, 2, \dots$ and suppose $\Phi \subseteq \text{FORM}_n$ is satisfiable. Then for any $\phi, \psi \in \text{FORM}_n$ we write $\phi \equiv_{\Phi} \psi$ iff $\Phi \vdash \phi \leftrightarrow \psi$. For each formula $\varphi \in \text{FORM}_n$ we denote by $\varphi/\equiv_{\Phi}$ the $\equiv_{\Phi}$ -equivalence class of φ . The set of $\equiv_{\Phi}$ -equivalence classes forms an MV-algebra $\text{FORM}_n/\equiv_{\Phi}$, called the *Lindenbaum algebra* of Φ and denoted LIND_{Φ} . Thus,

$$\text{LIND}_{\Phi} = \left\{ \frac{\psi}{\equiv_{\Phi}} \mid \psi \in \text{FORM}_n \right\}. \quad (1.7)$$

In case $\Phi = \{\theta\}$ for some $\theta \in \text{FORM}_n$, we write LIND_{θ} instead of $\text{LIND}_{\{\theta\}}$, and $\equiv_{\theta}$ instead of $\equiv_{\{\theta\}}$.

Lemma 1.12 Let $\theta = \theta(X_1, \dots, X_n)$ be a satisfiable formula. Then the map $\lambda: \varphi/\equiv_{\theta} \mapsto \hat{\varphi} \upharpoonright \text{Mod}(\theta)$ is an isomorphism of LIND_{θ} onto $\mathcal{M}(\text{Mod}(\theta))$.

Proof If $\hat{\psi} \upharpoonright \text{Mod}(\theta) \neq \hat{\varphi} \upharpoonright \text{Mod}(\theta)$ then by (1.5), $V_w(\theta) = 1$ and $V_w(\psi) \neq V_w(\varphi)$ for some $w \in [0, 1]^n$, whence $\text{Mod}(\theta) \not\subseteq \text{Mod}(\psi \leftrightarrow \varphi)$. By Theorem 1.7, $\theta \not\vdash \psi \leftrightarrow \varphi$, and hence $\psi/\equiv_{\theta} \neq \varphi/\equiv_{\theta}$, thus showing that λ is a homomorphism. Next suppose $\psi/\equiv_{\theta} \neq 0$, i.e., $\theta \not\vdash \neg\psi$. Again by Theorem 1.7, $\text{Mod}(\theta) \not\subseteq \text{Mod}(\neg\psi)$. In other words, for some valuation V_w we have $V_w(\theta) = 1$ and $V_w(\neg\psi) \neq 1$. Thus $V_w(\psi) > 0$, whence $\hat{\theta}(w) = 1$ and $\hat{\psi}(w) > 0$. As a consequence, $\hat{\psi} \upharpoonright \text{Mod}(\theta) \neq 0$, and λ is one-one. Finally, let $g \in \mathcal{M}(\text{Mod}(\theta))$, and write $g = h \upharpoonright \text{Mod}(\theta)$ for some $h \in \mathcal{M}([0, 1]^n)$. By McNaughton theorem (A21.48), we can write $h = \hat{\psi}$ for some formula $\psi(X_1, \dots, X_n)$, and conclude that $g = \lambda(\psi/\equiv_{\theta})$, which shows that λ is onto $\mathcal{M}(\text{Mod}(\theta))$. $\square$

Lemma 1.13 For any $z \in [0, 1]^n$ and open neighborhood N of z there is a function $g \in \mathcal{M}([0, 1]^n)$ such that $g(z) = 0$ and $g(y) = 1$ for each $y \in [0, 1]^n \setminus N$.

Proof There is an open cube N' such that $z \in N' \cap [0, 1]^n \subseteq N$, and whose faces are given by equations of the form $x_i = r_i$ for suitable rationals r_i . By (A21.18) there is a function $f \in \mathcal{M}([0, 1]^n)$ vanishing precisely on the closure of N' . The continuity

of f ensures that for some $\epsilon > 0$ we constantly have $f \geq \epsilon$ over the compact set $[0, 1]^n \setminus N$. For all suitably large m the McNaughton function $g = f \oplus \dots \oplus f$ (m times) will constantly take the value 1 on $[0, 1]^n \setminus N$. $\square$

Lemma 1.14 *For every nonempty closed subset W of $[0, 1]^n$, $W = \text{Mod}(\text{Th } W)$.*

Proof Since $\text{Mod}(\text{Th } W) = \{x \in [0, 1]^n \mid V_x(\theta) = 1 \text{ for all } \theta \in \text{Th } W\}$, the inclusion $W \subseteq \text{Mod}(\text{Th } W)$ is trivial.

For the converse inclusion, arguing by induction on the number of connectives in $\psi \in \text{FORM}_n$ and using Lemma 1.12 together with (1.3 and 1.4), we obtain

$$\begin{aligned} \psi \in \text{Th } W &\Leftrightarrow V_x(\psi) = 1 \quad \text{for all } x \in W \\ &\Leftrightarrow \hat{\psi}(x) = 1 \quad \text{for all } x \in W \\ &\Leftrightarrow \hat{\psi} \upharpoonright W = 1, \end{aligned}$$

whence $\text{Th } W = \{\psi \in \text{FORM}_n \mid \hat{\psi} \upharpoonright W = 1\}$. Therefore,

$$\begin{aligned} x \in \text{Mod}(\text{Th } W) &\Leftrightarrow V_x(\theta) = 1 \quad \text{for all } \theta \in \text{Th } W \\ &\Leftrightarrow \hat{\theta}(x) = 1 \quad \text{for all } \theta \in \text{Th } W \\ &\Leftrightarrow x \in \bigcap \{\hat{\theta}^{-1}(1) \mid \theta \in \text{Th } W\}. \end{aligned}$$

Suppose $x \in [0, 1]^n \setminus W$. Since W is closed, Lemma 1.13 yields an open neighborhood N of x disjoint from W , together with a formula $\psi \in \text{FORM}_n$ such that $\hat{\psi}(y) = 1$ for all $y \in [0, 1]^n \setminus N \supseteq W$. Thus $\psi \in \text{Th } W$ and $x \notin \text{Mod}(\psi) = \{y \in [0, 1]^n \mid \hat{\psi}(y) = 1\}$. A fortiori, $x \notin \text{Mod}(\text{Th } W)$, whence $W \supseteq \text{Mod}(\text{Th } W)$, as required to complete the proof. $\square$

Conclusion of the proof of Theorem 1.4.: By Lemma 1.14, $\beta \in \text{conv}(W)$ iff β is the restriction to $\{X_1, \dots, X_n\}$ of a convex combination of valuations all satisfying $\text{Th } W$. We have proved (iii $\Leftrightarrow$ iv) in Theorem 1.4. The proof of the theorem is complete. $\square$

1.6 Remarks

Definition 1.1 is a generalization of de Finetti's notion of coherent assessment for yes–no events, ([2, Sect. 7, p. 308], [3, pp. 6–7]). Item (ii) in Theorem 1.4 is de Finetti's alternative definition of incoherent assessment [4, footnote page 87]. Lemma 1.13 was first proved in [5, 4.17]. Theorem 1.4 was first proved in [6].

Theorem 1.7 and Corollary 1.9 combine results by Hay [7] and Wójcicki [8] with Pogorzelski's Local Deduction Theorem [9]. Wójcicki proved that the equivalence (i $\Leftrightarrow$ iii) no longer holds in general if θ is replaced by an infinite set of formulas [8, Theorem 2]. The generalization of Theorem 1.7 for arbitrary sets of formulas in each finite-valued Łukasiewicz logic was also established by Wójcicki (see [8, Lemma 1], [10, 4.3.3]).

See [11] for a geometric representation of the consequence relation $\vdash$.

Note that in [1, 4.6.8], Lindenbaum algebras are only defined for sets of formulas in the set of variables $\{X_1, X_2, \dots\}$.

Events and possible worlds from physical systems. As anticipated in Sect. 1.1, the final part of this chapter is devoted to giving “events” and “possible worlds” a sufficiently general definition within the commutative C^* -algebraic formulation of classical physical systems. This also works for quantum physical systems, by just removing the commutativity axiom [12, pp. 362, 378].

Readers not interested in the C^* -algebraic sources of events and possible worlds may safely skip the remainder of this section.

Let $\mathcal{C}$ be the C^* -algebra of a classical physical system S . We denote by $\mathcal{C}_{\text{sa}}$ the set of self-adjoint elements of $\mathcal{C}$. Any element of $\mathcal{C}_{\text{sa}}$ represents an observable of S . Further, we write $\mathcal{S}$ for the set of real-valued normalized positive linear functionals on $\mathcal{C}_{\text{sa}}$. By [13, VIII, 2.1 and pp. 224–225], $\mathcal{C}$ can be identified with the C^* -algebra $C(\Sigma)$ of all complex-valued continuous functions over the compact Hausdorff space Σ of maximal ideals of $\mathcal{C}$. Under this identification, Σ is the space of all possible phases of S , and $\mathcal{C}_{\text{sa}}$ is the set of all real-valued continuous functions on Σ .

$\mathcal{C}_{\text{sa}}$ typically includes such observables as position, energy, momentum, and each element of $\mathcal{S}$ is thought of as a convenient mathematical counterpart of a “mode of preparation” of S . The Riesz representation theorem (B21.64) yields a one–one correspondence between $\mathcal{S}$ and the set of regular Borel probability measures on Σ . For any $\rho \in \mathcal{S}$ and $A \in \mathcal{C}_{\text{sa}}$ the real number $\rho(A)$ is said to be the expectation value of the observable A whenever S is prepared in mode ρ .

Let now $E = \{X_1, \dots, X_m\}$ be a set of nonzero positive elements of $\mathcal{C}_{\text{sa}}$. Under our standing identification, each X_i is a continuous function over Σ such that $f(x) \geq 0$ for all $x \in \Sigma$. Let $\sup X_i$ denote the sup norm of X_i . As explained in [12, pp. 363–369], each preparation mode $\rho \in \mathcal{S}$ determines a map $w_\rho: E \rightarrow [0, 1]$ by the stipulation $w_\rho(X_i) = \rho(X_i)/(\sup X_i)$. The set $W = \{w_\rho \mid \rho \in \mathcal{S}\}$ is closed in the m -cube $[0, 1]^E$. Intuitively, the “event” X_i occurs if “the value of the observable X_i is high,” and the “possible world” $w_\rho \in W$ gives a precise “truth-value” in $[0, 1]$ to this event.

Altogether, C^* -algebras have extensive capabilities to model events and possible worlds.

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Chapter 2

Rational Polyhedra, Interpolation, Amalgamation

One can hardly understand the fine structure of finitely presented (especially of finitely generated free and projective) MV-algebras without a working knowledge of the basic properties of rational polyhedra and their regular triangulations. The simplexes of these triangulations provide the volume elements of the integrals that evaluate the average truth-value of formulas and compute the invariant Rényi conditional introduced later in this book. Rational polyhedra are the genuine algebraic varieties of the formulas of Łukasiewicz logic: for, the zeroset of a McNaughton function of n variables is the most general possible rational polyhedron P contained in $[0, 1]^n$, $n = 1, 2, \dots$. This chapter is an elementary introduction to rational polyhedra and their subdivisions into regular triangulations. The observation that rational polyhedra are preserved under projections onto rational hyperplanes gives us a way of proving the (deductive) interpolation property of L_∞ and the amalgamation property of MV-algebras.

2.1 Rational Polyhedra, Complexes, Fans

Fix $n = 1, 2, \dots$. A point $y \in \mathbb{R}^n$ is said to be *rational* if all its coordinates are rational numbers. A *rational hyperplane* H in $\mathbb{R}^n$ is a set $H = \{x \in \mathbb{R}^n \mid h \circ x = k\}$, for some nonzero $h \in \mathbb{Q}^n$ and $k \in \mathbb{Q}$. Equivalently, $0 \neq h \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$. When $k = 0$ we say that H is *homogeneous*. The two closed halfspaces H^+ and H^- of $\mathbb{R}^n$ determined by H are said to be *rational*.

For any rational point $y = (y_1, \dots, y_n) \in \mathbb{Q}^n$ we denote by $\text{den}(y)$ the least common denominator of its coordinates, and we say that $\text{den}(y)$ is the *denominator* of y . The integer vector

$$\tilde{y} = (\text{den}(y) \cdot y_1, \dots, \text{den}(y) \cdot y_n, \text{den}(y)) = \text{den}(y)(y, 1) \in \mathbb{Z}^{n+1}$$

is called the *homogeneous correspondent* of y . Then $\tilde{y}$ is *primitive*, that is, minimal (as a nonzero integer vector) along its *ray* $\langle \tilde{y} \rangle = \{\lambda \tilde{y} \in \mathbb{R}^{n+1} \mid \lambda \geq 0\} = \mathbb{R}_{\geq 0} \tilde{y}$.

For $0 \leq m \leq n$, an m -simplex in $\mathbb{R}^n$ is the convex hull $T = \text{conv}(v_0, \dots, v_m)$ of $m+1$ affinely independent points in n -dimensional euclidean space $\mathbb{R}^n$. The vertices $v_0, \dots, v_m$ are uniquely determined by T . We say that T is *rational* if each vertex of T is a rational point. The empty set is the only -1 -simplex.

By a *polyhedron* P in $\mathbb{R}^n$ we mean the union of finitely many simplexes T_i in $\mathbb{R}^n$. If all T_i are rational, P is said to be a *rational polyhedron*.

A *polyhedral complex* in $\mathbb{R}^n$ is a finite (always nonempty) family $\mathcal{K}$ of closed convex polyhedra in $\mathbb{R}^n$ such that the face of any $P \in \mathcal{K}$ is a member of $\mathcal{K}$, and the intersection of any two polyhedra in $P, Q \in \mathcal{K}$ is a common face of P and Q . (As the reader will recall, the empty set and P itself are among the faces of every convex polyhedron P). We say that $\mathcal{K}$ is a *simplicial complex* if all its elements are simplexes.

Examples 2.1 (i) Let $l_1, \dots, l_u$ be linear real-valued polynomials on $\mathbb{R}^n$. For every permutation π of the index set $\{1, \dots, u\}$ let

$$P_\pi = \{x \in [0, 1]^n \mid l_{\pi(1)}(x) \leq \dots \leq l_{\pi(u)}(x)\}.$$

By (B21.54) each P_π is a (possibly empty) closed convex polyhedron in $[0, 1]^n$. Arguing by induction on u , it follows that the P_π and their faces form a polyhedral complex $\mathcal{K}$ over $[0, 1]^n$. We say that $\mathcal{K}$ is obtained from $l_1, \dots, l_u$ by *stratification*. If the coefficients of each polynomial l_i are rational, then each P_π is a rational polyhedron.

(ii) Suppose we are given hyperplanes $H_1, \dots, H_u$ in $\mathbb{R}^n$, with their closed half-spaces $H_i^\pm, i = 1, \dots, u$. For every permutation π of the index set $J = \{1, \dots, u\}$, and any map $\iota: J \rightarrow \{+, -\}$ let

$$P_{\pi, \iota} = \left\{x \in [0, 1]^n \mid x \in H_{\pi(1)}^{\iota(1)} \cap \dots \cap H_{\pi(u)}^{\iota(u)}\right\}.$$

Then each $P_{\pi, \iota}$ is a (possibly empty) closed convex polyhedron in $[0, 1]^n$. If the hyperplanes H_i are rational, then $P_{\pi, \iota}$ is a rational polyhedron. Again, the $P_{\pi, \iota}$ and their faces form a polyhedral complex $\mathcal{K}$ over $[0, 1]^n$, and we again say that $\mathcal{K}$ is obtained from the hyperplanes $H_1, \dots, H_u$ by *stratification*.

For any polyhedral or simplicial complex $\mathcal{C}$ the point-set union of the elements of $\mathcal{C}$ is called the *support* of $\mathcal{C}$, and is denoted $|\mathcal{C}|$. We say that $\mathcal{C}$ is *rational* if so are all its members. $\mathcal{C}'$ is a *subdivision* of $\mathcal{C}$ if $|\mathcal{C}'| = |\mathcal{C}|$ and every member of $\mathcal{C}$ is a union of members of $\mathcal{C}'$.

In case $\mathcal{C}$ is a simplicial complex, $\mathcal{C}$ is said to be a *triangulation* of $|\mathcal{C}|$.

Lemma 2.2 (i) Let Q be a rational polyhedron in $\mathbb{R}^{n+1}$. Then the projection Q' of Q onto the hyperplane $x_{n+1} = 0$ is a rational polyhedron.

(ii) Suppose the rational polyhedron P lies in the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. Writing I for the unit real interval $[0, 1]$, let us identify the cartesian product $P \times I$ with the prism $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n, 0) \in P, x_{n+1} \in I\}$. Then the prism $P \times I$ is a rational polyhedron.

Proof (i) Write Q as a finite union of rational simplexes T_i . Letting T'_i be the projection of T_i onto the hyperplane $x_{n+1} = 0$, it follows that each T'_i is a convex polyhedron with rational vertices, and $Q' = \bigcup T'_i$. Without adding new vertices, by (B21.52–B21.53) we have a simplicial complex $\mathcal{T}_i$ with support T'_i . Then Q' is the finite union of the simplexes of all $\mathcal{T}_i$.

(ii) Let $\mathcal{T}$ be a rational simplicial complex with support P . For each i -dimensional simplex $T \in \mathcal{T}$, $T \times I$ is an $(i + 1)$ -dimensional prism P_i with basis T and rational vertices. Evidently, $P \times I = \bigcup \{T \times I \mid T \in \mathcal{T}\}$. Again by (B21.52–B21.53), without adding new vertices, we construct a simplicial complex $\mathcal{K}_i$ with support P_i . We conclude that $P \times I$ is the union of all simplexes of all complexes $\mathcal{K}_i$. $\square$

Remark 2.3 Lemma 2.2(ii) has a routine generalization to projections onto any rational hyperplane.

Let $m = 1, 2, \dots$. Given vectors $v_1, \dots, v_s \in \mathbb{R}^m$ we denote by $\langle v_1, \dots, v_s \rangle$ their *positive span* in $\mathbb{R}^m$. In symbols,

$$\langle v_1, \dots, v_s \rangle = \mathbb{R}_{\geq 0}v_1 + \dots + \mathbb{R}_{\geq 0}v_s. \quad (2.1)$$

For $t = 1, 2, \dots, m$, a *t-dimensional rational simplicial cone* in $\mathbb{R}^m$ is a set $\sigma \subseteq \mathbb{R}^m$ of the form

$$\sigma = \mathbb{R}_{\geq 0}d_1 + \dots + \mathbb{R}_{\geq 0}d_t = \langle d_1, \dots, d_t \rangle, \quad (2.2)$$

for linearly independent primitive integer vectors $d_1, \dots, d_t \in \mathbb{Z}^m$. The vectors $d_1, \dots, d_t$ are called the *primitive generating vectors* of σ . They are uniquely determined by σ . By a *face* of σ we mean the positive span of any subset S of $\{d_1, \dots, d_t\}$. For the sake of completeness we stipulate that the face of σ determined by the empty set is the singleton $\{0\}$. This is the only zerodimensional cone in $\mathbb{R}^m$. Given $k = 0, 1, \dots$ and a rational k -simplex $T = \text{conv}(v_0, \dots, v_k) \subseteq \mathbb{R}^n$, we denote by $T^\uparrow$ the positive span in $\mathbb{R}^{n+1}$ of the homogeneous correspondents of the vertices of T . In symbols,

$$T^\uparrow = \langle \tilde{v}_0, \dots, \tilde{v}_k \rangle = \mathbb{R}_{\geq 0}\tilde{v}_0 + \dots + \mathbb{R}_{\geq 0}\tilde{v}_k \subseteq \mathbb{R}^{n+1}. \quad (2.3)$$

We say that $T^\uparrow$ is the (*rational simplicial*) *cone of* T . Note that $\dim(T^\uparrow) = k + 1$.

A *simplicial fan* Φ in $\mathbb{R}^n$ is a *complex of rational simplicial cones* in $\mathbb{R}^n$: thus Φ is closed under taking faces of its cones, and the intersection of any two cones $\sigma, \tau \in \Phi$ is a common face of σ and τ . Note that the intersection of all cones of Φ is the singleton $\{0\}$. If $\mathcal{K}$ is a rational simplicial complex in $\mathbb{R}^n$, the set

$$\mathcal{K}^\uparrow = \{T^\uparrow \mid T \in \mathcal{K}\} \quad (2.4)$$

is a simplicial fan in $\mathbb{R}^{n+1}$.

2.2 Regularity

In rational polyhedral topology, as well as in Łukasiewicz logic and MV-algebra theory, the following notion is fundamental:

Definition 2.4 Let $n = 1, 2, \dots$ and $t = 1, \dots, n$. Then a t -dimensional rational simplicial cone $\sigma = \langle d_1, \dots, d_t \rangle \subseteq \mathbb{R}^n$ is said to be *regular* if the set $\{d_1, \dots, d_t\}$ of its primitive generating vectors can be extended to a basis of the free abelian group $\mathbb{Z}^n$ of integer points in $\mathbb{R}^n$.

A rational simplex T is said to be *regular* (or *unimodular*) if so is its cone $T^\uparrow$.

A rational simplicial complex (resp., a simplicial fan) is said to be *regular* if so is every simplex (resp., every cone) in it. Thus, a simplicial complex Δ is regular iff so is the simplicial fan $\Delta^\uparrow$.

The *multiplicity* $\text{mult}(\sigma)$ of $\sigma = \langle d_1, \dots, d_t \rangle \subseteq \mathbb{R}^n$ is the index of the group $\mathbb{Z}d_1 + \dots + \mathbb{Z}d_t$ in the subgroup $\sigma \cap \mathbb{Z}^n + (-\sigma \cap \mathbb{Z}^n)$ of $\mathbb{Z}^n$ generated by $\sigma \cap \mathbb{Z}^n$.

Lemma 2.5 Given integers $1 \leq k \leq n$, let $\sigma = \langle d_1, \dots, d_k \rangle$ be a k -dimensional rational simplicial cone in $\mathbb{R}^n$.

- (i) The set $\{d_1, \dots, d_k\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^n$ iff $\text{mult}(\sigma) = 1$.
- (ii) $\text{mult}(\sigma) > 1$ iff the half-open parallelepiped

$$P = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^k \lambda_i d_i, \quad 0 \leq \lambda_i < 1 \right\}$$

contains a nonzero integer point.

Proof (i) is an immediate consequence of the definition, (ii) is a particular case of Minkowski theorem in the Geometry of Numbers (B21.56), which can be checked by comparing the approximate number of integer points in the half-open parallelepiped $tP = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^k \lambda_i d_i, 0 \leq \lambda_i < t\}$ for large t , with the k -dimensional volume of tP . $\square$

Lemma 2.6 Given integers $0 \leq m \leq n$ with $n \geq 1$, let $T = \text{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ be an m -simplex, together with its half-open parallelepiped $P_T = \{x \in \mathbb{R}^{n+1} \mid x = \sum_{i=0}^m \lambda_i \tilde{v}_i, 0 \leq \lambda_i < 1\}$ and its cone $T^\uparrow \subseteq \mathbb{R}^{n+1}$. Then the following conditions are equivalent:

- (i) T is regular.
- (ii) $\mathbb{Z}^{n+1} \cap T^\uparrow = \mathbb{Z}_{\geq 0} \tilde{v}_0 + \dots + \mathbb{Z}_{\geq 0} \tilde{v}_m$.
- (ii') $\mathbb{Z}^{n+1} \cap (\mathbb{R} \tilde{v}_0 + \dots + \mathbb{R} \tilde{v}_m) = \mathbb{Z} \tilde{v}_0 + \dots + \mathbb{Z} \tilde{v}_m$.
- (iii) The half-open parallelepiped P_T contains no nonzero integer point.
- (iv) The multiplicity of $T^\uparrow$ is 1.

Proof In view of Lemma 2.5, we have only to check (i $\Leftrightarrow$ ii $\Leftrightarrow$ ii'). Trivially, $\mathbb{Z}^{n+1} \cap T^\uparrow \supseteq \mathbb{Z}_{\geq 0} \tilde{v}_0 + \dots + \mathbb{Z}_{\geq 0} \tilde{v}_m$ and $\mathbb{Z}^{n+1} \cap (\mathbb{R} \tilde{v}_0 + \dots + \mathbb{R} \tilde{v}_m) \supseteq \mathbb{Z} \tilde{v}_0 + \dots + \mathbb{Z} \tilde{v}_m$.

(i $\Leftrightarrow$ ii) If T is not regular, then P_T contains a nonzero integer point q ; evidently, q cannot be expressed as a linear combination of the $\tilde{v}_i$ with integer coefficients ≥ 0 . Thus, the set $\mathbb{Z}_{\geq 0}\tilde{v}_0 + \cdots + \mathbb{Z}_{\geq 0}\tilde{v}_m$ is strictly contained in $\mathbb{Z}^{n+1} \cap T^\uparrow$, and (ii) fails. Conversely, if T is regular then, by definition, every integer point of $T^\uparrow$ is a linear combination of the $\tilde{v}_i$ with uniquely determined integer coefficients c_i . By definition of $T^\uparrow$, $c_i \geq 0$ for each $i = 0, \dots, m$, whence (ii) holds.

The proof of (i $\Leftrightarrow$ ii') is similar. $\square$

The following characterization of the regularity of a rational simplex $T \subseteq \mathbb{R}^n$ does not mention the ambient space $\mathbb{R}^{n+1}$ of the cone $T^\uparrow$:

Lemma 2.7 *For every rational m -simplex $T = \text{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ the following conditions are equivalent:*

- (i) T is regular.
- (ii) *For every face F of T and every rational point r lying in the relative interior of F , the denominator of r is $\geq$ the sum of the denominators of the vertices of F .*

Proof For every face E of T let $E^\uparrow = \langle c_1, \dots, c_k \rangle$ and P_E respectively denote the cone and the half-open parallelepiped associated to E . Let further

$$q_E = \sum_{i=1}^k c_{i,n+1}$$

be the sum of the $(n+1)$ th coordinates of the primitive generating vectors of $E^\uparrow$. By construction, q_E coincides with the sum of the denominators of the vertices of E .

(ii $\Rightarrow$ i) If T is not regular, by Lemma 2.6 the half-open parallelepiped P_T contains a nonzero point $u = (u_1, \dots, u_{n+1}) \in \mathbb{Z}^{n+1}$. Without loss of generality, u is primitive. For a uniquely determined rational point $r \in T$ we can write $u = \tilde{r}$. It follows that $\text{den}(r) = u_{n+1}$. Let F be the smallest face of T such that r belongs to T . Then r lies in the relative interior of F , and u belongs to the half-open parallelepiped P_F . Therefore, $\text{den}(r) < q_F$.

(i $\Rightarrow$ ii) Conversely, suppose for some face $F = \text{conv}(w_0, \dots, w_l)$ of T and rational point t lying in the relative interior of F , $\text{den}(t)$ is strictly smaller than the sum d of the denominators of the vertices of F ,

$$\text{den}(t) < d. \tag{2.5}$$

The homogeneous correspondent $\tilde{t}$ lies in the relative interior of the cone $F^\uparrow$ and is a primitive vector.

We claim that the half-open parallelepiped P_F contains a nonzero integer point. Otherwise (absurdum hypothesis) by Lemma 2.6, F is regular and the set $\mathbb{Z}_{\geq 0}\tilde{w}_0 + \cdots + \mathbb{Z}_{\geq 0}\tilde{w}_l$ contains $\tilde{t}$ as a member, $\tilde{t} = n_0\tilde{w}_0 + \cdots + n_l\tilde{w}_l$, for suitable integers $n_i \geq 0$. The fact that $\tilde{t}$ lies in the relative interior of the cone $F^\uparrow$ is equivalent to writing $1 \leq n_0, \dots, n_l$. Thus the $(n+1)$ th coordinate $\tilde{t}_{n+1}$ of $\tilde{t}$ satisfies the inequality $\tilde{t}_{n+1} \geq q_F$, where $q_F = d$ denotes the sum of the $(n+1)$ th coordinates

of the primitive generating vectors $\tilde{w}_0, \dots, \tilde{w}_l$ of the cone $F^\uparrow$. Since $\tilde{t}$ is primitive and $\text{den}(t) = \tilde{t}_{n+1}$, then $\text{den}(t) \geq d$, against (2.5). Our claim is settled.

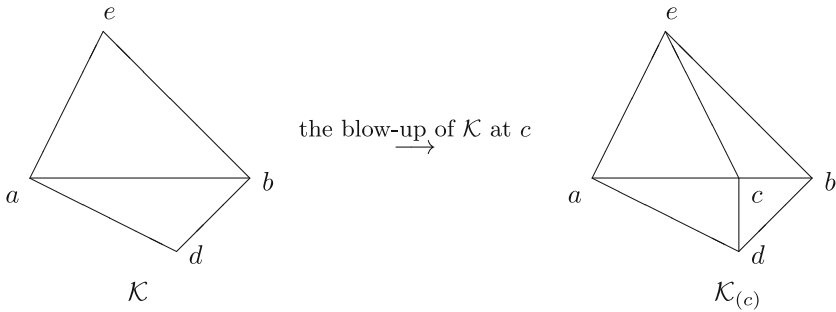
We have thus shown that P_F contains a nonzero integer point. By Lemma 2.6, F is not a regular simplex. A fortiori, T is not regular. $\square$

2.3 Blow-Up and Desingularization

For any $n = 1, 2, \dots$, simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$, and $c \in |\mathcal{K}|$, the *blow-up of $\mathcal{K}$ at c* is the following transformation:

Replace every simplex $C \in \mathcal{K}$ that contains c by the set of all simplexes of the form $\text{conv}(F \cup \{c\})$, where F is any face of C that does not contain c .

We then obtain a simplicial complex, denoted $\mathcal{K}_{(c)}$, which is a subdivision of $\mathcal{K}$. The inverse of a blow-up is called a *blow-down*. If $\mathcal{K}'$ is obtained from $\mathcal{K}$ via a finite sequence of blow-ups, we say that $\mathcal{K}'$ is a *stellar subdivision* of $\mathcal{K}$.



The following theorem establishes a main property of rational simplicial complexes in $\mathbb{R}^n$, for all $n = 1, 2, \dots$:

Theorem 2.8 [Desingularization] *Let $\mathcal{R}$ be a rational simplicial complex with support $|\mathcal{R}| \subseteq \mathbb{R}^n$. Then there exists a regular stellar subdivision Δ of $\mathcal{R}$.*

Proof We will construct a sequence $\mathcal{R} = \mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_z = \Delta$ of rational triangulations of $|\mathcal{R}|$, where $\mathcal{R}_{t+1}$ is obtained from $\mathcal{R}_t$ via a blow-up, ($t = 0, \dots, z-1$). The new vertex of $\mathcal{R}_{t+1}$ is obtained by the following procedure:

- (a) Choose a simplex $S = \text{conv}(w_1, \dots, w_r) \in \mathcal{R}_t$ of maximum multiplicity m .
- (b) Let the half-open parallelepiped P be defined by $P = \{x \in \mathbb{R}^{n+1} \mid x = v_1 \tilde{w}_1 + \dots + v_r \tilde{w}_r, 0 \leq v_i < 1, i = 1, \dots, r\}$.
- (c) Pick a primitive integer vector q in P , as given by Lemma 2.6.
- (d) Define $p \in S$ by $\tilde{p} = q$, and let $\mathcal{R}_{t+1}$ be obtained by blowing up $\mathcal{R}_t$ at p .

Observe that the multiplicity of each new simplex in $\mathcal{R}_{t+1}$ is strictly smaller than the maximum multiplicity m of the simplexes in $\mathcal{R}_t$. As the result of each blow-up $\mathcal{R}_t \mapsto \mathcal{R}_{t+1}$, either the number of simplexes having multiplicity m decreases, or the value itself of the maximum multiplicity decreases (this being the case when $\mathcal{R}_t$ has precisely one simplex of multiplicity m). A routine double induction argument shows that there can only be finitely many blow-ups in the procedure (a–d) above, and the sequence $\mathcal{R}_0, \mathcal{R}_1, \dots$ must terminate with the desired regular subdivision Δ of $\mathcal{R}$. $\square$

Desingularization yields a key tool to construct regular triangulations having special properties:

Corollary 2.9 *For any formulas $\psi_1, \dots, \psi_k \in \text{FORM}_n$ and rational polyhedra $Q_1, \dots, Q_m \subseteq [0, 1]^n$, there is a regular triangulation Δ of $[0, 1]^n$ having the following properties:*

- (i) *For every $i = 1, \dots, k$, the McNaughton function $\hat{\psi}_i \in \mathcal{M}([0, 1]^n)$ is linear over each simplex of Δ .*
- (ii) *For every $j = 1, \dots, m$, the set $\Delta_j = \{S \in \Delta \mid S \subseteq Q_j\}$ is a (necessarily regular) triangulation of Q_j .*

We say that each Δ_j is a regular $\{\hat{\psi}_1, \dots, \hat{\psi}_k\}$ -triangulation of Q_j .

Proof By (B21.54) there is a set $L = \{l_1, \dots, l_u\}$ of linear polynomials with integer coefficients such that every linear piece of each McNaughton function $\hat{\psi}_i$ is an element of L , and each rational polyhedron Q_j is a finite union of rational simplexes $S_{j1}, S_{j2}, \dots, S_{js(j)}$ of the form

$$S_{jt} = \bigcap_{l \in L_{jt}} \{x \in [0, 1]^n \mid l(x) \geq 0\}, \quad (t = 1, \dots, s(j))$$

for some subset L_{jt} of L . Without loss of generality the constant functions 0 and 1 are in L . Every permutation π of $\{1, \dots, u\}$ determines the rational convex polyhedron $Q_\pi = \{x \in [0, 1]^n \mid l_{\pi(1)} \leq \dots \leq l_{\pi(u)}\}$. As in Example 2.1(i), the Q_π determine a polyhedral complex $\mathcal{K}$ with support $[0, 1]^n$. There are enough elements in L to ensure that $\mathcal{K}$ satisfies conditions (i–ii) above. Next, without adding new vertices as in (B21.53), $\mathcal{K}$ can be subdivided into a rational triangulation $\mathcal{K}'$ which automatically satisfies the two conditions. A final application to $\mathcal{K}'$ of the desingularization Theorem 2.8 yields the desired regular triangulation Δ . $\square$

Desingularization is also the main ingredient in the following characterization of rational polyhedra:

Corollary 2.10 *Let $\emptyset \neq Y \subseteq [0, 1]^n$. Then the following conditions are equivalent:*

- (i) *Y is the support of some regular complex ∇ .*
- (ii) *$Y = f^{-1}(0)$ for some McNaughton function $f \in \mathcal{M}([0, 1]^n)$.*
- (iii) *$Y = \text{Mod}(\psi)$ for some formula $\psi \in \text{FORM}_n$.*
- (iv) *Y is a rational polyhedron.*

Proof To prove (iii $\Rightarrow$ ii) it is enough to write $f = \neg\hat{\psi}$ and recall (1.5) in Chap. 1. Combining McNaughton theorem (A21.48) with (1.5) we easily obtain (ii $\Rightarrow$ iii).

The nontrivial direction of (i $\Leftrightarrow$ iv) immediately follows from Corollary 2.9, because every rational polyhedron has a rational triangulation, (B21.52). To prove (ii $\Rightarrow$ i), using Corollary 2.9 we have a regular complex $\mathcal{K}$ with support $[0, 1]^n$ such that f is linear over each simplex of $\mathcal{K}$. Now let $\nabla = \{S \in \mathcal{K} \mid S \subseteq Y\}$.

Finally, to prove (i $\Rightarrow$ ii), let $S_1, \dots, S_u$ display the simplexes of ∇ . Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a set of rational closed half-spaces in $\mathbb{R}^n$ such that for each $j = 1, \dots, u$ the simplex S_j is the intersection of half-spaces of $\mathcal{H}$. The existence of $\mathcal{H}$ is ensured by (B21.54). In more detail, let us write $S_j = H_{j1} \cap \dots \cap H_{jt(j)}$. Proceeding as in Example 2.1(ii), from $\mathcal{H}$ we obtain a polyhedral complex $\mathcal{C}$ with support $[0, 1]^n$, such that each simplex S_j of ∇ is expressible as a union of polyhedra of $\mathcal{C}$. Using (B21.53) and Theorem 2.8, $\mathcal{C}$ can be subdivided into a regular complex Δ with support $[0, 1]^n$. By construction, the set $\{T \in \Delta \mid T \subseteq Y\}$ is a regular triangulation of Y having the additional property that for each $j = 1, \dots, u$, S_j is a union of simplexes of Δ . Let the continuous function $f: [0, 1]^n \rightarrow [0, 1]$ be uniquely determined by the following conditions:

- (a) f is linear over every simplex $T \in \Delta$,
- (b) $f(v) = 0$ for every vertex v of (any simplex of) Δ with $v \in Y$, and
- (c) $f(w) = 1$ for every vertex w of Δ with $w \notin Y$.

Using (a–c) together with the regularity of every simplex $T \in \Delta$, an elementary calculation shows that the function $f \upharpoonright T$ coincides with a linear polynomial with integer coefficients. Thus by definition, $f \in \mathcal{M}([0, 1]^n)$. Conditions (a–c) also ensure that $Y = f^{-1}(0)$, and the proof is complete. $\square$

The following result is a generalization of Lemma 2.2(ii):

Corollary 2.11 *Let $0 < m, n \in \mathbb{Z}$. If $P \subseteq [0, 1]^m$ and $Q \subseteq [0, 1]^n$ are rational polyhedra, then so is $P \times Q \subseteq [0, 1]^{m+n}$.*

Proof Corollary 2.10, provides formulas

$$\phi(X_1, \dots, X_m) \quad \text{and} \quad \psi(X_{m+1}, \dots, X_{m+n})$$

such that $P = \text{Mod}(\phi)$ and $Q = \text{Mod}(\psi)$. Now considering ϕ and ψ as members of FORM_{m+n} , we can write $P \times Q = \text{Mod}(\phi \odot \psi) = \text{Mod}(\phi \wedge \psi)$, which is a rational polyhedron, again by Corollary 2.10. $\square$

Corollary 2.12 *If P and Q are rational polyhedra in $[0, 1]^n$, then so is their intersection $P \cap Q$.*

Proof If $P \cap Q = \emptyset$ we have nothing to prove. So, assuming $P \cap Q \neq \emptyset$, let Δ be a regular triangulation of $[0, 1]^n$ such that the set $\Delta_P = \{T \in \Delta \mid T \subseteq P\}$ is a triangulation of P and $\Delta_Q = \{T \in \Delta \mid T \subseteq Q\}$ is a triangulation of Q . The existence of Δ is ensured by Corollary 2.9. It follows that $\Delta_P \cap \Delta_Q$ is a triangulation of $P \cap Q$. $\square$

2.4 Deductive Craig Interpolation in $\mathbb{L}_\infty$

Let $\vdash_2$ denote consequence in boolean logic. Let $\vdash_2 \psi$ mean that ψ is a tautology in boolean logic. Then Craig interpolation theorem has the following two equivalent formulations:

$$\text{If } \theta \vdash_2 \phi \text{ then } \theta \vdash_2 \gamma \text{ and } \gamma \vdash_2 \phi \text{ for some } \gamma \text{ with } \text{var}(\gamma) \subseteq \text{var}(\theta) \cap \text{var}(\phi) \quad (2.6)$$

and

$$\text{If } \vdash_2 \theta \rightarrow \phi \text{ then } \vdash_2 \theta \rightarrow \gamma \text{ and } \vdash_2 \gamma \rightarrow \phi \text{ for some } \gamma \text{ as in (2.6).} \quad (2.7)$$

The equivalence follows from the deduction theorem: $\alpha \vdash_2 \beta$ iff $\vdash_2 \alpha \rightarrow \beta$. Replacing $\vdash_2$ by the syntactic consequence relation $\vdash$ of $\mathbb{L}_\infty$, the tautology $X \wedge \neg X \rightarrow Y \vee \neg Y$ shows that the second version fails in $\mathbb{L}_\infty$.

However, the first version holds:

Theorem 2.13 [Deductive Craig interpolation] *Any two formulas ϕ and ψ such that $\phi \vdash \psi$ have an interpolant ι , i.e., a formula ι such that $\phi \vdash \iota$, $\iota \vdash \psi$, and $\text{var}(\iota) \subseteq \text{var}(\phi) \cap \text{var}(\psi)$. In case $\text{var}(\phi) \cap \text{var}(\psi) = \emptyset$, from $\phi \vdash \psi$ it follows that either ϕ is unsatisfiable or ψ is a tautology.*

Proof Let $X = \{X_1, \dots, X_m\}$, $Y = \{Y_1, \dots, Y_n\}$, and $Z = \{Z_1, \dots, Z_p\}$ be pairwise disjoint sets of variables. Let us suppose $\phi \in \text{FORM}_{X \cup Z}$ and $\psi \in \text{FORM}_{Y \cup Z}$, with the intent of constructing an interpolant $\iota \in \text{FORM}_Z$. By Corollary 2.10 (iii $\Rightarrow$ iv), $\text{Mod}(\phi)$ is a rational polyhedron P in $[0, 1]^{X \cup Z}$. By Corollary 2.11 the projection of P onto $\mathbb{R}^Z \subseteq \mathbb{R}^{X \cup Z}$ is a rational polyhedron Q in $[0, 1]^Z$. By Corollary 2.10 (iv $\Rightarrow$ iii) there is a formula $\iota \in \text{FORM}_Z$ such that $\text{Mod}(\iota) = Q$, and we can write $\text{Mod}(\iota) = \{z = (z_1, \dots, z_p) \in [0, 1]^Z \mid V_z(\iota) = 1\}$.

Claim 1 $\phi \vdash \iota$.

As a matter of fact, repeated applications of Lemma 2.2(ii) show that the cylindrication $Q' = [0, 1]^X \times Q$ is a rational polyhedron in $[0, 1]^{X \cup Z}$ containing P . Regarding ι as a formula of $\text{FORM}_{X \cup Z}$ and using the notation

$$\text{Mod}_{X \cup Z}(\iota) = \left\{ (x, z) = (x_1, \dots, x_m, z_1, \dots, z_p) \in [0, 1]^{X \cup Z} \mid V_{(x, z)}(\iota) = 1 \right\},$$

we have $Q' = \text{Mod}_{X \cup Z}(\iota)$. Since $\text{Mod}(\phi) = P \subseteq \text{Mod}_{X \cup Z}(\iota)$, a final application of Theorem 1.7 settles our first claim.

Claim 2 $[0, 1]^Y \times Q \subseteq \text{Mod}(\psi)$.

Let $R = \text{Mod}(\psi)$. For any $y \in [0, 1]^Y$ and $q \in Q$ let us denote by (y, q) the corresponding point in $[0, 1]^Y \times Q$. By definition of Q there is $x \in [0, 1]^X$ such that $(x, q) \in P$. Then $(x, q, y) \in P \times [0, 1]^Y$. From $\phi \vdash \psi$ we get $(y, q, x) \in R \times [0, 1]^X$. Since $R \subseteq [0, 1]^{Y \cup Z}$ then $(y, q) \in R$, and our second claim is settled.

Claim 3 $\iota \vdash \psi$.

Indeed, upon identifying ι with a formula of $\text{FORM}_{Y \cup Z}$, the set

$$\text{Mod}_{Y \cup Z}(\iota) = \{(y, z) \in [0, 1]^{Y \cup Z} \mid V_{(y, z)}(\iota) = 1\}$$

turns out to coincide with $[0, 1]^Y \times Q$, whence by Claim 2, $\text{Mod}_{Y \cup Z}(\iota) \subseteq \text{Mod}(\psi)$. One more application of Theorem 1.7 settles our third claim.

We have thus proved the general case of the deductive interpolation theorem for $\mathbb{L}_\infty$.

Concerning the special case $\text{var}(\psi) \cap \text{var}(\phi) = \emptyset$, by way of contradiction suppose ϕ is satisfied by some valuation $V \in \text{VAL}_{\text{var}(\phi)}$ and ψ fails to be satisfied by some valuation $V' \in \text{VAL}_{\text{var}(\psi)}$. The valuation $V \cup V' \in \text{VAL}_{\text{var}(\psi) \cup \text{var}(\phi)}$ satisfies ϕ , and does not satisfy ψ . By Theorem 1.7, this contradicts $\phi \vdash \psi$. $\square$

Remark 2.14 The proof has shown that ι is a *uniform interpolant* of ϕ , in the sense that for all $\chi \in \text{FORM}_{Y \cup Z}$, if $\phi \vdash \chi$ then $\phi \vdash \iota$ and $\iota \vdash \chi$, always with the same interpolant ι .

Corollary 2.15 *Let θ, ψ, ϕ be formulas such that $\{\theta, \psi\} \vdash \phi$ and $\text{var}(\phi) \subseteq \text{var}(\psi)$. Then there is a formula ι such that $\theta \vdash \iota$, $\{\psi, \iota\} \vdash \phi$ and $\text{var}(\iota) \subseteq \text{var}(\theta) \cap \text{var}(\psi)$.*

Proof We will repeatedly use Theorem 1.7. For some integer $t > 0$, $\theta \vdash \psi^t \rightarrow \phi$. By Theorem 2.13, there is a formula ι such that $\theta \vdash \iota$, $\iota \vdash \psi^t \rightarrow \phi$ and $\text{var}(\iota) \subseteq \text{var}(\theta) \cap (\text{var}(\psi) \cup \text{var}(\phi)) \subseteq \text{var}(\theta) \cap \text{var}(\psi)$. From $\iota \vdash \psi^t \rightarrow \phi$ it follows that $\{\iota, \psi\} \vdash \phi$. $\square$

2.5 Theories and Ideals

For any nonempty set $\mathcal{X}$ of variables, let $\text{TAUT}_{\mathcal{X}}$ denote the set of formulas of $\text{FORM}_{\mathcal{X}}$ which are tautologies.

Definition 2.16 By a *theory* Θ in $\mathcal{X}$ we mean a proper subset of $\text{FORM}_{\mathcal{X}}$ containing $\text{TAUT}_{\mathcal{X}}$ and having the additional property that if $\phi \in \text{FORM}_{\mathcal{X}}$ and $\Theta \vdash \phi$ then ϕ already belongs to Θ .

When Θ is a theory in $\mathcal{X}$, rather than a mere set of formulas of $\text{FORM}_{\mathcal{X}}$, then automatically $\text{var}(\Theta) = \mathcal{X}$. Further, syntactic consequence $\Theta \vdash \psi$ acquires a simpler form: as a matter of fact, if $\text{var}(\psi) \subseteq \mathcal{X}$ then $\Theta \vdash \psi$ iff $\psi \in \Theta$. In case ψ contains variables not in $\mathcal{X}$, we still have $\Theta \vdash \psi$ iff $\vdash \theta \rightarrow \psi$ for some $\theta \in \Theta$.

For every set $\mathcal{X}$ we denote by $\text{FREE}_{\mathcal{X}}$ the free MV-algebra over the free generating set $\mathcal{X}$. $\text{FREE}_{\mathcal{X}}$ has the form $\frac{\text{FORM}_{\mathcal{X}}}{\equiv_{\mathcal{X}}}$, where the equivalence relation $\equiv_{\mathcal{X}}$ is given by

$$\psi \equiv_{\mathcal{X}} \phi \text{ iff the formula } \psi \leftrightarrow \phi \text{ belongs to } \text{TAUT}_{\mathcal{X}}.$$

The MV-algebraic operations on $\text{FREE}_{\mathcal{X}}$ are naturally derived from the congruence properties of the connectives.

For $\mathcal{X} \subseteq \mathcal{X}'$ the free MV-algebra $\text{FREE}_{\mathcal{X}}$ is canonically identified with a subalgebra of $\text{FREE}_{\mathcal{X}'}$ via the unique (necessarily one-one) homomorphism extending the inclusion map $X \in \mathcal{X} \mapsto X \in \mathcal{X}'$.

Definition 2.17 By an *ideal* i of an MV-algebra A we mean the kernel of a homomorphism of A into an MV-algebra B . From our standing assumption about $0 \neq 1$ in both A and B , it follows that i is *proper*, i.e., $i \neq A$. Since $1 \notin i$, in the quotient MV-algebra A/i the zero and the unit element are different.

Unless otherwise specified, all ideals in this book will be proper, and the adjective “proper” will be omitted.

Notation Let A be an MV-algebra. Then for any $m = 1, 2, \dots$ and element $b \in A$ we use the notation

$$m \cdot b \quad \text{for} \quad \underbrace{b \oplus \dots \oplus b}_m \quad . \quad (2.8)$$

m occurrences of b

Throughout this book, $\mathbf{d}(x, y)$ will denote the *Chang distance* of $x, y \in A$,

$$\mathbf{d}(x, y) = (x \odot \neg y) \oplus (\neg x \odot y) = |x - y|,$$

where the absolute value and the subtraction operations are those of the unital ℓ -group $(G, 1)$ given by $A = \Gamma(G, 1)$.

Definition 2.18 For S a nonempty subset of A we denote by $\langle S \rangle$ the subset of A given by the following stipulation:

$$a \in \langle S \rangle \text{ iff } a \leq m \cdot (s_1 \oplus \dots \oplus s_k) \text{ for some } s_1, \dots, s_k \in S \text{ and } m = 1, 2, \dots \quad (2.9)$$

If $\langle S \rangle = A$ then by definition $A/\langle S \rangle$ is the singleton MV-algebra with $0 = 1$. Otherwise, $\langle S \rangle$ is an ideal of A , coinciding with the intersection of all ideals of A containing S . It follows that the zero and the unit elements of the quotient MV-algebra $A/\langle S \rangle$ are different. In this latter case we say that $\langle S \rangle$ is the ideal *generated by* S . There cannot be any danger of confusion with the notation used earlier in this chapter for the cone generated by a set of vectors.

2.6 MV-Algebras have the Amalgamation Property

Lemma 2.19 Let Z and A be MV-algebras with disjoint universes. Let $\alpha: Z \rightarrow A$ be an embedding (i.e., a one-one homomorphism), and $X = A \setminus \alpha(Z)$. Let σ_Z be the homomorphism of FREE_Z onto Z uniquely determined by the identity map $z \in Z \mapsto z$. Let σ_A be the homomorphism of $\text{FREE}_{Z \cup X}$ onto A uniquely determined by the map $z \in Z \mapsto \alpha(z) \in A$ and $x \in X \mapsto x \in A$. Then $\sigma_A \upharpoonright \text{FREE}_Z = \alpha \sigma_Z$ and $\ker \sigma_A \cap \text{FREE}_Z = \ker \sigma_Z$.

Proof The identity $\sigma_A \upharpoonright \text{FREE}_Z = \alpha\sigma_Z$ is immediately verified. For the second identity, we have the following diagram:

$$\begin{array}{ccc}
 A & \xleftarrow{\alpha} & Z \\
 \sigma_A \uparrow & & \sigma_Z \uparrow \\
 \text{FREE}_{X \cup Z} & \xleftarrow{\text{subalgebra}} & \text{FREE}_Z \\
 \text{inclusion} \uparrow & & \text{inclusion} \uparrow \\
 \ker \sigma_A & \xleftarrow{\text{inclusion}} & \ker \sigma_Z
 \end{array}$$

To prove the commutativity of the diagram we argue as follows: if $k \in \ker \sigma_A \cap \text{FREE}_Z$ then $\sigma_A(k) = 0$ and hence $\alpha(\sigma_Z(k)) = 0$, whence $\sigma_Z(k) = 0$ because α is one-one. Thus $k \in \ker \sigma_Z$. Conversely, if $h \in \ker \sigma_Z$ then $\sigma_Z(h) = 0$ and $\alpha(\sigma_Z(h)) = 0$. Therefore, $\sigma_A(h) = 0$, whence $h \in \ker \sigma_A$. Trivially, $h \in \text{FREE}_Z$. $\square$

Theorem 2.20 (MV-algebras have the amalgamation property) *Given one-one homomorphisms $A \xleftarrow{\alpha} Z \xrightarrow{\beta} B$ of MV-algebras, there is an MV-algebra D together with one-one homomorphisms $A \xrightarrow{\mu} D \xleftarrow{\nu} B$ such that $\mu\alpha = \nu\beta$.*

Proof Without loss of generality the universes of Z , A , B are pairwise disjoint. Let us define the sets $X = A \setminus \alpha(Z)$ and $Y = B \setminus \beta(Z)$, together with the surjections $\sigma_Z: \text{FREE}_Z \rightarrow Z$ and $\sigma_A: \text{FREE}_{X \cup Z} \rightarrow A$ as in Lemma 2.19. Let similarly $\sigma_B: \text{FREE}_{Y \cup Z} \rightarrow B$ be the homomorphism uniquely determined by the map $z \in Z \mapsto \beta(z) \in B$ and $y \in Y \mapsto y \in B$. Our hypothesis, together with Lemma 2.19, yields a commutative diagram

$$\begin{array}{ccccc}
 A & \xleftarrow{\alpha} & Z & \xrightarrow{\beta} & B \\
 \sigma_A \uparrow & & \sigma_Z \uparrow & & \sigma_B \uparrow \\
 \text{FREE}_{X \cup Z} & \xleftarrow{\text{subalgebra}} & \text{FREE}_Z & \xrightarrow{\text{subalgebra}} & \text{FREE}_{Y \cup Z} \\
 \text{inclusion} \uparrow & & \text{inclusion} \uparrow & & \text{inclusion} \uparrow \\
 \ker \sigma_A & \xleftarrow{\text{inclusion}} & \ker \sigma_Z & \xrightarrow{\text{inclusion}} & \ker \sigma_B
 \end{array}$$

Let $\mathfrak{i} = \langle \ker \sigma_A \cup \ker \sigma_B \rangle \subseteq \text{FREE}_{X \cup Y \cup Z}$ be defined as in (2.9) above. Let σ be the quotient homomorphism of $\text{FREE}_{X \cup Y \cup Z}$ onto $\text{FREE}_{X \cup Y \cup Z}/\mathfrak{i}$, and $D = \text{FREE}_{X \cup Y \cup Z}/\mathfrak{i}$. (We will see that $\mathfrak{i} \neq \text{FREE}_{X \cup Y \cup Z}$, whence $0 \neq 1$ in D .) We then have the following commutative diagram:

$$\begin{array}{ccccc}
\ker \sigma_A & \xrightarrow{\text{inclusion}} & \mathbf{i} = \langle \ker \sigma_A \cup \ker \sigma_B \rangle & \xleftarrow{\text{inclusion}} & \ker \sigma_B \\
\text{inclusion} \downarrow & & \text{inclusion} \downarrow & & \text{inclusion} \downarrow \\
\text{FREE}_{X \cup Z} & \xrightarrow{\text{subalgebra}} & \text{FREE}_{X \cup Y \cup Z} & \xleftarrow{\text{subalgebra}} & \text{FREE}_{Y \cup Z} \\
\sigma_A \downarrow & & \sigma \downarrow & & \sigma_B \downarrow \\
A & & D & & B
\end{array}$$

We next define the map $\mu: A \rightarrow D$ by the following stipulation: for any $a \in A$, letting a' be an arbitrarily chosen element of $\sigma_A^{-1}(a)$ the element $\mu(a) \in D$ is given by $a'/\mathbf{i}$. In more detail,

$$a \in A \xrightarrow{\sigma_A^{-1}} a' \in \text{FREE}_{X \cup Z} \xrightarrow{\text{inclusion}} a' \in \text{FREE}_{X \cup Y \cup Z} \xrightarrow{\text{quotient by } \mathbf{i}} a'/\mathbf{i} \in D.$$

Evidently, μ is a well-defined homomorphism of A into D : for, if a'' is another member of $\sigma_A^{-1}(a)$ then $\mathbf{d}(a', a'') = |a' - a''|$ belongs to $\ker \sigma_A$, whence $|a' - a''| \in \mathbf{i} \supseteq \ker \sigma_A$ and $a'/\mathbf{i} = a''/\mathbf{i}$.

Similarly, the homomorphism $\nu: B \rightarrow D$ is defined by

$$b \in B \xrightarrow{\sigma_B^{-1}} b' \in \text{FREE}_{Y \cup Z} \xrightarrow{\text{inclusion}} b' \in \text{FREE}_{X \cup Y \cup Z} \xrightarrow{\text{quotient by } \mathbf{i}} b'/\mathbf{i} \in D,$$

where b' is chosen arbitrarily in $\sigma_B^{-1}(b)$.

To prove that μ is one-one (whence in particular, $\mathbf{i} \neq \text{FREE}_{X \cup Y \cup Z}$), suppose that $a \in A$ satisfies $\mu(a) = 0$, with the intent of proving $a = 0$. Pick any $e \in \sigma_A^{-1}(a) \subseteq \text{FREE}_{X \cup Z}$. By definition of μ it follows that $e \in \mathbf{i}$, whence

$$e \leq f \oplus g \quad \text{for some } f \in \ker \sigma_A \quad \text{and} \quad g \in \ker \sigma_B.$$

Equivalently, $\neg e \geq \neg f \odot \neg g$. For some formulas $\epsilon, \phi \in \text{FORM}_{X \cup Z}$ and $\theta \in \text{FORM}_{Y \cup Z}$ we can write

$$e = \frac{\epsilon}{\equiv}, \quad f = \frac{\phi}{\equiv}, \quad g = \frac{\theta}{\equiv},$$

where $\equiv$ denotes equivalence in the free MV-algebra $\text{FORM}_{X \cup Y \cup Z}$. Thus,

$$\frac{\neg \epsilon}{\equiv} \geq \frac{\neg \phi}{\equiv} \odot \frac{\neg \theta}{\equiv}, \quad \text{whence} \quad \left(\frac{\neg \phi}{\equiv} \odot \frac{\neg \theta}{\equiv} \right) \rightarrow \frac{\neg \epsilon}{\equiv} = 1 \in \text{FREE}_{X \cup Y \cup Z}.$$

In other words, the equation $(\neg \phi \odot \neg \theta) \rightarrow \neg \epsilon = 1$ holds in $\text{FREE}_{X \cup Y \cup Z}$. By definition of $\text{FREE}_{X \cup Y \cup Z}$, $\vdash (\neg \phi \odot \neg \theta) \rightarrow \neg \epsilon$ and $\{\neg \phi, \neg \theta\} \vdash \neg \epsilon$. Corollary 2.15 yields a formula $\iota \in \text{FORM}_Z$ such that $\neg \theta \vdash \iota$ and $\{\iota, \neg \phi\} \vdash \neg \epsilon$. Equivalently, by [1, 4.2.7, 4.6.3], we can write

$$\frac{\neg \iota}{\equiv} \in \left\langle \frac{\theta}{\equiv} \right\rangle \quad \text{and} \quad \frac{\epsilon}{\equiv} \in \left\langle \left\{ \frac{\phi}{\equiv}, \frac{\neg \iota}{\equiv} \right\} \right\rangle.$$

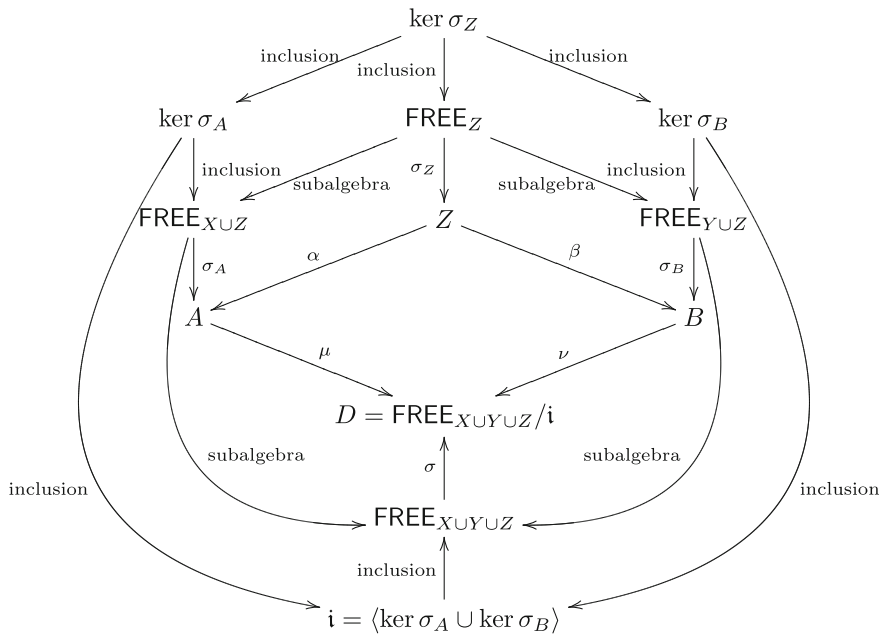
Denoting by $\neg \iota / \equiv_Z$ the equivalence class of $\neg \iota$ in FORM_Z , let the element $j \in \text{FREE}_Z$ be defined by $j = \neg \iota / \equiv_Z$. Then

$$j \in \langle g \rangle \subseteq \ker \sigma_B \subseteq \text{FREE}_{Y \cup Z} \quad \text{and} \quad e \in \langle \{f, j\} \rangle \subseteq \text{FREE}_{X \cup Z}.$$

By Lemma 2.19, $j \in \ker \sigma_Z \subseteq \text{FREE}_Z$ and $j \in \ker \sigma_A$. It follows that $e \in \langle j, f \rangle \subseteq \ker \sigma_A \subseteq \text{FREE}_{X \cup Z}$, and hence, $a = \sigma_A(e) = 0$ as desired.

We have proved that μ is one-one. A similar proof shows that ν is one-one.

In conclusion, arguing pointwise for each $z \in Z$, a routine verification shows that the following diagram is commutative:



□

2.7 Remarks

The definition of polyhedron is taken from [2], and so is the plural “simplexes” (rather than “simplices”).

The desingularization described in Theorem 2.8 is the affine version of the usual desingularization procedure for fans [3, p. 70].

For the blow-up operation see Alexander [4]. Synonyms of “blow-up” are “stellar subdivision” and “elementary subdivision” [3, III, 2.1].

Lemma 2.6 is a routine exercise in polyhedral and combinatorial topology (see [3, V, 1.11]).

Note that in [1, 4.6.1] only theories in the infinite set of variables $\{X_1, X_2, \dots\}$ are considered.

The amalgamation property for MV-algebras was first derived in [5, p. 91] from Pierce amalgamation theorem for abelian lattice-ordered groups, using the Γ functor. The first entirely MV-algebraic proof was given in [6, 6.1]. The present proof is inspired by the general result [7, 5.8], depending on a key result from [8, Theorem I, p. 25], which in turn relies on [9] (also see [10, 1.7]).

In [11] many results relating different amalgamation and interpolation properties are proved in the general framework of algebraizable logics. The amalgamation property of MV-algebras can be extracted from this general theory. Moreover, in view of [12] one may strengthen the amalgamation property as follows: Given MV-algebras A, Z, B and homomorphisms $A \xleftarrow{\alpha} Z \xrightarrow{\beta} B$ there is an MV-algebra D together with homomorphisms $A \xrightarrow{\mu} D \xleftarrow{\nu} B$ such that $\mu\alpha = \nu\beta$, $\ker \mu = \langle \alpha(\ker \beta) \rangle$, and $\ker \nu = \langle \beta(\ker \alpha) \rangle$.

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Chapter 3

The Galois Connection (Mod, Th) in $\mathbf{L}_\infty$

In the two previous chapters we have seen the tight relations between the arithmetic–geometric properties of rational polyhedra and the logic–algebraic properties of interpolation and amalgamation. In this chapter, we study the elementary properties of rational polyhedra as a category. In particular, two rational polyhedra P and Q are said to be $\mathbb{Z}$ -homeomorphic if Q is the image of P under a piecewise linear homeomorphism η such that all linear pieces of η and η^{-1} have integer coefficients. We will see that $\mathbb{Z}$ -homeomorphism is the appropriate notion of equivalence between rational polyhedra, by showing that the operators Mod and Th determine a one–one correspondence between $\mathbb{Z}$ -homeomorphism classes of rational polyhedra and equivalence classes of finitely axiomatizable theories, where two theories are said to be equivalent if their Lindenbaum algebras are isomorphic.

3.1 $\mathbb{Z}$ -Maps

Unless otherwise specified, all rational polyhedra in this chapter are nonempty.

Definition 3.1 Let m, n be integers > 0 . Given a rational polyhedron $P \subseteq [0, 1]^n$ and a continuous map $\zeta = (\zeta_1, \dots, \zeta_m): P \rightarrow [0, 1]^m$, we say that ζ is a $\mathbb{Z}$ -map (of P) if for each $j = 1, \dots, m$, ζ_j is *piecewise linear with integer coefficients*: in other words, there are linear polynomials with integer coefficients $l_{j1}, \dots, l_{jt_j}: [0, 1]^n \rightarrow \mathbb{R}$ such that

$$\text{for every } x \in P \text{ there is } i \in \{1, \dots, t_j\} \text{ with } \zeta_j(x) = l_{ji}(x).$$

The identity function on P , as well as the constant functions 0 and 1, are $\mathbb{Z}$ -maps. The projection of $P \in [0, 1]^{n+1}$ onto the hyperplane $x_{n+1} = 0$ is a $\mathbb{Z}$ -map. Compositions of $\mathbb{Z}$ -maps are $\mathbb{Z}$ -maps. Every McNaughton function $f \in \mathcal{M}([0, 1]^n)$ is a $\mathbb{Z}$ -map of $[0, 1]^n$ into $[0, 1]$.

Proposition 3.2 *Let $P \subseteq [0, 1]^n$ be a rational polyhedron and*

$$\zeta = (\zeta_1, \dots, \zeta_m): P \rightarrow [0, 1]^m$$

a continuous function. Then the following conditions are equivalent:

- (i) ζ is a $\mathbb{Z}$ -map.
- (ii) There are functions $g_1, \dots, g_m \in \mathcal{M}([0, 1]^n)$ such that $\zeta = (g_1, \dots, g_m) \upharpoonright P$.
- (iii) There is a regular triangulation Δ of P such that each ζ_i coincides on every $T \in \Delta$ with some linear polynomial with integer coefficients.

Proof (i $\Rightarrow$ iii) Let us suppose that $l_{j1}, \dots, l_{jt_j}$ are the linear constituents of ζ_j in Definition 3.1. Without loss of generality the constant functions 0 and 1 are among these constituents. Since each polynomial l_{ju} has integer coefficients, the stratification construction of Example 2.1(ii) yields a rational polyhedral complex $\mathcal{K}$ with support P . Using Corollary 2.9, we subdivide $\mathcal{K}$ into a regular ζ -triangulation Δ of P such that for each $j = 1, \dots, m$, the functions $l_{j1}, \dots, l_{jt_j}$ are *stratified* on S , in the sense that for some permutation $\pi = \pi_S$ of $\{1, \dots, t_j\}$ we have $l_{j\pi(1)} \leq l_{j\pi(2)} \leq \dots \leq l_{j\pi(t_j)}$ on S . Then necessarily, for some $i \in \{1, \dots, t_j\}$, the continuous function ζ_j coincides with l_{ji} on S .

(iii $\Rightarrow$ i) is trivial.

(ii $\Rightarrow$ iii) Let the function $g: [0, 1]^n \rightarrow [0, 1]^m$ be defined by $g = (g_1, \dots, g_m)$. Then Corollary 2.9 yields a regular g -triangulation Δ of P . It follows that ζ is linear on each simplex of Δ . Since each g_j belongs to $\mathcal{M}([0, 1]^n)$, $\zeta_j \upharpoonright T = g_j \upharpoonright T$ is expressible by some linear polynomial with integer coefficients.

(iii $\Rightarrow$ ii) By Corollary 2.9 there is a regular triangulation ∇ of $[0, 1]^n$ such that each simplex $T \in \Delta$ is a union of simplexes in ∇ . As a consequence, the set $\nabla_P = \{T \in \nabla \mid T \subseteq P\}$ is a triangulation of P . Further, every function ζ_j coincides on each $S \in \nabla_P$ with a linear polynomial with integer coefficients. In particular, for each vertex v of ∇_P , $\zeta_j(v)$ is an integer multiple of $1/\text{den}(v)$.

Let $g = (g_1, \dots, g_m): [0, 1]^n \rightarrow [0, 1]^m$ be defined by the following stipulations:

- (a) g is linear on every simplex T of ∇ ,
- (b) $g_j(v) = \zeta_j(v)$ for each vertex v of ∇_P , and
- (c) $g(w) = 0$ for every vertex w of ∇ not lying in P .

Evidently, $g \upharpoonright P = \zeta$. Since ∇ is regular, for every $j = 1, \dots, m$ the function g_j coincides on each simplex of ∇ with a linear polynomial with integer coefficients. Thus g_j belongs to $\mathcal{M}([0, 1]^n)$. $\square$

The following result is an immediate consequence of the definition:

Lemma 3.3 *Let $P \subseteq [0, 1]^n$ be a rational polyhedron and $\zeta: P \rightarrow [0, 1]^m$ a $\mathbb{Z}$ -map. Suppose $Q \subseteq P$ is a rational polyhedron. Then the restriction $\zeta \upharpoonright Q$ is a $\mathbb{Z}$ -map.*

The following result is a generalization of Lemma 2.2:

Lemma 3.4 *The image $\zeta(P)$ of a rational polyhedron $P \subseteq [0, 1]^n$ under a $\mathbb{Z}$ -map $\zeta: P \rightarrow [0, 1]^m$ is a rational polyhedron.*

Proof Corollary 2.9 yields a regular triangulation Δ of P such that on every simplex $T \in \Delta$, ζ coincides with a linear map ζ_T with integer coefficients. Then the image $\zeta_T(T)$ is a convex closed rational polyhedron in $[0, 1]^m$. It follows that $\zeta(P) = \bigcup \{\zeta_T(T) \mid T \in \Delta\}$ is a rational polyhedron. $\square$

The following lemma will be used in Corollary 6.7 for a proof that finitely presented MV-algebras have the amalgamation property, without using the interpolation property.

Lemma 3.5 *Suppose $P \subseteq [0, 1]^n$, $Q \subseteq [0, 1]^m$ and $R \subseteq [0, 1]^l$ are rational polyhedra, together with surjective $\mathbb{Z}$ -maps $\gamma: P \rightarrow R$ and $\delta: Q \rightarrow R$. Then there is a rational polyhedron $D \subseteq [0, 1]^{n+m}$ together with surjective $\mathbb{Z}$ -maps $\alpha: D \rightarrow P$ and $\beta: D \rightarrow Q$ such that $\gamma\alpha = \delta\beta$.*

Proof Let $D \subseteq P \times Q \subseteq [0, 1]^{n+m}$ be given by

$$D = \{(x, y) \in P \times Q \mid \gamma(x, y) = \delta(x, y)\}.$$

Let α be the projection of D onto P , and β be the projection of D onto Q . By Lemma 2.2 together with Corollary 2.11, $P \times Q$ is a rational polyhedron. By Corollary 2.10, the set $Z = \{(x, y) \in [0, 1]^{n+m} \mid \gamma(x, y) = \delta(x, y)\}$ is a rational polyhedron, coinciding with the zeroset of the McNaughton function $\mathbf{d}(\gamma, \delta)$. By Corollary 2.12, the intersection $D = Z \cap (P \times Q)$ is a rational polyhedron. Our hypothesis about γ and δ being surjective ensures that α is onto P and β is onto Q . Now α and β are restrictions to D of the canonical projections $P \times Q \rightarrow P$ and $P \times Q \rightarrow Q$, which are immediately seen to be $\mathbb{Z}$ -maps. By Lemma 3.3, α and β are $\mathbb{Z}$ -maps. Direct verification shows that $\gamma\alpha = \delta\beta$. $\square$

Lemma 3.6 *Let $f = (f_1, \dots, f_k)$ be a k -tuple of McNaughton functions of $\mathcal{M}([0, 1]^n)$, $P \subseteq [0, 1]^n$ a rational polyhedron, and $Q = f(P)$. Then the subalgebra M of $\mathcal{M}(P)$ generated by $f_1 \upharpoonright P, \dots, f_k \upharpoonright P$ is isomorphic to $\mathcal{M}(Q)$.*

Proof For every $g \in \mathcal{M}(Q)$ the composite function $h = gf \upharpoonright P$ is piecewise linear and every linear piece of h has integer coefficients. By Proposition 3.2(i $\Leftrightarrow$ ii), the function γ defined by $g \mapsto gf$ maps $\mathcal{M}(Q)$ into M , because f is a $\mathbb{Z}$ -map. A trivial verification shows that γ is a homomorphism of $\mathcal{M}(Q)$ onto M . To see that γ is one-one, suppose $g \neq 0 \in \mathcal{M}(Q)$, say, $g(z) \neq 0$ for some $z \in Q$. Then $gf(x) \neq 0$, where x is any point in P such that $f(x) = z$. $\square$

The following elementary result will provide a standard tool to construct $\mathbb{Z}$ -maps:

Lemma 3.7 *Let m, n be integers > 0 , and $k = 1, \dots, n + 1$.*

- (i) *Suppose $T = \text{conv}(x_1, \dots, x_k) \subseteq [0, 1]^n$ is a regular $(k - 1)$ -simplex and $y_1, \dots, y_k$ are rational points in $[0, 1]^m$ such that, for each $i = 1, \dots, k$, $\text{den}(y_i)$ is a divisor of $\text{den}(x_i)$. Then there is an integer $(m \times n)$ -matrix E and a vector $b \in \mathbb{Z}^m$ such that $Ex_i + b = y_i$, ($i = 1, \dots, k$).*

- (ii) The linear map $\eta: T \rightarrow \text{conv}(y_1, \dots, y_k)$ defined by $\eta(x) = Ex + b$ is uniquely determined by its values on $x_1, \dots, x_k$. The matrix E representing η on T is unique iff T is n -dimensional.
- (iii) Let $P \subseteq [0, 1]^n$ be a rational polyhedron and Δ a regular triangulation of P , with its set $\mathcal{V}$ of vertices. Suppose the map $d: \mathcal{V} \rightarrow [0, 1]^m$ has the property that, for each $v \in \mathcal{V}$, $\text{den}(d(v))$ is a divisor of $\text{den}(v)$. Then d uniquely extends to a $\mathbb{Z}$ -map $\eta: P \rightarrow [0, 1]^m$ which is linear on each simplex of Δ .

Proof (i) The regularity of T yields a basis $\{\tilde{x}_1, \dots, \tilde{x}_k, b_{k+1}, \dots, b_{n+1}\}$ of the free abelian group $\mathbb{Z}^{n+1}$, for suitable vectors $b_{k+1}, \dots, b_{n+1} \in \mathbb{Z}^{n+1}$. (In case $k = n + 1$, already $\tilde{x}_1, \dots, \tilde{x}_k$ form a basis.) Let N be the integer $((n + 1) \times (n + 1))$ -matrix whose columns coincide with the vectors $\tilde{x}_1, \dots, \tilde{x}_k, b_{k+1}, \dots, b_{n+1}$. Then N^{-1} is an integer matrix. Let us use the abbreviation $c_i = \text{den}(x_i)(y_i, 1)$, for each $i = 1, \dots, k$. By hypothesis, $c_i \in \mathbb{Z}^{m+1}$. Let $d_{k+1}, \dots, d_{n+1}$ be arbitrarily chosen vectors in $\mathbb{Z}^{m+1}$ such that for each $j = k + 1, \dots, n + 1$ the $(m + 1)$ th coordinate of d_j coincides with the $(n + 1)$ th coordinate of b_j . Let $R \in \mathbb{Z}^{(m+1) \times (n+1)}$ be the matrix whose columns are given by the vectors $c_1, \dots, c_k, d_{k+1}, \dots, d_{n+1}$. Since the $(n + 1)$ th row of N equals the $(m + 1)$ th row of R , we can write

$$RN^{-1} = \left(\begin{array}{c|c} E & b \\ \hline 0, \dots, 0 & 1 \end{array} \right)$$

for some integer $(m \times n)$ -matrix E and integer vector $b \in \mathbb{Z}^m$. For each $i = 1, \dots, k$ the following holds: $RN^{-1}\tilde{x}_i = RN^{-1}\text{den}(x_i)(x_i, 1) = \text{den}(x_i)(Ex_i + b, 1)$. The desired conclusion now follows by writing $RN^{-1}\tilde{x}_i = c_i = \text{den}(x_i)(y_i, 1)$, whence $Ex_i + b = y_i$.

(ii) is trivial.

(iii) By (ii), for each $S \in \Delta$ there is a unique linear map (with integer coefficients) $\eta_S: S \rightarrow [0, 1]^m$ such that $\eta_S(v) = d(v)$ for every $v \in \mathcal{V}$. The function $\eta = \bigcup \{\eta_S \mid S \in \Delta\}$ coincides with η_S on every simplex $S \in \Delta$, and provides the unique $\mathbb{Z}$ -map extending d and linear on each simplex of Δ . $\square$

Lemma 3.8 Let $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ be rational polyhedra, for $0 < m, n \in \mathbb{Z}$.

(i) Suppose σ is a $\mathbb{Z}$ -map of P onto Q . Let the map

$$\iota_\sigma: g \mapsto \iota_\sigma(g) = g\sigma$$

transform each $g \in \mathcal{M}(Q)$ into the composite function $g\sigma \in \mathcal{M}(P)$. Then ι_σ is a one-one homomorphism of $\mathcal{M}(Q)$ into $\mathcal{M}(P)$. The range of ι_σ coincides with the subalgebra of $\mathcal{M}(P)$ generated by the set of functions $\{\iota_\sigma(\pi_1 \upharpoonright Q), \dots, \iota_\sigma(\pi_m \upharpoonright Q)\}$.

(ii) Conversely, for any one-one homomorphism ι of $\mathcal{M}(Q)$ into $\mathcal{M}(P)$, letting $f_i = \iota(\pi_i \upharpoonright Q)$, $i = 1, \dots, m$, the function

$$\sigma_i: x \in P \mapsto (f_1(x), \dots, f_m(x)) \in [0, 1]^m$$

is a $\mathbb{Z}$ -map of P onto Q .

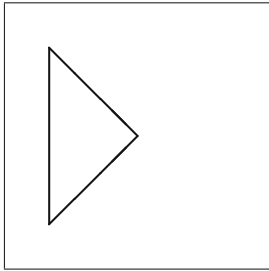
Proof (i) By Proposition 3.2, $g\sigma \in \mathcal{M}(P)$. Trivially, ι_σ is a homomorphism of $\mathcal{M}(Q)$ into $\mathcal{M}(P)$. Now suppose $0 \neq g \in \mathcal{M}(Q)$, i.e., $g(y) \neq 0$ for some $y \in Q$. Let x be an arbitrary element in $\sigma^{-1}(y)$, whose existence is ensured by hypothesis. Then $(\iota_\sigma(g))(x) = g(\sigma(x)) = g(y) \neq 0$, which proves that ι_σ is one-one. The final statement follows because $\{\pi_1 \upharpoonright Q, \dots, \pi_m \upharpoonright Q\}$ generates $\mathcal{M}(Q)$.

(ii) For any $g = \hat{\psi} \in \mathcal{M}(Q)$, induction on the number of connectives in ψ shows that $\iota(g) = g\sigma_i$. It follows that the range R of the $\mathbb{Z}$ -map $\sigma_i: P \rightarrow [0, 1]^m$ is contained in Q . For the converse inclusion, by way of contradiction suppose $y \in Q \setminus R$. Then $Q \setminus R$ is a relatively open subset of Q . By Lemma 1.13 there is $f \in \mathcal{M}(Q)$ constantly vanishing on R and taking value 1 at y . It follows that $f \neq 0$ is mapped by ι into $f\sigma_i = 0$. This contradicts the assumed injectivity of ι . $\square$

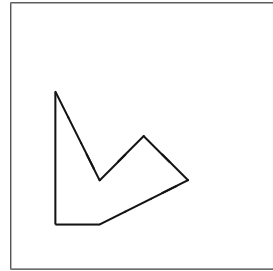
3.2 $\mathbb{Z}$ -Homeomorphism

Definition 3.9 Fix integers $m, n > 0$. Given rational polyhedra $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ we write $P \cong_{\mathbb{Z}} Q$ (read: “ P is $\mathbb{Z}$ -homeomorphic to Q ”) if there is a homeomorphism η of P onto Q such that both η and η^{-1} are $\mathbb{Z}$ -maps. We also say that η is a $\mathbb{Z}$ -homeomorphism.

The following picture shows the action of a $\mathbb{Z}$ -homeomorphism on a rational triangle:



The triangle $T = \text{conv}\left(\left(\frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{6}, \frac{5}{6}\right)\right)$



A $\mathbb{Z}$ -homeomorphic copy of T

Corollary 3.10 For any rational polyhedra $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$, $\mathcal{M}(P) \cong \mathcal{M}(Q)$ iff $P \cong_{\mathbb{Z}} Q$. In more detail,

- (i) If $\zeta: P \cong_{\mathbb{Z}} Q$ then the map $\iota_\zeta: f \mapsto f\zeta$ is an isomorphism of $\mathcal{M}(Q)$ onto $\mathcal{M}(P)$.

- (ii) Given an isomorphism $\iota: \mathcal{M}(Q) \cong \mathcal{M}(P)$, let $\pi_1, \dots, \pi_m$ be the coordinate functions on $\mathbb{R}^m$. Let the map $\zeta_i: P \rightarrow \mathbb{R}^m$ be defined by $\zeta_i = (\iota(\pi_1 \upharpoonright Q), \dots, \iota(\pi_m \upharpoonright Q))$. Then ζ_i is a $\mathbb{Z}$ -homeomorphism of P onto Q .

Proof Respectively from Lemma 3.8, (i) and (ii). $\square$

Lemma 3.11 Let $Q \subseteq [0, 1]^m$ and $R \subseteq P \subseteq [0, 1]^n$ be rational polyhedra. Suppose η is a $\mathbb{Z}$ -homeomorphism of Q onto R . Then the map $\omega: f \in \mathcal{M}(P) \mapsto f\eta \in \mathcal{M}(Q)$ is a homomorphism of $\mathcal{M}(P)$ onto $\mathcal{M}(Q)$.

Proof We have only to prove that ω is onto $\mathcal{M}(Q)$. Suppose $q \in \mathcal{M}(Q)$. Observe that the function $q\eta^{-1}$ is a $\mathbb{Z}$ -map defined on R . Therefore, by Proposition 3.2, $q\eta^{-1}$ belongs to $\mathcal{M}(R)$. In other words, there exists $f \in \mathcal{M}([0, 1]^n)$ with $f \upharpoonright R = q\eta^{-1}$. The function $p = f \upharpoonright P$ satisfies $p \in \mathcal{M}(P)$ and $p \upharpoonright R = f \upharpoonright R = q\eta^{-1}$. In conclusion, $\omega(p) = p\eta = (p \upharpoonright R)\eta = (q\eta^{-1})\eta = q$, and ω is onto $\mathcal{M}(Q)$. $\square$

Recall that π_i denotes the i th coordinate function defined on the n -cube.

For later use we record here the following result:

Lemma 3.12 Let $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ be rational polyhedra. Let α be a homomorphism of $\mathcal{M}(P)$ onto $\mathcal{M}(Q)$. For each $i = 1, \dots, n$ let $J_i = \alpha(\pi_i \upharpoonright P)$. Let the map $J: Q \rightarrow [0, 1]^n$ be defined by $J(y) = (J_1(y), \dots, J_n(y))$ for all $y \in Q$. Then for each $f \in \mathcal{M}(P)$, $\alpha(f) = fJ$. Further, $J(Q)$ is a rational polyhedron, and J is a $\mathbb{Z}$ -homeomorphism of Q onto $J(Q)$.

Proof Let $R = J(Q)$. By Lemma 3.4, R is a rational polyhedron. Arguing by induction on the number of MV-algebraic operation symbols necessary to obtain f from $\pi_1 \upharpoonright P, \dots, \pi_n \upharpoonright P$, we see that the composite map fJ coincides with $\alpha(f)$.

Evidently, J is a $\mathbb{Z}$ -map of Q onto R ; also, $R \subseteq P$, for otherwise $\alpha(f) = fJ$ would not be a member of $\mathcal{M}(Q)$. Thus $\mathcal{M}(P) \upharpoonright R = \mathcal{M}(R)$. Next suppose $x \neq y \in Q$. The set $\{J_1, \dots, J_n\}$ generates $\mathcal{M}(Q)$, because α is onto. By Lemma 1.13, the two n -tuples $(J_1(x), \dots, J_n(x))$ and $(J_1(y), \dots, J_n(y))$ are different, i.e., J is one-one.

Having thus shown that the $\mathbb{Z}$ -map J is a homeomorphism of Q onto R , there remains to be proved that J^{-1} is a $\mathbb{Z}$ -map.

Let the map θ be defined by $\theta: r \in \mathcal{M}(R) \mapsto rJ \in \mathcal{M}(Q)$. Then θ is one-one: as a matter of fact, if $r \in \mathcal{M}(R)$ is nonzero, say, $r(x) \neq 0$ then letting $y = J^{-1}(x)$ it follows that $rJ(y) \neq 0$. To see that θ is onto, for each $g \in \mathcal{M}(Q)$ let $h \in \mathcal{M}(P)$ be such that $\alpha(h) = g = hJ$. Then $(h \upharpoonright R)J = g$. Having thus proved that $\theta: \mathcal{M}(R) \cong \mathcal{M}(Q)$, arguing as in the first part of the proof we conclude that J^{-1} is a $\mathbb{Z}$ -map; actually, J^{-1} is a $\mathbb{Z}$ -homeomorphism of R onto Q . $\square$

Lemma 3.13 Suppose P is a rational polyhedron in $[0, 1]^n$, Q is a rational polyhedron in $[0, 1]^m$, and Δ is a regular triangulation of P . If η is a $\mathbb{Z}$ -homeomorphism of P onto Q and η is linear on every simplex of Δ , then the set $\nabla = \{\eta(T) \mid T \in \Delta\}$ is a regular triangulation of Q .

Proof Since each linear piece of every η_i has integer coefficients, ∇ is a rational triangulation of Q . Fix a simplex $S = \text{conv}(v_0, \dots, v_j)$ of Δ , with its image $\eta(S) =$

$T \in \nabla$. The linear map $\eta: x \in S \mapsto y \in T$ determines the homogeneous linear map $(x, 1) \mapsto (y, 1)$. In more detail, let M_S be the $(m+1) \times (n+1)$ integer matrix whose i th row ($i = 1, \dots, m$) is given by the coefficients of the linear polynomial $\eta_i \upharpoonright S$, and whose bottom row has the form $(0, 0, \dots, 0, 0, 1)$, with n zeros. Then $M_S(x, 1) = (y, 1)$. Letting the two cones $S^\uparrow$ and $T^\uparrow$ be defined by

$$S^\uparrow = \langle \tilde{v}_0, \dots, \tilde{v}_j \rangle \subseteq \mathbb{R}^{n+1} \quad \text{and} \quad T^\uparrow = \langle M_S \tilde{v}_0, \dots, M_S \tilde{v}_j \rangle,$$

it follows that M_S sends the set of integer points of $S^\uparrow$ one–one *into* the set of integer points of $T^\uparrow$. Interchanging the roles of P and Q one sees that M_S actually sends integer points of $S^\uparrow$ one–one *onto* integer points of $T^\uparrow$. From Lemma 2.6 it follows that S is regular iff the half-open parallelepiped

$$P_S = \{\mu_0 \tilde{v}_0 + \dots + \mu_j \tilde{v}_j \mid 0 \leq \mu_0, \dots, \mu_j < 1\}$$

contains no nonzero integer points, iff so does its M_S -image P_T iff T is regular. Thus the regularity of S is equivalent to the regularity of S' . This shows that ∇ is a regular triangulation of Q . $\square$

Recall from Corollary 2.9 the definition of f -triangulation.

Lemma 3.14 (i) *Let $T = \text{conv}(x_1, \dots, x_j) \subseteq [0, 1]^n$ and $S = \text{conv}(y_1, \dots, y_j) \subseteq [0, 1]^m$ be regular $(j-1)$ -simplexes. If $\text{den}(x_1) = \text{den}(y_1), \dots, \text{den}(x_j) = \text{den}(y_j)$ then there is a (necessarily unique) linear $\mathbb{Z}$ -homeomorphism η_T of T onto S such that $\eta_T(x_i) = y_i$ for all $i = 1, \dots, j$.*

(ii) *Let $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ be rational polyhedra with regular triangulations Δ and ∇ respectively. Suppose there exists a one–one map ι of the set $\mathcal{V}$ of vertices of Δ onto the set of vertices of ∇ such that $\text{den}(v) = \text{den}(\iota(v))$ for all $v \in \mathcal{V}$ and*

$$\text{conv}(v_1, \dots, v_k) \in \Delta \Leftrightarrow \text{conv}(\iota(v_1), \dots, \iota(v_k)) \in \nabla. \quad (3.1)$$

Then there is a unique $\mathbb{Z}$ -homeomorphism $\bar{\iota}$ of P onto Q which is linear on every simplex of Δ and extends ι .

(iii) *Suppose P is a rational polyhedron in $[0, 1]^n$, and $\eta: P \rightarrow [0, 1]^k$ is a one–one $\mathbb{Z}$ -map. Suppose Δ is a regular η -triangulation of P and for every vertex v of Δ , $\text{den}(v) = \text{den}(\eta(v))$. Suppose further the simplex $\eta(T)$ to be regular for each $T \in \Delta$. Then η is a $\mathbb{Z}$ -homeomorphism of P onto $\eta(P)$.*

Proof (i) Immediate from Lemma 3.7.

(ii) Suppose $\Delta \ni T = \text{conv}(v_1, \dots, v_k)$ and $T' = \text{conv}(\iota(v_1), \dots, \iota(v_k))$. By (i), ι extends to a unique linear $\mathbb{Z}$ -homeomorphism of T onto T' . From (3.1) it follows that $\iota_{T_1} \cap \iota_{T_2} = \iota_{T_1 \cap T_2}$. Now let $\bar{\iota} = \bigcup_{T \in \Delta} \iota_T$.

(iii) The set $\eta(\Delta) = \{\eta(T) \mid T \in \Delta\}$ is a regular triangulation of the rational polyhedron $Q = \eta(P)$. Now apply (ii). $\square$

Summing up, we have the following characterization of $\mathbb{Z}$ -homeomorphisms:

Proposition 3.15 *Given integers $m, n > 0$, let $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ be rational polyhedra, and η a one-one $\mathbb{Z}$ -map of P onto Q . Then the following conditions are equivalent:*

- (i) η is a $\mathbb{Z}$ -homeomorphism.
- (ii) *For some regular η -triangulation Δ of P , $\text{den}(\eta(v)) = \text{den}(v)$ for all vertices v of Δ , and the simplex $\eta(T)$ is regular for each $T \in \Delta$.*
- (iii) *For every regular η -triangulation Δ of P , $\text{den}(\eta(v)) = \text{den}(v)$ for all vertices v of Δ , and the simplex $\eta(T)$ is regular for each $T \in \Delta$.*
- (iv) *For every rational point $r \in P$, $\text{den}(r) = \text{den}(\eta(r))$.*

Proof (ii $\Rightarrow$ i) follows from Lemma 3.14(iii). (iii $\Rightarrow$ ii) trivially follows from Corollary 2.9. (i $\Rightarrow$ iii) follows from Lemma 3.13. (i $\Rightarrow$ iv) immediately follows by definition of $\mathbb{Z}$ -homeomorphism.

To prove (iv $\Rightarrow$ i) suppose η is not a $\mathbb{Z}$ -homeomorphism, but η preserves denominators of all rational points of P (absurdum hypothesis). Corollary 2.9 in combination with the present characterization (i $\Leftrightarrow$ ii $\Leftrightarrow$ iii) yields a regular η -triangulation Δ of P , and a simplex $S \in \Delta$ such that $T = \eta(S)$ is not regular. Without loss of generality T is minimal among all non-regular simplexes of Δ . Let $x_1, \dots, x_t$ be the vertices of S and $y_1, \dots, y_t$ be the vertices of T . By Lemma 2.6, the half-open parallelepiped $Q \subseteq \mathbb{R}^{m+1}$ associated to T contains a nonzero integer point q . Without loss of generality q is primitive. The assumed minimality of T is to the effect that q actually lies in the relative interior of Q . Let the rational point $y \in T$ be defined by $\tilde{y} = q$. Then y lies in the relative interior of T , and by Lemma 2.7, $\text{den}(y) < \text{den}(y_1) + \dots + \text{den}(y_t)$. The inverse image $x = \eta^{-1}(y)$ is a rational point satisfying $\text{den}(x) < \text{den}(x_1) + \dots + \text{den}(x_t)$, because η preserves denominators. The regularity of S ensures that the homogeneous correspondent $\tilde{x}$ is a positive integer combination of $\tilde{x}_1, \dots, \tilde{x}_t$. Since x lies in the relative interior of S , all the coefficients in this combination are ≥ 1 . It follows that $\text{den}(x) \geq \text{den}(x_1) + \dots + \text{den}(x_t)$, a contradiction. $\square$

3.3 $\mathbb{Z}$ -Homeomorphic Segments with Rational Endpoints

As we will see throughout this book, one-dimensional rational polyhedra enjoy very special properties. A first example is given by:

Proposition 3.16 *For $n = 1, 2, \dots$, let a, b be rational points in $[0, 1]^n$ and $E = \text{conv}(a, b)$.*

- (i) *Any $\mathbb{Z}$ -homeomorphism η of E onto E such that $a = \eta(a)$, necessarily coincides with identity.*
- (ii) *Given $m = 1, 2, \dots$ and rational points $a', b' \in [0, 1]^m$, let $E' = \text{conv}(a', b')$. If $\iota: E \cong_{\mathbb{Z}} E'$ is a $\mathbb{Z}$ -homeomorphism such that $\iota(a) = a'$ then ι is uniquely determined and is linear.*

- (iii) E is regular iff its relative interior does not contain any rational point of denominator $\leq \max(\text{den}(a), \text{den}(b))$.

Proof (i) By way of contradiction suppose η is a counterexample, and let Δ be a regular η -triangulation of E with vertices $a_0 = a < a_1 < \dots < a_k = b$. The existence of Δ is ensured by Corollary 2.9. Here the a_i are displayed in order of increasing distance from a . Let us write $a'_i = \eta(a_i)$, $(i = 0, \dots, k)$. By Proposition 3.15, $\text{den}(a_i) = \text{den}(a'_i)$ and $\text{conv}(a'_i, a'_{i+1})$ is regular. Now note that if $\eta(a_j) = a_j$ for all $j = 0, \dots, k$ in the regular η -triangulation Δ , then η would coincide with identity—against our absurdum hypothesis. So let j be the smallest index such that $a'_j \neq a_j$. Without loss of generality, $a'_j > a_j$. We then have distinct regular simplexes $\text{conv}(a_{j-1}, a_j)$ and $\text{conv}(a_{j-1}, a'_j) = \text{conv}(a'_{j-1}, a'_j)$. By Lemma 2.6 all rational points lying in the relative interior R of $\text{conv}(a_{j-1}, a'_j)$ have denominator $\geq \text{den}(a_{j-1}) + \text{den}(a'_j)$. But a_j lies in R and $\text{den}(a_j) = \text{den}(a'_j) < \text{den}(a_{j-1}) + \text{den}(a'_j)$, a contradiction.

(ii) Uniqueness immediately follows from (i). By way of contradiction, suppose ι is not linear. Let ∇ be a ι -regular triangulation of E , with vertices $c_0 = a < c_1 < \dots < c_k = b$. Let $c'_i = \iota(c_i)$. One more application of Proposition 3.15 yields a regular triangulation ∇' of E' with vertices $c'_0 = a' < c'_1 < \dots < c'_k = b'$. The restriction of ι to $\text{conv}(c_0, c_1)$ is the uniquely determined linear $\mathbb{Z}$ -homeomorphism $\bar{\iota}$ of $\text{conv}(c_0, c_1)$ onto $\text{conv}(c'_0, c'_1)$ of Lemma 3.7(ii). Let ι^* be the extension of $\bar{\iota}$ to the affine hull $\text{aff}(\text{conv}(c_0, c_1)) \supseteq E$. Arguing as in the final part of the proof of (i), we see that $\iota^*(c_2)$ must coincide with c'_2 , because by Proposition 3.2, both 1-simplexes $\text{conv}(c'_1, c'_2)$ and $\text{conv}(c'_1, \iota^*(c_2))$ are regular and $\text{den}(c'_2) = \text{den}(\iota^*(c_2))$. Proceeding inductively, each c'_i must coincide with $\iota^*(c_i)$. We have proved that $\iota^* \upharpoonright E = \iota$ and ι is linear.

- (iii) For definiteness, let us assume $\text{den}(a) \leq \text{den}(b)$. Let

$$P = \{x \in \mathbb{R}^{n+1} \mid x = \alpha \tilde{a} + \beta \tilde{b}, 0 \leq \alpha, \beta < 1\}$$

be the half-open parallelogram determined by E .

($\Rightarrow$) Suppose E is regular. Let c be a rational point lying in the relative interior of E . By Lemma 2.6, the regularity of the cone $E^\uparrow = \langle \tilde{a}, \tilde{b} \rangle$ ensures that $\tilde{c} = k\tilde{a} + l\tilde{b}$ for suitable integers $k, l \geq 0$. Actually, $k, l \geq 1$, because $\tilde{c}$ lies in the relative interior of $E^\uparrow$. As a consequence, the $(n+1)$ th coordinate of $\tilde{c}$ is strictly larger than the $(n+1)$ th coordinate of $\tilde{b}$. This is the same as saying that $\text{den}(c) > \text{den}(b)$.

($\Leftarrow$) Conversely, if E is not regular, again by Lemma 2.6 P contains a nonzero integer point s . If the $(n+1)$ th coordinate of s is $\leq$ the $(n+1)$ th coordinate of $\tilde{b}$, then the rational point r given by $\tilde{r} = s$ lies in the relative interior of E . Otherwise, we replace s by $s' = \tilde{a} + \tilde{b} - s$, and note that the rational point t given by $\tilde{t} = s'$ lies in the relative interior of E . $\square$

Remark 3.17 The above characterization (iii) is a refinement of Lemma 2.7.

For later use we record here the following consequence of Propositions 3.15 and 3.16(i):

Corollary 3.18 *Let S denote the unit square $[0, 1]^2$ and P its perimeter. Then for every $\mathbb{Z}$ -homeomorphism η of S onto S there is a linear (affine) $\mathbb{Z}$ -homeomorphism λ of S onto S such that $\lambda(\eta(x)) = x$ for each $x \in P$.*

Proof From Proposition 3.15 it follows that η maps P onto P and permutes the vertices of S ; further, η maps each edge E of S onto some edge of S : for otherwise, the number of points of denominator 1 in $\eta(E)$ would be different from 2, thus contradicting the assumption that η is a $\mathbb{Z}$ -homeomorphism. It follows that the two edges E' and E'' of S having the origin as a common vertex are mapped by η onto two consecutive edges of S . By Proposition 3.16(i), the restriction of η to $E' \cup E''$ is linear, whence so is the restriction of η to P . A final verification shows that $\eta \upharpoonright P$ extends to a linear $\mathbb{Z}$ -homeomorphism μ of S onto S . Letting now $\lambda = \mu^{-1}$, we have the desired conclusion. $\square$

3.4 Equivalent Theories and the Galois Connection (Mod, Th)

Given $n = 1, 2, \dots$ and $\Theta \subseteq \text{FORM}_n$ we say that Θ is *finitely axiomatizable* if for some (necessarily satisfiable) formula $\theta \in \text{FORM}_n$, Θ is the smallest theory in the variables $X_1, \dots, X_n$ containing θ .

Lemma 3.19 *For any theory $\Phi \subseteq \text{FORM}_n$ the following conditions are equivalent:*

- (i) Φ is finitely axiomatizable.
- (ii) For some $\phi \in \text{FORM}_n$, $\Phi = \{\psi \in \text{FORM}_n \mid \phi \vdash \psi\}$.
- (iii) For some $\phi \in \text{FORM}_n$, $\Phi = \{\psi \in \text{FORM}_n \mid \text{Mod}(\phi) \subseteq \text{Mod}(\psi)\}$.
- (iv) $\Phi = \text{Th}P$ for some rational polyhedron $P \subseteq [0, 1]^n$.
- (v) For some formula $\phi \in \text{FORM}_n$, $\Phi = \text{ThMod}(\phi)$.

Proof The equivalence (i $\Leftrightarrow$ ii) holds because the set $\{\psi \in \text{FORM}_n \mid \phi \vdash \psi\}$ is the smallest theory $\Phi \subseteq \text{FORM}_n$ such that $\phi \in \Phi$.

The equivalence (ii $\Leftrightarrow$ iii) is a reformulation of Theorem 1.7.

(iii $\Rightarrow$ iv) By Corollary 2.10, $\text{Mod}(\phi)$ coincides with some rational polyhedron $P \subseteq [0, 1]^n$. From (1.1) and (1.5) in Chap. 1 it follows that $\Phi = \text{Th Mod}(\phi)$.

The implication (iv $\Rightarrow$ iii) is proved by letting $\phi \in \text{FORM}_n$ satisfy $P = \text{Mod}(\phi)$. The existence of ϕ is ensured by Corollary 2.10.

(v $\Leftrightarrow$ iv) follows from Corollary 2.10. $\square$

The maps Mod and Th provide what is known as a *Galois connection*:

Theorem 3.20 *For each $n = 1, 2, \dots$ the map $P \mapsto \text{Th}P$ is a one-one correspondence between (always nonempty) rational polyhedra $P \subseteq [0, 1]^n$ and finitely axiomatizable theories in the variables $X_1, \dots, X_n$. The inverse map sends any such theory Φ to the rational polyhedron $\text{Mod}(\Phi)$. We further have:*

- (a) $P \subseteq P' \Rightarrow \text{Th}P \supseteq \text{Th}P'$.
- (b) $\Phi \subseteq \Phi' \Rightarrow \text{Mod}(\Phi) \supseteq \text{Mod}(\Phi')$.

(c) $\text{Th Mod}(\Phi) \supseteq \Phi$ and $\text{Mod}(\text{Th}P) = P$.

Proof For any rational polyhedron $P \subseteq [0, 1]^n$ let $\theta \in \text{FORM}_n$ be such that $P = \text{Mod}(\theta)$. The existence of θ is ensured by Corollary 2.10. It follows that $\text{Th}P = \{\psi \in \text{FORM}_n \mid \text{Mod}(\psi) \subseteq \text{Mod}(\theta)\}$. By Lemma 3.19, $\text{Th}P$ is a finitely axiomatizable theory and the map $P \mapsto \text{Th}P$ is onto. To prove that the map is one-one, let $Q \subseteq [0, 1]^n$ be a rational polyhedron with $Q \neq P$. Suppose first $z \in P \setminus Q$, with the intent of proving $\text{Th}Q \neq \text{Th}P$. By Lemma 1.13 there is a McNaughton function $f \in \mathcal{M}([0, 1]^n)$ such that $f(z) = 0$ and $f(x) = 1$ for all $x \in Q$. McNaughton theorem yields a formula $\phi \in \text{FORM}_n$ such that $f = \hat{\phi}$. Recalling (1.5), $z \notin \text{Mod}(\phi) \supseteq Q$. From (1.1) it follows that $\phi \in \text{Th}Q$ and $\phi \notin \text{Th}P$, whence $\text{Th}Q \neq \text{Th}P$. The case $z \in Q \setminus P$ similarly implies $\text{Th}Q \neq \text{Th}P$. Now conditions (a–c) are easily verified. In particular, $\text{Mod}(\text{Th}P) = P$ follows from Lemma 1.14. $\square$

Remark 3.21 Since Φ is finitely axiomatizable, the first inclusion in (c) above can be strengthened to $\text{ThMod}(\Phi) = \Phi$.

Definition 3.22 Theories $\Theta \subseteq \text{FORM}_n$ and $\Phi \subseteq \text{FORM}_m$ are said to be *equivalent* if their Lindenbaum algebras are isomorphic.

Theorem 3.23 (i) Let Φ and Φ' be finitely axiomatizable theories in the variables $Y_1, \dots, Y_n$ and $Y'_1, \dots, Y'_{n'}$ respectively. We then have: $\text{Mod}(\Phi) \cong_{\mathbb{Z}} \text{Mod}(\Phi')$ iff $\text{LIND}_{\Phi} \cong \text{LIND}_{\Phi'}$.

(ii) For each $n = 1, 2, \dots$, the maps $R \mapsto \text{Th}R$ and $\Psi \mapsto \text{Mod}(\Psi)$ determine a one-one correspondence between $\mathbb{Z}$ -homeomorphism classes of rational polyhedra contained in the cube $[0, 1]^n$ and equivalence classes of finitely axiomatizable theories in n variables.

Proof Theorem 3.20 yields $\phi \in \text{FORM}_n$, $\phi' \in \text{FORM}_{n'}$ and rational polyhedra $Q \subseteq [0, 1]^n$ and $Q' \subseteq [0, 1]^{n'}$ such that

$$\Phi = \{\psi \in \text{FORM}_n \mid \phi \vdash \psi\}, \quad \Phi' = \{\psi \in \text{FORM}_{n'} \mid \phi' \vdash \psi\}$$

and

$$Q = \text{Mod}(\phi) = \text{Mod}(\Phi), \quad Q' = \text{Mod}(\phi') = \text{Mod}(\Phi').$$

The last two identities follow by noting that the inclusion $\text{Mod}(\Phi) \subseteq \text{Mod}(\phi)$ is trivial, and the converse inclusion is an immediate consequence of Lemma 3.19(iii).

(i) $(\Rightarrow)$ Suppose $\zeta = (\zeta_1, \dots, \zeta_{n'})$ is a $\mathbb{Z}$ -homeomorphism of Q onto Q' . Since each linear piece of ζ and of ζ^{-1} has integer coefficients, a point $x \in Q$ is rational iff so is the point $\zeta(x) \in Q'$. By Proposition 3.2 there is a regular complex Δ with support Q , together with formulas $\psi_i(Y_1, \dots, Y_n)$, $i = 1, \dots, n'$, such that ζ is linear on every simplex of Δ , and ζ_i is the restriction of the McNaughton function $\hat{\psi}_i: [0, 1]^n \rightarrow [0, 1]$. Letting the map $l: [0, 1]^n \rightarrow [0, 1]^{n'}$ be defined by $l = (\hat{\psi}_1, \dots, \hat{\psi}_{n'})$, it follows that $\zeta = l \upharpoonright Q$. For every $g \in \mathcal{M}(Q')$, the composite map gl belongs to $\mathcal{M}([0, 1]^n)$, whence $gl \upharpoonright Q \in \mathcal{M}(Q)$. By McNaughton theorem,

g has the form $g = \hat{\gamma} \upharpoonright Q'$ for some formula $\gamma(Y'_1, \dots, Y'_n)$. Arguing inductively on the number of connectives in γ , it follows that the map λ given by $g \mapsto g\zeta$ is a homomorphism of $\mathcal{M}(Q')$ into $\mathcal{M}(Q)$. Reversing the roles of Q and Q' , the homomorphism $\mu: \mathcal{M}(Q) \rightarrow \mathcal{M}(Q')$ given by $f \in \mathcal{M}(Q) \mapsto f\zeta^{-1} \in \mathcal{M}(Q')$ is the inverse of λ . Thus $\mathcal{M}(Q)$ is isomorphic to $\mathcal{M}(Q')$. By Lemma 1.12, $\text{LIND}_\Phi \cong \text{LIND}_{\Phi'}$.

(i) ($\Leftarrow$) Conversely, suppose $\text{LIND}_\Phi \cong \text{LIND}_{\Phi'}$, whence, again by Lemma 1.12, we have an isomorphism $\omega: \mathcal{M}(Q') \cong \mathcal{M}(Q)$. For each $i = 1, \dots, n'$, letting $\pi'_i \upharpoonright Q'$ be the restriction to Q' of the i th coordinate function $\pi'_i: [0, 1]^{n'} \rightarrow [0, 1]$, we will write for short f_i for $\omega(\pi'_i \upharpoonright Q')$. There is a function $g_i \in \mathcal{M}([0, 1]^n)$ such that $f_i = g_i \upharpoonright Q$. Let the continuous function $J: [0, 1]^n \rightarrow [0, 1]^{n'}$ be defined by $J(x) = (g_1(x), \dots, g_{n'}(x))$ for all $x \in [0, 1]^n$. Let now $\iota = J \upharpoonright Q = (f_1, \dots, f_{n'})$. By McNaughton theorem, every function $h \in \mathcal{M}(Q')$ has the form $h = \hat{\theta} \upharpoonright Q'$ for some formula $\theta(x'_1, \dots, x'_{n'})$. Induction on the number of connectives in θ shows $\omega(h) = \omega(\hat{\theta} \upharpoonright Q') = \hat{\theta}\iota$ for all $h \in \mathcal{M}(Q')$.

We now *claim* that ι maps Q into Q' . Arguing by way of contradiction, let us suppose there exists a point $y' \in \text{range}(\iota) \setminus Q'$. Then Lemma 1.13 yields a formula $\psi(Y'_1, \dots, Y'_{n'})$ such that $\hat{\psi}(y') = 1$ and $\hat{\psi}(y) = 0$, for all $y \in Q'$. It follows that $\omega(\hat{\psi} \upharpoonright Q') = \hat{\psi}\iota \neq 0$, while $\hat{\psi} \upharpoonright Q'$ is the zero element of the MV-algebra $\mathcal{M}(Q')$, which contradicts the assumption that ω is a homomorphism. Our claim is settled.

The same argument with the roles of Q' and Q interchanged yields a function $\iota': Q' \rightarrow Q$ such that $\omega^{-1}(\hat{\chi} \upharpoonright Q) = \hat{\chi}\iota'$, for every formula $\chi(Y_1, \dots, Y_n) \in \text{FORM}_n$. Since ω is an isomorphism, ι and ι' are the inverse of each other. By construction, ι and ι' are continuous piecewise linear with integer coefficients. It follows that ι is a $\mathbb{Z}$ -homeomorphism of Q onto Q' .

(ii) The proof immediately follows by (i). $\square$

Equivalent finitely axiomatizable theories have the following purely syntactical characterization:

Corollary 3.24 *Let $Y = \{Y_1, \dots, Y_m\}$ and $Z = \{Z_1, \dots, Z_n\}$ be two sets of variables. Let $\Phi \subseteq \text{FORM}_Y$ and $\Psi \subseteq \text{FORM}_Z$ be finitely axiomatizable theories. Then Φ and Ψ are equivalent iff there are formulas $\phi_1, \dots, \phi_n \in \text{FORM}_Y$ and $\psi_1, \dots, \psi_m \in \text{FORM}_Z$ such that for all $i = 1, \dots, m$ and $j = 1, \dots, n$,*

$$\Phi \vdash Y_i \leftrightarrow \psi_i(\phi_1(Y), \dots, \phi_n(Y)), \quad \text{and} \quad \Psi \vdash Z_j \leftrightarrow \phi_j(\psi_1(Z), \dots, \psi_m(Z)). \quad (3.2)$$

Proof By Lemma 3.19, there are formulas $\phi \in \text{FORM}_Y$ and $\psi \in \text{FORM}_Z$ together with rational polyhedra $P \subseteq [0, 1]^m = [0, 1]^Y$ and $Q \subseteq [0, 1]^n = [0, 1]^Z$ such that

$$\Phi = \text{ThMod}(\phi) = \text{Th}P \quad \text{and} \quad \Psi = \text{ThMod}(\psi) = \text{Th}Q. \quad (3.3)$$

($\Leftarrow$) Let the maps $\alpha: [0, 1]^m \rightarrow [0, 1]^n$ and $\beta: [0, 1]^n \rightarrow [0, 1]^m$ be defined by

$$\alpha = (\hat{\phi}_1, \dots, \hat{\phi}_n) \quad \text{and} \quad \beta = (\hat{\psi}_1, \dots, \hat{\psi}_m).$$

By Theorem 1.7, our standing hypothesis (3.2) actually means that the composite map $\alpha\beta \upharpoonright Q$ (resp., the composite map $\beta\alpha \upharpoonright P$) is the identity function on Q (resp., is the identity function on P). The two $\mathbb{Z}$ -maps $\alpha \upharpoonright P$ and $\beta \upharpoonright Q$ are inverse of each other, and $\alpha \upharpoonright P$ is a $\mathbb{Z}$ -homeomorphism of P onto Q . By Theorem 3.23, Φ is equivalent to Ψ .

($\Rightarrow$) Theorem 3.23 yields $\mathbb{Z}$ -homeomorphisms $\iota : P \cong_{\mathbb{Z}} Q$ and $J = \iota^{-1} : Q \cong_{\mathbb{Z}} P$ such that

$$J\iota = \text{identity on } P \quad \text{and} \quad \iota J = \text{identity on } Q. \quad (3.4)$$

Proposition 3.2 yields functions $f_1, \dots, f_n \in \mathcal{M}([0, 1]^m)$ and $g_1, \dots, g_m \in \mathcal{M}([0, 1]^n)$ such that

$$\iota = (f_1, \dots, f_n) \upharpoonright P \quad \text{and} \quad J = (g_1, \dots, g_m) \upharpoonright Q. \quad (3.5)$$

McNaughton theorem yields formulas $\phi_1, \dots, \phi_n \in \mathbf{FORM}_Y$ and $\psi_1, \dots, \psi_m \in \mathbf{FORM}_Z$ such that for all $i = 1, \dots, m$ and $j = 1, \dots, n$, $g_i = \hat{\psi}_i$ and $f_j = \hat{\phi}_j$. Let $\pi_1, \dots, \pi_m$ be the coordinate functions on $[0, 1]^m$ and $\pi'_1, \dots, \pi'_n$ be the coordinate functions on $[0, 1]^n$. By (3.3–3.5) we can write $\hat{\psi}_i(\hat{\phi}_1, \dots, \hat{\phi}_n) \upharpoonright P = \pi_i \upharpoonright P$ and $\hat{\phi}_j(\hat{\psi}_1, \dots, \hat{\psi}_m) \upharpoonright Q = \pi'_j \upharpoonright Q$. By Theorem 1.7 and Corollary 1.10, condition (3.2) is easily verified. $\square$

3.5 Remarks

The basic properties of $\mathbb{Z}$ -homeomorphisms of $[0, 1]^n$ were established by Panti in his Ph.D. thesis [1], (see [2, 2.6] where $\mathbb{Z}$ -homeomorphisms are called McNaughton homeomorphisms).

The Galois connection between finitely axiomatizable theories and rational polyhedra is folklore (see, e.g., [3, 5.1, 6.4]). See [4] for a generalization to arbitrary theories.

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Chapter 4

The Spectral and the Maximal Spectral Space

Generalizing the construction of the Stone space of a boolean algebra, the set of prime ideals of every MV-algebra A is endowed with the hull-kernel (also known as Zariski, or spectral) topology. The resulting space is denoted $\text{Spec}(A)$. In contrast to the Stone space of a boolean algebra, $\text{Spec}(A)$ is generally not rich enough to uniquely characterize A up to isomorphism. Moreover, unless A is hyperarchimedean, $\text{Spec}(A)$ strictly contains the compact Hausdorff space $\mu(A)$ of maximal ideals. Despite MV-algebra theory cannot be reduced to topology, $\mu(A)$ is a main tool for the representation and classification of large classes of σ -complete MV-algebras, and for the definition of the fundamental notion of basis. This chapter introduces the main properties of $\mu(A)$.

4.1 Ideals of Free MV-Algebras

For any cardinal $\kappa > 0$, the Tychonov cube $[0, 1]^\kappa$ comes equipped with the product topology of $[0, 1]$, making it a compact Hausdorff space.

We denote by $\mathcal{M}([0, 1]^\kappa)$ the MV-algebra of all McNaughton functions on $[0, 1]^\kappa$. $\mathcal{M}([0, 1]^\kappa)$ is the free MV-algebra on κ generators, (A21.48). For any nonempty subset Y of $[0, 1]^\kappa$ we write $\mathcal{M}(Y)$ for the MV-algebra of restrictions to Y of the functions in $\mathcal{M}([0, 1]^\kappa)$.

As an immediate consequence of their piecewise linearity, McNaughton functions have all directional derivatives:

Proposition 4.1 *Let $f \in \mathcal{M}([0, 1]^\kappa)$, $x, y \in [0, 1]^\kappa$ and $u = y - x$. Then the (one-sided) directional derivative*

$$f'(x; u) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda u) - f(x)}{\lambda}$$

exists finite.

Strengthening Lemma 1.13, we record here some relevant separation properties of McNaughton functions:

Proposition 4.2

- (i) For every open set $U \subseteq [0, 1]^k$ and $x \in U$ there is a McNaughton function $f \in \mathcal{M}([0, 1]^k)$ such that f vanishes over some open neighborhood of x and is identically equal to 1 on $[0, 1]^k \setminus U$.
- (ii) For any two nonempty disjoint closed sets X, Y in $[0, 1]^k$, some function $l \in \mathcal{M}([0, 1]^k)$ vanishes on X and is identically equal to 1 on Y .
- (iii) For any two nonempty disjoint closed sets X, Y in $[0, 1]^k$ and functions $f, g \in \mathcal{M}([0, 1]^k)$ there are disjoint open sets $U \supseteq X, V \supseteq Y$ and a function $h \in \mathcal{M}([0, 1]^k)$ such that $h = f$ on U and $h = g$ on V .

Proof (i) Without loss of generality, there is an integer $n > 0$, and for each $i = 1, \dots, n$ an open interval with rational endpoints $I_i = (p_i/q_i, r_i/s_i)$, such that the basic open set $\{y \in [0, 1]^k \mid y_i \in I_i \forall i = 1, \dots, n\}$ is a neighborhood of x contained in U . Let the integers $0 < p'_i, q'_i, r'_i, s'_i$ satisfy $I_i \supseteq (p'_i/q'_i, r'_i/s'_i)$ for all $i = 1, \dots, n$. Recalling (A21.18) let the functions $f_i, g_i \in \mathcal{M}([0, 1])$ be defined by

$$f_i(y) = (0 \vee s'_i y_i - r'_i) \wedge 1, \quad \text{and} \quad g_i(y) = (0 \vee -q'_i x_i + p'_i) \wedge 1, \quad \text{for all } y \in [0, 1]^k.$$

Fix $i = 1, \dots, n$. With the notation of (2.8), for all sufficiently large integers m we can write

$$m \cdot f_i(y) = 1 \quad \text{for all } y \in [0, 1]^k \quad \text{such that } y_i \geq r_i/s_i$$

and

$$m \cdot g_i(y) = 1 \quad \text{for all } y \in [0, 1]^k \quad \text{such that } y_i \leq p_i/q_i.$$

As a consequence, the function $f = m \cdot (f_i \vee g_1 \vee \dots \vee f_n \vee g_n)$ has the required properties to settle (i).

The proof of (ii) easily follows from (i) by a routine compactness argument.

(iii) The normality of $[0, 1]^k$ yields disjoint closed sets X', Y' and disjoint open sets U, V such that $X \subseteq U \subseteq X'$ and $Y \subseteq V \subseteq Y'$. By (ii), there are functions $l', l'' \in \mathcal{M}([0, 1]^k)$ such that l' vanishes on X' and has value 1 on Y' , while l'' vanishes on Y' and has value 1 on X' . The function $h = (f \wedge l') \vee (g \wedge l'')$ has the required properties. $\square$

Corollary 4.3 Let $X_1, \dots, X_n \subseteq [0, 1]^k$ be pairwise disjoint nonempty closed sets. Then

$$\mathcal{M}(X_1 \cup \dots \cup X_n) \cong \mathcal{M}(X_1) \times \dots \times \mathcal{M}(X_n).$$

Proof The normality of $[0, 1]^k$ yields pairwise disjoint open sets $O_1, \dots, O_n$ with $O_i \supseteq X_i$. The map $J: f \in \mathcal{M}(X_1 \cup \dots \cup X_n) \mapsto (f|_{X_1}, \dots, f|_{X_n})$ is a one-one homomorphism of $\mathcal{M}(X_1 \cup \dots \cup X_n)$ into $\mathcal{M}(X_1) \times \dots \times \mathcal{M}(X_n)$.

To prove surjectivity, given $(f_1, \dots, f_n) \in \mathcal{M}(X_1) \times \dots \times \mathcal{M}(X_n)$, by definition, each f_i has an extension to a $g_i \in \mathcal{M}([0, 1]^\kappa)$. By Proposition 4.2(iii) there are $p_1, \dots, p_n \in \mathcal{M}([0, 1]^\kappa)$ such that $p_i = 1$ on some open set containing X_i , and $p_i = 0$ on some open set containing the union of the remaining X_j . Then the McNaughton function $f = (g_1 \wedge p_1) \vee \dots \vee (g_n \wedge p_n)$ coincides with f_i on X_i . Since $J(f|_{X_1 \cup \dots \cup X_n}) = (f_1, \dots, f_n)$ then J is surjective. $\square$

4.2 Zerosets

The *zeroset* $\mathcal{Z}f$ of a function $f: X \rightarrow \mathbb{R}$ is defined by

$$\mathcal{Z}f = f^{-1}(0) = \{x \in X \mid f(x) = 0\}.$$

More generally, for any set $j \subseteq \mathbb{R}^X$ the *zeroset* $\mathcal{Z}j$ of j is defined by

$$\mathcal{Z}j = \bigcap \{\mathcal{Z}f \mid f \in j\}. \quad (4.1)$$

In the particular case when j is a maximal ideal of $\mathcal{M}([0, 1]^\kappa)$, $\mathcal{Z}j$ is a singleton by Proposition 4.2(ii), and we write

$$\dot{\mathcal{Z}}j = \text{the only element of } \mathcal{Z}j. \quad (4.2)$$

Proposition 4.4

- (i) For every element $f \in \mathcal{M}([0, 1]^\kappa)$ with $\mathcal{Z}f \neq \emptyset$, the ideal $\langle f \rangle$ generated by f consists of all $g \in \mathcal{M}([0, 1]^\kappa)$ such that $\mathcal{Z}g \supseteq \mathcal{Z}f$.
- (ii) For every ideal $\mathfrak{i}$ of $\mathcal{M}([0, 1]^\kappa)$ and $h \in \mathcal{M}([0, 1]^\kappa)$, if h identically vanishes over some open neighborhood U of $\mathcal{Z}\mathfrak{i}$, then h belongs to $\mathfrak{i}$.
- (iii) Fix an element $l \in \mathcal{M}([0, 1]^\kappa)$ with $\mathcal{Z}l \neq \emptyset$. Then the map $\iota: p/\langle l \rangle \mapsto p|_{\mathcal{Z}l}$ is an isomorphism of the quotient MV-algebra $\mathcal{M}([0, 1]^\kappa)/\langle l \rangle$ onto the MV-algebra $\mathcal{M}(\mathcal{Z}l) = \mathcal{M}([0, 1]^\kappa)|_{\mathcal{Z}l}$ of restrictions to $\mathcal{Z}l$ of the functions in $\mathcal{M}([0, 1]^\kappa)$.
- (iv) The sets of the form $\mathcal{Z}l$, for $l \in \mathcal{M}([0, 1]^n)$, are a basis of closed sets for the topology of $[0, 1]^n$.

Proof (i) follows from (A21.24). (ii) The compactness of $[0, 1]^\kappa$ yields an element $k \in \mathfrak{i}$ such that $\mathcal{Z}h \supseteq U \supseteq \mathcal{Z}k \supseteq \mathcal{Z}\mathfrak{i}$. Now apply (i). (iii) By (i), for each $p \in \mathcal{M}([0, 1]^\kappa)$, $p \in \langle l \rangle \Leftrightarrow \mathcal{Z}p \supseteq \mathcal{Z}l \Leftrightarrow p|_{\mathcal{Z}l} = 0$. This shows that ι is a one-one homomorphism. Evidently, ι is onto $\mathcal{M}([0, 1]^\kappa)|_{\mathcal{Z}l}$. (iv) now follows from Corollary 2.10. $\square$

For later use, we give a more direct proof of the following special case of (i) above, when $\kappa > 0$ is finite:

Proposition 4.5 *Let $Q \subseteq [0, 1]^n$ be a rational polyhedron and $\mathfrak{i}$ an ideal of $\mathcal{M}(Q)$. Then a function $g \in \mathcal{M}(Q)$ belongs to $\mathfrak{i}$ iff for some $f \in \mathfrak{i}$ we have the inclusion $\mathcal{Z}g \supseteq \mathcal{Z}f$.*

Proof The $(\Rightarrow)$ direction is trivial. $(\Leftarrow)$ Suppose $\mathcal{Z}g \supseteq \mathcal{Z}f$ and $f \in \mathbf{i}$. Corollary 2.9 yields an $\{f, g\}$ -triangulation ∇ of Q ; in other words, on every $T \in \nabla$ both f and g are linear. Let $\{x_1, \dots, x_s\}$ be the vertices of ∇ . Fix $i = 1, \dots, s$. Since by hypothesis, $g(x_i) \neq 0 \Rightarrow f(x_i) \neq 0$, then there exists an integer $m_i > 0$ such that $m_i \cdot f(x_i) \geq g(x_i)$. Letting now $m = \max(m_1, \dots, m_s)$, the desired conclusion $m \cdot f \geq g$ follows from the linearity of f and g on each simplex of Δ . $\square$

Proposition 4.6 *Fix integers $m, n > 0$ together with a closed nonempty subset Y of $[0, 1]^m$ and functions $h_1, \dots, h_n \in \mathcal{M}(Y)$. Let the function $h: Y \rightarrow [0, 1]^n$ be defined by $h(y) = (h_1(y), \dots, h_n(y))$ for all $y \in Y$. Let $\gamma: \mathcal{M}([0, 1]^n) \rightarrow \mathcal{M}(Y)$ be the unique homomorphism extending the map $\pi_i \mapsto h_i$, ($i = 1, \dots, n$). It follows that*

- (i) *For each $f \in \mathcal{M}([0, 1]^n)$, $\gamma(f)$ coincides with the composite function $f h$.*
- (ii) *The range R of h coincides with the zero set $\mathcal{Z} \ker \gamma = \bigcap \{\mathcal{Z}l \mid l \in \ker \gamma\}$.*
- (iii) *The map $\iota: p \in \mathcal{M}(R) \mapsto ph$ is an isomorphism of $\mathcal{M}(R)$ onto the range E of γ . The inverse of ι sends each $q = gh \in E$ to $g \upharpoonright R$.*

Proof

(i) By McNaughton theorem, f coincides with $\hat{\phi}$ for some formula $\phi(X_1, \dots, X_n)$. Arguing by induction on the number of connectives in ϕ , it is easily verified that $\gamma(f) = fh$.

(ii) From the equivalences $f \in \ker \gamma \Leftrightarrow fh = 0 \Leftrightarrow f \upharpoonright R = 0$ it follows that $x \in \mathcal{Z} \ker \gamma \Leftrightarrow x \in \mathcal{Z}l$ for all $l \in \ker \gamma \Leftrightarrow l(x) = 0$ for all l vanishing on $R \Leftrightarrow x \in R$. The last equivalence follows from Proposition 4.2, together with the observation that R is closed, because h is continuous on the closed set $Y \subseteq [0, 1]^m$.

(iii) $p = \bar{p} \upharpoonright R$ for some $\bar{p} \in \mathcal{M}([0, 1]^n)$. Trivially, $ph = \bar{p}h$, and ι is a homomorphism. Further, ι is one-one. For, if $p(x) \neq 0$ for some $x \in R$ then $p(h(y)) \neq 0$ for all $y \in h^{-1}(x)$. Finally, ι is onto E , since every g belonging to E has the form $g = fh$ for some $f \in \mathcal{M}([0, 1]^n)$, and $fh = (f \upharpoonright R)h$, with $f \upharpoonright R \in \mathcal{M}(h(Y))$. The rest is clear. $\square$

4.3 Germinal Ideals

Definition 4.7 For any cardinal $\kappa > 0$ and nonempty set $X \subseteq [0, 1]^\kappa$, the ideals $\mathfrak{h}_X$ and $\mathfrak{o}_X$ are defined by

$$\mathfrak{h}_X = \{f \in \mathcal{M}([0, 1]^\kappa) \mid f = 0 \text{ on } X\}, \text{ and}$$

$$\mathfrak{o}_X = \{f \in \mathcal{M}([0, 1]^\kappa) \mid f = 0 \text{ on some open set in } [0, 1]^\kappa \text{ containing } X\}.$$

The ideal $\mathfrak{h}_X$ is known as the *hull* of X . When $X = \{x\}$ is a singleton we write $\mathfrak{h}_x$ and $\mathfrak{o}_x$ instead of $\mathfrak{h}_{\{x\}}$ and $\mathfrak{o}_{\{x\}}$. We also say that $\mathfrak{o}_x$ is the *germinal ideal* of $\mathcal{M}([0, 1]^k)$ at x .

For each $f \in \mathcal{M}([0, 1]^k)$, $f/\mathfrak{o}_x$ is called the *germ* of f at x .

Proposition 4.8 *Two functions $f, g \in \mathcal{M}([0, 1]^k)$ coincide on some open neighborhood of a point x in $[0, 1]^k$ iff $f(x) = g(x)$ and $f'(x; y - x) = g'(x; y - x)$ for all $y \in [0, 1]^k$.*

Proof By Proposition 4.1 all directional derivatives of f and g exist. For suitable variables $z_1, \dots, z_n$ we can write $f = f(z_1, \dots, z_n)$ and $g = g(z_1, \dots, z_n)$, and restrict attention to n -dimensional euclidean space $\mathbb{R}^n$. For the $(\Leftarrow)$ -direction, the piecewise linearity properties of f and g ensure that the function $|f(z_1, \dots, z_n) - g(z_1, \dots, z_n)|$ vanishes on an open neighborhood of $x_1, \dots, x_n$. By definition of product topology it follows that $|f - g|$ vanishes on an open neighborhood of x . The $(\Rightarrow)$ -direction is trivial. $\square$

Proposition 4.9 *Let X be a nonempty closed subset of $[0, 1]^k$. Among all ideals $\mathfrak{i}$ of $\mathcal{M}([0, 1]^k)$ such that $\mathcal{Z}\mathfrak{i} = X$, $\mathfrak{o}_X$ is the smallest and $\mathfrak{h}_X$ the largest.*

Proof Trivially, $X \subseteq \mathcal{Z}\mathfrak{h}_X \subseteq \mathcal{Z}\mathfrak{o}_X$. If $\mathfrak{i}$ is an ideal of $\mathcal{M}([0, 1]^k)$ such that $X = \mathcal{Z}\mathfrak{i}$ then necessarily $\mathfrak{h}_X \supseteq \mathfrak{i}$. By Proposition 4.4(ii), $\mathfrak{i} \supseteq \mathfrak{o}_X$. There remains to be shown that $\mathcal{Z}\mathfrak{o}_X \subseteq X$. For each $x \in [0, 1]^k \setminus X$, Proposition 4.4(iii) yields disjoint open sets $U \supseteq X$ and $V \ni x$ and a function $h \in \mathcal{M}([0, 1]^k)$ such that $h = 0$ on U and $h = 1$ on V . Thus $h \in \mathfrak{o}_X$ and $\mathcal{Z}h \subseteq \mathcal{Z}\mathfrak{o}_X$. From $h(x) = 1$ we conclude that $x \notin \mathcal{Z}h$ and $x \notin \mathcal{Z}\mathfrak{o}_X$. $\square$

Proposition 4.10 *Fix $n = 1, 2, \dots$ and $x, y \in [0, 1]^n$. Then the following conditions are equivalent:*

- (i) *There is an isomorphism $\iota: \mathcal{M}([0, 1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0, 1]^n)/\mathfrak{o}_y$.*
- (ii) *There exist rational polyhedra $X, Y \subseteq [0, 1]^n$ and a $\mathbb{Z}$ -homeomorphism $\eta: Y \xrightarrow{\cong} X$ such that $x \in \text{int}(X)$, $y \in \text{int}(Y)$ and $\eta(y) = x$.*

Proof (ii) $\Rightarrow$ (i) Without loss of generality we can assume $X = \text{cl}(\text{int}(X))$ and $Y = \text{cl}(\text{int}(Y))$. We will not distinguish between the germ of $f \in \mathcal{M}([0, 1]^n)$ at x , and the germ of $f|_X$ at x . Similarly, we will use the identification $\mathcal{M}(Y)/\mathfrak{o}_y = \mathcal{M}([0, 1]^n)/\mathfrak{o}_y$. Let us define the homomorphism $\gamma: \mathcal{M}([0, 1]^n) \rightarrow \mathcal{M}(Y)$ by $\gamma: f \mapsto f|_Y$. If f and g agree on an open neighborhood M of x , then, assuming without loss of generality $M \subseteq X$, it follows that $\gamma f = \gamma g$ on $\eta^{-1}(M)$, the latter being an open neighborhood of y . Since

$$\frac{f}{\mathfrak{o}_x} = \frac{g}{\mathfrak{o}_x} \Rightarrow \frac{\gamma(f)}{\mathfrak{o}_y} = \frac{\gamma(g)}{\mathfrak{o}_y},$$

upon writing $f/\mathfrak{o}_x \mapsto \gamma(f)/\mathfrak{o}_y$, we have a map $\iota: \mathcal{M}([0, 1]^n)/\mathfrak{o}_x \rightarrow \mathcal{M}(Y)/\mathfrak{o}_y = \mathcal{M}([0, 1]^n)/\mathfrak{o}_y$. If $\gamma(f)/\mathfrak{o}_y = 0$ then $\gamma(f)$ identically vanishes on some open neighborhood $N \subseteq Y$ of y . Thus f vanishes on the open neighborhood

$\eta(N)$ of x , and $f/o_x = 0$, which shows that ι is one-one. Reversing the roles of ι and ι^{-1} we have the desired isomorphism.

($\Rightarrow$ ii) For each $i = 1, \dots, n$ the germ π_i/o_x of the i th coordinate function π_i at x is mapped by ι to the germ q_i/o_y at y of some McNaughton function $q_i \in \mathcal{M}([0, 1]^n)$. We have a $\mathbb{Z}$ -map $q = (q_1, \dots, q_n): [0, 1]^n \rightarrow [0, 1]^n$. For each $f(x_1, \dots, x_n) \in \mathcal{M}([0, 1]^n)$, arguing by induction on the number of operations necessary to obtain f from $\pi_1, \dots, \pi_n$, it follows that the germ f/o_x is mapped by ι to the germ $\frac{fq}{o_y}$ of the composite map $f q \in \mathcal{M}([0, 1]^n)$,

$$\iota: \frac{f}{o_x} \mapsto \frac{fq}{o_y}.$$

Similarly, there are $p_1, \dots, p_n \in \mathcal{M}([0, 1]^n)$, and a $\mathbb{Z}$ -map $p = (p_1, \dots, p_n): [0, 1]^n \rightarrow [0, 1]^n$ such that for each $g \in \mathcal{M}([0, 1]^n)$,

$$\iota^{-1}: \frac{g}{o_y} \mapsto \frac{gp}{o_x}.$$

In particular, for each $i = 1, \dots, n$,

$$\iota^{-1}\iota: \frac{\pi_i}{o_x} \mapsto \frac{q_i p}{o_x} \quad \text{and} \quad \iota\iota^{-1}: \frac{\pi_i}{o_y} \mapsto \frac{p_i q}{o_y}.$$

Therefore, the composite map $qp: [0, 1]^n \rightarrow [0, 1]^n$ agrees with identity on some open neighborhood M of x . Without loss of generality, M is the interior (in $[0, 1]^n$) of some rational polyhedron. Similarly, the $\mathbb{Z}$ -map pq acts identically on some rational polyhedron containing y in its interior. For a suitably small rational n -dimensional cube $X \subseteq M$ with $x \in \text{int}(X)$, letting $Y = p(X)$ we have $\mathbb{Z}$ -homeomorphisms $q|Y: Y \cong_{\mathbb{Z}} X$ and $p|X: X \cong_{\mathbb{Z}} Y$. Evidently, y lies in the interior of Y and $p(x) = y$. Now let $\eta = q|Y$. $\square$

Example 4.11 Let the points $a, b \in [0, 1]^2$ be given by $a = (1/5, 1/5)$ and $b = (1/5, 2/5)$. We then have an isomorphism $\mathcal{M}([0, 1]^2)/o_a \cong \mathcal{M}([0, 1]^2)/o_b$.

As a matter of fact, let M be the square of side $1/10$ centered at a . Let $\alpha: M \rightarrow [0, 1]^2$ be the $\mathbb{Z}$ -map $x' = x, y' = x + y$. Then $\alpha(a) = b, a \in \text{int}(M), b \in \text{int}(\alpha(M))$ and $\alpha: M \cong_{\mathbb{Z}} \alpha(M)$. The inverse of α is the $\mathbb{Z}$ -map

$$x = x'; y = -x' + y' \quad \text{for all } (x', y') \in \alpha(M).$$

Now apply Proposition 4.10(ii $\Rightarrow$ i).

In Example 5.5 we will see that for $a = 1/5$ and $b = 2/5$, $\mathcal{M}([0, 1])/o_a$ is not isomorphic to $\mathcal{M}([0, 1])/o_b$.

4.4 The Spectral Topology

Definition 4.12 An ideal $\mathfrak{p}$ of an MV-algebra A is said to be *prime* if $\mathfrak{p} \neq A$ and for all $x, y \in A$, either $x \odot \neg y \in \mathfrak{p}$ or $y \odot \neg x \in \mathfrak{p}$.

Prime ideals can be characterized as follows:

Proposition 4.13 Let $\mathfrak{p} \neq A$ be an ideal of an MV-algebra A . Then the following conditions are equivalent:

- (i) $\mathfrak{p}$ is a prime ideal of A .
- (ii) $A/\mathfrak{p}$ is totally ordered.
- (iii) Whenever $\mathfrak{p}$ contains the intersection of two ideals $\mathfrak{i}$ and $\mathfrak{j}$, then either $\mathfrak{p} \supseteq \mathfrak{i}$ or $\mathfrak{p} \supseteq \mathfrak{j}$.
- (iv) Whenever $x, y \in A$ and $x \wedge y \in \mathfrak{p}$ then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Proof (i $\Leftrightarrow$ ii) by (A21.4)(iv) and (A21.6).

(i $\Rightarrow$ iv) Suppose $x \wedge y \in \mathfrak{p}$ and, without loss of generality, $x \odot \neg y \in \mathfrak{p}$. Then $\mathfrak{p} \ni (x \wedge y) \oplus (x \odot \neg y) = (x \oplus (x \odot \neg y)) \wedge (y \oplus (x \odot \neg y)) = y \vee x \geq x$, whence $x \in \mathfrak{p}$.

(iv $\Rightarrow$ iii) By way of contradiction suppose $\mathfrak{p} \supseteq \mathfrak{i} \cap \mathfrak{j}$ but there are elements $x \in \mathfrak{i} \setminus \mathfrak{p}$ and $y \in \mathfrak{j} \setminus \mathfrak{p}$. Since $x \wedge y \in \mathfrak{i} \cap \mathfrak{j} \subseteq \mathfrak{p}$ then either x or y belongs to $\mathfrak{p}$, a contradiction.

(iii $\Rightarrow$ i) Recall Definition 2.18 for the ideal $\langle a \rangle$ generated by a . Also recall the notation $n \cdot a$ of (2.8). For all $x, y \in A$ let $\mathfrak{i} = \langle y \odot \neg x \rangle$ and $\mathfrak{j} = \langle x \odot \neg y \rangle$. By (A21.2), $(y \odot \neg x) \wedge (x \odot \neg y) = 0$. By (A21.3), $n \cdot (y \odot \neg x) \wedge n \cdot (x \odot \neg y) = 0$ for all $n = 1, 2, \dots$. This shows that $\mathfrak{i} \cap \mathfrak{j} = \{0\} \subseteq \mathfrak{p}$. Thus either $\mathfrak{i}$ or $\mathfrak{j}$ is contained in $\mathfrak{p}$, which easily yields the desired conclusion. $\square$

We denote by $\text{Spec}(A)$ the set of prime ideals of A . For each ideal $\mathfrak{j}$ of A (including A itself) let us write

$$F_{\mathfrak{j}} = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq \mathfrak{j}\}.$$

Thus in particular, $F_A = \emptyset$ and $F_{\{0\}} = \text{Spec}(A)$. Using Proposition 4.13(i $\Leftrightarrow$ iii) one easily checks that $F_{\mathfrak{i}} \cup F_{\mathfrak{j}} = F_{\mathfrak{i} \cap \mathfrak{j}}$. For any family $\mathfrak{j}_i$ of ideals of A , letting $\bigvee \mathfrak{j}_i$ denote the smallest ideal of A containing all $\mathfrak{j}_i$, it follows that $\bigcap F_{\mathfrak{j}_i} = F_{\bigvee \mathfrak{j}_i}$.

Definition 4.14 For every MV-algebra A , the closed sets of the *spectral* (also known as *hull-kernel*, or *Zariski*) topology on $\text{Spec}(A)$ are all sets of the form $F_{\mathfrak{j}}$, letting $\mathfrak{j}$ range over ideals of A , including A itself. By abuse of notation, the resulting topological space is denoted $\text{Spec}(A)$. We say that $\text{Spec}(A)$ is the *spectral* space of A . The totality of subsets of $\text{Spec}(A)$ of the form

$$F_{\langle a \rangle} = \{\mathfrak{p} \in \text{Spec}(A) \mid \langle a \rangle \subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \text{Spec}(A) \mid a \in \mathfrak{p}\},$$

where a ranges over elements of A , constitutes a basis of closed sets for $\text{Spec}(A)$, called the *standard basis* of closed sets for $\text{Spec}(A)$.

We further denote by $\mu(A)$ the set of maximal ideals of A equipped with the topology inherited from $\text{Spec}(A)$ by restriction. The *standard basis* of closed sets of $\mu(A)$ is given by all sets of the form $F_a = F_{\langle a \rangle} \cap \mu(A) = \{\mathfrak{m} \in \mu(A) \mid a \in \mathfrak{m}\}$, letting a range over elements of A . Any such F_a is said to be a *basic closed set* of $\mu(A)$. We say that $\mu(A)$ is the *maximal spectral space* of A .

Proposition 4.15 *For any MV-algebra A , $\mu(A)$ is a nonempty compact Hausdorff space.*

Proof To see that $\mu(A)$ is nonempty let $\mathfrak{n}$ be a subset of A which is maximal for the following three properties: (a) $1 \notin \mathfrak{n}$, (b) $x, y \in \mathfrak{n} \Rightarrow x \oplus y \in \mathfrak{n}$, and (c) $z \in \mathfrak{n}, y \leq z \Rightarrow y \in \mathfrak{n}$. The existence of $\mathfrak{n}$ follows from the axiom of choice, together with the fact that $\{0\}$ is a subset of A satisfying (a–c), and the union of an increasing family of subsets of A satisfying (a–c) still satisfies (a–c). Then, by direct inspection, $\mathfrak{n}$ is a maximal ideal of A . Since any two distinct elements of $\mu(A)$ are incomparable, $\mu(A)$ is a Hausdorff space. To prove that $\mu(A)$ is compact, let $\mathcal{F}$ be a family of closed sets in $\mu(A)$ having the finite intersection property: in other words, every finite subset of $\mathcal{F}$ has nonempty intersection. We will show that $\bigcap \mathcal{F} \neq \emptyset$. Again using the axiom of choice we may assume $\mathcal{F}$ to be maximal. Let $\mathfrak{n}$ be the set of those elements $a \in A$ such that the basic closed set F_a is a member of $\mathcal{F}$. In symbols, $a \in \mathfrak{n} \Leftrightarrow \{\mathfrak{m} \in \mu(A) \mid a \in \mathfrak{m}\} \in \mathcal{F}$. It is easy to verify that $\mathfrak{n}$ is a maximal ideal of A , and $\mathfrak{n}$ is a common element of each set in $\mathcal{F}$, specifically, $\{\mathfrak{n}\} = \bigcap \mathcal{F}$. This shows that $\mu(A)$ is compact. $\square$

4.5 The MV-Algebra $C(X)$

Let $E \neq \emptyset$ be a set and $B \subseteq [0, 1]^E$ an MV-algebra of $[0, 1]$ -valued functions on E , with the pointwise operations of $[0, 1]$. We then say that B *separates points* (or, B is *separating*), if for any two distinct points $x, y \in E$ there is $f \in B$ such that $f(x) \neq f(y)$.

For any nonempty compact Hausdorff space $X \neq \emptyset$ we let $C(X)$ denote the MV-algebra of all continuous $[0, 1]$ -valued functions on X , with the pointwise operations of the MV-algebra $[0, 1]$.

Theorem 4.16

Let A be an MV-algebra.

(i) *For any maximal ideal $\mathfrak{m}$ of A there is a unique pair $(\overline{\mathfrak{m}}, I_{\mathfrak{m}})$ with $I_{\mathfrak{m}}$ an MV-subalgebra of $[0, 1]$ and $\overline{\mathfrak{m}}$ an isomorphism of the quotient MV-algebra $A/\mathfrak{m}$ onto $I_{\mathfrak{m}}$.*

(ii) *The map $\ker: \eta \mapsto \ker \eta$ is a one–one correspondence between $\text{hom}(A)$ and $\mu(A)$. The inverse map sends each $\mathfrak{m} \in \mu(A)$ to the homomorphism $\eta_{\mathfrak{m}}: A \rightarrow [0, 1]$ given by $a \mapsto \overline{\mathfrak{m}}(a/\mathfrak{m})$. For each $\theta \in \text{hom}(A)$ and $a \in A$,*

$$\theta(a) = \overline{\ker \theta} \left(\frac{a}{\ker \theta} \right). \quad (4.3)$$

(iii) The map $*$: $a \in A \mapsto a^* \in [0, 1]^{\mu(A)}$ defined by

$$a^*(\mathfrak{m}) = \overline{\mathfrak{m}}(a/\mathfrak{m}),$$

is a homomorphism of A onto a separating MV-subalgebra A^* of $C(\mu(A))$. The basic closed sets of $\mu(A)$ have the form $F_b = \{\mathfrak{m} \in \mu(A) \mid b^*(\mathfrak{m}) = 0\}$, where b ranges over all elements of A . The map $a \mapsto a^*$ is an isomorphism of A onto A^* iff A is semisimple.

(iv) Suppose $X \neq \emptyset$ is a compact Hausdorff space and B is a separating subalgebra of $C(X)$. Then the map

$$\iota: x \in X \mapsto \mathfrak{h}_x = \{f \in B \mid f(x) = 0\}$$

is a homeomorphism of X onto $\mu(B)$. The inverse map ι^{-1} sends each $\mathfrak{m} \in \mu(B)$ to the only element $\dot{Z}\mathfrak{m}$ of $Z\mathfrak{m}$.

(v) With the same hypotheses of (iv), for each $f \in B$, $f^*\iota = f$. Thus the map $f^* \in B^* \mapsto f^*\iota \in C(X)$ is the inverse of the isomorphism $*$: $B \cong B^*$ of (iii). In particular, $f(x) = f^*(\mathfrak{h}_x)$ for each $x \in X$.

Proof

(i) From the maximality of $\mathfrak{m}$ it follows that the quotient MV-algebra $A/\mathfrak{m}$ is simple, i.e., $\{0\}$ is the only (proper) ideal of A , (A21.7). A fortiori, $\{0\}$ is maximal. By (A21.26) $A/\mathfrak{m}$ is isomorphic to a subalgebra $I_{\mathfrak{m}}$ of $[0, 1]$. By (A21.45) $I_{\mathfrak{m}}$ is uniquely determined, and so is the isomorphism $\overline{\mathfrak{m}}$ of $A/\mathfrak{m}$ onto $I_{\mathfrak{m}}$.

(ii) As proved in (i), for every $\mathfrak{m} \in \mu(A)$, $\eta_{\mathfrak{m}}$ is a homomorphism of A into $[0, 1]$. Since $[0, 1]$ is simple, the only ideal $\{0\}$ of $[0, 1]$ is maximal. By (A21.12) $\ker \eta$ is a maximal ideal of A . To prove that $\ker \eta_{\mathfrak{m}} = \mathfrak{m}$, for each $a \in A$ we have

$$a \in \mathfrak{m} \Leftrightarrow a/\mathfrak{m} = 0 \Leftrightarrow \overline{\mathfrak{m}}(a/\mathfrak{m}) = 0 \Leftrightarrow a \in \ker \eta_{\mathfrak{m}}.$$

To prove that for different maximal ideals $\mathfrak{m}, \mathfrak{n} \in \mu(A)$ the homomorphisms $\eta_{\mathfrak{m}}$ and $\eta_{\mathfrak{n}}$ are different, let us choose $a \in \mathfrak{m} \setminus \mathfrak{n}$. Then $\eta_{\mathfrak{m}}(a) = 0$, but $\eta_{\mathfrak{n}}(a) \neq 0$. Finally, (4.3) is a reformulation of $\eta_{\ker \theta} = \theta$.

(iii) Evidently, the map $*$ is a homomorphism of A into $[0, 1]^{\mu(A)}$. If $\mathfrak{m}$ and $\mathfrak{n}$ are two distinct maximal ideals of A , letting $b \in \mathfrak{m} \setminus \mathfrak{n}$, by (i) we have $b^*(\mathfrak{m}) = 0 < b^*(\mathfrak{n})$. Thus the MV-algebra A^* separates points. To prove the continuity of each function a^* let $r, s \in \mathbb{Q}$ satisfy $0 \leq r < s \leq 1$, with the intent of proving that the inverse image $a^{*-1}([r, s])$ is a closed subset of $\mu(A)$. By (A21.18), we can write down an MV-term τ in one variable such that $\hat{\tau}^{-1}(0) = [r, s]$. With self-explanatory notation, τ determines an element $\tau(a) \in A$.

It follows that $(\tau(a))^* = \hat{\tau}(a^*) \in [0, 1]^{\mu(A)}$ and $a^{*-1}([r, s]) = (\hat{\tau}(a^*))^{-1}(0)$. Therefore, by (i)–(ii),

$$a^{*-1}([r, s]) = \{\mathfrak{m} \in \mu(A) \mid \hat{\tau}(a^*)(\mathfrak{m}) = 0\} = \{\mathfrak{m} \in \mu(A) \mid \tau(a) \in \mathfrak{m}\} = F_{\tau(a)},$$

a (basic) closed set in $\mu(A)$. A subset F of $\mu(A)$ is a basic closed set iff for some $b \in A$ we can write $F = \{\mathfrak{n} \in \mu(A) \mid b \in \mathfrak{n}\}$. By (i), this is the same as $F =$

$\{n \in \mu(A) \mid b^*(n) = 0\}$. Finally, if A is semisimple and $a \neq 0$, there is a maximal $m \in \mu(A)$ such that $a/m > 0$, whence, again by (i), $a^*(m) > 0$ and the function a^* is nonzero. So the map $a \mapsto a^*$ is an isomorphism of A onto A^* . Conversely, if this map is an isomorphism, then A is semisimple by (A21.33).

(iv) By (A21.20), ι sends X one-one onto $\mu(B)$, and $\iota^{-1}(m) = \dot{Z}m$. By (iii), ι is a homeomorphism.

(v) We first prove that, for all $f \in B$ and $m \in \mu(B)$,

$$\overline{m}(f/m) = f(\dot{Z}m). \quad (4.4)$$

Let $\eta_1: B \rightarrow [0, 1]$ be evaluation at $\dot{Z}m$, in symbols, $\eta_1(f) = f(\dot{Z}m)$. Let $\eta_2: B \rightarrow [0, 1]$ be the composition of the isomorphism $\overline{m}$ with the quotient map $f \in B \mapsto f/m \in B/m$. In symbols, $\eta_2(f) = \overline{m}(f/m)$. Direct inspection shows that $\ker \eta_1 = \ker \eta_2 = m$. Thus, by (ii), $\eta_1 = \eta_2$, and (4.4) is settled. Recalling (iv), for every $x \in X$ we can write

$$f(x) = f(\dot{Z}h_x) = \overline{h_x}(f/h_x) = f^*(h_x) = f^*(\iota(x)), \quad (4.5)$$

which completes the proof. $\square$

Notation 4.17 To increase readability, in the light of Theorem 4.16 we will introduce some notational simplifications. For every MV-algebra A , $a \in A$ and $m \in \mu(A)$ we will tacitly identify a/m with its corresponding real number $\overline{m}(a/m)$, and write

$$a/m = \overline{m}(a/m) = a^*(m). \quad (4.6)$$

For $\theta \in \text{hom}(A)$ and $a \in A$ we will also write

$$\theta(a) = \frac{a}{\ker \theta}. \quad (4.7)$$

Further, if B is a separating MV-subalgebra of $C(X)$ as in Theorem 4.16(iv–v), identifying B with B^* and X with $\mu(B)$, we will write without fear of ambiguity,

$$f(x) = f(h_x) = f/h_x \quad (4.8)$$

for each $x \in X$ and $f \in B$.

Combining Theorem 4.16(iv–v) with Proposition 4.2(ii) we obtain the following concrete visualization of the maximal spectral space of $\mathcal{M}(Y)$:

Corollary 4.18 *For every nonempty closed subset Y of $[0, 1]^K$, the map $\iota: x \in Y \mapsto h_x = \{f \in \mathcal{M}(Y) \mid f(x) = 0\}$ of Theorem 4.16(iv) is a homeomorphism of Y onto $\mu(\mathcal{M}(Y))$. The inverse map $m \mapsto x_m$ sends every maximal ideal m of $\mathcal{M}(Y)$ to the only element $\dot{Z}m$ of Zm .*

4.6 The Radical

Lemma 4.19 *For every MV-algebra B and ideal i of B let us write $\mu_i(B) = \{q \in \mu(B) \mid q \supseteq i\}$. Further, for every subset $D \subseteq B$ let $D/i = \{a/i \mid a \in D\}$.*

- (i) $\mu_i(B)$ is a compact subset of $\mu(B)$ coinciding with $\bigcap \{F_a \mid a \in i\}$.
- (ii) The map $\lambda: m \mapsto m/i$ is a homeomorphism of $\mu_i(B)$ onto $\mu(B/i)$. The inverse $\lambda^{-1}: \mu(B/i) \rightarrow \mu_i(B)$ is given by $\lambda^{-1}(p) = \{a \in B \mid a/i \in p\}$.
- (iii) For every $m \in \mu_i(B)$ and $a \in B$ we have identical real numbers

$$\overline{m} \left(\frac{a}{m} \right) = \overline{m/i} \left(\frac{a/i}{m/i} \right). \quad (4.9)$$

Proof

(i) Let $n \in \mu(B)$. Then n contains i iff for each $a \in i$, n belongs to the basic closed set F_a . This shows that $\mu_i(B)$ coincides with $\bigcap \{F_a \mid a \in i\}$, the latter being a closed subset of the compact set $\mu(B)$.

(ii) By (A21.7), λ maps $\mu_i(B)$ one-one onto $\mu(B/i)$. Conversely, for each $q \in \mu(B/i)$, $\lambda^{-1}(q) = \{a \in B \mid a/i \in q\}$. Every basic closed set in $\mu_i(B)$ has the form $F_a \cap \mu_i(B)$ for some $a \in B$. Its λ -image in $\mu(B/i)$ satisfies the identities

$$\lambda(F_a \cap \mu_i(B)) = \{\lambda(m) \mid m \in \mu_i(B), a \in m\} = \{m/i \in B/i \mid a/i \in m/i\},$$

because for all $m \in \mu_i(B)$, $a \in m \Leftrightarrow a/i \in m/i$. Thus the λ -image of any basic closed set of $\mu_i(B)$ is a (basic) closed set of $\mu(B/i)$. By (i) together with Proposition 4.15, both $\mu(B/i)$ and $\mu_i(B)$ are compact Hausdorff spaces. It follows that λ is a homeomorphism.

(iii) Let the map $\chi: B/i \rightarrow B/m$ be defined by $\chi(a/i) = a/m$. Then $\ker \chi$ coincides with the ideal m/i of B/i . Since χ is onto B/m , then the map

$$\frac{a/i}{m/i} \mapsto \frac{a}{m}$$

is an isomorphism. Now observe that the MV-algebras $(B/i)/(m/i)$ and B/m are isomorphic to the same MV-subalgebra of $[0, 1]$: for otherwise, there would exist two distinct isomorphic subalgebras of $[0, 1]$, thus contradicting (A21.45). By Theorem 4.16(i) we have the desired conclusion (4.9). $\square$

For any MV-algebra A we denote by $\text{Rad}(A)$, the *radical* of A , i.e., the intersection of all maximal ideals of A .

For every ideal i of A , and element $a \in A$ in this section we will write a' , A' and i' respectively for $a/\text{Rad}(A)$, $A/\text{Rad}(A)$ and $i/\text{Rad}(A)$.

Lemma 4.20 *Let A be an MV-algebra.*

- (i) $A' = A/\text{Rad}(A)$ is semisimple.

- (ii) The map $\lambda: \mathfrak{m} \mapsto \mathfrak{m}' = \mathfrak{m}/\text{Rad}(A)$ of Lemma 4.19(ii) is a homeomorphism of $\mu(A)$ onto $\mu(A')$, in symbols, $\lambda: \mu(A) \cong \mu(A')$.
- (iii) For some cardinal $\kappa > 0$ and closed nonempty set $X \subseteq [0, 1]^\kappa$, A' is isomorphic to $\mathcal{M}(X)$, and $\mu(A)$ is homeomorphic to X .

Proof (i) This follows from (A21.31). (ii) By Lemma 4.19(ii). (iii) By (i), A' is semisimple. By (A21.32) for some cardinal κ and closed set $X \subseteq [0, 1]^\kappa$ we can write $A'/\text{Rad}(A') \cong \mathcal{M}(X)$. Corollary 4.18 yields a homeomorphism $X \cong \mu(A'/\text{Rad}(A'))$. By (ii), X is homeomorphic to $\mu(A)$. $\square$

Given an ideal $\mathfrak{j}$ of $\mathcal{M}([0, 1]^n)$, $n = 1, 2, \dots$, let us write for short $\mu_{\mathfrak{j}} = \{\mathfrak{n} \in \mu(\mathcal{M}([0, 1]^n)) \mid \mathfrak{n} \supseteq \mathfrak{j}\}$. We then have:

Lemma 4.21 *The map $\iota: x \mapsto \mathfrak{h}_x$ is a homeomorphism of $\mathcal{Z}\mathfrak{j}$ onto $\mu_{\mathfrak{j}}$. The inverse map is given by $\mathfrak{n} \mapsto \check{\mathcal{Z}}\mathfrak{n}$.*

Proof By Proposition 4.2(ii) and Theorem 4.16(iv), ι maps $\mathcal{Z}\mathfrak{j}$ one-one into $\mu(\mathcal{M}([0, 1]^n))$. Whenever $x \in \mathcal{Z}(\mathfrak{j})$, every $f \in \mathfrak{j}$ satisfies $f(x) = 0$, whence ι maps $\mathcal{Z}\mathfrak{j}$ into $\mu_{\mathfrak{j}}$. For any maximal ideal $\mathfrak{n} \supseteq \mathfrak{j}$, the point $x_{\mathfrak{n}} = \check{\mathcal{Z}}\mathfrak{n}$ satisfies

$$\{x_{\mathfrak{n}}\} = \bigcap \{f^{-1}(0) \mid f \in \mathfrak{n}\} \subseteq \bigcap \{f^{-1}(0) \mid f \in \mathfrak{j}\} = \mathcal{Z}\mathfrak{j},$$

a compact subset of $[0, 1]^n$. It follows that ι maps $\mathcal{Z}\mathfrak{j}$ onto $\mu_{\mathfrak{j}}$. By Corollary 4.18, $\mathfrak{h}_{x_{\mathfrak{n}}} = \mathfrak{n}$, whence $\iota^{-1} = \check{\mathcal{Z}}$.

There remains to be proved that ι is a homeomorphism. Every basic closed set in $\mu_{\mathfrak{j}}$ has the form $\mu_{\mathfrak{j}} \cap F_g$ for some $g \in \mathcal{M}([0, 1]^n)$. Again recalling Corollary 4.18 and the notational conventions in 4.17, we can write

$$\begin{aligned} \iota(\mu_{\mathfrak{j}} \cap F_g) &= \iota(\{\mathfrak{m} \in \mu(\mathcal{M}([0, 1]^n)) \mid \mathfrak{m} \supseteq \mathfrak{j}, g \in \mathfrak{m}\}) \\ &= \{x_{\mathfrak{m}} \mid \mathfrak{m} \supseteq \mathfrak{j}, g \in \mathfrak{m}\} \\ &= \{x \in \mathcal{Z}\mathfrak{j} \mid g^*(\mathfrak{h}_x) = 0\} \\ &= \{x \in \mathcal{Z}\mathfrak{j} \mid g(x) = 0\} \\ &= \mathcal{Z}\mathfrak{j} \cap g^{-1}(0), \end{aligned}$$

a closed set in $[0, 1]^n$. By Lemma 4.19(i), $\mu_{\mathfrak{j}}$ is compact. Then ι is a homeomorphism. $\square$

4.7 Remarks

Theorem 4.16(i) is an MV-algebraic variant of the Hion-Hölder's theorem [1, 2.6, 12.2.1], (B21.71–B21.72). Theorem 4.16(iii–v) is the MV-algebraic counterpart of Yosida representation [2].

The spectral (Zariski) topology introduced in this chapter is frequently used for sheaf representations of MV-algebras and ℓ -groups [1, 3]. Each stalk has precisely one maximal ideal. Alternative sheaf representations with totally ordered stalks and a different (co-Zariski) spectral topology are considered in [4] and [5].

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Chapter 5

De Concini–Procesi Theorem and Schauder Bases

In this chapter we consider the dynamical properties of regular triangulations, arising from a special kind of regularity-preserving operation called Farey blow-up. The De Concini–Procesi theorem states that for any two rational triangulations Δ and ∇ with the same support and with Δ regular, one can obtain a subdivision of ∇ by applying to Δ finitely many consecutive Farey blow-ups. A self-contained proof will be given in this chapter. This result makes Farey blow-ups a key tool in the study of Schauder bases. The latter originate as MV-algebraic partitions of unity, provide disjunctive normal forms in Łukasiewicz logic, and are ubiquitous in MV-algebra theory. In this chapter Schauder bases will be used to classify prime ideals of free MV-algebras.

5.1 Farey Subdivisions

We continue our analysis of blow-ups started in [Sect. 2.3](#).

Let $0 < n \in \mathbb{Z}$. For any regular j -simplex $S = \text{conv}(v_0, \dots, v_j) \subseteq \mathbb{R}^n$, the *Farey mediant* of (the vertices of) S is the rational point c of S whose homogeneous correspondent $\tilde{c}$ coincides with $\tilde{v}_0 + \dots + \tilde{v}_j$.

Suppose S belongs to a regular complex Δ in $\mathbb{R}^n$ and c is the Farey mediant of S . Then the blow-up

$$\Delta \mapsto \Delta_{(c)}$$

is said to be a *Farey blow-up*. Its inverse is called a *Farey blow-down*. Since S is regular, the definition of the homogeneous correspondents $\Delta^\uparrow$ and $\Delta_{(c)}^\uparrow$ in [\(2.3\)](#) shows that the vector $\tilde{c}$ is primitive, and

$$\text{den}(c) = \text{den}(v_0) + \dots + \text{den}(v_j). \quad (5.1)$$

For every maximal simplex $T = \text{conv}(v_0, \dots, v_j, \dots, v_d)$ of Δ containing c , let $F_0 = \text{conv}(c, v_1, \dots, v_j, \dots, v_d)$, $F_j = \text{conv}(v_0, v_1, \dots, v_{j-1}, c, \dots, v_d)$, and

$$F_t = \text{conv}(v_0, \dots, v_{t-1}, c, v_{t+1}, \dots, v_j, \dots, v_d) \quad (5.2)$$

for each $t = 1, \dots, j-1$. Then, by definition of blow-up, T and its faces are replaced in $\Delta_{(c)}$ by the simplicial complex whose maximal simplices are the d -simplices $F_0, \dots, F_j$. The regularity of T entails the regularity of F_u for each $u = 0, \dots, j$. Thus regularity is preserved by Farey blow-ups.

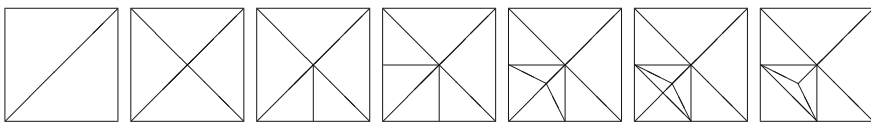
Definition 5.1 For every regular simplex $S \subseteq \mathbb{R}^n$ we let $\text{den}(S)$ be the product of the denominators of its vertices. We say that $\text{den}(S)$ is the *denominator of the regular simplex* S .

We then have

$$\text{den}(F_u) = \frac{\text{den}(T) \cdot \text{den}(c)}{\text{den}(v_u)}, \quad \text{whence} \quad \frac{1}{\text{den}(T)} = \sum_{u=0}^j \frac{1}{\text{den}(F_u)}. \quad (5.3)$$

We inductively define $\Delta_{(c_1, \dots, c_m)}$ as the final outcome of a sequence of Farey blow-ups, $\Delta = \Delta(0) \mapsto \Delta(1) = \Delta(0)_{(c_1)} \mapsto \dots \mapsto \Delta(m) = \Delta(m-1)_{(c_m)}$, where c_t is the Farey median of some simplex T_t of $\Delta(t-1)$, ($t = 1, \dots, m$). It follows that $\Delta_{(c_1, \dots, c_m)}$ is a regular simplex with the same support $|\Delta_{(c_1, \dots, c_m)}|$ as Δ . We say that $\Delta_{(c_1, \dots, c_m)}$ is a *Farey subdivision* of Δ .

Passing to homogeneous coordinates, for any regular cone $\sigma = \langle v_0, \dots, v_j \rangle$ the *Farey median* (of the primitive generating vectors) of σ is the primitive integer vector $v = v_0 + \dots + v_j$. For any regular fan Φ , and Farey median v of a cone $\sigma \in \Phi$, one similarly defines the *Farey blow-up* $\Phi_{(v)}$, and the *Farey subdivision* $\Phi_{(v_1, \dots, v_m)}$.



Five Farey blow-ups and one final Farey blow-down

5.2 De Concini–Procesi Theorem

We first prove the following simple case of Theorem 5.3:

Proposition 5.2 For $n = 1, 2, \dots$, let $T \subseteq [0, 1]^n$ be a regular simplex and $v \in T$ a rational point. Then there is a sequence $\Delta_0 = \{T \text{ and its faces}\}$, $\Delta_1, \dots, \Delta_u$ of regular complexes such that Δ_{i+1} is a Farey blow-up of Δ_i , and v is a vertex of (some simplex of) Δ_u .

Proof Let ω be a fixed but otherwise arbitrary well-ordering of the set of all pairs of distinct rational points in $[0, 1]^n$. We inductively define the regular triangulation Δ_{i+1} of T by

Δ_{i+1} = the blow-up of Δ_i at the Farey mediant of the ω -first 1-simplex $\text{conv}(w_1, w_2)$ of Δ_i such that $\text{den}(w_1) + \text{den}(w_2) \leq \text{den}(v)$.

This sequence must terminate after a finite number u of steps, just because there are only finitely many rational points w in $[0, 1]^n$ satisfying $\text{den}(w) \leq \text{den}(v)$. Let F be the *smallest* simplex of Δ_u containing v . In other words, F is the intersection of all simplexes of Δ_u containing v . It follows that v belongs to the relative interior of F . By way of contradiction, suppose v is not a vertex of F . Then $F \neq \{v\}$. By Lemma 2.7, for some $w_1, \dots, w_r \in [0, 1]^n$ with $r \geq 2$, we have $F = \text{conv}(w_1, \dots, w_r)$ and $\text{den}(v) \geq \text{den}(w_1) + \dots + \text{den}(w_r)$. It is easy to check that the inequality is strict, unless v is the Farey mediant of F . A fortiori, $\text{den}(v) \geq \text{den}(w_1) + \text{den}(w_2)$, whence the Farey blow-up Δ_{u+1} of Δ_u exists, against our assumption about u . $\square$

The following result is an affine variant of the De Concini–Procesi theorem:

Theorem 5.3 *Fix an integer $n > 0$ and a rational polyhedron $P \subseteq [0, 1]^n$. Then for any regular triangulation Δ of P and any rational (possibly non-regular) triangulation $\mathcal{T}$ of P , some Farey subdivision ∇ of Δ is a subdivision of $\mathcal{T}$.*

Proof Let $\Phi_0 = \Delta^\uparrow = \{T^\uparrow \mid T \in \Delta\}$. Then Φ_0 is a regular fan in $\mathbb{R}^{n+1}$ with support $|\Phi_0| = \bigcup \{T^\uparrow \mid T \in \Delta\}$. We will construct a sequence

$$\Phi_0 \mapsto \Phi_1 \mapsto \dots \mapsto \Phi_u \quad (5.4)$$

where each $\Phi_i \mapsto \Phi_{i+1}$ is a Farey blow-up, and we will define ∇ by $\Phi_u = \nabla^\uparrow$.

Let $\{\tau_1, \dots, \tau_r\} \subseteq \mathcal{T}^\uparrow$ be the list of all two-dimensional cones of $\mathcal{T}^\uparrow = \{T^\uparrow \mid T \in \mathcal{T}\}$. There is a set

$$\mathcal{H} = \{H_1, \dots, H_z\} \quad (5.5)$$

of rational homogeneous hyperplanes in $\mathbb{R}^{n+1}$ such that every τ_i is an intersection of half-spaces determined by the hyperplanes of $\mathcal{H}$. For each $j = 1, \dots, z$ let us choose, once and for all, a nonzero vector $h_j \in \mathbb{Z}^{n+1}$ perpendicular to H_j .

Each homomorphism $\eta: \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}$ of the free abelian group $\mathbb{Z}^{n+1}$ into $\mathbb{Z}$ is identified with a row vector $h \in \mathbb{Z}^{n+1}$, and for every column vector $v \in \mathbb{Z}^{n+1}$ the value $h(v)$ is given by the scalar product $h \circ v$.

Given a nonzero row vector $h \in \mathbb{Z}^{n+1}$ and a two-dimensional cone $\sigma = \langle v, w \rangle \subseteq \mathbb{R}^{n+1}$, we define the *h -temperature of σ* to be the integer $(h \circ v)(h \circ w)$. There is no ambiguity in this definition, because the primitive generating vectors of σ are uniquely determined by σ .

Geometrically, the h -temperature of σ is ≥ 0 iff the hyperplane $H = h^\perp \subseteq \mathbb{R}^{n+1}$ has empty intersection with the relative interior of σ . In other words, letting H^+ and H^- be the closed half-spaces of H , the cone σ is contained in exactly one of H^+ and H^- . Thus, if the h -temperatures of all two-dimensional cones in a regular fan Ψ are ≥ 0 , then $H \cap |\Psi|$ must be a union of cones of Ψ . In this case we say that H is *triangulated in Ψ* .

The following claim states that, once H has been triangulated in Ψ , it remains triangulated in any Farey subdivision Ψ' of Ψ :

Claim 1 For Ψ a regular fan and σ a two-dimensional cone of Ψ , suppose the h -temperature of σ is ≥ 0 . Let Ψ' be a Farey subdivision of Ψ , and τ a two-dimensional cone of Ψ' contained in σ . Then the h -temperature of τ is ≥ 0 .

The claim is immediately proved by induction on the number of Farey blow-ups from Ψ to Ψ' .

Our next aim is to ensure that all hyperplanes in the list (5.5) above are eventually triangulated in some Farey subdivision Φ_u of Φ_0 . Proceeding by induction, let us suppose the first s hyperplanes $H_1, \dots, H_s$ are triangulated in Φ_l , with the intent of triangulating H_{s+1} in some Farey subdivision $\Phi_{l'}$ of Φ_l . Claim 1 ensures that each of $H_1, \dots, H_s$ is still triangulated in $\Phi_{l'}$.

As above, let $h = h_{s+1}$ be a perpendicular vector to H_{s+1} . By hypothesis, the h -temperature of some two-dimensional cone of Φ_l is < 0 . So let v be the number of h -coolest cones in Φ_l , those having the smallest h -temperature μ . Now pick a coolest cone $\tau = \langle v, w \rangle \in \Phi_l$, and blow-up Φ_l at its Farey median $c = v + w$, thereby obtaining the regular fan Φ_{l+1} .

Claim 2 The following alternative holds: either the minimum h -temperature μ' of all two-dimensional cones of Φ_{l+1} is $> \mu$; or $\mu' = \mu$, and the number v' of cones in Φ_{l+1} whose h -temperature is μ satisfies $v' < v$.

As a matter of fact, let τ be a new maximal cone in Φ_{l+1} . Then τ has either form $\langle c, w, u_1, \dots, u_k \rangle$ or $\langle v, c, u_1, \dots, u_k \rangle$, with $u_1, \dots, u_k$ primitive generating vectors of cones of Φ_l . We must show that the h -temperature of every new two-dimensional face of τ is $> \mu$. First of all, the cone $\langle v, c \rangle$ satisfies

$$(h \circ v)(h \circ c) = (h \circ v)(h \circ v + h \circ w) = (h \circ v)^2 + (h \circ v)(h \circ w) > (h \circ v)(h \circ w) = \mu.$$

Similarly, the h -temperature of $\langle c, w \rangle$ is $> \mu$. There remains to be checked that the h -temperature of $\langle c, u_r \rangle$ is $> \mu$ for each $r = 1, \dots, k$. Since by hypothesis $h^\perp$ has nonempty intersection with the relative interior of $\langle v, w \rangle$, then necessarily $(h \circ u_r)(h \circ c) = (h \circ u_r)(h \circ v + h \circ w) > \mu$, because the two h -temperatures $(h \circ u_r)(h \circ v)$ and $(h \circ u_r)(h \circ w)$ have opposite signs, and are both $\geq \mu$, unless $(h \circ u_r) = 0$, in which case the inequality $(h \circ u_r)(h \circ c) > \mu$ is trivial. Our second claim is settled.

Therefore, after a finite number of Farey blow-ups $\Phi_l \mapsto \Phi_{l+1} \mapsto \Phi_{l+2} \mapsto \dots$, we obtain a regular fan $\Phi_{l'}$ having the property that the h -temperature of all its two-dimensional cones is ≥ 0 . Geometrically speaking, the hyperplane H_{s+1} has been triangulated in $\Phi_{l'}$. Iterated application of Claims 1 and 2 to the remaining hyperplanes $H_{s+2}, H_{s+3}, \dots$ of (5.5), finally yields a regular fan Φ_u such that each hyperplane of $\mathcal{H}$ is triangulated in Φ_u . Upon defining ∇ by $\Phi_u = \nabla^\uparrow$, we obtain the desired Farey subdivision of Δ which is also a subdivision of $\mathcal{T}$. $\square$

Remark 5.4 Direct inspection of the proofs of Proposition 5.2 and Theorem 5.3 shows that ∇ is obtained from Δ by a sequence of *binary* Farey blow-ups, i.e., Farey blow-ups at 1-simplexes.

The following example should be contrasted with Example 4.11:

Example 5.5 Let $a = 1/5$ and $b = 2/5$. Then $\mathcal{M}([0, 1])/\mathfrak{o}_a$ is not isomorphic to $\mathcal{M}([0, 1])/\mathfrak{o}_b$.

As a matter of fact, from Proposition 5.2 it follows that every regular 1-simplex of the form $[1/5, p/q]$ (resp., of the form $[r/s, 1/5]$) satisfies the condition $q \equiv 4 \pmod{5}$ (resp., $s \equiv 1 \pmod{5}$). On the other hand, every regular 1-simplex of the form $[2/5, t/u]$ satisfies $u \equiv 2 \pmod{5}$. By way of contradiction, suppose there are open intervals $X, Y \subseteq [0, 1]$ such that $1/5$ lies in the interior of X , $2/5$ lies in the interior of Y and $\eta: Y \xrightarrow{\cong} X$ satisfies $\eta(2/5) = 1/5$. By Proposition 3.15, every suitably small regular 1-simplex $[2/5, t'/u']$ is mapped by η one-one onto a regular 1-simplex of either form $[1/5, p'/q']$ or $[r'/s', 1/5]$. Since $u' \equiv 2 \pmod{5}$, then either $q' \equiv 2 \pmod{5}$ or $s' \equiv 2 \pmod{5}$, a contradiction showing that η does not exist. An application of Proposition 4.10 concludes the proof.

5.3 Schauder Bases

Partitions of unity have an interesting MV-algebraic formulation in terms of Schauder bases, to be defined in the present section. Specifically, let A be an MV-algebra with its associated unital ℓ -group, $A = \Gamma(G, 1)$. Then the fact that elements $a_1, \dots, a_z \in A$ sum up to the unit 1 of A , is expressible in A without resorting to the addition operation of G :

Lemma 5.6 *Let A be an MV-algebra with its associated unital ℓ -group $(G, 1)$. Then for all $a_1, \dots, a_z \in A$,*

$$a_1 + \dots + a_z = 1 \Leftrightarrow \neg a_i = \bigoplus_{j \neq i} a_j \quad (i = 1, \dots, z). \quad (5.6)$$

Proof ($\Rightarrow$) By (A21.16) the lattice order of A agrees with the restriction of the lattice order of G to its unit interval, and $a_1 \oplus \dots \oplus a_z = 1 \wedge (a_1 + \dots + a_z) = 1$. For each $i = 1, \dots, z$ we have

$$\neg a_i = 1 \wedge (1 - a_i) = 1 \wedge \sum_{j \neq i} a_j = \bigoplus_{j \neq i} a_j.$$

($\Leftarrow$) By hypothesis, $1 \wedge \sum_{j \neq i} a_j = 1 - a_i$ for each $i = 1, \dots, z$. Since the lattice order of G is translation invariant, then

$$(1 - (1 - a_i)) \wedge \left(\left(\sum_{j \neq i} a_j \right) - (1 - a_i) \right) = \left(1 \wedge \sum_{j \neq i} a_j \right) - (1 - a_i) = 0,$$

i.e., $0 = a_i \wedge \left(-1 + \sum_{j=1}^z a_j\right)$, whence

$$0 = \left(\sum_{j=1}^z a_j\right) \wedge \left(-1 + \sum_{j=1}^z a_j\right) = -1 + \sum_{j=1}^z a_j,$$

which completes the proof. $\square$

Definition 5.7 Let Δ be a regular triangulation of a rational polyhedron $P \subseteq [0, 1]^n$, $n = 1, 2, \dots$. For each vertex v of (a simplex of) Δ , let $h_v: P \rightarrow [0, 1]$ be defined by

- (a) h_v is linear over every simplex $T \in \Delta$;
- (b) $h_v(v) = 1/\text{den}(v)$;
- (c) $h_v(w) = 0$ for any vertex $w \neq v$ of Δ .

Each function h_v is said to be the (Schauder) *hat* of Δ at vertex v . We also say that v is the vertex of h_v . The set $\mathcal{H}_\Delta$ of all Schauder hats of Δ is called the *Schauder basis* of Δ .

By construction, each $h_v \in \mathcal{H}_\Delta$ is continuous. Further properties of $\mathcal{H}_\Delta$ are given by the following

Theorem 5.8 Let $P \subseteq [0, 1]^n$ be a rational polyhedron, Δ a regular triangulation of P , and $\mathcal{H}_\Delta = \{h_1, \dots, h_s\}$ its associated Schauder basis.

- (i) $\mathcal{H}_\Delta$ is a generating set of nonzero elements of $\mathcal{M}(P)$.
- (ii) For each $k = 1, 2, \dots$ and k -element subset C of $\mathcal{H}_\Delta$ satisfying $\bigwedge \{b \mid b \in C\} \neq 0$, the set $\{\mathfrak{m} \in \mu(\mathcal{M}(P)) \mid \mathfrak{m} \supseteq \mathcal{H}_\Delta \setminus C\}$ is homeomorphic to a $(k-1)$ -simplex.
- (iii) There are uniquely determined integers $1 \leq m_1, \dots, m_s$ such that

$$-h_i = (m_i - 1) \cdot h_i \oplus \bigoplus_{j \neq i} m_j \cdot h_j, \quad \text{for all } i = 1, \dots, s, \quad (5.7)$$

i.e., $\sum_{i=1}^s m_i h_i = 1$ in the unital ℓ -group corresponding to $\mathcal{M}(P)$.

As the reader will recall from (2.8), $m \cdot h$ is short for $h \oplus \dots \oplus h$ (m times).

Proof (i) The affine variant of (B21.57) yields a regular triangulation Δ' of $[0, 1]^n$ such that Δ is a subset of Δ' . Stated otherwise, Δ' is a regular extension of Δ over $[0, 1]^n$ that leaves unchanged all simplexes of Δ . Then every $h \in \mathcal{H}_\Delta$ can be extended to a function $h': [0, 1]^n \rightarrow [0, 1]$ which is linear over every simplex of Δ' , and takes value 0 at each vertex of Δ' lying outside P . By (A21.47), the regularity of Δ' ensures that all linear pieces of h' have integer coefficients, whence $h' \in \mathcal{M}([0, 1]^n)$. Thus $h = h'|P$ belongs to $\mathcal{M}(P)$, and $\mathcal{H}_\Delta$ is a subset of $\mathcal{M}(P)$.

In order to prove that $\mathcal{H}_\Delta$ generates $\mathcal{M}(P)$, let $f \in \mathcal{M}(P)$. Then for some $\varphi \in \text{FORM}_n$ we can write $f = \hat{\varphi}|P$. Let Φ be a regular f -triangulation of P , as given by Corollary 2.9. Let ∇ be a Farey subdivision of Δ which is also a subdivision of Φ , as given by the De Concini–Procesi Theorem 5.3. Let

$$\Delta(0) = \Delta \mapsto \Delta(1) \mapsto \cdots \mapsto \Delta(z) = \nabla$$

be a sequence of Farey blow-ups transforming Δ into ∇ .

Claim 1 In the associated unital ℓ -group $(G, 1)$ of $\mathcal{M}(P)$, the function f can be expressed as a linear combination of the hats $h_1, \dots, h_u$ of $\mathcal{H}_\nabla$ with integer coefficients ≥ 0 , $f = n_1 h_1 + \cdots + n_u h_u$. Thus $f = n_1 h_1 \oplus \cdots \oplus n_u h_u$ belongs to the MV-algebra generated by $\mathcal{H}_\nabla$.

As a matter of fact, f is linear over each simplex of ∇ , and its value at each vertex v of ∇ is an integer multiple n_v of $1/\text{den}(v)$, because each linear piece of f has integer coefficients. Letting l_v be the hat of $\mathcal{H}_\nabla$ at v , we then have $f(v) = n_v l_v(v)$. The sum $g = \sum n_v l_v(v)$ for all vertices of ∇ coincides with f at all vertices, and g is linear over each simplex of ∇ . So $f = g$ and the claim is proved.

Claim 2 For each $t = 0, \dots, z-1$, $\mathcal{H}_{\Delta(t+1)}$ is generated by $\mathcal{H}_{\Delta(t)}$.

As noted in Remark 5.4 we need only consider binary blow-ups. So, let us suppose that $\Delta(t+1)$ arises via a Farey blow-up of $\Delta(t)$ at the Farey median c of the 1-simplex $\text{conv}(p, q)$ of $\Delta(t)$. Then in $\Delta(t+1)$ the two hats h_p and h_q of $\mathcal{H}_{\Delta(t)}$ are replaced by three new hats h'_p, h'_q, h'_c . Direct inspection (compare with [1, 9.2.1.]) shows

$$h'_p = h_p - (h_p \wedge h_q) = h_p \odot \neg(h_p \wedge h_q), \quad h'_q = h_q \odot \neg(h_p \wedge h_q), \quad \text{and} \quad h'_c = h_p \wedge h_q.$$

Our second claim is settled and the proof of (i) is complete.

(iii) Letting $m_1, \dots, m_s$ be the denominators of the vertices $v_1, \dots, v_s$ of Δ , and recalling condition (b) in Definition 5.7, it is easy to verify that $\sum_{i=1}^s m_i h_{v_i} = 1$ in G . By Lemma 5.6, $\mathcal{H}_\Delta$ satisfies (5.7).

(ii) Following the notational conventions 4.17 in the light of Theorem 4.16, we will write $f(\mathfrak{m}) = f/\mathfrak{m}$ for every $f \in \mathcal{M}(P)$ and maximal ideal $\mathfrak{m} \in \boldsymbol{\mu}(\mathcal{M}(P))$. Let us display $C \subseteq \mathcal{H}_\Delta$ as $C = \{h_1, \dots, h_k\}$, with $h_1 \wedge \cdots \wedge h_k \neq 0$. For each $j = 1, \dots, k$ let w_j be the vertex of h_j . Then $h_j(w_j) = 1/\text{den}(w_j)$. Let $D = \mathcal{H}_\Delta \setminus C$. By (iii), for all $\mathfrak{m} \in \boldsymbol{\mu}(\mathcal{M}(P))$ we have

$$\mathfrak{m} \supseteq D \Leftrightarrow h(\mathfrak{m}) = 0 \forall h \in D \Leftrightarrow \text{den}(w_1) \cdot h_1(\mathfrak{m}) + \cdots + \text{den}(w_k) \cdot h_k(\mathfrak{m}) = 1.$$

A direct inspection now shows that the homeomorphism $x \in P \mapsto \mathfrak{h}_x \in \boldsymbol{\mu}(\mathcal{M}(P))$ of Theorem 4.16 restricts to a homeomorphism of the compact topological space $\{\mathfrak{n} \in \boldsymbol{\mu}(\mathcal{M}(P)) \mid \mathfrak{n} \supseteq D\}$ onto the $(k-1)$ -simplex $\text{conv}(w_1, \dots, w_k) = \{x \in P \mid \text{den}(w_1) \cdot h_1(\mathfrak{h}_x) + \cdots + \text{den}(w_k) \cdot h_k(\mathfrak{h}_x) = 1\}$. $\square$

Remark 5.9 The first claim in the proof above shows that for every formula $\psi \in \text{FORM}_n$ there is a regular triangulation Δ of $[0, 1]^n$ such that ψ can be written as a $\oplus$ -disjunction of formulas representing the hats of $\mathcal{H}_\Delta$. In this sense, Schauder bases provide disjunctive normal forms in Łukasiewicz logic. Compare with [1, p. 184].

5.4 The Prime Ideals of FREE_2 and FREE_1

The main results of this section will only be used in [Chaps. 9](#) and [18](#), and may be skipped on a first reading.

Let $n = 1, 2, \dots$ and $0 \leq t \leq n$. By an *index* $\mathcal{U} = (u_0, u_1, \dots, u_t)$ (in $\mathbb{R}^n$) we understand a $(t+1)$ -tuple of elements of $\mathbb{R}^n$ such that $u_1, \dots, u_t$ are pairwise orthogonal unit vectors, and for some $\epsilon_1, \dots, \epsilon_t > 0$ the simplex

$$T = \text{conv}(u_0, u_0 + \epsilon_1 u_1, u_0 + \epsilon_1 u_1 + \epsilon_2 u_2, \dots, u_0 + \epsilon_1 u_1 + \dots + \epsilon_t u_t)$$

is contained in $[0, 1]^n$. Any such T is said to be a $\mathcal{U}$ -simplex.

Proposition 5.10 *Fix an index $\mathcal{U} = (u_0, u_1, \dots, u_t)$ in $\mathbb{R}^n$.*

- (i) *For any two $\mathcal{U}$ -simplexes S and T their intersection contains a $\mathcal{U}$ -simplex.*
- (ii) *If the union of two (rational) polyhedra P and Q of $[0, 1]^n$ contains a $\mathcal{U}$ -simplex T , then either P or Q contains some $\mathcal{U}$ -simplex.*

Proof (i) Let us write $S = \text{conv}(u_0, u_0 + \epsilon_1 u_1, u_0 + \epsilon_1 u_1 + \epsilon_2 u_2, \dots, u_0 + \epsilon_1 u_1 + \dots + \epsilon_t u_t)$ and $T = \text{conv}(u_0, u_0 + \eta_1 u_1, u_0 + \eta_1 u_1 + \eta_2 u_2, \dots, u_0 + \eta_1 u_1 + \dots + \eta_t u_t)$, with the intent of showing that $S \cap T$ contains a $\mathcal{U}$ -simplex. Without loss of generality $u_0 = 0 = \text{the origin of } \mathbb{R}^n$. The proof is by induction on t . The cases $t = 0$ and $t = 1$ are trivial. For the induction step, letting

$$S' = \text{conv}(0, \epsilon_1 u_1, \epsilon_1 u_1 + \epsilon_2 u_2, \dots, \epsilon_1 u_1 + \dots + \epsilon_{t-1} u_{t-1})$$

and

$$T' = \text{conv}(0, \eta_1 u_1, \eta_1 u_1 + \eta_2 u_2, \dots, \eta_1 u_1 + \dots + \eta_{t-1} u_{t-1}),$$

by induction there are reals $\omega_i > 0$ such that the simplex

$$R = \text{conv}(0, \omega_1 u_1, \omega_1 u_1 + \omega_2 u_2, \dots, \omega_1 u_1 + \dots + \omega_{t-1} u_{t-1})$$

is contained in $S' \cap T'$. For any simplex U , let $\text{relint } U$ denote the relative interior of U . Since the point $c = \frac{\omega_1}{2} u_1 + \dots + \frac{\omega_{t-1}}{2} u_{t-1}$ lies in $\text{relint } R \subseteq \text{relint } S' \cap \text{relint } T'$, there is $\omega_t > 0$ such that $c + \omega_t u_t \in S \cap T$. Therefore the $\mathcal{U}$ -simplex $Q = \text{conv}(0, \frac{\omega_1}{2} u_1, \frac{\omega_1}{2} u_1 + \frac{\omega_2}{2} u_2, \dots, c, c + \omega_t u_t)$ is contained in $S \cap T$.

(ii) Let $\mathcal{T}$ be a triangulation of $[0, 1]^n$ such that both P and Q are unions of simplexes of $\mathcal{T}$. The existence of $\mathcal{T}$ is ensured by Corollary 2.9. Induction on t shows that some simplex T of $\mathcal{T}$ contains a $\mathcal{U}$ -simplex. Such T either belongs to P or to Q . $\square$

For any index $\mathcal{U}$ in $\mathbb{R}^n$ we define $\mathfrak{p}_{\mathcal{U}} \subseteq \text{FREE}_n$ by

$$f \in \mathfrak{p}_{\mathcal{U}} \text{ iff the zeroset } \mathcal{Z}f = f^{-1}(0) \text{ contains some } \mathcal{U}\text{-simplex.} \quad (5.8)$$

Proposition 5.11 *For any index $\mathcal{U}$ in $\mathbb{R}^n$, $\mathfrak{p}_{\mathcal{U}}$ is a prime ideal of FREE_n .*

Proof Trivially, $0 \in \mathfrak{p}_{\mathcal{U}}$. By (A21.24), $g \leq f \in \mathfrak{j}_{\mathcal{U}} \Rightarrow g \in \mathfrak{p}_{\mathcal{U}}$, because $\mathcal{Z}g \supseteq \mathcal{Z}f$. To complete the proof that $\mathfrak{p}_{\mathcal{U}}$ is an ideal, suppose $f, g \in \mathfrak{p}_{\mathcal{U}}$, and let the $\mathcal{U}$ -simplexes T' and T'' be such that $\mathcal{Z}f \supseteq T'$ and $\mathcal{Z}g \supseteq T''$. By Proposition 5.10(i), T' and T'' contain some $\mathcal{U}$ -simplex T . Since $\mathcal{Z}(f \oplus g) = \mathcal{Z}f \cap \mathcal{Z}g \supseteq T' \cap T'' \supseteq T$, then $f \oplus g \in \mathfrak{p}_{\mathcal{U}}$.

In order to prove that $\mathfrak{p}_{\mathcal{U}}$ is prime, suppose $f \wedge g \in \mathfrak{p}_{\mathcal{U}}$, whence $\mathcal{Z}(f \wedge g) = \mathcal{Z}f \cup \mathcal{Z}g \supseteq R$ for some $\mathcal{U}$ -simplex R . By Proposition 5.10(ii), either $\mathcal{Z}f$ or $\mathcal{Z}g$ contains some $\mathcal{U}$ -simplex, whence either f or g belongs to $\mathfrak{p}_{\mathcal{U}}$. $\square$

Conversely, every prime ideal $\mathfrak{p}$ of FREE_n has the form $\mathfrak{p} = \mathfrak{p}_{\mathcal{U}}$ for some index $\mathcal{U}$. For simplicity, here we will only give the (altogether lengthy) proof of the case $n = 2$.

By a rational line in $[0, 1]^2$ we mean a line L given by the equation $ax_1 + bx_2 + c = 0$ where $a, b, c \in \mathbb{Q}$. L is irrational if it is not rational.

Theorem 5.12 *Every prime ideal of $\text{FREE}_2 = \mathcal{M}([0, 1]^2)$ has the form $\mathfrak{p} = \mathfrak{p}_{\mathcal{U}}$ for some index $\mathcal{U}$ in $\mathbb{R}^2$. Specifically, for each $x \in [0, 1]^2$ we have:*

- (I) *If no rational line crosses x then the maximal ideal $\mathfrak{p}_x = \mathfrak{h}_x$ is the only ideal $\mathfrak{i}$ of $\mathcal{M}([0, 1]^2)$ such that $\mathcal{Z}\mathfrak{i} = \{x\}$.*
- (II) *If precisely one rational line L crosses x , say $L = x + \mathbb{R}u$ for some unit vector $u \in [0, 1]^2$, then $\mathfrak{p}_{x, u^\perp}$ and $\mathfrak{p}_{x, -u^\perp}$ are the only two prime ideals properly contained in $\mathfrak{p}_x$ —unless x lies on an edge of the unit square, in which case $\mathfrak{p}_x$ properly contains exactly one of $\mathfrak{p}_{x, u^\perp}$ and $\mathfrak{p}_{x, -u^\perp}$. Here $u^\perp$ and $-u^\perp$ are the two unit vectors of $\mathbb{R}^2$ perpendicular to u .*
- (III) *If two distinct rational lines cross x , (i.e., x is a rational point) then*
 - *For any irrational half-line $x + \mathbb{R}_{\geq 0}u$ intersecting $[0, 1]^2$ in at least two points, $\mathfrak{p}_{x, u}$ is the only prime ideal strictly contained in $\mathfrak{p}_x$.*
 - *For any rational half-line $x + \mathbb{R}_{\geq 0}v$ intersecting $[0, 1]^2$ in at least two points, $\mathfrak{p}_{x, v}$, $\mathfrak{p}_{x, v, v^\perp}$, $\mathfrak{p}_{x, v, -v^\perp}$ are the only three prime ideals strictly contained in $\mathfrak{p}_x$.*

Proof For brevity we will only argue in case x lies in the interior of the unit square. The case when x lies on the border is treated in a similar way. Let $\mathfrak{p} \subseteq \mathfrak{p}_x$ be a prime ideal.

Case (I) No rational line crosses x .

Let $\mathfrak{p} \subseteq \mathfrak{p}_x$ be a prime ideal. Suppose $f \in \mathcal{M}([0, 1]^2)$. Since each linear piece of f has integer coefficients, $f \in \mathfrak{p}_x \Rightarrow f \in \mathfrak{o}_x$, whence $\mathfrak{p} \subseteq \mathfrak{o}_x$. By Proposition 4.9, $\mathfrak{p} = \mathfrak{o}_x = \mathfrak{p}_x$.

Case (II) Exactly one rational line L crosses x .

Let $f \in \mathfrak{p}_x$. Let Δ be a regular f -triangulation of $[0, 1]^2$. Since x lies in the relative interior of the common side BC of two regular triangles ABC and DBC

of Δ , both directional derivatives $f'(x; u)$ and $f'(x; -u)$ vanish. For otherwise (absurdum hypothesis), by direct inspection of the zeroset of f , there would exist a rational line $M \neq L$ crossing x , which is impossible. We have proved

$$\mathfrak{p} \subseteq \mathfrak{p}_{x,u} \cap \mathfrak{p}_{x,-u}. \quad (5.9)$$

The hats h_A and h_D of the Schauder basis $\mathcal{H}_\Delta$ satisfy $h \wedge k = 0 \in \mathfrak{o}_x$, but neither belongs to $\mathfrak{o}_x$. It follows that the germinal ideal $\mathfrak{o}_x$ is not prime.

We now prove

$$\text{there is } b \in \mathfrak{p} \text{ such that either } b'(x; u^\perp) > 0 \text{ or } b'(x; -u^\perp) > 0. \quad (5.10)$$

For otherwise, by (5.9), all directional derivatives at x of every $b \in \mathfrak{p}$ would vanish, and $\mathfrak{p} \subseteq \mathfrak{o}_x$, whence again by Proposition 4.9, $\mathfrak{p} = \mathfrak{o}_x$, would not be prime, a contradiction.

Subcase (II.1) There are $f, g \in \mathfrak{p}$ such that $f'(x; u^\perp) > 0$ and $g'(x; -u^\perp) > 0$. Then the function $h = f \vee g$ has zero directional derivative at x precisely along the two directions $\pm u$. Fix now $l \in \mathfrak{p}_x$, with the intent of showing $l \in \mathfrak{p}$. Let Δ be a regular $\{l, h\}$ -triangulation of $[0, 1]^2$ such that $L \cap [0, 1]^2$ is a union of simplexes of Δ . Exactly two triangles S, T of Δ contain the point x . Evidently, x lies in the relative interior of the common side of S and T . By Lemma 1.13, some $p \in \mathfrak{o}_x \subseteq \mathfrak{p}$ takes constant value 1 on $[0, 1]^2 \setminus (S \cup T)$. Further, there is $m = 1, 2, 3, \dots$ such that $m \cdot h \geq l$ over $S \cup T$. Thus $p \vee m \cdot h \geq l$, and $l \in \mathfrak{p}$ as desired. Recalling (5.9) we conclude that $\mathfrak{p} = \mathfrak{p}_x = \mathfrak{p}_{x,u} = \mathfrak{p}_{x,-u}$.

Subcase (II.2) There is $f \in \mathfrak{p}$ such that $f'(x; u^\perp) > 0$ and $f'(x; -u^\perp) = 0$. Then let Δ be a regular f -triangulation of $[0, 1]^2$ such that $L \cap [0, 1]^2$ is a union of simplexes of Δ .

Subsubcase (II.2.1) There is $g \in \mathfrak{p}$ such that $g'(x; -u^\perp) > 0$. Then $\mathfrak{p} = \mathfrak{p}_{x,u}$ as in Subcase (II.1).

Subsubcase (II.2.2) Every $g \in \mathfrak{p}$ has $g'(x; -u^\perp) = 0$. Then $\mathfrak{p} \subseteq \mathfrak{p}_{x,u,-u^\perp}$. For the converse inclusion, by way of contradiction suppose $k \in \mathfrak{p}_{x,u,-u^\perp} \setminus \mathfrak{p}$. Let L_1, L_2 be the intersections of the unit square with the two closed half-spaces of L . By (A21.18) there are $f_i \in \mathcal{M}([0, 1]^2)$, ($i = 1, 2$) with $\mathcal{Z} f_i = L_i$. Thus without loss of generality we can assume $k'(x; u^\perp) > 0$. Let $h \in \mathfrak{p}_x$ satisfy $h'(x; u^\perp) = 0$ and $h'(x; -u^\perp) > 0$, whence $h \notin \mathfrak{p}$. Then $h \wedge k \in \mathfrak{o}_x \subseteq \mathfrak{p}$, and $\mathfrak{p}$ is not prime, a contradiction showing that $\mathfrak{p} = \mathfrak{p}_{x,u,-u^\perp} = \mathfrak{p}_{x,-u^\perp}$.

Subcase (II.3) There is $g \in \mathfrak{p}$ such that $g'(x; -u^\perp) > 0$ and $g'(x; u^\perp) = 0$. Arguing as in Subcase (II.2) we either have $\mathfrak{p} = \mathfrak{p}_x$ or $\mathfrak{p} = \mathfrak{p}_{x,u,u^\perp} = \mathfrak{p}_{x,u^\perp}$.

Case (III) x is rational.

Subcase (III.1) For every unit vector $u \in \mathbb{R}^2$ some $f \in \mathfrak{p}$ satisfies $f'(x; u) > 0$.

Then by continuity, for all unit vectors v sufficiently close to u we also have $f'(x; v) > 0$. The compactness of the unit circumference centered at x yields $g =$

$f_1 \vee \cdots \vee f_l \in \mathfrak{p}$ such that $g'(x; w) > 0$ for all unit vectors $w \in \mathbb{R}^2$. It follows that the ideal $\langle g \rangle$ generated by g coincides with $\mathfrak{p}_x$.

Subcase (III.2) There is a unit vector $u \in \mathbb{R}^2$ such that $f'(x; u) = 0$ for all $f \in \mathfrak{p}$.

Then u is unique: For, if also $v \neq u$ satisfies $f'(x; v) = 0$ for all $f \in \mathfrak{p}$, then some rational line M crossing x separates the two points $x + u$ and $x + v$, in the sense that each of the two open half-planes M_1 and M_2 determined by M contains precisely one of these two points. By (A21.18), there are $p, q \in \mathcal{M}([0, 1]^2)$ such that $p = 0$ on M_1 and $p > 0$ on M_2 , while $q = 0$ on M_2 and $q > 0$ on M_1 . Then $p \wedge q = 0 \in \mathfrak{p}$ while neither p nor q belongs to $\mathfrak{p}$, thus contradicting the primality of $\mathfrak{p}$. As a consequence,

$$\mathfrak{p} \subseteq \mathfrak{p}_{x,u} \text{ for a unique unit vector } u \in \mathbb{R}^2. \quad (5.11)$$

Subsubcase (III.2.1) The half-line $L = x + \mathbb{R}_{\geq 0}u$ is irrational.

Then each $f \in \mathfrak{p}$ satisfies $f'(x; v) = 0$ for all unit vectors sufficiently close to u . It follows that $\mathfrak{p} \subseteq \mathfrak{p}_{x,u,u^\perp} = \mathfrak{p}_{x,u,-u^\perp}$. For the converse inclusion, by way of contradiction suppose $k \in \mathfrak{p}_{x,u,-u^\perp} \setminus \mathfrak{p}$. Let Δ be a regular k -triangulation containing x as a vertex. Then k vanishes on some triangle $T \in \Delta$ having x as a vertex and such that L has nonempty intersection with the interior of T . Let A, B be the other two vertices of T , and C their Farey mediant. Let Δ' be the Farey subdivision of Δ at C . Let h be the hat of $\mathcal{H}_{\Delta'}$ at C . Since $h'(x; u) > 0$, $h \notin \mathfrak{p}$. On the other hand, $h \wedge k$ vanishes on an open neighborhood of x , so it belongs to $h \wedge k \in \mathfrak{o}_x \subseteq \mathfrak{p}$. Since neither k nor h belong to $\mathfrak{p}$, we have contradicted the primality of $\mathfrak{p}$, and settled the identity $\mathfrak{p} = \mathfrak{p}_{x,u,u^\perp} = \mathfrak{p}_{x,u,-u^\perp}$.

Subsubcase (III.2.2) The half-line $L = x + \mathbb{R}_{\geq 0}u$ is rational.

Let ∇ be any regular triangulation such that x is a vertex of ∇ , and $L \cap [0, 1]^2$ is a union of simplexes of ∇ . Then x is the vertex of exactly two triangles of ∇ , denoted S_∇ and T_∇ , intersected by L along their common edge $S_\nabla \cap T_\nabla \in \nabla$. Let us write $S_\nabla \cap T_\nabla = xA_\nabla$, and denote by B_∇ and C_∇ the third vertex of S_∇ and T_∇ , respectively.

With reference to (5.11), let $\mathfrak{p} \subseteq \mathfrak{p}_{x,u}$ be a prime ideal of $\mathcal{M}([0, 1]^2)$.

Claim 1 Any $h \in \mathfrak{p}_{x,u,u^\perp} \cap \mathfrak{p}_{x,u,-u^\perp}$ automatically belongs to $\mathfrak{p}$.

As a matter of fact, let ∇ be a regular h -triangulation of $[0, 1]^2$ such that x is a vertex of ∇ and $L \cap [0, 1]^2$ is a union of simplexes of ∇ . Then, by hypothesis, $h(C_\nabla) = h(D_\nabla) = 0$. Let k be the hat of $\mathcal{H}_\nabla$ at vertex A_∇ . Then by (5.11), $k \notin \mathfrak{p}_{x,u} \supseteq \mathfrak{p}$. On the other hand, $k \wedge h \in \mathfrak{o}_x \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, necessarily $h \in \mathfrak{p}$.

Claim 2 If some $f \in \mathfrak{p}$ satisfies $f'(x; u^\perp) > 0$ and $f'(x; -u^\perp) > 0$ then $\mathfrak{p} = \mathfrak{p}_{x,u}$.

Indeed, let $g \in \mathfrak{p}_{x,u}$. Let Δ be a regular $\{f, g\}$ -triangulation of $[0, 1]^2$ such that x is a vertex of Δ , and $L \cap [0, 1]^2$ is a union of simplexes of Δ . With the notation above,

$f(C_\Delta) > 0$, $f(D_\Delta) > 0$; by (5.11), $f(A_\Delta) = g(A_\Delta) = 0$. For some $m = 1, 2, \dots$ the inequality $m \cdot f \geq g$ holds on the quadrilateral $S_\Delta \cup T_\Delta$. A suitable sequence of Farey blow-ups will produce a subdivision $\bar{\Delta}$ of Δ such that the three points

$$A = A_{\bar{\Delta}}, \quad B = B_{\bar{\Delta}}, \quad C = C_{\bar{\Delta}}$$

lie in the interior of $S_\Delta \cup T_\Delta$. Let h_x, h_A, h_B, h_C be the hats in $\mathcal{H}_{\bar{\Delta}}$ at vertices x, A, B, C , with their respective denominators d, a, b, c . Let the hats $h_1, \dots, h_4 \in \mathcal{H}_{\bar{\Delta}}$ be defined by

$$h_1 = d \cdot h_x, \quad h_2 = a \cdot h_A, \quad h_3 = b \cdot h_B, \quad h_4 = c \cdot h_C.$$

Then the McNaughton function $h = \neg(h_1 \oplus h_2 \oplus h_3 \oplus h_4) = 1 - (h_1 + h_2 + h_3 + h_4)$ is linear over each simplex of $\bar{\Delta}$, has value 0 precisely at the vertices x, A, B, C , and value 1 at all remaining vertices of $\bar{\Delta}$. By Claim 1, $h \in \mathfrak{p}$. Direct inspection shows that $h \oplus m \cdot f \geq g$, whence $g \in \mathfrak{p}$, as required to settle our claim.

Claim 3 $\mathfrak{p}_{x,u,u^\perp} \cap \mathfrak{p}_{x,u,-u^\perp}$ is not a prime ideal. Thus $\mathfrak{p}$ is not contained in $\mathfrak{p}_{x,u,u^\perp} \cap \mathfrak{p}_{x,u,-u^\perp}$.

Let L_1 and L_2 be the two closed half-planes determined by L . One more application of (A21.18) yields McNaughton functions f, g with $f \wedge g = 0$, $\mathcal{Z}f = L_1 \cap [0, 1]^2$, $\mathcal{Z}g = L_2 \cap [0, 1]^2$. Neither f nor g belongs to $\mathfrak{p}_{x,u,u^\perp} \cap \mathfrak{p}_{x,u,-u^\perp}$. This shows that $\mathfrak{p}_{x,u,u^\perp} \cap \mathfrak{p}_{x,u,-u^\perp}$ is not a prime ideal. For the second statement, by (A21.8) the prime ideal $\mathfrak{p}$ cannot be contained in the non-prime ideal $\mathfrak{p}_{x,u,u^\perp} \cap \mathfrak{p}_{x,u,-u^\perp}$. Our third claim is settled.

Assume now the prime ideal $\mathfrak{p}$ is properly contained in $\mathfrak{p}_{x,u}$, with the intent of proving that either $\mathfrak{p} = \mathfrak{p}_{x,u,u^\perp}$ or $\mathfrak{p} = \mathfrak{p}_{x,u,-u^\perp}$. For each $f \in \mathfrak{p}$ our three claims above are to the effect that either $f'(x + \epsilon u; u^\perp) > 0 = f'(x + \epsilon u; -u^\perp)$, or

$$f'(x + \epsilon u; u^\perp) = 0 \quad \text{and} \quad f'(x + \epsilon u; -u^\perp) > 0. \quad (5.12)$$

In the first case, by Claim 2 for no $g \in \mathfrak{p}$ we can have $g'(x + \epsilon u; -u^\perp) > 0$. Thus $\mathfrak{p} \subseteq \mathfrak{p}_{x,u,-u^\perp}$. For the converse inclusion, suppose $g \in \mathfrak{p}_{x,u,-u^\perp} \setminus \mathfrak{p}$. Let Δ be a g -regular triangulation such that x is a vertex of Δ , and $L \cap [0, 1]^2$ is a union of simplexes of Δ . With the above notation, g vanishes on, say, S_Δ . To contradict the primality of $\mathfrak{p}$ it suffices to find $k \notin \mathfrak{p}_{x,u,-u^\perp} \supseteq \mathfrak{p}$ with $g \wedge k = 0$. To this purpose, a finite sequence of Farey blow-ups again yields a subdivision $\bar{\Delta}$ of Δ such that the vertex $B_{\bar{\Delta}}$ of $S_{\bar{\Delta}}$ not lying on L belongs to the interior of S_Δ . The hat k of $\mathcal{H}_{\bar{\Delta}}$ at $B_{\bar{\Delta}}$ does not belong to $\mathfrak{p}_{x,u,-u^\perp}$ and vanishes over $[0, 1]^2 \setminus S_{\bar{\Delta}}$. Thus $g \wedge k = 0 \in \mathfrak{p}$, but neither g nor k are members of $\mathfrak{p}$, which contradicts the primality of $\mathfrak{p}$, as desired. We have shown that $\mathfrak{p} = \mathfrak{p}_{x,u,-u^\perp}$. Similarly, from (5.12) we obtain $\mathfrak{p} = \mathfrak{p}_{x,u,u^\perp}$.

In conclusion, the three ideals $\mathfrak{p}_{x,u}$, $\mathfrak{p}_{x,u,u^\perp}$, $\mathfrak{p}_{x,u,-u^\perp}$ are distinct. Assuming $\mathfrak{p}$ to be properly contained in $\mathfrak{p}_{x,u}$, we have shown that $\mathfrak{p}$ either coincides with $\mathfrak{p}_{x,u,u^\perp}$ or with $\mathfrak{p}_{x,u,-u^\perp}$.

Since by (A21.9) every $\mathfrak{p} \in \text{Spec}(\mathcal{M}([0, 1]^2))$ is contained in exactly one maximal ideal of $\mathcal{M}([0, 1]^2)$, we have exhausted all possible cases. $\square$

Corollary 5.13 *Every prime ideal of $\text{FREE}_1 = \mathcal{M}([0, 1])$ has the form $\mathfrak{p} = \mathfrak{p}_{\mathcal{U}}$ for some index $\mathcal{U}$ in $\mathbb{R}$. Specifically, for each $x \in [0, 1]$,*

- (i) *If x is irrational then the maximal ideal $\mathfrak{p}_x$ is the only ideal $\mathfrak{i}$ of FREE_1 such that $\mathcal{Z}\mathfrak{i} = \{x\}$. In other words, $\mathfrak{p}_x = \mathfrak{h}_x =$ the hull of x .*
- (ii) *If $0 < x < 1$ is rational, let e be the unit vector of $\mathbb{R}$ in the positive direction. It follows that $\mathfrak{p}_{x,e}$, and $\mathfrak{p}_{x,-e}$ are the only two prime ideals strictly contained in $\mathfrak{p}_x$. Further, $\mathfrak{p}_{0,e}$ is the only prime ideal strictly contained in $\mathfrak{p}_0$; similarly, $\mathfrak{p}_{1,-e}$ is the only prime ideal strictly contained in $\mathfrak{p}_1$.*

5.5 Remarks

Proposition 5.2 follows from Cauchy's 1816 analysis of the Farey sequence, (*Oeuvres*, II Série, Tome VI, 1887, pp. 146–148, or Tome II, 1958, pp. 207–209).

MV-algebraic Schauder bases were introduced in [2].

The De Concini–Procesi theorem on the elimination of points of indeterminacy in toric varieties was first proved in [3]. For the fan-theoretic version see [4, p. 252]. The theorem holds, with the same proof, for every rational polyhedron $P \subseteq \mathbb{R}^n$.

In this chapter, by suitably modifying the proof in [5, 3.6], we have established an affine version of the theorem, which is more suitable for our analysis of Schauder bases. The first proof that any Schauder basis of A generates A was given in [6, 4.4].

Using the Γ functor, the classification of prime ideals of FREE_n follows from Panti's analysis of the prime ideals of the unital ℓ -group corresponding to FREE_n , [7, Corollary 4.9]. See [8] for further details.

Corollary 5.13 has applications in the analysis of the prime spectrum of the AF C^* -algebra $\mathfrak{M}_1$ of [2], denoted $\mathfrak{A}$ in [9]. See [10] and [11] for details.

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Chapter 6

Bases and Finitely Presented MV-Algebras

Up to isomorphism, every finitely presented MV-algebra A is the Lindenbaum algebra LIND_θ of some satisfiable formula θ . The finite string of symbols needed to write θ is a “presentation” of A . Equivalently, A can be presented as $\mathcal{M}(P)$ for some rational polyhedron $P \subseteq [0, 1]^n$, ($n = 1, 2, \dots$). Writing $P = \bigcup_1^k T_i$ for suitable rational simplexes T_i , the string of symbols needed to list the vertices of each T_i is an alternative finite presentation of A . A first problem is to recognize presentations of isomorphic algebras (Problem 1 in the list of problems in [Chap. 20](#)). In [Chap. 18](#) we will describe various algorithms dealing with finitely presented MV-algebras. As is often the case, the algorithmic theory implements the algebraic theory. This chapter is devoted to bases, a central MV-algebraic notion. Our proof that A is finitely presented iff it has a basis, will show that bases are just isomorphic copies of Schauder bases. Bases will then be used to characterize large classes of finitely generated MV-algebras, such as free and projective MV-algebras. Bases also give a way of proving a purely MV-algebraic version of the De Concini–Procesi theorem, and are a fundamental ingredient of conditionals in L_∞ and MV-algebraic Lebesgue integration, to be developed in [Chaps. 15](#) and [16](#).

6.1 Bases

Just as a $\mathbb{Z}$ -homeomorphism of a rational polyhedron P onto itself in general distorts a triangulation of P into something that is no longer a triangulation, similarly an automorphic copy of a Schauder basis of $\mathcal{M}(P)$ need not be a Schauder basis.

In the light of Theorem 5.8, the following definition provides a representation-free generalization of Schauder bases:

Definition 6.1 A *basis* of an MV-algebra A is a (necessarily nonempty) finite set $\mathcal{B} = \{b_1, \dots, b_n\}$ of nonzero elements of A such that

- (i) $\mathcal{B}$ generates A .

- (ii) For each $k = 1, 2, \dots$ and k -element subset C of $\mathcal{B}$ with $\bigwedge \{b \mid b \in C\} \neq 0$, the set $\{\mathfrak{m} \in \mu(A) \mid \mathfrak{m} \supseteq \mathcal{B} \setminus C\}$ is homeomorphic to a $(k - 1)$ -simplex.
- (iii) There are integers (called *multipliers*) $1 \leq m_1, \dots, m_n$ such that, letting as usual $m \cdot b$ be short for $b \oplus \dots \oplus b$ (m times),

$$\neg b_i = (m_i - 1) \cdot b_i \oplus \bigoplus_{j \neq i} m_j \cdot b_j, \quad \text{for all } i = 1, \dots, n. \quad (6.1)$$

Writing $A = \Gamma(G, 1)$, and using Lemma 5.6, condition (iii) can be more intuitively expressed in G by the single identity

$$(iii') \quad \sum_{i=1}^n m_i b_i = 1.$$

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis of an MV-algebra A with multipliers $m_1, \dots, m_n$. Fix $k = 1, 2, \dots, n$. Then by a k -cluster of $\mathcal{B}$ we understand a k -element subset C of $\mathcal{B}$ such that $\bigwedge C \neq 0$. We denote by $\mathcal{B}^{\triangleright}$ the set of all clusters of $\mathcal{B}$,

$$\mathcal{B}^{\triangleright} = \left\{ \emptyset \neq C \subseteq \mathcal{B} \mid \bigwedge C \neq 0 \right\}.$$

For every k -cluster $C = \{b_{i_1}, \dots, b_{i_k}\}$ of $\mathcal{B}$ and maximal ideal $\mathfrak{m} \in \mu(A)$ we have by (iii'):

$$\mathfrak{m} \supseteq \mathcal{B} \setminus C \Leftrightarrow 0 = b/\mathfrak{m} \forall b \in \mathcal{B} \setminus C \Leftrightarrow (m_{i_1} b_{i_1} + \dots + m_{i_k} b_{i_k})/\mathfrak{m} = 1. \quad (6.2)$$

The set of all maximal ideals $\mathfrak{m}$ of A obeying these equivalent conditions will be called the *apogee* of C . Then condition (ii) in the definition of $\mathcal{B}$ has the following shorter reformulation:

(ii') *The apogee of every k -cluster of $\mathcal{B}$ is homeomorphic to a $(k - 1)$ -simplex.*

In particular, when $k = 1$ the apogee of the cluster b_j is a singleton $\{\mathfrak{a}_j\}$. By (6.2) and (iii'), for each $i \neq j$, $b_i(\mathfrak{a}_j) = 0$. Here we are using the notation of 4.17. As a consequence, the elements $b_1, \dots, b_n$ are independent in the $\mathbb{Z}$ -module G , whence

(iii'') *the multipliers m_i of $\mathcal{B}$ are uniquely determined.*

Recall the definition of Chang's distance function $\mathbf{d}$ given in Sect. 2.5.

Lemma 6.2

- (i) *If $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of $A = \Gamma(G, 1)$ with multipliers $m_1, \dots, m_n$, then for each $i = 1, 2, \dots, n$,*

$$\mathbf{d} \left(\neg b_i, (m_i - 1) \cdot b_i \oplus \bigoplus_{j \neq i} m_j \cdot b_j \right) = \left| \neg b_i - (m_i - 1)b_i + \sum_{j \neq i} m_j b_j \right| = 0. \quad (6.3)$$

- (ii) *For every cluster C , letting $\mathcal{B} \setminus C = \{b_{j_1}, \dots, b_{j_t}\}$ and $b_C = b_{j_1} \vee \dots \vee b_{j_t}$, (with $b_{\mathcal{B}} = 0$ in case $\mathcal{B} \in \mathcal{B}^{\triangleright}$), it follows that*

$$\bigwedge_{C \in \mathcal{B}^{\triangleright}} b_C = 0. \quad (6.4)$$

Proof (i) Follows from condition (iii) in the definition of $\mathcal{B}$. (ii) By way of contradiction let us suppose $\bigwedge_{C \in \mathcal{B}^{\triangleright\triangleleft}} b_C \neq 0$. Thus $\mathcal{B} \notin \mathcal{B}^{\triangleright\triangleleft}$. The subdirect representation theorem (A21.13) yields a prime ideal $\mathfrak{p}$ of A such that the totally ordered MV-algebra $A/\mathfrak{p}$ satisfies the condition $b_C/\mathfrak{p} \neq 0$ for each cluster $C \in \mathcal{B}^{\triangleright\triangleleft}$. Thus, for each $C \in \mathcal{B}^{\triangleright\triangleleft}$ there is an element $b(C) \in \mathcal{B} \setminus C$ such that $b(C)/\mathfrak{p} \neq 0$. Letting $K = \{b(C) \mid C \in \mathcal{B}^{\triangleright\triangleleft}\}$ we have $\bigwedge K/\mathfrak{p} \neq 0$, which shows that K is a cluster of $\mathcal{B}$. Yet, K is different from every cluster C of $\mathcal{B}$, since $b(C)$ is an element of K but not of C . Having thus reached a contradiction, the proof is complete. $\square$

6.2 Finitely Presented MV-Algebras

An MV-algebra A is said to be *finitely presented* if it is isomorphic to $\text{FREE}_n/\mathfrak{i}$ for some $n = 1, 2, \dots$, and some finitely generated (= singly generated = principal) ideal $\mathfrak{i}$ of FREE_n .

Theorem 6.3 *For any MV-algebra A the following conditions are equivalent:*

- (i) A is finitely presented.
- (ii) For some rational polyhedron $P \neq \emptyset$, A is isomorphic to $\mathcal{M}(P)$.
- (iii) A is isomorphic to LIND_θ for some satisfiable formula θ .
- (iv) A has a basis $\mathcal{B}$.

Proof The equivalence of (i), (ii), (iii) routinely follows from Theorems 3.20 and 3.23.

(ii $\Rightarrow$ iv) Without loss of generality, $A = \mathcal{M}(P)$, where $P \subseteq [0, 1]^n$ for some $n = 1, 2, \dots$. Using Corollary 2.9 we equip P with a regular triangulation Δ . By Theorem 5.8, the Schauder basis $\mathcal{H}_\Delta$ is a basis.

(iv $\Rightarrow$ i) By assumption, A has a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ with multipliers $m_1, \dots, m_n$. Let $(G, 1)$ be the associated unital ℓ -group of A , and write $A = \Gamma(G, 1)$. For each $i = 1, \dots, n$ let $\pi_i: [0, 1]^n \rightarrow [0, 1]$ be the i th coordinate function. By Theorem 1.5, $\{\pi_1, \dots, \pi_n\}$ is a free generating set of $\mathcal{M}([0, 1]^n)$. The map $\pi_i \mapsto b_i$ uniquely extends to a homomorphism $\lambda: \mathcal{M}([0, 1]^n) \rightarrow A$ which turns out to be onto A , because $\mathcal{B}$ generates A [condition (i) in the definition of $\mathcal{B}$]. Let $\mathfrak{i} = \ker \lambda$. By (A21.6) we can write without loss of generality

$$A = \mathcal{M}([0, 1]^n)/\mathfrak{i}, \quad \lambda(\pi_i) = \pi_i/\mathfrak{i} = b_i, \quad (i = 1, \dots, n). \quad (6.5)$$

We will prove that $\mathfrak{i}$ is a principal ideal of A .

To this purpose, proceeding as in Lemma 6.2(i), let the formula $\alpha(X_1, \dots, X_n)$ be defined by

$$\bigvee_{i=1}^n \mathbf{d} \left(\neg X_i, (m_i - 1) \cdot X_i \oplus \bigoplus_{j \neq i} m_j \cdot X_j \right)$$

By condition (iii) in the definition of $\mathcal{B}$, $\alpha(b_1, \dots, b_n) = 0$. Since $\lambda(\pi_i) = b_i$, the McNaughton function $a = \alpha(\pi_1, \dots, \pi_n)$ belongs to $\mathbf{i}$. Then the zeroset $\mathcal{Z}a$ is readily seen to coincide with the set of points $x = (x_1, \dots, x_n) \in [0, 1]^n$ satisfying

$$1 - x_i = (m_i - 1)x_i + \sum_{j \neq i} m_j x_j, \quad (i = 1, \dots, n).$$

In other words, $\mathcal{Z}a = \text{conv}(e_1/m_1, \dots, e_n/m_n)$.

Let $C = \{b_{k_1}, \dots, b_{k_t}\}$ be a cluster of $\mathcal{B}$. We define the simplex $\mathcal{T}_C \subseteq [0, 1]^n$ by

$$\mathcal{T}_C = \text{conv}\left(\frac{e_{k_1}}{m_{k_1}}, \dots, \frac{e_{k_t}}{m_{k_t}}\right), \quad (6.6)$$

where, as the reader will recall, $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. Next, let $\mathcal{B} \setminus C = \{b_{j_1}, \dots, b_{j_s}\}$ be the complementary set of C in $\mathcal{B}$. We then define the element $b_C \in A$ by

$$b_C = b_{j_1} \vee \dots \vee b_{j_s} \quad (\text{with } b_C = 0 \text{ in case } C = \mathcal{B} \in \mathcal{B}^{\triangleright\triangleleft}).$$

Finally, the McNaughton functions f_C and $f^{\natural}_C$ are defined by

$$f_C = a \vee \pi_{j_1} \vee \dots \vee \pi_{j_s} \quad \text{and} \quad f^{\natural}_C = \bigwedge_{D \in \mathcal{B}^{\triangleright\triangleleft}} f_D.$$

By Lemma 6.2, $f^{\natural}_C \in \mathbf{i}$. Moreover,

$$\mathcal{Z}f_C = \text{conv}(e_1/m_1, \dots, e_n/m_n) \cap \{x \in [0, 1]^n \mid x_{j_1} = \dots = x_{j_s} = 0\},$$

whence by condition (iii'), $\mathcal{Z}f_C = \mathcal{T}_C$. Summing up, we have the inclusion

$$\mathcal{Z}\mathbf{i} \subseteq \mathcal{Z}f^{\natural}_C = \bigcup_{C \in \mathcal{B}^{\triangleright\triangleleft}} \mathcal{T}_C. \quad (6.7)$$

For the converse inclusion we will settle the following

Claim For each $C \in \mathcal{B}^{\triangleright\triangleleft}$, $\mathcal{T}_C \subseteq \mathcal{Z}\mathbf{i}$.

The proof is by induction on the number l of elements of C .

Basis $l = 1$.

Then for some $i = 1, \dots, n$, $C = \{b_i\}$ and $\mathcal{T}_C = \{e_i/m_i\}$. By way of contradiction, suppose $e_i/m_i \notin \mathcal{Z}\mathbf{i}$. Then $m_i \pi_i(x) \neq 1$ for all $x \in \mathcal{Z}\mathbf{i} \subseteq \mathcal{Z}f^{\natural}_C$. By Lemmas 4.21 and 4.19, for each $\mathbf{m} \in \mu(A)$ with $\mathbf{m} \supseteq \mathbf{i}$,

$$\frac{m_i \cdot \pi_i}{\mathbf{m}} = \frac{m_i \cdot \pi_i / \mathbf{i}}{\mathbf{m} / \mathbf{i}} \neq 1,$$

whence $m_i \cdot b_i / \mathbf{n} \neq 1$ for all $\mathbf{n} \in \mu(A)$. Thus in particular, the apogee $\{\mathbf{m} \in \mu(A) \mid m_i \cdot b_i / \mathbf{m} = 1\}$ of the one-cluster $\{b_i\}$ is empty, against condition (ii') in the definition of $\mathcal{B}$.

Induction step: $C = \{b_{k_1}, \dots, b_{k_l}\} \in \mathcal{B}^{\infty}$.

By way of contradiction, suppose $\mathcal{T}_C$ is not contained in $\mathcal{Z}i$. By induction hypothesis, the inclusion $\mathcal{T}_D \subseteq \mathcal{Z}i$ holds for each u -cluster D with $u < l$. Thus the set $Q = \{x \in \mathcal{Z}i \mid m_{k_1}\pi_{k_1}(x) + \dots + m_{k_l}\pi_{k_l}(x) = 1\}$ is a proper subset of $\mathcal{T}_C$ containing all proper faces of $\mathcal{T}_C$. In (B21.76) it is proved that Q is not homeomorphic to $\mathcal{T}_C$: the latter is contractible, the former is not. Intuitively, $\mathcal{T}_C$ can be continuously shrunk to a point, whereas Q cannot. On the other hand, by Lemma 4.21, Q is homeomorphic to the set

$$\left\{ m \in \mu(A) \mid \frac{m_{k_1}\pi_{k_1} + \dots + m_{k_l}\pi_{k_l}}{m} = 1 \text{ and } m \geq i \right\}.$$

By Lemma 4.19 we have homeomorphisms

$$\begin{aligned} Q &\cong \left\{ \frac{m}{i} \mid m \in \mu(A), \frac{(m_{k_1}\pi_{k_1} + \dots + m_{k_l}\pi_{k_l})/i}{m/i} = 1 \right\} \\ &\cong \left\{ n \in \mu(A) \mid \frac{m_{k_1}b_{k_1} + \dots + m_{k_l}b_{k_l}}{n} = 1 \right\}. \end{aligned}$$

As a consequence, the apogee of C is not homeomorphic to $\mathcal{T}_C$, against condition (ii') in the definition of $\mathcal{B}$.

Our claim is settled.

Recalling (6.7) we have proved $\mathcal{Z}i = \mathcal{Z}f^\natural$. Since $f^\natural$ belongs to i , i is contained in the ideal $\langle f^\natural \rangle$ generated by $f^\natural$. Conversely, if $f \in i$ then $\mathcal{Z}f \supseteq \mathcal{Z}i = \mathcal{Z}f^\natural$, whence $f \in \langle f^\natural \rangle$, by (A21.24). Thus

$$\langle f^\natural \rangle = i, \tag{6.8}$$

and i is principal. □

6.3 Further Properties of Finitely Presented MV-Algebras

Let us recall from Theorem 4.16 the notation $\mathfrak{h}_x$ for the hull of x . Also, for any MV-algebra A , basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of A with multipliers $m_1, \dots, m_n$, and cluster $C = \{b_{k_1}, \dots, b_{k_l}\}$ of $\mathcal{B}$, the simplex $\mathcal{T}_C$ is defined as in (6.6).

The following corollary of Theorem 6.3 shows that bases are Schauder bases up to isomorphism:

Corollary 6.4 *Let A be an MV-algebra and $\mathcal{B} = \{b_1, \dots, b_n\}$ a basis of A with multipliers $m_1, \dots, m_n$. Let $\Delta = \{\mathcal{T}_C \subseteq [0, 1]^n \mid C \in \mathcal{B}^{\infty}\}$ and $P = \bigcup \{\mathcal{T}_C \mid C \in \mathcal{B}^{\infty}\}$. It follows that*

- (i) Δ is a regular triangulation of P .
- (ii) The map $b_i \mapsto \pi_i \restriction P$ uniquely extends to an isomorphism ω of A onto $\mathcal{M}(P)$. We have $\omega(\mathcal{B}) = \mathcal{H}_\Delta$.

- (iii) The map $v: x \in P \mapsto \mathfrak{h}_x \in \mu(\mathcal{M}(P)) \mapsto \omega^{-1}(\mathfrak{h}_x)$ is a homeomorphism of P onto the maximal spectral space of A such that, for every $C \in \mathcal{B}^{\triangleright\triangleleft}$, $v(\mathcal{T}_C)$ coincides with the apogee of C .

Proof

- (i) For the proof it is enough to check that the simplex $\mathcal{T}_C$ is regular for each $C \in \mathcal{B}^{\triangleright\triangleleft}$.
(ii) Uniqueness immediately follows because $\mathcal{B}$ generates A . To prove the existence of ω , combining Proposition 4.4(iii) with (6.5) and (6.8) we can write

$$\begin{aligned} A &\cong \mathcal{M}([0, 1]^n)/i \\ &= \mathcal{M}([0, 1]^n)/\langle f^\natural \rangle \\ &\cong \mathcal{M}(\mathcal{Z}f^\natural) \\ &= \mathcal{M}\left(\bigcup\{\mathcal{T}_C \mid C \in \mathcal{B}^{\triangleright\triangleleft}\}\right) \\ &= \mathcal{M}(P). \end{aligned}$$

- (iii) Follows from the proof of the claim in Theorem 6.3. $\square$

The following example shows that bases of $\mathcal{M}(Q)$ need not be Schauder bases up to automorphism:

Example 6.5 (L. Cabrer) Let $Q = \{(x, y) \in [0, 1]^2 \mid y = \min(x, 1 - x)\}$. Let $f = \pi_1|_Q$ and $g = 1 - \pi_1|_Q$. Then $B = \{f, g\}$ is a basis of $\mathcal{M}(Q)$: as a matter of fact, projection onto the x -axis is a $\mathbb{Z}$ -homeomorphism σ of Q onto $[0, 1]$, and the associated isomorphism ι_σ of Lemma 3.8 sends $\mathcal{M}([0, 1])$ one-one onto $\mathcal{M}(Q)$ in such a way that $\{\iota_\sigma^{-1}(f), \iota_\sigma^{-1}(g)\}$ is a Schauder basis of $\mathcal{M}([0, 1])$. However, for no automorphism α of $\mathcal{M}(Q)$ onto $\mathcal{M}(Q)$, the set $\{\alpha(f), \alpha(g)\}$ can be a Schauder basis of $\mathcal{M}(Q)$ —just because any such Schauder basis has at least three hats.

Corollary 6.6 *Let A be a finitely presented MV-algebra and B a subalgebra of A . Then B is finitely presented iff it is finitely generated.*

Proof ($\Rightarrow$) Trivially, every finitely presented MV-algebra is finitely generated.

($\Leftarrow$) In the light of Theorem 6.3 let us write $A = \mathcal{M}(P)$ for some rational polyhedron $P \subseteq [0, 1]^n$. Suppose $B \subseteq \mathcal{M}(P)$ is generated by $g_1, \dots, g_k \in \mathcal{M}(P)$. Let $g = (g_1, \dots, g_k): P \rightarrow [0, 1]^k$. Let $Q = g(P)$. By Lemma 3.6, $B \cong \mathcal{M}(Q)$. By Lemma 3.4, Q is a rational polyhedron. By Theorem 6.3, B is finitely presented. $\square$

The following result can be immediately obtained from the proof of the amalgamation property of MV-algebras, Theorem 2.20 above. However, we include here a short proof that does not make use of interpolation:

Corollary 6.7 *Finitely presented MV-algebras have the amalgamation property.*

Proof Suppose A, B, Z are finitely presented MV-algebras, with embeddings $\alpha: Z \rightarrow A$ and $\beta: Z \rightarrow B$. Theorem 6.3 yields rational polyhedra P, Q, R such that $A = \mathcal{M}(P)$, $B = \mathcal{M}(Q)$, $Z = \mathcal{M}(R)$. By Lemma 3.8(ii) we have surjective

$\mathbb{Z}$ -maps $\bar{\alpha}: P \rightarrow R$ and $\bar{\beta}: Q \rightarrow R$. By Lemma 3.5 we have a rational polyhedron D and surjective $\mathbb{Z}$ -maps $\bar{\gamma}: D \rightarrow P$ and $\bar{\delta}: D \rightarrow Q$ such that $\bar{\alpha}\bar{\gamma} = \bar{\beta}\bar{\delta}$. By Lemma 3.8(i) we have embeddings $\gamma: A \rightarrow \mathcal{M}(D)$ and $\delta: B \rightarrow \mathcal{M}(D)$. Direct inspection in the definitions of the maps of Lemma 3.8 shows that $\gamma\alpha = \delta\beta$. $\square$

Theorem 6.8 *Suppose A and A' are MV-algebras with bases $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{B}' = \{b'_1, \dots, b'_n\}$, with the same multipliers $m_1, \dots, m_n$. Suppose further*

$$\{b_{i_1}, \dots, b_{i_n}\} \in \mathcal{B}^{\bowtie} \Leftrightarrow \{b'_{i_1}, \dots, b'_{i_n}\} \in \mathcal{B}'^{\bowtie}. \quad (6.9)$$

Then there is a unique isomorphism of A onto A' extending the map $b_i \mapsto b'_i$, ($i = 1, \dots, n$).

Proof Corollary 6.4 yields rational polyhedra P and P' contained in $[0, 1]^n$ such that, without loss of generality, $A = \mathcal{M}(P)$, $A' = \mathcal{M}(P')$. Letting the regular triangulations Δ of P and Δ' of P' be defined by

$$\Delta = \{\mathcal{T}_C \mid C \in \mathcal{B}^{\bowtie}\} \quad \text{and} \quad \Delta' = \{\mathcal{T}_{C'} \mid C' \in \mathcal{B}'^{\bowtie}\},$$

we can identify each b_i with a Schauder hat of Δ , and each b'_i with a hat of Δ' , in such a way that corresponding hats have equal multipliers.

For any $C = \{b_{i_1}, \dots, b_{i_n}\} \in \mathcal{B}^{\bowtie}$ let $\eta_{\mathcal{T}_C}$ be the unique linear $\mathbb{Z}$ -homeomorphism $\eta_{\mathcal{T}_C}: \mathcal{T}_C \cong \mathcal{T}_{C'}$ of Lemma 3.14(i). By hypothesis (6.9), the map

$$\eta = \bigcup_{C \in \mathcal{B}^{\bowtie}} \eta_{\mathcal{T}_C}$$

is well defined, and by Lemma 3.14(ii) is a $\mathbb{Z}$ -homeomorphism of P onto P' . The isomorphism $\iota_\eta: \mathcal{M}(P) \cong \mathcal{M}(P')$ of Corollary 3.10 satisfies $\iota_\eta(b_i) = b'_i$. Uniqueness follows because $\mathcal{B}$ generates $\mathcal{M}(P)$. $\square$

6.4 The Characteristic Bases of Finitely Generated Free MV-Algebras

Let a, b, n be integers with $n = 1, 2, \dots$ and $0 \leq a < b \leq 2^n - 1$. We then write $a \sqsubset_n b$ iff there is $k \in \{0, \dots, n-1\}$ such that $b - a = 2^k$ and the integer part of the fraction $a/2^k$ is even. In other words, upon writing a and b in binary notation

$$a = \alpha_{n-1}\alpha_{n-2}\dots\alpha_0, \quad b = \beta_{n-1}\beta_{n-2}\dots\beta_0, \quad \alpha_i, \beta_i \in \{0, 1\}, \quad (i = 0, \dots, n-1),$$

it follows that $\alpha_k = 0$, $\beta_k = 1$ and $\alpha_j = \beta_j$ for all $j \neq k$.

Theorem 6.9 *Fix $n = 1, 2, \dots$. Then an MV-algebra A is isomorphic to FREE_n iff A has a basis $\mathcal{B} = \{b_0, \dots, b_{2^n-1}\}$ with each multiplier equal to 1, and with exactly $n!$ maximal clusters $C = \{b_{i_0}, \dots, b_{i_n}\}$, one for each sequence*

$$0 = i_0 \sqsubset_n i_1 \sqsubset_n \cdots \sqsubset_n i_n = 2^n - 1.$$

Proof Let $\{e_1, \dots, e_n\}$ be the standard basis vectors in the euclidean space $\mathbb{R}^n$. For any permutation π of the set $\{1, 2, \dots, n\}$ let the n -simplex T_π be the convex hull of the $n + 1$ points

$$0, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, e_{\pi(1)} + e_{\pi(2)} + e_{\pi(3)}, \dots, e_{\pi(1)} + e_{\pi(2)} + \cdots + e_{\pi(n)}.$$

It is not hard to verify that the family of all simplexes T_π 's together with their faces forms a regular triangulation $\mathcal{S}$ of the n -cube, called the *standard triangulation*. Equivalently, $\mathcal{S}$ can be obtained by stratification of the coordinate functions $\pi_1, \dots, \pi_n$ as in Example 2.1(i).

The Schauder basis $\mathcal{H}_\mathcal{S}$ has 2^n hats $h_0, \dots, h_{2^n-1}$, respectively at the vertices $v_0, \dots, v_{2^n-1}$ of $[0, 1]^n$. We assume the index l of h_l coincides with the integer coded by the binary string of the coordinates of v_l .

Each maximal cluster $C = \{h_{l_0}, \dots, h_{l_n}\}$ of $\mathcal{H}_\mathcal{S}$ is a set of $n + 1$ overlapping hats, whose vertices $v_{l_0}, \dots, v_{l_n}$ are the vertices of a maximal simplex S_C of $\mathcal{S}$ so that S_C coincides with T_π for a unique permutation π . All $n!$ maximal simplexes of $\mathcal{S}$ arise in this way. If we arrange the vertices of S_C in order of increasing distance from the origin, we can write

$$0 = l_0 \sqsubset_n l_1 \sqsubset_n \cdots \sqsubset_n l_n = 2^n - 1. \quad (6.10)$$

As a matter of fact, the vertices of S_C are obtainable as the nodes of a path with n steps starting from the origin and ending at $(1, 1, \dots, 1)$, where exactly one coordinate of v_{l_i} flips from 0 to 1 in the step $v_{l_i} \mapsto v_{l_{i+1}}$.

($\Rightarrow$) By (A21.48) we can write $A = \mathcal{M}([0, 1]^n)$ without loss of generality. Then an application of Theorem 5.8 shows that the Schauder basis $\mathcal{H}_\mathcal{S}$ has all the required properties.

($\Leftarrow$) By Corollary 6.4, for some rational polyhedron $Q \subseteq [0, 1]^{2^n}$ and regular triangulation Δ of Q we can safely assume $A = \mathcal{M}(Q)$ and $\mathcal{B} = \mathcal{H}_\Delta$. Let $\mathcal{H}_\Delta = \{b_0, \dots, b_{2^n-1}\}$, with corresponding vertices $\{u_0, \dots, u_{2^n-1}\}$. By assumption, these vertices and hats can be so indexed that each maximal cluster C has the form $C = \{b_{i_0}, \dots, b_{i_n}\}$ with $0 = i_0 \sqsubset_n i_1 \sqsubset_n \cdots \sqsubset_n i_n = 2^n - 1$. By definition of Schauder basis, for any such cluster C , the set $b_{i_0}^{-1}(1) \cap \cdots \cap b_{i_n}^{-1}(1)$ is an n -simplex $R_C \in \Delta$ whose vertices are $u_{i_0}, \dots, u_{i_n}$. Since all multipliers are equal to 1, $\text{den}(u_{i_0}) = \text{den}(u_{i_1}) = \cdots = \text{den}(u_{i_n}) = 1$. Then R_C is a regular and maximal simplex in Δ all of whose vertices have unit denominator.

Having preliminarily indexed each vertex v_l of $[0, 1]^n$ in such a way that l is the integer coded by the binary string of bits given by the coordinates of v_l , we obtain a one-one correspondence $\gamma: u_l \mapsto v_l$. Arguing as in the proof of Theorem 6.8, and using Lemma 3.14, for each maximal cluster C of $\mathcal{H}_\Delta$ we are given a linear $\mathbb{Z}$ -homeomorphism η_C of R_C onto the maximal simplex $S_C \in \mathcal{S}$ whose vertices have the same indexes as those of R_C . For any two maximal clusters C and D of $\mathcal{H}_\Delta$, η_C

and η_D agree on their common domain $R_C \cap R_D$, and $R_C \cap R_D$ is mapped one-one onto $S_C \cap S_D$. The map $\eta = \bigcup \{\eta_C \mid C \in \mathcal{H}_\Delta^{\infty}, C \text{ maximal}\}$ is a $\mathbb{Z}$ -homeomorphism of Q onto $[0, 1]^n$. By Theorem 3.23(i), $A = \mathcal{M}([0, 1]^n) \cong \text{FREE}_n$. $\square$

Corollary 6.10 (i) *An MV-algebra is isomorphic to FREE_1 iff it has a basis consisting of two elements of multiplicity 1 forming a cluster.*

(ii) *An MV-algebra is isomorphic to FREE_2 iff it has a basis with four elements $b_1, \dots, b_4$, each of multiplicity 1, with maximal clusters $\{b_1, b_2, b_3\}$ and $\{b_4, b_3, b_2\}$.*

6.5 Algebraic Farey Blow-Ups and De Concini–Procesi Theorem

Definition 6.11 Let $A = \Gamma(G, 1)$ be an MV-algebra with a basis $\mathcal{B} = \{b_1, \dots, b_n\}$. Let $C = \{h_1, \dots, h_q\}$ be a cluster of $\mathcal{B}$. Let $h = h_1 \wedge \dots \wedge h_q$. Then the *algebraic blow-up* $\mathcal{B}_{(C)}$ of $\mathcal{B}$ at C is defined by

$$\mathcal{B}_{(C)} = (\mathcal{B} \setminus C) \cup \{h, h_1 \odot \neg h, \dots, h_q \odot \neg h\}. \quad (6.11)$$

Equivalently, $\mathcal{B}_{(C)} = (\mathcal{B} \setminus C) \cup \{h, h_1 - h, \dots, h_q - h\}$, where “ $-$ ” denotes the subtraction operation of G .

Definition 6.12 For every basis $\mathcal{B}$ of an MV-algebra A , the *algebraic subdivision* $\mathcal{B}_{(C_1, C_2, \dots, C_z)}$ of $\mathcal{B}$ is defined by induction on z as follows:

$$\mathcal{B}_{(C_1, C_2, \dots, C_z)} = (\mathcal{B}_{(C_1, C_2, \dots, C_{z-1})})_{(C_z)}.$$

Recall the notational conventions in 4.17 and the definition of the Farey blow-up $\Delta_{(C)}$ in Sect. 5.1.

Lemma 6.13 *Suppose $\mathcal{B} = \mathcal{H}_\Delta \subseteq \mathcal{M}(P)$ is the Schauder basis of Δ , for Δ a regular triangulation of the rational polyhedron $P \subseteq [0, 1]^n$. Suppose further $C \in \mathcal{B}^{\infty}$, and c is the Farey median of the regular simplex whose vertices are the vertices of the hats in C . Then $\mathcal{B}_{(C)}$ is the Schauder basis of $\Delta_{(C)}$.*

Proof The proof amounts to verifying that $\mathcal{H}_{(\Delta_{(C)})} = (\mathcal{H}_\Delta)_{(C)}$. $\square$

Theorem 6.14 (Representation-free De Concini–Procesi theorem). *Let $A = \Gamma(G, 1)$ be an MV-algebra with its corresponding unital ℓ -group $(G, 1)$. Let $\mathcal{B}$ be a basis of A .*

- (i) *If C is a cluster of $\mathcal{B}$ then $\mathcal{B}_{(C)}$ is a basis of A .*
- (ii) *For each $b \in A$, $\mathcal{B}$ has an algebraic subdivision $\mathcal{B}' = \mathcal{B}_{(C_1, C_2, \dots, C_z)}$ such that b belongs to the submonoid of G^+ generated by $\mathcal{B}'$.*
- (iii) *Suppose $\mathcal{D}$ is another basis of A . Then $\mathcal{B}$ has an algebraic subdivision $\mathcal{B}'$ such that $\mathcal{D}$ is contained in the submonoid of G^+ generated by $\mathcal{B}'$.*

Proof (i) From Lemma 6.13 and Corollary 6.4. (ii) From Theorem 5.3 and Corollary 6.4. (iii) is an immediate consequence of (ii). $\square$

Theorem 6.15 (i) *Let A be an MV-algebra with a basis $\mathcal{B} = \{b_1, \dots, b_n\}$, and γ an arbitrary isomorphism of A onto $\mathcal{M}(Q)$ for some rational polyhedron $Q \subseteq [0, 1]^m$, $m = 1, 2, \dots$. At least one such isomorphism exists by Corollary 6.4. Then there is an algebraic subdivision $\mathcal{B}^* = \{b_1^*, \dots, b_t^*\}$ of $\mathcal{B}$ such that $\{\gamma(b_1^*), \dots, \gamma(b_t^*)\}$ is a Schauder basis of $\mathcal{M}(Q)$.*

(ii) *In particular, for each $n = 1, 2, \dots$ and rational polyhedron $P \subseteq [0, 1]^n$, every basis of $\mathcal{M}(P)$ is a Schauder basis, up to finitely many binary algebraic blow-ups.*

Proof (i) Without loss of generality we can write $A = \mathcal{M}(Q)$. For each $i = 1, \dots, m$ let $\pi_i: [0, 1]^m \rightarrow [0, 1]$ be the i th coordinate function. Corollary 6.4 yields a rational polyhedron $P \subseteq [0, 1]^n$, an isomorphism $j: \mathcal{M}(Q) \cong \mathcal{M}(P)$, and a regular triangulation Δ of P such that

$$j(b_i) = \pi_i|_P = i\text{th hat of } \mathcal{H}_\Delta.$$

Let $f_i = j(\pi_i|_Q)$. By Lemma 3.8(ii), the map $\eta: P \rightarrow Q$ given by

$$\eta(x) = (f_1(x), \dots, f_m(x)) \quad \text{for all } x \in P$$

is a $\mathbb{Z}$ -homeomorphism of P onto Q .

We must construct a regular triangulation Δ^* such that η is linear on each simplex of Δ^* . To this purpose, Theorem 5.3 yields a sequence

$$\Delta_0 = \Delta \mapsto \Delta_1 \mapsto \dots \mapsto \Delta_k = \Delta^*$$

of binary Farey blow-ups and a regular triangulation Δ^* of P such that η is linear on each simplex of Δ^* . Letting $\mathcal{H}_i$ be the Schauder basis of Δ_i , each Farey blow-up $\Delta_t \mapsto \Delta_{t+1}$ determines the *algebraic* blow-up

$$\mathcal{H}_{\Delta_t} \mapsto \mathcal{H}_{\Delta_{t+1}}.$$

Upon setting $\mathcal{B}_i = j^{-1}(\mathcal{H}_{\Delta_i})$ we have a corresponding sequence of algebraic blow-ups of bases of $\mathcal{M}(Q)$

$$\mathcal{B}_0 = \mathcal{B} \mapsto \mathcal{B}_1 \mapsto \dots \mapsto \mathcal{B}_k = \mathcal{B}^*.$$

As an isomorphic copy of a Schauder basis, each $\mathcal{B}_i$ is a basis. By construction, $\mathcal{B}^* = j^{-1}(\Delta^*)$ is an algebraic subdivision of $\mathcal{B}$. By Lemma 3.13, the set $\nabla = \{\zeta(T) \mid T \in \Delta^*\}$ is a regular triangulation of Q . It is easy to see that $\mathcal{B}^* = j^{-1}(\mathcal{H}_{\nabla^*})$ coincides with $\mathcal{H}_\nabla$.

(ii) is a particular case of (i). □

6.6 Remarks

Bases were originally defined in Marra's thesis [1] as isomorphic copies of Schauder bases. Their properties were explored in his thesis and in Manara's thesis [2].

Definition 6.1 is an equivalent simplified reformulation of [3, Definition 4.3].

Theorem 6.3 was proved in [3, Theorem 4.5] in the equivalent context of unital ℓ -groups, under the additional hypothesis that $(G, 1)$ is *archimedean*, i.e., $\Gamma(G, 1)$ is semisimple. This hypothesis was eliminated in [4].

By Corollary 6.6, finitely presented MV-algebras are preserved under *finitely generated* subalgebras; they are also preserved under *finite* products—because a disjoint union of finitely many rational polyhedra in $[0, 1]^n$ is a rational polyhedron in $[0, 1]^n$; further, they are preserved under *principal* quotients: this easily follows from Corollary 2.12. In Chap. 7 it will be shown that finitely presented MV-algebras are preserved under *finite* free products.

Theorem 6.8 was first proved in [3, Theorem 6.5] for semisimple MV-algebras and unital ℓ -groups.

Theorem 6.9 was first proved in [5]. Marra [6] proved a generalization to free MV-algebras over a finite distributive lattice. Lacava [7] gave a characterization of free generating sets in FREE_n that does not mention the universal property of free MV-algebras. For details on the standard triangulation of the n -cube see [8, 3.4.1].

Theorem 6.14 was first proved in [3, Corollary 5.4] for semisimple MV-algebras and unital ℓ -groups. Theorem 6.15(ii) is due to Marra.

While finitely presented boolean algebras are (trivially) classified by the cardinality of their maximal spectral spaces, it is not known if there exists a computable complete set of invariants for finitely presented MV-algebras. This is the first problem in the list of Sect. 20.3.

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Chapter 7

The Free Product of MV-Algebras

The free product $A \amalg B$ of two MV-algebras A and B is the most general joint embedding $A \rightarrow A \amalg B \leftarrow B$, in a sense that will be made precise in this chapter. It turns out that $A \amalg B$ is uniquely determined up to isomorphism. By setting $Z = \{0, 1\}$ in the statement of Theorem 2.20, the MV-algebra D constructed in the proof is the free product of A and B . However, in this chapter we will give a more direct construction of $A \amalg B$, that will enable us to work out many concrete computations. We will then see, e.g., that if A and B are finite then so is $A \amalg B$. On the other hand, such properties as being totally ordered, simple, or semisimple, are not preserved under free products. Finally, it will be shown that free products distribute over cartesian products.

7.1 The Construction of Free Products

The *free product* $A_1 \amalg A_2$ of two MV-algebras A_1 and A_2 is an MV-algebra $A = A_1 \amalg A_2$ together with one–one homomorphisms $\mu_i : A_i \rightarrow A$ having the following *universal property*:

$\mu_1(A_1) \cup \mu_2(A_2)$ generates A , and for any MV-algebra E and homomorphisms $\xi_i : A_i \rightarrow E$ there is a (necessarily unique) homomorphism $\xi : A \rightarrow E$ such that $\xi_1 = \xi \mu_1$ and $\xi_2 = \xi \mu_2$.

The embeddings μ_1 and μ_2 will be explicitly mentioned only when they are not evident from the context. As an immediate consequence of the definition, free products are unique up to isomorphism, and the notation $A = A_1 \amalg A_2$ will never cause confusion.

As the reader will recall, unless otherwise specified, all MV-algebras considered in this book have $0 \neq 1$.

Theorem 7.1 *Any two MV-algebras A_1 and A_2 have a free product. In more detail, let us assume X_1 and X_2 are nonempty disjoint sets, and for $i = 1, 2$, j_i is*

an ideal of FREE_{X_i} such that $A_i = \mathcal{M}([0, 1]^{X_i})/j_i = \text{FREE}_{X_i}/j_i$. Then, letting $j = \langle j_1 \cup j_2 \rangle \subseteq \mathcal{M}([0, 1]^{X_1 \cup X_2})$ it follows that

$$A_1 \sqcup A_2 = \mathcal{M}([0, 1]^{(X_1 \cup X_2)})/j = \text{FREE}_{X_1 \cup X_2}/j,$$

with embeddings $\mu_i: f/j_i \mapsto f/j$ induced by the inclusions $\text{FREE}_{X_i} \subseteq \text{FREE}_{X_1 \cup X_2}$.

Proof For each $i = 1, 2$ let us write in abbreviated form $F_i = \text{FREE}_{X_i}$, $F = \text{FREE}_{X_1 \cup X_2}$ and $A = F/j$. Since $j_i \subseteq j$, the map $\mu_i: A_i \rightarrow A$ defined by $\mu_i(f/j_i) = f/j$, ($f \in F_i$) is a well defined homomorphism.

To prove that μ_1 is one-one, suppose the McNaughton function $f \in F_1 \subseteq F$ satisfies $f/j = 0$, with the intent of showing $f/j_1 = 0$. Let us identify f with a McNaughton function over $[0, 1]^{X_1 \cup X_2}$ under the inclusion $\text{FREE}_{X_1} \subseteq \text{FREE}_{X_1 \cup X_2}$. Then the condition $f/j = 0$ is equivalent to writing $f \leq f_1 \vee f_2$ for some $f_1, f_2 \in j$ satisfying $f_1 \in j_1$ and $f_2 \in j_2$. Identifying f_2 with a McNaughton function defined on $[0, 1]^{X_2}$, there is a point $y \in [0, 1]^{X_2}$ such that $f_2(y) = 0$. Since both f and f_1 depend on (a finite subset of) variables other than those of f_2 , then necessarily $f \leq f_1$, whence $\mathcal{Z}f \supseteq \mathcal{Z}f_1 \subseteq [0, 1]^{X_1}$. By Proposition 4.4(i), $f \in \langle f_1 \rangle \subseteq j_1$, whence $f/j_1 = 0$, as required to prove that μ_1 is one-one. Similarly, μ_2 is one-one.

To prove that $\mu_1(A_1) \cup \mu_2(A_2)$ generates A , we first observe that the set $\{x/j \mid x \in X_1\} \cup \{y/j \mid y \in X_2\}$ generates A . By definition of μ_1 and μ_2 , the set $\{\mu_1(x/j_1) \mid x \in X_1\} \cup \{\mu_2(y/j_2) \mid y \in X_2\}$ generates A .

Finally, to prove the universal property of (A, μ_1, μ_2) , let E be an MV-algebra together with homomorphisms $\xi_i: A_i \rightarrow E$, for each $i = 1, 2$. Restricting, for the moment, our attention to A_1 and F_1 we have the following commutative diagram:

$$\begin{array}{ccc}
 j_1 & \xrightarrow{\text{inclusion}} & j \\
 \text{inclusion} \downarrow & & \downarrow \text{inclusion} \\
 F_1 & \xrightarrow{\text{subalgebra}} & F \\
 \text{quotient by } j_1 \downarrow & & \downarrow \text{quotient by } j \\
 A_1 & \xrightarrow{\mu_1} & A \\
 \xi_1 \downarrow & & \\
 E & &
 \end{array}$$

Let the map $\delta: X \rightarrow E$ be defined by $p \in X_1 \mapsto p/j_1 \mapsto \xi_1(p/j_1)$ and $q \in X_2 \mapsto q/j_2 \mapsto \xi_2(q/j_2)$. Since $X_1 \cap X_2 = \emptyset$, the map δ is well defined and uniquely extends to a homomorphism $\bar{\delta}: F \rightarrow E$. In particular, $\bar{\delta}(r) = \xi_1(r/j_1)$ for all $r \in F_1$, and similarly, $\bar{\delta}(s) = \xi_2(s/j_2)$ for all $s \in F_2$. The desired homomorphism

$\xi: A \rightarrow E$ is obtained by the stipulation $\xi: f/j \in F/j \mapsto \bar{\delta}(f)$. To see that ξ is well defined, recalling that $\mathbf{d}$ denotes Chang distance, suppose $f/j = f'/j$, i.e., $\mathbf{d}(f - f') \in j$. It follows that $\mathbf{d}(f - f') \leq g_1 \vee g_2$ for some $g_i \in j_i$. We then have

$$\mathbf{d}(\bar{\delta}(f) - \bar{\delta}(f')) = \bar{\delta}(\mathbf{d}(f - f')) \leq \bar{\delta}(g_1) \vee \bar{\delta}(g_2) = \xi_1(g_1/j_1) \vee \xi_2(g_2/j_2) = 0,$$

whence $\bar{\delta}(f) = \bar{\delta}(f')$, as desired. The identities $\xi_1 = \xi\mu_1$ and $\xi_2 = \xi\mu_2$ are easily verified. $\square$

7.2 Free Product Computations

Recall that for each $m = 1, 2, \dots$, L_m denotes the $(m + 1)$ -element MV-chain $\mathbb{Z}_{\frac{1}{m}} \cap [0, 1]$ with the operations inherited as a subalgebra of $[0, 1]$.

By (A21.45), any simple MV-algebra A is uniquely isomorphic to a subalgebra of the standard MV-algebra $[0, 1]$. Thus any two simple MV-algebras $A, A' \subseteq [0, 1]$ uniquely embed into a smallest subalgebra of $[0, 1]$, denoted $\text{gen}(A, A')$.

Proposition 7.2 *If A and A' are subalgebras of $[0, 1] \cap \mathbb{Q}$ then $A \amalg A' = \text{gen}(A, A')$. In particular, $A \amalg A = A = A \amalg \{0, 1\}$. Further, $L_m \amalg L_n = L_{\text{lcm}(m, n)}$.*

Proof Let $r_0 = 1/d_0, r_1 = 1/d_1, \dots$ and $r'_0 = 1/d'_0, r'_1 = 1/d'_1, \dots$ be generating sequences of A and A' , respectively. Here d_i (resp., d'_i) is the denominator of r_i (resp., of r'_i). For distinct variables $x_0, x_1, \dots$, let the functions $f_0(x_0), f_1(x_1), \dots$ be defined by

$$f_i(x_i) = ((d_i x_i - 1) \vee (1 - d_i x_i) \vee 0) \wedge 1.$$

By (A21.18), f_i is a McNaughton function. Then $\mathcal{Z}f_i = \{r_i\}$. Let ω denote the first infinite cardinal, and $r = (r_0, r_1, \dots) \in [0, 1]^\omega$. The set $\{f_0, f_1, \dots\}$ generates an ideal j of $\mathcal{M}([0, 1]^\omega)$ such that $\mathcal{Z}j$ is isomorphic to the singleton $\{r\}$. By Proposition 4.9, $j \subseteq \mathfrak{h}_r$. For the converse inclusion, assume $g \in \mathfrak{h}_r$. This means that $g(r) = 0$. Now g depends only on the finite set of variables $x_0, \dots, x_m$, and can be naturally identified with some McNaughton function of $\mathcal{M}([0, 1]^{m+1}) \subseteq \mathcal{M}([0, 1]^\omega)$. Letting $k \in \mathcal{M}([0, 1]^{m+1})$ be defined by $k(x_0, \dots, x_m) = f_0 \vee \dots \vee f_m$, it follows that $\mathcal{Z}k = \{(r_0, \dots, r_m)\} \subseteq \mathcal{Z}g$. By (A21.24), $g \in \langle k \rangle \subseteq j$, whence $j = \mathfrak{h}_r$, as desired. By Theorem 4.16, in the notation of (4.8), the quotient $\mathcal{M}([0, 1]^\omega)/j$ can be identified with the MV-algebra of restrictions to r of the functions in $\mathcal{M}([0, 1]^\omega)$,

$$A \cong \mathcal{M}(\{r\}) \cong \mathcal{M}([0, 1]^\omega)/j.$$

Similarly, using different variables $x'_0, x'_1, \dots$ to form a disjoint copy $\mathcal{M}([0, 1]^\omega)'$ of $\mathcal{M}([0, 1]^\omega)$, and letting j' be the ideal of $\mathcal{M}([0, 1]^\omega)'$ generated by the functions $f'_i(x'_i) = ((d'_i x'_i - 1) \vee (1 - d'_i x'_i) \vee 0) \wedge 1$, we obtain

$$A' \cong \mathcal{M}(\{r'\}) \cong \mathcal{M}([0, 1]^\omega)/j',$$

where $r' = (r'_0, r'_1, \dots)$.

To conclude the proof, let 2ω represent the disjoint union of two copies of ω , and $\mathbf{r} = (r_0, r_1, \dots; r'_0, r'_1, \dots) \in [0, 1]^{2\omega}$. The zeroset of the ideal

$$i = \langle \mathfrak{h}_r \cup \mathfrak{h}_{r'} \rangle \subseteq \mathcal{M}([0, 1]^{2\omega})$$

coincides with $\mathbf{r}$. Arguing as in the above proof of $j = \mathfrak{h}_r$ we now get $i = \mathfrak{h}_{\mathbf{r}}$ and

$$\mathcal{M}(\{\mathbf{r}\}) \cong \mathcal{M}([0, 1]^{2\omega})/i.$$

Again by Theorem 4.16 and (4.8), we conclude

$$A \sqcup A' \cong \frac{\mathcal{M}([0, 1]^\omega)}{j} \sqcup \frac{\mathcal{M}([0, 1]^\omega)'}{j'} \cong \frac{\mathcal{M}([0, 1]^{2\omega})}{i} = \mathcal{M}(\{\mathbf{r}\}) \cong \text{gen}(A, A').$$

The rest is clear. $\square$

In sharp contrast with Proposition 7.2 we have

Proposition 7.3 *Let A_ξ denote the MV-subalgebra of $[0, 1]$ generated by an irrational $\xi \in [0, 1]$. Then $A_\xi \sqcup A_\xi$ is not totally ordered and is not semisimple.*

Proof By Corollary 5.13, the maximal ideal $\mathfrak{h}_\xi$ of $\mathcal{M}([0, 1])$ coincides with the germinal $\mathfrak{o}_\xi$. Further,

$$A_\xi = \mathcal{M}([0, 1])/\mathfrak{o}_\xi = \mathcal{M}(\{\xi\}).$$

Let $\mathcal{M}([0, 1])'$ be a disjoint copy of $\mathcal{M}([0, 1])$ with its germinal ideal $\mathfrak{o}'_\xi$. Let

$$j = \langle \mathfrak{o}_\xi \cup \mathfrak{o}'_\xi \rangle$$

be the ideal of $\mathcal{M}([0, 1]^2)$ generated by the union of these two germinal ideals. Since all elements of j vanish on (the closure of an) open rectangular neighborhood of (ξ, ξ) , then

$$j \subseteq \mathfrak{o}_{(\xi, \xi)}.$$

For the converse inclusion, if $f \in \mathfrak{o}_{(\xi, \xi)}$ then there exists an open rectangle $[x_1, x_2] \times [y_1, y_2]$ contained in the zeroset of f and containing (ξ, ξ) . By (A21.18) there are two McNaughton functions $a(x)$ (resp., $b(y)$) whose zeroset contains ξ and is contained in $[x_1, x_2]$ (resp., is contained in $[y_1, y_2]$), and constantly takes the value 1 on $[0, 1] \setminus [x_1, x_2]$ (resp., on $[0, 1] \setminus [y_1, y_2]$). Thus $a \in \mathfrak{h}_\xi, b \in \mathfrak{h}'_\xi$ and $f \leq a \vee b \in j$. It follows that $f \in j$ and

$$A_\xi \sqcup A_\xi \cong \mathcal{M}([0, 1]^2)/j \cong \mathcal{M}([0, 1]^2)/\mathfrak{o}_{(\xi, \xi)}.$$

Now, $\mathcal{Z}j = \{(\xi, \xi)\}$, and precisely one rational line D in R^2 crosses (ξ, ξ) , namely the diagonal. The two McNaughton functions $f = x \odot \neg y$ and $g = y \odot \neg x$ satisfy $f \wedge g = 0 \in j$ but neither belongs to j . Therefore, j is not prime and $A_\xi \sqcup A_\xi$ is not totally ordered.

The non-semisimplicity of $A_\xi \sqcup A_\xi$ follows from $\mathfrak{o}_{(\xi, \xi)} \subsetneq \mathfrak{h}_{(\xi, \xi)}$ and $\mathcal{Z}\mathfrak{o}_{(\xi, \xi)} = \{(\xi, \xi)\}$. $\square$

Corollary 7.4 *Each of the following classes of MV-algebras is not preserved under free products:*

- *Totally ordered.*
- *Hyperarchimedean.*
- *Simple.*
- *Semisimple.*

7.3 Distributivity of Free Products Over Products

Lemma 7.5 *Let B and C be MV-algebras. Then there is a set Y together with ideals i and j of FREE_Y such that*

$$B \cong \frac{\text{FREE}_Y}{i}, \quad C \cong \frac{\text{FREE}_Y}{j}, \quad B \times C = \frac{\text{FREE}_Y}{i \cap j}, \quad \mathcal{Z}(i) \cap \mathcal{Z}(j) = \emptyset.$$

Proof Since every MV-algebra is a quotient of a free MV-algebra, there are disjoint sets V and W and ideals $\mathfrak{l}$ of FREE_V and $\mathfrak{k}$ of FREE_W such that $B \cong \text{FREE}_V/\mathfrak{l}$ and $C \cong \text{FREE}_W/\mathfrak{k}$. Letting Y be a suitably large set containing $V \cup W$, there certainly exist ideals i and j of FREE_Y such that $B \cong \text{FREE}_Y/i$, $C \cong \text{FREE}_Y/j$, $\mathcal{Z}(i) \cap \mathcal{Z}(j) = \emptyset$. There remains to be proved $B \times C \cong \frac{\text{FREE}_Y}{i \cap j}$. To this purpose, let the map

$$\xi: \frac{\text{FREE}_Y}{i \cap j} \rightarrow \frac{\text{FREE}_Y}{i} \times \frac{\text{FREE}_Y}{j}$$

be defined by

$$\xi \left(\frac{f}{i \cap j} \right) = \left(\frac{f}{i}, \frac{f}{j} \right), \quad \text{for all } f \in \text{FREE}_Y = \mathcal{M}([0, 1]^Y).$$

Then ξ is a well-defined homomorphism, because $f/(i \cap j) = g/(i \cap j)$ implies $f/i = g/i$ and $f/j = g/j$.

Further, ξ is one-one, because $(f/(i \cap j)) = (0, 0)$ implies $f \leq i$ and $f \leq j$ for some $i \in i$ and $j \in j$, whence $f \leq i \wedge j \in i \cap j$ and $f/(i \cap j) = 0$.

In order to prove that ξ is onto $\text{FREE}_Y/i \times \text{FREE}_Y/j$, given $f, g \in \text{FREE}_Y = \mathcal{M}([0, 1]^Y)$, Proposition 4.2 yields $p, q \in \mathcal{M}([0, 1]^Y)$ such that

$$\begin{aligned}
p &= 1 && \text{on some open set} && \supseteq \mathcal{Z}(i) \\
p &= 0 && \text{on some open set} && \supseteq \mathcal{Z}(j) \\
q &= 1 && \text{on some open set} && \supseteq \mathcal{Z}(j) \\
q &= 0 && \text{on some open set} && \supseteq \mathcal{Z}(i).
\end{aligned}$$

Let the function $l \in \mathcal{M}([0, 1]^Y)$ be defined by $l = (f \wedge p) \vee (g \wedge q)$. By (A21.24), $l/i = f/i$ and $l/j = g/j$. It follows that

$$\xi\left(\frac{l}{i \cap j}\right) = \left(\frac{f}{i}, \frac{g}{j}\right),$$

whence ξ is onto.

Having thus proved that $B \times C$ is isomorphic to $\frac{\text{FREE}_Y}{i \cap j}$, the proof is complete. $\square$

Lemma 7.6 *Suppose i and j are ideals of FREE_Y with $\mathcal{Z}(i) \cap \mathcal{Z}(j) = \emptyset$. Let $X \neq \emptyset$ be a set disjoint from Y , and $\mathfrak{k}$ be an ideal of FREE_X . Then*

$$\langle \mathfrak{k} \cup (i \cap j) \rangle = \langle \mathfrak{k} \cup i \rangle \cap \langle \mathfrak{k} \cup j \rangle.$$

Proof $(\subseteq) \langle \mathfrak{k} \cup (i \cap j) \rangle = \langle \langle \mathfrak{k} \cup i \rangle \cap \langle \mathfrak{k} \cup j \rangle \rangle \subseteq \langle \langle \mathfrak{k} \cup i \rangle \cap \langle \mathfrak{k} \cup j \rangle \rangle = \langle \mathfrak{k} \cup i \rangle \cap \langle \mathfrak{k} \cup j \rangle$.

$(\supseteq)$ Suppose $f \in \langle \mathfrak{k} \cup i \rangle \cap \langle \mathfrak{k} \cup j \rangle$. Then there are $a \in \mathfrak{k}$, $b \in i$ and $c \in j$ such that $f \leq a \oplus b$ and $f \leq a \oplus c$, whence $f \leq a \oplus (b \wedge c)$. Now note that $b \wedge c$ belongs to $i \cap j$. It follows that for some $a \in \mathfrak{k}$ and $d \in i \cap j$, $f \leq a \oplus d$. In other words, f belongs to $\langle \mathfrak{k} \cup (i \cap j) \rangle$. $\square$

Theorem 7.7 *For any MV-algebras A, B, C ,*

$$A \amalg (B \times C) \cong (A \amalg B) \times (A \amalg C). \quad (7.1)$$

Proof By Lemma 7.5 there are suitably large disjoint sets X and Y and ideals $i, j, \mathfrak{k}$ such that $\mathcal{Z}(i) \cap \mathcal{Z}(j) = \emptyset$ and

$$A \cong \frac{\text{FREE}_X}{\mathfrak{k}}, \quad B \cong \frac{\text{FREE}_Y}{i}, \quad C \cong \frac{\text{FREE}_Y}{j}, \quad B \times C \cong \frac{\text{FREE}_Y}{i \cap j}.$$

By the construction of Theorem 7.1,

$$A \amalg (B \times C) = \frac{\text{FREE}_{X \cup Y}}{\langle \mathfrak{k} \cup (i \cap j) \rangle}.$$

By Lemma 7.6,

$$(A \amalg B) \times (A \amalg C) \cong \frac{\text{FREE}_{X \cup Y}}{\langle \mathfrak{k} \cup i \rangle} \times \frac{\text{FREE}_{X \cup Y}}{\langle \mathfrak{k} \cup j \rangle} \cong \frac{\text{FREE}_{X \cup Y}}{\langle \mathfrak{k} \cup i \rangle \cap \langle \mathfrak{k} \cup j \rangle}.$$

$\square$

Corollary 7.8 Let $m(1), \dots, m(h), n(1), \dots, n(k)$ be integers > 0 . Then

$$(L_{m(1)} \times \cdots \times L_{m(h)}) \amalg (L_{n(1)} \times \cdots \times L_{n(k)}) = \prod_{i=1}^h \prod_{j=1}^k L_{\text{lcm}(m(i), n(j))}.$$

Proof A combination of Theorem 7.7 and Proposition 7.2. $\square$

Corollary 7.9 Each of the following classes of MV-algebras is preserved under (finite) free products:

- (i) *Finitely generated.*
- (ii) *Free.*
- (iii) *Finite.*
- (iv) *Finitely presented: specifically, writing $A_i = \mathcal{M}(P_i)$, for suitable rational polyhedra P_i ($i = 1, 2$), it follows that $A_1 \amalg A_2$ coincides with the finitely presented MV-algebra $\mathcal{M}(P_1 \times P_2)$.*

Proof (i) is trivial. For (ii), given two disjoint sets X and Y , we immediately have $\text{FREE}_X \amalg \text{FREE}_Y = \text{FREE}_{X \cup Y}$. (iii) is proved in Corollary 7.8.

(iv) Let $P_1 \subseteq [0, 1]^{\{x_1, \dots, x_n\}}$ and $P_2 \subseteq [0, 1]^{\{x_{n+1}, \dots, x_{n+m}\}}$, so that the cartesian product $P = P_1 \times P_2$ is a rational polyhedron in $[0, 1]^{n+m}$, by Corollary 2.11. By Proposition 4.4(iii), $\mathcal{M}(P_1)$ can be identified with the quotient $\mathcal{M}([0, 1]^n)/j_1$, where the ideal $j_1 = \{f \in \mathcal{M}([0, 1]^n) \mid f \upharpoonright P_1 = 0\}$ is principal, say $j_1 = \langle f_1 \rangle$ for some McNaughton function vanishing over P_1 . Similarly, for some McNaughton function f_2 vanishing on P_2 , letting $j_2 = \langle f_2 \rangle$ be the principal ideal of all McNaughton functions $f(x_{n+1}, \dots, x_{n+m})$ that vanish on P_2 , we can identify $\mathcal{M}(P_2)$ with the quotient by j_2 of the free MV-algebra $\mathcal{M}([0, 1]^m) = \text{FREE}_{\{x_{n+1}, \dots, x_{n+m}\}}$. The ideal j generated in $\mathcal{M}([0, 1]^{n+m})$ by $f_1 \vee f_2$ is principal, where each f_i is naturally identified with a McNaughton function on the cube $[0, 1]^{n+m}$. By Theorem 7.1,

$$A_1 \amalg A_2 = \mathcal{M}([0, 1]^{n+m})/j = \mathcal{M}(\mathcal{Z}(f_1 \vee f_2)).$$

It follows that $\mathcal{Z}(f_1 \vee f_2) = \mathcal{Z}f_1 \times \mathcal{Z}f_2 = P_1 \times P_2 = P$, whence $A_1 \amalg A_2$ is finitely presented. $\square$

7.4 Remarks

For our terminology in this chapter we have followed [1, II, p. 85]. If in the definition of free product one omits the requirement that the homomorphisms μ_i are one-one, then one gets the definition of coproduct. As shown, e.g., in [1, II, 2.11], coproducts exist in any variety of algebras.

Essentially the same proof of Theorem 7.1 yields (see [2, 1.1] for details):

For any family $\{A_i \mid i \in I\}$ of MV-algebras there is an MV-algebra A together with embeddings $\mu_i: A_i \rightarrow A$ such that A is generated by the union of the $\mu_i(A_i)$,

and for any MV-algebra E and homomorphisms $\xi_i: A_i \rightarrow E$ there is a (necessarily unique) homomorphism $\xi: A \rightarrow E$ such that $\xi_i = \xi \mu_i$ for all $i \in I$.

In Proposition 8.11 we will prove that free products of locally finite MV-algebras are locally finite. In Proposition 17.3 free products of projective MV-algebras will be shown to be projective. A moment's reflection shows that (finite, as well as infinite) free products of free MV-algebras are free.

Using the Γ functor, further computations of MV-algebraic free products are carried over in [2].

References

1. Jacobson, N. (1989). *Basic Algebra*, Vol. II, 2nd edn. New York: Freeman.
2. Mundici, D. (1988). Free products in the category of abelian ℓ -groups with strong unit. *Journal of Algebra*, 113, 89–109.

Chapter 8

Direct Limits, Confluence and Multisets

In this chapter we will study the direct limit $A = \varinjlim A_i$ of a direct system of MV-algebras A_i . Every MV-algebra is the direct limit of its finitely generated subalgebras. Every finitely generated MV-algebra is the direct limit of a sequence of finitely presented MV-algebras. We will prove that confluence is necessary and sufficient for the direct limits of any two such sequences to be isomorphic. While sufficiency is routinely checked, the necessity of confluence critically relies on the polyhedral theory developed in earlier sections.

We will then go on with direct limits of finite MV-algebras, i.e., locally finite MV-algebras. Equivalently, an MV-algebra A is locally finite if every finite subset of A generates a finite subalgebra of A . We will discuss the natural correspondence between locally finite MV-algebras and multisets, generalizing the correspondence between finite MV-algebras and finite multisets. Knowledge of the rudiments of category theory is useful for a full understanding of this chapter.

8.1 Preliminaries on Direct Limits of MV-Algebras

Let $(I, \leq)$ be a nonempty partially ordered set such that any two elements of I have a common upper bound. Let $\mathcal{D} = \{(A_i, \eta_{ij}) \mid i, j \in I, i \leq j\}$ be a family of MV-algebras A_i together with homomorphisms $\eta_{ij}: A_i \rightarrow A_j$ satisfying the following conditions:

- (i) η_{ii} is the identity map on A_i ;
- (ii) $\eta_{ik} = \eta_{jk} \eta_{ij}$ whenever $i \leq j \leq k \in I$.

We then say that $\mathcal{D}$ is a *direct system* of MV-algebras. On the disjoint union $\bigsqcup_i A_i$ of the A_i we define the equivalence relation $\bar{\sim}$ by stipulating that, for arbitrary $a_i \in A_i$ and $b_j \in A_j$,

$$a_i \bar{\sim} b_j \quad \text{iff} \quad \eta_{ik}(a_i) = c_k = \eta_{jk}(b_j) \text{ for some } c_k \in A_k \text{ with } k \geq i, j.$$

We then let A_∞ be the set of equivalence classes

$$A_\infty = \left(\bigsqcup_i A_i \right) / \bar{\sim}$$

and we denote by $\eta_{i\infty}: A_i \rightarrow A_\infty$ the *telescopic* (also known as the “canonical” or “colimit”) map sending each $a \in A_i$ to its equivalence class $a/\bar{\sim}$. The telescopic maps naturally make A_∞ into an MV-algebra, also denoted A_∞ , in such a way that each $\eta_{i\infty}$ turns out to be a homomorphism. We then say that A_∞ is *direct limit* of the system $\mathcal{D}$, and we write

$$A_\infty = \varinjlim \{(A_i, \eta_{ij}) \mid i, j \in I, i \leq j\}.$$

In category theory, direct limits are also known as filtered colimits. For the sake of brevity, the telescopic maps $\eta_{i\infty}$ are tacitly included in the definition of direct limit, just as the projection maps are built in the definition of cartesian product.

The following are routine exercises:

Lemma 8.1 *Each MV-algebra B is the direct limit of the direct system of its finitely generated subalgebras B_i , where each homomorphism $\epsilon_{ij}: B_i \rightarrow B_j$ is the inclusion of B_i in B_j , whenever B_i is a subalgebra of B_j .*

Lemma 8.2 *Fix $n = 1, 2, \dots$. For each sequence $\mathcal{P} = \{P_i \subseteq [0, 1]^n \mid i = 1, 2, \dots\}$ of nonempty rational polyhedra $P_1 \supseteq P_2 \supseteq P_3 \supseteq \dots$, generalizing the notation of Sect. 4.3, let us write*

$$\mathfrak{h}_{\mathcal{P}} = \{f \in \mathcal{M}([0, 1]^n) \mid \mathcal{Z}(f) \supseteq P_i \text{ for some } i = 1, 2, \dots\}.$$

Then $\mathfrak{h}_{\mathcal{P}}$ is an ideal of $\mathcal{M}([0, 1]^n)$. Further, every ideal of $\mathcal{M}([0, 1]^n)$ has this form.

Proof The first statement is easily verified. The second follows from Proposition 4.4(iv), because rational polyhedra are a basis of closed sets for the topology of $[0, 1]^n$. $\square$

Lemma 8.3 *Every n -generated MV-algebra A is the direct limit of a countable direct system of finitely presented MV-algebras $A_i \cong \mathcal{M}([0, 1]^n)/\mathfrak{j}_i$, ($i = 1, 2, \dots$), with each $\mathfrak{j}_i$ a principal ideal, and surjective connecting homomorphisms. Conversely, the direct limit of any such sequence is a finitely generated MV-algebra.*

Proof For some ideal $\mathfrak{i}$ of $\mathcal{M}([0, 1]^n)$, A is isomorphic to $\mathcal{M}([0, 1]^n)/\mathfrak{i}$. Identifying A with $\mathcal{M}([0, 1]^n)/\mathfrak{i}$, the zerosets of the McNaughton functions in $\mathfrak{i}$ form a sequence $\mathcal{Q}$ of nonempty rational polyhedra in $[0, 1]^n$. There is a decreasing sequence $\mathcal{P} = \{P_i \mid i = 1, 2, \dots, P_i \supseteq P_{i+1}\}$ such that any $Q \in \mathcal{Q}$ contains some $P \in \mathcal{P}$. By (A21.24) and Lemma 8.2, $\mathfrak{i} = \mathfrak{h}_{\mathcal{P}}$. Letting now

$$j_i = \mathfrak{h}_{P_i} = \{f \in \mathcal{M}([0, 1]^n) \mid \mathcal{Z}(f) \supseteq P_i\},$$

a direct inspection shows that $A = \lim \mathcal{M}([0, 1]^n)/j_i$. The converse is trivial. $\square$

In what follows, the symbol $\twoheadrightarrow$ denotes a surjective homomorphism.

Definition 8.4 Two sequences of MV-algebras

$$A_0 \twoheadrightarrow A_1 \twoheadrightarrow \cdots \quad \text{and} \quad B_0 \twoheadrightarrow B_1 \twoheadrightarrow \cdots \quad (8.1)$$

are said to be *confluent* if there are indices $i(1) < j(1) < i(2) < j(2) < \cdots$ and surjective homomorphisms

$$\eta_{i(k)}: A_{i(k)} \twoheadrightarrow B_{j(k)} \quad \text{and} \quad \theta_{j(k)}: B_{j(k)} \twoheadrightarrow A_{i(k+1)}$$

such that the composite map $\theta_{j(k)}\eta_{i(k)}$ coincides with the map $A_{i(k)} \twoheadrightarrow A_{i(k+1)}$ and conversely, $\eta_{i(k+1)}\theta_{j(k)}$ coincides with $B_{j(k)} \twoheadrightarrow B_{j(k+1)}$.

The proof of the following result is a routine exercise on direct limits:

Lemma 8.5 *Given direct systems $\mathcal{D}$ and $\mathcal{E}$ of finitely presented MV-algebras with surjective connecting homomorphisms*

$$\mathcal{D} = A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \dots \quad \text{and} \quad \mathcal{E} = B_0 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} B_2 \dots,$$

let A and B denote their respective direct limits. If $\mathcal{D}$ and $\mathcal{E}$ are confluent then $A \cong B$.

8.2 Finitely Generated MV-Algebras and Confluence

By Lemma 8.5, the confluence of two sequences is *sufficient* for their direct limits to be isomorphic. While in general, confluence is not a necessary condition for direct limits to be isomorphic, in Corollary 8.8 it is proved that any two sequences of finitely presented MV-algebras and surjective homomorphisms having isomorphic direct limits are necessarily confluent.

Let $0 < n \in \mathbb{Z}$. For $P \subseteq [0, 1]^n$ a rational polyhedron and $\mathfrak{i}$ an ideal of $\mathcal{M}(P)$ we write $\mathfrak{Z}(\mathfrak{i})$ for the family of zerosets of the functions in $\mathfrak{i}$,

$$\mathfrak{Z}(\mathfrak{i}) = \{\mathcal{Z}(f) \mid f \in \mathfrak{i}\}.$$

Lemma 8.6 *Let $\mathfrak{i}$ be an ideal of $\mathcal{M}([0, 1]^n)$ and $P \in \mathfrak{Z}(\mathfrak{i})$. Observe that the set $\mathfrak{i} \upharpoonright P = \{f \upharpoonright P \mid f \in \mathfrak{i}\}$ is an ideal of $\mathcal{M}(P)$. Now let us introduce the notation*

$$\mathfrak{Z}(\mathfrak{i} \upharpoonright P) = \{\mathcal{Z}(f \upharpoonright P) \mid f \in \mathfrak{i}\} \quad \text{and} \quad \mathfrak{Z}_P(\mathfrak{i}) = \{X \cap P \mid X \in \mathfrak{Z}(\mathfrak{i})\}.$$

Then $\mathfrak{Z}(\mathfrak{i} \upharpoonright P) = \mathfrak{Z}_P(\mathfrak{i})$.

Proof An immediate consequence of the definitions. $\square$

Isomorphic finitely generated MV-algebras have the following geometric characterization:

Theorem 8.7 *For any integers $m, n > 0$ and ideals $\mathfrak{p}$ of $\mathcal{M}([0, 1]^m)$ and $\mathfrak{q}$ of $\mathcal{M}([0, 1]^n)$ the following conditions are equivalent:*

- (i) $\mathcal{M}([0, 1]^m)/\mathfrak{p}$ is isomorphic to $\mathcal{M}([0, 1]^n)/\mathfrak{q}$.
- (ii) *For some rational polyhedra $P \in \mathfrak{Z}(\mathfrak{p})$, $Q \in \mathfrak{Z}(\mathfrak{q})$ and $\mathbb{Z}$ -homeomorphism η of P onto Q , the map $X \mapsto \eta(X)$ is a one-one correspondence between $\mathfrak{Z}_P(\mathfrak{p})$ and $\mathfrak{Z}_Q(\mathfrak{q})$.*

Proof (ii $\Rightarrow$ i) As above, let $\mathfrak{h}_P$ denote the hull of P , i.e., the set of all functions of $\mathcal{M}([0, 1]^m)$ identically vanishing on P . Then $\mathfrak{h}_P$ is an ideal of $\mathcal{M}([0, 1]^m)$. Similarly, the hull $\mathfrak{h}_Q$ of Q is an ideal of $\mathcal{M}([0, 1]^n)$. By (A21.24) we have the inclusions

$$\mathfrak{p} \supseteq \mathfrak{h}_P \quad \text{and} \quad \mathfrak{q} \supseteq \mathfrak{h}_Q. \quad (8.2)$$

By Corollary 2.10, both $\mathfrak{h}_P$ and $\mathfrak{h}_Q$ are principal. Proposition 4.4(iii) yields isomorphisms

$$\iota_1: \mathcal{M}(P) \cong \mathcal{M}([0, 1]^m)/\mathfrak{h}_P, \quad \text{where } \iota_1(\mathfrak{p} \upharpoonright P) = \mathfrak{p}/\mathfrak{h}_P = \{f/\mathfrak{h}_P \mid f \in \mathfrak{p}\}$$

and

$$\iota_2: \mathcal{M}(Q) \cong \mathcal{M}([0, 1]^n)/\mathfrak{h}_Q, \quad \text{where } \iota_2(\mathfrak{q} \upharpoonright Q) = \mathfrak{q}/\mathfrak{h}_Q = \{g/\mathfrak{h}_Q \mid g \in \mathfrak{q}\}.$$

From (8.2) it follows that the map

$$\frac{f/\mathfrak{h}_P}{\mathfrak{p}/\mathfrak{h}_P} \mapsto \frac{f}{\mathfrak{p}}$$

is an isomorphism of $\frac{\mathcal{M}([0, 1]^m)/\mathfrak{h}_P}{\mathfrak{p}/\mathfrak{h}_P}$ onto $\frac{\mathcal{M}([0, 1]^m)}{\mathfrak{p}}$. We then have isomorphisms

$$\frac{\mathcal{M}([0, 1]^m)}{\mathfrak{p}} \cong \frac{\mathcal{M}([0, 1]^m)/\mathfrak{h}_P}{\mathfrak{p}/\mathfrak{h}_P} \cong \frac{\mathcal{M}(P)}{\mathfrak{p} \upharpoonright P} \quad (8.3)$$

and

$$\frac{\mathcal{M}([0, 1]^n)}{\mathfrak{q}} \cong \frac{\mathcal{M}([0, 1]^n)/\mathfrak{h}_Q}{\mathfrak{q}/\mathfrak{h}_Q} \cong \frac{\mathcal{M}(Q)}{\mathfrak{q} \upharpoonright Q}. \quad (8.4)$$

The map

$$J: k \in \mathcal{M}(P) \mapsto k\eta^{-1} \in \mathcal{M}(Q)$$

is an isomorphism of $\mathcal{M}(P)$ onto $\mathcal{M}(Q)$. By Lemma 8.6, the map $Y \mapsto \eta^{-1}(Y)$ sends $\mathfrak{Z}_Q(\mathfrak{q}) = \mathfrak{Z}(\mathfrak{q} \upharpoonright Q)$ one-one onto $\mathfrak{Z}_P(\mathfrak{p}) = \mathfrak{Z}(\mathfrak{p} \upharpoonright P)$. Similarly, for each

$l \in \mathcal{M}(P)$ if $l \in \mathfrak{p} \upharpoonright P$ then $\mathcal{Z}(l) \in \mathfrak{Z}(\mathfrak{p} \upharpoonright P) = \mathfrak{Z}_P(\mathfrak{p})$. Thus by definition of J , $\mathcal{Z}(J(l)) = \mathcal{Z}(l\eta^{-1}) = \eta(\mathcal{Z}(l)) \in \mathfrak{Z}_Q(q)$. By Proposition 4.5, $J(l) \in \mathfrak{q} \upharpoonright Q$. Arguing in a similar way for J^{-1} , it follows that J maps $\mathfrak{p} \upharpoonright P$ one-one onto $\mathfrak{q} \upharpoonright Q$. Thus the map

$$\frac{f}{\mathfrak{p} \upharpoonright P} \mapsto \frac{J(f)}{J(\mathfrak{p} \upharpoonright P)} \quad \text{for each } f \in \mathcal{M}(P)$$

is an isomorphism of $\mathcal{M}(P)/(\mathfrak{p} \upharpoonright P)$ onto $\mathcal{M}(Q)/(\mathfrak{q} \upharpoonright Q)$. Recalling now (8.3–8.4), we conclude

$$\frac{\mathcal{M}([0, 1]^m)}{\mathfrak{p}} \cong \frac{\mathcal{M}(P)}{\mathfrak{p} \upharpoonright P} \cong \frac{\mathcal{M}(Q)}{\mathfrak{q} \upharpoonright Q} \cong \frac{\mathcal{M}([0, 1]^n)}{\mathfrak{q}},$$

which settles (ii $\Rightarrow$ i).

(i $\Rightarrow$ ii) We have an isomorphism $\iota: \mathcal{M}([0, 1]^m)/\mathfrak{p} \cong \mathcal{M}([0, 1]^n)/\mathfrak{q}$. Let π' denote the identity $(\pi_1, \dots, \pi_m)$ on the m -cube, and $\pi^\wedge$ the identity on the n -cube. Each element $\pi_i/\mathfrak{p} \in \mathcal{M}([0, 1]^m)/\mathfrak{p}$ is sent by ι to some element $q_i/\mathfrak{q}$ of $\mathcal{M}([0, 1]^n)/\mathfrak{q}$. Thus for a suitable m -tuple $q = (q_1, \dots, q_m)$ of functions $q_i \in \mathcal{M}([0, 1]^n)$ we have a map $q: [0, 1]^n \rightarrow [0, 1]^m$. Similarly, for some map $p = (p_1, \dots, p_n): [0, 1]^m \rightarrow [0, 1]^n$ with $p_1, \dots, p_n \in \mathcal{M}([0, 1]^m)$, we can write

$$\iota: \pi'/\mathfrak{p} \mapsto q/\mathfrak{q} \quad \text{and} \quad \iota^{-1}: \pi^\wedge/\mathfrak{q} \mapsto p/\mathfrak{p}. \quad (8.5)$$

For any $f \in \mathcal{M}([0, 1]^m)$ and $g \in \mathcal{M}([0, 1]^n)$, arguing by induction on the number of operations in f and g , and using McNaughton theorem we obtain the following generalization of (8.5):

$$\iota: \frac{f}{\mathfrak{p}} \mapsto \frac{fq}{\mathfrak{q}} \quad \text{and} \quad \iota^{-1}: \frac{g}{\mathfrak{q}} \mapsto \frac{gp}{\mathfrak{p}}, \quad (8.6)$$

where fq denotes the composition of f and q . As a consequence,

$$\frac{f}{\mathfrak{p}} = \iota^{-1} \left(\iota \left(\frac{f}{\mathfrak{p}} \right) \right) = \iota^{-1} \left(\frac{fq}{\mathfrak{q}} \right) = \frac{fq p}{\mathfrak{p}}.$$

By (A21.5), for each $i = 1, \dots, m$ the element $\mathbf{d}(\pi_i, q_i p) = \mathbf{d}(\pi_i, \pi_i q p)$ belongs to $\mathfrak{p}$. Let $s \in \mathfrak{p}$ be defined by $s = \mathbf{d}(\pi_1, q_1 p) \oplus \dots \oplus \mathbf{d}(\pi_m, q_m p)$. The rational polyhedron $P = \mathcal{Z}(s) \in \mathfrak{Z}(\mathfrak{p})$ coincides with the set $\{x \in [0, 1]^m \mid q(p(x)) = x\}$. Similarly, the set $Q = \{y \in [0, 1]^n \mid p(q(y)) = y\}$ belongs to $\mathfrak{Z}(\mathfrak{q})$. Let η be the restriction of p to P . Then η is a $\mathbb{Z}$ -homeomorphism of P onto Q , whose inverse η^{-1} is the restriction of q to Q . We now assume $X \in \mathfrak{Z}_P(\mathfrak{p})$, with the intent of proving $\eta(X) \in \mathfrak{Z}_Q(\mathfrak{q})$. By Lemma 8.6, $X = \mathcal{Z}(l \upharpoonright P)$ for some McNaughton function $l \in \mathfrak{p}$. By (8.6), $lq \in \mathfrak{q}$. Thus,

$$\eta(X) = \eta(\mathcal{Z}(l \upharpoonright P)) = \mathcal{Z}((l \upharpoonright P)\eta^{-1}) = \mathcal{Z}(lq \upharpoonright Q) = Q \cap \mathcal{Z}(lq) \in \mathfrak{Z}_Q(\mathfrak{q}).$$

Reversing the roles of η and η^{-1} we finally obtain the required one–one correspondence $X \mapsto \eta(X)$ between $\mathfrak{Z}_P(\mathfrak{p})$ and $\mathfrak{Z}_Q(\mathfrak{q})$. $\square$

Corollary 8.8 *Let $\mathcal{D} = A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \dots$ and $\mathcal{E} = B_0 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} B_2 \dots$, be direct systems of finitely presented MV-algebras with surjective homomorphisms. Let A and B denote their respective direct limits. Then the following conditions are equivalent:*

- (i) $A \cong B$.
- (ii) $\mathcal{D}$ and $\mathcal{E}$ are confluent.

Proof For (ii $\rightarrow$ i) see Lemma 8.5. (i $\rightarrow$ ii) By Theorem 6.3 there exist rational polyhedra $P_0, P_1, \dots, (i = 0, 1, 2, \dots)$ such that $\mathcal{M}(P_i) \cong A_i$. Let the $\mathbb{Z}$ -homeomorphism $\zeta_i: P_i \cong_{\mathbb{Z}} \zeta_i(P_i) \subseteq P_{i-1}$ be as given by Lemma 3.12. Let $\mathcal{P} = P'_0 \supseteq P'_1 \supseteq P'_2 \supseteq \dots$, where $P'_0 = P_0 \subseteq [0, 1]^m$ and $P'_i = \zeta_1 \dots \zeta_i(P_i)$, ($i = 1, 2, \dots$). By Corollary 3.10,

$$A_i \cong \mathcal{M}(P_i) \cong \mathcal{M}(P'_i) \quad \text{and} \quad A \cong \mathcal{M}([0, 1]^m)/\mathfrak{h}_{\mathcal{P}}. \quad (8.7)$$

Arguing in the same way for $\mathcal{E}$ we obtain a sequence

$$[0, 1]^n \supseteq Q_0 \xleftarrow{\epsilon_1} Q_1 \xleftarrow{\epsilon_2} Q_2 \xleftarrow{\epsilon_3} \dots,$$

where for each i , $B_i \cong \mathcal{M}(Q_i)$ and ϵ_i is a $\mathbb{Z}$ -homeomorphism of Q_i onto $\epsilon_i(Q_i) \subseteq Q_{i-1}$. Let $\mathcal{Q} = Q'_0 \supseteq Q'_1 \supseteq Q'_2 \supseteq \dots$, where it is assumed that $Q'_0 = Q_0$ and $Q'_i = \epsilon_1 \dots \epsilon_i(Q_i)$ for each $i = 1, 2, \dots$. It follows that

$$B_i \cong \mathcal{M}(Q_i) \cong \mathcal{M}(Q'_i) \quad \text{and} \quad B \cong \mathcal{M}([0, 1]^n)/\mathfrak{h}_{\mathcal{Q}}. \quad (8.8)$$

By hypothesis,

$$\mathcal{M}([0, 1]^m)/\mathfrak{h}_{\mathcal{P}} \cong A \cong B \cong \mathcal{M}([0, 1]^n)/\mathfrak{h}_{\mathcal{Q}}.$$

By Theorem 8.7, there exist $P \in \mathfrak{h}_{\mathcal{P}}$, $Q \in \mathfrak{h}_{\mathcal{Q}}$ and a $\mathbb{Z}$ -homeomorphism $\eta: P \cong_{\mathbb{Z}} Q$ sending $\mathfrak{Z}_P(\mathfrak{h}_{\mathcal{P}})$ one–one onto $\mathfrak{Z}_Q(\mathfrak{h}_{\mathcal{Q}})$. By definition of $\mathfrak{h}_{\mathcal{P}}$ and $\mathfrak{h}_{\mathcal{Q}}$, there exist rational polyhedra P'_k and Q'_l such that $P'_k \subseteq P$ and $Q'_l \subseteq Q$. Thus, for each $i \geq k$ there exists i' such that $\eta^{-1}(Q'_{i'}) \subseteq P'_i$.

Interchanging the roles of η and η^{-1} , it follows that for each $j \geq l$ there exists j' such that $\eta(P'_{j'}) \subseteq Q'_j$. In conclusion, there are indices

$$i(1) < j(1) < i(2) < j(2) < \dots$$

such that $\eta(P'_{i(k)}) \subseteq Q'_{j(k)}$ and $\eta^{-1}(Q'_{j(k)}) \subseteq P'_{i(k+1)}$ for each $k = 1, 2, \dots$. The desired confluence property now follows combining (8.7) and (8.8) with Lemma 3.11. $\square$

8.3 Locally Finite MV-Algebras

An MV-algebra is *locally finite* iff all its finitely generated subalgebras are finite.

For any topological space $X \neq \emptyset$ we denote by $C_{\mathbb{Q}}(X)$ the MV-subalgebra of $C(X)$ consisting of all functions having rational range.

Further, we denote by $B(A)$ the *center* of A , i.e., the set of all elements $a \in A$ satisfying $a \oplus a = a$.

By a *boolean space*, we mean a totally disconnected compact Hausdorff space.

Theorem 8.9 *For every MV-algebra A the following conditions are equivalent:*

- (i) A is locally finite.
- (ii) For each prime ideal $\mathfrak{p}$ of A , $A/\mathfrak{p}$ is isomorphic to a subalgebra of $\mathbb{Q} \cap [0, 1]$.
- (iii) $\mu(A)$ is a boolean space and the correspondence $b \mapsto b^*$ of Theorem 4.16 is an isomorphism of A onto a separating subalgebra of $C_{\mathbb{Q}}(\mu(A))$, such that each b^* has a finite range.
- (iv) For some boolean space Y , A is isomorphic to a separating subalgebra of the MV-algebra $C_{\mathbb{Q}}(Y)$ consisting of functions of finite range.
- (v) For some set J , there is an isomorphism ι of A onto a subalgebra $\iota(A)$ of the MV-algebra $(\mathbb{Q} \cap [0, 1])^J$, where each $f \in \iota(A)$ has a finite range.
- (vi) A is the direct limit of a direct system $\{(A_{ij}, \iota_{ij}) \mid i, j \in J, i \leq j\}$ of finite MV-algebras with injective homomorphisms $\iota_{ij}: A_i \rightarrow A_j$.

Proof The equivalence (i $\Leftrightarrow$ vi) immediately follows from Lemma 8.1.

(i $\Rightarrow$ ii). Let $a \in A$. Then for every integer $m = 1, 2, \dots$, the element $m \cdot a$ belongs to the subalgebra $\text{gen}(a)$ of A generated by a . (Notation of (2.8)). Since $\text{gen}(a)$ is finite and $a \leq 2 \cdot a \leq 3 \cdot a \leq \dots$, then necessarily for some integer $n \geq 1$ we have $n \cdot a = (n + 1) \cdot a$. By (A21.35) and (A21.36), A is hyperarchimedean, and all its prime ideals are maximal. By Theorem 4.16, for each prime ideal $\mathfrak{p}$ of A there is a unique embedding $\eta: A/\mathfrak{p} \rightarrow [0, 1]$. For any $c \in A$, let $\text{gen}(c)$ be the subalgebra of A generated by c . Then $\eta(\text{gen}(c))$ is a finite subalgebra of $[0, 1]$, whence, by (A21.28), $\eta(c) \in \eta(\text{gen}(c)) \subseteq \mathbb{Q} \cap [0, 1]$.

(ii $\Rightarrow$ iii). From (A21.7) it follows that every prime ideal of A is maximal. By (A21.36) and (A21.37), A is hyperarchimedean and $\mu(A)$ is a boolean space. The map $*$ of Theorem 4.16 embeds A onto a separating subalgebra A^* of $C(\mu(A))$, in such a way that for each $\mathfrak{m} \in \mu(A)$ the quotient algebra $A/\mathfrak{m}$ is uniquely isomorphic to a subalgebra of $\mathbb{Q} \cap [0, 1]$, and $a^*(\mathfrak{m}) = a/\mathfrak{m} \in \mathbb{Q} \cap [0, 1]$. Thus for each $a \in A$ the range of a^* is a subset of $\mathbb{Q} \cap [0, 1]$.

We *claim* that a^* has a finite range.

By way of contradiction, let us suppose that a^* has infinite range R . Let $r \in [0, 1]$ be an accumulation point of R . Since R is closed in $[0, 1]$, r is an element of R , and r is rational. By (A21.18) there is a McNaughton function $f \in \mathcal{M}([0, 1])$ such that $f(x) = 1 \Leftrightarrow x = r$. Then the composite map $b^* = fa^*$ belongs to A^* , and 0 is an accumulation point of its range S . As a consequence, for no $m = 1, 2, \dots$ the element $m \cdot b^*$ belongs to the center of A^* , and $A \cong A^*$ is not hyperarchimedean.

Having thus reached a contradiction we have settled our claim, and completed the proof of (ii $\Rightarrow$ iii).

The implications (iii $\Rightarrow$ iv) and (iv $\Rightarrow$ v) are trivial.

(v $\Rightarrow$ i). Without loss of generality, ι is the inclusion map, $A = \iota(A)$. By hypothesis, for every finite subset $\{f_1, \dots, f_n\}$ of A there is a finite partition of J , say $J = I_1 \cup \dots \cup I_k$, such that every function f_i is constant on each I_j . Thus for each $1 \leq i \leq n$ and $1 \leq j \leq k$ there is a rational $\frac{m_{ij}}{d_{ij}} \in [0, 1]$ such that $f_i(x) = \frac{m_{ij}}{d_{ij}}$ for all $x \in I_j$. Let $\text{gen}(f_1, \dots, f_n)$ be the subalgebra of A generated by $f_1, \dots, f_n$. Let d_j be the least common multiple of $\{d_{1j}, \dots, d_{nj}\}$. Over the set I_j the possible values of the functions in $\text{gen}(f_1, \dots, f_n)$ are integer multiples of $\frac{1}{d_j}$. It follows that the MV-algebra $\text{gen}(f_1, \dots, f_n) \upharpoonright I_j$ obtained by restricting to I_j all the functions in $\text{gen}(f_1, \dots, f_n)$ is isomorphic to an MV-subalgebra of the finite chain L_{d_j} . As a consequence, $\text{gen}(f_1, \dots, f_n)$ is isomorphic to a subalgebra of the finite MV-algebra $\prod_{j=1}^k L_{d_j}$. We have thus shown that A is locally finite. $\square$

From the foregoing theorem it follows that subalgebras and homomorphic images of locally finite algebras are locally finite. Concerning products, the MV-algebra $\prod_{n \geq 1} L_n$ is not hyperarchimedean, whence it is not locally finite (see [1, 116–117]).

While locally finite MV-algebras are not closed under cartesian products, the following result shows that locally finite MV-algebras do have products in the natural sense of category theory:

Theorem 8.10 *For any set $J \neq \emptyset$ and family $\{A_i \mid i \in J\}$ of locally finite MV-algebras, there exists a locally finite MV-algebra P , together with homomorphisms*

$$\rho_i: P \rightarrow A_i \quad (i \in J)$$

with the following universal property: for any locally finite MV-algebra E and homomorphisms $\eta_i: E \rightarrow A_i$ ($i \in J$), there is a unique homomorphism $\theta: E \rightarrow P$ such that the following diagram commutes, for each $i \in J$:

$$\begin{array}{ccc} E & \xrightarrow{\theta} & P \\ \eta_i \searrow & & \swarrow \rho_i \\ & A_i & \end{array} \quad (8.9)$$

Proof Let $A = \prod_{i \in J} A_i$, and for each $i \in J$, let $\pi_i: A \rightarrow A_i$, $i \in J$, be the i th projection function. Theorem 8.9(v $\Leftrightarrow$ i) allows us to identify each A_i with a subalgebra of $(\mathbb{Q} \cap [0, 1])^{Y_i}$ whose elements are functions of finite range on a suitable set Y_i . Letting Y denote the disjoint union of the Y_i , it follows that A is isomorphic to a subalgebra of $(\mathbb{Q} \cap [0, 1])^Y$. Let the subalgebra P of A be defined by

$$P = \{f \in A \mid A \text{ has a finite range}\}.$$

P is the subalgebra of A of all functions of finite range. Another application of Theorem 8.9(v $\Leftrightarrow$ i) shows that P is a locally finite MV-algebra.

Claim Every locally finite subalgebra F of A is a subalgebra of P .

By way of contradiction, suppose the locally finite subalgebra F is a counterexample. Then there is $g \in F \setminus P$, and necessarily g has an infinite range. Let $\text{gen}(g)$ be the subalgebra of A generated by g . Then $\text{gen}(g)$ is also a subalgebra of F , and hence it is locally finite. For each point $y \in Y$ the evaluation map

$$\bar{y}: f \mapsto f(y), (f \in \text{gen}(g))$$

is a homomorphism of $\text{gen}(g)$ into $\mathbb{Q} \cap [0, 1]$. Since g has an infinite range there are infinitely many homomorphisms from $\text{gen}(g)$ into $\mathbb{Q} \cap [0, 1]$. From Theorem 4.16(ii) it follows that the maximal spectrum $\mu(\text{gen}(g))$ is infinite, and so is $\text{gen}(g)$. Thus $\text{gen}(g)$ is not locally finite, a contradiction. Our claim is settled.

To conclude the proof, suppose E is a locally finite MV-algebra, together with homomorphisms $\eta_i: E \rightarrow A_i$ for each $i \in J$. By the universal property of cartesian products *in the category of MV-algebras*, there is a unique homomorphism $\theta: E \rightarrow A$ making the following diagram commutative, for each $i \in J$:

$$\begin{array}{ccc} E & \xrightarrow{\theta} & A \\ \eta_i \searrow & & \swarrow \pi_i \\ & A_i & \end{array}$$

Since $\theta(E)$ is a locally finite subalgebra of A , by our claim $\theta(E) \subseteq P$. Letting $\rho_i = \pi_i \upharpoonright P$, we conclude that P together with the maps $\{\rho_i \mid i \in J\}$ satisfies (8.9).

□

Proposition 8.11 *If A and B are locally finite MV-algebras then so is their free product $A \amalg B$.*

Proof For suitably directed partially ordered sets I and J let us write

$$A = \varinjlim \{(A_i, \alpha_{ih}) \mid i \leq h\} \quad \text{and} \quad B = \varinjlim \{(B_j, \beta_{jk}) \mid j \leq k\}$$

for some finite MV-algebras A_i, B_j and injective homomorphisms $\alpha_{ih}: A_i \rightarrow A_h$ and $\beta_{jk}: B_j \rightarrow B_k$, as given by Theorem 8.9. Without loss of generality we can identify A_i and B_j with subalgebras of A and B . Let

$$\alpha_i: A_i \rightarrow A_i \amalg B_j, \beta_j: B_j \rightarrow A_i \amalg B_j, \alpha: A \rightarrow A \amalg B \quad \text{and} \quad \beta: B \rightarrow A \amalg B$$

be the canonical embeddings. Let $c_1, \dots, c_n \in A \amalg B$, with the intent of proving the finiteness of the subalgebra

$$C = \text{gen}(c_1, \dots, c_n) \subseteq A \amalg B$$

generated by $c_1, \dots, c_n$. Since by definition of free product, $\text{gen}(\alpha(A) \cup \beta(B)) = A \amalg B$, the definition of direct limit yields elements $a_1, \dots, a_p \in A$ and $b_1, \dots, b_q \in B$ such that $C \subseteq \text{gen}(\alpha(a_1), \dots, \alpha(a_p), \beta(b_1), \dots, \beta(b_q)) = D$. To complete the proof it suffices to settle the following

Claim D is finite.

As a matter of fact, there are $i \in I$ and $j \in J$ such that $\{a_1, \dots, a_p\} \subseteq A_i$ and $\{b_1, \dots, b_q\} \subseteq B_j$. We have the following commutative diagram:

$$\begin{array}{ccccc} A_i & \xrightarrow{\alpha_i} & A_i \amalg B_j & \xleftarrow{\beta_j} & B_j \\ \text{inclusion} \downarrow & & & & \downarrow \text{inclusion} \\ A & \xrightarrow{\alpha} & A \amalg B & \xleftarrow{\beta} & B \end{array}$$

Let $c \in C$, say, $c = \tau(\alpha(a_1), \dots, \alpha(a_p), \beta(b_1), \dots, \beta(b_q))$ for some MV-term τ in $p + q$ variables. The universal property of the free product $A_i \amalg B_j$ yields a homomorphism $\eta: A_i \amalg B_j \rightarrow A \amalg B$ such that $\eta\alpha_i = \alpha \upharpoonright A_i$ and $\eta\beta_j = \beta \upharpoonright B_j$. We have the commutative diagram

$$\begin{array}{ccccc} A_i & \xrightarrow{\alpha_i} & A_i \amalg B_j & \xleftarrow{\beta_j} & B_j \\ \text{inclusion} \downarrow & & \eta \downarrow & & \downarrow \text{inclusion} \\ A & \xrightarrow{\alpha} & A \amalg B & \xleftarrow{\beta} & B \end{array}$$

It follows that $c = \tau(\eta\alpha(a_1), \dots, \eta\alpha(a_p), \eta\beta(b_1), \dots, \eta\beta(b_q))$, and hence c belongs to $\eta(A_i \amalg B_j)$. By Corollary 7.8, $A_i \amalg B_j$ is finite, whence so is D .

Having thus settled our claim, we have also completed the proof that $A \amalg B$ is locally finite. $\square$

8.4 Remarks

Direct limits exist in any variety of algebras. While each such direct limit is uniquely determined by an appropriate universal property, the construction of A_∞ given in the first section of this chapter provides an explicit description of the direct limit object in many interesting cases.

Lemma 8.1 is an instance of a general result in universal algebra (see, for instance, [2, p. 72]). Lemma 8.5 is a tedious but straightforward exercise on direct limits. For a proof see, e.g., [3, VIII, 4.13–4.15].

Theorem 8.7 and Corollary 8.8 were first proved in [4] in the equivalent context of unital ℓ -groups. Theorems 8.9 and 8.10 were first proved in [5].

In this book, “locally finite” has its usual algebraic meaning. Yet in the literature on MV-algebras—beginning with [6, 3.10]—“locally finite” is sometimes used as a

synonym of “simple”, i.e., “having no proper homomorphic images”. This latter use is incompatible with the standard usage: for, there exist infinite simple MV-algebras that are generated by one element [see, for instance, (A21.46)]. These MV-algebras are not locally finite in the usual sense.

From Theorem 8.7 and Proposition 4.10 the following result can be readily proved (as usual, $\text{int}(P)$ denotes the interior of P):

Let $X \subseteq [0, 1]^n$ and $Y \subseteq [0, 1]^m$ be nonempty closed sets.

(i) $\mathcal{M}([0, 1]^m)/\mathfrak{h}_X \cong \mathcal{M}([0, 1]^n)/\mathfrak{h}_Y$ *iff there exist rational polyhedra $X \subseteq P \subseteq [0, 1]^n$, $Y \subseteq Q \subseteq [0, 1]^m$ and a $\mathbb{Z}$ -homeomorphism η of P onto Q , such that $\eta(X) = Y$.*

(ii) $\mathcal{M}([0, 1]^m)/\mathfrak{o}_X \cong \mathcal{M}([0, 1]^n)/\mathfrak{o}_Y$ *iff there exist rational polyhedra $P \subseteq [0, 1]^n$, $Q \subseteq [0, 1]^m$ and a $\mathbb{Z}$ -homeomorphism η of P onto Q , such that $X \subseteq \text{int}(P)$, $Q \subseteq \text{int}(Q)$ and $\eta(X) = Y$.*

* * *

The notion of multiset is commonplace in mathematics [7]. Yet “multiset morphisms” are not so frequently considered. Upon thinking of finite MV-algebras as duals of finite multisets, locally finite MV-algebras suggest themselves as the dual of a category of multisets allowing the familiar constructions and closure properties of sets.

This program is carried out in detail in [5], with the help of “supernatural numbers”. A *supernatural number* is a function

$$v: \text{prime numbers} \rightarrow \{0, 1, \dots, \infty\}.$$

We say that v is *finite* iff ∞ does not belong to the range of v , and $v(p)$ is zero for all primes p , up to a finite number of possible exceptions. Under various denominations, supernatural numbers are frequently used in the literature as classifiers ([8–10], and [11, Chap. XIII, §85]).

Regarding v as a list of exponents for the sequence of prime numbers, we may safely identify each number $n = 1, 2, \dots$ with its corresponding finite supernatural number v_n . In particular, v_1 constantly takes value 0 on every prime number. Upon writing $v \leq \mu$ iff $v(p) \leq \mu(p)$ for each prime number p , the set of supernatural numbers forms a complete lattice denoted $\mathcal{N}$.

The one–one correspondence

$$n \leftrightarrow v_n \tag{8.10}$$

identifies the sub-lattice of finite supernatural numbers with the distributive lattice $\mathcal{N}_{\text{div}}$ of all integers $n > 0$ equipped with divisibility. Further, $\mathcal{N}$ is isomorphic to the lattice of all ideals of $\mathcal{N}_{\text{div}}$, in such a way that finite supernatural numbers correspond to principal ideals of $\mathcal{N}_{\text{div}}$.

In [5] the set $\mathcal{N}$ is further equipped with the topology having as an open basis all sets of the form

$$U_n = \{v \in \mathcal{N} \mid v \geq v_n\}, \quad n = 1, 2, \dots$$

This is known as the Scott topology of $\mathcal{N}$, [12].

The *category of multisets* is now defined as follows:

Objects: All pairs (S, σ) , where S is a boolean space, and σ is a continuous map from S into $\mathcal{N}$.

Morphisms: Given objects (T, τ) and (U, ν) , a morphism $\varphi : (T, \tau) \rightarrow (U, \nu)$ is a continuous function $\varphi : T \rightarrow U$ such that $\tau \geq \nu\varphi$.

Naturally enough, a *finite multiset* is defined as a pair (X, σ) where X is a discrete finite set, and $\sigma : X \rightarrow \mathcal{N}_{\text{div}}$ assigns a finite supernatural number to each $x \in X$. Under the identification (8.10), one immediately sees that a morphism $\varphi : (T, \tau) \rightarrow (U, \nu)$ of finite multisets is a multiplicity decreasing function, in the sense that for all $x \in T$, (the number corresponding to) $\nu(\varphi(x))$ is a divisor of $\tau(x)$.

The following natural properties of sets were shown in [5] to hold also for finite and infinite multisets:

- (i) There is a natural equivalence between the opposite of the category of multisets and the category of locally finite MV algebras.
- (ii) Under this equivalence, finite multisets correspond to finite MV algebras.
- (iii) The equivalence extends Stone duality between boolean spaces and boolean algebras.
- (iv) The category of multisets has all limits and colimits. Hence, in particular, it has all infinite products and coproducts.
- (v) Every multiset is the inverse limit of an inverse system of finite multisets.

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Chapter 9

Tensors

While Łukasiewicz logic does not have a multiplication connective, it is often the case that an MV-algebra A possesses a monoidal map $\diamond$ that distributes over truncated subtraction $\ominus$. The most notable example is multiplication in the standard MV-algebra $[0, 1]$. If A has such an operation and is semisimple, the isomorphism $*$: $A \cong A^* \subseteq C(\mu(A^*))$ of Theorem 4.16 automatically transforms $\diamond$ into pointwise multiplication: $(a \diamond b)^* = a^* \cdot b^*$. Thus any semisimple MV-algebra A has at most one $\ominus$ -distributive monoidal map.

Given MV-algebras A and B one can force a sort of multiplication between elements of A and B by introducing the tensor product $A \otimes_{MV} B$ as the universal bilinear map of $A \times B$, in a sense that will be made precise in this chapter. Even in case A and B are MV-algebras of $[0, 1]$ -valued functions, their tensor product $A \otimes_{MV} B$ is usually quite hard to calculate. By contrast, once the universal property is restricted to semisimple MV-algebras, the tensor product of $A \subseteq C(\mu(A))$ and $B \subseteq C(\mu(B))$ is isomorphic to the MV-subalgebra of $C(\mu(A) \times \mu(B))$ generated by the set of functions $a(x) \cdot b(y)$ for $a \in A$ and $b \in B$. Semisimple tensor products can be used to characterize those semisimple MV-algebras A possessing a $\ominus$ -distributive monoidal map. They also provide a substitute for multiplication in the MV-algebraic description of functions defined by partially overlapping cases.

Throughout this chapter we allow the trivial singleton MV-algebra $\{0\} = \{1\}$. The symbols A, B, C will range over MV-algebras with $0 \neq 1$. The symbols S and T will range over all MV-algebras, including the trivial one. For any MV-algebra S there is precisely one homomorphism of S into $\{0\}$. On the other hand, for any MV-algebra A there is no homomorphism of $\{0\}$ into A . For any two MV-algebras S and T , in this chapter $S \times T$ will denote the cartesian product of their underlying sets.

9.1 $\ominus$ -Distributive Monoidal Maps and Multiplicative MV-Algebras

A $\ominus$ -distributive monoidal map on an MV-algebra S is an associative commutative map $\diamond$: $S^2 \rightarrow S$ such that $x \diamond 1 = x$ and $x \diamond (y \ominus z) = (x \diamond y) \ominus (x \diamond z)$ for all $x, y, z \in S$.

Examples 9.1 The $\wedge$ operation is a $\ominus$ -distributive monoidal map in every boolean algebra. Multiplication is a $\ominus$ -distributive monoidal map in the standard MV-algebra $[0, 1]$, as well as in its subalgebra $\mathbb{Q} \cap [0, 1]$. More generally, for any nonempty set X , pointwise multiplication is a $\ominus$ -distributive monoidal map in $C(X)$ and in $[0, 1]^X$. A non-semisimple example is obtained by equipping Chang's algebra $C = \{0, \epsilon, 2\epsilon, \dots, 1, 1 - \epsilon, 1 - 2\epsilon, \dots\}$ with the following operation:

$$\begin{cases} n\epsilon \diamond m\epsilon = 0 \\ (1 - n\epsilon) \diamond (1 - m\epsilon) = 1 - (n + m)\epsilon \\ n\epsilon \diamond (1 - m\epsilon) = n\epsilon. \end{cases}$$

To increase readability of formulas, throughout this chapter we will assume that $\diamond$ is more binding than any MV-algebraic operation.

For later use we collect here some elementary properties of $\diamond$.

Lemma 9.2 *Let S be an MV-algebra with a $\ominus$ -distributive monoidal map $\diamond$. Then for all elements $x, y, z, u, v \in S$ the following conditions hold:*

- (i) $x \diamond 0 = 0$.
- (ii) $x \leq y \Rightarrow x \diamond z \leq y \diamond z$.
- (iii) $x \diamond y \leq x \wedge y$.
- (iv) $x \diamond y \geq x \odot y$.
- (v) $y \odot z = 0 \Rightarrow x \diamond (y \oplus z) = x \diamond y \oplus x \diamond z$.
- (vi) $\mathbf{d}(x \diamond u, y \diamond v) \leq \mathbf{d}(x, y) \oplus \mathbf{d}(u, v)$.

Proof (i–iii) are immediate consequences of the definition of $\diamond$.

(iv) Using the monotonicity properties of $\ominus$ and of $\diamond$ [as given by (ii)] we get $x \odot y = x \ominus \neg y \leq x \ominus (x \diamond \neg y) = x \diamond 1 \ominus x \diamond \neg y = x \diamond \neg \neg y = x \diamond y$.

(v) Let the unital ℓ -group $(G, 1)$ be given by $S = \Gamma(G, 1)$. We first prove

$$p \geq q \geq r \Rightarrow p \ominus (q \ominus r) = (p \ominus q) \oplus r \quad \text{and} \quad (p \ominus q) \oplus (q \ominus r) = p \ominus r. \quad (9.1)$$

This is promptly verified, because, by (A21.16), the order of S agrees with the restriction of the order of G to its unit interval S , and for our present p, q, r , the $\ominus$ operation coincides with the subtraction operation of G , and $\oplus$ coincides with addition.

Next we write

$$\begin{aligned} x \diamond (y \oplus z) &= x \diamond \neg(\neg z \odot \neg y) \\ &= x \diamond (1 \ominus ((1 \ominus z) \ominus y)) \\ &= x \ominus (x \diamond (1 \ominus z) \ominus x \diamond y) \\ &= (x \ominus x \diamond (1 \ominus z)) \oplus x \diamond y, \text{ by (9.1), because } x \geq x \diamond (1 \ominus z) \geq x \diamond y \\ &= x \diamond (1 \ominus (1 \ominus z)) \oplus x \diamond y \\ &= x \diamond z \oplus x \diamond y. \end{aligned}$$

(vi) For all $a, b \in S$, using the absolute value and the subtraction operation of G , we can write

$$\mathbf{d}(a, b) = |a - b| = (a \vee b) - (a \wedge b) = (a \vee b) \ominus (a \wedge b).$$

As a consequence,

$$\begin{aligned} & \mathbf{d}(u, v) \oplus \mathbf{d}(x, y) \\ &= |u - v| \oplus |x - y| \\ &= ((u \vee v) \ominus (u \wedge v)) \oplus ((x \vee y) \ominus (x \wedge y)) \\ &\geq (x \vee y) \diamond ((u \vee v) \ominus (u \wedge v)) \oplus (u \wedge v) \diamond ((x \vee y) \ominus (x \wedge y)) \\ &= ((x \vee y) \diamond (u \vee v) \ominus (x \vee y) \diamond (u \wedge v)) \oplus ((u \wedge v) \diamond (x \vee y) \ominus (u \wedge v) \diamond (x \wedge y)) \\ &= (x \vee y) \diamond (u \vee v) \ominus (u \wedge v) \diamond (x \wedge y), \text{ by (9.1)} \\ &= ((x \vee y) \diamond (u \vee v) \ominus (u \wedge v) \diamond (x \wedge y)) \oplus ((u \wedge v) \diamond (x \wedge y) \ominus (x \vee y) \diamond (u \vee v)) \\ &\geq ((x \diamond u) \ominus (y \diamond v)) \oplus ((y \diamond v) \ominus (x \diamond u)) \\ &= \mathbf{d}(x \diamond u, y \diamond v). \end{aligned}$$

□

From (vi) we immediately have

Lemma 9.3 *Suppose A is an MV-algebra with a $\ominus$ -distributive monoidal map $\diamond$. Let $\mathfrak{i}$ be an ideal of A . Then the map*

$$\left(\frac{a}{\mathfrak{i}}, \frac{b}{\mathfrak{i}} \right) \mapsto \frac{a \diamond b}{\mathfrak{i}} \quad \text{for all } a, b \in A$$

is a $\ominus$ -distributive monoidal map on the quotient MV-algebra $A/\mathfrak{i}$.

Proposition 9.4 *Let A be an MV-subalgebra of $[0, 1]$ with a $\ominus$ -distributive monoidal map $\diamond$. Then (A is closed under multiplication and) $\diamond$ coincides with multiplication. As a consequence, $\diamond$ is the only $\ominus$ -distributive monoidal map on A .*

Proof We first verify the zero product law, stating that whenever $a, b \in A$ are nonzero, then so is $a \diamond b$. By way of contradiction suppose $a \diamond b = 0$. In case $b \leq \neg b$, by Lemma 9.2(v) we have $a \diamond (b \oplus b) = a \diamond b \oplus a \diamond b = 0$. Thus without loss of generality we may assume $a, b > 1/2$. From Lemma 9.2(iv) it follows that $0 < a \odot b \leq a \diamond b$, a contradiction.

Using the zero product law we get for all $a, b, c \in A$ with $0 \neq a$ and $b < c$,

$$0 < a \diamond (c \ominus b) = a \diamond c \ominus a \diamond b \quad \text{and hence, } a \diamond b < a \diamond c. \quad (9.2)$$

Claim A is closed under multiplication.

For otherwise, there are nonzero elements $a, b \in A$ such that $a \cdot b \notin A$. Let A' be the set $\{a \diamond x \in [0, 1] \mid x \in A\}$ equipped with the operations $\neg' y = a \ominus y$ and

$p \oplus' q = (p \oplus q) \wedge a$. Then by Lemma 9.2(v), A' is an MV-algebra whose unit element coincides with a . By (9.2), the map

$$\iota': x \in A \mapsto a \diamond x \in A'$$

is an isomorphism of A onto A' . Let A'' be the set $\{a \cdot x \in [0, 1] \mid x \in A\}$ equipped with the operations $\neg'' y = a \odot y$ and $p \oplus'' q = (p + q) \wedge a$. Then A'' is an MV-algebra whose unit element coincides with a . The map

$$\iota'': x \in A \mapsto a \cdot x \in A''$$

is an isomorphism of A onto A'' . Thus $A' \cong A''$. Let (G', a) and (G'', a) be the associated unital ℓ -groups of A' and A'' given by $A' = \Gamma(G', a)$ and $A'' = \Gamma(G'', a)$. A direct verification shows that (G', a) and (G'', a) are ℓ -subgroups of the additive group of real numbers with the same order-unit a . The properties of the Γ functor yield a unital ℓ -isomorphism ι of (G', a) onto (G'', a) corresponding to the isomorphism of A' onto A'' . The proof of (A21.72) in [1, 7.2.5] ensures that ι is the identity map. Thus $A' = A''$, $\iota''\iota'^{-1} = \iota'\iota''^{-1}$ is the identity on A' and $a \cdot b = a \diamond b \in A$, a contradiction. Our claim is settled.

The proof has also shown that $\diamond$ coincides with multiplication. The rest is clear. $\square$

Recall from Theorem 4.16 the definition of the map $A \mapsto A^*$.

Definition 9.5 An MV-algebra A is said to be *multiplicative* if it is semisimple, and its isomorphic copy $A^* \subseteq C(\mu(A))$ is closed under pointwise multiplication.

Corollary 9.6 Let A be a semisimple MV-algebra. Then A has at most one $\ominus$ -distributive monoidal map. Further, A has a $\ominus$ -distributive monoidal map $\diamond$ iff it is multiplicative. When this is the case, $(a \diamond b)^* = a^* \cdot b^*$.

Proof This follows by combining Lemma 9.3 and Proposition 9.4. $\square$

Using Theorem 4.16 we easily get:

Corollary 9.7 Suppose B is an MV-algebra of continuous $[0, 1]$ -valued functions on some compact Hausdorff space X . Then B is multiplicative iff it is closed under pointwise multiplication.

9.2 Interval MV-Algebras and Bimorphisms

Definition 9.8 For every element u in an MV-algebra S , let $[0, u]$ denote the set $\{x \in S \mid 0 \leq x \leq u\}$ equipped with the operations $\neg_u x = u \odot \neg x$ and $x \oplus_u y = u \wedge (x \oplus y)$. Then $[0, u]$ is a (possibly singleton) MV-algebra with zero element 0 and unit element u . We say that $[0, u]$ is the *interval MV-algebra* of S . The ambient MV-algebra S will always be clear from the context.

Definition 9.9 Let A, B, S be MV-algebras. A *bimorphism* $\flat$ of the set $A \times B$ into S is a function $\flat: A \times B \rightarrow S$ satisfying the following conditions, for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$

- (1) $\flat(1, 1) = 1$.
- (2) $\flat(a, 0) = 0$.
- (3) $\flat(0, b) = 0$.
- (4) $\flat(a, b_1 \vee b_2) = \flat(a, b_1) \vee \flat(a, b_2)$.
- (5) $\flat(a, b_1 \wedge b_2) = \flat(a, b_1) \wedge \flat(a, b_2)$.
- (6) $\flat(a_1 \vee a_2, b) = \flat(a_1, b) \vee \flat(a_2, b)$.
- (7) $\flat(a_1 \wedge a_2, b) = \flat(a_1, b) \wedge \flat(a_2, b)$.
- (8) $\flat(a, b_1) \odot \flat(a, b_2) = 0$ whenever $b_1 \odot b_2 = 0$.
- (9) $\flat(a, b_1 \oplus b_2) = \flat(a, b_1) \oplus \flat(a, b_2)$ whenever $b_1 \odot b_2 = 0$.
- (10) $\flat(a_1, b) \odot \flat(a_2, b) = 0$ whenever $a_1 \odot a_2 = 0$.
- (11) $\flat(a_1 \oplus a_2, b) = \flat(a_1, b) \oplus \flat(a_2, b)$ whenever $a_1 \odot a_2 = 0$.

Examples 9.10 The trivial map $A \times B \rightarrow \{0\}$ is a bimorphism. Multiplication is a bimorphism of $[0, 1] \times [0, 1]$ onto $[0, 1]$. More generally, for any multiplicative MV-algebra A , the map $(a, b) \mapsto a \odot b$ is a bimorphism of $A \times A$ onto A .

The list (1–11) is not the most economical one. We have written all these conditions in the above form for their intuitive content.

We denote by $\text{bim}(A, B, S)$ the set of all bimorphisms $\flat: A \times B \rightarrow S$.

Lemma 9.11 Let A, B, S be MV-algebras, $\flat \in \text{bim}(A, B, S)$, $a \in A$ and $b \in B$.

- (i) For all $u, v \in B$, if $u \leq v$ then $\flat(a, u) \leq \flat(a, v)$.
- (ii) $\flat(a, \neg b) = \flat(a, 1) \odot \neg \flat(a, b)$.
- (iii) For all $u, v \in B$, $\flat(a, u \oplus v) = \flat(a, 1) \wedge (\flat(a, u) \oplus \flat(a, v))$.
- (iv) Let the functions $\flat_a: B \rightarrow S$ and $\flat^b: A \rightarrow S$ be defined by $\flat_a(y) = \flat(a, y)$ and $\flat^b(x) = \flat(x, b)$ for all $x \in A, y \in B$. Then $\flat_a$ is a homomorphism of B into $[0, \flat(a, 1)]$, and $\flat^b$ is a homomorphism of A into $[0, \flat(1, b)]$.

Proof

- (i) is an immediate consequence of Definition 9.9(4).
- (ii) From Definition 9.9(10) we get $0 = \flat(a, b) \odot \flat(a, \neg b)$. Thus, $\flat(a, \neg b) \leq \neg \flat(a, b)$, i.e.,

$$\flat(a, \neg b) \odot \flat(a, b) = 0. \quad (9.3)$$

From Definition 9.9(9) we get

$$\flat(a, \neg b) \oplus \flat(a, b) = \flat(a, 1). \quad (9.4)$$

From (i) it follows that $\flat(a, b) \leq \flat(a, 1)$, i.e., $\neg \flat(a, 1) \odot \flat(a, b) = 0$. By definition of Chang distance, for any two elements $x \leq y$ in an MV-algebra S we have $\mathbf{d}(x, y) = y \odot \neg x$ and $y = x \oplus \mathbf{d}(x, y)$. Therefore, $\flat(a, b) \oplus \mathbf{d}(\flat(a, 1), \flat(a, b)) = \flat(a, 1)$,

and hence, $\flat(a, b) \oplus (\flat(a, 1) \odot \neg \flat(a, b)) \oplus (\neg \flat(a, 1) \odot \flat(a, b)) = \flat(a, 1)$. Since $\neg \flat(a, 1) \odot \flat(a, b) = 0$ then

$$(\flat(a, 1) \odot \neg \flat(a, b)) \oplus \flat(a, b) = \flat(a, 1). \quad (9.5)$$

We now recall the *MV-algebraic cancellation law*, stating that, for any elements x, y, z of any MV-algebra E ,

$$x \oplus z = y \oplus z \quad \text{and} \quad x \odot z = y \odot z = 0 \Rightarrow x = y.$$

Indeed, by (A21.16)(i), in the unital ℓ -group $(G, 1)$ corresponding to E via Γ we have $x \oplus z = x + z$ and $y \oplus z = y + z$, whence $x = y$.

Using the cancellation law together with (9.3)–(9.5) and the trivial equation $\flat(a, 1) \odot \neg \flat(a, b) \odot \flat(a, b) = 0$, we finally obtain $\flat(a, \neg b) = \flat(a, 1) \odot \neg \flat(a, b)$.

(iii) Repeated application of Definition 9.9 yields

$$\begin{aligned} \flat(a, u \oplus v) &= \flat(a, u \vee (u \oplus v)) \\ &= \flat(a, u \oplus (\neg u \odot (u \oplus v))) \\ &= \flat(a, u) \oplus \flat(a, \neg u \odot (u \oplus v)) \\ &= \flat(a, u) \oplus \flat(a, \neg u \wedge v) \\ &= \flat(a, u) \oplus (\flat(a, \neg u) \wedge \flat(a, v)) \\ &= (\flat(a, u) \oplus \flat(a, \neg u)) \wedge (\flat(a, u) \oplus \flat(a, v)) \\ &= \flat(a, 1) \wedge (\flat(a, u) \oplus \flat(a, v)). \end{aligned}$$

(iv) Immediate from (ii)–(iii). □

9.3 The MV-Algebraic Tensor Product

Let A, B, S, T be MV-algebras. If $\eta: S \rightarrow T$ is a homomorphism and $\flat \in \text{bim}(A, B, S)$ then $\eta \flat \in \text{bim}(A, B, T)$.

A bimorphism $\flat \in \text{bim}(A, B, S)$ is said to be *universal* if for every MV-algebra T and $\flat' \in \text{bim}(A, B, T)$ there is a unique homomorphism $\lambda: S \rightarrow T$ such that $\lambda \flat = \flat'$.

The construction of a universal bimorphism of two MV-algebras A and B follows a routine pattern, mimicking Definition 9.9:

Construction 9.12 For any MV-algebras A and B , let $\text{FREE}_{A \times B}$ be the free MV-algebra over the free generating set $A \times B$. Let $\mathfrak{t}$ be the ideal of $\text{FREE}_{A \times B}$ generated by the following elements, for every $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$

- (1) $\mathbf{d}((1, 1), 1)$.
- (2) $\mathbf{d}((a, 0), 0)$.

- (3) $\mathbf{d}((0, b), 0)$.
- (4) $\mathbf{d}((a, b_1 \vee b_2), (a, b_1) \vee (a, b_2))$.
- (5) $\mathbf{d}((a, b_1 \wedge b_2), (a, b_1) \wedge (a, b_2))$.
- (6) $\mathbf{d}((a_1 \vee a_2, b), (a_1, b) \vee (a_2, b))$.
- (7) $\mathbf{d}((a_1 \wedge a_2, b), (a_1, b) \wedge (a_2, b))$.
- (8) $\mathbf{d}((a, b_1) \odot (a, b_2), 0)$ whenever $b_1 \odot b_2 = 0$.
- (9) $\mathbf{d}((a, b_1 \oplus b_2), (a, b_1) \oplus (a, b_2))$ whenever $b_1 \odot b_2 = 0$.
- (10) $\mathbf{d}((a_1, b) \odot (a_2, b), 0)$ whenever $a_1 \odot a_2 = 0$.
- (11) $\mathbf{d}((a_1 \oplus a_2, b), (a_1, b) \oplus (a_2, b))$ whenever $a_1 \odot a_2 = 0$.

For the sake of uniformity, we have resorted to Chang distance in all items (1–11), although some items have a simpler reformulation. Thus for instance, (1) is equivalent to $\neg(1, 1) \in \mathfrak{t}$; (2) just says $(a, 0) \in \mathfrak{t}$ for each $a \in A$. We now set

$$A \otimes_{MV} B = \frac{\mathbf{FREE}_{A \times B}}{\mathfrak{t}}.$$

Without danger of confusion, let the map $\otimes_{MV}: (a, b) \mapsto a \otimes_{MV} b$ be defined by

$$a \otimes_{MV} b = \frac{(a, b)}{\mathfrak{t}}, \quad \text{for all } (a, b) \in A \times B. \quad (9.6)$$

Then $\otimes_{MV} \in \text{bim}(A, B, A \times B)$. Since $A \times B$ is a generating set of $\mathbf{FREE}_{A \times B}$, the MV-algebra $A \otimes_{MV} B$ is generated by the set of elements of the form $a \otimes_{MV} b$. Any element of $A \otimes_{MV} B$ has the form

$$\tau(a_1 \otimes_{MV} b_1, \dots, a_k \otimes_{MV} b_k)$$

for some $a_i \in A, b_i \in B$ and MV-term $\tau(X_1, \dots, X_k)$.

Intuitively, the ideal $\mathfrak{t}$ is a set of axioms $\{\neg\phi \mid \phi \in \mathfrak{t}\}$ (in the variables given by all elements of $A \times B$) stating that $\otimes_{MV}$ is a bimorphism of $A \times B$ into $\mathbf{FREE}_{A \times B}/\mathfrak{t}$.

On the other hand, the following proposition shows that $\mathfrak{t}$ is the smallest possible ideal $\mathfrak{i}$ of $\mathbf{FREE}_{A \times B}$ ensuring that the map $(a, b) \mapsto (a, b)/\mathfrak{i}$ is a bimorphism of $A \times B$ into $\mathbf{FREE}_{A \times B}/\mathfrak{i}$:

Proposition 9.13 (Universality of $\otimes_{MV}$) *For any bimorphism $\flat: A \times B \rightarrow S$ there is precisely one homomorphism $\lambda: A \otimes_{MV} B \rightarrow S$ such that $\lambda(a \otimes_{MV} b) = \flat(a, b)$ for all $(a, b) \in A \times B$.*

Proof Let $\iota: A \times B \rightarrow \mathbf{FREE}_{A \times B}$ be the inclusion map, and $\sigma: \mathbf{FREE}_{A \times B} \rightarrow \mathbf{FREE}_{A \times B}/\mathfrak{t}$ the quotient map. By Theorem 1.5 the map sending each free generator (a, b) of $\mathbf{FREE}_{A \times B}$ into the element $\flat(a, b) \in S$ uniquely extends to a homomorphism $\eta: \mathbf{FREE}_{A \times B} \rightarrow S$. Let $\mathbf{d}(p, q)$ be a generator of $\mathfrak{t}$ as given by (1–11) in 9.12. Since by assumption $\flat$ satisfies the bilinearity conditions (1–11) in Definition 9.9, then $\eta(\mathbf{d}(p, q)) = 0$, i.e., $\ker(\eta) \supseteq \mathfrak{t}$. As a consequence,

the stipulation $\lambda: z/\mathfrak{t} \mapsto \eta(z)$ for all $z \in \text{FREE}_{A \times B}$, yields a homomorphism $\lambda: \text{FREE}_{A \times B}/\mathfrak{t} \rightarrow S$ satisfying $\lambda\sigma = \eta$, whence $\lambda \otimes_{MV} = \flat$. The following diagram commutes:

$$\begin{array}{ccccc}
 A \times B & \xrightarrow{\iota} & \text{FREE}_{A \times B} & \xrightarrow{\sigma} & \text{FREE}_{A \times B}/\mathfrak{t} = A \otimes_{MV} B \\
 \searrow \flat & & \downarrow \eta & & \swarrow \lambda \\
 & & S & &
 \end{array}$$

Since $A \otimes_{MV} B$ is generated by all elements of the form $(a, b)/\mathfrak{t}$, and $\lambda((a, b)/\mathfrak{t}) = \flat(a, b)$ for all $(a, b) \in A \times B$, then λ is uniquely determined. $\square$

Definition 9.14 $A \otimes_{MV} B$ is uniquely determined up to isomorphism. We say that $A \otimes_{MV} B$ is the *MV-algebraic tensor product* of A and B .

Recall that L_m denotes the MV-chain with $m + 1$ elements.

Example 9.15 ($L_2 \otimes_{MV} L_2 \not\cong L_2 \amalg L_2$). Since there are nine elements in $L_2 \times L_2$,

$$L_2 \otimes_{MV} L_2 = \frac{\text{FREE}_9}{\mathfrak{t}},$$

where the ideal $\mathfrak{t}$ is given by (1–11) in the construction above. Since $\mathfrak{t}$ is finitely generated, by Proposition 4.4(iii),

$$L_2 \otimes_{MV} L_2 = \mathcal{M}([0, 1]^9) \upharpoonright \mathcal{Z}\mathfrak{t}.$$

To give a concrete representation of $\mathcal{Z}\mathfrak{t}$, we will move from nine-dimensional to two-dimensional space. First of all, in the light of Theorem 1.5 each free generator $(a, b) \in \{0, 1/2, 1\}^2 = L_2 \times L_2$ of $\text{FREE}_9 = \mathcal{M}([0, 1]^9)$ is identified with the coordinate function $\pi_{a,b}: [0, 1]^9 \rightarrow [0, 1]$,

$$\pi_{a,b}: x = (x_{0,0}, x_{0,1/2}, \dots, x_{1,1}) \in [0, 1]^9 \mapsto x_{a,b} \in [0, 1].$$

The properties of Chang distance function ensure that the zeroset of each generator of $\mathfrak{t}$ in (1–11) coincides with a hyperplane in nine-dimensional space. Thus, e.g., the zeroset of $\mathbf{d}((0, 1), (1, 0))$ is the set

$$\{x \in [0, 1]^9 \mid \pi_{0,1}(x) = \pi_{1,0}(x)\} = \{x \in [0, 1]^9 \mid x_{0,1} = x_{1,0}\} \supseteq \mathcal{Z}\mathfrak{t}.$$

Proceeding in this way with the other generators of $\mathfrak{t}$, it is easy to see that $\mathcal{Z}\mathfrak{t}$ is contained in the following set:

$$\begin{aligned}
 & \{x \in [0, 1]^9 \mid x_{0,1} = x_{1,0} = x_{1/2,0} = x_{0,1/2} = x_{0,0} = 0\} \\
 & \cap \{x \in [0, 1]^9 \mid x_{1,1/2} = x_{1/2,1}\} \cap \{x \in [0, 1]^9 \mid x_{1,1} = 1\} \\
 & \cap \{x \in [0, 1]^9 \mid x_{1,1} = 2x_{1,1/2} = 4x_{1/2,1/2}\}.
 \end{aligned}$$

Writing x, y, z respectively for $x_{1/2, 1/2}, x_{1, 1/2}, x_{1, 1}$, it follows that $\mathcal{Z}\mathfrak{t}$ is $\mathbb{Z}$ -homeomorphic to the set $\{(x, y, z) \in [0, 1]^3 \mid 4x = 2y = z \text{ and } z = 1\}$, i.e., to the singleton $\{(1/4, 1/2, 1)\}$. The latter is $\mathbb{Z}$ -homeomorphic to the singleton $\{(1/4, 1/2)\} \subseteq [0, 1]^2$. By Corollary 3.10, $\mathbb{Z}$ -homeomorphic rational polyhedra determine isomorphic finitely presented MV-algebras. In conclusion, we have an isomorphism

$$L_2 \otimes_{MV} L_2 \cong \mathcal{M}([0, 1]^9) \upharpoonright \mathcal{Z}\mathfrak{t} \cong \mathcal{M}([0, 1]^2) \upharpoonright \left\{ \left(\frac{1}{4}, \frac{1}{2} \right) \right\} \cong \left\{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \right\} = L_4,$$

whence by Corollary 7.8, $L_2 \otimes_{MV} L_2 \cong L_4 \not\cong L_2 \cong L_2 \sqcup L_2$.

9.4 The Semisimple Tensor Product

For any semisimple MV-algebras A and B we write

$$A \otimes B = \frac{A \otimes_{MV} B}{\text{Rad}(A \otimes_{MV} B)}$$

and say that $A \otimes B$ is the *semisimple tensor product* of A and B .

Without danger of confusion we let the map $\otimes: A \times B \rightarrow A \otimes B$ be defined by

$$a \otimes b = \frac{(a \otimes_{MV} b)}{\text{Rad}(A \otimes_{MV} B)} \quad \text{for all } (a, b) \in A \times B.$$

Then $\otimes$ is a bimorphism of $A \times B$ into $A \otimes B$ whose range generates $A \otimes B$.

Each element e of $A \otimes B$ has the form

$$e = \tau(a_1 \otimes b_1, \dots, a_k \otimes b_k), \quad (9.7)$$

for suitable elements $a_i \in A, b_j \in B$ and MV-term $\tau(X_1, \dots, X_k)$

The universal property of the bimorphism $\otimes$ is as follows:

Proposition 9.16 *Let A and B be semisimple MV-algebras. Then $A \otimes B$ is a semisimple MV-algebra. Further, for every bimorphism $\flat$ of $A \times B$ into a semisimple MV-algebra C there is a unique homomorphism $\lambda: A \otimes B \rightarrow C$ such that $\lambda(a \otimes b) = \flat(a, b)$ for all $(a, b) \in A \times B$.*

Proof $A \otimes B$ is a semisimple MV-algebra by Lemma 4.20(i). By Proposition 9.13 there a unique homomorphism $\eta: A \otimes_{MV} B \rightarrow C$ such that $\flat$ coincides with the composite map $\eta \otimes_{MV}$. Since C is semisimple, $\ker(\eta) \supseteq \text{Rad}(A \otimes_{MV} B)$. Thus the map

$$\frac{a \otimes_{MV} b}{\text{Rad}(A \otimes_{MV} B)} \mapsto \eta(a \otimes_{MV} b) \quad \text{for all } (a, b) \in A \times B$$

uniquely extends to a homomorphism λ of $A \otimes B$ into C satisfying $\lambda(a \otimes b) = \flat(a, b)$; uniqueness follows because elements of the form $(a \otimes_{MV} b)/\text{Rad}(A \otimes_{MV} B)$ generate $A \otimes B$. $\square$

Using Theorem 4.16 and the notation of 4.17, we next provide a concrete representation of semisimple tensor products:

Theorem 9.17 *For X and Y nonempty compact Hausdorff spaces, let the MV-algebras A and B be separating subalgebras of $C(X)$ and $C(Y)$, respectively. Let us write $X = \mu(A)$ and $Y = \mu(B)$ without loss of generality.*

(i) *For each $\flat \in \text{bim}(A, B, [0, 1])$ there is a unique pair of maximal ideals $(\flat_b, \flat_a) \in \mu(A) \times \mu(B)$ such that $\flat(f, g) = f(\flat_b) \cdot g(\flat_a)$ for all $(f, g) \in A \times B$. The map $\flat \mapsto (\flat_b, \flat_a)$ is a one-one correspondence of $\text{bim}(A, B, [0, 1])$ onto $\mu(A) \times \mu(B)$.*

(ii) *Let the map π transform each pair $(f, g) \in A \times B$ into the function $\pi(f, g) \in C(\mu(A) \times \mu(B))$ given by*

$$(\pi(f, g))(l, m) = f(l) \cdot g(m) \quad \text{for all } (l, m) \in \mu(A) \times \mu(B). \quad (9.8)$$

Let $A\pi B$ denote the MV-algebra generated by the range of π in $C(\mu(A) \times \mu(B))$. Then $\pi \in \text{bim}(A, B, A\pi B)$ has the universal property of Proposition 9.16. Further, $A\pi B$ is a separating subalgebra of $C(\mu(A) \times \mu(B))$, and π coincides with the composite map $\mathbf{j} \otimes$ for a uniquely determined isomorphism $\mathbf{j}$ of $A \otimes B$ onto $A\pi B$. Each element $e \in A \otimes B$ of the form (9.7) is sent by $\mathbf{j}$ to the element $\tau(a_1 \cdot b_1, \dots, a_k \cdot b_k)$.

Proof (i) For fixed, but otherwise arbitrary $f \in A$, let $\flat_f: B \rightarrow [0, 1]$ be the homomorphism of Lemma 9.11(iv).

We claim that a maximal ideal $\flat_f \in \mu(B)$ and a real number $m_f \geq 0$ such that

$$\flat_f(g) = m_f \cdot g(\flat_f) \quad \text{for all } g \in B. \quad (9.9)$$

As a matter of fact, let $w = \flat_f(1)$. Let the unital ℓ -groups $(G, 1)$ and (L, w) be defined by $B = \Gamma(G, 1)$ and $[0, w] = \Gamma(L, w)$. By writing $\Gamma(\eta) = \flat_f$ we obtain a uniquely determined ℓ -homomorphism $\eta: G \rightarrow L$ such that $\eta(1) = w$. Now, in case $w = 0$ we set $m_f = 0$, and arbitrarily choose $\flat_f$ in $\mu(A)$. In case $w \neq 0$, by (B21.71) there is a real number $m_f > 0$ such that the map $\psi: x \mapsto \frac{1}{m_f}x$ is the only ℓ -homomorphism of L into $\mathbb{R}$ sending w to 1. As a consequence, the composite map $\psi\eta: G \rightarrow \mathbb{R}$ is a unital ℓ -homomorphism of G into $\mathbb{R}$, and $\Gamma(\psi\eta)$ is a homomorphism of B into $[0, 1]$. By Theorem 4.16(i), there is a unique maximal ideal $\flat_f \in \mu(B)$ satisfying (9.9). Our claim is settled.

Similarly, for each $g \in B$ there is $\flat_g \in \mu(A)$ and $0 \leq n_g \in \mathbb{R}$ such that

$$\flat_g(f) = n_g \cdot f(\flat_g) \quad \text{for all } f \in A. \quad (9.10)$$

In particular, for $f = 1 \in A$ and $g = 1 \in B$, from (9.9–9.10) we get $1 = \flat_1(1) = m_1 \cdot 1 = m_1$ and $1 = \flat^1(1) = n_1$, and hence, $m_1 = n_1 = 1$. It follows that

$f(i_1) = n_1 \cdot f(i_1) = b^1(f) = b(f, 1) = b_f(1) = m_f$, and $g(j_1) = m_1 \cdot g(j_1) = b_1(g) = b(1, g) = b^g(1) = n_g$, whence

$$f(i_1) \cdot g(j_f) = m_f \cdot g(j_f) = b(f, g) = n_g \cdot f(i_g) = g(j_1) \cdot f(i_g).$$

We now *claim* that $b(f, g) = f(i_1) \cdot g(j_1)$ for all $f \in A$ and $g \in B$.

As a matter of fact, in case $b(f, 1) = 0$, by Lemma 9.11(i), $b(f, g) = 0$ for all $g \in B$. Further, $f(i_1) = b(f, 1) = 0$. In case $b(f, 1) \neq 0$, our analysis above yields an ideal $j_f \in \mu(B)$ such that $b(f, g) = b(f, 1) \cdot g(j_f)$ for all $g \in B$. By way of contradiction, let us suppose $j_f \neq j_1$. Since B is a separating subalgebra of $C(\mu(B))$, there exists $p \in B$ be such that $p(j_f) > 0$ and $p(j_1) = 0$. Evidently, $b(f, p) > 0$. But $b(1, p) = p(j_1) = 0$, against the monotonicity property $b(f, p) \leq b(1, p)$. In conclusion, $j_f = j_1$ and $b(f, g) = b(f, 1) \cdot g(j_1) = f(i_1) \cdot g(j_1)$, as required to settle our second claim.

Upon defining $i_b = i_1$ and $j_b = j_1$ we have thus obtained the desired one-one map of $\text{bim}(A, B, [0, 1])$ onto $\mu(A) \times \mu(B)$: injectivity follows because A and B are separating subalgebras; to prove surjectivity, it is enough to note that for each $(i, j) \in \mu(A) \times \mu(B)$, the map $(f, g) \mapsto f(i) \cdot g(j)$ is a bimorphism of $A \times B$ into $[0, 1]$.

(ii) It is no loss of generality to identify $A \otimes B$ with the separating subalgebra $(A \otimes B)^* \subseteq C(\mu(A \otimes B))$ of Theorem 4.16, and write

$$A \otimes B \subseteq C(\mu(A \otimes B)). \quad (9.11)$$

The universal property of $\otimes$ yields a unique homomorphism $J: A \otimes B \rightarrow A\pi B$ such that

$$\pi = J \otimes. \quad (9.12)$$

Since the range of π generates $A\pi B$, J is *onto* the semisimple MV-algebra $A\pi B \subseteq C(\mu(A) \times \mu(B))$. By (A21.12), for every maximal ideal $\mathfrak{p} \in \mu(A\pi B)$ we have a maximal ideal $J^{-1}(\mathfrak{p}) \in \mu(A \otimes B)$. Using the notational simplifications in 4.17, for every $l \in A \otimes B$ we have equal real numbers

$$\frac{J(l)}{\mathfrak{p}} = \frac{l}{J^{-1}(\mathfrak{p})}. \quad (9.13)$$

As a matter of fact, in the light of Theorem 4.16(ii) it suffices to verify that the two homomorphisms $l \mapsto \frac{J(l)}{\mathfrak{p}}$ and $l \mapsto \frac{l}{J^{-1}(\mathfrak{p})}$ have the same kernel.

Let the closed set $J \subseteq \mu(A \otimes B)$ be defined by

$$J = \mathcal{Z} \ker(J) = \bigcap \{l^{-1}(0) \mid l \in \ker(J)\}. \quad (9.14)$$

Since $\frac{A \otimes B}{\ker(J)} \cong A\pi B$ is semisimple, then by (A21.31), $\ker(J)$ is the intersection of maximal ideals of $A \otimes B$. By (A21.21), there is an isomorphism

$$\omega: (A \otimes B) \upharpoonright J \cong A\pi B \quad (9.15)$$

such that

$$J(l) = \omega(l \upharpoonright J) \quad \text{for all } l \in A \otimes B. \quad (9.16)$$

Claim J is one-one.

For otherwise (absurdum hypothesis), by (9.15–9.16), the set J is strictly contained in $\mu(A \otimes B)$. Let us fix, once and for all,

$$n \in \mu(A \otimes B) \setminus J. \quad (9.17)$$

Let $\sigma_n: A \otimes B \rightarrow [0, 1]$ be the quotient map by n . Combining (9.11) with Theorem 4.16, we obtain

$$\sigma_n(l) = \frac{j}{n} = l(n) \quad \text{for all } l \in A \otimes B. \quad (9.18)$$

The composite map $\sigma_n \otimes$ is a bimorphism of $A \times B$ into $[0, 1]$. By (i), there is a unique pair $(i, j) \in \mu(A) \times \mu(B)$ such that

$$(\sigma_n \otimes)(f, g) = f(i) \cdot g(j) \quad \text{for all } (f, g) \in A \times B. \quad (9.19)$$

Let the homomorphism $v: A\pi B \rightarrow [0, 1]$ be defined by

$$v(k) = k(i, j) \quad \text{for all } k \in A\pi B. \quad (9.20)$$

Then for all $(f, g) \in A \times B$

$$vJ \otimes (f, g) = v\pi(f, g), \text{ by (9.12)} \quad (9.21)$$

$$= (\pi(f, g))(i, j), \text{ by (9.20)} \quad (9.22)$$

$$= f(i) \cdot g(j), \text{ by (9.8)} \quad (9.23)$$

$$= (\sigma_n \otimes)(f, g), \text{ by (9.19).} \quad (9.24)$$

In other words, $vJ \otimes = (\sigma_n \otimes)$. Since the range of $\otimes$ generates $A \otimes B$,

$$\sigma_n = vJ. \quad (9.25)$$

Recalling (9.12) we have the following commutative diagram:

$$\begin{array}{ccccc}
 A \times B & \xrightarrow{\otimes} & A \otimes B & \xrightarrow{\sigma_n} & [0, 1] \\
 \pi \searrow & & J \downarrow & & \nearrow v \\
 & & A\pi B & &
 \end{array}$$

By Theorem 4.16, $\ker(v)$ is a maximal ideal $\mathfrak{p}$ of $A\pi B$. By (A21.12)(i), $J^{-1}(\mathfrak{p})$ is a maximal ideal $\mathfrak{m}$ of $A \otimes B$ containing $\ker(J)$. Thus by (9.14)

$$\mathfrak{m} \in J. \quad (9.26)$$

Combining (9.21–9.23) with (9.18) and (9.25), for each $l = (f \otimes g) \in A \otimes B$ we obtain identical real numbers

$$l(\mathfrak{n}) = \sigma_n(l) = (vJ)(l) = f(i) \cdot g(j).$$

The commutative diagram above yields

$$\begin{aligned}
 l(\mathfrak{m}) &= l(J^{-1}(\mathfrak{p})) \\
 &= (J(l))(\mathfrak{p}), \text{ by (9.13)} \\
 &= (J(l))(\ker(v)) \\
 &= (J(l))/\ker(v) \\
 &= v(J(l)), \text{ by Theorem 4.16(ii)} \\
 &= f(i) \cdot g(j), \text{ by (9.21)–(9.23).}
 \end{aligned}$$

Since the elements of the form $f \otimes g$ generate $A \otimes B$, then $k(\mathfrak{m}) = k(\mathfrak{n})$ for all $k \in A \otimes B$. Since $A \otimes B$ is a separating subalgebra of $C(\mu(A \otimes B))$, then necessarily $\mathfrak{m} = \mathfrak{n}$, which is incompatible with (9.17) and (9.26).

Our claim is proved.

It follows that J is an isomorphism of $A \otimes B$ onto $A\pi B$. The universal property of $\otimes$ ensures that J is unique. Since $A \otimes B$ is a separating MV-algebra of $[0, 1]$ -valued functions on its own maximal spectral space, then $A\pi B$ is a separating MV-algebra of $[0, 1]$ -valued functions on $\mu(A\pi B) \cong \mu(A) \times \mu(B)$.

The rest is clear. □

Corollary 9.18 *For any semisimple MV-algebras A and B , let the map $(*, *) : A \times B \rightarrow A^* \times B^*$ be defined by*

$$(a, b) \mapsto (a^*, b^*) \text{ for all } (a, b) \in A \times B.$$

Let $A^\pi B^*$ denote the separating subalgebra of $C(\mu(A) \times \mu(B))$ generated by the range of the bimorphism π , as in Theorem 9.17.*

- (i) *There is a unique isomorphism $\rho : A \otimes B \rightarrow A^*\pi B^*$ such that the following diagram commutes:*

$$\begin{array}{ccc}
A \times B & \xrightarrow{\otimes} & A \otimes B \\
\downarrow (*, *) & & \downarrow \rho \\
A^* \times B^* & \xrightarrow{\pi} & A^* \pi B^*
\end{array}$$

- (ii) The map $j \mapsto \rho(j) = \{\rho(f) \mid f \in j\}$ is an isomorphism of the lattice of ideals of $A \otimes B$ onto the lattice of ideals of $A^* \pi B^*$.
- (iii) Suppose the ideal i of $A \otimes B$ is the intersection of maximal ideals of $A \otimes B$. Then for a unique isomorphism $\rho_i: (A \otimes B)/i \cong (A^* \pi B^*) \upharpoonright \mathcal{Z}(\rho(i))$ we have a commutative diagram

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\text{quotient}} & (A \otimes B)/i \\
\downarrow \rho & & \downarrow \rho_i \\
A^* \pi B^* & \xrightarrow{\text{restriction}} & (A^* \pi B^*) \upharpoonright \mathcal{Z}(\rho(i))
\end{array}$$

In particular, for each $(a, b) \in A \times B$,

$$\rho_i \left(\frac{a \otimes b}{i} \right) = (a^* \cdot b^*) \upharpoonright \mathcal{Z}(\rho(i)).$$

Proof (i) Immediately follows from foregoing theorem. (ii) is then a consequence of (A21.7). (iii) is a consequence of (i–ii) together with (A21.21). $\square$

We finally show that the MV-algebraic tensor product of two semisimple MV-algebras need not coincide with their semisimple tensor product.

Proposition 9.19 *For each irrational $\xi \in [0, 1]$, let A be the subalgebra of $[0, 1]$ generated by ξ . Then $A \otimes_{MV} A$ is not semisimple.*

Proof We will use the free MV-algebra $\mathcal{M}([0, 1])$. Theorem 5.12 with Corollary 5.13 shows that the quotient MV-algebra $\mathcal{M}([0, 1])/\mathfrak{o}_\xi = \mathcal{M}([0, 1])/\mathfrak{h}_\xi$ is simple and coincides with A . Let the map $\mathfrak{b} : \mathcal{M}([0, 1]) \times \mathcal{M}([0, 1]) \rightarrow C([0, 1]^2)$ be defined by $\mathfrak{b}(f, g) = f(x) \cdot g(y)$ for all $f, g \in \mathcal{M}([0, 1])$. Let $\mathfrak{o}_{(\xi, \xi)}$ be the germinal ideal of $\mathcal{M}([0, 1]^2)$ at (ξ, ξ) and $\bar{\mathfrak{o}}_{(\xi, \xi)} \supseteq \mathfrak{o}_{(\xi, \xi)}$ the germinal ideal of $C([0, 1]^2)$ at (ξ, ξ) . Let the MV-algebra H be defined by $H = C([0, 1]^2)/\bar{\mathfrak{o}}_{(\xi, \xi)}$. Then H is the MV-algebra of germs at (ξ, ξ) of continuous $[0, 1]$ -valued functions on $[0, 1]^2$. By Proposition 7.3, the free product MV-algebra $A \amalg A = \mathcal{M}([0, 1]^2)/\mathfrak{o}_{(\xi, \xi)}$ can be identified with a subalgebra of H . From $\mathfrak{b}$ we obtain a map $\mathfrak{b} : A \times A \rightarrow H$ sending each pair of germs $(f/\mathfrak{o}_\xi, g/\mathfrak{o}_\xi)$ into the germ at (ξ, ξ) of the function $f(x) \cdot g(y)$. Let K denote the subalgebra K of H generated by the range of $\mathfrak{b}$. Direct inspection shows that $\mathfrak{b}$ is a bimorphism of $A \times A$ into K .

Let π_1 and π_2 be the coordinate functions on $[0, 1]^2$. The germ $\pi_1/\mathfrak{o}_{(\xi, \xi)} = \pi_1/\bar{\mathfrak{o}}_{(\xi, \xi)}$ at (ξ, ξ) is a member of K , and so is the germ of π_2 at (ξ, ξ) . Therefore, the free product MV-algebra $A \amalg A$ is a subalgebra of K . By Proposition 7.3, $A \amalg A = \mathcal{M}([0, 1]^2)/\mathfrak{o}_{(\xi, \xi)}$ is not semisimple, and hence, K is not semisimple.

To conclude the proof, by way of contradiction, assume $A \otimes_{MV} A$ is semisimple. Then we *claim* that $A \otimes_{MV} A$ has precisely one maximal ideal. For otherwise, by Theorem 4.16 there would exist at least two homomorphisms η_1, η_2 of $A \otimes_{MV} A$ into $[0, 1]$. Composition of $\otimes_{MV} \in \text{bim}(A, A, A \otimes_{MV} A)$ with η_1 and η_2 yields two distinct bimorphisms of $A \times A$ into $[0, 1]$. By Theorem 9.17(i), $\mu(A) \times \mu(A)$ has at least two elements, whence so does $\mu(A)$, which is impossible.

Having thus settled our claim, combining Theorem 4.16 with our standing absurdum hypothesis, it follows that $A \otimes_{MV} A$ is simple, i.e., it is isomorphic to a subalgebra of $[0, 1]$. There is a unique (necessarily one–one) homomorphism $\lambda: A \otimes_{MV} A \rightarrow K$ such that $\lambda \otimes_{MV} = \flat$. Since K is generated by the range of $\flat$, λ is onto K . Thus, $A \otimes_{MV} A$ is isomorphic to the non-semisimple MV-algebra K , a contradiction. $\square$

9.5 Remarks

9.20. Generalizing Example 9.15 one can easily prove $L_m \otimes_{MV} L_n = L_m \otimes L_n = L_{mn}$ and more generally,

$$\prod_i L_{m_i} \otimes_{MV} \prod_j L_{j_i} = \prod_i L_{m_i} \otimes \prod_j L_{j_i} = \prod_{ij} L_{m_i n_j}.$$

See [2, p. 235] for details. Routine verifications also yield

$$S \otimes_{MV} \{0, 1\} \cong S \otimes \{0, 1\} \cong S \quad \text{for every MV-algebra } S.$$

For every simple multiplicative MV-algebra A , $A \otimes A \cong A$, whence in particular

$$[0, 1] \otimes [0, 1] \cong [0, 1] \quad \text{and} \quad [0, 1] \cap \mathbb{Q} \otimes [0, 1] \cap \mathbb{Q} \cong [0, 1] \cap \mathbb{Q}.$$

In sharp contrast with semisimple tensor products, MV-algebraic tensor products are not easily visualized, already for elementary examples such as $[0, 1] \otimes_{MV} [0, 1]$ or $C \otimes_{MV} C$, (with C the MV-algebra introduced by Chang in [3, p. 474]).

9.21. Let A be a semisimple MV-algebra. The isomorphism **flip**: $A \otimes B \cong B \otimes A$ transforms each element $e \in A \otimes B$ in (9.7) into the element

$$\text{flip}(e) = \tau(b_1 \otimes a_1, \dots, b_k \otimes a_k).$$

Let flip_A denote flip automorphism of $A \otimes A$. A subset J of $A \otimes A$ is *invariant* if $J = \{\text{flip}_A(x) \mid x \in J\}$. The *diagonal ideal* $\mathfrak{d}$ of $A \otimes A$ is the intersection of all maximal invariant ideals of $A \otimes A$. By (A21.31), the quotient MV-algebra

$(A \otimes A)/\mathfrak{d}$ is semisimple. The *diagonal map* $\delta: A \rightarrow (A \otimes A)/\mathfrak{d}$ is defined by $\delta(a) = (a \otimes 1)/\mathfrak{d}$ for all $a \in A$. Then repeated application of Corollary 9.18 yields the following characterization:

$$A \text{ is multiplicative} \Leftrightarrow \delta \text{ is onto } (A \otimes A)/\mathfrak{d}.$$

See [2] for details. Note that the ideal I of [2, Corollary 4.4(ii)] must be assumed to coincide with an intersection of maximal ideals, as we have done in Corollary 9.18. All applications of this corollary in [2] remain true, because the corollary is only applied to ideals which are intersections of maximal ideals.

9.22. One may add a new symbol $\diamond$ to the language of MV-algebras, together with equations stating that $\diamond$ is a monoidal operation that distributes over $\ominus$. Members of the resulting equational class are known as PMV-algebras.

It is known (see, e.g., [4] and [5, p. 98]) that the equational class K generated by the unit interval equipped with multiplication is strictly contained in the class of PMV-algebras. Other results can be found in [6, 7]. Much work will be needed for a better understanding of free PMV-algebras, their principal quotients and the deductive-algorithmic machinery of their underlying logic. Even less is known about K .

Further applications of multiplicative MV-algebras will be discussed in later chapters. It should be noted that the Slovak school defines “product MV-algebras” as PMV-algebras satisfying an additional condition, [8].

* * *

Semisimple tensor products naturally arise in the MV-algebraic formulation of “if then else” definitions by cases. To see this, let us start from the familiar situation when a boolean function f on the domain $\{0, 1\}^k$ is defined by cases using the formula

$$(\alpha_1 \rightarrow \beta_1) \wedge \cdots \wedge (\alpha_t \rightarrow \beta_t). \quad (9.27)$$

To ensure the single-valuedness of f , the formulas $\alpha_1, \dots, \alpha_t$ must be pairwise incompatible. To ensure that the domain of f covers all possible truth-value assignments in $\{0, 1\}^k$, the formula $\alpha_1 \vee \cdots \vee \alpha_t$ must be a tautology. A routine exercise in boolean propositional logic then shows that formula (9.27) is equivalent to the following disjunction of conjunctions:

$$(\alpha_1 \wedge \beta_1) \vee \cdots \vee (\alpha_t \wedge \beta_t). \quad (9.28)$$

Here, the $\wedge$ connective between each α_i and β_i is conveniently read as “and then”.

Next suppose the cases $\alpha_1, \dots, \alpha_t$ in the definition of f are not sharply distinguishable, and may partially overlap. Then Schauder bases yield a suitable generalization of the boolean partition of $\{0, 1\}^k$ induced by the α_i . To this purpose, assuming the domain P of f to be a rational polyhedron in $[0, 1]^k$, we prepare a regular triangulation Δ of P . The boolean algebra generated by the α_i is replaced by the MV-algebra $\mathcal{M}(P)$ generated by the Schauder basis $\mathcal{H}_\Delta$, as in Theorem 5.8. Let $v_1, \dots, v_l$ be

the vertices of Δ , with their denominators $m_1, \dots, m_l$ and Schauder hats $h_i = h_{v_i}$. For each $i = 1, \dots, l$ let $a_i = m_i h_i \in \mathcal{M}(P)$.

Just as $\alpha_1 \vee \dots \vee \alpha_t$ is a tautology expressing the exhaustiveness of the cases α_i , the McNaughton functions a_i form a partition of unity: in other words, in the unital ℓ -group $(G, 1)$ corresponding to $\mathcal{M}(P)$ the following identity holds:

$$\sum_{i=1}^l a_i(x) = 1 \quad \text{for all } x \in P. \quad (9.29)$$

By Lemma 5.6 this property can be expressed in the language of MV-algebras as a conjunction of finitely many identities. The pairwise incompatibility of the α_j now takes the form $a_i \odot a_j = 0$ for all $i \neq j$, which is an immediate consequence of (9.29). To give an MV-algebraic generalization of the boolean algebra generated by the formulas β_i in (9.29) we can, e.g., fix once and for all some semisimple MV-algebra B (only depending on $\mathbf{f}$) and let, for each $i = 1, \dots, l$, some element $b_i \in B$ be linked to each a_i via a suitable $\otimes$ operation generalizing the “and then” connective of (9.28). More light is cast on the nature of $\otimes$, upon noting that $\otimes$ must distribute over $\oplus$, as well as over sums $x + y$ taken in $(G, 1)$ for all $x, y, x + y \in \mathcal{M}(P)$. As a matter of fact, let us suppose $b_i = b_j = b$ for two distinct i, j . In view of (9.29) we may reasonably expect that

$$(a_i \otimes b) \oplus (a_j \otimes b) \quad \text{is the same as} \quad (a_i \oplus a_j) \otimes b. \quad (9.30)$$

Because of the distributivity properties (9) and (11) in Definition 9.9, a natural choice for $\otimes$ is given by the semisimple tensor $\otimes$: should we replace $\otimes$ by $\wedge$ or $\odot$, condition (9.30) would no longer hold in general.

Summing up, an appropriate MV-algebraic generalization of formula (9.28) is given by the following element c of $\mathcal{M}(P) \otimes B$:

$$c = (a_1 \otimes b_1) \oplus \dots \oplus (a_l \otimes b_l). \quad (9.31)$$

Let us suppose the same function $\mathbf{f}$ is described by four elements $c_1, \dots, c_4$ of $\mathcal{M}(P) \otimes B$ satisfying $c_1, c_2 \leq c_3, c_4$. Then one may ask under what conditions there is an element $c \in \mathcal{M}(P) \otimes B$ of the form (9.31) satisfying $c_1, c_2 \leq c \leq c_3, c_4$. This naturally leads to the problem of constructing a Schauder partition which is a joint refinement of the four Schauder partitions of unity of $c_1, \dots, c_4$. The theory of bases and their algebraic subdivisions developed in Theorem 6.14(iii) can be used for this purpose (See [9] for details).

The special role of MV-algebras in the treatment of imprecisely defined functions is also discussed in [10].

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Chapter 10

States and the Kroupa–Panti Theorem

The equivalence proved in Theorem 1.4 at the outset of this book, between the coherence of a map $\beta: \{X_1, \dots, X_n\} \rightarrow [0, 1]$ and its extendability to a convex combination of valuations in Łukasiewicz logic $\mathbb{L}_\infty$, is a prelude to the rich theory of states of MV-algebras, regular Borel probability measures on their maximal spectral spaces, invariant Rényi conditionals in Łukasiewicz logic, and abstract integration in finitely presented MV-algebras. In this chapter, states and Borel probability measures will be used to give further characterizations of coherent assessments on $[0, 1]$ -valued events. We will also prove that the extreme states of any MV-algebra A coincide with the homomorphisms of A into the standard MV-algebra $[0, 1]$. This result provides an interpretation of valuations in Łukasiewicz logic as extremal coherent probability assessments of the possible outcomes of measurements of continuous observables of physical systems.

This chapter is largely independent of the earlier chapters. The reader is only required to have some familiarity with maximal spectral spaces and with the Γ functor. Appendix B collects the basic prerequisites of functional analysis needed for the understanding of the main results: the Stone–Weierstrass, the Krein–Milman, and the Riesz representation theorem.

10.1 States

Definition 10.1 A *state* of an MV-algebra A is a map $s: A \rightarrow [0, 1]$ such that $s(1) = 1$ and $s(x \oplus y) = s(x) + s(y)$ whenever $x, y \in A$ and $x \odot y = 0$.

Every $\eta \in \text{hom}(A)$ is a state of A , and so is every convex combination of states of A . We let $\text{ST}(A)$ denote the convex set of states of A , equipped with the topology of the Tychonov cube $[0, 1]^A$. It is easy to see that $\text{ST}(A)$ is compact.

Given a unital ℓ -group (G, u) , by a *state* t of (G, u) we mean a positive normalized additive map $t: G \rightarrow \mathbb{R}$, i.e., a group homomorphism such that $t(G^+) \subseteq \mathbb{R}^+$ and

$t(u) = 1$. We similarly let $\mathbf{ST}(G, u) \subseteq \mathbb{R}^G$ denote the convex set of states of (G, u) . From Proposition 10.3 it will follow that $\mathbf{ST}(G, u)$ is a compact subset of $\mathbb{R}^G$.

Let D denote either of A or (G, u) . Then a state s of D is *extreme* if $s \notin \text{conv}(\mathbf{ST}(D) \setminus \{s\})$. We denote by $\text{ext}(\mathbf{ST}(D))$ the set of extreme states of D .

Recall our precedence rules for formulas: $\neg$ is more binding than $\odot$, and the latter is more binding than $\oplus$.

Proposition 10.2 *For every state s of an MV algebra A we have:*

- (i) (Monotony) *If $x \leq y$ then $s(x) \leq s(y)$.*
- (ii) (Additivity) *$s(x) + s(y) = s(x \oplus y) + s(x \odot y)$ for all $x, y \in A$.*
- (iii) *$s(0) = 0$.*

Proof To prove (i) let $d = \neg x \odot y$. Then $x \odot d = x \odot \neg x \odot y = 0$, and $x \oplus d = x \oplus (\neg x \odot y) = \neg(x \oplus \neg y) \oplus x = \neg(\neg x \oplus y) \oplus y = \neg 1 \oplus y = y$. It follows that $s(y) = s(x \oplus d) = s(x) + s(d) \geq s(x)$.

To prove (ii), we first note the trivial identity $(x \oplus y) \odot \neg y \odot x \odot y = 0$. Next, replacing y by $\neg y$ in (A21.15) we get $x \odot y \oplus (x \oplus y) \odot \neg y = x$, whence $s(x) = s((x \oplus y) \odot \neg y) + s(x \odot y)$. Interchanging the roles of x and y we similarly have $s(y) = s((x \oplus y) \odot \neg x) + s(x \odot y)$. Arguing by cases according as $x \leq y$ or $x > y$, the following equation is easily verified to hold in the standard MV-algebra $[0, 1]$:

$$((x \oplus y) \odot \neg y) \odot ((x \oplus y) \odot \neg x) = 0. \quad (10.1)$$

Arguing by cases according as $x + y < 1$ or $x + y \geq 1$, a tedious but straightforward verification shows that the following identities hold in $[0, 1]$:

$$((x \oplus y) \odot \neg y \oplus (x \oplus y) \odot \neg x) \odot x \odot y = 0, \quad (10.2)$$

$$(x \oplus y) \odot \neg y \oplus (x \oplus y) \odot \neg x \oplus x \odot y = x \oplus y. \quad (10.3)$$

By Chang's completeness theorem (A21.17), identities (10.1–10.3) hold in A . Summing up,

$$\begin{aligned} s(x) + s(y) &= s((x \oplus y) \odot \neg y) + s((x \oplus y) \odot \neg x) + s(x \odot y) + s(x \odot y) \\ &= s((x \oplus y) \odot \neg y \oplus (x \oplus y) \odot \neg x) + s(x \odot y) + s(x \odot y) \\ &= s((x \oplus y) \odot \neg y \oplus (x \oplus y) \odot \neg x \oplus x \odot y) + s(x \odot y) \\ &= s(x \oplus y) + s(x \odot y), \end{aligned}$$

as required to complete the proof of (ii).

(iii) easy. □

Given vector spaces $\mathcal{V}$ and $\mathcal{W}$ and convex subsets $K \subseteq \mathcal{V}$ and $L \subseteq \mathcal{W}$, by an *affine isomorphism* between K and L we mean a one–one map of K onto L that preserves convex combinations.

Proposition 10.3 *Let $A = \Gamma(G, 1)$ be an MV-algebra with its associated unital ℓ -group $(G, 1)$. Then for every state t of $(G, 1)$ the restriction of t to A is a state of A . The map $t \rightarrow t \upharpoonright A$ is an affine isomorphism of $\mathbf{ST}(G, 1) \subseteq \mathbb{R}^G$ onto $\mathbf{ST}(A) \subseteq [0, 1]^A$. Thus, the extreme states of $(G, 1)$ are in one-one correspondence with the extreme states of A .*

Proof Let us write for short $\dot{t}$ instead of $t \upharpoonright A$. By (A21.16)(i), for all $a, b \in A$ we have $a + b = (a \oplus b) + (a \odot b)$. As a consequence,

$$\begin{aligned} \dot{t}(a) + \dot{t}(b) &= t(a) + t(b) \\ &= t(a + b) \\ &= t(a \oplus b + a \odot b) \\ &= t(a \oplus b) + t(a \odot b) \\ &= \dot{t}(a \oplus b) + \dot{t}(a \odot b), \end{aligned}$$

whence $\dot{t}$ is a state of A .

Conversely, for every state s of A we will extend s to a monoid homomorphism $\bar{s}: G^+ \rightarrow \mathbb{R}^+$, and further extend $\bar{s}$ to a state of $(G, 1)$. Without loss of generality, G^+ can be identified with the partially ordered monoid M_A of good sequences of A (see [1, 2.2–2.3]).

Construction of $\bar{s}$.

For every $x \in G^+$ let $(g_1, \dots, g_n)$ be the good sequence of x . Thus each g_i belongs to A , $g_i \oplus g_{i+1} = g_i$ and $x = g_1 + \dots + g_n$. By [1, 2.2], the n -tuple $(g_1, \dots, g_n)$ is uniquely determined up to a finite final list of zeros. We then define

$$\bar{s}(x) = \bar{s}((g_1, \dots, g_n)) = s(g_1) + \dots + s(g_n).$$

To show that $\bar{s}$ is additive, it suffices to show $\bar{s}(x + h) = \bar{s}(x) + \bar{s}(h)$ for each $h \in A$. To this purpose, following [1, 2.2.4], we define the $(n + 1)$ -tuple $(g'_1, \dots, g'_{n+1})$ by

$$g'_1 = g_1 \oplus h, g'_2 = g_2 \oplus g_1 \odot h, g'_3 = g_3 \oplus g_2 \odot h, \dots, g'_n = g_n \oplus g_{n-1} \odot h, g'_{n+1} = g_n \odot h.$$

Arguing as in [1, p. 37] (or, in [2, 3.8.2]) and recalling our identification $G^+ = M_A$ it follows that $(g'_1, \dots, g'_{n+1})$ is the good sequence of $x + h$. A straightforward computation using Proposition 10.2(ii) yields:

$$\begin{aligned} \bar{s}(h + x) &= \bar{s}((g'_1, \dots, g'_{n+1})) \\ &= s(g_1 \oplus h) + s(g_2 \oplus g_1 \odot h) + s(g_3 \oplus g_2 \odot h) \\ &\quad + \dots + s(g_n \oplus g_{n-1} \odot h) + s(g_n \odot h) \end{aligned}$$

$$\begin{aligned}
&= s(h) + s(g_1) - s(g_1 \odot h) + s(g_2) + s(g_1 \odot h) - s(g_2 \odot h) + s(g_3) \\
&\quad + s(g_2 \odot h) - s(g_3 \odot h) + \cdots + s(g_n) + s(g_{n-1} \odot h) - s(g_n \odot h) \\
&\quad + s(g_n \odot h) \\
&= s(h) + s(g_1) + \cdots + s(g_n) = \bar{s}(h) + \bar{s}((g_1, \dots, g_n)) = \bar{s}(h) + \bar{s}(x),
\end{aligned}$$

as required to prove the additivity of $\bar{s}$. The identity $\bar{s}(0) = 0$ follows from Proposition 10.2(iii). Since $\bar{s}$ is a monoid homomorphism of G^+ into $\mathbb{R}^+$, and $G = G^+ - G^+$, upon setting $\check{s}(x - y) = \bar{s}(x) - \bar{s}(y)$ we extend $\bar{s}$ to a state $\check{s}$ of $(G, 1)$.

The mutually inverse maps $t \mapsto \check{t}$ and $s \mapsto \check{s}$ provide the desired affine isomorphism between $\mathbf{ST}(A)$ and $\mathbf{ST}(G, 1)$. $\square$

10.2 The Kroupa–Panti Theorem

As we did in Sect. 4.6, for every MV-algebra A we will use the abbreviated notation $A' = A/\text{Rad}(A)$.

Every state s of A satisfies $s(r) = 0$ whenever $r \in \text{Rad}(A)$: for otherwise, with the notation of (2.8), $s(n \cdot r) > s(\neg r)$ for all large n , and r cannot be infinitesimal [1, 3.6.3, 3.6.4]. Therefore, the map $\rho: s \in \mathbf{ST}(A) \mapsto s' \in \mathbf{ST}(A')$ defined by

$$s'(a) = s(a/\text{Rad}(A)) \quad \text{for all } a \in A \quad (10.4)$$

is an affine isomorphism of $\mathbf{ST}(A) \subseteq [0, 1]^A$ onto $\mathbf{ST}(A') \subseteq [0, 1]^{A'}$, in symbols,

$$\rho: \mathbf{ST}(A) \cong \mathbf{ST}(A/\text{Rad}(A)). \quad (10.5)$$

For any compact Hausdorff space X we denote by $P(X)$ the convex set of regular Borel probability measures ν on X , where ν is said to be *regular* if for all Borel sets $Y \subseteq X$

$$\nu(Y) = \sup\{\nu(K) \mid K \subseteq Y, K \text{ closed}\} = \inf\{\nu(U) \mid U \supseteq Y, U \text{ open}\}. \quad (10.6)$$

There cannot be any danger of confusion between regular Borel measures and regular simplexes and cones.

If A is a countable MV-algebra, the maximal spectral space $\mu(A)$ has the property that every open set is a countable union of compact sets. Then by (B21.65), every Borel probability measure on $\mu(A)$ is regular.

Definition 10.4 A *real vector lattice* (also known as a *Riesz space*) is a real vector space V which is also a lattice by a partial order relation $x \leq y$ in such a way that for all $x, y, t \in V$ and $0 \leq \gamma \in \mathbb{R}$,

$$x \leq y \Rightarrow x + t \leq y + t \quad \text{and} \quad \gamma x \leq \gamma y.$$

The definition of *rational vector lattice* is the same, with V a rational vector space. Recall from Theorem 4.16 the definition of the map $f \mapsto f^*$.

Theorem 10.5 (Kroupa–Panti theorem). *For any MV-algebra A there is an affine isomorphism $\nu \mapsto s_\nu$ of the convex set of regular Borel probability measures on the maximal spectral space $\mu(A)$, onto the state space of A . For every element $f \in A$, and $\mathfrak{m} \in \mu(A)$,*

$$s_\nu(f) = \int_{\mu(A)} f^*(\mathfrak{m}) \, d\nu(\mathfrak{m}). \quad (10.7)$$

Proof In view of (10.5), it is sufficient to prove the theorem for semisimple A . By (A21.32) we can write $A = \mathcal{M}(X)$ for some nonempty closed $X \subseteq [0, 1]^\kappa$ and cardinal $0 < \kappa$. By Corollary 4.18 we can identify the topological spaces X and $\mu(A)$ in such a way that for every $g = g^* \in A$ and $x \in X$,

$$g(x) = g(\mathfrak{h}_x) = g/\mathfrak{h}_x.$$

Let $(G, 1)$ be defined by $\Gamma(G, 1) = \mathcal{M}(X)$. $(G, 1)$ is the unital ℓ -group of all continuous piecewise linear functions $f: X \rightarrow \mathbb{R}$ where each linear piece of f has integer coefficients, and f only depends on finitely many variables. Let $P(X)$ denote the convex set of regular Borel probability measures on X .

Let $\mathcal{C}(X)$ be the Banach algebra of real-valued continuous functions on X , with the sup norm $\|\cdot\|$. Let $\mathcal{C}(X)_\ell$ denote the underlying divisible abelian lattice-ordered group of $\mathcal{C}(X)$ with the distinguished order-unit given by the constant function 1 and with multiplication by rational scalars, which makes $\mathcal{C}(X)_\ell$ into a rational vector lattice. Then $\mathbf{ST}(\mathcal{C}(X)_\ell) \subseteq \mathbb{R}^{\mathcal{C}(X)}$ coincides with the convex set of normalized positive linear functionals on $\mathcal{C}(X)$.

By the Riesz representation theorem (B21.64), $\mathbf{ST}(\mathcal{C}(X)_\ell)$ is affinely isomorphic to $P(X)$ via the integral map $\nu \in P(X) \mapsto t_\nu \in \mathbf{ST}(\mathcal{C}(X)_\ell)$ given by

$$t_\nu(f) = \int_Y f \, d\nu \quad \text{for all } f \in \mathcal{C}(X)_\ell. \quad (10.8)$$

Since Proposition 10.3 provides an affine isomorphism of $\mathbf{ST}(G, 1)$ onto $\mathbf{ST}(A)$, there remains to exhibit an affine isomorphism of $\mathbf{ST}(G, 1)$ onto $\mathbf{ST}(\mathcal{C}(X)_\ell)$.

To this purpose, let $\mathbb{Q}G = \{cf \mid f \in G, c \in \mathbb{Q}\}$ be the divisible hull of G , as in (B21.70). We can identify $\mathbb{Q}G$ with a rational vector sub-lattice of $\mathcal{C}(X)_\ell$ with the same order-unit 1 as $(G, 1)$. We then have natural unital ℓ -embeddings $G \hookrightarrow \mathbb{Q}G \hookrightarrow \mathcal{C}(X)_\ell$. Every $s \in \mathbf{ST}(G, 1)$ uniquely extends to a state s' of $(\mathbb{Q}G, 1)$ by stipulating that, for all $f \in G$, $s'(cf) = cs(f)$. Then s' has the following boundedness property, for every $g \in \mathbb{Q}G$:

$$\|g\| \leq 1 \Rightarrow |s'(g)| \leq s'(1) = 1.$$

It follows that s' is continuous: for all $f \in \mathbb{Q}G$

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall g \in \mathbb{Q}G, \quad (\|f - g\| < \delta \Rightarrow |s'(f) - s'(g)| < \epsilon).$$

By the Stone–Weierstrass theorem (B21.63), $\mathbb{Q}G$ is norm-dense in $\mathcal{C}(X)$. Therefore, the stipulation

$$s''(\lim f_i) = \lim s'(f_i), \quad (f_i \in \mathbb{Q}G)$$

yields a unique extension $s'': \mathcal{C}(X) \rightarrow [0, 1]$ of s' . It is easy to see that $s'' \in \mathbf{ST}(\mathcal{C}(X)_\ell)$. We will only verify positivity: if $0 \leq f \in \mathcal{C}(X)$ then f can be written as the (norm) limit of elements of the positive cone $(\mathbb{Q}G)^+$ of $\mathbb{Q}G$. This is an immediate consequence of the trivial fact that every $0 \leq f \in \mathcal{C}(X)$ is obtainable as a limit of functions $0 \leq g_j \in \mathcal{C}(X)_\ell$.

In conclusion, $s \in \mathbf{ST}(G, 1)$ uniquely extends to $s'' \in \mathbf{ST}(\mathcal{C}(X)_\ell)$. Every state of $\mathcal{C}(X)_\ell$ arises in this way as an extension of its restriction to G . The map $s \mapsto s''$ is the desired affine isomorphism of $\mathbf{ST}(G, 1)$ onto $\mathbf{ST}(\mathcal{C}(X)_\ell)$. The integral formula (10.7) follows from (10.8). $\square$

Corollary 10.6 *The extreme states of any MV-algebra A coincide with the homomorphisms of A into $[0, 1]$. In symbols, $\text{hom}(A) = \text{ext}(\mathbf{ST}(A))$.*

Proof By Proposition (10.2)(ii), $\text{hom}(A) \subseteq \mathbf{ST}(A)$.

Conversely, in view of formula (10.4) it is no loss of generality to assume that A is semisimple, $A = A/\text{Rad}(A)$. By (A21.32), for some cardinal κ and closed subset X of $[0, 1]^\kappa$ we can write $A = \mathcal{M}(X)$. In the notation of 4.17 we can write $X = \mu(A)$ and $f(x) = f(\mathfrak{h}_x) = f/\mathfrak{h}_x$ for every $x \in X$.

Claim Extreme elements of $P(A)$ coincide with *point masses* at $\mathfrak{m} \in X$, i.e., those Borel probability measures $\nu_{\mathfrak{m}}$ of the form $\nu_{\mathfrak{m}}(Z) = 1$ iff $\mathfrak{m} \in Z$, (for Z an arbitrary Borel set of X).

For the sake of completeness we give here the routine proof. First of all, for each $\mathfrak{m} \in X$ the point mass $\nu_{\mathfrak{m}}$ is extreme: for otherwise, if $\mu, \nu \in P(A)$ are such that $\mu_{\mathfrak{m}} = \lambda\mu + (1 - \lambda)\nu$ (for some $0 < \lambda < 1$), then

$$1 = \mu_{\mathfrak{m}}(\{\mathfrak{m}\}) = \lambda\mu(\{\mathfrak{m}\}) + (1 - \lambda)\nu(\{\mathfrak{m}\}),$$

which is < 1 unless $\mu(\{\mathfrak{m}\}) = \nu(\{\mathfrak{m}\}) = 1$, i.e., $\mu_{\mathfrak{m}} = \mu = \nu$, a contradiction.

Conversely, suppose ν is different from all point masses $\nu_{\mathfrak{m}}$ and extreme in $P(A)$, (absurdum hypothesis). Then

$$\text{for no closed set } Y \in \mu(A), 0 < \nu(Y) < 1. \quad (10.9)$$

For otherwise, letting $Z = X \setminus Y$, we define the regular Borel probability measures ν_Y and ν_Z by stipulating that for all Borel sets $B \subseteq X$

$$\nu_Y(B) = \frac{\nu(B \cap Y)}{\nu(Y)} \quad \text{and} \quad \nu_Z(B) = \frac{\nu(B \cap Z)}{\nu(Z)}.$$

We obtain $\nu = \nu(Y)\nu_Y + (1 - \nu(Y))\nu_Z = \nu(Y)\nu_Y + \nu(Z)\nu_Z$, against the assumption that ν is extreme. This settles (10.9).

To conclude the proof of the claim, let $\mathcal{J}$ be the set of all closed subsets F of X such that $\nu(F) = 1$. Observe that $\mathcal{J}$ has the finite intersection property, and pick an $\mathfrak{n} \in \bigcap \mathcal{J}$. We cannot have $\nu(\{\mathfrak{n}\}) = 1$; for otherwise, ν would be the point mass at $\mathfrak{n}$. Further, we cannot have $\nu(\{\mathfrak{n}\}) > 0$, for otherwise, the closed set $\{\mathfrak{n}\}$ would contradict (10.9). Thus we can only have $\nu(\{\mathfrak{n}\}) = 0$. By the assumed regularity of ν , (condition 10.6 above), there is an open neighborhood $N \ni \mathfrak{n}$ with $0 < \nu(N) < 1/2$. The complement $X \setminus N$ is a closed set with $0 < \nu(X \setminus N) < 1$, again contradicting (10.9). Our claim is proved.

From the Kroupa–Panti theorem it follows that all extreme states of A arise from point masses via the integral formula (10.7). By Theorem 4.16, for every point mass $\nu_{\mathfrak{m}}$ the corresponding state $s_{\mathfrak{m}}$ is the evaluation map at $\mathfrak{m}$ of all functions $f \in A \subseteq C(X)$,

$$s_{\mathfrak{m}}(a) = a/\mathfrak{m} \quad \text{for all } a \in A.$$

Different point masses give different extreme states and different homomorphisms. We then have a one–one map of the set of extreme states of A into $\text{hom}(A)$. To see that the map is onto $\text{hom}(A)$ let $\eta \in \text{hom}(A)$ and $\mathfrak{m} = \ker(\eta)$. By (A21.12), $\mathfrak{m}$ is a maximal ideal of A , and $\eta(a) = a/\mathfrak{m}$ for all $a \in A$. By Theorem 4.16 we can write $\eta(A) = A/\mathfrak{m} \subseteq [0, 1]$. Thus η coincides with the quotient map $A \mapsto A/\mathfrak{m}$. $\square$

10.3 Further Characterizations of de Finetti Coherence Criterion

Theorem 10.7 (Continuation of Theorem 1.4) *With the notation of Theorem 1.4, we have the following further equivalences:*

- (i) β is W -coherent.
- (vi) There is a state $s \in \text{ST}(\mathcal{M}(W))$ such that $\beta(X_i) = s(\pi_i \upharpoonright W)$ for all $i = 1, \dots, n$.
- (vii) There is a (necessarily regular) Borel probability measure ν on W such that for all $i = 1, \dots, n$,

$$\beta(X_i) = \int_W \pi_i d\nu.$$

Proof W is a compact Hausdorff space in which every open set is a countable union of compact sets. By (B21.65), every Borel measure on W is regular.

(i $\Rightarrow$ vi) From Lemma 1.14 it follows that $W = \text{Mod}(\text{Th } W)$. By Theorem 1.4, $\beta \in \text{conv}(\text{Mod}(\text{Th } W))$, and there is a convex combination of valuations $V_1, \dots, V_t \in \text{VAL}_n$, each V_i satisfying $\text{Th } W$, with coefficients $\gamma_1, \dots, \gamma_t \geq 0$, such that $\beta = (\gamma_1 V_1 + \dots + \gamma_t V_t) \upharpoonright E$ and $\gamma_1 + \dots + \gamma_t = 1$. For each $j = 1, \dots, t$, let the function

$s_j: \mathcal{M}(W) \rightarrow [0, 1]$ be defined by $s_j(\hat{\phi} \upharpoonright W) = V_j(\phi)$, for all $\phi \in \mathbf{FORM}_n$. Then s_j is well defined: for all formulas $\psi, \chi \in \mathbf{FORM}_n$ we have $\hat{\psi} \upharpoonright W = \hat{\chi} \upharpoonright W$ iff $V(\psi) = V(\chi)$ for every $V \in \mathbf{VAL}_n$ satisfying $\text{Th } W$. Actually, s_j is a homomorphism of $\mathcal{M}(W)$ into $[0, 1]$, whence s_j is a state of $\mathcal{M}(W)$, and so is the convex combination $s = \gamma_1 s_1 + \cdots + \gamma_t s_t$. Obviously, $\beta(X_i) = s(\pi_i \upharpoonright W)$.

(vi $\Rightarrow$ i) By the Krein–Milman theorem (B21.66),

$$\mathbf{ST}(\mathcal{M}(W)) = \text{cl conv ext}(\mathbf{ST}(\mathcal{M}(W))) \subseteq [0, 1]^{\mathcal{M}(W)}.$$

Thus a map $r: \mathcal{M}(W) \rightarrow [0, 1]$ belongs to $\mathbf{ST}(\mathcal{M}(W))$ iff r is the limit, in the product topology of $[0, 1]^{\mathcal{M}(W)}$, of convex combinations of extreme states of $\mathcal{M}(W)$. From Corollary 10.6 it follows that $\text{ext}(\mathbf{ST}(\mathcal{M}(W))) = \text{hom}(\mathcal{M}(W))$. Letting $\mathcal{E} = \{\pi_1 \upharpoonright W, \dots, \pi_n \upharpoonright W\}$, from Carathéodory theorem (B21.55) it follows that for all $\epsilon > 0$ there is a convex combination $s' = \sum_{j=0}^n \lambda_j \eta_j$ ($0 \leq \lambda_j, \sum \lambda_j = 1$) of homomorphisms $\eta_j \in \text{hom}(\mathcal{M}(W))$, such that, for all $i = 1, \dots, n$,

$$|s'(\pi_i \upharpoonright W) - s(\pi_i \upharpoonright W)| = |s'(\pi_i \upharpoonright W) - \beta(X_i)| < \epsilon. \quad (10.10)$$

Writing K as an abbreviation of $\text{hom}(\mathcal{M}(W))$, the continuity of the MV-algebraic operations $\neg, \oplus, \odot$ on $[0, 1]$ guarantees that K is a closed subset of $[0, 1]^{\mathcal{M}(W)}$, whence $K \upharpoonright \mathcal{E}$ is a closed subset of $[0, 1]^{\mathcal{E}}$. By (B21.50), $(\text{conv}(K)) \upharpoonright \mathcal{E} = \text{conv}(K \upharpoonright \mathcal{E})$ is a closed subset of $[0, 1]^{\mathcal{E}} = [0, 1]^{\{1, \dots, n\}}$. Without loss of generality we can assume $s' \upharpoonright \mathcal{E} = s \upharpoonright \mathcal{E}$, and strengthen (10.10) to $s'(\pi_i \upharpoonright W) = s(\pi_i \upharpoonright W) = \beta(X_i)$. For each $j = 0, \dots, n$, let the valuation V_j be defined by $V_j(\psi) = \eta_j(\hat{\psi} \upharpoonright W)$, for every $\psi \in \mathbf{FORM}_n$. Fix $\theta \in \mathbf{FORM}_n$. By Theorem 1.5(v),

$$\theta \in \text{Th } W \Rightarrow \hat{\theta} \upharpoonright W = 1 \Rightarrow \eta_j(\hat{\theta} \upharpoonright W) = 1 \Rightarrow V_j(\theta) = 1.$$

Since θ is an arbitrary formula in $\text{Th } W$, we conclude that V_j satisfies $\text{Th } W$ and $V_j \upharpoonright E = (V_j(X_1), \dots, V_j(X_n)) \in \text{Mod}(\text{Th } W)$. Let $U = \sum_{j=0}^n \lambda_j V_j$. It follows that $\beta(X_i) = s'(\pi_i \upharpoonright W) = s(\pi_i \upharpoonright W) = U(X_i)$ for all $i = 1, \dots, n$. We conclude that $\beta \in \text{conv}(\text{Mod}(\text{Th } W))$. Now apply Theorem 1.4.

(vi $\Leftrightarrow$ vii) By Corollary 4.18 we can identify W with the maximal spectral space $\mu(\mathcal{M}(W))$, and write $f(\mathfrak{m}) = f/\mathfrak{m}$ for each $f \in \mathcal{M}(W)$ and maximal ideal $\mathfrak{m}$ of $\mathcal{M}(W)$. Now formula (10.7) in the statement of the Kroupa–Panti theorem yields the desired conclusion. $\square$

10.4 Coherent Assessments of Infinite Sets of Events

For any nonempty set $\mathcal{E}$ and nonempty convex subset K of $[0, 1]^{\mathcal{E}}$, we denote by $\text{ext}(K)$ the set of all *extreme points* of K .

Generalizing Definition 1.1 we stipulate:

Definition 10.8 Suppose we are given a nonempty set $\mathcal{E}$, a subset $\mathcal{W}$ of $[0, 1]^\mathcal{E}$ and a map $\beta: \mathcal{E} \rightarrow [0, 1]$. Let $\mathcal{E}' = \{a_1, \dots, a_n\}$ be a finite subset of $\mathcal{E}$. Then β is said to be $\mathcal{W}$ -coherent on $\mathcal{E}'$ if for all $\sigma_1, \dots, \sigma_n \in \mathbb{R}$ there is $V \in \mathcal{W}$ such that $\sum_{i=1}^n \sigma_i (\beta(a_i) - V(a_i)) \geq 0$. Further, β is said to be $\mathcal{W}$ -coherent on $\mathcal{E}$ if it is coherent on every finite nonempty subset $\mathcal{E}'$ of $\mathcal{E}$. We let $\mathcal{E}_\mathcal{W}$ denote the set of all $\mathcal{W}$ -coherent maps on $\mathcal{E}$.

For the sake of readability, whenever no confusion may arise, we will write $\text{cl } X$, $\text{conv } X$ and $\text{int } X$ instead of $\text{cl}(X)$, $\text{conv}(X)$ and $\text{int}(X)$.

Theorem 10.9

(i) For any nonempty set $\mathcal{E}$ and closed set $\mathcal{W} \subseteq [0, 1]^\mathcal{E}$, $\mathcal{E}_\mathcal{W}$ is the closure of the convex hull of $\mathcal{W}$ in $[0, 1]^\mathcal{E}$, $\mathcal{E}_\mathcal{W} = \text{cl conv } \mathcal{W}$.

(ii) For any MV-algebra A , $\emptyset \neq \mathbf{ST}(A) = \text{cl conv}(\text{hom}(A)) = \mathcal{E}_{\text{hom}(A)}$ = the set of $\text{hom}(A)$ -coherent maps on $\mathcal{E}$.

Proof (i) We first settle

Claim 1 $\text{conv } \mathcal{W} \subseteq \mathcal{E}_\mathcal{W}$.

As a matter of fact, suppose $V_1, \dots, V_h$ are elements of $\mathcal{W}$ and $\beta = \sum_{i=1}^h \gamma_i V_i \in \text{conv } \mathcal{W}$, where $0 \leq \gamma_i$ for all $i = 1, \dots, h$ and $\sum_{i=1}^h \gamma_i = 1$. If $\beta \notin \mathcal{E}_\mathcal{W}$ (absurdum hypothesis), there are elements $a_1, \dots, a_k \in \mathcal{E}$ and real numbers $\sigma_1, \dots, \sigma_k$ such that, for all $U \in \mathcal{W}$, $\sum_{j=1}^k \sigma_j \beta(a_j) < \sum_{j=1}^k \sigma_j U(a_j)$. In particular, for each $i = 1, \dots, h$, $\sum_{j=1}^k \sigma_j \beta(a_j) < \sum_{j=1}^k \sigma_j V_i(a_j)$. Thus, $\sum_{j=1}^k \sigma_j \beta(a_j) < \sum_{i=1}^h \gamma_i \sum_{j=1}^k \sigma_j V_i(a_j)$, and $\sum_{j=1}^k \sigma_j \sum_{i=1}^h \gamma_i V_i(a_j) < \sum_{i=1}^h \gamma_i \sum_{j=1}^k \sigma_j V_i(a_j)$, a contradiction. (The hypothesis that $\mathcal{W}$ is closed has not been used in the proof of this claim.)

We now strengthen Claim 1 as follows:

Claim 2 $\text{cl conv } \mathcal{W} \subseteq \mathcal{E}_\mathcal{W}$.

For otherwise, if $\beta \in (\text{cl conv } \mathcal{W}) \setminus \mathcal{E}_\mathcal{W}$, there exist $b_1, \dots, b_k \in \mathcal{E}$ together with $\sigma_1, \dots, \sigma_k \in \mathbb{R}$ such that $\sum_i \sigma_i \beta(b_i) < \sum_i \sigma_i V(b_i)$ for all $V \in \mathcal{W}$. Identifying, as above, $\mathcal{W} \upharpoonright \{b_1, \dots, b_k\}$ with a set of points in $[0, 1]^k$, from the hypothesis that $\mathcal{W}$ is closed it follows that the set $\mathcal{W} \upharpoonright \{b_1, \dots, b_k\}$ is closed. By continuity there is $U \in \mathcal{W}$ such that $\sum_i \sigma_i U(b_i) = \min_{V \in \mathcal{W}} \sum_i \sigma_i V(b_i)$. For some $0 < \lambda \in \mathbb{R}$ we have $\lambda = \sum_{i=1}^k \sigma_i U(b_i) - \sum_{i=1}^k \sigma_i \beta(b_i)$. Since $\beta \in \text{cl conv } \mathcal{W}$ we have

$$\forall \epsilon > 0 \exists \beta_\epsilon \in \text{conv}(\mathcal{W}) \text{ such that } |\beta(b_i) - \beta_\epsilon(b_i)| < \epsilon \quad (i = 1, \dots, k).$$

For all sufficiently small $\epsilon > 0$, we then have $\sum_i \sigma_i \beta_\epsilon(b_i) < \lambda/2 + \sum_i \sigma_i \beta(b_i) < \sum_i \sigma_i U(b_i) \leq \sum_i \sigma_i V(b_i)$ for every $V \in \mathcal{W}$. In other words, β_ϵ is not coherent on $\{b_1, \dots, b_k\}$ whence $\beta_\epsilon \notin \mathcal{E}_\mathcal{W}$, which contradicts Claim 1.

In order to prove the converse inclusion $\mathcal{E}_\mathcal{W} \subseteq \text{cl conv } \mathcal{W}$, we make the following

Claim 3 For every subset $\mathcal{A} = \{c_1, \dots, c_k\}$ of $\mathcal{E}$ there exists $\beta' \in \text{conv } \mathcal{W}$ such that $\beta(c_i) = \beta'(c_i)$ for all $i = 1, \dots, k$.

Let $\mathcal{W} \upharpoonright \mathcal{A} = \{v \in [0, 1]^k \mid v = (V(c_1), \dots, V(c_k)) \text{ for some } V \in \mathcal{W}\}$. Since $\mathcal{W}$ is closed, $\mathcal{W} \upharpoonright \mathcal{A}$ is a compact subset of the cube $[0, 1]^k$, whence so is its convex hull $\text{conv } \mathcal{W} \upharpoonright \mathcal{A}$ in $[0, 1]^k$, by (B21.50). By way of contradiction, suppose no $\beta' \in \text{conv } \mathcal{W}$ satisfies $\beta = \beta'$ on $\mathcal{A}$. Let the vector $w \in [0, 1]^k$ be defined by $w = (\beta(c_1), \dots, \beta(c_k)) \in [0, 1]^k$. By (B21.51), for some vector $s \in \mathbb{R}^k$ and $\tau \in \mathbb{R}$, the hyperplane $H = \{x \in \mathbb{R}^k \mid s \circ x = \tau\}$ strongly separates the point w from the set $\text{conv } \mathcal{W} \upharpoonright \mathcal{A}$. In other words, $s \circ w < \tau$ and $s \circ v > \tau$ for all $v \in \mathcal{W} \upharpoonright \mathcal{A}$. As a consequence, $s \circ (w - v) = s \circ w - s \circ v < 0$ for all $v \in \mathcal{W} \upharpoonright \mathcal{A}$. Letting $\sigma_1, \dots, \sigma_k$ be the coordinates of s , it follows that $\sum_{i=1}^k \sigma_i (\beta(c_i) - V(c_i)) < 0$ for every valuation $V \in \mathcal{W}$. This contradicts the assumption that β is coherent on $\mathcal{A}$. Our third claim is settled.

For any finite subset $\mathcal{A}$ of $\mathcal{E}$ we now define

$$\mathcal{C}_{\mathcal{A}} = \{\gamma \in \text{cl conv } \mathcal{W} \mid \gamma = \beta \text{ on } \mathcal{A}\}.$$

Since by Claim 3, $\text{conv } \mathcal{W} \cap \mathcal{C}_{\mathcal{A}} \neq \emptyset$, then $\mathcal{C}_{\mathcal{A}}$ is a nonempty closed subset of $\text{cl conv } \mathcal{W} \subseteq [0, 1]^{\mathcal{E}}$. From

$$\mathcal{C}_{\mathcal{A}_1} \cap \dots \cap \mathcal{C}_{\mathcal{A}_m} = \mathcal{C}_{\mathcal{A}_1 \cup \dots \cup \mathcal{A}_m}$$

it follows that the family $\{\mathcal{C}_{\mathcal{A}} \mid \mathcal{A} \text{ a finite subset of } \mathcal{E}\}$ has the finite intersection property. As a consequence, some (necessarily unique) $\gamma \in \text{cl conv } \mathcal{W}$ belongs to the intersection of all $\mathcal{C}_{\mathcal{A}}$. Then $\gamma = \beta$, and hence, $\beta \in \text{cl conv } \mathcal{W}$, which completes the proof of (i).

(ii) Let $\mathcal{W} = \text{hom}(A)$. By Proposition 4.15, A has a maximal ideal $\mathfrak{m}$. The map $a \in A \mapsto a/\mathfrak{m}$ is a homomorphism of A into $[0, 1]$. By Corollary 10.6, $\text{ext}(\text{ST}(A)) = \mathcal{W}$. From the continuity of the MV-algebraic operations in the MV-algebra $[0, 1]$ it follows that $\mathcal{W}$ is a closed subset of the compact cube $[0, 1]^A$. As already noted, every homomorphism of A into $[0, 1]$ is a state, $\mathcal{W} \subseteq \text{ST}(A)$. By the Krein–Milman theorem (B21.66),

$$\text{ST}(A) = \text{cl conv ext}(\text{ST}(A)) = \text{cl conv } \mathcal{W} \subseteq \text{cl conv ST}(A) = \text{ST}(A),$$

because $\text{ST}(A)$ is a closed convex subset of $[0, 1]^A$. Thus by (i), $\mathcal{E}_{\mathcal{W}} = \text{cl conv } \mathcal{W} = \text{ST}(A)$. $\square$

10.5 Remarks

For the special case when the events $X_1, \dots, X_n$ belong to a boolean algebra A , and $\mathcal{W}$ is the set of homomorphisms of A into $\{0, 1\}$, de Finetti showed that coherence is necessary and sufficient for the existence of a state (called by him a “prevision”) ν on A such that $\nu(X_i) = \beta(X_i)$ for all i , (see [3, 311–312], [4, Chap. 1], [5, 85–90]).

Over the last decade, de Finetti's characterization has been extended to (events represented by) formulas in various nonclassical logics, [6–8]. In [9, 10] one can find an algebraic characterization of de Finetti's criterion in terms of a class of structures where states are defined as internal operators. Non-reversible bookmaking is considered in [11].

Maybe it is time to pause for a comment on the frequently heard claim that Finetti's coherence criterion deals with finitely additive probability measures. This may follow the wrong impression that the theory of finitely additive measures on boolean algebras is more general than the theory of regular Borel measures. Actually, following [12, 18.7], it would be more accurate to say the reverse: finitely additive measures (i.e., states) on boolean algebras correspond to regular Borel measures on the totally disconnected compact Hausdorff spaces given by their dual Stone spectra. Moreover, the results of this chapter show that de Finetti's coherence criterion (Definition 1.1) can be used to characterize regular probability Borel measures on any compact Hausdorff space.

For the Kroupa–Panti theorem see [13] and [14, 1.1].

In combination with Theorem 1.4, Theorem 10.7 shows the universal role of Łukasiewicz logic to interpret coherent probability assessments on any set E of continuously valued events. Both theorems were first proved in [15].

Theorem 10.9 was first proved in [7].

For a detailed account on vector lattices see [16] and [12].

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Chapter 11

The MV-Algebraic Loomis–Sikorski Theorem

An MV-algebra A is said to be σ -complete if its underlying lattice is closed under countable suprema. It follows that A is semisimple, whence it is isomorphic to an MV-algebra A^* of continuous $[0, 1]$ -valued functions defined on some compact Hausdorff space X . In the representation $A \cong A^*$ the image a^* of the supremum a of a countable sequence of elements $a_i \in A$ is in general different from the pointwise supremum of the functions $a_i^* \in A^*$. The MV-algebraic Loomis–Sikorski theorem, however, extends A^* to a σ -complete MV-algebra $\mathcal{T} \subseteq [0, 1]^X$ of functions, closed under pointwise suprema of countable sequences in $\mathcal{T}$, in such a way that A is the image of $\mathcal{T}$ under a homomorphism that also preserves countable suprema. This is the main result of the present chapter, which is largely independent of the previous ones.

11.1 Basically Disconnected Spaces

An MV-algebra A is σ -complete if every countable set of elements a_i of A has a least upper bound (also known as a sup, or supremum) a , with respect to the underlying lattice order of A . Thus $a \geq a_i$ for each $i = 0, 1, \dots$, and whenever $b \in A$ is an upper bound of all the a_i , it follows that $b \geq a$. We write $a = \bigvee_i a_i$ to mean that a is the sup of the a_i in A . The ambient MV-algebra A will always be clear from the context.

By (A21.38) every σ -complete MV-algebra is semisimple.

Given two σ -complete MV-algebras A and B , a homomorphism $\rho: A \rightarrow B$ is called a σ -homomorphism if for each sequence $a_0, a_1, \dots \in A$, the image $\rho(\bigvee_i a_i)$ of the sup in A of the a_i coincides with the sup $\bigvee_i \rho(a_i)$ in B of the sequence $\rho(a_i), \rho(a_2), \dots$.

Every finite MV-algebra A is σ -complete—indeed, A is *complete*, in the sense that every nonempty subset of A has a supremum in A . The standard example of

a complete MV-algebra is $[0, 1]$. Further examples of σ -complete MV-algebras are given by σ -complete boolean algebras.

A trivial adaptation of the proof of (A21.39) in [1, p. 130] shows that every σ -complete MV-algebra A satisfies the *countable distributivity law*

$$b \wedge (b_1 \vee b_2 \vee \cdots) = (b \wedge b_1) \vee (b \wedge b_2) \vee \cdots. \quad (11.1)$$

A subset of a topological space X is said to be an F_σ if it is the union of countably many closed subsets of X . A subset of X is a G_δ if it is the intersection of countably many open subsets of X .

Definition 11.1 A topological space X is said to be *basically disconnected* if the closure of every open F_σ subset of X is open.

To stress the relevance of this topological property we record here the following result:

Lemma 11.2 *For any topological space $X \neq \emptyset$ the following conditions are equivalent:*

- (i) X is homeomorphic to the maximal spectral space of a σ -complete boolean algebra B .
- (ii) X is a totally disconnected compact Hausdorff space such that the closure of any countable union of clopen subsets of X is open.
- (iii) X is a basically disconnected compact Hausdorff space.

Proof (i $\Rightarrow$ ii) By Stone duality, X is a totally disconnected compact Hausdorff space. Further, we can identify B with a σ -complete boolean algebra of clopen subsets of X . For any countable family $C_i \in B$ the clopen $C = \bigvee_i C_i \in B$ need not coincide with the open F_σ set $D = \bigcup_i C_i \subseteq C$. However, $C \setminus D$ cannot contain any nonempty clopen set E ; for otherwise, $C \setminus E$ is an upper bound of the C_i strictly contained in C , against the definition of C . The clopen set $\text{cl}(D)$ is still contained in C , and cannot be strictly contained in C , for otherwise, $C \setminus \text{cl}(D)$ contains a nonempty clopen set, which again contradicts the definition of C . Thus the closure of $\bigcup_i C_i$ coincides with the (cl)open set C .

(ii $\Rightarrow$ i) Let B be the boolean algebra of clopen subsets of X , with the intent of showing that B is σ -complete. Given a countable family of clopen sets C_i , the clopen set $C = \text{cl}(\bigcup_i C_i) \in B$ is an upper bound of each C_i . To show that C is the least upper bound in B , by way of contradiction suppose $B \ni D \supsetneq \bigcup_i C_i$ and $x \in C \setminus D$. Then the clopen set $C \setminus D$ contains a clopen neighborhood N of x , and $C \setminus N$ is a closed set of B strictly contained in C and containing $\bigcup_i C_i$, thus contradicting the definition of C . Thus C is the sup of the C_i in B , and B is σ -complete. By Stone duality, X is homeomorphic to $\mu(B)$.

(iii $\Leftrightarrow$ i) This is a well-known classical result, (B21.68). $\square$

An element b of an MV-algebra A is said to be *boolean* if $b \oplus b = b$. Recall that $B(A)$ denotes the center of A , i.e., the MV-subalgebra of A given by all boolean elements of A .

For every set E and function $f : E \rightarrow [0, 1]$ the (open) support of f is the set

$$\text{supp}(f) = \{x \in E \mid f(x) > 0\}.$$

There cannot be any danger of confusion with the support $|\Delta|$ of a simplicial complex.

Lemma 11.3 *If A is a σ -complete MV-algebra then $B(A)$ is a σ -complete boolean algebra, and for any sequence $b_i \in B(A)$ the supremum of the b_i in $B(A)$ coincides with their supremum in A .*

Proof A routine adaptation of the proof of (A21.40) in [1, p.131]. $\square$

Lemma 11.4 *Let A be a σ -complete MV-algebra.*

- (i) *If $\mathfrak{m} \neq \mathfrak{n}$ are maximal ideals of A then some element $b \in B(A)$ satisfies $b/\mathfrak{m} = 0$ and $b/\mathfrak{n} = 1$.*
- (ii) *The map $\iota : \mathfrak{m} \mapsto \mathfrak{m} \cap B(A)$ is a homeomorphism of $\mu(A)$ onto $\mu(B(A))$.*
- (iii) *$\mu(A)$ is a basically disconnected compact Hausdorff space.*

Proof By (A21.38), A is semisimple and by Theorem 4.16(ii), without loss of generality, we can identify A with the separating MV-subalgebra A^* of $C(\mu(A))$.

(i) Using Proposition 4.2 we let $f \in A$ be such that $f = 0$ on an open neighborhood N of $\mathfrak{m}$ and $f(\mathfrak{n}) = 1$.

Claim 1 Let $b = \bigvee_n f_n$ be the supremum in A of the sequence $f_1 = f$, $f_2 = f \oplus f$, $f_3 = f \oplus f \oplus f$, $\dots$. Then b identically vanishes on N , and in particular, $b(\mathfrak{m}) = 0$.

By way of contradiction, suppose $q \in N$ satisfies $b(q) > 0$. Then b is > 0 on some open neighborhood $U \subseteq N$ of q . By definition of the topology of $\mu(A)$, there is $g \in A$ such that $q \in \text{supp}(g) \subseteq U$. Trivially, each f_i identically vanishes over N . It follows that $b \ominus g \in A$ is an upper bound of the f_i and $b \ominus g < b$, (because $b \ominus g \leq b$ and $(b \ominus g)(q) < b(q)$), thus contradicting the definition of b .

Claim 2 b is a boolean function.

For otherwise, $b(\mathfrak{p}) \notin \{0, 1\}$ for some $\mathfrak{p} \in \mu(A)$. By continuity, there is an open neighborhood W of $\mathfrak{p}$ such that $b(W) \cap \{0, 1\} = \emptyset$. Let S be the common open support of each f_i . Since $b(x) = 1$ for every $x \in S$, then W is disjoint from S , whence every f_i identically vanishes on W . By definition of the topology of $\mu(A)$, there exists $l \in A$ with $\mathfrak{p} \in \text{supp}(l) \subseteq W$. It follows that the function $b \ominus l$ belongs to A and is an upper bound in A of the f_i satisfying $b \ominus l < b$, against the definition of b .

(ii) Under our identification $A = A^*$, $B(A)$ is the MV-subalgebra of A consisting of all $\{0, 1\}$ -valued functions of A . By (A21.12)(ii), ι sends every maximal ideal of A into a maximal ideal of $B(A)$. By (i), $\mu(A)$ is mapped by ι one-one into $\mu(B(A))$.

Fix now a maximal ideal $\mathfrak{b} \in \mu(B(A))$, and identify $\mathfrak{b}$ with a point $x_{\mathfrak{b}}$ of $\mu(B(A)) \subseteq \mu(A)$. The intersection Z of the zerosets of all functions in $\mathfrak{b}$ is a nonempty closed subset of $\mu(A)$. By (i), Z is a singleton $\{y\}$. The set of elements of A vanishing at y is a maximal ideal $\mathfrak{h}_y$ of A containing $\mathfrak{b}$. It follows that $\iota(\mathfrak{h}_y) = \mathfrak{b}$, and ι is onto $\mu(B(A))$.

Finally, to prove that ι is a homeomorphism, let us identify $\mu(A)$ and $\mu(B(A))$ as sets via the one–one correspondence ι . A clopen basis of the topology of $\mu(B(A))$ is given by the zerosets of the $\{0, 1\}$ -valued functions of A . Let O be an arbitrarily chosen nonempty open set in $\mu(A)$, and $\mathfrak{p}$ a maximal ideal in O . Let $\mathcal{C}$ be the set of all clopens of $\mu(A)$ containing $\mathfrak{p}$. By (i), the intersection $\bigcap \mathcal{C}$ is the singleton $\{\mathfrak{p}\}$. Since $\mu(A)$ is compact, some clopen $C \in \mathcal{C}$ will satisfy $C \subseteq O$. We have just shown that the (ι^{-1}) -images of the clopens of $\mu(B(A))$ constitute a basis for the topology of $\mu(A)$. As a consequence, ι is a homeomorphism of $\mu(A)$ onto $\mu(B(A))$.

(iii) By Lemma 11.3, $B(A)$ is a σ -complete boolean algebra. By (B21.68), the maximal ideal space of $B(A)$ is a basically disconnected compact Hausdorff space. Now apply (ii). $\square$

Lemma 11.5 *Let A be a σ -complete MV-algebra, identified with the separating subalgebra $A^* \subseteq C(\mu(A))$ of Theorem 4.16.*

- (i) *For every clopen $C \subseteq \mu(A^*)$ the characteristic function $\chi_C: \mu(A) \rightarrow \{0, 1\}$ belongs to A^* .*
- (ii) *For every nonzero ideal $\mathfrak{j}$ of the MV-algebra $C(\mu(A))$, $\mathfrak{j} \cap B(A)$ is a nonzero ideal of $B(A)$, and $\mathfrak{j} \cap A$ is a nonzero ideal of A .*
- (iii) *If a is the supremum in A of a sequence of elements $a_i \in A$, then the supremum of the a_i in $C(\mu(A))$ exists and equals a .*

Proof (i) is an immediate consequence of Lemma 11.4(ii).

(ii) Evidently, $\mathfrak{j} \cap B(A)$ is an ideal of $B(A)$ and $\mathfrak{j} \cap A$ is an ideal of A . To prove that these ideals are nonzero, arguing as in Lemma 11.4(ii) and using Theorem 4.16(iii), we can write without loss of generality

$$\mu(B(A)) = \mu(A) = \mu(C(\mu(A))).$$

Fix a function $f \in \mathfrak{j}$ other than the constant 0. By assumption, the zeroset $\mathcal{Z}f$ is a closed proper subset of $\mu(A)$. By Lemmas 11.4(iii) and 11.2, some nonempty clopen C is contained in $\mu(A) \setminus \mathcal{Z}f$. Let the function $b: \mu(A) \rightarrow \{0, 1\}$ be defined by $b(n) = 1$ iff $n \in C$. Then b is continuous and vanishes over an open set containing $\mathcal{Z}f$. Because $C \subseteq \text{supp}(f)$ is compact, there is an integer $m > 0$ such that $m \cdot f \geq b$, where as usual, $m \cdot f = f \oplus f \oplus f \oplus \cdots \oplus f$ (m times). As a consequence, b belongs to the ideal generated by f in $C(\mu(A))$, whence b belongs to $\mathfrak{j}$. By (i), b belongs to $B(A)$. Thus both $\mathfrak{j} \cap B(A)$ and $\mathfrak{j} \cap A$ are nonzero ideals.

(iii) We will prove the equivalent formulation of (iii) stating that whenever 0 is the infimum in A of a sequence of elements $a_0, a_1, \dots \in A$, then 0 is also the infimum of the a_i in $C(\mu(A))$.

By way of contradiction, let $0 \neq l \in C(\mu(A))$ be any arbitrary lower bound of the a_i . Let $\{0\} \neq \mathfrak{l}$ be the ideal generated by l in $C(\mu(A))$. Let $0 \neq a \in \mathfrak{l} \cap A$ be as given by (ii). There is a fixed integer $n > 0$ such that $a \leq n \cdot l \leq n \cdot a_i$ for all $i = 0, 1, \dots$. Recalling that $n \cdot \bigwedge a_i = \bigwedge a_i$ in A , the countable distributivity law (11.1) shows that the infimum of the sequence $n \cdot a_0, n \cdot a_1, \dots$ is 0, whence $a = 0$, a contradiction. $\square$

11.2 The MV-Algebraic Loomis–Sikorski Theorem

Definition 11.6 By a *tribe* on a nonempty set Ω we mean an MV-algebra $\mathcal{T}$ of $[0, 1]$ -valued functions on Ω closed under pointwise countable sups; in other words, for any sequence $f_0, f_1, \dots \in \mathcal{T}$, the function $f = \sup_i f_i$ defined by $f(x) = \sup_i f_i(x)$ (for all $x \in \Omega$) also belongs to $\mathcal{T}$.

Any tribe is a σ -complete MV-algebra. A partial converse will be given in Theorem 11.7. To this purpose, a subset Z of a (compact Hausdorff) topological space Y is said to be *meager*, (or in alternative terminology, a *set of the first category*) if it is the union of a countable family of subsets of Y whose closure has empty interior. We say that two functions $f, g \in [0, 1]^Y$ *essentially coincide*, and we write

$$f \stackrel{\circ}{=} g,$$

if the set $\{x \in X \mid f(x) \neq g(x)\}$ is meager.

Let us denote by $*$ the inverse of the isomorphism $A \rightarrow A^* \subseteq C(X)$ of Theorem 4.16(ii).

Theorem 11.7 (Loomis–Sikorski theorem, MV-algebraic formulation) *Let A be a σ -complete MV-algebra with its maximal spectral space $X = \mu(A)$. Let $\mathcal{T} \subseteq [0, 1]^X$ be the set of functions l essentially coinciding with some function of A^* .*

- (i) *Each $l \in \mathcal{T}$ essentially coincides with exactly one function $\rho(l) \in A^*$. Further, $\mathcal{T}$ is a tribe and ρ is a σ -homomorphism of $\mathcal{T}$ onto A^* , whence the map $l \mapsto \rho(l)_*$ is a σ -homomorphism of $\mathcal{T}$ onto A .*
- (ii) *If, in addition, A has a $\ominus$ -distributive monoidal operation $\diamond$, then $\mathcal{T}$ is closed under pointwise multiplication, and for all $f, g \in \mathcal{T}$, $\rho(f \cdot g)_* = \rho(f)_* \diamond \rho(g)_*$.*

Proof Without loss of generality we can assume that A coincides with the MV-algebra $A^* \subseteq C(X)$ of Theorem 4.16(ii).

(i) The first statement is an immediate consequence of the definition of meager set. It is easy to see that $\mathcal{T}$ is an MV-algebra and that ρ is a homomorphism of $\mathcal{T}$ onto A .

There remains to be shown that $\mathcal{T}$ is a tribe and that ρ is a σ -homomorphism.

For $f_0, f_1, \dots$ a sequence of elements of $\mathcal{T}$, let as above $\sup_i f_i \in [0, 1]^X$ denote the pointwise sup of the f_i . Since for each $i = 0, 1, \dots$, $f_i \stackrel{\circ}{=} \rho(f_i)$, by definition of meager set we immediately obtain $\sup_i f_i \stackrel{\circ}{=} \sup_i \rho(f_i) \stackrel{\circ}{=} \bigvee_i \rho(f_i)$. It follows that $\sup_i f_i$ belongs to $\mathcal{T}$, and $\mathcal{T}$ is a tribe on X .

For the proof that ρ is a σ -homomorphism, let $f = \sup_i \rho(f_i)$ be the pointwise sup of the sequence of functions $\rho(f_i)$. Let $a = \bigvee_i \rho(f_i) \geq f$ be the sup in A of this sequence. For each $n = 0, 1, \dots$, let $a_n = \bigvee_{i=0}^n \rho(f_i) = \max_{i=0}^n \rho(f_i)$. Each a_n is a continuous $[0, 1]$ -valued function belonging to A . Evidently, the sequence $a_0 \leq a_1 \leq \dots$ converges pointwise to f . By Baire category theorem (B21.60), the subset Y of X given by $Y = \{y \in X \mid f \text{ is discontinuous at } y\}$ is meager.

Claim $f \doteq a$.

For otherwise (absurdum hypothesis) assume the set $D = \{y \in X \mid f(y) \neq a(y)\}$ is not meager. Then $D \cap (X \setminus Y)$ contains at least one element z , because D cannot be contained in the meager set Y . Let $0 < \epsilon = a(z) - f(z)$. Since both f and a are continuous at z , there is an open neighborhood U of z such that $a(x) - f(x) > 2\epsilon/3$ for all $x \in U$. Since by Lemma 11.4(iii) X is a basically disconnected compact Hausdorff space, U may be assumed clopen. Let the function $h: X \rightarrow [0, 1]$ be defined by $h(x) = 0$ for each $x \in X \setminus U$, and $h(x) = \epsilon/2$ for each $x \in U$. Since U is clopen, h is continuous. Thus the function $k = a - h$ is continuous, strictly smaller than a , and for all $i = 1, 2, \dots$, $\rho(f_i) \leq f \leq k$. It follows that a is not the sup in $C(X)$ of the sequence $\rho(f_i)$, which contradicts Lemma 11.5(iii). Our claim is settled, and the proof of (i) is complete.

(ii) By Lemma 9.3 and Corollaries 9.6–9.7, A is multiplicative, and so is every homomorphic image of A . Under the present hypothesis about A , the tribe $\mathcal{T}$ of (i) is closed under pointwise multiplication. Further, $\rho(f \cdot g) = \rho(f) \cdot \rho(g)$, whence $\rho(f \cdot g)_* = \rho(f)_* \diamond \rho(g)_*$. $\square$

For $\mathcal{S}$ a σ -field of sets on a nonempty set Ω (See Sect. 13.1 for more on fields and σ -fields of sets), let $\mathcal{F}$ be the set of all $\mathcal{S}$ -measurable functions $f: \Omega \rightarrow [0, 1]$. Trivially, every function constantly taking value ξ on Ω belongs to $\mathcal{F}$. Conversely, let us record the following result, for later use in this book.

Lemma 11.8 *Let $\mathcal{F}$ be a tribe on a nonempty set Ω . Let $\mathcal{S}$ be the family of subsets X of Ω whose characteristic function χ_X belongs to $\mathcal{F}$.*

(i) *$\mathcal{S}$ is a σ -field of sets on Ω , and each $f \in \mathcal{F}$ is $\mathcal{S}$ -measurable.*

(ii) *If, in addition, for each $\xi \in [0, 1]$ the function constantly taking value ξ on Ω belongs to $\mathcal{F}$, then $\mathcal{F}$ coincides with the set of all $\mathcal{S}$ -measurable functions $f: \Omega \rightarrow [0, 1]$.*

Proof For every function $g: \Omega \rightarrow [0, 1]$ let us agree to call $g^{-1}(1)$ the *oneset* of g . We also let

$$g_\infty = \inf_n \underbrace{g \odot \cdots \odot g}_{n \text{ times}} = \bigwedge \underbrace{\{g \odot \cdots \odot g \mid n = 1, 2, \dots\}}_{n \text{ times}},$$

where infima are taken pointwise. If g belongs to $\mathcal{F}$ then so does g_∞ ; further, g_∞ is a boolean function, and its oneset belongs to $\mathcal{S}$, and coincides with the oneset of g .

(i) Clearly, $\mathcal{S}$ is a σ -field of sets on Ω . For the second assertion, it is enough to show that for every rational $s \in [0, 1]$ the set $f^{-1}([s, 1])$ belongs to $\mathcal{S}$. This is trivially true of $s \in \{0, 1\}$. In case $0 < s < 1$ and $s \in \mathbb{Q}$, by (A21.18) there is a McNaughton function $f_s \in \mathcal{M}([0, 1])$ whose oneset coincides with $[s, 1]$. It follows that the composite function $q(x) = f_s(f(x))$ belongs to $\mathcal{F}$, and its oneset coincides with $f^{-1}([s, 1])$. Further, q_∞ is a boolean function of $\mathcal{F}$, its oneset belongs to $\mathcal{S}$ and coincides with the oneset of $f^{-1}([s, 1])$.

(ii) It suffices to show that every $\mathcal{S}$ -measurable $[0, 1]$ -valued function f with finite range belongs to $\mathcal{F}$. So let

$$f = \bigoplus_{i=1}^n \lambda_i \chi_{P_i} = \sum_{i=1}^n \lambda_i \chi_{P_i}$$

for a suitable partition $\{P_1, \dots, P_n\}$ of Ω into Borel sets P_i , and coefficients $\lambda_i \in [0, 1]$. Since by assumption, for each $i = 1, 2, \dots$ the constant function $\lambda_i: \Omega \rightarrow [0, 1]$ belongs to $\mathcal{F}$, then $\chi_{P_i} \in \mathcal{F}$, whence $\lambda_i \wedge \chi_{P_i} \in \mathcal{F}$. We conclude that $\bigoplus_{i=1}^n \lambda_i \chi_{P_i}$ is a member of $\mathcal{F}$. $\square$

11.3 Further Properties of Basically Disconnected Spaces

For later use in this book we record here some relevant material on basically disconnected spaces.

Lemma 11.9 *Let $X \neq \emptyset$ be a basically disconnected compact Hausdorff space. For each $x \in X$ let $\text{Clop}(x) = \{C \subseteq X \mid x \in C, C \text{ clopen}\}$. Given a sequence $l_1, l_2, \dots$ of functions in $C(X)$ let $l, \ell \in [0, 1]^X$ be defined by $l(x) = \sup\{l_i(x) \mid i = 1, 2, \dots\}$ and*

$$\ell(x) = \inf_{C \in \text{Clop}(x)} \sup_{y \in C} l(y) \quad \text{for all } x \in X.$$

It follows that $\ell \in C(X)$, and ℓ is the sup of the l_i in $C(X)$. Thus $C(X)$ is a σ -complete MV-algebra.

Proof Fix $0 < \alpha < 1$.

We first claim that the set $\ell^{-1}[0, \alpha) = \{x \in X \mid \ell(x) < \alpha\}$ is open. Indeed, for each $x \in \ell^{-1}[0, \alpha)$ there is $C \in \text{Clop}(x)$ such that $\sup_{y \in C} l(y) < \alpha$. Now for every $z \in C$, $C \in \text{Clop}(z)$ and $\ell(z) \leq \sup_{y \in C} l(y) < \alpha$. We have shown that every $x \in \ell^{-1}[0, \alpha)$ has a (cl)open neighborhood C contained in $\ell^{-1}[0, \alpha)$, and our first claim is settled.

Next we claim that also $\ell^{-1}(\alpha, 1] = \{x \in X \mid \ell(x) > \alpha\}$ is open. To prove this, for each $i = 1, 2, \dots$ we prepare the three sets

$$A_i = \ell^{-1}[\alpha + 1/i, 1], \quad B_i = l^{-1}[\alpha + 1/i, 1], \quad C_i = l^{-1}(\alpha + 1/i, 1].$$

Trivially, $C_i \subseteq B_i \subseteq C_{i+1}$, whence $\text{cl } C_i \subseteq \text{cl } B_i \subseteq \text{cl } C_{i+1}$. For each $x \in X$ we have

$$\ell(x) \geq \alpha + 1/i \Leftrightarrow \forall C \in \text{Clop}(x) \exists y \in C \text{ such that } l(y) \geq \alpha + 1/i.$$

It follows that $A_i = \text{cl } B_i$. Summing up,

$$\ell^{-1}(\alpha, 1] = \bigcup_i A_i = \bigcup_i \text{cl } B_i = \bigcup_i \text{cl } C_i.$$

Fix now $i = 1, 2, \dots$. For every $n = 1, 2, \dots$, the assumed continuity of l_n is to the effect that $l_n^{-1}(\alpha + 1/i, 1]$ is an open F_σ subset of X , whence so is $l^{-1}(\alpha + 1/i, 1] = C_i$. Since X is basically disconnected, $\text{cl } C_i$ is open, whence $\ell^{-1}(\alpha, 1]$ is open.

We have proved that ℓ is continuous.

We finally claim that ℓ is the sup of the l_i in $C(X)$. Indeed, by definition, $\ell \geq l_i$, whence $\ell \geq l_i$ for each $i = 1, 2, \dots$. Suppose now $h \in C(X)$ to be an upper bound of each l_i . For each $x \in X$ and $0 < \epsilon \in \mathbb{R}$ there is $C \in \text{Clop}(x)$ such that $h(y) < h(x) + \epsilon$ for all $y \in C$. Therefore,

$$\ell(x) \leq \sup_{y \in C} l(y) \leq \sup_{y \in C} h(y) \leq h(x) + \epsilon,$$

which yields the desired inequality $\ell \leq h$. □

Proposition 11.10 *Let X be a nonempty compact Hausdorff space.*

- (i) *X is basically disconnected iff $C(X)$ is a σ -complete MV-algebra.*
- (ii) *X is extremally disconnected iff $C(X)$ is a complete MV-algebra.*

Proof The $(\Rightarrow)$ direction is proved in Lemma 11.9. The $(\Leftarrow)$ direction follows from Theorem 4.16 and Lemma 11.4. Statement (ii) is a routine variant of (i). □

Let X be a compact Hausdorff space and $Y \subseteq X$. We then define the MV-algebra $C_Y(X)$ by

$$C_Y(X) = \{f \in C(X) \mid f(y) \in \{0, 1\} \text{ for all } y \in Y\}. \quad (11.2)$$

Definition 11.11 A subset Y of a basically disconnected compact Hausdorff space X is said to be *special* if whenever $C_1, C_2, \dots$ is a countable sequence of clopen subsets of X , all disjoint from Y , then $\text{cl}(\bigcup_i C_i)$ is disjoint from Y .

Trivial examples of special closed sets of X are $\emptyset$ and X .

Proposition 11.12 *Let $\emptyset \neq Y$ be a basically disconnected compact Hausdorff space, and Y be a special closed subset of X . Let $f_1, f_2, \dots$ be a sequence of functions in the MV-algebra $C_Y(X)$, and f be their sup in $C(X)$ as given by Lemma 11.9. Then $f \in C_Y(X)$, and f is the sup of the f_i in $C_Y(X)$. Thus $C_Y(X)$ is a σ -complete MV-algebra.*

Proof Without loss of generality the sequence $f_1, f_2, \dots$ is ascending. By way of contradiction, suppose

$$f \notin C_Y(X). \quad (11.3)$$

Then there is an element $y^* \in Y$ such that $f(y^*) = \xi \notin \{0, 1\}$. Since f is continuous there is $\epsilon > 0$ and a clopen C such that $y^* \in C$ and

$$f(C) \subseteq [\xi - \epsilon, \xi + \epsilon] \quad \text{and} \quad [\xi - \epsilon, \xi + \epsilon] \cap \{0, 1\} = \emptyset. \quad (11.4)$$

Let $J_1 \subseteq J_2 \subseteq \dots$ be a sequence of closed intervals in $[0, 1]$ whose union coincides with $[0, 1] \setminus \{0, 1\}$. For all $k = 1, 2, \dots$ let O_k be the interior of J_k and

$$C_{ik} = \text{cl}(f_i^{-1}(O_k)).$$

Since each f_i is continuous, our assumptions about X ensure that $f_i^{-1}(O_k)$ is an open F_σ in X , and C_{ik} is clopen. Evidently, $C_{ik} \subseteq f_i^{-1}(J_k) \subseteq X \setminus Y$. Let the clopen set $K \subseteq X$ be defined by

$$K = \text{cl} \left(\bigcup_{i,k} C_{ik} \right).$$

Claim $Y \cap K \neq \emptyset$.

Suppose $Y \cap K = \emptyset$, absurdum hypothesis. Then K is a clopen subset of X containing each point $z \in X$ such that for some $i = 1, 2, \dots$, $f_i(z) \notin \{0, 1\}$. By (11.4) there is a clopen subset $D \subseteq X$ such that $y^* \in D \subseteq X \setminus K$ and $f(D) \subseteq [f(y^*) - \epsilon, f(y^*) + \epsilon]$. From $f_i \leq f$ and $[f(y^*) - \epsilon, f(y^*) + \epsilon] \cap \{0, 1\} = \emptyset$ it follows that $f_i(x) < f(y^*) - \epsilon$ for all $x \in D$ and $i = 1, 2, \dots$. Let $l : X \rightarrow \mathbb{R}$ be the function taking constantly the value $f(y^*) - \epsilon$ on D , and the value 1 on $X \setminus D$. Then l is continuous, and the continuous function $l \wedge f$ is an upper bound in $C(X)$ of the sequence $f_1, f_2, \dots$ satisfying $g \wedge f < f$, which contradicts the definition of f . Our claim is settled.

By our claim, Y is not special. This contradiction shows that (11.3) is impossible. Having thus proved the first statement, the second follows from Lemma 11.5. The rest is clear. $\square$

A converse of Proposition 11.12 will be proved in Lemma 12.9.

11.4 Remarks

The present definition of tribe is taken from [2, 8.1.1] and [3, 2.25].

For the origins of basically disconnected compact Hausdorff spaces see [4, p.186] and [5].

For historical remarks and credits concerning the classical Loomis–Sikorski theorem see [6, 29.1, p. 93].

The MV-algebraic Loomis–Sikorski theorem was independently proved in [7] and [8].

A generalization of Lemma 11.8 can be found in [2, 8.1.6]. Also see [9, 3.2, 3.3].

In [6, 21.6] special closed sets are discussed with a different terminology. A generalization of Proposition 11.12 is proved in [10, 2.7].

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Chapter 12

The MV-Algebraic Stone–von Neumann Theorem

The two prototypical examples of multiplicative σ -complete MV-algebras are $C(X)$, for X a basically disconnected compact Hausdorff space, and its center $B(C(X))$. The latter turns out to be the most general possible σ -complete boolean algebra. In this chapter, any multiplicative σ -complete MV-algebra M will be represented as the MV-algebra of *all* continuous functions f on $\mu(M)$ having the property that $f/m \in \{0, 1\}$ for all $m \in \mu(M)$ such that $M/m = \{0, 1\}$. The class of multiplicative σ -complete MV-algebras will be completely classified in terms of pairs (X, F) where X ranges over all basically disconnected compact Hausdorff spaces, and F is an arbitrary special closed subset of X . For multiplicative *complete* MV-algebras M , this classification leads to the Stone–von Neumann theorem,

$$M \cong C(X) \times B(C(Y))$$

for suitable extremally disconnected compact spaces X and Y .

12.1 Basic Properties

In this chapter, M will denote a multiplicative σ -complete MV-algebra.

In the light of Theorem 4.16 we will identify M with M^* without explicit mention. M^* is a separating σ -complete MV-algebra of continuous functions on the basically disconnected compact Hausdorff space $X = \mu(M) = \mu(M^*)$. By Proposition 4.4(iv) the basic open subsets of X can be identified with the subsets of X having the form

$$\text{supp}(f) = \{m \in X \mid f(m) > 0\},$$

where f ranges over elements of M^* . By Corollary 9.6, the only $\ominus$ -distributive monoidal operation of M^* coincides with pointwise multiplication.

We say that a maximal ideal $\mathfrak{m}$ of M has *finite rank* if the quotient MV-algebra $M/\mathfrak{m}$ is finite. In this case, $\text{rank}(\mathfrak{m})$ stands for the number of elements of $M/\mathfrak{m}$. By Lemma 9.3, $M/\mathfrak{m}$ is multiplicative and

$$M/\mathfrak{m} = M(\mathfrak{m}) = \{0, 1\}. \quad (12.1)$$

In case $M/\mathfrak{m}$ is infinite we say that $\mathfrak{m}$ has *infinite rank*, and we write $\text{rank}(\mathfrak{m}) = \infty$.

Lemma 12.1 *Suppose M is a multiplicative σ -complete MV-algebra. Let $\mathfrak{m}$ be a maximal ideal of M of infinite rank. Then $M/\mathfrak{m}$ is uniquely isomorphic to $[0, 1]$, via the map $\overline{\mathfrak{m}}$.*

Proof Let us write $M/\mathfrak{m} = I$ for a uniquely determined MV-subalgebra of $[0, 1]$, via the map $\overline{\mathfrak{m}}$ of Theorem 4.16. Fix $\xi \in [0, 1]$ with the intent of proving $\xi \in I$. By (A21.27), I is a dense subchain of $[0, 1]$. Let $\alpha_1 < \alpha_2 < \alpha_3 < \dots$ be an ascending sequence of elements of I whose sup in $\mathbb{R}$ is ξ . Similarly let $\omega_1 > \omega_2 > \omega_3 > \dots$ be a descending sequence of elements of I whose inf is ξ . Let $M \ni a_1, a_2, \dots, z_1, z_2, \dots$ satisfy $a_i(\mathfrak{m}) = \alpha_i$, and $z_i(\mathfrak{m}) = \omega_i$. For all $i, j = 1, 2, \dots$ there are real numbers $\epsilon_1, \epsilon_2, \epsilon_3, \dots > 0$ such that $\alpha_i + \epsilon_i < \xi < \omega_j - \epsilon_j$. By Lemma 11.5(i), for every clopen $C \subseteq \mu(M)$ the characteristic function χ_C , as well as its negation, both belong to M . As a consequence, for suitable clopen neighborhoods $C_1, C_2, C_3, \dots$ of $\mathfrak{m}$, the functions a_i and z_j can be further assumed to satisfy the following conditions:

- (i) $a_i(C_i) \subseteq [\alpha_i - \epsilon_i, \alpha_i + \epsilon_i]$ and $a_i(\mu(M) \setminus C_i) = 0$.
- (ii) $z_j(C_j) \subseteq [\omega_j - \epsilon_j, \omega_j + \epsilon_j]$ and $z_j(\mu(M) \setminus C_j) = 1$.

It follows that $a_i < z_j$ for all i, j . Let a be the sup in M of the sequence $a_1, a_2, \dots$. Then $a \leq z_j$ for all j , whence $a(\mathfrak{m}) = \xi$ and $\xi \in I$. The isomorphism $[0, 1] \cong I$ is unique by Theorem 4.16 and (A21.45). $\square$

Definition 12.2 Let X be a topological space. A function $r: X \rightarrow [0, 1]$ is said to be *rectangular* if there is a clopen $C \subseteq X$ such that r is constant on C and zero on $X \setminus C$.

Definition 12.3 Let M be a multiplicative σ -complete MV-algebra, and $a \in M$. We then say that a is *admissible* if for each $\mathfrak{m} \in \mu(M)$ of finite rank, $a/\mathfrak{m} \in \{0, 1\}$.

Remark 12.4 For each $f \in M = M^*$, identifying $f/\mathfrak{m}$ with $f(\mathfrak{m})$ we immediately see that all elements of M are (continuous and) admissible. The converse will be proved in Theorem 12.7.

Lemma 12.5 *Suppose $M = M^*$ is a multiplicative σ -complete MV-algebra. Then every rectangular admissible function $f: \mu(M) \rightarrow [0, 1]$ belongs to M .*

Proof It is enough to argue in case $\text{range}(f) \subseteq \mathbb{Q}$. Then there is a rational $\rho \in [0, 1]$ and a clopen $K \subseteq \mu(M)$ such that $f(K) = \rho$ and $f(\mu(M) \setminus K) = 0$. If $\rho = 1$ the desired result immediately follows from Lemma 11.4(ii). In the general case, the admissibility of f together with Lemma 12.1 are to the effect that for each $\mathfrak{m} \in K$ there is $g_{\mathfrak{m}} \in M$ such that $g_{\mathfrak{m}}(\mathfrak{m}) = f(\mathfrak{m}) = \rho$. By (A21.18) there exists a McNaughton function $t \in \mathcal{M}([0, 1])$ having the following properties: t coincides

with the identity function on the closed interval $[0, \rho]$, t linearly decreases from the value ρ to 0 in some interval $[\rho, \tau]$, and vanishes over $[\tau, 1]$.

Let h_m denote the composite function tg_m . Then $h_m \in M$. Further, for each $x \in K$, $h_m(x) \leq g_m(x)$ and $h_m(x) \leq \rho$. Let χ_K denote the characteristic function of K . Again using Lemma 11.4 and replacing, if necessary, h_m by $h_m \wedge \chi_K$, we can further assume $h_m(\mu(M) \setminus K) = 0$. Summing up, for each $m \in K$ there exists $h_m \in M$ such that $h_m(m) = \rho = f(m)$ and $h_m \leq f$.

Fix now an integer $j = 1, 2, \dots$. Since both f and h_m are continuous, Lemma 11.4(iii) yields a clopen neighborhood $C_{j,m}$ of m such that $0 \leq f(y) - h_m(y) < 1/j$ for all $y \in C_{j,m}$. Since K is compact, there is a finite cover of K by pairwise disjoint clopen sets $C_{j,m(1)}, \dots, C_{j,m(u)}$ such that, letting $h_j = h_{m(1)} \vee \dots \vee h_{m(u)}$, it follows that $f \geq h_j$ and $f - h_j < 1/j$. Letting now h be the pointwise supremum of $h_1, h_2, \dots$, we have $h = f$. In conclusion, the function $h \in C(\mu(M))$ is the supremum in $C(\mu(M))$ of the sequence $h_1, h_2, \dots$. Letting l be the supremum in M of this sequence, by Lemma 11.5(iii) we have the desired conclusion $f = h = l$. $\square$

12.2 Representation

Lemma 12.6 *Let $M = M^*$ be a multiplicative σ -complete MV-algebra with its maximal spectral space X . In view of Lemma 11.4(ii) let us write $X = \mu(B(M)) = \mu(M) = \mu(C(X))$. Suppose $f: X \rightarrow [0, 1]$ is a continuous admissible function such that*

$$\text{cl}(f^{-1}(0, 1)) = X. \quad (12.2)$$

Then f belongs to M , and f is the supremum in M (= the supremum in $C(X)$) of a countable sequence of rational-valued, rectangular, admissible functions.

Proof For each $t = 3, 4, \dots$ let the open interval $I_t \subseteq [0, 1]$, and the subsets $O_t, C_t \subseteq X$ be defined by

$$I_t = \left(\frac{1}{t}, 1 - \frac{1}{t}\right), \quad O_t = f^{-1}(I_t), \quad C_t = \text{cl}(O_t).$$

Since O_t is an open F_σ and by Lemma 11.4(iii) X is basically disconnected, then C_t is clopen. Arguing by way of contradiction, it is not hard to see that C_t is contained in O_{t+1} . It follows that, for each $m \in C_t$ and $t = 3, 4, \dots$, $\text{rank}(m) = \infty$.

Now the same patching construction of the proof of Lemma 12.5 using the compactness of each clopen C_t yields a function $f_t \in M$ which is the sup of finitely many rational-valued rectangular functions of M , and also satisfies

$$0 \leq f - f_t < \frac{1}{t} \text{ on } C_t \text{ and } f_t = 0 \text{ on } X \setminus C_t. \quad (12.3)$$

Let $f^\sharp$ be the sup of $f_3, f_4, \dots$ in M , existing because M is σ -complete.

Claim $f = f^\sharp$.

By way of contradiction, let $x \in X$ be such that $f(x) \neq f^\sharp(x)$. Since both f and $f^\sharp$ are continuous, recalling (12.2), x can be assumed to lie in $f^{-1}(0, 1)$. There is a clopen $C \subseteq f^{-1}(0, 1)$ and a real $\delta > 0$ such that

$$|f - f^\sharp| > \delta \text{ on } C.$$

Case 1 $f^\sharp > f$ on C . Then by Lemma 12.5 the rectangular function $r: X \rightarrow [0, 1]$ constantly taking value δ on C and vanishing on $X \setminus C$, belongs to M , whence so does the function $f^\sharp \ominus r \geq f$. We have thus exhibited an upper bound in M for the sequence f_t strictly smaller than $f^\sharp$, against the definition of $f^\sharp$.

Case 2 $f^\sharp < f$ on C . By (12.3), for all sufficiently large t we have the inequality $f - f_t < \delta$, whence $f^\sharp$ is not an upper bound to f_t , against the definition of $f^\sharp$.

The identity between countable sups in M and in $C(X)$ follows from Lemma 11.5. $\square$

Theorem 12.7 *Suppose M is a multiplicative σ -complete MV-algebra, identified with the separating subalgebra $M^* \subseteq C(\mu(M))$. Let us write X for $\mu(M)$.*

- (i) *A $[0, 1]$ -valued function f defined on X belongs to M iff f is continuous and admissible.*
- (ii) *Every $f \in M$ is the supremum in M of a countable sequence of rational-valued, rectangular, admissible functions, and f also coincides with the supremum of these functions in $C(X)$.*

Proof (i) The $(\Rightarrow)$ direction was noted in Remark 12.4. $(\Leftarrow)$ The two sets $f^{-1}(0)$ and $f^{-1}(1)$ are closed G_δ in X . By Lemma 11.4, X is basically disconnected, and the two sets $C_0 = \text{int}(f^{-1}(0))$ and $C_1 = \text{int}(f^{-1}(1))$ are clopen. By Lemma 12.5 there is a boolean function $b \in M$ coinciding with f on $C = C_0 \cup C_1$, and vanishing on the complementary clopen $D = X \setminus C = \text{cl}(f^{-1}(0, 1))$. By Lemma 12.6, some function $f^\sharp \in M$ coincides with f on D and vanishes on C . In conclusion, the function $b \vee f^\sharp \in M$ coincides with f on X .

The proof of (ii) follows by combining the proofs of (i) and Lemma 12.6. $\square$

Corollary 12.8 *Let M be a multiplicative σ -complete MV-algebra. Let*

$$M' = \{f \in C(\mu(M)) \mid f(\mathfrak{m}) \in \{0, 1\} \text{ for all } \mathfrak{m} \in \mu(M) \text{ of finite rank}\}.$$

Then $M' = M^$.*

12.3 Classification

Lemma 12.9 *Suppose M is a multiplicative σ -complete MV-algebra. Let*

$$F_M \subseteq \mu(M)$$

be the set of all maximal ideals of M having finite rank. Then F_M is a special closed set of $\mu(M)$.

Proof For brevity let us write F instead of F_M . For each $f \in M = M^*$ the set $E_f = \{\mathfrak{m} \in \mu(M) \mid f = f \oplus f\}$ is closed. Then so is the set $\bigcap \{E_f \mid f \in M\}$, which in view of (12.1) is promptly seen to coincide with F .

Trivially, $\emptyset$ is a special closed set. So suppose F is nonempty. By way of contradiction, suppose F is not special, and let $C_1, C_2, \dots$ be a sequence of clopen sets all contained in $\mu(M) \setminus F$, such that $F \cap \text{cl}(\bigcup_j C_j)$ is nonempty. Let $\mathfrak{b} \in F \cap \text{cl}(\bigcup_j C_j)$. Without loss of generality, the C_j is pairwise disjoint. By hypothesis, for each $j = 1, 2, \dots$ and $x \in C_j$ the rank of the maximal ideal $\mathfrak{h}_x$ is infinite. By Lemma 12.1, $M/\mathfrak{h}_x = M(\mathfrak{h}_x) = [0, 1]$. Thus there is $f_x \in M$ and a clopen $C_x \ni x$ such that $f_x(C_x) \subseteq [5/11, 6/11]$. A routine compactness argument yields a function $f_j \in M$ such that $f_j(C_j) \subseteq [5/11, 6/11]$. By Lemma 12.5, M contains the characteristic function of C_j , whence we can safely assume that f_j identically vanishes on $\mu(M) \setminus C_j$. Summing up,

$$f_j \in M, \quad f_j(C_j) \subseteq [5/11, 6/11], \quad f_j(\mu(M) \setminus C_j) = 0.$$

Let f be the sup of $f_1, f_2, \dots$ in M . Since $\mathfrak{b}$ belongs to F , $f(\mathfrak{b}) \in \{0, 1\}$. Now, $f(\mathfrak{b}) \neq 0$, for otherwise, for some $k = 1, 2, \dots$ and point $z \in C_k$ we would have $f(z) < 1/11 < f_k(z)$, against the definition of f . So $f(\mathfrak{b}) = 1$. The continuity of f yields an index $i = 1, 2, \dots$ and a clopen $D \subseteq \mu(M)$ having nonempty intersection E with C_i such that $f(D) \subseteq [10/11, 1]$. Arguing as in the first part of the proof, there is a function $0 \neq g \in M$ having the following properties: $g(E) \subseteq [0, 1/11]$ and $g(\mu(M) \setminus E) = 0$. The function $f \ominus g$ belongs to M is an upper bound of $f_1, f_2, \dots$ and $f \ominus g < f$, again contradicting the definition of f .

We conclude that F is special. □

For each $i = 1, 2$ let (X_i, F_i) be a pair of topological spaces where $\emptyset \neq X_i$ is basically disconnected compact Hausdorff, and F_i is a special closed subset of X_i . We then say that these two pairs are *equivalent* if there is a homeomorphism η of X_1 onto X_2 such that $\eta(F_1) = F_2$.

By Lemma 12.9 and Corollary 12.8 we have

Theorem 12.10 *The map*

$$M \mapsto (\mu(M), F_M)$$

induces a one-one correspondence between isomorphism classes of multiplicative σ -complete MV-algebras and equivalence classes of pairs (X, F) such that $\emptyset \neq X$ is basically disconnected compact Hausdorff, and F is a special closed subset of X . The inverse map sends any such pair (X, F) to the MV-algebra $C_F(X)$.

12.4 Reconstruction from the Underlying Involutive Lattice Structure

Corollary 12.11 *Let M and N be multiplicative σ -complete MV-algebras. Suppose there exists a one–one map κ of M onto N such that, for all $x, y \in M$,*

$$\kappa(\neg x) = \neg \kappa(x) \quad \text{and} \quad \kappa(x \vee y) = \kappa(x) \vee \kappa(y).$$

Then κ is an isomorphism of M onto N .

Proof It is easy to see that $\kappa(1) = 1$, $\kappa(0) = 0$, and $\kappa(x \wedge y) = \kappa(x) \wedge \kappa(y)$. Similarly, also the inverse map κ^{-1} preserves the involutive lattice structure of M and N . Let us identify M with the separating MV-subalgebra M^* of $C(\mu(M))$. Then

$$f \in B(M) \Leftrightarrow (f \wedge \neg f = 0 \quad \text{and} \quad f \vee \neg f = 1).$$

As a consequence, the property of being a boolean element of M only depends on the negation and the underlying lattice operations of M . It follows that $\kappa \upharpoonright B(M)$ is an isomorphism of the boolean algebras $B(M)$ and $B(N)$. We will now repeatedly apply Lemma 11.4, and identify $\mu(M)$ and $\mu(B(M))$. By Stone duality, κ yields a homeomorphism of $\mu(B(M))$ onto $\mu(B(N))$ by the stipulation

$$\mathfrak{b} \mapsto \kappa(\mathfrak{b}) = \{\kappa(f) \mid f \in \mathfrak{b}\}.$$

For each $\mathfrak{b} \in \mu(B(M))$ let $\mathfrak{b}^\ddagger \in \mu(M)$ be the only maximal ideal of M corresponding to $\mathfrak{b}$ via the inverse of the map $\mathfrak{m} \mapsto \mathfrak{m} \cap B(M)$ of Lemma 11.4(ii). Let $\mathfrak{q}^\ddagger$ be similarly defined for each $\mathfrak{q} \in \mu(B(N))$. Then the maximal spectral spaces of $\mu(M)$ and $\mu(N)$ are homeomorphic via the map

$$\mathfrak{m} \mapsto \mathfrak{m} \cap B(M) \mapsto \kappa(\mathfrak{m} \cap B(M)) \mapsto (\kappa(\mathfrak{m} \cap B(M)))^\ddagger.$$

Claim For each $\mathfrak{b} \in \mu(B(M))$,

$$\text{rank}(\mathfrak{b}^\ddagger) = \infty \Leftrightarrow \neg b \in \mathfrak{b} \text{ and } f \wedge b = \neg f \wedge b \text{ for some } b \in B(M) \text{ and } f \in M = M^*.$$

As a matter of fact, the condition on the right hand side of this equivalence states that $b(\mathfrak{b}^\ddagger) = 1$ and f takes the constant value $1/2$ on the clopen support of b . By Lemmas 12.1 and 12.5, the condition is equivalent to $\text{rank}(\mathfrak{b}^\ddagger) = \infty$.

We have just shown that the property “the rank of $\mathfrak{b}^\ddagger$ is infinite” only depends on the involutive lattice structure of M . As a consequence, the homeomorphism induced by the map κ above sends the maximal ideals of M of infinite rank one–one onto the maximal ideals of N of infinite rank.

The desired conclusion now follows from Theorem 12.10. □

12.5 The MV-Algebraic Stone–von Neumann Theorem

Theorem 12.12 *Let M be a multiplicative complete MV-algebra. Then exactly one of the following cases occurs:*

- (i) $M \cong C(X)$ for some extremally disconnected compact Hausdorff space X .
- (ii) $M \cong B(C(Y))$ for some extremally disconnected compact Hausdorff space Y .
- (iii) $M \cong C(X) \times B(C(Y))$ for some nonempty extremally disconnected compact Hausdorff spaces X and Y . Further, the pair (X, Y) is uniquely determined up to homeomorphism.

Conversely, if an MV-algebra M satisfies any of the conditions (i–iii) then M is multiplicative and complete.

Proof Evidently, the cases (i–iii) are mutually exclusive. Proposition 11.10 in combination with Corollary 9.7 shows that, if M satisfies any of (i–iii) then M is multiplicative and complete. Assuming now that M satisfies neither condition (i) nor (ii), there remains to be proved that M satisfies (iii).

Let M be identified with a separating subalgebra of $C(\mu(M))$. Let $O \subseteq \mu(M)$ be the open set of maximal ideals of infinite rank. Since M is complete, then so is its center $B(M)$ (see (A21.40)), whence $\mu(B(M)) \cong \mu(M)$ is an extremally disconnected compact Hausdorff space. From Lemma 11.4(ii) it follows that $\text{cl}(O)$ is open.

Claim $\text{cl}(O) = O$.

Otherwise, there is a maximal ideal $\mathfrak{m} \in \text{cl}(O)$ having finite rank (absurdum hypothesis). Then $M/\mathfrak{m} = \{0, 1\}$. For each nonempty clopen $N \subseteq \text{cl}(O)$, Lemma 12.5 yields a nonempty clopen $C_N \subseteq N \cap O$ and a function $f_N \in M$ constantly taking the value $1/4$ on C_N and 0 on $\mu(M) \setminus C_N$. Let the function $f \in M$ be defined by

$$f = \bigvee \{f_N \mid \emptyset \neq N \subseteq \text{cl}(O), N \text{ clopen}\}. \quad (12.4)$$

Then $f(\mathfrak{m}) = 1$, for otherwise continuity yields a clopen neighborhood U of $\mathfrak{m}$ such that $f < 1/5$ on U , whence f cannot be an upper bound of the rectangular function f_W , for $W \subseteq O$ a clopen neighborhood of some $\mathfrak{n} \in O$ sufficiently close to $\mathfrak{m}$. Having thus shown that $f(\mathfrak{m}) = 1$, continuity now yields a clopen neighborhood $K \subseteq \text{cl}(O)$ of $\mathfrak{m}$ such that $f(K) \subseteq [3/4, 1]$. There is a nonempty clopen $C_K \subseteq K \cap O$ and a function $f_K \in M$ constantly taking value $1/4$ on C_K and vanishing on $\mu(M) \setminus C_K$. As a consequence, the function $f \ominus f_K \in M$ is an upper bound of all f_N in (12.4), and $f \ominus f_K < f$ on C_K , thus contradicting the definition of f . Our claim is settled.

In conclusion, $\mu(M)$ splits into two disjoint clopen sets $X = O$ and $Y = \mu(M) \setminus O$. Since M satisfies neither condition (i) nor condition (ii), both X and Y are nonempty. By Theorem 12.7, the assumed properties of O ensure that

$$M \cong C(X) \times B(C(Y)).$$

By Theorem 12.10, the pair (X, Y) is uniquely determined up to homeomorphism. This settles (iii). $\square$

12.6 Remarks

A generalization of Theorem 12.7 is proved in [1].

A strong form of Theorem 12.10 is proved in [2] in a categorical framework.

Corollary 12.11 was first proved in [3].

Other reconstruction theorems are considered in [4] and in [5].

The MV-subalgebras of $[0, 1]$ generated by any two distinct irrationals $\xi, \xi' \in [0, 1/2]$ are not isomorphic (see [6, Example, p. 149]), but their underlying posets are order-isomorphic.

See [7, p. 407] for the original Stone–von Neumann theorem.

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Chapter 13

Recurrence, Probability, Measure

In earlier chapters we studied the relationship between Łukasiewicz logic and probability theory for continuously valued events, with particular reference to de Finetti's coherence criterion, states, and the Kroupa–Panti theorem. This chapter is a brief excursion into three different domains of MV-algebraic probability and measure theory:

- (i) Riečan's MV-algebraic Poincaré recurrence theorem.
- (ii) Probability MV-algebras, as a main ingredient of the generalization of Carathéodory boolean algebraic probability theory to events described by formulas in $\mathbb{L}_\infty$.
- (iii) The Jordan representation of each bounded measure of an MV-algebra A as the difference between uniquely determined positive scalar multiples of two states of A .

13.1 Riečan's MV-Algebraic Poincaré Recurrence Theorem

Throughout this section, Ω will denote a nonempty set.

A *field of sets* on Ω is a nonempty collection $\mathcal{S}$ of subsets of Ω such that

$$A, B \in \mathcal{S} \Rightarrow A \cap B, \quad A \cup B, \quad \Omega \setminus A \in \mathcal{S}.$$

A *σ -field of sets* on Ω is a field of sets closed under countable unions.

Identifying each $A \in \mathcal{S}$ with its characteristic function $\chi_A: \Omega \rightarrow \{0, 1\}$, we will make no distinction between $\mathcal{S}$ and its associated tribe $\{\chi_A \mid A \in \mathcal{S}\}$.

For $\mathcal{S}$ a σ -field of sets on Ω let us assume we are given a map $P: \mathcal{S} \rightarrow [0, 1]$ satisfying $P(\Omega) = 1$ and the *σ -additivity condition* $P(\bigcup_i X_i) = \sum_i P(X_i)$ whenever $X_0, X_1, \dots$ is a sequence of pairwise disjoint subsets of Ω . The triple $(\Omega, \mathcal{S}, P)$ is known as a *probability space*.

Any function $T: \Omega \rightarrow \Omega$ satisfying $T^{-1}(X) \in \mathcal{S}$ and $P(T^{-1}(X)) = P(X)$ for all $X \in \mathcal{S}$ is said to be a *measure-preserving map*.

Poincaré's recurrence theorem states that, up to a P -negligible set of exceptions, every $x \in X$ will revisit X under the action of T , in symbols, $P(X \setminus \bigcup_{i=1}^{\infty} T^{-i}(X)) = 0$. It follows that

$$\lim_{t \rightarrow \infty} P\left(X \setminus \bigcup_{i=1}^t T^{-i}(X)\right) = 0.$$

The following result is an MV-algebraic counterpart of Poincaré's recurrence theorem:

Theorem 13.1 *Let $A = \Gamma(G, 1)$ be an MV-algebra with a state s and an endomorphism $\tau: A \rightarrow A$ such that $s(\tau(a)) = s(a)$ for all $a \in A$. Then for every $a \in A$,*

$$\lim_{t \rightarrow \infty} s\left(a \ominus \bigvee_{i=1}^t \tau^i(a)\right) = 0. \quad (13.1)$$

If, in addition, A is σ -complete then

$$s\left(a \ominus \bigvee_{i=1}^{\infty} \tau^i(a)\right) = 0. \quad (13.2)$$

Proof We first settle the following

Claim Let $c_0, c_1, \dots$ be a sequence of elements of A , and $n \geq 1$ an integer. Then $c_0 \ominus (c_1 \vee \dots \vee c_n) + c_1 \ominus (c_2 \vee \dots \vee c_n) + \dots + c_{n-1} \ominus c_n + c_n \leq c_0 \vee \dots \vee c_n$. (13.3)

Here $+$ is the sum operation of G , which is assumed to be less binding than the truncated subtraction operation $\ominus$ of A ; by (A21.16)(iii), $x \ominus y = (x - y) \vee 0$ for all $x, y \in A$.

The proof is by induction on n .

Basis We must show

$$c_0 \ominus c_1 + c_1 \leq c_0 \vee c_1.$$

We first note that $(c_0 \ominus c_1) \odot c_1 = c_0 \odot \neg c_1 \odot c_1 = 0$, whence $c_0 \ominus c_1 + c_1 = (c_0 \ominus c_1) \oplus c_1$. As a corollary of the subdirect representation theorem (A21.14), without loss of generality A is totally ordered. For all $c_0, c_1 \in A$ we then have

$$c_0 \geq c_1 \Rightarrow (c_0 \ominus c_1) \oplus c_1 = (c_0 - c_1) \oplus c_1 \leq (c_0 - c_1) + c_1 \leq c_0 \vee c_1$$

and

$$c_0 \leq c_1 \Rightarrow (c_0 \ominus c_1) \oplus c_1 = 0 \oplus c_1 \leq c_0 \vee c_1.$$

Induction step

$$\begin{aligned} c_0 \ominus (c_1 \vee \cdots \vee c_n) + c_1 \ominus (c_2 \vee \cdots \vee c_n) + \cdots + c_{n-1} \ominus c_n + c_n \\ \leq c_0 \ominus (c_1 \vee \cdots \vee c_n) + (c_1 \vee \cdots \vee c_n) \leq c_1 \vee \cdots \vee c_n. \end{aligned}$$

Our claim is settled.

For every integer $N > 0$ let us now use the notation

$$d_N = \tau^0(d_N) = a \ominus \bigvee_{i=1}^N \tau^i(a).$$

Then for each $j = 0, 1, \dots, N-1$, the endomorphism τ satisfies the following identity:

$$\tau^j(d_N) = \tau^j(a) \ominus \tau^j\left(\bigvee_{i=1}^N \tau^i(a)\right) = \tau^j(a) \ominus \left(\bigvee_{i=j+1}^N \tau^i(a)\right).$$

As a consequence, for each $t = 0, 1, \dots, N-1$ we have

$$\begin{aligned} \tau^0(d_N) + \tau^1(d_N) + \cdots + \tau^t(d_N) \\ \leq \tau^0(a) \ominus \left(\bigvee_{i=1}^N \tau^i(a)\right) + \tau^1(a) \ominus \left(\bigvee_{i=2}^N \tau^i(a)\right) + \cdots + \tau^t(a) \ominus \left(\bigvee_{i=t+1}^N \tau^i(a)\right) \\ \leq \tau^0(a) \ominus \left(\bigvee_{i=1}^{t+1} \tau^i(a)\right) + \tau^1(a) \ominus \left(\bigvee_{i=2}^{t+1} \tau^i(a)\right) + \cdots + \tau^t(a) \ominus \left(\bigvee_{i=t+1}^{t+1} \tau^i(a)\right) \\ \leq \tau^0(a) \ominus \left(\bigvee_{i=1}^{t+1} \tau^i(a)\right) + \tau^1(a) \ominus \left(\bigvee_{i=2}^{t+1} \tau^i(a)\right) + \cdots + \tau^t(a) \ominus \tau^{t+1}(a) + \tau^{t+1}(a). \end{aligned}$$

Thus, by our claim,

$$\tau^0(d_N) + \tau^1(d_N) + \cdots + \tau^t(d_N) \leq \tau^0(a) \vee \tau^1(a) \vee \cdots \vee \tau^{t+1}(a) \leq 1. \quad (13.4)$$

By Proposition 10.2, the limit in (13.1) exists finite. By way of contradiction suppose this limit is $\lambda > 0$. Let $\bar{s}$ be the unique extension of s to a state of $(G, 1)$ as given by Proposition 10.3. Since $s(\tau^i(d_N)) = s(\tau^{i+1}(d_N))$, the additivity of $\bar{s}$ yields, for sufficiently large N and $t < N$,

$$\bar{s}(d_N + \tau(d_N) + \cdots + \tau^t(d_N)) = \bar{s}(d_N) + \bar{s}(\tau(d_N)) + \cdots + \bar{s}(\tau^t(d_N)) > 1.$$

On the other hand, from the monotonicity property of $\bar{s}$ and (13.4) it follows that

$$\bar{s}(d_N + \tau(d_N) + \cdots + \tau^t(d_N)) \leq \bar{s}(\tau^0(a) \vee \tau^1(a) \vee \cdots \vee \tau^{t+1}(a)) \leq 1,$$

a contradiction. This settles (13.1).

If in addition, A is σ -complete, the proof of (13.2) easily follows from the monotonicity properties of s . $\square$

Example 13.2 Given the probability space $(\Omega, \mathcal{S}, \mu)$ let the σ -complete MV-algebra $\mathcal{F}$ be defined by

$$\mathcal{F} = \{f: \Omega \rightarrow [0, 1] \mid f \text{ is } \mathcal{S}\text{-measurable}\}.$$

Let the state m of $\mathcal{F}$ be defined by stipulating that for all $f \in \mathcal{F}$,

$$m(f) = \int_{\Omega} f d\mu.$$

For T a measure-preserving map, let the endomorphism τ of $\mathcal{F}$ be defined by $\tau(f) = f \circ T$. By Theorem 13.1,

$$\int_{\Omega} \left(\left(f - \bigvee_{i=1}^{\infty} f \circ T^i \right) \vee 0 \right) d\mu = 0.$$

Thus, μ -almost everywhere,

$$\left(f - \bigvee_{i=1}^{\infty} f \circ T^i \right) \vee 0 = 0, \quad \text{i.e.,} \quad f \leq \bigvee_{i=1}^{\infty} f \circ T^i.$$

13.2 Probability MV-Algebras

Given a sequence $X_1 \subseteq X_2 \subseteq \dots$ of subsets of a set E , $X_n \nearrow X$ stands for $X_1 \subseteq X_2 \subseteq \dots$ and $X = \bigcup_n X_n$. More generally, for any sequence $d_1, d_2, \dots$ of elements in a partially ordered set S , $d_n \nearrow d$ stands for

$$d_1 \leq d_2 \leq \dots \quad \text{and } d \text{ is the supremum of this sequence.}$$

Definition 13.3 A σ -state s of a σ -complete MV-algebra N is a state such that $s(x_n) \nearrow s(x)$ whenever $x, x_1, x_2, \dots \in N$ and $x_n \nearrow x$.

Lemma 13.4 Let N be a σ -complete MV-algebra with its center $B(N)$. Then the map $\gamma: s \mapsto s \upharpoonright B(N)$ is a one-one correspondence between the σ -states of N and the σ -states of $B(N)$.

Proof The verification that $s \upharpoonright B(N)$ is a σ -state of $B(N)$ easily follows by combining Lemma 11.3 with (A21.40).

The proof that γ is one-one follows from Theorem 12.7 together with the assumed σ -completeness of both s and its restriction to $B(N)$.

Finally, to prove that γ is onto, given a σ -state t of $B(N)$ we extend t to the state t' of N given by the stipulation

$$t'(a) = \sup(a_i), \quad \text{for all } a \in N,$$

where the $a_i \in B(N)$ are as in Theorem 12.7(ii). It is not hard to verify that t' does not depend on the choice of the approximating set. It is easy to verify that t' is σ -complete. $\square$

The proof of the following proposition is immediate:

Lemma 13.5 *Let N be a σ -complete MV-algebra with a distinguished σ -state t . Let $\mathfrak{j} = t^{-1}(0)$. Then $\mathfrak{j}$ is an ideal of N closed under countable sups, the quotient MV-algebra $N/\mathfrak{j}$ is σ -complete, and there is a unique σ -state s of $N/\mathfrak{j}$ such that $s(a/\mathfrak{j}) = t(a)$ for all $a \in N$. Further, s is a faithful σ -state.*

This motivates the following

Definition 13.6 A probability MV-algebra is a σ -complete MV-algebra M equipped with a distinguished faithful σ -state s .

Examples 13.7 Let $D = x_1, x_2, \dots$ be a countable set with a sequence $0 < \rho_1, \rho_2, \dots \in \mathbb{R}$ such that $\sum_i \rho_i = 1$. Let us equip the set $M = [1, 0]^D$ with termwise MV-algebraic operations. Let $s: M \rightarrow [0, 1]$ be defined by $s(f) = \sum_i f(x_i)\rho_i$, for each $f \in M$. Then s is a faithful σ -state on the complete MV-algebra M .

Let $\mathcal{F}$ be a tribe on a set Ω with a state l . Let $\mathfrak{s}$ be the set of all $g \in \mathcal{F}$ such that $l(g) = 0$. Then $\mathfrak{s}$ is an ideal of $\mathcal{F}$, and $\mathfrak{s}$ is closed under countable suprema. The quotient MV-algebra $M = \mathcal{F}/\mathfrak{s}$ is σ -complete, and the quotient map $q: \mathcal{F} \rightarrow M$ is a σ -homomorphism. Two functions $f', f'' \in \mathcal{F}$ have the same image under q iff $l(|f' - f''|) = 0$. Let the map $s: M \rightarrow [0, 1]$ be defined by $s(q(f)) = l(f)$ for each element $q(f) \in M$. Then as in Lemma 13.5, s is a well-defined faithful state on M , and (M, s) is a probability MV-algebra.

Theorem 11.7 shows that this is the most general possible example.

Theorem 13.8 *Let (M, s) be a probability MV-algebra.*

- (i) *M is complete.*
- (ii) *(M, s) is weakly σ -distributive. That is, if a_{ij} is a double sequence of elements of M such that for all i, j , $a_{ij} \geq a_{ij+1}$, and for each fixed i^* , $\bigwedge_j a_{i^*j} = 0$, then*

$$0 = \bigwedge_{\phi \in \Phi} \bigvee_i a_{i\phi(i)}, \quad \text{where } \Phi = \{1, 2, \dots\}^{\{1, 2, \dots\}}.$$

Proof (i) Suppose D is a set of pairwise disjoint elements of the center $B = B(M)$ of M . Since s is faithful, for each $n = 2, 3, \dots$ the set $D_n \subseteq D$ of elements satisfying $s(x) \geq 1/n$ is finite, and $D = \bigcup_n D_n$, must be (finite or) countable. We have just

proved that B satisfies the *countable chain condition*, i.e., B does not contain any uncountable set of pairwise disjoint elements.

Let E be a subset of B , with the intent of proving that $\sup(E)$ exists in B . Without loss of generality, E is an ideal of B . Let F be a maximal family of nonzero pairwise disjoint elements of E , as given by the axiom of choice. Trivially, any upper bound of E is also an upper bound of F . Conversely, the maximality of F ensures that any upper bound of F is an upper bound of E . Since B satisfies the countable chain condition, F is countable. By Lemma 11.4 and Proposition 11.10, B is σ -complete, whence $\sup(F)$ exists in B . Since E and F have the same upper bounds in B , $\sup(E)$ exists in B and equals $\sup(F)$. We have just proved that B is complete.

A final application of Proposition 11.10, completes the proof.

(ii) Let $a_{ij} \in M$ be a double sequence ($i, j = 1, 2, \dots$) satisfying the hypotheses of (ii). We will prove

$$\bigwedge_{\phi \in \Phi} \bigvee_i a_{i\phi(i)} = 0.$$

By way of contradiction, assume $0 < a \in M$ is a lower bound for $\bigvee_i a_{i\phi(i)}$, for every $\phi \in \Phi$. Since s is faithful $s(a) > 0$. For any fixed $i = 1, 2, \dots$, we have $\bigwedge_j (a \wedge a_{ij}) = 0$. Since s is a σ -state,

$$\lim_{j \rightarrow \infty} s(a \wedge a_{ij}) = 0.$$

There is an element $\psi \in \Phi$ such that $s(a \wedge a_{i\psi(i)}) \leq s(a)/2^{i+1}$. By our standing absurdum hypothesis, $a \leq \bigvee_i a_{i\psi(i)}$. From the distributivity law (11.1) we get

$$s(a) = s\left(a \wedge \bigvee_i a_{i,\psi(i)}\right) = s\left(\bigvee_i (a \wedge a_{i,\psi(i)})\right) \leq \sum_i s(a \wedge a_{i,\psi(i)}) < s(a),$$

a contradiction. □

13.3 Bounded Measures on MV-Algebras

Definition 13.9 A *measure* of an MV-algebra A is a map $m: A \rightarrow \mathbb{R}$ such that $m(a \oplus b) = m(a) + m(b)$ whenever $a \odot b = 0$. We say that m is *positive* if $m(A) \subseteq \mathbb{R}_{\geq 0}$. If $m(1) = 1$ we say that m is *normalized*. Finally, m is *bounded* if for some integer $j > 0$ the range $m(A)$ is contained in $[-j, j]$. We denote by $\mathbf{BM}(A)$ the set of bounded measures of A .

It follows that the states of A coincide with the normalized positive (automatically bounded) measures of A . For any two states s and t of A and real numbers $0 \leq \sigma, \tau$,

the difference $\sigma s - \tau t$ is a bounded measure of A . In Corollary 13.12 it will be shown that this is the most general bounded measure of A .

Examples 13.10 For an example of an unbounded measure, given the Chang algebra $C = \{0, \epsilon, 2\epsilon, \dots, 1 - 2\epsilon, 1 - \epsilon, 1\}$, let the function $u: C \rightarrow \mathbb{R}$ be defined by: $u(j\epsilon) = j$ and $u(1 - j\epsilon) = -j$ for each $j = 0, 1, 2, \dots$.

Let $\mathbb{R}_{\mathcal{B}}$ denote the real line $\mathbb{R}$ as a *rational* vector space with a distinguished Hamel basis $\mathcal{B}$. Using $\mathcal{B}$ one easily obtains a $\mathbb{Q}$ -linear (discontinuous) function $n: \mathbb{R} \rightarrow \mathbb{R}$, whose restriction $m = n \upharpoonright [0, 1]$ is an unbounded measure of $[0, 1]$.

Evidently, $\mathbf{BM}(A)$ is a (real) vector space. Let $l, m \in \mathbf{BM}(A)$. Upon defining

$$l \leq m \quad \text{iff} \quad l(x) \leq m(x) \quad \text{for all} \quad x \in A,$$

$\mathbf{BM}(A)$ becomes a *partially ordered vector space*: in other words, $\leq$ is a partial order such that, for all $l, m, n \in \mathbf{BM}(A)$ and $0 \leq \gamma \in \mathbb{R}$,

$$l \leq m \Rightarrow (l + n \leq m + n \text{ and } \gamma l \leq \gamma m).$$

The following proposition shows that much more is true:

Proposition 13.11 *Let A be an MV-algebra. Let us equip $\mathbf{BM}(A)$ with the following operations, for all $m, n \in \mathbf{BM}(A)$ and $x \in A$,*

$$(m \vee n)(x) = \sup\{m(x_1) + n(x_2) \mid x_1, x_2 \in A, \quad x = x_1 \oplus x_2, \quad x_1 \odot x_2 = 0\}$$

and

$$(m \wedge n)(x) = \inf\{m(x_1) + n(x_2) \mid x_1, x_2 \in A, \quad x = x_1 \oplus x_2, \quad x_1 \odot x_2 = 0\}.$$

Then $(m \wedge n)(x) = -(-m \vee -n)$, both $m \vee n$ and $m \wedge n$ are bounded measures of A , and the operations $\vee$ and $\wedge$ turn $\mathbf{BM}(A)$ into a vector lattice.

Proof Let $(G, 1)$ be the unital ℓ -group associated to A by the stipulation $\Gamma(G, 1) = A$. By (A21.16), for any $y, z \in A \subseteq G$ the condition $y \odot z = 0$ is equivalent to $y + z \in A$. Further, whenever $x \in A$ is written as a sum $x = y + z$ with $0 \leq y, z \in G$ then automatically $y, z \in A$.

Since both m and n are bounded, the map $p: A \rightarrow \mathbb{R}$ defined by

$$p(x) = \sup\{m(x_1) + n(x_2) \mid x_1, x_2 \in A, \quad x = x_1 \oplus x_2, \quad x_1 \odot x_2 = 0\}$$

is bounded.

Fix now $x, y \in A$ with $x \odot y = 0$, with the intent of showing $p(x + y) = p(x) + p(y)$. Let $z = x \oplus y = x + y$.

Claim 1 $p(x \oplus y) \geq p(x) + p(y)$.

Any decomposition

$$x = x_1 + x_2, \quad y = y_1 + y_2, \quad (x_i, y_i \in A, \quad i = 1, 2)$$

trivially yields a decomposition $z = (x_1 + y_1) + (x_2 + y_2)$. Any element of G which is obtainable as a sum in G of distinct elements of the set $\{x_1, x_2, y_1, y_2\}$ already belongs to A . The additive property of m and n now yields

$$m(x_1) + n(x_2) + m(y_1) + n(y_2) = m(x_1 + y_1) + n(x_2 + y_2) \leq p(x + y) = p(x \oplus y),$$

which settles our claim.

Claim 2 $p(x \oplus y) \leq p(x) + p(y)$.

Suppose we have a decomposition $z = u + v$, for suitable elements $u, v \in A$. The Riesz decomposition property (B21.69) yields elements $a_{ij} \in A$, $i, j \in \{1, 2\}$ such that

$$u = a_{11} + a_{21}, \quad v = a_{12} + a_{22}, \quad x = a_{11} + a_{12}, \quad y = a_{21} + a_{22}.$$

Then

$$\begin{aligned} m(u) + n(v) &= m(a_{11} + a_{21}) + n(a_{12} + a_{22}) \\ &= m(a_{11}) + m(a_{21}) + n(a_{12}) + n(a_{22}) \\ &= m(a_{11}) + n(a_{12}) + m(a_{21}) + n(a_{22}) \\ &\leq p(x) + p(y), \end{aligned}$$

and our second claim is settled.

We have proved that p is a bounded measure of A . It is easy to see that p is an upper bound of m and n , and that every bounded measure $q \geq m, n$ satisfies $q \geq p$.

The identity $m \wedge n = -(-m \vee -n)$ is routinely verified.

A final verification now shows that $\vee, \wedge$ make $\mathbf{BM}(A)$ into a vector lattice. $\square$

Corollary 13.12 *For every MV-algebra A and $m \in \mathbf{BM}(A)$, let us define $m^+ = 0 \vee m$ and $m^- = 0 \vee -m$, in the light of Proposition 13.11. It follows that*

- (i) m^+ and m^- are the unique elements of $\mathbf{BM}(A)$ such that $m = m^+ - m^-$ and $m^+ \wedge m^- = 0$.
- (ii) Thus there are states s and t of A together with real numbers $0 \leq \sigma, \tau$ such that $m = \sigma s - \tau t$.
- (iii) The ℓ -group $\mathbf{BM}(A)$ is a Dedekind complete vector lattice, in the sense that any set of elements of A having an upper bound in $\mathbf{BM}(A)$ also has a least upper bound in $\mathbf{BM}(A)$.

Proof (i) Jordan decomposition (B21.73).

(ii) For some $0 \leq \sigma \in \mathbb{R}$ and state s of A we can write $m^+ = \sigma s$. As a matter of fact, if $m^+ = 0$ we have nothing to prove. Otherwise, let $\sigma = m^+(1)$ and $s = m^+/m^+(1)$. Similarly, $m^- = m^-/m^-(1) = m^-/\tau$ (unless $m^- = 0$). In any case, condition (ii) holds.

(iii) A routine verification. $\square$

Corollary 13.13 *Let A be an MV-algebra and $(G, 1)$ its associated unital ℓ -group. For every function $f: A \rightarrow \mathbb{R}$ the following conditions are equivalent:*

- (i) $f \in \mathbf{BM}(A)$.

- (ii) *There is a group homomorphism $h: G \rightarrow \mathbb{R}$ such that $f = h \upharpoonright A$ and $h(A)$ is a bounded subset of $\mathbb{R}$.*
- (iii) *There are two states p, q of $(G, 1)$ and real numbers $0 \leq \sigma, \tau$ such that $f(x) = (\sigma p - \tau q)(x)$ for all $x \in A$.*

Proof (iii $\Rightarrow$ ii) and (ii $\Rightarrow$ i) are trivial. (i $\Rightarrow$ iii) Using Corollary 13.12 there are $0 \leq \sigma, \tau$ and states s, t of A such that $f = \sigma s - \tau t$. Now let p and q be the unique extensions of s and t to states of $(G, 1)$, as given by Proposition 10.3. $\square$

13.4 Remarks

For terminology on fields and σ -fields of sets we have followed [1].

The MV-algebraic Poincaré's theorem is due to Riečan [2].

Early papers on boolean algebraic probability and measure theory include [3, Appendix], [4, XI, 5, p. 260], [5]–[9].

Our σ -states are called “states” in [10] and in the paper [11] where they were first introduced.

Weak σ -distributivity, and completeness are mutually independent properties; see [12, (B) p. 105 and (D) p. 106].

As shown by Chaps. 20–23 in the second volume of the *Handbook of Measure Theory* [13], MV-algebraic probability and measure theory is multifaceted and well developed. The monograph [14], the final section of [15] and the bibliographies therein show the fundamental contribution of the Slovak School.

In every multiplicative probability MV-algebra M one has an interesting theory of “independent random variables”. As a generalization of Sikorski's real homomorphisms [12, p. 152ff], and Varadarajan's observables [16, p. 14ff], MV-algebraic random variables are suitable maps from the Borel sets of $\mathbb{R}^n$ to M . As a joint effect of completeness, weak σ -distributivity, and multiplicativity, one can investigate random variables in M by first splitting M into its boolean part and its $C(X)$ part (via Theorem 12.12), and then applying to the two pieces of M classical techniques, respectively from measure boolean algebras [17, 18] and vector lattices [19, 20], via the Γ functor.

Proposition 13.11 is the starting point of a rich MV-algebraic measure theory, for which we refer to [21] and references therein.

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Chapter 14

Measuring Polyhedra and Averaging Truth-Values

This chapter introduces a special integration theory for McNaughton functions $f(x_1, \dots, x_n)$ over rational polyhedra $P \subseteq [0, 1]^n$, $n = 1, 2, \dots$. For each $i = 0, 1, \dots, n$ we preliminarily define the rational i -dimensional measure $\lambda^{(i)}(P)$. When P is n -dimensional, $\lambda^{(n)}(P)$ coincides with n -dimensional Lebesgue measure. In contrast to Lebesgue measure, $\lambda^{(\dim(P))}(P) > 0$ even if $0 \leq \dim(P) < n$. The volume elements of our integral $\int_P f \, dx$ are the rational volumes $\lambda^{(\dim(P))}(T)$ of all simplexes T of a regular triangulation Δ of P such that f is linear on each simplex of Δ . The value of $\int_P f \, dx$ does not depend on Δ , and is invariant under isomorphisms. Similarly, for each i , the rational measure $\lambda^{(i)}(P)$ is invariant under $\mathbb{Z}$ -homeomorphisms. If P happens to be n -dimensional, $\int_P f \, dx$ coincides with the Lebesgue integral of f over P . The rational measure and integral introduced in this chapter will be the key tool for the development of a purely algebraic integration theory on every finitely presented MV-algebra A , independently of any representation of A as an algebra of McNaughton functions, and for the introduction of an invariant Rényi conditional in Łukasiewicz logic.

14.1 The Rational Measure of a Rational Polyhedron

Fix $0 < n \in \mathbb{Z}$ and let $Q \subseteq \mathbb{R}^n$ be a polyhedron. For any (not necessarily rational) triangulation $\mathcal{T}$ of Q we denote by $\mathcal{T}^{\max}$ the set of maximal simplexes in $\mathcal{T}$. For all $i = 0, 1, 2, \dots$ we let

$$Q^{(i)} = \bigcup \{T \in \mathcal{T}^{\max} \mid \dim(T) = i\}. \quad (14.1)$$

We say that $Q^{(i)}$ is the i -dimensional part of Q . $Q^{(i)}$ is a (possibly empty) polyhedron and does not depend on the triangulation $\mathcal{T}$ of Q : indeed, any two triangulations of Q have a joint subdivision. If $Q^{(i)}$ is nonempty, then it is an i -dimensional polyhedron. Further, the j -dimensional part of $Q^{(i)}$ is empty iff $j \neq i$.

As the reader will recall from Definition 5.1, for every regular simplex $S \subseteq \mathbb{R}^n$ the integer $\text{den}(S)$ is defined by

$$\text{den}(S) = \text{den}(v_0) \cdots \text{den}(v_m), \quad (14.2)$$

and is said to be the denominator of S .

Theorem 14.1 *For each $i = 0, 1, \dots$ and regular triangulation Δ of a rational polyhedron $P \subseteq \mathbb{R}^n$, let*

$$\lambda_{\Delta}^{(i)}(P) = \sum \left\{ \frac{1}{i! \text{den}(S)} \mid S \in \Delta^{\max}, \dim(S) = i \right\}, \quad (14.3)$$

where the sum equals zero if there are no maximal i -simplexes in Δ . Then $\lambda_{\Delta}^{(i)}(P) = \lambda_{\nabla}^{(i)}(P)$ for any regular triangulation ∇ of P . Accordingly, we will write $\lambda^{(i)}(P)$ instead of $\lambda_{\Delta}^{(i)}(P)$, and call $\lambda^{(i)}(P)$ the i -dimensional rational measure of P .

Proof The existence of Δ is ensured by (B21.52) and Theorem 2.8. We first suppose ∇ to be obtained from Δ by a blow-up at the Farey median c of some j -simplex $S = \text{conv}(v_0, \dots, v_j) \in \Delta$. Then as in (5.1), $\text{den}(c) = \text{den}(v_0) + \dots + \text{den}(v_j)$. Fix $d = 0, 1, \dots$

Case 1 There exists a d -simplex $T = \text{conv}(v_0, \dots, v_j, \dots, v_d)$ of $\Delta^{\max}$ with $c \in T$.

Then let $F_0 = \text{conv}(c, v_1, \dots, v_d)$, $F_j = \text{conv}(v_0, v_1, \dots, v_{j-1}, c, \dots, v_d)$, and for each $t = 1, \dots, j-1$, $F_t = \text{conv}(v_0, \dots, v_{t-1}, c, v_{t+1}, \dots, v_j, \dots, v_d)$. By definition of Farey blow-up, T and its faces are replaced in ∇ by the simplicial complex whose maximal simplexes are the d -simplexes $F_0, \dots, F_j$. Since T is regular, then so is F_u for each $u = 0, \dots, j$, and $\text{den}(F_u) = \text{den}(T) \cdot \text{den}(c) / \text{den}(v_u)$. As a consequence, $1/\text{den}(T) = \sum_{u=0}^j 1/\text{den}(F_u)$. Since

$$\sum_{T \in \Delta^{\max}, \dim(T)=d} \frac{1}{d! \text{den}(T)} = \sum_{U \in \nabla^{\max}, \dim(U)=d} \frac{1}{d! \text{den}(U)}, \quad (14.4)$$

we have just shown that $\lambda_{\Delta}^{(d)}(P) = \lambda_{\nabla}^{(d)}(P)$.

Case 2 There is no d -simplex T of $\Delta^{\max}$ with $c \in T$.

Then the d -simplexes of $\Delta^{\max}$ are exactly those of $\nabla^{\max}$.

Thus under the standing hypothesis about Δ and ∇ , from (14.4) it follows that $\lambda_{\Delta}^{(i)}(P) = \lambda_{\nabla}^{(i)}(P)$ for all $i = 0, 1, \dots$

In the general case, when ∇ is an arbitrary regular triangulation of P , the solution of the weak Oda conjecture (B21.58) yields a sequence of regular triangulations $\Delta_0 = \Delta, \Delta_1, \dots, \Delta_{s-1}, \Delta_s = \nabla$, where each Δ_{t+1} is obtained from Δ_t by a Farey blow-up or vice versa, Δ_t is obtained from Δ_{t+1} by a Farey blow-up. Then the desired conclusion follows by induction on s . $\square$

Henceforth, $\mathcal{L}^{(n)}$ will denote n -dimensional Lebesgue measure.

Proposition 14.2 *For every $n = 1, 2, \dots$ and rational polyhedron $P \subseteq \mathbb{R}^n$,*

$$\lambda^{(n)}(P) = \mathcal{L}^{(n)}(P). \quad (14.5)$$

Proof If $\dim(P) < n$ then $\mathcal{L}^{(n)}(P) = \lambda^{(n)}(P) = 0$. If $\dim(P) = n$, without loss of generality we may assume $P = P^{(n)}$, whence $\lambda^{(m)}(P) = 0$ for all $m \neq n$. Let ∇ be a regular triangulation of P as given by (B21.52) and Theorem 2.8. Let S be an arbitrarily chosen n -simplex of ∇ , with its vertices $w_0, \dots, w_n$, and their respective denominators $m_0, \dots, m_n$. It is enough to prove $\lambda^{(n)}(S) = \mathcal{L}^{(n)}(S)$.

To this purpose, let $T \subseteq \mathbb{R}^{n+1}$ be the $(n+1)$ -simplex with vertices $0, (w_0, 1), \dots, (w_n, 1)$. Then T is contained in the closed $(n+1)$ -dimensional parallelepiped $E = \{\alpha_0(w_0, 1) + \dots + \alpha_n(w_n, 1) \in \mathbb{R}^{n+1} \mid \alpha_0, \dots, \alpha_n \in [0, 1]\}$. In turn, E is contained in the parallelepiped $U = \{\alpha_0\tilde{w}_0 + \dots + \alpha_n\tilde{w}_n \in \mathbb{R}^{n+1} \mid \alpha_0, \dots, \alpha_n \in [0, 1]\}$. By Lemmas 2.5–2.6, the regularity of S amounts to writing $\mathcal{L}^{(n+1)}(U) = 1$. Since $\tilde{w}_0 = m_0(w_0, 1), \dots, \tilde{w}_n = m_n(w_n, 1)$, then $\mathcal{L}^{(n+1)}(E) = (m_0 \cdots m_n)^{-1}$. We then have

$$\frac{\mathcal{L}^{(n)}(S) \times 1}{n+1} = \mathcal{L}^{(n+1)}(T) = \frac{\mathcal{L}^{(n+1)}(E)}{(n+1)!}.$$

The first identity is the classical formula for the volume of the $(n+1)$ -dimensional pyramid T' with base S and height 1. Evidently, T' and T have the same basis S and the same height. Since E can be triangulated by $(n+1)!$ simplexes, all having the same Lebesgue measure as T , the second identity follows. Summing up, $\mathcal{L}^{(n)}(S) = \mathcal{L}^{(n+1)}(E)/n! = (n!m_0 \cdots m_n)^{-1} = \lambda^{(n)}(S)$. $\square$

Lemma 14.3 *Suppose $P \subseteq [0, 1]^n$ is a rational polyhedron.*

- (i) *If $Q \subseteq [0, 1]^n$ is a rational polyhedron and $P^{(i)} \subseteq Q^{(i)}$ for some $i = 0, 1, \dots$, then $\lambda^{(i)}(P) \leq \lambda^{(i)}(Q)$.*
- (ii) *If for some $m = 1, 2, \dots$, $R \subseteq [0, 1]^m$ is a rational polyhedron and η is a $\mathbb{Z}$ -homeomorphism of P onto R then for each $d = 0, 1, 2, \dots$, $\lambda^{(d)}(P) = \lambda^{(d)}(R)$.*

Proof (i) Let ∇ be a regular triangulation of $[0, 1]^n$ such that $\nabla_P = \{T \in \nabla \mid T \subseteq P\}$ and $\nabla_Q = \{T \in \nabla \mid T \subseteq Q\}$ are triangulations of P and Q , respectively. The existence of ∇ is ensured by Corollary 2.9. Now observe that every maximal m -simplex of ∇_P is also a maximal m -simplex of ∇_Q , and apply Theorem 14.1.

(ii) Let $\eta_1, \dots, \eta_m$ be the components of η . By Proposition 3.15 there are regular triangulations Δ of P and ∇ of R such that η is linear on each simplex of Δ , η^{-1} is linear over every simplex of ∇ , and η sends simplexes of Δ one-one onto simplexes of ∇ . Since η preserves denominators, the conclusion follows by Theorem 14.1. $\square$

14.2 The Natural Measure of Simplexes and Polyhedra

Definition 14.4 For each $0 < n \in \mathbb{Z}$ let $P \subseteq \mathbb{R}^n$ be a polyhedron. Then for every $d = 0, 1, 2, \dots$, the *natural d -dimensional measure* $\mathcal{N}^{(d)}(P)$ of P is defined in successive stages as follows:

Full-dimensional simplex: if $P = \text{conv}(v_0, \dots, v_n)$ is an n -simplex then let M be the $(n \times n)$ -matrix whose i th row is given by the vector $v_i - v_0$, ($i = 1, \dots, n$). Then $\mathcal{N}^{(n)}(P) = |\det(M)|/n! = \mathcal{L}^{(n)}(P)$. We also stipulate that $\mathcal{N}^{(d)}(P) = 0$ for all $d \neq n$.

Low-dimensional simplex: if P is an m -simplex with $0 < m < n$, we first map P onto a copy P' by means of an isometry ι sending the affine hull of P onto the subspace $\mathbb{R}^m$ of $\mathbb{R}^n$ spanned by the first m standard basis vectors. We then define $\mathcal{N}^{(m)}(P) = \mathcal{L}^{(m)}(P')$, and $\mathcal{N}^{(d)}(P) = 0$ for all $d \neq m$. This does not depend on the isometry ι used. In case $m = 0$ we stipulate that $\mathcal{N}^{(0)}(P) = 1$ and $\mathcal{N}^{(d)}(P) = 0$ for $d > 0$.

Polyhedron: If $P \neq \emptyset$ is a polyhedron, recalling the definition of $P^{(i)}$, we let $\mathcal{T}$ be a triangulation of the d -dimensional part $P^{(d)}$ of P , with its d -simplexes $T_1, \dots, T_k$. Then we set $\mathcal{N}^{(d)}(P) = \mathcal{N}^{(d)}(P^{(d)}) = \sum_{j=1}^k \mathcal{N}^{(d)}(T_j)$, and note that the final result does not depend on the actual triangulation used.

Empty set: if $P = \emptyset$ then $\mathcal{N}^{(d)}(P) = 0$ for all $d = 0, 1, \dots$.

We list here some elementary properties of the natural measure $\mathcal{N}^{(d)}$ that will be tacitly used below. They are trivial consequences of the well known corresponding properties of Lebesgue measure:

Lemma 14.5 (i) For integers $0 \leq m \leq n$ let $B = \text{conv}(v_0, \dots, v_m)$ be an m -simplex in $\mathbb{R}^{n+1}$ lying on the hyperplane H given by $x_{n+1} = 0$. For each $j = 0, \dots, m$ let $I_j = \text{conv}(v_j, w_j)$ be the 1-simplex perpendicular to H , of height h_j in the positive half-space $x_{n+1} \geq 0$. Let $P \subseteq \mathbb{R}^{n+1}$ be the truncated prism $P = \text{conv}(B, w_0, \dots, w_m)$. Then

$$\mathcal{N}^{(m+1)}(P) = \mathcal{N}^{(m)}(B) \frac{\mathcal{N}^{(1)}(I_0) + \dots + \mathcal{N}^{(1)}(I_m)}{m+1} = \mathcal{N}^{(m)}(B) \frac{h_0 + \dots + h_m}{m+1}.$$

(ii) For $0 \leq m < n$ suppose $T = \text{conv}(v_0, \dots, v_m)$, $T' = \text{conv}(v'_0, \dots, v'_m) \subseteq \mathbb{R}^n$ are m -simplexes satisfying the coplanarity condition $\text{aff}(T) = \text{aff}(T')$. Pick an arbitrary point v in $\mathbb{R}^n \setminus \text{aff}(T)$ and let $U = \text{conv}(T, v)$ and $U' = \text{conv}(T', v)$. Then $\mathcal{N}^{(m+1)}(U')/\mathcal{N}^{(m+1)}(U) = \mathcal{N}^{(m)}(T')/\mathcal{N}^{(m)}(T)$.

Thus, for any n -simplexes $W = \text{conv}(v_0, \dots, v_m, v_{m+1}, \dots, v_n)$ and $W' = \text{conv}(v'_0, \dots, v'_m, v_{m+1}, \dots, v_n) \subseteq \mathbb{R}^n$, we have the identity

$$\frac{\mathcal{N}^{(n)}(W')}{\mathcal{N}^{(n)}(W)} = \frac{\mathcal{N}^{(m)}(T')}{\mathcal{N}^{(m)}(T)}.$$

(iii) Given integers $0 \leq i \leq n$, suppose the i -dimensional rational polyhedron P lies in the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. Writing I for the unit real interval $[0, 1]$, let us identify the cartesian product $P \times I$ with the prism

$$\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n, 0) \in P, x_{n+1} \in I\}.$$

Then $\mathcal{N}^{(i+1)}(P \times I) = \mathcal{N}^{(i)}(P)$.

By a *rational affine subspace* of $\mathbb{R}^n$ we mean a finite intersection of rational hyperplanes in $\mathbb{R}^n$.

Lemma 14.6 *Let $S = \text{conv}(x_0, \dots, x_d)$ and $S' = \text{conv}(x'_0, \dots, x'_d)$ be regular d -simplexes in $[0, 1]^n$, with $0 \leq d < n$. Suppose that the affine hulls $\text{aff}(S)$ and $\text{aff}(S')$ coincide. Then there exist rational points $x_{d+1}, \dots, x_n \in [0, 1]^n$ such that both $\text{conv}(x_0, \dots, x_d, x_{d+1}, \dots, x_n)$ and $\text{conv}(x'_0, \dots, x'_d, x_{d+1}, \dots, x_n)$ are regular n -simplexes in $[0, 1]^n$.*

Proof Recall the notation introduced in Sect. 2.1 for the homogeneous correspondent $\tilde{a} \in \mathbb{Z}^{n+1}$ of a rational point $a \in \mathbb{R}^n$. By Lemma 2.6(i $\Leftrightarrow$ ii'), the regularity of S means that the set $\{\tilde{x}_0, \dots, \tilde{x}_d\}$ is a basis of the free abelian group $F = \mathbb{Z}^{n+1} \cap (\mathbb{R}\tilde{x}_0 + \dots + \mathbb{R}\tilde{x}_d)$ of integer points in the $(d+1)$ -dimensional linear space spanned in $\mathbb{R}^{n+1}$ by $\tilde{x}_0, \dots, \tilde{x}_d$. Since the regular d -simplex S' lies in $\text{aff}(S)$, also $\{\tilde{x}'_0, \dots, \tilde{x}'_d\}$ is a basis of F . Upon writing each $\tilde{x}_i$ and $\tilde{x}'_j$ as a column vector, let E be the $(n+1) \times (d+1)$ matrix whose i th row coincides with $\tilde{x}_i$. Let similarly E' be the $(n+1) \times (d+1)$ matrix whose j th row is $\tilde{x}'_j$. Let the $(d+1) \times (d+1)$ integer matrix U be defined by $EU = E'$. Then the $(d+1) \times (d+1)$ integer matrix V defined by $E'V = E$ coincides with U^{-1} , whence $|\det(U)| = |\det(U^{-1})| = 1$, i.e., U is unimodular. Letting I_{n-d} denote the $(n-d) \times (n-d)$ identity matrix, we now define the $(n+1) \times (n+1)$ unimodular integer matrix N by

$$N = \begin{pmatrix} U & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

The assumed regularity of S yields rational points $x_{d+1}, \dots, x_n \in \mathbb{R}^n$ such that $\text{conv}(x_0, \dots, x_d, x_{d+1}, \dots, x_n)$ is a regular n -simplex R . Without loss of generality, $R \cap \text{int}([0, 1]^n) \neq \emptyset$. By applying, if necessary, suitably many Farey blow-ups, each point $x_{d+1}, \dots, x_n$ can be safely assumed to lie in $[0, 1]^n$. The points $\tilde{x}_0, \dots, \tilde{x}_n$ form a basis of the free abelian group $\mathbb{Z}^{n+1}$ of integer points in $\mathbb{R}^{n+1}$. Let W (resp., W') be the $(n+1) \times (n+1)$ integer matrix whose first $d+1$ columns are those of E (resp., those of E'), and whose last $n-d$ columns are given by the column vectors $\tilde{x}_{d+1}, \dots, \tilde{x}_n$. From $WN = W'$, it follows that the vectors $\tilde{x}'_0, \dots, \tilde{x}'_d, \tilde{x}_{d+1}, \dots, \tilde{x}_n$ constitute a basis of the free abelian group $\mathbb{Z}^{n+1}$. We conclude that also $\text{conv}(x'_0, \dots, x'_d, x_{d+1}, \dots, x_n)$ is a regular n -simplex in $[0, 1]^n$. $\square$

The main relationship between rational and natural measure is given by:

Theorem 14.7 *Let the integers m, n satisfy the conditions $n \geq 1$ and $0 \leq m \leq n$. Let further A be an m -dimensional rational affine subspace of $\mathbb{R}^n$. Then there is a constant $\kappa_A > 0$, only depending on A , such that $\lambda^{(m)}(P) = \kappa_A \cdot \mathcal{N}^{(m)}(P)$ for every rational m -dimensional polyhedron $P \subseteq A \cap [0, 1]^n$.*

Proof By Corollary 2.9, P has a regular triangulation. It suffices to argue in case P is a regular m -simplex. If $m = n$ the result follows from Proposition 14.2 since, by Definition 14.4, $\mathcal{N}^{(n)}(P) = \mathcal{L}^{(n)}(P)$. In this case $\kappa_A = 1$.

Next suppose $m < n$. It suffices to prove that for any two regular m -simplexes $T = \text{conv}(v_0, \dots, v_m)$, $T' = \text{conv}(v'_0, \dots, v'_m) \subseteq A$,

$$\lambda^{(m)}(T)/\lambda^{(m)}(T') = \mathcal{N}^{(m)}(T)/\mathcal{N}^{(m)}(T').$$

To this purpose let $U = \text{conv}(v_0, \dots, v_m, v_{m+1}, \dots, v_n)$ be a regular n -simplex in $[0, 1]^n$ having T as a face. The *proof* of Lemma 14.6 shows that the simplex $U' = \text{conv}(v'_0, \dots, v'_m, v_{m+1}, \dots, v_n)$ is regular. Let $d_i = \text{den}(v_i)$, ($i = 0, \dots, n$) and $d'_j = \text{den}(v'_j)$, ($j = 0, \dots, m$). Since U and U' are regular we can write the identities

$$\frac{\lambda^{(m)}(T)}{\lambda^{(m)}(T')} = \frac{(m! d_0 \cdots d_m)^{-1}}{(m! d'_0 \cdots d'_m)^{-1}} = \frac{(n! d_0 \cdots d_m d_{m+1} \cdots d_n)^{-1}}{(n! d'_0 \cdots d'_m d_{m+1} \cdots d_n)^{-1}} = \frac{\lambda^{(n)}(U)}{\lambda^{(n)}(U')}.$$

By Lemma 14.5(ii) and Proposition 14.2,

$$\frac{\lambda^{(n)}(U)}{\lambda^{(n)}(U')} = \frac{\mathcal{L}^{(n)}(U)}{\mathcal{L}^{(n)}(U')} = \frac{\mathcal{N}^{(n)}(U)}{\mathcal{N}^{(n)}(U')} = \frac{\mathcal{N}^{(m)}(T)}{\mathcal{N}^{(m)}(T')},$$

which concludes the proof. $\square$

Corollary 14.8 *Given integers $0 \leq m < n$, let $T = \text{conv}(v_0, \dots, v_m) \subseteq [0, 1]^n$ be a regular m -simplex lying on the hyperplane $x_n = 0$. Let B be the $(m + 1)$ -dimensional affine subspace in $\mathbb{R}^n$ obtained by cylindrifying $\text{aff}(T)$ along the x_n -axis,*

$$B = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}, 0) \in \text{aff}(T)\}.$$

Then $\kappa_{\text{aff}(T)} = \kappa_B$.

Proof Let $d_0 = \text{den}(v_0), \dots, d_m = \text{den}(v_m)$ be the denominators of the vertices of T . Let $v'_0 \in [0, 1]^n$ have the same coordinates as v_0 , except the last one, which is assumed to be equal to $1/d_0$. Then $\text{den}(v'_0) = d_0$. Let the $(m + 1)$ -simplex S in $[0, 1]^n$ be defined by $S = \text{conv}(v_0, \dots, v_m, v'_0)$. From the assumed regularity of T it follows that S is regular. In the light of Theorem 14.7 it suffices to prove

$$\lambda^{(m)}(T)/\lambda^{(m+1)}(S) = \mathcal{N}^{(m)}(T)/\mathcal{N}^{(m+1)}(S).$$

By Definition 14.4,

$$\mathcal{N}^{(m+1)}(S) = \frac{\mathcal{N}^{(m)}(T) \cdot \mathcal{N}^{(1)}(\text{conv}(v_0, v'_0))}{m + 1} = \frac{\mathcal{N}^{(m)}(T)}{d_0(m + 1)}.$$

Thus, $\mathcal{N}^{(m)}(T)/\mathcal{N}^{(m+1)}(S) = d_0(m + 1)$. On the other hand, using the regularity of S and T we obtain

$$\frac{\lambda^{(m)}(T)}{\lambda^{(m+1)}(S)} = \frac{(m! d_0 \cdots d_m)^{-1}}{((m + 1)! d_0 \cdots d_m d_0)^{-1}} = d_0(m + 1),$$

which completes the proof. $\square$

14.3 The Rational Integral of McNaughton Functions

Definition 14.9 Let P be a rational polyhedron in $[0, 1]^n$ and $f \in \mathcal{M}([0, 1]^n)$, for $n = 1, 2, \dots$. Let

$$\text{sub } f \upharpoonright P = \bigcup \{ \text{conv}((x, 0), (x, f(x))) \mid x \in X \} \quad (14.6)$$

denote the portion of space in $[0, 1]^{n+1}$ between P and the graph of $f \upharpoonright P$. Then the rational integral $\int_P f$ is defined by

$$\int_P f = \lambda^{(1+\dim(P))} (\text{sub } f \upharpoonright P). \quad (14.7)$$

Proposition 14.10 $\int_P f$ is a rational number. In case P is n -dimensional, the rational integral coincides with the Lebesgue integral of f on P .

Proof An immediate consequence of Proposition 14.2 and of the elementary properties of Lebesgue integral. $\square$

For every $f \in \mathcal{M}([0, 1]^n)$ and simplex $T = \text{conv}(v_0, \dots, v_d)$ of a regular triangulation Δ in $[0, 1]^n$ such that f is linear on T , we write

$$\overline{f \upharpoonright T} = \frac{f(v_0) + \dots + f(v_d)}{d + 1} \quad (14.8)$$

for the average value of f on T . Letting $f = \hat{\psi}$ for some formula $\psi \in \text{FORM}_n$, the quantity $\overline{f \upharpoonright T}$ is the average truth-value of ψ over the set of valuations $\text{Mod}(\text{Th } T)$.

The following formula for $\int_P f$ will find repeated use in the sequel:

Theorem 14.11 For every $f \in \mathcal{M}([0, 1]^n)$ and rational polyhedron $P \subseteq [0, 1]^n$ with $m = \dim(P)$, we have the identity

$$\int_P f = \sum \left\{ \lambda^{(m)}(T) \cdot \overline{f \upharpoonright T} \mid T \in \Delta, \dim(T) = m \right\}, \quad (14.9)$$

where Δ is an arbitrary regular f -triangulation of P .

Proof We identify $[0, 1]^n$ with $\{x \in [0, 1]^{n+1} \mid x_{n+1} = 0\}$. Since

$$\int_P f = \sum_{T \in \Delta, \dim(T)=m} \int_T f,$$

it is enough to prove

$$\lambda^{(m+1)}(\text{sub } f \upharpoonright T) = \lambda^{(m)}(T) \cdot \overline{f \upharpoonright T},$$

for every m -simplex $T \in \Delta$. If f vanishes on T , then both members are zero. On the other hand, if $f(x) > 0$ for some $x \in T$ then $\text{sub } f \upharpoonright T$ is $(m+1)$ -dimensional and $\lambda^{(m+1)}(\text{sub } f \upharpoonright T) > 0$. As in Corollary 14.8, let B be the cylindrification of $\text{aff}(T)$ along the $(n+1)$ th axis, $B = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n, 0) \in \text{aff}(T)\}$. We then have:

$$\begin{aligned} \lambda^{(m+1)}(\text{sub } f \upharpoonright T) &= \kappa_B \cdot \mathcal{N}^{(m+1)}(\text{sub } f \upharpoonright T), \text{ Theorem 14.7} \\ &= \kappa_B \cdot \mathcal{N}^{(m)}(T) \cdot \overline{f \upharpoonright T}, \text{ Lemma 14.5(i)} \\ &= \frac{\kappa_B \cdot \lambda^{(m)}(T) \cdot \overline{f \upharpoonright T}}{\kappa_{\text{aff}(T)}}, \text{ Theorem 14.7} \\ &= \lambda^{(m)}(T) \cdot \overline{f \upharpoonright T}, \text{ Corollary 14.8.} \end{aligned}$$

The proof is complete. □

In particular, when f is the constant 1 on P we have

Corollary 14.12 *For $0 \leq m \leq n$, let $P \subseteq [0, 1]^{n+1}$ be an m -dimensional rational polyhedron lying on the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. Writing I for the unit real interval $[0, 1]$, let us identify $P \times I$ with the prism*

$$\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n, 0) \in P, x_{n+1} \in I\}.$$

We then have $\lambda^{(m+1)}(P \times I) = \lambda^{(m)}(P)$.

14.4 Remarks

The weak Oda conjecture [1] [and its equivalent reformulation (B21.58)] was proved independently by Włodarczyk [2] and Morelli [3].

From the isodiametric inequality (see, e.g., [4, 2.10.33]) it follows that $N^{(d)}$ agrees with Hausdorff d -dimensional measure on rational d -dimensional polyhedra.

The rational measure was introduced in [5] for all rational polyhedra in $\mathbb{R}^n$; in the same paper its main properties were established.

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Chapter 15

A Rényi Conditional in Łukasiewicz Logic

A conditional $(\phi, \theta) \mapsto \mathbf{P}(\phi \mid \theta)$ in Łukasiewicz logic $\mathbf{L}_\infty$ is supposed to provide a coherent probability assessment of the occurrence of event ϕ “given” θ . From our analysis of coherence it follows that $\mathbf{P}(\theta \mid \theta) = 1$ and $\mathbf{P}(\psi \oplus \phi \mid \theta) = \mathbf{P}(\psi \mid \theta) + \mathbf{P}(\phi \mid \theta)$ whenever $\theta \vdash \neg(\phi \odot \psi)$. If one also requires that $\mathbf{P}(\phi \mid \theta) = \mathbf{P}(\phi' \mid \theta')$ whenever $\theta \equiv \theta'$ and $\phi \equiv_\theta \phi'$, we have a map $\theta \mapsto s_\theta$ sending each θ to a state s_θ of $\mathbf{LIND}_\theta$. Classically, the family of states s_θ is tied together by the principle of compound probabilities and by various kinds of invariance properties. Among others, we will consider the following (fairly neglected) substitutivity property:

$$\mathbf{P}(\phi \mid \psi) = \mathbf{P}(X \mid \psi \odot (X \leftrightarrow \phi))$$

whenever X is a fresh variable.

Our $\mathbf{P}(\phi \mid \theta)$ is defined even in case $\text{Mod}(\theta)$ is a very thin subset of $[0, 1]^n$, so as to circumvent Borel’s paradox about the probability for a point to lie in the Western Hemisphere, given that it lies on the equator. To this purpose we will use the rational integration theory of the previous chapter.

15.1 Statement of the Main Result and Proof of (I–III)

Theorem 15.1 *For all $n = 1, 2, \dots$ and $\phi, \theta \in \text{FORM}_n$ with θ satisfiable let $\mathbf{P}: (\phi, \theta) \mapsto \mathbf{P}(\phi \mid \theta)$ be defined by*

$$\mathbf{P}(\phi \mid \theta) = \frac{\int_{\text{Mod}(\theta)} \hat{\phi} \, dx}{\int_{\text{Mod}(\theta)} dx}, \quad (15.1)$$

where $\int$ is the rational integral of Definition 14.9. Then $\mathbf{P}$ has the following properties:

(I) (Normalization): $\mathbf{P}(\phi \mid \theta) \geq 0$ and $\mathbf{P}(\theta \mid \theta) = 1$.

- (II) (Additivity): If $\neg(\phi \odot \psi)$ is a tautology, $P(\phi \oplus \psi \mid \theta) = P(\phi \mid \theta) + P(\psi \mid \theta)$.
 (III) (Lebesgue integral): Let $\hat{\phi}$ and $\hat{\theta}$ be the McNaughton functions of ϕ and θ . If $\dim(\text{Mod}(\theta)) = n$ then

$$P(\phi \mid \theta) = \frac{\text{the Lebesgue integral of } \hat{\phi} \text{ on } \text{Mod}(\theta)}{\text{the } n\text{-dimensional Lebesgue measure of } \text{Mod}(\theta)},$$

i.e., the average value of $\hat{\phi}$ on $\text{Mod}(\theta)$.

- (IV) (Invariance under isomorphisms): If $\phi', \theta' \in \text{FORM}_{n'}$, and ϕ and ϕ' correspond via an isomorphism of the Lindenbaum algebras of θ and θ' , then $P(\phi \mid \theta) = P(\phi' \mid \theta')$.
 (V) (Invariance under substitutions): Let $\text{var}(\theta)$ denote the set of variables in θ . If $X \notin \text{var}(\theta) \cup \text{var}(\phi)$ then $P(\phi \mid \theta) = P(X \mid \theta \odot (\phi \leftrightarrow X))$.
 (VI) (Rényi's multiplication law): If $\psi \odot \theta$ is satisfiable, then

$$P((\phi \odot \psi)^\infty \mid \theta) = P(\phi^\infty \mid \psi \odot \theta) \cdot P(\psi^\infty \mid \theta),$$

where $P(\alpha^\infty \mid \beta) = \lim_{t \rightarrow \infty} P(\alpha^t \mid \beta)$ and $\alpha^t = \underbrace{\alpha \odot \dots \odot \alpha}_t$.
t occurrences of α

- (VII) (Independence): If $\text{var}(\phi) \cap \text{var}(\theta) = \emptyset$ then

$$P(\phi \mid \theta) = P(\phi \mid \phi \leftrightarrow \phi) = \int_{[0,1]^n} \hat{\phi}(x) dx.$$

The first statement of (I) immediately follows by definition. The second statement follows upon noting that $\hat{\theta} = 1$ on $\text{Mod } \theta$, whence

$$\int_{\text{Mod}(\theta)} dx = \lambda^{(d)}(\text{Mod}(\theta)).$$

Concerning (II), let Δ be a $\{\hat{\phi}, \hat{\psi}\}$ -regular triangulation of P , as given by Corollary 2.9. By Theorem 1.7, the hypothesis of (II) means that $V(\phi \odot \psi) = 0$ for every valuation $V \in \text{VAL}_n$. Thus, the identity

$$\frac{\phi}{\equiv} \odot \frac{\psi}{\equiv} = \frac{\phi \odot \psi}{\equiv} = 0$$

holds in the free MV-algebra FREE_n . Lemma 1.12 yields an isomorphism $\text{FREE}_n \cong \mathcal{M}([0, 1]^n)$, whence $\widehat{\phi \odot \psi} = \hat{\phi} \odot \hat{\psi} = 0$, and $\widehat{\phi \oplus \psi} = \hat{\phi} + \hat{\psi}$. We now easily obtain (II) from (14.9), since

$$\widehat{\phi \oplus \psi \upharpoonright T} = \widehat{(\hat{\phi} + \hat{\psi}) \upharpoonright T} = \widehat{\hat{\phi} \upharpoonright T} + \widehat{\hat{\psi} \upharpoonright T}$$

for every d -simplex $T \in \Delta$.

Proposition 15.2 *For every satisfiable formula θ let the map $P_\theta : \text{LIND}_\theta \rightarrow [0, 1]$ be defined by $P_\theta(\frac{\psi}{\equiv}) = P(\psi \mid \theta)$. Then P_θ is a state of LIND_θ .*

Proof It is easy to see that $P_\theta(\psi) = 1$. In order to prove that P_θ has the additivity property, suppose $\theta \vdash \neg(\psi \odot \phi)$. By Theorem 1.7 and Lemma 1.12 we can write $\widehat{\psi \odot \phi} \upharpoonright P = (\hat{\psi} \odot \hat{\phi}) \upharpoonright P = 0$. In other words, the sum $\hat{\psi} + \hat{\phi}$ is ≤ 1 on P . Corollary 2.9 yields a regular triangulation Φ of P such that both $\hat{\psi}$ and $\hat{\phi}$ (whence their sum) are linear on each simplex T of Φ . Since $(\hat{\psi} + \hat{\phi}) \upharpoonright T$ is uniquely determined by its values at the vertices of T , using Theorem 14.11 and arguing as in the above proof of (II), we easily obtain $P_\theta(\psi \oplus \phi) = P_\theta(\psi) + P_\theta(\phi)$. $\square$

The proof of (III) follows from Proposition 14.2.

15.2 Proof of (IV) and (V)

Proposition 15.3 *P is invariant under isomorphisms.*

Proof Given $\theta' \in \text{FORM}_{n'}$ and an isomorphism $\eta : \text{LIND}_{\theta'} \cong \text{LIND}_\theta = \mathcal{M}(P)$, let $P' = \text{Mod}(\theta') \subseteq [0, 1]^{n'}$. By Lemma 1.12 we may write without loss of generality $\text{LIND}_{\theta'} = \mathcal{M}(P')$. Corollary 3.10 yields a $\mathbb{Z}$ -homeomorphism ζ of P onto P' . It follows that a point $z \in P$ is rational iff so is the point $\zeta(z) \in P'$, and for all $y \in P \cap \mathbb{Q}^n$, $\text{den}(\zeta(y)) = \text{den}(y)$.

Suppose $\psi' \in \text{FORM}_{n'}$, and $\psi / \equiv_\theta = \eta(\psi' / \equiv_{\theta'})$, with the intent of proving $P_\theta(\psi) = P_{\theta'}(\psi')$. By Lemma 1.12 we can write $\psi / \equiv_\theta = \hat{\psi} \upharpoonright P$ and $\psi' / \equiv_{\theta'} = \hat{\psi}' \upharpoonright P'$. Let Σ be a regular $\hat{\zeta}$ -triangulation of P . Suppose Σ has the additional property that $\hat{\psi}$ is linear on every simplex of Σ . The existence of Σ follows from Corollary 2.9. By Lemma 3.13, the image $\Sigma' = \zeta(\Sigma)$ is a regular $\hat{\psi}'$ -triangulation of P' . For each d -simplex $S \in \Sigma$ its correspondent $\zeta(S) = S'$ is a d -simplex of Σ' . Since $\mathbb{Z}$ -homeomorphisms preserve denominators (Proposition 3.15) we can write

$$\lambda^{(d)}(S) = (d! \text{den}(S))^{-1} = (d! \text{den}(\zeta(S)))^{-1} = \lambda^{(d)}(S'). \quad (15.2)$$

For all $e \neq d$ both $\lambda^{(e)}(S)$ and $\lambda^{(e)}(S')$ vanish. Maximal simplexes of Σ correspond via ζ to maximal simplexes of Σ' . Computing $\lambda^{(d)}(P')$ with the help of the regular triangulation Σ' , by Theorem 14.1 we obtain

$$\lambda^{(d)}(P) = \lambda^{(d)}(P'). \quad (15.3)$$

For all $x \in P$ we can write

$$\hat{\psi}'(\zeta(x)) = (\eta^{-1}(\hat{\psi}))(\zeta(x)) = \hat{\psi}(\zeta^{-1}(\zeta(x))) = \hat{\psi}(x),$$

whence the values of $\hat{\psi}$ and $\hat{\psi}'$ at corresponding points $x \in P$ and $x' = \zeta(x) \in P'$ coincide. For every d -simplex T of Σ , letting $T' = \zeta(T)$, by (14.8) and (15.2) we obtain the identity $\lambda^{(d)}(T) \times \hat{\psi} \upharpoonright T = \lambda^{(d)}(T') \times \hat{\psi}' \upharpoonright T'$. From (15.3) we get

$$\int_T \hat{\psi} = \int_{T'} \hat{\psi}',$$

whence

$$\sum_{T \in \nabla, \dim(T)=d} \int_T \hat{\psi} = \sum_{T' \in \nabla', \dim(T')=d} \int_{T'} \hat{\psi}'.$$

We conclude that $\mathbf{P}$ is invariant, and (IV) is proved. $\square$

To prove (V), in the light of Lemma 1.12, we again identify LIND_θ with $\mathcal{M}(P)$. Similarly, we can write $\text{LIND}_{\theta \odot (\phi \leftrightarrow X_{n+1})} = \mathcal{M}(Q)$, where the rational polyhedron Q is the set of all points $x \in [0, 1]^{n+1}$ such that the McNaughton function f of the formula $\theta \odot (\phi \leftrightarrow X)$ satisfies $f(x) = 1$. Upon identifying $[0, 1]^n$ with $\{x \in [0, 1]^{n+1} \mid x_{n+1} = 0\}$, it follows that $Q = f^{-1}(1)$ is the graph of $f \upharpoonright P$,

$$Q = \{(x_1, \dots, x_{n+1}) \in [0, 1]^{n+1} \mid \hat{\theta}(x_1, \dots, x_n) = 1 \text{ and } x_{n+1} = \hat{\phi}(x_1, \dots, x_n)\}.$$

The map $\eta: (x_1, \dots, x_n) \in P \mapsto (x_1, \dots, x_n, \hat{\phi}(x_1, \dots, x_n))$ is a homeomorphism of P onto Q . The inverse map η^{-1} projects Q onto P along the $(n+1)$ th axis of $\mathbb{R}^{n+1}$, by just forgetting the last coordinate of each point of Q . Both maps η and η^{-1} are piecewise linear with integer coefficients. The map $\iota: g \in \mathcal{M}(Q) \mapsto g(\eta) \in \mathcal{M}(P)$ is a homomorphism of $\mathcal{M}(Q)$ into $\mathcal{M}(P)$. The map $j: h \in \mathcal{M}(P) \mapsto h(\eta^{-1}) \in \mathcal{M}(Q)$ is a homomorphism of $\mathcal{M}(P)$ into $\mathcal{M}(Q)$. One easily sees that $\iota = j^{-1}$, whence we can write $j: \text{LIND}_\theta = \mathcal{M}(P) \cong \mathcal{M}(Q) = \text{LIND}_{\theta \odot (\phi \leftrightarrow X)}$. Letting $\xi_{n+1}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ denote the $(n+1)$ th coordinate function, the two elements

$$\hat{\phi} \upharpoonright P = \frac{\phi}{\equiv_\theta} \quad \text{and} \quad \xi_{n+1} \upharpoonright Q = \frac{X}{\equiv_{\theta \odot (X \leftrightarrow \phi)}}$$

correspond under the isomorphism j . An application of 15.1(IV) completes the proof of (V).

Remark 15.4 In particular, if the variable X does not occur in ϕ ,

$$\mathbf{P}(X \mid (\phi \leftrightarrow X)) = \mathbf{P}(\phi \mid (\phi \leftrightarrow \phi)) = \int_{[0,1]^m} \hat{\phi}(x) dx. \quad (15.4)$$

The second identity follows from (III).

15.3 Preparatory Material for the Proof of (VI)

Lemma 15.5 *Given $\tau, \phi, \chi, \psi, \theta \in \text{FORM}_n$, let the rational polyhedron $P \subseteq [0, 1]^n$ be defined by $P = \hat{\theta}^{-1}(1)$. Suppose $\dim(P) = d \geq 0$. We then have:*

- (i) $P(\neg\phi \mid \theta) = 1 - P(\phi \mid \theta)$. Thus $P(\tau \mid \theta) = 0$ whenever $\neg\tau$ is a tautology. Further, $P(\phi \mid \theta) \leq 1$.
- (ii) If $\theta \odot \chi$ is satisfiable then so is $\theta \wedge \chi$, and $P(\phi \mid \theta \odot \chi) = P(\phi \mid \theta \wedge \chi)$.
- (iii) $P(\phi \mid \theta) = P(\phi \mid \theta \odot \theta) = P(\phi \mid \theta^n)$, for all $n = 2, 3, \dots$
- (iv) $P(\phi \mid \theta) = P(\phi \odot \theta \mid \theta) = P(\phi \wedge \theta \mid \theta)$.
- (v) If $\theta \vdash \phi \rightarrow \psi$ then $P(\phi \mid \theta) \leq P(\psi \mid \theta)$.
- (vi) If $\theta \vdash \phi \leftrightarrow \psi$ then $P(\phi \mid \theta) = P(\psi \mid \theta)$.
- (vii) (Strong form of (I)) If $\theta \vdash \phi$ then $P(\phi \mid \theta) = 1$.
- (viii) (Strong form of (II)) If $\theta \vdash \neg(\phi \odot \psi)$ then $P(\phi \oplus \psi \mid \theta) = P(\phi \mid \theta) + P(\psi \mid \theta)$.

Proof

(i) Trivial.

(ii) The two rational polyhedra $(\widehat{\chi \odot \theta})^{-1}(1)$ and $(\widehat{\chi \wedge \theta})^{-1}(1)$ coincide with $\widehat{\chi}^{-1}(1) \cap \widehat{\theta}^{-1}(1)$.

(iii) Immediate from (ii), because $\widehat{\theta \wedge \theta} = \widehat{\theta}$.

(iv) We first observe that the three functions $\widehat{\phi}, \widehat{\phi \odot \theta}$ and $\widehat{\phi \wedge \theta}$ coincide on P . In the light of Theorem 14.11 it is enough to evaluate

$$\sum \left\{ \lambda^{(\dim(P))} \cdot \overline{f \upharpoonright T} \mid T \in \Delta, \dim(T) = \dim(P) \right\}$$

with the help of some regular $\{\widehat{\phi}, \widehat{\phi \wedge \theta}, \widehat{\phi \odot \theta}\}$ -triangulation Δ of P as given by Corollary 2.9.

(v) Let ∇ be a regular $\{\widehat{\phi}, \widehat{\psi}\}$ -triangulation of P as given by Corollary 2.9. By Theorem 1.7, our standing assumption states that $V(\phi) \leq V(\psi)$ for every valuation V satisfying θ . Thus the inequality $\frac{\phi}{\equiv_{\theta}} \leq \frac{\psi}{\equiv_{\theta}}$ holds in LIND_{θ} . Lemma 1.12 yields an isomorphism $\mathcal{M}(P) \cong \text{LIND}_{\theta}$, whence $\widehat{\phi}(x) \leq \widehat{\psi}(x)$ for all $x \in P$. By construction of ∇ , for each d -simplex $T = \text{conv}(v_0, \dots, v_d) \in \nabla$ and for all $i = 0, \dots, d$ we have $\widehat{\phi}(v_i) \leq \widehat{\psi}(v_i)$. Therefore, $\widehat{\phi} \upharpoonright T \leq \widehat{\psi} \upharpoonright T$, for every $T \in \nabla$ with $\dim(T) = d$. The desired conclusion now follows from Theorem 14.11.

(vi) Immediate from (v).

(vii) The hypothesis actually states that $\widehat{\phi} \upharpoonright P = 1$. By Theorem 14.11, the numerator and the denominator of the fraction in the definition of $\mathbf{P}$ coincide.

(viii) The hypothesis states that the functions $\widehat{\phi \oplus \psi}$ and $\widehat{\phi} + \widehat{\psi}$ coincide on P . Using Corollary 2.9, let Δ be a regular $\{\widehat{\phi}, \widehat{\psi}\}$ -triangulation of P . Then the desired conclusion follows arguing as in the proof of (II) above. $\square$

In the definition of the McNaughton function $\widehat{\psi}$ of a formula $\psi \in \text{FORM}_n$, the dependence on n was tacitly understood for the sake of readability. If there is danger of confusion we will write $\widehat{\psi}_{(n)}$ to denote the McNaughton function of ψ as a formula of FORM_n , and $\widehat{\psi}_{(n+1)}$ for the McNaughton function of ψ as a formula of FORM_{n+1} . These two functions are linked by the trivial relation

$$\widehat{\psi}_{(n+1)}(x_1, \dots, x_{n+1}) = \widehat{\psi}_{(n)}(x_1, \dots, x_n), \text{ for all } x_1, \dots, x_{n+1} \in [0, 1]^{n+1}. \quad (15.5)$$

One might wonder if a similar distinction should be made in the definition of $P(\phi \mid \theta)$, according as $\phi, \theta \in \text{FORM}_n$ or else $\phi, \theta \in \text{FORM}_{n+1}$.

The following proposition shows that no such notational distinction is necessary:

Proposition 15.6 *For $\phi, \theta \in \text{FORM}_n$, let us suppose the set $P = \hat{\theta}_{(n)}^{-1}(1)$ is m -dimensional, for some $0 \leq m \leq n$. Let Δ be a regular $\{\hat{\phi}_{(n)}\}$ -triangulation of P . Let ∇ be a regular $\{\hat{\phi}_{(n+1)}\}$ -triangulation of the $(m+1)$ -dimensional rational polyhedron $P' = \hat{\theta}_{(n+1)}^{-1}(1) \subseteq [0, 1]^{n+1}$. Then*

$$\int_P \hat{\phi}_{(n)} = \int_{P'} \hat{\phi}_{(n+1)}.$$

Proof In view of Theorem 14.11 we must show that

$$\frac{\sum \left\{ \lambda^{(m)}(T) \cdot \overline{\hat{\phi}_{(n)} \upharpoonright T} \mid T \in \Delta, \dim(T) = m \right\}}{\lambda^{(m)}(P)} \quad (15.6)$$

coincides with

$$\frac{\sum \left\{ \lambda^{(m+1)}(S) \cdot \overline{\hat{\phi}_{(n+1)} \upharpoonright S} \mid S \in \nabla, \dim(S) = m+1 \right\}}{\lambda^{(m+1)}(P')}. \quad (15.7)$$

To this purpose let us identify $\mathbb{R}^n$ with the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. Let $I = [0, 1]$ be identified with the segment

$$I = \text{conv}((0, \dots, 0, 0), (0, \dots, 0, 1)) \subseteq \mathbb{R}^{n+1}. \quad (15.8)$$

We will show that the two fractions (15.6 and 15.7) have the same value. As a matter of fact, repeated applications of Theorem 14.11 together with Corollary 14.12 yields

$$\begin{aligned} & \sum \left\{ \lambda^{(m)}(T) \cdot \overline{\hat{\phi}_{(n)} \upharpoonright T} \mid T \in \Delta, \dim(T) = m \right\} \\ &= \sum \left\{ \lambda^{(m+1)}(\text{sub } \hat{\phi}_{(n)} \upharpoonright T) \mid T \in \Delta, \dim(T) = m \right\} \\ &= \sum \left\{ \lambda^{(m+2)}(I \times \text{sub } \hat{\phi}_{(n)} \upharpoonright T) \mid T \in \Delta, \dim(T) = m \right\} \\ &= \sum \left\{ \lambda^{(m+2)}(\text{sub } \hat{\phi}_{(n+1)} \upharpoonright (T \times I)) \mid T \in \Delta, \dim(T) = m \right\} \\ &= \sum \left\{ \lambda^{(m+1)}(S) \cdot \overline{\hat{\phi}_{(n+1)} \upharpoonright S} \mid S \in \nabla, \dim(S) = m+1 \right\}. \end{aligned}$$

We have just proved that the numerators coincide. Replacing $\hat{\phi}_{(n)}$ and $\hat{\phi}_{(n+1)}$ respectively by the constant functions 1 on $[0, 1]^n$ and on $[0, 1]^{n+1}$, a similar computation shows that also the denominators of (15.6) and (15.7) coincide. $\square$

15.4 End of Proof of (VI)

Proposition 15.7 For $\phi, \theta \in \text{FORM}_n$ let us write $P = \hat{\theta}^{-1}(1)$ and $Q = \hat{\phi}^{-1}(1)$. Suppose $0 \leq \dim(P) = m$. We then have:

- (i) If $\dim(Q \cap P) < \dim(P)$ then $P(\phi^\infty \mid \theta) = 0$.
- (ii) If $\dim(Q \cap P) = \dim(P)$ then $P(\phi^\infty \mid \theta) = \lambda^{(m)}(Q \cap P) / \lambda^{(m)}(P)$.

Proof (i) We must show that the numerator of $P(\phi^t \mid \theta)$ computed with the help of Theorem 14.11 tends to zero as t tends to infinity. To this purpose, for each $t = 1, 2, \dots$ let Δ_t be a regular $\hat{\phi}^t$ -triangulation of P . Since P is a rational polyhedron, the m -simplexes of Δ_t span only finitely many affine spaces $A_1, \dots, A_u$ with corresponding constants $\kappa_1, \dots, \kappa_u$ as given by Theorem 14.7. This list does not depend on t . Let $\kappa^* = \max(\kappa_1, \dots, \kappa_u)$. Further, let $R_1, \dots, R_v$ be the m -simplexes of Δ_1 . It follows that

$$P^{(m)} = R_1 \cup \dots \cup R_v = \bigcup \{T \in \Delta_t \mid \dim(T) = m\}, \text{ for all } t = 1, 2, \dots \quad (15.9)$$

We then have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \sum \left\{ \lambda^{(m)}(T) \cdot \overline{\hat{\phi}^t \upharpoonright T} \mid T \in \Delta_t, \dim(T) = m \right\} \\ &= \lim_{t \rightarrow \infty} \sum \left\{ \lambda^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright T) \mid T \in \Delta_t, \dim(T) = m \right\}, \text{ by 14.11} \\ &= \lim_{t \rightarrow \infty} \sum \left\{ \kappa_{\text{aff}(T)} \cdot \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright T) \mid T \in \Delta_t, \dim(T) = m \right\}, \text{ by 14.7} \\ &\leq \kappa^* \cdot \lim_{t \rightarrow \infty} \sum \left\{ \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright T) \mid T \in \Delta_t, \dim(T) = m \right\} \\ &= \kappa^* \cdot \lim_{t \rightarrow \infty} \mathcal{N}^{(m+1)} \left(\bigcup \{ \text{sub } \hat{\phi}^t \upharpoonright T \mid T \in \Delta_t, \dim(T) = m \} \right), \text{ by 14.4} \\ &= \kappa^* \cdot \lim_{t \rightarrow \infty} \mathcal{N}^{(m+1)} \left(\bigcup \{ \text{sub } \hat{\phi}^t \upharpoonright R \mid R \in \Delta_1, \dim(R) = m \} \right), \text{ by (15.9)} \\ &= \kappa^* \cdot \lim_{t \rightarrow \infty} \sum_{j=1}^u \mathcal{N}^{(m+1)} \left(\text{sub } \hat{\phi}^t \upharpoonright R_j \right). \end{aligned}$$

Claim For each $j = 1, \dots, u$, $\mathcal{N}^{(m+1)} \left(\text{sub } \hat{\phi}^t \upharpoonright R_j \right)$ tends to zero as t tends to infinity.

As a matter of fact, for every $t = 1, 2, \dots$ let $S_{jt} = \{x \in R_j \mid \phi^t(x) > 0\}$. Note that $S_{j,t+1} \subseteq S_{jt}$. In case $\phi \upharpoonright R_j$ has a maximum value < 1 (equivalently, $R_j \cap Q = \emptyset$) it is easy to see that ϕ^t vanishes on R_j and the set S_{jt} is empty, for all large t . Thus, $\mathcal{N}^{(m)}(\bigcap_t S_{jt}) = 0$. On the other hand, if $R_j \cap Q \neq \emptyset$ let $\{v_1, \dots, v_p\}$ be the list of all vertices of R_j lying in Q . Then not all vertices of R_j are in this list. For otherwise, from the assumption that Δ_1 is a $\hat{\phi}$ -triangulation it would follow that R_j is a subset of Q , and hence $\dim(Q \cap P) = \dim(P) = m$, which is impossible. As a consequence, $1 \leq p \leq m$. For all large t the set S_{jt} shrinks to the set $\text{conv}(v_1, \dots, v_p)$,

in the sense that for every open neighborhood N of $\text{conv}(v_1, \dots, v_p)$ there is t_N such that $S_{jt} \subseteq N$ for all $t \geq t_N$. This follows by definition of ϕ^t and S_{jt} . We then have $\bigcap_t S_{jt} = \text{conv}(v_1, \dots, v_p)$ with $\dim(\text{conv}(v_1, \dots, v_p)) = p - 1 < m$. By definition of $\mathcal{N}^{(m)}$ and of dimension, also in this case, $\mathcal{N}^{(m)}(\bigcap_t S_{jt}) = 0$. Lemma 14.3 yields a sequence $0 \leq \mathcal{N}^{(m)}(S_{j_{t+1}}) \leq \mathcal{N}^{(m)}(S_{jt})$, whence $\lim_{t \rightarrow \infty} \mathcal{N}^{(m)}(S_{jt})$ exists. Since the sequence of polyhedra S_{jt} shrinks to a polyhedron of dimension $< m$, by the elementary properties of the natural measure $\mathcal{N}$, (Definition 14.4), we obtain $\lim_{t \rightarrow \infty} \mathcal{N}^{(m)}(S_{jt}) = 0$. Let $I \subseteq [0, 1]^{n+1}$ be the unit interval along the $(n+1)$ th axis, $I = \text{conv}((0, \dots, 0, 0), (0, \dots, 0, 1))$. Then $\lim_{t \rightarrow \infty} \mathcal{N}^{(m+1)}(I \times S_{jt}) = 0$. The claim is settled upon noting that $\mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright R_j) \leq \mathcal{N}^{(m+1)}(I \times \text{cl } S_{jt}) = \mathcal{N}^{(m+1)}(I \times S_{jt})$, once I is identified with the interval $[0, 1]$.

The proof of (i) is complete.

(ii) For each t we let ∇_t be a regular $\hat{\phi}^t$ -triangulation of P such that $Q \cap P$ is a union of simplexes of ∇_t . This exists by Corollary 2.9. We must only consider the numerator in the definition of $\mathbf{P}(\phi \mid \theta)$. Arguing as in (i) with the help of Theorems 14.11 and 14.7, we obtain

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sum \left\{ \lambda^{(m)}(T) \cdot \overline{\hat{\phi}^t \upharpoonright T} \mid T \in \nabla_t, \dim(T) = m \right\} \\
&= \lim_{t \rightarrow \infty} \sum \left\{ \lambda^{(m)}(T) \cdot \overline{\hat{\phi}^t \upharpoonright T} \mid T \in \nabla_t, \dim(T) = m, T \subseteq Q \right\} \\
&\quad + \lim_{t \rightarrow \infty} \sum \left\{ \lambda^{(m)}(T) \cdot \overline{\hat{\phi}^t \upharpoonright T} \mid T \in \nabla_t, \dim(T) = m, T \not\subseteq Q \right\} \\
&= \lim_{t \rightarrow \infty} \sum \left\{ \lambda^{(m)}(T) \cdot 1 \mid T \in \nabla_t, \dim(T) = m, T \subseteq Q \right\} \\
&\quad + \lim_{t \rightarrow \infty} \sum \left\{ \lambda^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright T) \mid T \in \nabla_t, \dim(T) = m, T \not\subseteq Q \right\} \\
&= \lambda^{(m)}(Q \cap P) \\
&\quad + \lim_{t \rightarrow \infty} \sum \left\{ \kappa_{\text{aff}(T)} \cdot \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright T) \mid T \in \nabla_t, \dim(T) = m, T \not\subseteq Q \right\} \\
&\leq \lambda^{(m)}(Q \cap P) \\
&\quad + \kappa^* \lim_{t \rightarrow \infty} \sum \left\{ \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright T) \mid T \in \nabla_t, \dim(T) = m, T \not\subseteq Q \right\} \\
&= \lambda^{(m)}(Q \cap P) \\
&\quad + \kappa^* \lim_{t \rightarrow \infty} \sum \left\{ \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright T) \mid T \in \nabla_1, \dim(T) = m, T \not\subseteq Q \right\} \\
&= \lambda^{(m)}(Q \cap P) + \kappa^* \lim_{t \rightarrow \infty} \left(\mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright R_1) + \dots + \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright R_v) \right) \\
&= \lambda^{(m)}(Q \cap P),
\end{aligned}$$

because, as in the claim above, for each $j = 1, \dots, v$, $\mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}^t \upharpoonright R_j)$ tends to zero as t tends to infinity. $\square$

To conclude the proof of (VI), let the rational polyhedra $A, B, P, Q, R \subseteq [0, 1]^n$ be defined by

$$A = \hat{\phi}^{-1}(1), B = \hat{\psi}^{-1}(1), P = \hat{\theta}^{-1}(1), Q = (\hat{\psi} \odot \hat{\theta})^{-1}(1), R = (\hat{\phi} \odot \hat{\psi} \odot \hat{\theta})^{-1}(1).$$

By hypothesis, $Q = B \cap P$ is nonempty, whence so is P . By Lemma 15.5(iv), condition (VI) is equivalent to

$$\mathbf{P}((\phi \odot \psi \odot \theta)^\infty \mid \theta) = \mathbf{P}((\phi \odot \psi \odot \theta)^\infty \mid \psi \odot \theta) \cdot \mathbf{P}((\psi \odot \theta)^\infty \mid \theta). \quad (15.10)$$

To prove (15.10) we argue by cases:

Case 1 $\dim(Q) < \dim(P)$.

Then by Lemma 15.5(iv) and Proposition 15.7(i), $\mathbf{P}(\psi^\infty \mid \theta) = \mathbf{P}((\psi \odot \theta)^\infty \mid \theta) = 0$, whence a fortiori, by Lemma 15.5(v), $\mathbf{P}((\phi \odot \psi \odot \theta)^\infty \mid \theta) = 0$.

Case 2 $\dim(Q) = \dim(P) = m$.

Subcase 2.1 $\dim(R) < \dim(Q)$. Then again by Lemma 15.5(iv) and Proposition 15.7(i), $\mathbf{P}((\phi \odot \psi \odot \theta)^\infty \mid \theta) = 0 = \mathbf{P}((\phi \odot \psi \odot \theta)^\infty \mid \psi \odot \theta)$ and (15.10) is satisfied.

Subcase 2.2 $\dim(R) = \dim(Q)$. Then, by Proposition 15.7(ii), $0 < \lambda^{(m)}(Q) \leq \lambda^{(m)}(P)$, and we can write $\mathbf{P}((\phi \odot \psi \odot \theta)^\infty \mid \theta) = \frac{\lambda^{(m)}(R)}{\lambda^{(m)}(P)} = \frac{\lambda^{(m)}(R)}{\lambda^{(m)}(Q)} \cdot \frac{\lambda^{(m)}(Q)}{\lambda^{(m)}(P)} = \mathbf{P}((\phi \odot \psi \odot \theta)^\infty \mid \psi \odot \theta) \cdot \mathbf{P}((\psi \odot \theta)^\infty \mid \theta)$. The proof of (VI) is now complete.

15.5 Independence, Proof of (VII)

Definition 15.8 Given $\psi, \theta \in \text{FORM}_n$ with θ satisfiable, we say that ψ is *independent of* θ if $\mathbf{P}(\psi \mid \theta) = \mathbf{P}(\psi \mid \psi \leftrightarrow \psi)$. We will write $\mathbf{P}(\psi)$ as an abbreviation of the unconditional probability $\mathbf{P}(\psi \mid \psi \leftrightarrow \psi)$.

We then have

Proposition 15.9 (i) *For any two formulas $\psi, \theta \in \text{FORM}_n$ with θ satisfiable, the following conditions are equivalent:*

- (a) ψ is independent of θ ;
- (b) $\mathbf{P}(\psi \mid \theta) = \mathbf{P}(\psi \mid \theta \leftrightarrow \theta)$;
- (c) $\mathbf{P}(\psi \mid \theta) = \int_{[0,1]^n} \hat{\psi}(x) dx$;
- (d) $\mathbf{P}(\psi \mid \theta) = \mathbf{P}(X \mid (X \leftrightarrow \psi))$ whenever $X \notin \text{var}(\psi)$;
- (e) $\mathbf{P}(\psi \mid \theta) = \mathbf{P}(\psi \mid \tau)$, where $\tau \in \text{FORM}_n$ is an arbitrary tautology.

(ii) If ψ is independent of θ then $\lim_{t \rightarrow \infty} (\mathbf{P}(\psi^t \odot \theta^t) - \mathbf{P}(\psi^t) \cdot \mathbf{P}(\theta^t)) = 0$.

Proof (i) The proof routinely follows combining 14.11, 15.1(III), 15.6, (15.4), and 15.4.

(ii) Let $\tau \in \mathbf{FORM}_n$ be an arbitrary tautology. By (VI) we have

$$\begin{aligned} \mathbf{P}(\psi^\infty \odot \theta^\infty) &= \mathbf{P}((\psi \odot \theta)^\infty) = \mathbf{P}((\psi \odot \theta)^\infty \mid \tau) = \mathbf{P}(\psi^\infty \mid \theta \odot \tau) \cdot \mathbf{P}(\theta^\infty \mid \tau) \\ &= \mathbf{P}(\psi^\infty \mid \theta) \cdot \mathbf{P}(\theta^\infty \mid \tau) = \mathbf{P}(\psi^\infty \mid \psi \leftrightarrow \psi) \cdot \mathbf{P}(\theta^\infty \mid \theta \leftrightarrow \theta) = \mathbf{P}(\psi^\infty) \cdot \mathbf{P}(\theta^\infty). \end{aligned}$$

□

We are now in a position to prove (VII). To this purpose, given formulas

$$\phi = \phi(Y_1, \dots, Y_m) \text{ and } \theta = \theta(Z_1, \dots, Z_n),$$

we will use the abbreviations

$$Y = \{Y_1, \dots, Y_m\}, \quad Z = \{Z_1, \dots, Z_n\}, \quad P = \hat{\theta}^{-1}(1) \subseteq [0, 1]^Z = [0, 1]^n, \quad d = \dim(P).$$

By hypothesis, $0 \leq d \leq n$. The set $P' = P \times [0, 1]^Y$ is a $(d + m)$ -dimensional rational polyhedron in the cube $[0, 1]^Z \times [0, 1]^Y = [0, 1]^{n+m}$. With reference to (15.5), we will use the notation $\hat{\phi}_{(m)}$ (resp., $\hat{\phi}_{(n+m)}$) for the McNaughton function of $\phi \in \mathbf{FORM}_m$ (resp., for the McNaughton function of $\phi \in \mathbf{FORM}_{n+m}$). Let Δ be a regular triangulation of P , and ∇ a regular $\hat{\phi}_{(m+n)}$ -triangulation of P' , as given by Corollary 2.9. For T any arbitrary d -simplex of Δ we will write $T' = T \times [0, 1]^Y$. Then $|\nabla| = \bigcup \{T' \mid T \in \Delta\}$. Recalling Proposition 15.6, we compute $\mathbf{P}(\phi \mid \theta)$ considering ϕ, θ as elements of $\mathbf{FORM}_{m+n}$. We then have

$\mathbf{P}(\phi \mid \theta)$

$$\begin{aligned} &= \frac{\sum \left\{ \lambda^{(d+m)}(R) \cdot \overline{\hat{\phi}_{(m+n)} \upharpoonright R} \mid R \in \nabla, \dim(R) = d + m \right\}}{\lambda^{(d+m)}(P')} \\ &= \frac{\lambda^{(d+m+1)}(\text{sub } \hat{\phi}_{(m+n)} \upharpoonright P')}{\lambda^{(d+m)}(P')}, \text{ Theorem 14.11} \\ &= \frac{\sum \left\{ \lambda^{(d+m+1)}(\text{sub } \hat{\phi}_{(m+n)} \upharpoonright (T')) \mid T \in \Delta, \dim(T) = d \right\}}{\sum \left\{ \lambda^{(d+m)}(T') \mid T \in \Delta, \dim(T) = d \right\}}, \end{aligned}$$

by definition of λ and P'

$$= \frac{\sum \left\{ \kappa_{\text{aff}(T')} \cdot \mathcal{N}^{(m+d+1)}(\text{sub } \hat{\phi}_{(m+n)} \upharpoonright T') \mid T \in \Delta, \dim(T) = d \right\}}{\sum \left\{ \kappa_{\text{aff}(T')} \cdot \mathcal{N}^{(m+d)}(T') \mid T \in \Delta, \dim(T) = d \right\}},$$

Theorem 14.7

$$= \frac{\sum \left\{ \kappa_{\text{aff}(T')} \cdot \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}_{(m)} \upharpoonright [0, 1]^m) \cdot \mathcal{N}^{(d)}(T) \mid T \in \Delta, \dim(T) = d \right\}}{\sum \left\{ \kappa_{\text{aff}(T')} \cdot \mathcal{N}^{(m)}([0, 1]^Y) \cdot \mathcal{N}^{(d)}(T) \mid T \in \Delta, \dim(T) = d \right\}},$$

by definition of T' and multiplicativity of $\mathcal{N}$ for cartesian products

$$= \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}_{(m)} \upharpoonright [0, 1]^m) \frac{\sum \left\{ \kappa_{\text{aff}(T)} \mathcal{N}^{(d)}(T) \mid T \in \Delta, \dim(T) = d \right\}}{\sum \left\{ \kappa_{\text{aff}(T)} \cdot 1 \cdot \mathcal{N}^{(d)}(T) \mid T \in \Delta, \dim(T) = d \right\}},$$

$$\text{because } \mathcal{N}^{(m)}([0, 1]^m) = \mathcal{L}^{(m)}([0, 1]^m) = 1$$

$$= \mathcal{N}^{(m+1)}(\text{sub } \hat{\phi}_{(m)} \upharpoonright [0, 1]^m)$$

$$= \mathcal{L}^{(m+1)}(\text{sub } \hat{\phi}_{(m)} \upharpoonright [0, 1]^m), \text{ Definition 14.4}$$

$$= \lambda^{(m+1)}(\text{sub } \hat{\phi}_{(m)} \upharpoonright [0, 1]^m) = \frac{\lambda^{(m+1)}(\text{sub } \hat{\phi}_{(m)} \upharpoonright [0, 1]^m)}{\lambda^{(m)}([0, 1]^m)}, \text{ Proposition 14.2}$$

$$= \frac{\sum \left\{ \lambda^{(m)}(T) \cdot \overline{\hat{\phi}_{(m)} \upharpoonright T} \mid T \in \Delta, \dim(T) = d \right\}}{\lambda^{(m)}([0, 1]^m)}, \text{ Theorem 14.11}$$

$$= \mathbf{P}(\phi \mid \phi \leftrightarrow \phi).$$

An application of (III) concludes the proof of (VII).

The proof of Theorem 15.1 is now complete. $\square$

15.6 Remarks

Conditions (I–III) are a many-valued generalization of Rényi's axioms I–III [1, p. 289] for random yes–no events.

In his treatment of the boolean algebraic case [1, pp. 289–290], Rényi observed that a conditional probability is nothing else than a set of states (called by him “ordinary probability spaces”) which are connected with each other in conformity with the usual definition of conditional probability. In his Axiom II, Rényi also assumed countable additivity, whereas states are only finitely additive.

However, by the Kroupa–Panti theorem for every MV-algebra M , the states of M are in one–one correspondence with the regular Borel probability measures on the maximal spectral space of M equipped with the hull-kernel topology. Since LIND_θ is countable, its maximal spectral space is separable, and regularity is automatically ensured by (B21.65). In conclusion, we have a map $\theta \mapsto \mathbf{P}_\theta \mapsto \mu_\theta$ where μ_θ is a regular Borel probability measure on the maximal spectral space of LIND_θ which in turn can be identified with the space of valuations of formulas in FORM_n with their natural topology as a subspace of the cube $[0, 1]^{\text{FORM}_n}$. The (countably additive) probability measures μ_θ are connected by the usual defining properties of (countably

additive) conditional probabilities, precisely as their corresponding (finitely additive) states are.

The conditional probability $P(\phi \mid \theta)$ is defined for all formulas ϕ and all satisfiable formulas θ in Łukasiewicz logic L_∞ . In particular, $P(\phi \mid \theta)$ exists even if the set $\hat{\theta}^{-1}(1) \subseteq [0, 1]^n$ of valuations satisfying θ is Lebesgue negligible. This takes care of Borel's problem about the probability that a point lies on the western hemisphere, given that it lies on the equator. See [1, Sect. 2.5, p. 310]. Naturally enough, in our present context the earth surface is to be approximated by a suitable two-dimensional rational polyhedron $E \subseteq [0, 1]^3$.

Lemma 15.5(i) is a generalization of [1, Theorems 2–3, p. 290]. Lemma 15.5 (iv) is a generalization of [1, Theorem 1, p. 290].

The literature where conditional is taken as a primitive notion is multifarious, beginning with the papers [1]–[5], until the recent generalizations to many-valued conditional probability, [6]–[8].

Our definition of P simplifies the definition in [9, p. 226], where the conditional probability $\mathcal{P}(\phi \mid \theta)$ was equally distributed over all nonempty j -dimensional parts $W^{(0)}, W^{(1)}, \dots$ of W , whereas our present P only depends on $W^{(\dim(W))}$.

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Chapter 16

The Lebesgue State and the Completion of FREE_n

In this chapter we establish a lower bound for the dimension of the convex set of faithful invariant states in every finitely presented MV-algebra, and we give a purely MV-algebraic generalization of the Lebesgue integral of McNaughton functions on rational polyhedra. Since every faithful state of an MV-algebra A induces a metric on A , we describe the (Cauchy) completion of the free MV-algebra FREE_n with respect to the metric induced by the Lebesgue state.

16.1 Faithful, Invariant, and Lebesgue States

A state s of an MV-algebra A is said to be *invariant* if $s(\alpha(x)) = s(x)$ for every automorphism α of A and $x \in A$. We say that s is *faithful* if $s(y) > 0$ for every $0 \neq y \in A$.

For P a rational polyhedron in $[0, 1]^n$, ($n = 1, 2, \dots$), recalling (14.1), let us write $\delta(P)$ for the set of dimensions such that $P^{(i)}$ is nonempty, in symbols,

$$\delta(P) = \{j \in \{0, \dots, n\} \mid P^{(j)} \neq \emptyset\}.$$

For any set X let $\text{card}(X)$ denote its cardinality.

Theorem 16.1 *For any finitely presented MV-algebra A let us write $A \cong \mathcal{M}(P)$ for some rational polyhedron $P \subseteq [0, 1]^n$, as given by Theorem 6.3. Let d be the dimension of the set of faithful invariant states of A , as a convex subset of the vector space $\mathbb{R}^A$. Then $d \geq -1 + \text{card}(\delta(P))$.*

Proof It is no loss of generality to identify A with $\mathcal{M}(P)$. Recalling Definition 14.9, for each $i \in \delta(P)$ we define the state $s^{(i)}: A \rightarrow [0, 1]$ by

$$s^{(i)}(f) = \frac{\int_{P^{(i)}} f dx}{\lambda^{(i)}(P^{(i)})} = \frac{\int_{P^{(i)}} f dx}{\int_{P^{(i)}} dx}, \quad \text{for all } f \in \mathcal{M}(P). \quad (16.1)$$

The integral in the numerator coincides with the $(i + 1)$ -dimensional rational measure of the portion of $[0, 1]^{n+1}$ between $P^{(i)}$ and the graph of f . A routine generalization to arbitrary $i = 0, \dots, n$ of the argument used in the proof of (V) of Theorem 15.1 shows that both the numerator and the denominator of (16.1) are invariant under automorphisms of A . Any convex combination

$$s = \sum_{i \in \delta(P)} v_i s^{(i)}, \quad 0 \leq v_i, \quad \sum_{i \in \delta(P)} v_i = 1$$

of the $s^{(i)}$ will also be an invariant state of A . In particular, if each coefficient v_i is > 0 then s is faithful: for, whenever $f \in \mathcal{M}([0, 1]^n)$ is nonzero, there will be an $i \in \delta(P)$ and an $x \in P^{(i)}$ such that $f(x) > 0$. The definition of $s^{(i)}$ ensures that $s^{(i)}(f) > 0$, whence $s(f) > 0$ and s is faithful. By Lemma 1.13 the set of states $S = \{s^{(i)} \mid i \in \delta(P)\}$ is linearly independent in the vector space $\mathbb{R}^A$. The dimension d of the set of convex combinations of the $s^{(i)}$ with coefficients > 0 will then satisfy the inequality $d \geq -1 + \text{card}(S)$. $\square$

Lemma 16.2 *Let $P \subseteq [0, 1]^n$ be a rational polyhedron, Δ a regular triangulation of P , and $\mathcal{H}_\Delta$ its Schauder basis. Fix $T = \text{conv}(v_0, \dots, v_t) \in \Delta$. For each $i = 1, \dots, t$ let $d_i = \text{den}(v_i)$ and define the point $w_i \in [0, 1]^{n+1}$ by $w_i = (v_i, 0)$. Let the point $w_{t+1} \in [0, 1]^{n+1}$ be defined by $w_{t+1} = (v_0, 1/d_0)$. Then the simplex $U = \text{conv}(w_0, \dots, w_{t+1}) \subseteq [0, 1]^{n+1}$ is regular.*

Proof Without loss of generality, $t = n$. For each $i = 0, \dots, n$ let us display v_i as $v_{i1}, \dots, v_{in}$. Passing to homogeneous coordinates, T determines the integer $(n + 1) \times (n + 1)$ matrix M_T whose columns are the vectors $\tilde{v}_0, \dots, \tilde{v}_n$. Each $\tilde{v}_i$ has the form $\tilde{v}_i = d_i(v_i, 1) = (d_i v_{i1}, \dots, d_i v_{in}, d_i)$. Similarly, the first $n + 1$ columns of the integer matrix M_U have the form $(d_i v_{i1}, \dots, d_i v_{in}, 0, d_i)$, ($i = 0, \dots, n$), and the last column is $(d_0 v_{01}, \dots, d_0 v_{0n}, 1, d_0)$. The regularity of Δ implies the regularity of T . As an easy consequence of Lemma 2.6, the regularity of T is equivalent to the unimodularity condition $\det(M_T) = \pm 1$. Direct inspection shows that $\det(M_U) = \pm 1$, whence U is regular. $\square$

Bases yield a representation-free generalization of the averaging process given by the Lebesgue integral of McNaughton functions.

Let A be a finitely presented MV-algebra and $\mathcal{B} = \{b_1, \dots, b_n\}$ a basis of A with multipliers $m_1, \dots, m_n$. See Sect. 6.2 for the definition of the simplex $\mathcal{T}_C$ associated to a cluster C of $\mathcal{B}$. Recall from (14.2) the definition of $\text{den}(T)$, for T a regular simplex.

Let $d = \dim(\mu(A))$. For each $i = 1, \dots, n$, we set

$$\Theta_i = \{\mathcal{T}_C \mid C \text{ is a } (d + 1)\text{-cluster of } \mathcal{B} \text{ and } b_i \in C\}. \quad (16.2)$$

and

$$\Theta = \{\mathcal{T}_C \mid C \text{ is a } (d + 1)\text{-cluster of } \mathcal{B}\}. \quad (16.3)$$

Theorem 16.3 *Let A be a finitely presented MV-algebra and $d = \dim(\mu(A))$. Then there exists precisely one state s^* of A such that for any basis $\mathcal{B} = \{b_1, \dots, b_n\}$ with multipliers $m_1, \dots, m_n$ the following identity is satisfied for each $i = 1, \dots, n$:*

$$s^*(b_i) = \frac{\sum_{T \in \Theta_i} (m_i \text{den}(T))^{-1}}{(d+1) \sum_{T \in \Theta} (\text{den}(T))^{-1}}. \quad (16.4)$$

Proof Note that the denominator is > 0 . By Corollary 6.4, the family of simplexes $\Delta = \{\mathcal{T}_C \mid C \in \mathcal{B}^{\leq n}\}$ is a regular triangulation of $P = \bigcup \Delta \subseteq [0, 1]^n$. Without loss of generality we can assume A to coincide with $\mathcal{M}(P)$ in such a way that $\mathcal{B}$ coincides with $\mathcal{H}_\Delta$. Letting $w_1, \dots, w_n$ be the vertices of Δ and $h_1, \dots, h_n$ their corresponding hats, the maximum value of h_i equals m_i and is attained precisely at w_i .

The quantity

$$\frac{1}{m_i} \cdot \frac{1}{d+1} \cdot \frac{1}{d! \text{den}(T)}$$

equals the rational $(d+1)$ -dimensional volume of the pyramid in $[0, 1]^{n+1}$ with basis T and height $1/m_i$. By Lemma 16.2, this pyramid is regular. Therefore, the quantity

$$\sum_{T \in \Theta_i} \frac{1}{m_i (d+1) d! \text{den}(T)}$$

is the rational $(d+1)$ -dimensional volume of the portion of space in $\mathbb{R}^{n+1}$ between the graph of the Schauder hat h_i and the set

$$\bigcup \{T \mid T \in \Theta_i\}$$

of all d -simplexes T of Δ such that w_i is a vertex of T . The quantity

$$\sum_{T \in \Theta} (d! \text{den}(T))^{-1}$$

is the rational d -dimensional volume of the d -dimensional part $P^{(d)}$ of P , and we can write

$$\sum_{T \in \Theta} (d! \text{den}(T))^{-1} = \lambda^{(d)}(P^{(d)}) = \lambda^{(d)}(P).$$

Then

$$s^*(h_i) = s^*(b_i) = \left(\sum_{T \in \Theta_i} \frac{1}{m_i (d+1) d! \text{den}(T)} \right) / \left(\sum_{T \in \Theta} \frac{1}{d! \text{den}(T)} \right). \quad (16.5)$$

Given $f \in \mathcal{M}(P)$ let us now suppose Δ to be a regular f -triangulation of P (in the usual sense that f is linear on each simplex of Δ). For suitable integers $0 \leq l_i \leq m_i, i = 1, \dots, n$, we have the identity

$$f = \sum_{i=1}^n l_i h_i, \quad (16.6)$$

where the sum is taken in the associated unital ℓ -group $(G, 1)$ of A given by $\Gamma(G, 1) = A$. We then set

$$s^*(f) = \sum_{i=1}^n l_i s^*(h_i).$$

If f happens to coincide with some b_i , the two formulas (16.6) and (16.4) agree.

Claim The value of $s^*(f)$ does not depend on regular f -triangulation of P used to compute (16.6).

As a matter of fact, suppose we compute (16.6) with the help of another regular f -triangulation $\Delta' = \{h'_1, \dots, h'_{n'}\}$ of P , with vertices $w'_1, \dots, w'_{n'}$ and denominators $m'_1, \dots, m'_{n'}$. For suitable integers $l'_1, \dots, l'_{n'}$ we can write

$$f = \sum_{j=1}^{n'} l'_j h'_j.$$

Recalling (16.6) we can write $s^*(f) = \sum_{j=1}^{n'} l'_j s^*(h'_j)$ and

$$s^*(f) = \left(\sum_j l'_j \sum_{R \in \Theta'_j} \frac{1}{m'_j(d+1)d! \text{den}(R)} \right) / \left(\sum_{R \in \Theta'} \frac{1}{d! \text{den}(R)} \right), \quad (16.7)$$

where Θ'_j and Θ' are defined by (16.2–16.3). Now the numerator of (16.7) is the $(d+1)$ -dimensional rational volume of the portion of space in $\mathbb{R}^{n+1}$ between the graph of f and $P^{(d)}$. By Theorem 14.1, this volume does not depend on which f -triangulation Δ or Δ' is used to compute it. The denominator of (16.7) is the rational volume of $P^{(d)}$.

We must now check that $s^*(1) = 1$. Any regular triangulation of P will do, so let us take Δ' . Again, the denominator of (16.7) coincides with the rational measure $\lambda^{(d)}(P)$. As for the numerator, since the sum $\sum_{j=1}^{n'} m'_j h'_j$ taken in G constantly evaluates to 1 on $P^{(d)}$ we can write

$$\sum_j m'_j \sum_{R \in \Theta'_j} \frac{1}{m'_j(d+1)d! \text{den}(R)} = (d+1) \cdot \lambda^{(d)}(P)/(d+1) = \lambda^{(d)}(P),$$

because

$$\sum_j \sum_{R \in \Theta'_j} \frac{1}{d! \operatorname{den}(R)} = (d+1) \sum_{R \in T'} \frac{1}{d! \operatorname{den}(R)} = \lambda^{(d)}(P).$$

Now we check additivity:

$$f \odot g = 0 \Rightarrow s^*(f \oplus g) = s^*(f) + s^*(g).$$

Note that $f \oplus g = f + g$. The proof follows the same pattern as the proof of Proposition 15.2, using some regular $\{f, g\}$ -triangulation of P . One may observe that the rational d -dimensional volume of the region between the graph of $f + g$ and P equals the sum of the rational d -dimensional volumes of the region between P and the graphs of f and of g .

The uniqueness of s^* follows from its additivity, because every $f \in A$ can be represented as in (16.6). $\square$

In combination with Proposition 14.2, the proof above also shows:

Corollary 16.4 *In the particular case when $A = \mathcal{M}(P)$ with $P \subseteq [0, 1]^d$ and $1 \leq d = \dim(P)$, it follows that $d = \dim(\mu(A))$ and for all $f \in \mathcal{M}(P)$,*

$$s^*(f) = \frac{\int_P f(x) dx}{\int_P dx} = \text{mean value of } f \text{ on } P.$$

Since in this case, $\int_P f(x) dx$ coincides with the Lebesgue integral of f on P , s^* is called the *Lebesgue state* of A .

Further motivation for this terminology will be given in the next two sections.

16.2 Invariance and Faithfulness of the Lebesgue State

Corollary 16.5 *For every finitely presented MV-algebra A , automorphism α of A , and element $a \in A$, we have the identity $s^*(\alpha(a)) = s^*(a)$.*

Proof Particular case of Theorem 15.1(IV). $\square$

As the reader will recall, for each $n = 1, 2, \dots$, a *topological n -manifold with boundary* is a Hausdorff space in which every point has a neighborhood homeomorphic either to $\mathbb{R}^n$ or to $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$.

Corollary 16.6 *Let A be a finitely presented MV-algebra, and d the dimension of $\mu(A)$.*

- (i) *If A is finite then $d = 0$ and the Lebesgue state s^* of A is faithful.*
- (ii) *If A is infinite then s^* is faithful iff $\mu(A)$ is a topological d -manifold with boundary.*

Proof

(i) Let us write $A = \mathcal{M}(Q)$ for some rational polyhedron $Q \subseteq [0, 1]^n$. By (A21.30), A is a finite product of finite chains, $d = 0$, and Q is the union of finitely many rational points $x_1, \dots, x_k$ in $[0, 1]^n$. For each $0 \neq f \in A$ some hat h_i of the only possible (necessarily regular) triangulation Δ of Q will satisfy $h_i \leq f$. This shows that s^* is faithful.

(ii) For some integer $n \geq d$ we can safely identify A with $\mathcal{M}(P)$, where $P \subseteq [0, 1]^n$ is a d -dimensional rational polyhedron. By Lemma 1.13 and Theorem 4.16(iv), P is homeomorphic to $\mu(A)$. Since A is infinite and finitely presented, $d \geq 1$.

($\Leftarrow$) Since by assumption, P is a topological d -manifold with boundary, $P = P^{(d)}$. As an application of Corollary 2.9, for every $0 \neq f \in \mathcal{M}(P)$ let Δ be a regular f -triangulation of P . Let $v_1, \dots, v_k$ be the vertices of Δ . Let $\mathcal{H}_\Delta$ be the Schauder basis of Δ , with its Schauder hats $h_1, \dots, h_k$. For suitable integers $0 \leq m_1, \dots, m_k$ we can write

$$f = \sum_{i=1}^k m_i h_i = \bigoplus_{i=1}^k m_i h_i.$$

Evidently, not all m_i are zero. Thus some hat h_j is dominated by f . Recalling now that $P = P^{(d)}$, by (16.4), we conclude $s^*(f) > 0$.

($\Rightarrow$) The assumption that P is not a topological n -manifold with boundary is to the effect that for some $0 \leq d' < d$ the rational polyhedron $P^{(d')}$ is nonempty. Let ∇ be a regular triangulation of P , and let T be a d' -dimensional simplex of ∇ . Let v be the Farey mediant of the vertices of T , and ∇' the Farey blow-up of ∇ at v . The Schauder hat h_v of $\mathcal{H}_{\nabla'}$ constantly vanishes outside T . By definition of the Lebesgue measure, $s^*(h_v) = 0$, whence the Lebesgue state s^* is not faithful. $\square$

16.3 The Cauchy Completion of FREE_n for the Lebesgue Metric

As already noted at the beginning of Sect. 10.2, if s is a state of an MV-algebra A , and A happens to contain an infinitesimal element a then the additivity property of s ensures that $s(a) = 0$. Recalling (A21.29) we conclude that whenever A has a faithful state then A is semisimple.

Every faithful state s of A induces a *metric* $\delta_s: A \times A \rightarrow \mathbb{R}_{\geq 0}$ on A by the stipulation

$$\delta_s(a, b) = s(\mathbf{d}(a, b)) \quad \text{for all } a, b \in A, \quad (16.8)$$

where, as the reader will recall, $\mathbf{d}$ is Chang distance.

We are naturally interested in the (Cauchy) metric completion of A . Let l^* denote the Lebesgue state of $\text{FREE}_n = \mathcal{M}([0, 1]^n)$. Let λ be the Borel probability measure

on $\mu(\text{FREE}_n) = [0, 1]^n$ associated to l^* by the Kroupa–Panti theorem 10.5. Then λ is regular, and by Corollary 16.4, λ coincides with Lebesgue measure $\mathcal{L}^{(n)}$ on the Borel sets of $[0, 1]^n$. Let δ^* be the associated metric of l^* , as given by (16.8). The dependence on $n = 1, 2, \dots$ is tacitly understood.

Theorem 16.7 *The completion of FREE_n with respect to the metric δ^* is isomorphic to the MV-algebra of $[0, 1]$ -valued Lebesgue integrable functions on $[0, 1]^n$, provided we identify any two that are equal λ -almost everywhere.*

Proof In view of (B21.61–B21.62) for any $f \in C([0, 1]^n)$ and $\epsilon > 0$ we must only exhibit a McNaughton function $h \in \mathcal{M}([0, 1]^n)$ such that

$$\epsilon > \int_{[0,1]^n} |f - h| dx = \delta^*(f, h).$$

The uniform continuity of f yields a real number $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow |f(x) - f(y)| < \epsilon/1000, \quad \text{for all } x, y \in [0, 1]^n,$$

where $\|\cdot\|$ is euclidean distance in n -space.

Let us agree to say that a rational point $x \in [0, 1]^n$ is *fine* if its denominator is $> 1000/\epsilon$. Analogously, a rational n -simplex S will be said to be *fine* if all its vertices are fine.

Starting from an arbitrary regular triangulation ∇ of $[0, 1]^n$ as given by Corollary 2.9, as the result of a finite number of Farey blow-ups we will obtain a regular triangulation Δ of $[0, 1]^n$ with the following two properties:

- (a) The diameter of every simplex of Δ is $< \delta$.
- (b) The Lebesgue measure of the union of all fine simplexes of Δ is $> 1 - \epsilon/1000$.

The existence of Δ follows from Theorem 5.3.

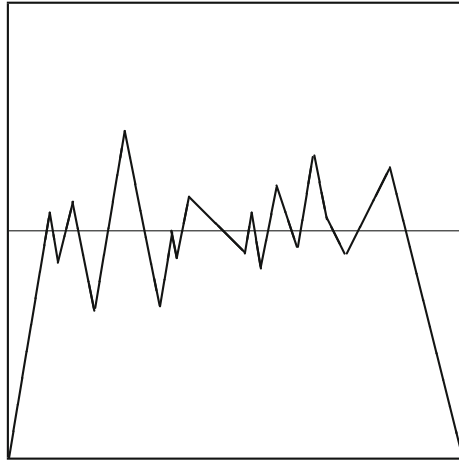
Let $v_1, \dots, v_u$ be the vertices of Δ . Let $\mathcal{H}_\Delta$ be the Schauder basis of Δ . Using the operations of the associated unital ℓ -group $(G, 1)$ of $\mathcal{M}([0, 1]^n)$ we now let $h \in \mathcal{M}([0, 1]^n)$ be defined by $h = \sum_{i=1}^u k_i v_i$ where k_i is the integer $0 \leq k \leq \text{den}(v_i)$ that minimizes the error $|f(v_i) - k/\text{den}(v_i)|$. (For definiteness, in case two distinct k' and k'' minimize the error we let $k = \min(k', k'')$). One recalls here that for every $g \in \mathcal{M}([0, 1]^n)$ the value $g(v_i)$ is an integer multiple of $1/\text{den}(v_i)$. It follows that $|f(v_i) - h(v_i)| < \epsilon/100$ for each fine vertex $v_i \in \Delta$.

From condition (a) together with the uniform continuity of f it follows that, if T is a fine simplex of Δ and x belongs to T , then $|f(x) - h(x)| < \epsilon/100$. Let F be the union of all fine simplexes of Δ . Now we compute the Lebesgue distance between f and h :

$$\begin{aligned}
\delta^*(f, h) &= \int_{[0,1]^n} |f(x) - h(x)| dx \\
&= \int_F |f(x) - h(x)| dx + \int_{[0,1]^n \setminus F} |f(x) - h(x)| dx \\
&\leq \frac{\epsilon}{100} + \int_{[0,1]^n \setminus F} dx \leq \frac{\epsilon}{100} + \frac{\epsilon}{1000} \leq \epsilon.
\end{aligned}$$

The proof is complete. $\square$

The picture below describes the approximation process of the proof of Theorem 16.7, in the particular case $n = 1$ and $f =$ the constant function $1/2$.



Approximating the constant function $1/2$
by a McNaughton function of $\mathcal{M}([0, 1])$

A variant of the proof of Theorem 16.7 shows that, whenever P is a rational polyhedron which is also a topological d -manifold, the Lebesgue metric completion of $\mathcal{M}(P)$ coincides with $\Gamma(L^1(P))$. As it happens, the metric completion of $\mathbf{FREE}_n$ is equipped with a much richer structure than $\mathbf{FREE}_n$: multiplication, multiplication by real scalars in $[0, 1]$, and a very special topology, making its associated unital ℓ -group a real Banach space.

16.4 Remarks

Particular cases of the Lebesgue state are considered in [1] and in [2].

From Panti's characterization [3, Theorem 2.3] of the Lebesgue integral on $\mathcal{M}([0, 1]^n)$ it follows that the Lebesgue state of $\mathbf{FREE}_m$, ($m = 2, 3, \dots$) is *ergodic*, i.e., extreme among the invariant states of $\mathbf{FREE}_m$.

For general information on metric completions induced by states see [4].

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Chapter 17

Finitely Generated Projective MV-Algebras

In classical first-order logic, a unifier of two expressions (terms or atomic formulas) $A(x_1, \dots, x_n)$ and $B(x_1, \dots, x_n)$ is a substitution $\nu: x_i \mapsto t_i$, where each t_i is a term in the variables $x_1, \dots, x_n$, and $A(t_1, \dots, t_n) = B(t_1, \dots, t_n)$. For any two unifiable expressions A and B , the celebrated unification algorithm provides an idempotent unifier ν^* (in the sense that $\nu^* \nu^* = \nu^*$) having the “maximum generality” property with respect to any other unifier of A and B . In Łukasiewicz logic $\mathbb{L}_\infty$, by an idempotent unifier of two formulas $\phi(X_1, \dots, X_n)$ and $\psi(X_1, \dots, X_n)$ we mean an n -tuple of formulas $\sigma = (\sigma_1, \dots, \sigma_n)$ of FORM_n such that $\phi\sigma \equiv \psi\sigma$ and $\sigma\sigma = \sigma$, i.e., $\sigma_i\sigma = \sigma_i$ for all $i = 1, \dots, n$. If we let ϕ and ψ range over formulas having an idempotent unifier, the Lindenbaum algebra $\text{LIND}_{\phi \leftrightarrow \psi}$ yields, up to isomorphism, every finitely generated projective MV-algebra A . It follows that A is finitely presented, whence $A \cong \mathcal{M}(P)$ for some rational polyhedron $P \subseteq [0, 1]^n$.

This naturally leads to the problem of characterizing those rational polyhedra $P \subseteq [0, 1]^n$ such that the MV-algebra $\mathcal{M}(P)$ is projective. Using elementary algebraic topology, we will see that P satisfies three *necessary* conditions: (i) P is contractible—equivalently, P is the retract of $[0, 1]^n$, (ii) P contains a vertex of $[0, 1]^n$ and (iii) P has a regular triangulation Δ such that for every maximal simplex $T \in \Delta$, the greatest common divisor of the denominators of the vertices of T is 1. In case P is one-dimensional, these three conditions will be shown to be *sufficient* for $\mathcal{M}(P)$ to be projective. Thus, in contrast to ℓ -groups (for which finitely presented = finitely generated projective), finitely generated projective MV-algebras are only a small subclass of finitely presented MV-algebras. All the elementary prerequisites from algebraic topology needed in this chapter are collected in Appendix B.

17.1 Finitely Generated Projective MV-Algebras and $\mathbb{Z}$ -Retractions

Definition 17.1 An MV-algebra F is said to be *projective* if whenever $\psi: A \rightarrow B$ is a surjective homomorphism and $\phi: F \rightarrow B$ is a homomorphism, there is a homomorphism $\theta: F \rightarrow A$ such that $\phi = \psi\theta$.

Example 17.2 For every cardinal $\kappa > 0$, the free MV-algebra $\mathcal{M}([0, 1]^\kappa)$ is projective: given $\psi: A \rightarrow B$ and $\phi: \mathcal{M}([0, 1]^\kappa) \rightarrow B$ as in the above definition, the map ϕ factors as $\psi\theta$, where θ is the only homomorphism of $\mathcal{M}([0, 1]^\kappa)$ into A extending the map $\pi_\alpha \mapsto a_\alpha$, with a_α an arbitrarily chosen element of $\psi^{-1}(\phi(\pi_\alpha))$. Here, for each ordinal $\alpha < \kappa$, π_α is the α th projection function.

Proposition 17.3 *Free products of projective MV-algebras are projective.*

Proof Suppose we are given two MV-algebras A_1, A_2 and one–one homomorphisms $\eta_i: A_i \rightarrow A$ ($i = 1, 2$), where $A = A_1 \sqcup A_2$. Let B, C be MV-algebras with $\sigma: B \rightarrow C$ a surjective homomorphism and $\mu: A \rightarrow C$ a homomorphism. Since A_i is projective there are homomorphisms $\rho_i: A_i \rightarrow B$ such that $\mu\eta_i = \sigma\rho_i$ for each $i = 1, 2$. The universal property of free products yields a homomorphism $\psi: A \rightarrow B$ such that $\rho_i = \psi\eta_i$. As a consequence, $\mu\eta_i = \sigma\rho_i = \sigma\psi\eta_i$, ($i = 1, 2$). Thus μ coincides with $\sigma\psi$ on the generating set $\eta_1(A_1) \cup \eta_2(A_2)$ of A . Therefore, $\mu = \sigma\psi$. $\square$

Let X be a topological space and $Y \subseteq X$. Then a continuous map r from X onto Y is said to be a *retraction* of X onto Y if the restriction of r to Y is the identity map on Y . Equivalently, r is *idempotent*, $rr = r$. If there exists a retraction of X onto Y we say that Y is a *retract* of X .

Definition 17.4 For each $n = 1, 2, \dots$, by a $\mathbb{Z}$ -*retraction* of a rational polyhedron $P \subseteq [0, 1]^n$ onto $R \subseteq P$ we understand a $\mathbb{Z}$ -map $\mu: P \rightarrow R$ which is also a retraction of P onto R . If there exists a $\mathbb{Z}$ -retraction of P onto R we say that R is a $\mathbb{Z}$ -*retract* of P . By Lemma 3.4, R is a rational polyhedron.

If $U \supseteq V \supseteq W$ are rational polyhedra in $[0, 1]^n$, μ is a $\mathbb{Z}$ -retraction of U onto V , and ν is a $\mathbb{Z}$ -retraction of V onto W , then the composite map $\nu\mu$ is a $\mathbb{Z}$ -retraction of U onto W .

In (B21.74) several equivalent conditions are given for a rational polyhedron $P \subseteq [0, 1]^n$ to be a retract of $[0, 1]^n$. Among them, a notable property is *contractibility*, meaning that the identity map on P is nullhomotopic. Contractibility already had a crucial role in the proof of Theorem 6.3.

The relationship between $\mathbb{Z}$ -retracts of cubes $[0, 1]^n$ and finitely generated projective MV-algebras is given by the following:

Proposition 17.5

- (i) *Suppose F is an n -generated projective MV-algebra, $n = 1, 2, \dots$. Then there is a $\mathbb{Z}$ -retract Q of $[0, 1]^n$ such that F is isomorphic to $\mathcal{M}(Q)$. Thus in particular, F is finitely presented.*
- (ii) *Conversely, if R is a $\mathbb{Z}$ -retract of $[0, 1]^n$ and the MV-algebra D is isomorphic to $\mathcal{M}(R)$, then D is n -generated projective.*

Proof (i) Let $\{a_1, \dots, a_n\}$ be a generating set of F . The coordinate functions $\pi_1, \dots, \pi_n \in \mathcal{M}([0, 1]^n)$ form a free generating set of the free MV-algebra $\mathcal{M}([0, 1]^n)$. Let η be the unique homomorphism of $\mathcal{M}([0, 1]^n)$ onto F extending the map $\pi_i \mapsto a_i$. Upon setting in Definition 17.1, $B = F$, $\phi = \text{identity}_F$ and

$\psi = \eta$, we get an isomorphism of F onto a subalgebra A of $\mathcal{M}([0, 1]^n)$. Identifying F with this subalgebra, it follows that F is the range of the endomorphism $\sigma = \theta\eta$ of $\mathcal{M}([0, 1]^n)$. Since $\eta\theta = \text{identity}_F$ then $\theta\eta\theta\eta = \theta\eta$, and σ is idempotent. By Lemma 3.4 and Theorem 6.3, F is finitely presented. For each $i = 1, \dots, n$ let $g_i = \sigma(\pi_i) = \theta(a_i) \in \mathcal{M}([0, 1]^n)$ be the image of π_i under the endomorphism σ . Let $g: [0, 1]^n \rightarrow [0, 1]^n$ be defined by $g(x) = (g_1(x), \dots, g_n(x))$ for all $x \in [0, 1]^n$. For any arbitrary McNaughton function $h = \hat{\psi}$ in $\mathcal{M}([0, 1]^n)$, arguing by induction on the number of connectives in ψ , it follows that $\sigma(h) = hg$. Since σ is idempotent, $hgg = hg$. It follows that the range of g coincides with the set of points $y \in [0, 1]^n$ such that $g(y) = y$, and g is idempotent. By definition, the range Q of g is a $\mathbb{Z}$ -retract of $[0, 1]^n$. By Theorem 3.23, $F \cong \mathcal{M}(Q)$.

(ii) Without loss of generality, $D = \mathcal{M}(R)$. Trivially, the set $\{\pi_1 \upharpoonright R, \dots, \pi_n \upharpoonright R\}$ generates $\mathcal{M}(R)$. Let $\gamma: [0, 1]^n \rightarrow R$ be a $\mathbb{Z}$ -retraction onto R . The map $\iota: f \in \mathcal{M}(R) \mapsto f\gamma \in \mathcal{M}([0, 1]^n)$ is an embedding of $\mathcal{M}(R)$ into $\mathcal{M}([0, 1]^n)$. Let $\rho: g \in \mathcal{M}([0, 1]^n) \mapsto g \upharpoonright R \in \mathcal{M}(R)$ be the restriction homomorphism. Then the composition $\rho\iota$ is the identity function of D . To show that D is projective, let $\psi: A \rightarrow B$ and $\phi: D \rightarrow B$ be as in Definition 17.1, with D in place of F . Since $\mathcal{M}([0, 1]^n)$ is projective, Example 17.2 shows that the homomorphism $\phi\rho: \mathcal{M}([0, 1]^n) \rightarrow B$ factors through ψ , in the sense that $\phi\rho = \psi\nu$ for some homomorphism $\nu: \mathcal{M}([0, 1]^n) \rightarrow A$. The desired factorization of ϕ is given by $\phi = \phi\rho\iota = \psi\nu\iota$. $\square$

Lemma 17.6 ($\mathbb{Z}$ -retracts of unit cubes are invariant under $\mathbb{Z}$ -homeomorphisms.)
Let $\rho: [0, 1]^n \rightarrow Q$ be a $\mathbb{Z}$ -retraction onto Q , and $\zeta: Q \rightarrow R \subseteq [0, 1]^m$ a $\mathbb{Z}$ -homeomorphism of Q onto R . Then R is a $\mathbb{Z}$ -retract of $[0, 1]^m$.

Proof By Corollary 2.9, there is a regular triangulation ∇ of $[0, 1]^m$ such that the set $\nabla_R = \{T \in \nabla \mid T \subseteq R\}$ is a ζ^{-1} -triangulation of R , meaning that ζ^{-1} is linear on each simplex of ∇_R . Let O denote the origin of $\mathbb{R}^n$. By Lemma 3.7(iii) there exists a uniquely determined $\mathbb{Z}$ -map $\eta: [0, 1]^m \rightarrow [0, 1]^n$ such that, for each vertex v of ∇ ,

$$\eta(w) = \begin{cases} \zeta^{-1}(w) & \text{if } w \in R \\ O & \text{if } w \notin R. \end{cases}$$

Since $\eta \upharpoonright R = \zeta^{-1}$, then $\eta(R) = Q$. Since $\rho \upharpoonright Q$ is the identity on Q , then the composite map $\zeta\rho\eta \upharpoonright R$ coincides with the identity on R , and $\zeta\rho\eta([0, 1]^m) = \zeta(Q) = R$. In conclusion, the map $\zeta\rho\eta: [0, 1]^m \rightarrow R$ is a $\mathbb{Z}$ -retraction of $[0, 1]^m$ onto R . $\square$

As usual, integers $a_1, \dots, a_n$ are said to be *relatively prime* (or, *coprime*) if their greatest common divisor $\gcd(a_1, \dots, a_n)$ is 1.

Lemma 17.7 *For $i = 1, 2$ let e_i denote the i th standard basis vector of the cartesian plane $\mathbb{R}^2$. Let O denote the origin in $\mathbb{R}^2$. For some $0 < m_i \in \mathbb{Z}$ with $\gcd(m_1, m_2) = 1$ let $v_i = e_i/m_i$. Then there is a $\mathbb{Z}$ -retraction ρ of the triangle $\text{conv}(O, v_1, v_2)$ onto the broken line $\text{conv}(v_1, v_2) \cup \text{conv}(O, v_1)$.*

Proof Observe that the rational 2-simplex $\text{conv}(O, v_1, v_2)$ is regular. Define inductively $c_0 = O$ and c_{t+1} to be the Farey median of the regular 1-simplex $\text{conv}(c_t, v_2)$. The denominator of c_t is $1 + tm_2$. Since m_1 and m_2 are relatively prime, then for some $q > 1$ the denominator of c_q is a multiple of m_1 . Let the regular triangulation Δ_0 be given by the 2-simplex $\text{conv}(O, v_1, v_2)$ together with its sides and vertices. For each $t = 0, 1, \dots, q-1$, let us inductively define $\Delta_t \mapsto \Delta_{t+1}$ to be the Farey blow-up of Δ_t at c_{t+1} . Then by Lemma 3.7, the map $c_q \mapsto v_1$ uniquely extends to a linear $\mathbb{Z}$ -retraction ρ_q of $\text{conv}(v_1, c_q, v_2)$ onto $\text{conv}(v_1, v_2)$. Also, the map $c_q \mapsto v_1, c_{q-1} \mapsto O$ uniquely extends to a linear $\mathbb{Z}$ -retraction ρ_{q-1} of $\text{conv}(v_1, c_q, c_{q-1})$ onto $\text{conv}(v_1, O)$. Further, for each $p = q-1, q-2, \dots, 1$ the map $c_p \mapsto O, c_{p-1} \mapsto O$ uniquely extends to a linear $\mathbb{Z}$ -retraction ρ_{p-1} of $\text{conv}(v_1, c_p, c_{p-1})$ onto $\text{conv}(v_1, O)$. Any two maps ρ_i and ρ_j agree on their common domain. The map $\rho = \rho_q \cup \rho_{q-1} \cup \dots \cup \rho_0$ is the desired $\mathbb{Z}$ -retraction. $\square$

17.2 Common Properties of All $\mathbb{Z}$ -Retracts of $[0, 1]^n$

Definition 17.8 Let $n = 1, 2, \dots$. Then a triangulation Δ of a rational polyhedron $Q \subseteq [0, 1]^n$ is said to be *strongly regular* if it is regular and the greatest common divisor of the denominators of the vertices of each maximal simplex of Δ is equal to 1.

Strong regularity is actually a property of Q :

Lemma 17.9 Let Δ and ∇ be regular triangulations of a rational polyhedron $Q \subseteq [0, 1]^n$. Then Δ is strongly regular iff ∇ is.

Proof The solution of the weak Oda conjecture (B21.58) shows that Δ and ∇ are connected by a finite path of Farey blow-ups and blow-downs. As a consequence, without loss of generality, we can assume Δ to be the blow-up at the Farey median c of some m -simplex $T = \text{conv}(v_0, \dots, v_m) \in \nabla$. Let

$$U = \text{conv}(v_0, \dots, v_m, w_1, \dots, w_k) \in \nabla$$

be a maximal simplex such that $T \subseteq U$. Since $\text{den}(c) = \sum_{j=0}^m \text{den}(v_j)$, the integer $1 = \gcd(\text{den}(v_0), \dots, \text{den}(v_m), \text{den}(w_1), \dots, \text{den}(w_k))$ coincides with the greatest common divisor of

$$\text{den}(v_0), \dots, \text{den}(v_{i-1}), \text{den}(c), \text{den}(v_{i+1}), \dots, \text{den}(v_m), \text{den}(w_1), \dots, \text{den}(w_k),$$

as required to complete the proof. $\square$

Theorem 17.10 Suppose the polyhedron Q is a $\mathbb{Z}$ -retract of $[0, 1]^n$. Then

- (i) Q is a retract of $[0, 1]^n$.
- (ii) Q contains a vertex of $[0, 1]^n$.
- (iii) Q has a strongly regular triangulation.

Proof Trivially (i) holds. Since every $\mathbb{Z}$ -retraction ρ of $[0, 1]^n$ onto Q is piecewise linear with integer coefficients, then automatically ρ sends every vertex of $[0, 1]^n$ into some vertex of $[0, 1]^n$. We have proved that (ii) holds.

Finally, to prove (iii), using Corollary 2.9 let ∇ be a regular triangulation of Q . Let $T = \text{conv}(w_0, \dots, w_r)$ be a maximal r -simplex of ∇ , and

$$d = \gcd(\text{den}(w_0), \dots, \text{den}(w_r)).$$

The Farey median w^* of T lies in the relative interior of T . The assumed maximality of T yields an open set $U \subseteq [0, 1]^n$ such that $w^* \in U$ and $U \cap Q \subseteq T$. Let $\zeta: [0, 1]^n \rightarrow Q$ be a $\mathbb{Z}$ -retraction onto Q . Then $\zeta^{-1}(U)$ is an open set. Let x be a rational point in $\zeta^{-1}(U)$ with a prime denominator $\text{den}(x) > d$. By construction, the rational point $\zeta(x)$ belongs to the regular simplex T . By Proposition 5.2 there is a sequence of regular complexes

$$\Delta_0, \Delta_1, \dots, \Delta_t,$$

such that Δ_0 is the regular complex given by T and its faces, $\zeta(x)$ is a vertex of Δ_t but not of Δ_{t-1} , and each Δ_{i+1} is the blow-up of Δ_i at the Farey median a_i of some 1-simplex of Δ_i . One inductively verifies that d is a divisor of the denominator of each a_i . In particular, d is a divisor of $\text{den}(\zeta(x))$. Since $\text{den}(x)$ is prime and all linear pieces of ζ have integer coefficients, then necessarily $\text{den}(\zeta(x)) \in \{1, \text{den}(x)\}$. Since $\text{den}(x) > d$, then $d = 1$. We have proved that every regular triangulation ∇ of Q is strongly regular. $\square$

Remark 17.11 By Corollary 2.9 and Lemma 17.9, condition (iii) above can be replaced by

(iii') Every regular triangulation of Q is strongly regular.

Corollary 17.12 *If the MV-algebra A is finitely generated projective then*

- (j) *The maximal spectral space $\mu(A)$ is contractible.*
- (jj) *For some maximal ideal $\mathfrak{m} \in \mu(A)$, $A/\mathfrak{m} = \{0, 1\}$.*
- (jjj) *For some basis $\mathcal{B}$ of A , each maximal cluster of $\mathcal{B}$ has relatively prime multipliers.*

Proof By Proposition 17.5 we can write $A = \mathcal{M}(P)$ for some $\mathbb{Z}$ -retract P of $[0, 1]^n$. By Theorem 4.16(iv), P is homeomorphic to the maximal spectral space $\mu(\mathcal{M}(P))$, whence by Theorem 17.10(i), $\mu(\mathcal{M}(P))$ is homeomorphic to a retract of some cube. By (B21.74), $\mu(A)$ is contractible, and condition (j) holds.

Let v be a vertex of $[0, 1]^n$ as given by Theorem 17.10(ii). With reference to Theorem 4.16, let $\mathfrak{h}_v \in \mu(A)$ be its corresponding maximal ideal. It follows that $A/\mathfrak{h}_v = \{0, 1\}$, and condition (jj) holds.

Theorem 17.10(iii) yields a strongly regular triangulation Δ of P , with its Schauder basis $\mathcal{H}_\Delta$. Every maximal simplex $S = \text{conv}(w_0, \dots, w_t)$ of Δ determines the maximal cluster $C_S = \{h_{w_0}, \dots, h_{w_t}\}$ of $\mathcal{H}_\Delta$. The denominator of each vertex w_i coincides with the multiplier of $\mathcal{H}_\Delta$ corresponding to the hat h_{w_i} . Conversely,

every maximal cluster C of $\mathcal{H}_\Delta$ has the form C_T for a unique maximal simplex T of Δ . The strong regularity of Δ is to the effect that $\mathcal{H}_\Delta$ satisfies condition (jjj). $\square$

The proof above shows that condition (jjj) can be replaced by

(jjj') *Each maximal cluster of every basis $\mathcal{B}$ of A has relatively prime multipliers.*

17.3 Star-Like Polyhedra and Their Algebraic Counterparts

Definition 17.13 A set $X \subseteq [0, 1]^n$ is said to be *star-like at a point* $q \in X$ if for every $y \in X$, the segment $\text{conv}(q, y)$ is contained in X .

Theorem 17.14 *Suppose the rational polyhedron $L \subseteq [0, 1]^n$ is star-like at a point $q = (q_1, \dots, q_n) \in \{0, 1\}^n$. Then L is a $\mathbb{Z}$ -retract of $[0, 1]^n$, and $\mathcal{M}(L)$ is a projective MV-algebra.*

Proof Corollary 2.9 yields a regular triangulation Δ of $[0, 1]^n$ such that $\Delta_L = \{S \in \Delta \mid S \subseteq L\}$ is a triangulation of L . By (B21.59) it is no loss of generality to assume that Δ_L is a *full* subcomplex of Δ , in the sense that every simplex $S \in \Delta$ with all vertices lying in Δ_L , is already a simplex of Δ_L . For each simplex T of Δ let $\rho_T: T \rightarrow [0, 1]^n$ be the unique linear extension of the map

$$\rho_T(v) = \begin{cases} v & \text{if } v \text{ is a vertex of } T \text{ lying in } L \\ q & \text{if } v \text{ is a vertex of } T \text{ not lying in } L. \end{cases}$$

In view of Lemma 3.7, ρ_T is expressible by a linear polynomial with integer coefficients. Further, for any two simplexes T and U of Δ , the maps ρ_T and ρ_U agree (with $\rho_{T \cap U}$) on their common domain $T \cap U$. It follows that $\rho = \bigcup \{\rho_S \mid S \in \Delta\}$ is a continuous map of $[0, 1]^n$ into $[0, 1]^n$. By construction, every $S \in \Delta_L$ satisfies $\rho_S(S) = S$, whence $\rho \upharpoonright L$ is the identity function on L and $L \subseteq \rho([0, 1]^n)$. For the converse inclusion, let $v \in [0, 1]^n$ with $v \notin L$ and let $S \in \Delta$ be such that $v \in S$. Since $S \not\subseteq L$ and Δ_L is full in Δ , the set $\text{ext}(S)$ of vertices of S is not a subset of L . Let $\{v_1, \dots, v_m\} = \text{ext}(S) \setminus L$ and $\{w_1, \dots, w_k\} = \text{ext}(S) \cap L$. There are reals $\mu_1, \dots, \mu_m, v_1, \dots, v_k \geq 0$ such that

$$v = \sum_{i=1}^m \mu_i v_i + \sum_{j=1}^k v_j w_j \quad \text{and} \quad \sum_{i=1}^m \mu_i + \sum_{j=1}^k v_j = 1.$$

Since ρ_S is linear over S ,

$$\rho(v) = \rho_S(v) = \sum_{i=1}^m \mu_i \rho_S(v_i) + \sum_{j=1}^k v_j \rho_S(w_j) = \sum_{i=1}^m \mu_i q + \sum_{j=1}^k v_j w_j.$$

Let $\mu = \sum_{i=1}^m \mu_i$ and $v = \sum_{j=1}^k v_j$. Let $S' = \text{conv}(\text{ext}(S) \cap L) \in \Delta$. Since Δ_L is full in Δ and $\text{ext}(S') = \text{ext}(S) \cap L$, then S' belongs to Δ_L and $S' \subseteq L$. In case

$v \neq 0$, let

$$\bar{v} = \sum_{j=1}^k (v_j/v) w_j.$$

Since $\bar{v} \in \text{conv}(\text{ext}(S) \cap L) = S'$, then $\bar{v} \in L$. It follows that $\rho(v) = \mu q + v\bar{v}$, whence $\rho(v) \in \text{conv}(q, \bar{v})$. Since L is star-like at q , $\rho(v)$ belongs to L . In case $v = 0$ then $\rho(v) = q \in L$. Thus $\rho([0, 1]^n) \subseteq L$, whence $\rho: [0, 1]^n \rightarrow L$. As a consequence, $\rho([0, 1]^n) = L$, $\rho\rho = \rho$ and ρ is a $\mathbb{Z}$ -retraction of $[0, 1]^n$ onto L . The rest follows from Proposition 17.5. $\square$

Corollary 17.15 *Suppose A has a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ with multipliers $m_1, \dots, m_n$ such that all maximal clusters of $\mathcal{B}$ intersect in a common element, say b_1 , with $m_1 = 1$. Then A is projective.*

Proof By Corollary 6.4, we can identify A with $\mathcal{M}([0, 1]^n) \upharpoonright P$ for some rational polyhedron equipped with a regular triangulation Δ , in such a way that $\mathcal{B} = \mathcal{H}_\Delta$ is the Schauder basis of Δ . By hypothesis, all hats have a common vertex v which, necessarily, is a vertex of the cube, $v \in \{0, 1\}^n$. Further, all maximal simplexes of Δ have v as a common vertex. Thus P is star-like at v . From Theorem 17.14 it follows that P is a $\mathbb{Z}$ -retract of the cube and A is projective. $\square$

17.4 The Case of MV-Algebras with One-Dimensional Maximal Spectrum

For one-dimensional rational polyhedra P , the three conditions (i–iii) in Theorem 17.10 turn out to be sufficient for P to be the $\mathbb{Z}$ -retract of some cube:

Theorem 17.16 *For $n = 1, 2, \dots$, let $Q \subseteq [0, 1]^n$ be a one-dimensional polyhedron. Suppose*

- (i) Q is a retract of $[0, 1]^n$.
- (ii) Q contains a vertex v of $[0, 1]^n$.
- (iii) Q has a strongly regular triangulation ∇ .

Then Q is a $\mathbb{Z}$ -retract of $[0, 1]^n$. Further, condition (i) is equivalent to Q being connected and simply connected, i.e., Q is a tree.

Proof Let ∇ be a strongly regular triangulation of Q with vertices $v_1, \dots, v_k$. Let us write $\text{den}(v_i) = m_i$. Without loss of generality $\text{den}(v_k) = 1$. By Lemma 3.7(ii), the map $v_i \in [0, 1]^n \mapsto e_i/m_i \in [0, 1]^k$ uniquely extends to a $\mathbb{Z}$ -homeomorphism ι of Q onto a one-dimensional rational polyhedron $P \subseteq [0, 1]^k$, with ι linear over every simplex of ∇ . By Proposition 3.2, the image $\Delta = \iota(\nabla)$ is a strongly regular triangulation of P . Since the property of being a $\mathbb{Z}$ -retract of the ambient unit cube is preserved under $\mathbb{Z}$ -homeomorphisms, it is no loss of generality to argue for P and

Δ instead of Q and ∇ . Let O be the origin in $\mathbb{R}^k$. Let us say that the 1-simplexes of Δ are its *edges*, and its 0-simplexes are its *nodes*. For each edge $E \in \Delta$ let OE denote the triangle $\text{conv}(O, E)$. We define the two-dimensional rational polyhedron OP by

$$OP = \bigcup \{\text{conv}(OE) \mid E \in \Delta\}.$$

Then the set of all simplexes of Δ , together with all triangles OE and their faces determines a regular triangulation $O\Delta$ of OP .

Since OP is star-like at the origin O , by Theorem 17.14 there is a $\mathbb{Z}$ -retraction γ_0 of $[0, 1]^k$ onto OP .

We will now construct a $\mathbb{Z}$ -retraction of OP onto P .

To this purpose we inductively define two lists of vertices $w_1, \dots, w_{k-1}$ and $u_1, \dots, u_{k-1}$, together with a list of edges $E_1, \dots, E_{k-1}$ and simplicial complexes $\Delta_0 = \Delta, \Delta_1, \dots, \Delta_{k-1}$, according to the following stipulations:

- w_1 is a *terminal* node of Δ_0 , i.e., a node of exactly one edge of Δ_0 , which can be assumed to be distinct from v_k ;
- E_1 the only edge of Δ_0 having w_1 as a vertex;
- u_1 the other vertex of E_1 ;
- $\Delta_1 = \Delta_0 \setminus \{E_1, w_1\}$,

and inductively, for $t = 2, 3, \dots, k-1$,

- w_t is a terminal node of Δ_{t-1} other than v_k ;
- E_t the only edge of Δ_{t-1} having w_t as a vertex;
- u_t the other vertex of E_t ;
- $\Delta_t = \Delta_{t-1} \setminus \{E_t, w_t\}$.

Then necessarily, $E_{k-1} = \text{conv}(w_{k-1}, v_k) = \text{conv}(w_{k-1}, u_{k-1})$ is the only edge of Δ_{k-2} . Further, $\Delta_{k-1} = \{v_k\}$. Lemma 17.7 yields a $\mathbb{Z}$ -retraction γ_1 of $OP = |O\Delta_0| = P \cup |O\Delta_0|$ onto $P \cup |O\Delta_1|$. Inductively, for each $t = 2, \dots, k-1$, the same lemma yields a $\mathbb{Z}$ -retraction γ_t of $P \cup |O\Delta_{t-1}|$ onto $P \cup |O\Delta_t|$. For $t = k-1$ there is a $\mathbb{Z}$ -retraction γ_{k-1} of $P \cup |O\Delta_{k-2}|$ onto $P \cup |O\Delta_{k-1}| = P \cup \text{conv}(Ov_k)$. Since $\text{den}(v_k) = 1$, it is easy to exhibit a $\mathbb{Z}$ -retraction γ_k of $P \cup \text{conv}(O, v_k)$ onto P . The composition $\gamma_k \gamma_{k-1} \cdots \gamma_1 \gamma_0$ finally yields the desired $\mathbb{Z}$ -retraction of $[0, 1]^k$ onto P .

The final statement is proved in (B21.75). □

The following is an algebraic counterpart of Theorem 17.16:

Corollary 17.17 *Let A be an MV-algebra with one-dimensional maximal spectral space $\mu(A)$. Then A is finitely generated projective iff the following conditions are satisfied:*

- (i) $\mu(A)$ is connected and simply connected.
- (ii) A has a basis $\mathcal{B}$ such that at least one of the multipliers of $\mathcal{B}$ is equal to 1.

- (iii) For every 2-cluster $\{b', b''\} \in \mathcal{B}^{\bowtie}$ the multipliers m' and m'' of b' and b'' are relatively prime.

Proof ($\Rightarrow$) By Proposition 17.5, there is $n = 1, 2, \dots$ and a $\mathbb{Z}$ -retract P of $[0, 1]^n$ such that $A = \mathcal{M}(P)$. As in Corollary 2.9, let Δ be a regular triangulation of P . By (iii'), Δ is strongly regular. By Corollary 4.18, P is a one-dimensional rational polyhedron having the contractibility property. From (B21.74) and (B21.75), it follows that P is connected and simply connected. By Theorem 17.10(ii), P contains some vertex of the cube. Direct inspection shows that the Schauder basis $\mathcal{H}_\Delta$ satisfies all the three conditions: specifically, condition (i) follows from Corollary 6.4 because P is a tree; (ii) is satisfied because P contains some vertex of the cube; finally, (iii) follows from the strong regularity of Δ and the identity between denominators of vertices of Δ and multipliers of their corresponding hats of $\mathcal{H}_\Delta$ (Corollary 6.4).

($\Leftarrow$) By Theorem 6.3, A is finitely presented, and we can write $A = \mathcal{M}(P)$ for some rational polyhedron P . Corollary 6.4 allows us to identify $\mathcal{B}$ with the Schauder basis of some regular triangulation ∇ of P . From our hypotheses and (B21.75) it follows that the support of ∇ is a tree having nonempty intersection with the set of vertices of the cube. Further, ∇ is strongly regular. By Theorem 17.16, P is a $\mathbb{Z}$ -retract of the cube. By Proposition 17.5, A is projective. Letting m be the number of elements of $\mathcal{B}$, it follows that A is m -generated. $\square$

17.5 Remarks

For ℓ -groups, Baker [1] and Beynon [2]–[5, 3.1] (also see [6, Corollary 5.2.2]) gave the following characterization: *An ℓ -group G is finitely generated projective iff it is finitely presented.* For unital ℓ -groups the ($\Rightarrow$)-direction holds (see [7, Proposition 5] for details). An application of the Γ functor together with Corollary 17.12 shows that the converse direction fails in general, for unital ℓ -groups as well as for MV-algebras. For instance, if the rational polyhedron $P \subseteq [0, 1]$ is disconnected, or if P has empty intersection with the set $\{0, 1\}$, then $\mathcal{M}([0, 1]) \upharpoonright P$ is finitely presented but not projective.

The transition from Δ_{t-1} to Δ_t in Theorem 17.16 is called an *elementary collapse* in [8, III, 7.2] (“elementary contraction” in [9, p. 247]; also see [10, 6.6]). A triangulation is *collapsible* iff it collapses to (the simplicial complex of) one of its vertices. For every one-dimensional polyhedron P , contractibility is equivalent to the collapsibility of some triangulation of P [11, Example 5, p. 50].

Collapsibility implies contractibility [11, p. 49], but not vice versa [11, Example 6, p. 50]. Collapsibility is not invariant under homeomorphisms, [12]. In [13] it is proved that Conditions (ii) and (iii) of Theorem 17.10, together with collapsibility, are sufficient for a polyhedron $P \subseteq [0, 1]^n$ to be a $\mathbb{Z}$ -retract of $[0, 1]^n$.

$\mathbb{Z}$ -retracts and strongly regular triangulations were introduced in [14] and [13], respectively.

Propositions 17.5 and 17.3, as well as Theorem 17.14, were first proved in [14]. Theorems 17.10 and 17.16 (in a much more general form), were first proved in [13].

A different approach to projective MV-algebras is taken by [15, 16].

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Chapter 18

Effective Procedures for $\mathbb{L}_\infty$ and MV-Algebras

Łukasiewicz propositional logic $\mathbb{L}_\infty$ comes equipped with the same parsing algorithms of boolean propositional logic, and with its own arithmetic–geometric algorithms to recognize tautologies and to decide if a formula ϕ is a consequence of ψ . Building on this basic algorithmic structure, and pulling the threads of earlier chapters together, the present chapter describes a rich array of methods for constructing algorithms for finitely presented MV-algebras and their associated rational polyhedra. This is perhaps the main service offered to algebra and geometry by the formulas of $\mathbb{L}_\infty$ and their semantics. Naturally enough, the algorithmic theory developed in this chapter relies upon the algebraic–geometric theory developed in the earlier chapters, with particular reference to bases and desingularization. To keep the length of this book within reasonable bounds, we will proceed at a faster pace than in all previous chapters, assuming the reader to be familiar with effective (= Turing) computability. The first chapters of the classical textbooks [1] or [2] provide all necessary background.

18.1 Preliminary Algorithms

For later use we record here several algorithms and decision procedures for $\mathbb{L}_\infty$ and rational polyhedra.

Lemma 18.1 *The following are effectively computable procedures:*

Tautology *The recognition of tautologies of $\mathbb{L}_\infty$.*

Vertices *Input: The affine hulls of the maximal proper faces of a rational simplex $T \subseteq [0, 1]^n$ (with $\dim(T) \geq 1$), each affine hull being presented as an intersection of finitely many rational hyperplanes.*

Output: The list of vertices of T .

Faces *Input: The list of vertices of a rational simplex $T \subseteq [0, 1]^n$ with $\dim(T) \geq 1$.*

Output: The list of affine hulls of all maximal proper faces of T , each affine hull being presented as an intersection of finitely many rational hyperplanes.

Regular Input: A rational simplex $T \subseteq [0, 1]^n$.

Question: Is T regular?

Pieces Input: A formula $\theta \in \text{FORM}_n$.

Output: A finite list containing all linear pieces $l_1, \dots, l_l$ of the McNaughton function $\hat{\theta}$, where each linear piece is represented by a linear polynomial with integer coefficients.

Mod Input: A formula $\theta \in \text{FORM}_n$.

Output: A regular triangulation of the rational polyhedron $\text{Mod}(\theta) \subseteq [0, 1]^n$.

Triangulation Input: A formula $\psi \in \text{FORM}_n$, together with a rational polyhedron P , presented as the union of rational simplexes $T_1, \dots, T_k$ in $[0, 1]^n$.

Output: A regular triangulation Δ of P such that $\hat{\psi}$ is linear on each simplex of Δ .

Th Input: A rational polyhedron $P = \bigcup_{i=1}^k T_i$, where each T_i is a rational simplex in $[0, 1]^n$.

Output: A formula $\psi \in \text{FORM}_n$ such that $\text{Mod}(\psi) = P$.

Part Input: A rational polyhedron $P \subseteq [0, 1]^n$ and an integer $d = 0, 1, 2, \dots$.

Output: The d -dimensional part $P^{(d)}$ of P , as a finite union of d -dimensional rational simplexes.

Satisfiable The recognition of satisfiable formulas of $\mathbb{L}_\infty$.

One-one Input: A $\mathbb{Z}$ -map $\zeta: [0, 1]^n \rightarrow [0, 1]^k$ presented as $\zeta = (\hat{\phi}_1, \dots, \hat{\phi}_k)$ for suitable formulas $\phi_1, \dots, \phi_k \in \text{FORM}_n$.

Question: Is ζ one-one?

Proof The bulk of this proof is the verification that the proofs of Theorem 2.8 and Corollaries 2.9–2.10 are constructive.

Tautology From (A21.34).

Vertices Let $F_0, \dots, F_k$ be the maximal proper faces of T . For each $j = 0, \dots, k$ the vertex of T not lying in F_j is given by the intersection of the hyperplanes listed in the presentations of all maximal proper faces of T other than F_j .

Faces Let $v_0, \dots, v_m \in \mathbb{Q}^n$ be affinely independent rational points. Then an elementary computation in linear algebra provides rational hyperplanes $H_1, \dots, H_{n-m}$ such that $\text{aff}(\{v_0, \dots, v_m\}) = H_1 \cap \dots \cap H_{n-m}$.

Regular By Lemma 2.6, non-regularity is equivalent to the existence of an integer point in the half-open parallelepiped P given by the homogeneous correspondents of the vertices of T . A trivial exhaustive search of all integer vectors of length not exceeding the diameter of P will provide the required decision problem.

Pieces One proceeds by induction on the number of connectives in the subformulas of $\theta(X_1 \dots, X_n)$. The only linear piece of the McNaughton function $\widehat{X_j}$ associated to the variable X_j is the coordinate function π_j . If S contains the linear pieces of $\hat{\rho}$ then the set $\{1 - l \mid l \in S\}$ contains the linear pieces of $\widehat{\neg\rho}$. Finally, a set containing all linear pieces of $\hat{\rho} \oplus \hat{\sigma}$ is given by the constant function 1 along with

all sums $f + g$, where f ranges over linear pieces of $\hat{\rho}$, and g ranges over linear pieces of $\hat{\sigma}$.

Mod We will verify that all steps of the proof of Corollary 2.10 (ii $\Rightarrow$ i) are effective. To this purpose, using algorithm **Pieces** we effectively list the linear pieces $l_1, \dots, l_u$ of $\hat{\theta}$ as linear polynomials with integer coefficients. For every permutation π of the index set $\{1, \dots, u\}$ we construct the rational polyhedron $P_\pi \subseteq \{x \in [0, 1]^n \mid l_{\pi(1)}(x) \leq l_{\pi(2)} \leq \dots \leq l_{\pi(u)}(x)\}$, thus obtaining a complex $\mathcal{C}$ whose elements are convex rational polyhedra, and whose support is $[0, 1]^n$. Since $\mathcal{C}$ is a rational polyhedral complex, the classical proof of (B21.53) yields an effective procedure to subdivide $\mathcal{C}$ into a triangulation $\mathcal{T}$ of $[0, 1]^n$ without adding new vertices. To obtain a further subdivision of $\mathcal{T}$ into a regular triangulation ∇ of $[0, 1]^n$ we will now construct the sequence $\mathcal{T} = \nabla_0, \nabla_1, \dots, \nabla_z = \nabla$ of rational triangulations of $[0, 1]^n$ defined in the proof of Theorem 2.8, and check that all steps are effective. By construction, ∇_{t+1} is a subdivision of ∇_t having one more vertex p than ∇_t . This new vertex is obtained by

- (i) Choosing a maximal non-regular simplex $N = \text{conv}(w_1, \dots, w_r) \in \nabla_t$.
- (ii) Writing down the homogeneous correspondents $\tilde{w}_1, \dots, \tilde{w}_r$ of the vertices of N .
- (iii) Picking an integer point q in the half-open parallelepiped

$$\{x \in \mathbb{R}^{n+1} \mid x = v_1 \tilde{w}_1 + \dots + v_r \tilde{w}_r, 0 \leq v_i < 1, \quad i = 1, \dots, r\}.$$

By Lemma 2.6, the existence of q is an equivalent reformulation of the non-regularity of N .

- (iv) Letting $p \in N$ be defined by $\tilde{p} = q$.
- (v) Letting ∇_{t+1} be the result of the stellar subdivision of ∇_t at p .

Each step (i–v) of the blow-up $\nabla_t \mapsto \nabla_{t+1}$ is effective. The sequence $\nabla_0, \nabla_1, \dots$ terminates after a certain finite number z of steps with a regular triangulation $\nabla = \nabla_z$ of $[0, 1]^n$. We present ∇ as the list of its simplexes. By construction, $\hat{\theta}$ is linear over each simplex of ∇ . Let $Q = \hat{\theta}^{-1}(1) = \text{Mod}(\theta)$. The set $\nabla_Q = \{S \in \nabla \mid S \subseteq Q\}$ is the desired regular triangulation of Q . A simplex $S \in \nabla$ belongs to ∇_Q iff $\hat{\theta}(v) = 1$ for each vertex v of S .

Triangulation The argument is a variant of the foregoing proof, and amounts to checking that all steps of the proof of Corollary 2.9 are effective.

Th First of all, using algorithm **Triangulation** we can assume without loss of generality that P is presented by a regular triangulation $\Delta = \{R_1, \dots, R_l\}$. Indeed, using Corollary 2.9 we may insist that there is a regular triangulation $\Delta' = \{T_1, \dots, T_v\}$ of $[0, 1]^n$ such that Δ is a subset of Δ' . Let $w_1, \dots, w_u$ be the vertices of Δ' , with their denominators $m_1, \dots, m_u$. Without loss of generality, the first l of these vertices belong to Δ . Following [3, 9.1.4–9.1.5], from Δ' we effectively compute formulas $\omega_1, \dots, \omega_u$ such that $\mathcal{H}_{\Delta'} = \{\hat{\omega}_1, \dots, \hat{\omega}_u\}$ is the Schauder basis of Δ' . By (A21.47), the McNaughton function $\hat{\omega} = \sum_{i=1}^l m_i \hat{\omega}_i$ takes value 1 precisely on P . Therefore, the formula

$$\psi = m_1 \cdot \omega_1 \oplus \cdots \oplus m_l \cdot \omega_l$$

provides the desired axiomatization of $\text{Th}(P)$. [Notation of (2.8)].

Part Using algorithm **Triangulation** we construct a regular triangulation ∇ of P , and we extract the maximal d -dimensional simplexes of ∇ .

Satisfiable For every input formula $\theta \in \text{FORM}_n$, a direct inspection of the proof of the co-NP-completeness of the tautology problem of L_∞ given in [3, 9.3.3] shows that the maximum value of $\hat{\theta}$ is a rational number that can be effectively computed from the list of linear pieces of $\hat{\theta}$; the latter are given by algorithm **Pieces**.

One-one Again using algorithm **Pieces**, we first list all linear pieces of all McNaughton functions $\hat{\phi}_j$, and compute the absolute value a of the largest coefficient of these linear polynomials. Since n is given, we can now effectively compute an integer $d = d(n, a)$ having the property that, whenever ζ is not injective, $\zeta(x) = \zeta(y)$ for two distinct rational points x, y of denominator $\leq d$. The injectivity of ζ will then be decided by an exhaustive search over all pairs of rational points in $[0, 1]^n$ of denominator $\leq d$.

Remark 18.2 The foregoing lemma relieves us of the task of specifying, e.g., if simplexes are to be presented by their vertices or by their faces. The notions of “effectively given” rational polyhedron, regular triangulation, and the like, are now unambiguously defined. A different matter would be if we were concerned with polynomial complexity issues. (For a trivial example, since the n -cube has $2n$ faces and 2^n vertices, the complexity of algorithms involving $[0, 1]^n$ critically depends on the way it is presented.) Likewise, we are exempted from giving dreary details on how variables, integers, rational numbers and rational points in $[0, 1]^n$ are written down as finite strings of symbols.

18.2 Deciding $\phi \vdash \psi$ and Computing the Conditional Probability $P(\phi \mid \theta)$

Theorem 18.3 *The consequence problem is defined by*

Input: Formulas $\phi, \psi \in \text{FORM}_n$.

Question: $\phi \vdash \psi$?

Then the problem is decidable.

Proof By Theorem 1.7, if $\phi \vdash \psi$ then for some m the formula $\phi^m \rightarrow \psi$ will be a tautology. Since for each $k = 1, 2, \dots$, algorithm **Tautology** checks if $\phi^k \rightarrow \psi$ is a tautology, then the consequence problem is recursively enumerable.

On the other hand, in order to check $\phi \not\vdash \psi$, arguing as in the proof of Corollary 2.9, we compute a regular triangulation Δ of $[0, 1]^n$ such that the set $\Delta_\phi = \{T \in \Delta \mid T \subseteq \text{Mod}(\phi)\}$ is a triangulation of $\text{Mod}(\phi)$ and $\Delta_\psi = \{T \in \Delta \mid T \subseteq \text{Mod}(\psi)\}$ is a triangulation of $\text{Mod}(\psi)$. By Theorem 1.7, there is a simplex

$R \in \Delta_\phi \setminus \Delta_\psi$, whence not all vertices of R belong to $\text{Mod}(\psi)$. Let $r_1, r_2, \dots$ be the list of all rational points of $[0, 1]^n$ in the order of increasing denominators. Then for some i we will have $\hat{\phi}(r_i) = 1 > \hat{\psi}(r_i)$. We have shown that the complementary problem $\phi \not\vdash \psi$ is recursively enumerable. Thus the consequence problem is decidable. $\square$

For the sake of completeness we give here a short proof of the following routine exercise in computational geometry:

Corollary 18.4 *Consider the following problem:*

Input: Two rational polyhedra $P, Q \subseteq [0, 1]^n$, presented as finite unions of rational simplexes, $P = \bigcup S_i$ and $Q = \bigcup T_j$.

Question: Does P coincide with Q ?

Then the problem is decidable.

Proof Using algorithm **Th** we first write down two formulas $\phi, \psi \in \text{FORM}_n$ such that $P = \text{Mod}(\phi)$ and $Q = \text{Mod}(\psi)$. Then the identity of P and Q is equivalent to $\phi \vdash \psi$ and $\psi \vdash \phi$. An application of Theorem 18.3 settles the decidability of the problem. $\square$

Theorem 18.5 *The map*

$$(\phi, \theta) \mapsto P(\phi \mid \theta)$$

for $\phi, \theta \in \text{FORM}_n$ with θ satisfiable, is effectively computable.

Proof Using algorithm **Satisfiable** we preliminarily check if θ is satisfiable. Next, the rationality of $P(\phi \mid \theta)$ immediately follows by the definition of **P** in Theorem 15.1. Further, algorithm **Mod** yields a presentation of the rational polyhedron $Q = \hat{\theta}^{-1}(1) = \text{Mod}(\theta)$ as the union of the simplexes of a regular triangulation Δ . From Δ we immediately get the dimension d of Q , as well as the rational volume $\lambda^{(d)}(Q)$. Algorithm **Pieces** lists linear polynomials $g_1, \dots, g_v$, with integer coefficients for the linear pieces of $\hat{\phi}$. Algorithm **Triangulation** computes a regular $\{\hat{\phi}\}$ -triangulation ∇ of $[0, 1]^n$ such that the set $\nabla_Q = \{S \in \nabla \mid S \subseteq Q\}$ is a triangulation of Q . Using ∇_Q we straightforwardly compute $P(\phi \mid \theta)$. $\square$

18.3 Recognizing $\mathbb{Z}$ -Homeomorphic Copies of the Unit Interval

Theorem 18.6 *The recognition problem of the real unit interval $[0, 1]$ is as follows:*

Input: A formula $\theta \in \text{FORM}_n$.

Question: Is the real unit interval $\mathbb{Z}$ -homeomorphic to the rational polyhedron $\text{Mod}(\theta)$?

The problem is decidable.

Proof By induction on the number of connectives in θ , using algorithm **Mod** we effectively write down the rational polyhedron $P = \text{Mod}(\theta)$ as a finite union of rational simplexes I_j and promptly decide if P is *homeomorphic* to $[0, 1]$, as a preliminary test. If the test is passed, we fix once and for all an orientation of P and denote by a and b its first and last endpoint. Using algorithm **Triangulation** we now compute a regular triangulation ∇ of P and let

$$a_0 = a < a_1 < \cdots < a_k = b$$

be the list of vertices of ∇ , written in order of increasing distance from a . We also let $d = \max(\text{den}(a_0), \dots, \text{den}(a_k))$. An exhaustive search procedure now lists all rational points in P whose denominator is $\leq d$,

$$b_0 = 0 < b_1 < \cdots < b_m = b.$$

For later reference let us call this process the *d-saturation* of ∇ . Proposition 3.16(iii) ensures that each interval $\text{conv}(b_j, b_{j+1})$ is regular. The set of vertices $b_0, \dots, b_m$ determines a regular triangulation Δ which is a subdivision of ∇ . We say that Δ is *d-saturated*, meaning that the set of vertices of Δ coincides with the set of all rational points of P whose denominator is $\leq d$.

Let now

$$c_0 = 0 < c_1 < \cdots < c_t = 1$$

display the elements of the d th Farey sequence $\mathfrak{F}_d$. As the reader will recall, e.g., from [5, p. 23], $\mathfrak{F}_d$ is the set of all rational numbers in $[0, 1]$ whose denominator is $\leq d$. Farey conjectured, and Cauchy proved, that $\mathfrak{F}_d$ is the set of vertices of the d -saturated regular triangulation of $[0, 1]$.

Claim $P \cong_{\mathbb{Z}} [0, 1]$ iff $t = m$ and the two sequences $\text{den}(b_0), \dots, \text{den}(b_m)$ and $\text{den}(c_0), \dots, \text{den}(c_m)$ coincide.

($\Leftarrow$) Then a $\mathbb{Z}$ -homeomorphism is given by Lemma 3.14(ii).

($\Rightarrow$) Suppose $\eta: P \cong_{\mathbb{Z}} [0, 1]$ and, without loss of generality, $\eta(b_0) = c_0 = 0$. For some rational $d_1 \in [0, 1]$ the restriction η_1 of η to the interval $\text{conv}(b_0, b_1)$ is $\mathbb{Z}$ -homeomorphic to the interval $[0, d_1] \subseteq [0, 1]$. By Proposition 3.16(ii), η_1 is linear. Since $\mathbb{Z}$ -homeomorphisms preserve denominators, $\text{den}(b_1) = \text{den}(d_1)$. Since Δ is d -saturated, there are no rational points of denominator $\leq d$ in $\text{conv}(b_0, b_1)$ and $[0, d_1]$. Thus d_1 must be the first nonzero element of $\mathfrak{F}_d$, $d_1 = c_1$, and $\text{den}(d_1) = \text{den}(c_1) = \text{den}(b_1)$. Proceeding inductively, the restriction η_t of η to the interval $\text{conv}(b_{t-1}, b_t)$ is a linear $\mathbb{Z}$ -homeomorphism onto $[c_{t-1}, c_t]$, and the claim is proved.

Summing up, in order to decide if P is $\mathbb{Z}$ -homeomorphic to $[0, 1]$, we must check if the sequence of denominators of a regular d -saturated triangulation Δ of P (with d effectively computable from θ) coincides with the sequence of denominators of the d th Farey sequence $\mathfrak{F}_d$. Therefore, the recognition problem for $[0, 1]$ is decidable. $\square$

A routine generalization of the above proof shows:

Proposition 18.7 *The $\mathbb{Z}$ -homeomorphism problem for one-dimensional rational polyhedra $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ is decidable.*

Proof We build regular triangulations Δ of P and ∇ of Q , and let d be the maximum of all denominators of the vertices of Δ and ∇ . Arguing as in the foregoing proof, by adding all possible points of denominator $\leq d$ we obtain d -saturated (automatically regular) subdivisions Δ' of Δ and ∇' of ∇ . By Lemma 3.14(ii), P is $\mathbb{Z}$ -homeomorphic to Q iff among the finitely many bijections of the set $\mathcal{V}$ of vertices of Δ' onto the set $\mathcal{W}$ of vertices of ∇' , there is at least one, say ι , satisfying the following conditions:

- (i) $\text{den}(\iota(v)) = \text{den}(v)$ for all $v \in \mathcal{V}$;
- (ii) a subset V of $\mathcal{V}$ is the set of vertices of some simplex of Δ' iff the corresponding set $\iota(V) \subseteq \mathcal{W}$ is the set of vertices of some simplex of ∇' .

This completes the proof. $\square$

The recognizability of $[0, 1]$ and FREE_1 was first proved in [6].

Notwithstanding the unrecognizability result by A.A. Markov for PL-homeomorphism of rational polyhedra, [7], [8, pp. 143–144], the problem of deciding if a rational polyhedron $Q \subseteq [0, 1]^n$ is $\mathbb{Z}$ -homeomorphic to the cube $[0, 1]^m$ is still open. The problem is open already for $m = 2$; it looks harder than the easily solved problem of deciding if Q is just homeomorphic to $[0, 1]^2$.

18.4 The Recognition of Free Generating Sets in FREE_n

Proposition 18.8 *Consider the following problem:*

Input: Formulas $\psi_1, \dots, \psi_k \in \text{FORM}_n$.

Question: Do the McNaughton functions $\hat{\psi}_1, \dots, \hat{\psi}_k$ generate FREE_n ?

Then this problem is decidable.

Proof Let $\eta: [0, 1]^n \rightarrow [0, 1]^k$ be defined by $\eta = (\hat{\psi}_1, \dots, \hat{\psi}_k)$. Let R be the range of η . Then by Lemma 3.6, $\text{gen}(\hat{\psi}_1, \dots, \hat{\psi}_k) \cong \mathcal{M}(R)$. By Corollary 3.10, our problem is equivalent to checking if $[0, 1]^n$ is $\mathbb{Z}$ -homeomorphic to R via η . Running algorithm **Triangulation**, we compute a regular η -triangulation Δ of $[0, 1]^n$. By Proposition 3.15, a necessary and sufficient set of conditions for η to be a $\mathbb{Z}$ -homeomorphism is that η is one-one, η preserves denominators of vertices of Δ , and η also preserves regularity of simplexes of Δ . All these three conditions can be effectively checked: the first condition, by algorithm **One-one**; the second, by direct inspection; the third, using algorithm **Regular**. $\square$

More generally, a routine variant of the above proof shows

Proposition 18.9 *Let $P \subseteq [0, 1]^n$ be a rational polyhedron and $\psi_1, \dots, \psi_k \in \text{FORM}_n$. Then the problem whether the McNaughton functions $\hat{\psi}_1 \upharpoonright P, \dots, \hat{\psi}_k \upharpoonright P$ generate $\mathcal{M}(P)$ is decidable.*

Corollary 18.10 *Let $P \subseteq [0, 1]^n$ be a rational polyhedron and $\psi_1, \dots, \psi_k \in \text{FORM}_n$. Let $\eta: P \rightarrow [0, 1]^k$ be the restriction to P of the $\mathbb{Z}$ -map*

$$(\hat{\psi}_1, \dots, \hat{\psi}_k): [0, 1]^n \rightarrow [0, 1]^k.$$

Then the problem whether $\eta(P)$ is $\mathbb{Z}$ -homeomorphic to P is decidable.

Proof Immediate from the proof of Proposition 18.8. $\square$

Proposition 18.11 *Suppose $P \subseteq [0, 1]^n$ is $\mathbb{Z}$ -homeomorphic to $[0, 1]^n$. Then $P = [0, 1]^n$.*

Proof If not (absurdum hypothesis) the Lebesgue n -dimensional measure $\mathcal{L}^{(n)}(U) = \mathcal{L}^{(n)}(\text{cl}(U))$ of the open set $U = [0, 1]^n \setminus P$ is > 0 . By Proposition 14.2, this coincides with the n -dimensional rational measure of $\text{cl}(U)$. Thus the rational n -dimensional measure of P is strictly less than 1. On the other hand, by Lemma 14.3(ii) the assumed $\mathbb{Z}$ -homeomorphism of $[0, 1]^n$ onto P guarantees that the rational measure of P equals 1, a contradiction. $\square$

Proposition 18.12 *Consider the following problem:*

Input: A formula $\phi \in \text{FORM}_n$.

Question: Is the Lindenbaum algebra LIND_ϕ isomorphic to $\mathcal{M}([0, 1]^n)$?

This problem is decidable.

Proof By Lemma 3.19 and Theorem 3.23, the problem equivalently asks if the set $\hat{\phi}^{-1}(1) \subseteq [0, 1]^n$ is $\mathbb{Z}$ -homeomorphic to $[0, 1]^n$. By Proposition 18.11 this is the same as deciding if $\hat{\phi}^{-1}(1) = [0, 1]^n$, i.e., ϕ is a tautology. By (A21.34), the latter problem is decidable. $\square$

Theorem 18.13 *The following problem is decidable:*

Input: Formulas $\psi_1, \dots, \psi_m \in \text{FORM}_n$.

Question: Is $\{\hat{\psi}_1, \dots, \hat{\psi}_m\}$ a free generating set of the free MV-algebra $\mathcal{M}([0, 1]^n)$?

Proof Let $f: [0, 1]^n \rightarrow [0, 1]^m$ be defined by $f(x) = (\hat{\psi}_1(x), \dots, \hat{\psi}_m(x))$ for each $x \in [0, 1]^n$. Let $R \subseteq [0, 1]^m$ be the range of f . We will use the isomorphism of Lemma 3.6 between $\text{gen}\{\hat{\psi}_1, \dots, \hat{\psi}_m\}$ and $\mathcal{M}(R)$.

Claim 1 The cardinality k of the set $\{\hat{\psi}_1, \dots, \hat{\psi}_m\}$ is effectively computable from the input list $I = \{\psi_1, \dots, \psi_m\}$.

We will compute a sublist $L = \{\psi_{i(1)}, \dots, \psi_{i(k)}\}$ of I such that the McNaughton functions of the formulas in L are all different, and each element of $\{\hat{\psi}_1, \dots, \hat{\psi}_m\}$ has exactly one copy in $\{\hat{\psi}_{i(1)}, \dots, \hat{\psi}_{i(k)}\}$. To this purpose, using algorithm **Tautology**, for each pair (ψ_i, ψ_j) of elements of I with $i < j$, we check if $\psi_i \leftrightarrow \psi_j$ is a tautology. Whenever this is the case, we delete ψ_j from the list. Evidently, the desired cardinality k is the number of formulas in the list $L = \{\psi_{i(1)}, \dots, \psi_{i(k)}\}$ obtained when no more deletions are possible.

Claim 2 A necessary condition for $\{\hat{\psi}_{i(1)}, \dots, \hat{\psi}_{i(k)}\}$ to be a free generating set of $\mathcal{M}([0, 1]^n)$ is that $k = n$.

To help the reader we give a proof of this well known general fact. Suppose $\{\hat{\psi}_{i(1)}, \dots, \hat{\psi}_{i(k)}\}$ is a free generating in $\mathcal{M}([0, 1]^n)$. Let $\{\pi_1, \dots, \pi_k\}$ be the free generating set of $\mathcal{M}([0, 1]^k)$ given by the coordinate functions. The one-one correspondence

$$\pi_1 \mapsto \hat{\psi}_{i(1)}, \dots, \pi_k \mapsto \hat{\psi}_{i(k)}$$

uniquely extends to a homomorphism of $\mathcal{M}([0, 1]^k)$ onto $\mathcal{M}([0, 1]^n)$ which turns out to be an isomorphism. By Corollary 3.10 the two unit squares $[0, 1]^k$ and $[0, 1]^n$ are $\mathbb{Z}$ -homeomorphic, which occurs iff $k = n$: this follows from a crude counting of all points of denominator 1 in the two cubes (no need of algebraic topology).

Having thus settled our claim, let us suppose the input verifies the necessary condition $k = n$. By Lemma 3.8, $\{\hat{\psi}_{i(1)}, \dots, \hat{\psi}_{i(n)}\}$ is a free generating set in $\mathcal{M}([0, 1]^n)$ iff the $\mathbb{Z}$ -map $\zeta(x) = (\hat{\psi}_{i(1)}(x), \dots, \hat{\psi}_{i(n)}(x))$ is a $\mathbb{Z}$ -homeomorphism of $[0, 1]^n$ onto $[0, 1]^n$. To decide if this is the case, algorithm **Triangulation** yields a regular ζ -triangulation Δ of $[0, 1]^n$. Using algorithm **One-one** we first check if ζ is one-one. Next, we check if ζ preserves the denominators of all vertices of Δ . Finally, using algorithm **Regular** we check that ζ preserves the regularity of every simplex of Δ . If these three checks are successful then automatically ζ is onto, because ζ preserves rational measure, coinciding with Lebesgue measure [Lemmas 14.2 and 14.3(ii)]. By Proposition 3.15, these three conditions are necessary and sufficient for ζ to be a $\mathbb{Z}$ -homeomorphisms of $[0, 1]^n$ onto itself. $\square$

Theorem 18.13 was first proved in Marra's thesis, [9]. The variant asking if FREE_n is isomorphic to the Lindenbaum algebra of an input formula $\phi \in \text{FORM}_{n+1}$ is open.

18.5 There is no Gödel Incomplete Prime Theory $\Theta \subseteq \text{FORM}_n$

A theory $\Theta \subseteq \text{FORM}_n$ is *prime* if for any two formulas $\phi, \psi \in \text{FORM}_n$ we either have $\Theta \vdash \phi \rightarrow \psi$ or $\Theta \vdash \psi \rightarrow \phi$.

In boolean propositional logic, prime theories coincide with maximally consistent theories. In Łukasiewicz propositional logic the two notions differ: for instance, Chang's algebra C introduced in [10, p. 474] is the Lindenbaum algebra of a prime not maximally consistent theory. Up to isomorphism, MV-chains are exactly the Lindenbaum algebras of prime theories.

Theorem 18.14 *Let $\Theta \subseteq \text{FORM}_2$ be a recursively enumerable prime theory. Then Θ is decidable.*

Proof Θ determines the prime ideal $\mathfrak{j} = \{\neg\hat{\phi} \mid \phi \in \Theta\}$ of $\mathcal{M}([0, 1]^2)$. We will argue by cases using the classification Theorem 5.12.

Case 1 j is maximal.

Then the following is a decision procedure for membership of $\psi \in \mathbf{FORM}_2$ in Θ : Wait until either ψ or $\neg\psi \oplus \dots \oplus \neg\psi$ appears in the recursive enumeration of the elements of Θ . Exactly one of these alternatives will occur, by the characterization of simple MV-algebras (A21.26).

Case 2 For some rational $x \in [0, 1]^2$ and rational direction u , $j = j_{x,u}$.

For each $\psi \in \mathbf{FORM}_2$ we compute the value of $\hat{\psi}$ at x and the value of the directional derivative $\hat{\psi}'(x; u)$. Then ψ belongs to Θ iff both values are zero.

Case 3 For some rational $x \in [0, 1]^2$ and rational direction (=unit vector) u , with $u^\perp$ a perpendicular direction to u , $j = j_{x,u,u^\perp}$.

Then for each $\psi \in \mathbf{FORM}_2$ we compute the value of $\hat{\psi}$ at x , the value of the directional derivative $\hat{\psi}'(x; u)$, and the value of the directional derivative $\hat{\psi}'(x + \epsilon u; u^\perp)$ for some rational $\epsilon > 0$ such that $\hat{\psi}$ is linear on the segment $[x, x + \epsilon u]$. It follows that ψ belongs to Θ iff all these three values are zero.

Case 4 $j = j_{x,u}$ for some rational x and direction u such that no rational line $x + \mathbb{R}u$ exists (for short, u is an *irrational direction*).

Then on input $\psi \in \mathbf{FORM}_2$, using algorithm *Triangulation*, let us construct a $\hat{\psi}$ -triangulation Δ of the square $[0, 1]^2$ containing x among its vertices. Let $T_1, \dots, T_k$ be the list of triangles of Δ having x among their vertices. For each $i = 1, \dots, k$ there is a McNaughton function $f_i: [0, 1]^2 \rightarrow [0, 1]$ whose zeroset coincides with T_i . The function $f = f_1 \wedge \dots \wedge f_k$ belongs to the germinal ideal $\mathfrak{o}_x$, whence it belongs to $j_{x,u}$ by Proposition 4.9. Since $j_{x,u}$ is prime, for some $j = 1, \dots, k$ necessarily f_j belongs to $j_{x,u}$. Evidently, such j is unique, because the irrationality of u and the definition of the triangles T_i ensure that for $i \neq j$ the directional derivative of f_i at x in direction u is > 0 . To effectively obtain such j , let $\theta_1, \theta_2, \dots$ be a recursive enumeration of the formulas in Θ . By (A21.24), for some t the zeroset $Z = \mathcal{Z}(\hat{\theta}_1 \wedge \dots \wedge \hat{\theta}_t)$ is contained in $\mathcal{Z}f_j$ and is not contained in $\mathcal{Z}f_i$ for $i \neq j$. Once j has been obtained, there remains to be checked that $\hat{\theta}$ constantly vanishes on the triangle T_j , i.e., $\hat{\theta}$ vanishes at each vertex of T_j .

Case 5 Precisely one rational line $L = x + \mathbb{R}u$ crosses x , and $j = j_{x,u^\perp}$, where $u^\perp$ is orthogonal to u .

It follows that x is not a rational point. Using Corollary 2.9, on input $\psi \in \mathbf{FORM}_2$, let us construct a $\hat{\psi}$ -triangulation Δ of $[0, 1]^2$ such that the segment $L \cap [0, 1]^2$ is a union of 1-simplexes of Δ . Let $T_1, T_2, \dots, T_{2s-1}, T_{2s}$ be the finite set of triangles of Δ having an edge on L . For each odd i , T_i shares with T_{i+1} an edge lying on L . (The case when L lies on the border of the square requires trivial modifications.) For exactly one pair $(i, i+1)$, the point x lies in the interior of the union $T_i \cup T_{i+1}$. As in Case 4, let f_i (resp., f_{i+1}) be a McNaughton function of $\mathcal{M}([0, 1]^2)$ whose zeroset coincides with T_i (resp., with T_{i+1}). Since $f_i \wedge f_{i+1}$ belongs to the germinal ideal $\mathfrak{o}_x$ at least one of f_i, f_{i+1} belongs to $j_{x,u}$, say f_i . Then $f_{i+1} \notin j_{x,u}$ because

$$\frac{\partial f_{i+1}}{\partial u^\perp}(y) > 0$$

for all $y \in (T_i \cap T_{i+1})$ suitably close to x . To effectively obtain such i , again let $\theta_1, \theta_2, \dots$ be a recursive enumeration of the formulas in Θ . By (A21.24), for some t the zeroset $Z = \mathcal{Z}(\hat{\theta}_1 \wedge \dots \wedge \hat{\theta}_t)$ is contained in T_i and is not contained in any other triangle T_j . Having thus obtained i , there remains to be checked that $\hat{\theta}$ constantly vanishes on the triangle T_i . $\square$

In [11] this result was extended to all recursively enumerable prime theories in finitely many variables.

In the above proof it is not claimed that, given a Turing machine $\mathcal{E}$ enumerating Θ , we can effectively construct a Turing machine $\mathcal{D}_{\mathcal{E}}$ deciding Θ . Given $\mathcal{E}$ we are unable to effectively decide which of 1–5 above will be the case for Θ . We only know that the prime ideal $\mathfrak{j}$ determined by $\mathcal{E}$ will fall in precisely one of these cases, and in each case *there exists* a Turing machine $\mathcal{D}$ deciding $\mathcal{E}$. Thus the decidability result above is highly non-uniform.

See [11] for a proof that there is an undecidable recursively enumerable prime theory T on an infinite set of variables.

18.6 Recognizing Coherent Books

In Chaps. 1 and 10 we have seen the universal role of Łukasiewicz propositional logic in the interpretation of coherent probability assessments

$$\beta: \{X_1, \dots, X_n\} \rightarrow [0, 1]$$

of $[0, 1]$ -valued events X_i , given a set W of possible worlds. In case $W = \text{Mod}(\theta)$ for some formula $\theta \in \text{FORM}_n$ and $\beta(X_i)$ is a rational number for each $i = 1, \dots, n$ we have:

Theorem 18.15 *The following problem is decidable:*

Input: A satisfiable formula θ in the variables $X_1, \dots, X_n$, together with a function

$$\beta: \{X_1, \dots, X_n\} \rightarrow [0, 1] \cap \mathbb{Q}.$$

Question: Is β coherent with respect to the set $\text{Mod}(\theta)$ of possible worlds?

Proof Let $W = \text{Mod}(\theta)$. Using Algorithms **Pieces**, and **Faces**, by induction on the number of connectives in θ we compute a finite list

$$\mathcal{H} = \{H_1, \dots, H_h\}$$

of closed rational half-spaces in $\mathbb{R}^n$ having the following property: W is a finite union of simplexes $T_1, \dots, T_m$ such that for each $j = 1, \dots, m$

$$T_j = K_{j1} \cap \dots \cap K_{jr(j)}, K_{j1}, \dots, K_{jr(j)} \in \mathcal{H}.$$

By definition, the book β is coherent iff the following sentence $\chi_{\beta,\theta}$ of *first-order logic* is satisfied in the totally ordered field of real numbers:

$$\forall \sigma_1, \dots, \sigma_n \exists v = (v_1, \dots, v_n) (v \in T_1 \vee \dots \vee v \in T_m) \wedge \sum_{i=1}^n (\sigma_i (\beta(X_i) - v_i)) \geq 0.$$

The real v_i is the truth-value assigned by the possible world $v \in W$ to the i th event X_i . For each $j = 1, \dots, m$ there is a formula $\gamma_j(v_1, \dots, v_n)$ in the language of ordered fields stating that $v \in T_j$; γ_j is a conjunction of linear inequalities with rational coefficients, corresponding to the half-spaces $K_{j1} \cap \dots \cap K_{jr(j)}$ whose intersection is the simplex T_j . The formula

$$\gamma_1(v) \vee \dots \vee \gamma_m(v)$$

states that the n -tuple v belongs to $\text{Mod}(\theta)$, i.e., v is a possible world of W . The inequality

$$\sum_{i=1}^n (\sigma_i (\beta(X_i) - v_i)) \geq 0$$

states that in the possible world v , Blaise's stakes $\sigma_1, \dots, \sigma_n$ do not ensure him a net profit, i.e., Ada's book is coherent. Given β and θ , the formula $\chi_{\beta,\theta}$ can be effectively written down (as a string of symbols) in the first-order language of real closed fields. The decidability of $\chi_{\beta,\theta}$ is a corollary of the Tarski–Seidenberg decision procedure for real closed fields [12–14]. $\square$

In the statement of the foregoing theorem, the assumption that β is defined on a set of variables, rather than on a set of formulas, is no loss of generality. Indeed, by Theorem 15.1(V) the coherence of a book on events $\psi_1, \dots, \psi_n$ given the set of possible worlds $W = \text{Mod}(\theta)$ is the same problem as the coherence of the same book on the fresh variables $X_1, \dots, X_n$ given the set of possible worlds $W' = \text{Mod}(\theta')$, where

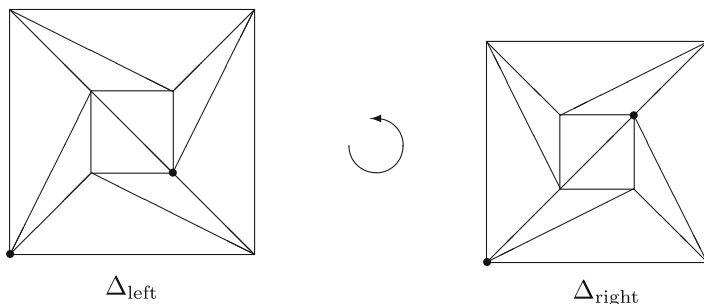
$$\theta' = \theta \wedge (X_1 \leftrightarrow \psi_1) \wedge \dots \wedge (X_n \leftrightarrow \psi_n).$$

Theorem 18.15 was first proved in [15], then strengthened by the authors of [16] who proved that the coherent book problem is in PSPACE, and further strengthened in [17], where the author proved that the problem is coNP-complete.

18.7 Automorphisms of Free MV-algebras, Bases, Schauder Bases

Fix $d = 1, 2, \dots$. Then a crude counting argument shows that there are only finitely many regular triangulations Δ of $[0, 1]^n$ such that $\text{den}(v) \leq d$ for each vertex of Δ .

In contrast, for $n \geq 2$ there are infinitely many $\mathbb{Z}$ -homeomorphic copies of any such triangulation—and each of them is a basis. To see this, consider the two simplicial complexes:



In both complexes Δ_{left} and Δ_{right} , the outer square is assumed to coincide with $[0, 1]^2$, while the innermost square has its center at the point $(1/2, 1/2)$ and side length equal to $1/3$. Direct inspection shows that both Δ_{left} and Δ_{right} are regular. Let $\eta: [0, 1]^2 \cong [0, 1]^2$ be the (necessarily unique) homeomorphism obtained by

- (i) rotating the inner square of Δ_{left} counterclockwise by 90° ,
- (ii) fixing the perimeter of the outer square, and
- (iii) mapping linearly each triangle of Δ_{left} onto its associated rotated triangle of Δ_{right} .

An application of Proposition 3.15. shows that η is in fact a $\mathbb{Z}$ -homeomorphism of $[0, 1]^2$ onto itself. By Lemma 3.8, the $\mathbb{Z}$ -homeomorphisms

$$\eta, \eta^2, \eta^3, \dots$$

yield corresponding automorphisms

$$\alpha, \alpha^2, \alpha^3, \dots$$

of $\mathcal{M}([0, 1]^2)$. By contrast with Schauder bases, bases are invariant under isomorphisms. Thus, for each fixed regular triangulation Δ of $[0, 1]^2$, the Schauder basis $\mathcal{H}_\Delta$ is transformed by the automorphisms α^i into infinitely many distinct (non-Schauder) bases

$$\alpha^1(\mathcal{H}_\Delta), \alpha^2(\mathcal{H}_\Delta), \dots$$

We refer to [18, Sect.3] for further details of the automorphism group G of $\mathcal{M}([0, 1]^2)$. Interestingly enough, the free abelian group over ω generators is embeddable into G .

In Example 6.5 we have seen that a basis of $\mathcal{M}(Q)$ need not be a Schauder basis of $\mathcal{M}(Q)$ up to automorphism. Given the wealth of automorphisms of the free MV-algebra $\mathcal{M}([0, 1]^2)$, and the fact that bases are Schauder bases up to isomorphism (Corollary 6.4), one might wonder if the situation is different when $Q = [0, 1]^2$.

(iii) By definition of $\mathcal{H}_\Delta$ we have

$$1 = h_{001} + h_{011} + h_{111} + 2 \cdot h_{112} + 2 \cdot h_{122} + 3 \cdot h_{233} + 3 \cdot h_{223} \\ + 4 \cdot h_{344} + h_{101} + 5 \cdot h_{335},$$

whence

$$1 = h_{001} + h_{011} + h_{111} + 2 \cdot h_{112} + 2 \cdot h_{122} + 3 \cdot h_{233} + 3 \cdot h_{223} + 4 \cdot h'_{344} + h'_{101}.$$

Having thus shown that B is a basis of $\mathcal{M}([0, 1]^2)$, we *claim* that there is no automorphism α of $\mathcal{M}([0, 1]^2)$ such that $\alpha(B)$ is a Schauder basis. For otherwise, suppose $\alpha(B) = \mathcal{H}_\nabla$ for some regular triangulation ∇ of the unit square. By Corollary 3.18 we can safely assume that the $\mathbb{Z}$ -homeomorphism σ_α of Lemma 3.8 fixes each point of the perimeter of $[0, 1]^2$. We also have $\sigma_\alpha(1/2, 1/2) = (1/2, 1/2)$. Our standing absurdum hypothesis ensures that σ_α sends the apogee of C onto some simplex U of ∇ . Then necessarily U coincides with the 2-simplex $T = \text{conv}((1/2, 1/2), (3/4, 1), (1, 0))$. A routine inspection shows that T is not regular, which contradicts the assumed regularity of ∇ .

In conclusion, B is a basis of $\mathcal{M}([0, 1]^2)$ such that no automorphism transforms B into a Schauder basis of $\mathcal{M}([0, 1]^2)$.

To further stress the difference between bases and Schauder bases of $\mathcal{M}(P)$, we finally prove:

Proposition 18.17 *Consider the following problem:*

Input: Formulas $\phi_1, \dots, \phi_k$ in FORM_n and integers $m_1, \dots, m_k \geq 1$.

Question: Is $\{\hat{\phi}_1, \dots, \hat{\phi}_k\}$ a Schauder basis in the free MV-algebra $\mathcal{M}([0, 1]^n)$ with multipliers $m_1, \dots, m_k$?

Then this problem is decidable.

Proof Using Proposition 18.8 we first check if $\hat{\phi}_1, \dots, \hat{\phi}_k$ is a generating set. Next we check if the sum

$$\sum_{i=1}^k m_i \hat{\phi}_i$$

constantly equals 1 in the associated unital ℓ -group $(G, 1)$ of $\mathcal{M}([0, 1]^n)$ given by $\Gamma(G, 1) = \mathcal{M}([0, 1]^n)$. By Lemma 5.6, this amounts to checking the tautologousness of certain formulas effectively computable from $\phi_1, \dots, \phi_k$ and $m_1, \dots, m_k$. By (A21.34) the tautology problem is decidable. Finally, whenever C is a k -element subset of F such that $\bigwedge C \neq 0$ (a condition that can be checked using algorithm **Tautology**) we compute the intersection T_C of the zerosets of the McNaughton functions in the complementary set $S \setminus C$ using a variant of algorithm **Mod**. The set $\{\hat{\phi}_1, \dots, \hat{\phi}_n\}$ will be a basis of $\mathcal{M}([0, 1]^n)$ iff, after all these successful tests,

- (a) each rational polyhedron T_C turns out to be a regular $(k - 1)$ -simplex, and
- (b) the family of all T_C is a triangulation of $[0, 1]^n$.

For the verification of condition (a), we proceed as follows: we compute a regular triangulation Δ_C of T_C using algorithm **Triangulation**. Letting $\mathcal{V}_C$ be the set of vertices of Δ_C , we let $V_1, \dots, V_u$ be the list of all affinely independent k -element subsets of $\mathcal{V}_C$. We then hope that T_C coincides with a simplex $\text{conv}(V_i)$ for some $i = 1, \dots, u$. If (and only if) this is the case, using algorithm **Regular** we check if T_C is regular. If all T_C are regular simplexes, a routine final computation will decide if the T_C form a triangulation (condition (b)). $\square$

The proof would not work if in the statement of Proposition 18.17 “Schauder basis” were replaced by “basis”. For, instead of checking if T_C is *equal* to a $(k - 1)$ -simplex, we must now check if T_C is *homeomorphic* to some $(k - 1)$ -simplex T —an undecidable problem by Markov unrecognizability theorem mentioned above. Thus for a proof (if any) of the recognizability of bases of $\mathcal{M}([0, 1]^n)$, new techniques must be developed.

Using Theorem 6.15 it is not hard to see that there exists a recursive enumeration of the set of bases of $\mathcal{M}([0, 1]^n)$. So we are left with the problem of finding a recursive enumeration of the set of finite sets of formulas whose corresponding McNaughton functions do *not* form a basis.

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Chapter 19

A First-Order Łukasiewicz Logic with $[0, 1]$ -Identity

In classical first-order logic with identity, a model $\mathfrak{M}$ is a nonempty set U equipped with extra machinery to interpret constant, function and relation symbols as elements, functions and relations on the universe U of $\mathfrak{M}$. Even when $\mathfrak{M}$ has no such symbols to interpret, U comes equipped with the identity relation. For the introduction of appropriate models in a first-order Łukasiewicz logic with identity, it seems advisable to replace the universe U of $\mathfrak{M}$ by a richer universe, called “MV-set”, having a natural built-in $[0, 1]$ -valued identity. Sets X of unit vectors forming angles $\leq \pi/2$ in a Hilbert space $\mathcal{H}$ and whose linear span is dense in $\mathcal{H}$, will be our MV-sets: the identity degree of two vectors of X is their scalar product in $\mathcal{H}$. Universes of first-order models are recovered in the special case when X is an orthonormal basis: for, scalar product then coincides with the characteristic function of the $\{0, 1\}$ -valued identity relation on X .

We will introduce a first-order Łukasiewicz logic $\mathbb{L}_{\omega\omega}$: universes of models in $\mathbb{L}_{\omega\omega}$ are MV-sets with $[0, 1]$ -valued identity given by scalar product; each model is further equipped with functions and relations satisfying suitable continuity conditions. Formulas of $\mathbb{L}_{\omega\omega}$ essentially coincide with skolemized first-order formulas. We will not duplicate the proof, given in [1], that $\mathbb{L}_{\omega\omega}$ is compact and the unsatisfiable formulas of $\mathbb{L}_{\omega\omega}$ form a recursively enumerable set. Rather, we will give all necessary prerequisites for the proof of these two results. We will also equip $\mathbb{L}_{\omega\omega}$ with a finitary recursively enumerable consequence relation.

This chapter requires no prerequisites from the previous chapters. On the other hand, the reader is supposed to know the basic syntactic and semantic definitions of classical first-order logic $\mathbb{L}_{\omega\omega}$. Given the symbiotic relationship between Hilbert space and MV-sets, the main techniques used for $\mathbb{L}_{\omega\omega}$ differ significantly from those used in classical model theory and in all the pre-existing literature on other first-order Łukasiewicz logics. For instance, the usual transitivity property of identity on sets now takes the more general form of positive semidefiniteness of the Gram matrix $\langle v_i, v_j \rangle$ of the elements of the MV-set X . The compactness property of $\mathbb{L}_{\omega\omega}$ follows from Kolmogorov dilation theorem for positive semidefinite functions—a standard construction in Hilbert space theory. The recursive enumerability of the consequence relation of $\mathbb{L}_{\omega\omega}$ introduced in this chapter follows from Kolmogorov

dilation theorem together with the Tarski–Seidenberg decidability theorem for real closed fields.

19.1 MV-Sets, MV-Functions, MV-Relations

A set in a Hilbert space $\mathcal{H}$ is said to be *fundamental* if its linear span is a dense subspace of $\mathcal{H}$. An *MV-set* is a pair $(\mathcal{H}, X)$ where $\mathcal{H}$ is a complex Hilbert space and X is a fundamental set of unit vectors in $\mathcal{H}$, such that for all $v, w \in X$, $\langle v, w \rangle \geq 0$. The scalar product $\langle v, w \rangle$ is called the *identity degree* of v and w . By (B21.67), $\langle v, w \rangle$ belongs to the real unit interval $[0, 1]$.

When the enveloping Hilbert space $\mathcal{H}$ is clear from the context we will write X instead of $(\mathcal{H}, X)$. For any two elements $v, w \in X$ we have the trivial equivalence

$$v = w \Leftrightarrow \langle v, w \rangle = 1. \quad (19.1)$$

For each $n = 1, 2, 3, \dots$, an *n-ary MV-function* on $(\mathcal{H}, X)$ is a function $f: X^n \rightarrow X$ having the following *congruence property*: For all $v_1, \dots, v_n, w_1, \dots, w_n \in X$,

$$\min(\langle v_1, w_1 \rangle, \dots, \langle v_n, w_n \rangle) \leq \langle f(v_1, \dots, v_n), f(w_1, \dots, w_n) \rangle. \quad (19.2)$$

When X is an orthonormal basis in $\mathcal{H}$, this inequality becomes the familiar congruence property of functions.

Proposition 19.1 *Let $(\mathcal{H}, X)$ be an MV-set and n, k integers > 0 .*

- (i) *Compositions of MV-functions are MV-functions. Specifically, let $l: X^k \rightarrow X$ be a k -ary MV-function, and $g_i: X^n \rightarrow X$, $i = 1, \dots, k$ be n -ary MV-functions. Let $f: X^n \rightarrow X$ be defined by $f(v) = l(g_1(v), \dots, g_k(v))$ for all $v = (v_1, \dots, v_n) \in X^n$. Then f is an MV-function.*
- (ii) *Every MV-function $f: X^n \rightarrow X$ is uniformly continuous, with respect to the usual topology induced on $\mathcal{H}$ by the norm $\|x\| = \langle x, x \rangle^{1/2}$.*

Proof (i) We must prove $\min(\langle v_1, w_1 \rangle, \dots, \langle v_n, w_n \rangle) \leq \langle f(v), f(w) \rangle$. Since each g_i is an MV-function,

$$\min(\langle v_1, w_1 \rangle, \dots, \langle v_n, w_n \rangle) \leq \langle g_i(v), g_i(w) \rangle.$$

It follows that $\min(\langle v_1, w_1 \rangle, \dots, \langle v_n, w_n \rangle) \leq \min_i(\langle g_i(v), g_i(w) \rangle)$. Since l is an MV-function,

$$\min_i(\langle g_i(v), g_i(w) \rangle) \leq \langle l(g_1(v), \dots, g_k(v)), l(g_1(w), \dots, g_k(w)) \rangle = \langle f(v), f(w) \rangle,$$

whence f is an MV-function.

(ii) It suffices to prove

$$||x_1 - y_1|| < \delta, \dots, ||x_n - y_n|| < \delta \Rightarrow ||f(x_1, \dots, x_n) - f(y_1, \dots, y_n)|| < \delta,$$

for all $x_i, y_i \in X$. To this purpose, for each $i = 1, \dots, n$ we write

$$\begin{aligned} ||x_i - y_i|| < \delta &\Leftrightarrow \langle x_i - y_i, x_i - y_i \rangle^{1/2} < \delta \\ &\Leftrightarrow (1 + 1 - 2\langle x_i, y_i \rangle)^{1/2} < \delta \\ &\Leftrightarrow \langle x_i, y_i \rangle > \frac{2 - \delta^2}{2}. \end{aligned}$$

Writing x instead of $(x_1, \dots, x_n)$, it follows that

$$\begin{aligned} ||f(x) - f(y)|| &= \langle f(x) - f(y), f(x) - f(y) \rangle^{1/2} \\ &= (2 - 2\langle f(x), f(y) \rangle)^{1/2} \\ &\leq (2 - 2\langle x_i, y_i \rangle)^{1/2} \quad (i = 1, \dots, n) \\ &< \left(2 - 2\frac{2 - \delta^2}{2}\right)^{1/2} \\ &= \delta. \end{aligned} \quad \square$$

An n -ary MV-relation R on the MV-set $(\mathcal{H}, X)$ is a map $R: X^n \rightarrow [0, 1]$ such that for all $v_1, \dots, v_n, w_1, \dots, w_n \in X$,

$$\min(\langle v_1, w_1 \rangle, \dots, \langle v_n, w_n \rangle) \leq 1 - |R(v_1, \dots, v_n) - R(w_1, \dots, w_n)|. \quad (19.3)$$

A variant of Proposition 19.1(ii) shows that every MV-relation is uniformly continuous.

When X is an orthonormal basis in Hilbert space, the above congruence property becomes the usual congruence property of relations.

19.2 The Syntax and Semantics of $\mathbf{L}_{\omega\omega}$

Syntax We will use the same stock of variable, constant, function and relation symbols

$$x, y, z, \dots, \quad a, b, c, \dots, \quad f, g, h, \dots, R, S, T, \dots$$

of classical first-order logic. Each function and relation symbol carries an integer ≥ 1 called *arity*. By a *language* we mean a set $\mathbf{L}$ of constant, function and relation symbols. *Terms* have the usual definition.

Atomic formulas have either form $\tau_1 \approx \tau_2$, or $S(\tau_1, \dots, \tau_n)$, where $\tau_1, \dots, \tau_n$ are terms, and S is an n -ary relation symbols.

Formulas are obtained from the atomic formulas using the connectives $\neg, \oplus, \odot$ of Łukasiewicz propositional logic (as well as the derived connectives $\rightarrow$ and $\leftrightarrow$).

By an *L-term* (resp., *L-formula*) we mean a term (resp., a formula) whose constant, function and relation symbols belong to L . For ϵ a term or a formula, we write $\epsilon(x_1, \dots, x_n)$ to mean that the variables occurring in ϵ belong to the set $\{x_1, \dots, x_n\}$. We say that ϵ is *ground* if no variables occur in it.

Assuming the set of constant symbols in L nonempty, we denote by H the *Herbrand universe* of L , i.e., the set of all ground terms obtained from L .

Semantics By an *L-model* we mean a pair $\mathfrak{M} = ((\mathcal{H}, X), \iota)$, where $(\mathcal{H}, X)$ is an MV-set and ι maps each constant symbol $c \in L$ into an element $c^\iota \in X$, each n -ary function symbol $f \in L$ into an n -ary MV-function f^ι on $(\mathcal{H}, X)$, and each n -ary relation symbol $S \in L$ into an n -ary MV-relation S^ι on $(\mathcal{H}, X)$. X is the *universe* of $\mathfrak{M}$, and $\mathcal{H}$ is its (enveloping) *Hilbert space*.

Let V be a set of variables. Then an *instantiation* λ of V in X is a function $\lambda: V \rightarrow X$. If $\{x_1, \dots, x_n\} \subseteq V$ and $\tau(x_1, \dots, x_n)$ is an *L-term*, then the instantiation $\lambda: x_i \in V \mapsto v_i \in X$ transforms τ into an element $\tau^{\mathfrak{M}[\lambda]}$ of X according to the usual definition:

- (i) If τ is x_i then $\tau^{\mathfrak{M}[\lambda]} = v_i$.
- (ii) If τ is a constant symbol c in L then $\tau^{\mathfrak{M}[\lambda]} = c^\iota$.
- (iii) If τ is $f(\tau_1, \dots, \tau_n)$ where $f \in L$ is an n -ary function symbol and $\tau_1, \dots, \tau_n$ are *L-terms*, then $\tau^{\mathfrak{M}[\lambda]} = f^\iota(\tau_1^{\mathfrak{M}[\lambda]}, \dots, \tau_n^{\mathfrak{M}[\lambda]})$.

If τ is ground we write $\tau^{\mathfrak{M}}$ instead of $\tau^{\mathfrak{M}[\lambda]}$ without fear of ambiguity.

For any *L-formula* $\phi = \phi(x_1, \dots, x_n)$, *L-model* $\mathfrak{M} = ((\mathcal{H}, X), \iota)$, and instantiation $\lambda: \{x_1, \dots, x_n\} \rightarrow X$, the *truth-value* $0 \leq \phi^{\mathfrak{M}[\lambda]} \leq 1$ is defined by the following stipulation:

- $(\sigma \approx \tau)^{\mathfrak{M}[\lambda]} = \langle \sigma^{\mathfrak{M}[\lambda]}, \tau^{\mathfrak{M}[\lambda]} \rangle$, whenever $\sigma(x_1, \dots, x_n)$ and $\tau(x_1, \dots, x_n)$ are *L-terms*.
- For any n -ary relation S , $S(\tau_1, \dots, \tau_n)^{\mathfrak{M}[\lambda]} = S^\iota(\tau_1^{\mathfrak{M}[\lambda]}, \dots, \tau_n^{\mathfrak{M}[\lambda]})$.
- Finally, by induction on the length of formulas,
- $(\neg\psi)^{\mathfrak{M}[\lambda]} = 1 - (\psi^{\mathfrak{M}[\lambda]}).$
- $(\psi \oplus \chi)^{\mathfrak{M}[\lambda]} = \min(1, \psi^{\mathfrak{M}[\lambda]} + \chi^{\mathfrak{M}[\lambda]}).$
- $(\psi \odot \chi)^{\mathfrak{M}[\lambda]} = \max(0, \psi^{\mathfrak{M}[\lambda]} + \chi^{\mathfrak{M}[\lambda]} - 1).$

Definition 19.2 For ϕ a formula in the variables $x_1, \dots, x_n$, we write $\mathfrak{M} \models \phi$ if $\phi^{\mathfrak{M}[\lambda]} = 1$ for every instantiation $\lambda \in X^{\{x_1, \dots, x_n\}}$. Intuitively, all variables of ϕ are thought of as being universally quantified. For any set Φ of *L-formulas* and *L-model* $\mathfrak{M}$, we write $\mathfrak{M} \models \Phi$ (read: “ $\mathfrak{M}$ satisfies Φ ”) if $\mathfrak{M} \models \phi$ for each $\phi \in \Phi$.

If no model satisfies Φ we say that Φ is *unsatisfiable*, otherwise Φ is *satisfiable*. We say that Φ is *finitely satisfiable* if all finite subsets of Φ are satisfiable.

The following is an immediate consequence of the definitions:

Proposition 19.3 *Let $\mathcal{L}$ be a language, $f \in \mathcal{L}$ an n -ary function symbol, and $S \in \mathcal{L}$ an n -ary relation symbol. Then for any $\mathcal{L}$ -model $\mathfrak{M}$,*

$$\mathfrak{M} \models (x_1 \approx y_1 \wedge \cdots \wedge x_n \approx y_n) \rightarrow f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n)$$

and

$$\mathfrak{M} \models (x_1 \approx y_1 \wedge \cdots \wedge x_n \approx y_n) \rightarrow (S(x_1, \dots, x_n) \longleftrightarrow S(y_1, \dots, y_n)).$$

More generally,

Proposition 19.4 *Let $\mathcal{L}$ be a language, $\tau(x_1, \dots, x_n)$ an $\mathcal{L}$ -term and ϕ an $\mathcal{L}$ -formula in the variables $x_1, \dots, x_n$. Then for any $\mathcal{L}$ -model $\mathfrak{M}$,*

$$\mathfrak{M} \models (x_1 \approx y_1 \wedge \cdots \wedge x_n \approx y_n) \rightarrow \tau(x_1, \dots, x_n) \approx \tau(y_1, \dots, y_n)$$

and

$$\mathfrak{M} \models (x_1 \approx y_1 \wedge \cdots \wedge x_n \approx y_n) \rightarrow (\phi(x_1, \dots, x_n) \longleftrightarrow \phi(y_1, \dots, y_n)).$$

Proof Induction on the number of function symbols in τ , and the number of connectives in ϕ . The basis is given by Proposition 19.3. The induction step uses Proposition 19.1(i). $\square$

19.3 [0, 1]-Valued Identity and Positive Semidefinite Matrices $\langle v_i, v_j \rangle$

Let $(\mathcal{H}, X)$ be an MV-set. Then for every model $\mathfrak{M}$ we have

Reflexivity: $\mathfrak{M} \models x \approx x$.

Symmetry: $\mathfrak{M} \models x \approx y \rightarrow y \approx x$.

Transitivity in either formulation $x \approx y \odot y \approx z \rightarrow x \approx z$, or $x \approx y \wedge y \approx z \rightarrow x \approx z$, usually fails. For instance, let X consist of the three vectors in $\mathbb{R}^2$ given by $a = (1, 0)$, $b = (2^{-1/2}, 2^{-1/2})$, $c = (0, 1)$. Then

$$\langle a, b \rangle \wedge \langle b, c \rangle > \langle a, b \rangle \odot \langle b, c \rangle > 2^{-1/2} \odot 2^{-1/2} > 0 = \cos \pi/2 = \langle a, c \rangle$$

To introduce the appropriate generalization of transitivity for MV-sets, letting $\mathbb{C}$ denote the set of complex numbers we prepare:

Definition 19.5 Let Y be a nonempty set. Then a function $\kappa: Y^2 \rightarrow \mathbb{C}$ is said to be *positive semidefinite* if for all $n = 1, 2, \dots$, complex numbers $\gamma_1, \dots, \gamma_n$ and elements $(y_1, \dots, y_n) \in Y^n$, we have the inequality

$$\sum_{i=1}^n \sum_{j=1}^n \kappa(y_i, y_j) \bar{\gamma}_i \gamma_j \geq 0, \quad (19.4)$$

where $\bar{\gamma}$ is the complex conjugate of γ .

The following result shows the *necessity* of the condition of positive semidefiniteness in the construction of the Gram matrix $\langle v_i, v_j \rangle$ of an MV-set:

Proposition 19.6 *Suppose R is an $(n \times n)$ -matrix with real entries. Then the following conditions are equivalent:*

- (i) R is positive semidefinite.
- (ii) R is symmetric and for all $\rho_1, \dots, \rho_n \in \mathbb{R}$, $\sum_{i,j} R_{ij} \rho_i \rho_j \geq 0$.
- (iii) There is an $(n \times n)$ -matrix Q with real entries such that $R = Q^T Q$, where Q^T denotes the transpose of Q .

Proof (i $\Rightarrow$ ii) By [2, p. 397], every positive semidefinite (not necessarily real) matrix N satisfies the identity $N = N^*$, where N^* is the conjugate transpose of N .

(ii $\Rightarrow$ iii), By [2, 7.2.1], every eigenvalue of R is ≥ 0 . Let E be the diagonal matrix with the eigenvalues of R in its diagonal. Let $E^{1/2}$ denote the diagonal matrix whose elements are the square roots of the diagonal elements of E . By [2, 2.5.6], there is an orthonormal $(n \times n)$ -matrix B with real entries such that $B^T R B = E$. Then $Q = E^{1/2} B^T$ will satisfy (iii).

Finally, the implication (iii $\Rightarrow$ i) is trivial. $\square$

Conversely, the following result shows that positive semidefiniteness (jointly with symmetry and the unit-diagonal property) is *sufficient* for the existence of the Gram matrix $\langle v_i, v_j \rangle$ of an MV-set:

Theorem 19.7 (Kolmogorov dilation theorem) *Let H be a nonempty set and $\wp: H^2 \rightarrow \mathbb{C}$ a positive semidefinite function. Then there is a Hilbert space $\mathcal{H}$ and a map $j: H \rightarrow \mathcal{H}$ such that $j(H)$ is a fundamental set in $\mathcal{H}$ and, letting $\langle \cdot, \cdot \rangle$ denote scalar product, $\langle j(y), j(z) \rangle = \wp(y, z)$ for all $y, z \in H$.*

Proof Let $\mathcal{V}_H$ be the vector space of all functions $f: H \rightarrow \mathbb{C}$ which are nonzero only at finitely many points of H . ($\mathcal{V}_H$ is known as the *free vector space* over H). The stipulation

$$(f, g) = \sum_{y, z \in H} \wp(y, z) f(y) g(z) \quad \text{for all } f, g \in \mathcal{V}_H$$

equips $\mathcal{V}_H$ with a sesquilinear product. The set $\mathcal{Z} = \{f \in \mathcal{V}_H \mid (f, f) = 0\}$ is a linear subspace of $\mathcal{V}_H$. The desired Hilbert space $\mathcal{H}$ is now obtained as the completion of the pre-Hilbert (= inner product) space $\mathcal{V}_H / \mathcal{Z}$.

To conclude the proof, let the map $\kappa: H \rightarrow \mathcal{V}_H$ be defined by $(\kappa(y))(y) = 1$, and $(\kappa(y))(z) = 0$ for $z \neq y$. The desired map $j: H \rightarrow \mathcal{H}$ is now defined by $j(y) = \kappa(y) / \mathcal{Z}$. $\square$

Theorem 19.8

- (i) A set Φ of formulas is satisfiable in $\mathbb{L}_{\omega\omega}$ iff it is finitely satisfiable.
- (ii) For any countable language $\mathbb{L}$, the set of unsatisfiable $\mathbb{L}$ -formulas in $\mathbb{L}_{\omega\omega}$ is recursively enumerable.

Proof [1]. □

In analogy with the case of first-order logic with equality, given a finitely satisfiable set Φ of formulas, the first step to show the satisfiability of Φ in $\mathcal{L}_{\omega\omega}$, is the construction of an MV-set $(\mathcal{H}, X)$, whose elements are equivalence classes in the Herbrand universe H of Φ . To this purpose, the proof in [1] constructs an infinite positive semidefinite, unit-diagonal, symmetric, $[0, 1]$ -valued matrix M indexed by the elements of H . Kolmogorov dilation theorem then yields an MV-set $(\mathcal{H}, X)$ and a model on $(\mathcal{H}, X)$ satisfying Φ . The fact that positive semidefiniteness can be checked by restricting to finite-dimensional sub-matrices, is also a main ingredient of the proof of the recursive enumerability of the set of unsatisfiable formulas [1, 6.1]. The other main ingredient is the Tarski–Seidenberg decision algorithm for the first-order theory of real-closed fields [3, §8], [4, 5].

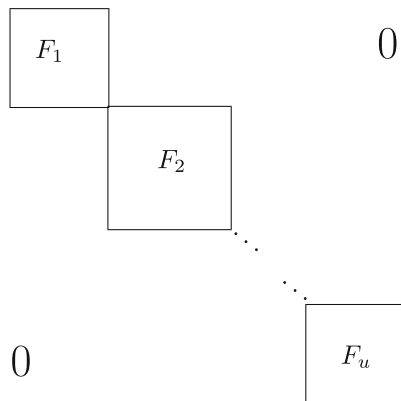
Closing a circle of ideas, the following proposition shows that positive semidefiniteness is the natural generalization to MV-sets of the transitivity property of classical equivalence relations.

Proposition 19.9 *Let M be a symmetric $(n \times n)$ -matrix with unit diagonal. If in addition M is boolean (i.e., $\{0, 1\}$ -valued) then the following conditions are equivalent:*

- (i) *M is positive semidefinite.*
- (ii) *There is an equivalence relation $\sim$ on the set $\{1, \dots, n\}$ such that M is the characteristic function of $\sim$: for all $i, j \in \{1, \dots, n\}$, $M_{ij} = 1$ iff $i \sim j$.*

Proof (i $\Rightarrow$ ii) By Proposition 19.6, $M = N^T N$ for a suitable real $(n \times n)$ -matrix N . For each $i = 1, \dots, n$ the i th column of N displays the coordinates of a vector v_i with respect to the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. The unit-diagonal property of M ensures that v_i is a unit vector. By (19.1), $\langle v_i, v_j \rangle = 1$ iff $v_i = v_j$. This shows that M is the characteristic function of the equivalence relation $\sim$ on $\{1, \dots, n\}$ given by $v_i = v_j$ iff $i \sim j$.

(ii $\Rightarrow$ i) After a suitable rearrangement of the rows and columns of M , without loss of generality M can be assumed to have the form of the block diagram below.



The block diagram matrix M

For each $i = 1, \dots, u$, F_i is a $(k_i \times k_i)$ -matrix with entries all equal to 1. The integers k_i sum up to n . Every entry of M not belonging to any of $F_1, \dots, F_u$ is zero. The set $\{1, \dots, n\}$ is partitioned into pairwise disjoint sets $D_1, \dots, D_u$, called blocks, each D_i of size k_i , where $D_1 = \{1, \dots, k_1\}$ and for each $1 < i \leq u$,

$$D_i = \{k_1 + \dots + k_{i-1} + 1, k_1 + \dots + k_{i-1} + 2, \dots, k_1 + \dots + k_i\}.$$

For each $j = 1, \dots, n$ let β_j be the block to which j belongs, $j \in D_{\beta(j)}$. For each $i = 1, \dots, u$ let the vector $v_i \in \mathbb{R}^n$ be defined by

$$v_i = k_i^{-1/2} \sum_{j \in \beta(i)} e_j.$$

Let B be the $(n \times n)$ -matrix whose j th column coincides with $v_{\beta(j)}$, ($j = 1, \dots, n$). Then $M = B^T B$. By Proposition 19.6, M is positive semidefinite. $\square$

19.4 Consequence Relations

Despite Łukasiewicz propositional logic $\mathbb{L}_\infty$ can be equipped with various kinds of consequence relations, throughout this book we have almost exclusively considered the (syntactic) consequence relation $\vdash$, because of its wide applicability. As a matter of fact, $\vdash$ leads to a definition of “theory” that finds use, e.g., in the proof of the Amalgamation Theorem 2.20. From the algorithmic viewpoint, Theorem 18.3 yields a decision procedure for $\phi \vdash \psi$. See Sect. 20.1 for a panoramic view of $\vdash$ in the framework of Abstract Algebraic Logic.

In a similar way, a main motivation of the following definition of consequence in $\mathbb{L}_{\omega\omega}$, is Theorem 19.11 below:

Definition 19.10 We say that an $\mathbb{L}$ -formula ϕ is a *consequence* (in $\mathbb{L}_{\omega\omega}$) of a set Φ of $\mathbb{L}$ -formulas, in symbols, $\Phi \vdash \phi$, if for every $\mathbb{L}$ -model $\mathfrak{M}$ if $\mathfrak{M} \models \Phi$ then $\phi^{\mathfrak{M}}[\lambda] > 0$ for every instantiation λ . More generally, for every rational number $0 \leq \xi < 1$ we write $\Phi \mid_{>\xi} \phi$ if for every $\mathfrak{M}$, if $\mathfrak{M} \models \Phi$ then $\phi^{\mathfrak{M}}[\lambda] > \xi$ for every instantiation λ .

Theorem 19.11 Let $0 \leq \xi = a/b < 1$ be a rational number and $\mathbb{L}$ a language.

- (i) If Φ is a set of $\mathbb{L}$ -formulas and ϕ is an $\mathbb{L}$ -formula, then $\Phi \mid_{>\xi} \phi$ iff $\Psi \mid_{>\xi} \phi$ for some finite subset Ψ of Φ .
- (ii) If $\mathbb{L}$ is countable, the set $\{(\theta, \psi) \in \mathbb{L}^2 \mid \theta \mid_{>\xi} \psi\}$ is recursively enumerable.

Proof Let the McNaughton function $g: [0, 1] \rightarrow [0, 1]$ be defined by $g(z) = \max(0, \min(1, -bz + a + 1))$, $z \in [0, 1]$. Observe that $g^{-1}(1) = [0, \xi]$. Let $\gamma \in \mathbf{FORM}_1$ be a propositional formula of $\mathbb{L}_\infty$ in one variable Y , whose associated McNaughton function $\hat{\gamma}$ coincides with g . The existence of γ is given by (A21.18) together with McNaughton theorem, (A21.48). Let us replace every occurrence of Y in γ by the formula ϕ of (i), and denote by $\gamma(\phi)$ the resulting formula of $\mathbb{L}_{\omega\omega}$.

Then, for every model $\mathfrak{M}$ and valuation ν , we have the equivalence $\phi^{\mathfrak{M}}[\nu] \leq \xi$ iff $(\gamma(\phi))^{\mathfrak{M}}[\lambda] = 1$.

Claim $\Phi \mid_{>\xi} \phi$ iff the set $\Phi \cup \{\gamma(\phi)\}$ is unsatisfiable. As a matter of fact,

$$\begin{aligned} & \forall \mathfrak{M}(\mathfrak{M} \models \Phi \Rightarrow \forall \lambda \phi^{\mathfrak{M}}[\lambda] > \xi) \\ \Leftrightarrow & \forall \mathfrak{M}(\mathfrak{M} \models \Phi \Rightarrow \forall \lambda (\gamma(\phi))^{\mathfrak{M}}[\lambda] < 1) \\ \Leftrightarrow & \forall \mathfrak{M}(\mathfrak{M} \models \Phi \Rightarrow \mathfrak{M} \not\models \gamma(\phi)) \\ \Leftrightarrow & \Phi \cup \{\gamma(\phi)\} \text{ is unsatisfiable.} \end{aligned}$$

Having thus settled our claim, from Theorem 19.8(i) we immediately obtain (i). Finally, (ii) follows from Theorem 19.8(ii). $\square$

19.5 Remarks

Belluce and Chang introduced a first-order Łukasiewicz logic $\mathbb{L}_{BC}$ without identity and function symbols. They proved that $\mathbb{L}_{BC}$ is compact [6, Theorem 7], [7, 5.4.24]. From a general result of Mostowski [8, §4], it follows that the set of unsatisfiable formulas of $\mathbb{L}_{BC}$ is recursively enumerable: see [7, 5.4.24, 6.3.17(1)] and [9, Theorem 3(2)]. Hájek [7, 5.6] equips each model of $\mathbb{L}_{BC}$ with a similarity relation r satisfying a suitable transitivity axiom [7, 5.6.1]. Except in the classical case when the MV-set X is an orthonormal basis in $\mathcal{H}$, the relations considered in [7, 5.6.5] are different from MV-relations: the uniform continuity property of Proposition 19.1(ii) requires a Hilbert space norm, that has no role in $\mathbb{L}_{BC}$ logic with similarity relations. In the monograph [7] function symbols are not used. The paper [10] is devoted to the proof-theory of a first-order Łukasiewicz logic with relations and functions, but identity and congruence are not considered.

For fundamental sets in Hilbert space see [11, p. 9]. Theorem 19.7 is a well-known result in Hilbert space theory, (see, e.g., [11, 3, Exercise 3, p. 117–118], or [12, 3.3.1, p. 81]).

Dilations of positive (semi)definite kernels—also known as positive kernels, or self-reproducing kernels, or (auto)correlation matrices—find several applications in various parts of mathematics, ranging from positive semidefinite programming [13, 14] to covariant phase observables in quantum mechanics [15] and Hilbert space representations of groups and C^* -algebras [16].

The counterpart of cartesian products for MV-sets is given by Hilbert space tensor products. In particular, for any MV-set $(\mathcal{H}, X)$ we define the n -fold product of the MV-set $(\mathcal{H}, X)$ by

$$(\mathcal{H}, X)^n = \underbrace{(\mathcal{H} \otimes \cdots \otimes \mathcal{H}, X^n)}_{n \text{ times}},$$

Table 19.1 From $L_{\omega\omega}$ to $L_{\omega\omega\omega}$

$L_{\omega\omega}$	$L_{\omega\omega\omega}$
Point	Unit vector in a fundamental set of a Hilbert space
Nonempty set	MV-set
(In)equality of two points	Scalar product of two unit vectors forming acute angles
Function	MV-function
Relation	MV-relation
Identity	Positive semidefinite, symmetrical, unit-diagonal matrix
Congruence of functions	The continuity property of MV-functions in 19.1(ii)
Gödel compactness theorem	Compactness theorem, 19.8(i)
Herbrand theorem	Kolmogorov dilation theorem, 19.7
Refutation of $L_{\omega\omega}$ -formulas	Tarski–Seidenberg algorithm for real closed fields
Unsatisfiable formulas are r.e.	Theorem 19.8(ii)
Consequence $\vdash$	$L_{\omega\omega\omega}$ -consequence $\mid_{>\xi}$

where $\otimes$ denotes tensor product. Then $(\mathcal{H}, X)^n$ is an MV-set. As a matter of fact, the linear span of X^n is norm dense in the Hilbert space

$$\underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}.$$

Given tensors $x = x_1 \otimes \cdots \otimes x_n$ and $y = y_1 \otimes \cdots \otimes y_n$, their scalar product $\langle x, y \rangle$ in $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ satisfies the identity

$$\langle x, y \rangle = \langle x_1, y_1 \rangle \cdots \langle x_n, y_n \rangle.$$

See [17, pp. 125 ff] for further information on tensor products of Hilbert spaces.

$L_{\omega\omega}$ provides an equational (more generally, a universally quantified) logic for Hilbert spaces enriched with relations and functions, generalizing Birkhoff–Maltsev equational–quasiequational logic for classical algebraic structures [18].

The restriction to universally quantified formulas is not essential: one can straightforwardly extend $L_{\omega\omega}$ to every variable free L-formula ψ in prenex normal form. Let $s_1, \dots, s_t$ be the new (constant and function) symbols arising from the t steps of the skolemization of ψ . Let $L^+ = L \cup \{s_1, \dots, s_t\}$. Let $\text{sk}(\psi)$ denote the Skolem normal form of ψ ; in $\text{sk}(\psi)$ all quantifiers are deleted. Then we stipulate that an L-model $\mathfrak{M}$ satisfies ψ iff some *expansion* $\mathfrak{M}^+$ of $\mathfrak{M}$ satisfies $\text{sk}(\psi)$ according to Definition 19.2. By definition of “expansion”, the L^+ -model $\mathfrak{M}^+$ is assumed to coincide with $\mathfrak{M}$ on each symbol of L . Evidently, the satisfiability of ψ does not depend on which sequence $s_1, \dots, s_t$ of new constant and function symbols is used in the skolemization of ψ .

Various notions and properties of first-order logic $L_{\omega\omega}$ and its models, together with their generalizations for $L_{\omega\omega\omega}$, are summarized in Table 19.1.

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Chapter 20

Applications, Further Reading, Selected Problems

20.1 Logic or Algebra? The Standpoint of Abstract Algebraic Logic

The borderline between Łukasiewicz logic and MV-algebras is vague. Proof-theoretic methods are fundamental in MV-algebra theory—and algebraic techniques pervade Łukasiewicz logic. Interpolation and amalgamation are symbiotic. Finite presentations of MV-algebras are finite axiomatizations of theories in $\mathbb{L}_\infty$. States are coherent probability assessments of continuously valued events described by $\mathbb{L}_\infty$ -formulas; extremal states are truth-valuations in the semantics of $\mathbb{L}_\infty$. The input of most MV-algebraic decision problems consists of $\mathbb{L}_\infty$ -formulas, and the output is obtained by running the natural built-in deductive machinery of $\mathbb{L}_\infty$; the coNP-completeness of the tautology problem for $\mathbb{L}_\infty$ follows from the arithmetic–algebraic properties of the elements of free MV-algebras; more generally, every nontrivial decision problem for $\mathbb{L}_\infty$ relies, in some way or another, upon the fine structure of finitely presented MV-algebras, featuring bases, rational polyhedra and desingularization.

Abstract algebraic logic provides a general framework for a further discussion of the synergy between Łukasiewicz logic and MV-algebras. We give here only a brief account. We will use the monograph [1] as our standard background reference. We will also assume familiarity with [2, Sect. 4].

Throughout this section the class of MV-algebras will be denoted by MV . We also let $\mathcal{C}$ denote the usual consequence relation of Łukasiewicz propositional logic (the “syntactic consequence” of [2, 4.3.2]). In view of the Local Deduction Theorem 1.7 and Corollary 1.9($a \Leftrightarrow c$), instead of saying that formula θ is a $\mathcal{C}$ -consequence of a set Θ of formulas one can equivalently write $\Theta \vdash \theta$ as in Definition 1.8. An algebraic counterpart of Theorem 1.7 is Proposition 4.4(i).

Let further $\mathcal{C}_{MV}$ denote the consequence relation determined by MV : thus θ is a $\mathcal{C}_{MV}$ -consequence of Θ iff for every $A \in MV$, every A -valuation that A -satisfies every formula of Θ also A -satisfies θ . (In the terminology of [2, 4.1.2], θ is said to be a “semantic A -consequence” of Θ .)

From Chang's completeness theorem it follows that $\mathcal{C} = \mathcal{C}_{MV}$. The following result can be obtained from [1, 4.6, 5.1] (see in particular [1, p. 319, p. 349]):

20.1. *$\mathcal{C}$ is a regularly algebraizable logic with an equivalence formula $x \leftrightarrow y$ and defining equation $x = 1$. Further, MV is the largest equivalent algebraic semantics for $\mathcal{C}$.*

From [1, 4.6.5, 5.1.2] it follows that for every MV-algebra A the lattice of congruences of A is isomorphic to the lattice of deductive (called "implicative" in [2, 4.2.6]) filters on A arising from $\mathcal{C}$. Filters are the $\neg$ -duals of ideals [2, 1.2.6, 4.2.7].

From 20.1 together with [1, Q.6.1 p. 260 and 2.6.11] we get

20.2. *MV does not have first-order definable principal congruences. A fortiori, principal congruences are not equationally definable in MV .*

For further information see [3, 8.5–8.6]. In [3, 7.6] one can find a short proof of the following result:

20.3. *MV is an arithmetical variety, in the sense that the congruences in every MV-algebra A permute and form a distributive lattice.*

To see that the congruences of A permute it is sufficient to note that for any $a, b \in A$ the translation $J(x) = (a \odot \neg x \oplus b) \wedge (b \odot \neg x \oplus a)$ satisfies $J(a) = b$ and $J(b) = a$ [4, 2.12].

In [3, 8.2] it is proved:

20.4. *MV has the congruence extension property.*

An alternative proof is obtainable from 20.1 together with [1, Q.5.2 p. 259] and Theorem 1.7 (see [1, Example 2.1.4]).

Next let $\mathcal{C}_{[0,1]}$ be the consequence relation determined by the standard MV-algebra $[0, 1]$. In [2, p. 81] $\mathcal{C}_{[0,1]}$ is called "semantic consequence". Then Theorem 1.7($i \Leftrightarrow v$) amounts to the following crucial result due to Hay [5] (also see Wójcicki's paper [6]):

20.5. *$\mathcal{C}$ agrees with $\mathcal{C}_{[0,1]}$ on finite sets.*

We assume familiarity with the operators **H**, **S**, **P** ([1, p. 34]). Further, for any class K of structures, $\mathbf{P}_U(K)$ is the class of all isomorphic copies of ultraproducts of families of members of K . Chang completeness theorem amounts to the identity $MV = \mathbf{HSP}([0, 1])$, more pedantically, $MV = \mathbf{HSP}(\{[0, 1]\})$. Actually,

20.6. *MV is generated by $[0, 1]$ as a quasivariety, in symbols,*

$$MV = \mathbf{SPP}_U([0, 1]).$$

Arguing as in [3, 7.2], a model-theoretic proof follows combining Di Nola's theorem [2, 9.5] with the characterization [2, 3.5.1] of simple MV-algebras as subalgebras of $[0, 1]$. For a quicker proof using the methods of abstract algebraic logic,

let $K = \mathbf{P}_U([0, 1])$. Then the consequence relation $\mathcal{C}_K$ determined by K is finitary, because K is closed under ultraproducts [1, 0.4.6]. Combining Chang completeness theorem with 20.5 we get $\mathcal{C} = \mathcal{C}_{MV} = \mathcal{C}_K$. In view of the second statement in 20.1 we finally obtain $MV = \mathbf{SPP}_U(K) = \mathbf{SPP}_U([0, 1])$, which concludes the proof.

As shown by Wójcicki in his paper [6], $\mathcal{C}$ differs from $\mathcal{C}_{[0,1]}$ on infinite sets of formulas (see [2, 4.6.6] for a characterization of those sets of formulas where the two consequence relations coincide). However, the following result shows that $\mathcal{C}$ is determined by any single MV-algebra which is a nonprincipal ultrapower of $[0, 1]$:

20.7. *Suppose the MV-algebra A is the ultraproduct of $[0, 1]$ via a nonprincipal ultrafilter over $\{0, 1, 2, \dots\}$. Then the consequence relation determined by A coincides with $\mathcal{C}$.*

The proof follows by combining [7] with [6] (also see [1, Exercise 0.1.19, p. 61]). As a corollary, $MV = \mathbf{SPP}_U(A)$. See [8] for general results along these lines.

As a particular case of a general definition, an MV-algebra A is *finitely subdirectly irreducible* if whenever A is isomorphic to a subdirect product of finitely many MV-algebras, then at least one of these MV-algebras is isomorphic to A . Let MV_{fsi} denote the class of all finitely subdirectly irreducible MV-algebras, and MV_{lin} the class of all MV-chains.

The following result is a strengthening of the Subdirect Representation Theorem (A21.13):

20.8.

$$MV_{\text{fsi}} = \mathbf{SP}_U([0, 1]) = MV_{\text{lin}}.$$

The identity $MV_{\text{fsi}} = MV_{\text{lin}}$ was first proved in [9, Theorem 15]. Trivially, $\mathbf{SP}_U([0, 1]) \subseteq MV_{\text{lin}}$. For the inclusion $MV_{\text{fsi}} \subseteq \mathbf{SP}_U([0, 1])$ one recalls 20.6 and uses the fact that the class of finitely subdirectly irreducible members of $MV = \mathbf{SPP}_U([0, 1])$ is contained in $\mathbf{SP}_U([0, 1])$. The latter inclusion is a particular case of a general result of [10]. An alternative proof of $MV_{\text{fsi}} \subseteq \mathbf{SP}_U([0, 1])$ is obtained from the proof of Di Nola's theorem in [2, 9.5]. In the special case when B is an MV-chain, the proof yields $MV_{\text{lin}} \subseteq \mathbf{SP}_U([0, 1])$, whence $MV_{\text{fsi}} \subseteq \mathbf{SP}_U([0, 1])$.

In Theorem 7.1, the joint embedding property of MV is obtained as a particular case of the amalgamation property established in Theorem 2.20. A different proof follows from 20.7 using [11] and [1, pp. 233–236]. In a similar way one also proves that MV is closed under free products.

The quoted results by Hay [5] and Wójcicki [6] on the two consequence relations $\mathcal{C}$ and $\mathcal{C}_{[0,1]}$ are key unifying tools in the study of the fundamental properties of Łukasiewicz logic and MV-algebras. They have been repeatedly used in several chapters of this book.

Altogether, Łukasiewicz logic and MV-algebras are one mathematical symphony with two main themes.

20.2 Applications and Further Reading

As we have seen throughout this book, Łukasiewicz logic and MV-algebras draw from various areas of mathematics, including polyhedral geometry, topology, measure theory, functional analysis, and lattice-ordered abelian groups. Conversely, MV-algebras find interesting applications in many of these areas.

Applications to abelian lattice-ordered groups with and without unit. Originating as disjunctive normal forms in Łukasiewicz logic, Schauder bases are a key tool for constructive proofs of McNaughton theorem [12], of Chang completeness theorem, [13], and of many other results of this book. They also have several applications to ℓ -groups including:

- The proof in [14] that every finitely presented (unital, as well as non-unital) ℓ -group is ultrasimplicial. In his paper [15], Marra extended this result to all ℓ -groups.
- The proof in [16] that an ℓ -group G is finitely generated projective iff it is presented by a lattice word. Baker and Beynon had previously proved that G is finitely generated projective iff it is presented by a lattice-group word [17–19].
- The proof in [20] that every finitely presented unital ℓ -group has an invariant faithful state.
- As shown in [21], the bases of a finitely presented unital ℓ -group (G, u) determine a direct system of simplicial groups with one–one positive unital homomorphisms, whose limit is (G, u) . For any such (G, u) , this yields a simplified proof of the Effros–Handelman–Shen representation theorem [22], Grillet’s theorem [23, 2.1], and Marra’s theorem [15].

Applications to probability and measure theory. The monograph [24], as well as Chaps. 20–23 in the *Handbook of Measure Theory* [25], show that MV-algebraic probability and measure theory have reached a mature level. Most results in boolean algebraic measure theory à la Carathéodory have an MV-algebraic extension, well beyond the Loomis–Sikorski or the Poincaré recurrence theorem presented in this book, or the various Cantor–Bernstein theorems proved in [26–29]. As a matter of fact, the list of classical results in measure theory having MV-algebraic generalizations includes:

- Kolmogorov probability space and its main properties [24, 2.3];
- The central limit theorem [24, 2.12];
- The law of large numbers [24, 2.6];
- The individual ergodic theorem [24, 3.4];
- Hahn and Riesz decomposition [30, 3.1.3–3.1.5], [31, Sect. 8–9];
- Darboux measures [31, Sect. 11, 12];
- Lebesgue and Hewitt–Yosida decomposition [30, 3.1.10, 3.1.12, 5.2.2];
- The Vitali–Hahn–Saks theorem and the Nikodym boundedness theorem [30, 5.4];
- Lyapunov theorem [31, §13];
- Representation of measures by Markov kernels [32, §§4–5].

For further results also see [25, 31, 33, 34]. MV-algebras also afford a straightforward generalization of Kolmogorov–Sinai entropy [35].

Along a different line of research, de Finetti’s coherence criterion provides a guiding principle to introduce “states” in various $[0, 1]$ -valued logics, including all finite-valued logics and, more generally, all logics whose connectives are continuous [36]. Extensions to discontinuous logics and non-reversible bookmaking are discussed in [37, 38].

Further relevant applications include Panti’s MV-algebraic ergodic theory [39] and MV-algebraic Lebesgue integration theory [40, 41].

Applications to AF C^ -algebras.* C^* -algebras have already appeared in the first pages of this book, as mathematical counterparts of physical systems, and sources of continuously valued events. Any such event arises from the measurement of a normalized observable.

In this section we discuss the correspondence between MV-algebras and a class of C^* -algebras which finds use in the mathematical treatment of infinite systems in quantum statistical mechanics, namely the Glimm–Dixmier–Bratteli AF C^* -algebras. An AF C^* -algebra is the limit of an ascending sequence of unital finite-dimensional C^* -algebras [42].

In 1976, Elliott classified every AF C^* -algebra in terms of a suitable partial monoid E [43]. Subsequent work [22] replaced E by a countable partially ordered unital abelian group with the Riesz decomposition property. The classifying functor turned out to be a suitable order-theoretic enrichment of Grothendieck group K_0 .

Since all unital ℓ -groups have the Riesz decomposition property, the composite functor ΓK_0 classifies AF C^* -algebras whose Murray–von Neumann order of projections is a lattice, in terms of countable MV-algebras. More precisely, letting $AF\ell$ denote the class of all AF C^* -algebras whose Murray–von Neumann order of projections is a lattice, the composite functor ΓK_0 induces a one–one correspondence between isomorphism classes in $AF\ell$ and isomorphism classes of countable MV-algebras.

Thus in particular,

- countable boolean algebras correspond to commutative AF C^* -algebras,
- finite MV-algebras correspond to finite-dimensional C^* -algebras,
- MV-subalgebras of $[0, 1] \cap \mathbb{Q}$ correspond to Glimm’s UHF algebras described, e.g., in [44],
- the simple MV-algebra generated by an irrational $\theta \in [0, 1]$ corresponds to the Effros–Shen algebra $\mathfrak{F}_\theta$ of [44],
- Chang’s MV-algebra C of [45, p. 474] corresponds to the Behnke–Leptin C^* -algebra $A_{1,0}$ with a two-point dual [46] (also see [47]).
- FREE_ω corresponds to the AF C^* -algebra $\mathfrak{M}$ of [48, §8].
- FREE_1 corresponds to the algebra $\mathfrak{M}_1$ introduced in [14] and rediscovered in [49].

As a first application, for every $\mathfrak{B} \in AF\ell$ there is a theory Θ in Łukasiewicz logic such that $\Gamma(K_0(\mathfrak{B})) \cong \text{LIND}_\Theta$. The algebra $\mathfrak{B}$ is presented by the list of formulas of Θ , in symbols, $\mathfrak{B} = \mathfrak{B}_\Theta$. In this way, isomorphism problems for Lindenbaum

algebras of theories Θ in Łukasiewicz logic are turned into isomorphism problems for their corresponding AF C^* -algebras $\mathfrak{B}_\Theta$. Further, in [48, Sect. 6] a nonsimplicity criterion for an arbitrary AF C^* -algebra $\mathfrak{B}_\Theta$ is given in terms of the Gödel incompleteness of Θ . For results in algorithmic AF C^* -algebra theory also see [50, 51, 52–54].

As a second application, for each $\kappa = 1, 2, \dots, \omega$ let $\mathfrak{M}_\kappa$ be the AF C^* -algebra corresponding to the free MV-algebra FREE_κ via K_0 and Γ , in symbols, $\Gamma(K_0(\mathfrak{M}_\kappa)) = \text{FREE}_\kappa$. As expected, the universal properties of FREE_κ have a counterpart for $\mathfrak{M}_\kappa$. For $\mathfrak{M}_1$ see [14, 54] and [49] (where $\mathfrak{M}_1$ is denoted $\mathfrak{A}$). For $\mathfrak{M}_\omega$ see [48], where $\mathfrak{M}_\omega$ is denoted $\mathfrak{M}$. In [48, Sect. 8] it is proved that every primitive ideal of $\mathfrak{M}_\omega$ is essential, and that every AF C^* -algebra $\mathfrak{C}$ with comparability of projections has the form $\mathfrak{C} \cong \mathfrak{M}_\omega/\mathfrak{j}$ for some primitive ideal $\mathfrak{j}$ of $\mathfrak{M}_\omega$.

As a third application, the spectral spaces of various AF C^* -algebras can be investigated by looking at the spectral spaces of their corresponding MV-algebras [48, 55–57].

Miscellaneous applications. MV-algebraic notions and results often lift to various classes of algebras arising in axiomatic quantum mechanics [33, 58, 59], classes of residuated lattices [60, 61], variously enriched MV-algebras and Łukasiewicz logics [62–67], MV-modules and MV-algebras with operators [68, 69], semirings, and Bézout domains [70–72].

The papers [73, 74] deal with generalizations of McNaughton theorem. The coNP-completeness of the tautology problem for Łukasiewicz logic, (A21.49) has been generalized to all continuous t-norm logics, [75]. See [76] and [77] for a general investigation of interpolation and amalgamation in many-valued logics and residuated lattices. See [61, 78], and [79] for extensions of the Γ functor.

Komori's classification [2, Sect. 8.4] is the starting point of an exploration of equational classes of noncommutative unital lattice-ordered groups, [80]. In [81] one finds a proof of Banaschewski theorem for a noncommutative generalization of MV-algebras.

Relationships with Moisil's determination principle are investigated in [82]. Completions and canonicity issues are the main topic of [83–85].

While Łukasiewicz logic is one of the oldest nonclassical logics, the mathematical literature on this theme has expanded so vigorously over the last 25 years that a comprehensive account of all the developments would probably require several volumes. Quite a few areas of current research have only been mentioned in the list of applications above and in the bibliographical remarks collected at the end of each chapter. What else was left out? Two main omissions are proof theory [86–88] and game-theoretic semantics [89–91]. Both might constitute relevant material for advanced courses on Łukasiewicz logic and MV-algebras. One additional merit of these two subjects is that they can be adjusted to other many-valued logics by just altering the rules of the game.

20.3 Eleven Problems

1. Does there exist a decision procedure for the isomorphism problem of finitely presented MV-algebras?
2. Which topological spaces are homeomorphic to $\text{Spec}(A)$ for some MV-algebra A ?
3. Is the category of locally finite MV-algebras equivalent to an equational class?
4. (*Strong Oda conjecture, MV-algebraic formulation*): Any two bases of a finitely presented MV-algebra A have a common algebraic subdivision.
5. *Conjecture*: For any $n = 2, 3, \dots$ and rational points $x, y \in \text{int}([0, 1]^n)$, if $\text{den}(x) = \text{den}(y)$ then $\mathcal{M}([0, 1]^n)/\mathfrak{o}_x \cong \mathcal{M}([0, 1]^n)/\mathfrak{o}_y$.
6. Are the necessary conditions (i–iii) in Theorem 17.10 also sufficient for a rational polyhedron $P \subseteq [0, 1]^n$ to be a $\mathbb{Z}$ -retract of $[0, 1]^n$?
7. Suppose $P \subseteq [0, 1]^n$ is a rational polyhedron such that the automorphism group of $\mathcal{M}(P)$ is finite. Does it follow that $\dim(P) \leq 1$?
8. For any satisfiable formula $\psi \in \bigcup_n \text{FORM}_n$ let m_ψ be the smallest integer such that the Lindenbaum algebra LIND_ψ is m_ψ -generated. Is the map $\psi \mapsto m_\psi$ effectively computable?
9. Supposing A and B to be finitely generated subalgebras of FREE_n , does it follow that $A \cap B$ is finitely generated?
10. Does the amalgamation property hold for projective MV-algebras?
11. Develop the equational and the quasi-equational logic of $\mathbb{L}_{\omega\omega}$.

Comments to these problems

1. Already the effective recognizability of FREE_2 is an open problem. For FREE_1 , see Theorem 18.6. Also see Proposition 18.7 and Corollary 18.10. As shown in [92], the undecidability of the isomorphism problem of ℓ -groups follows from Markov's theorem on the unrecognizability of PL-homeomorphic rational polyhedra (see [93] and [94, pp. 143–144].)
2. By (A21.42), the problem is equivalent to giving a characterization of the spaces of prime ideals (also known as irreducible ℓ -ideals) of unital ℓ -groups. The paper [95] gives a characterization in terms of topological conditions, *plus* an algebraic condition. In the final section of [96], one finds an example of a completely normal spectral space that cannot arise as the prime spectrum of any MV-algebra. A characterization of the *poset* of prime ideals of MV-algebras is given in [97].
3. As shown in Theorems 8.9 and 8.10, locally finite MV-algebras are closed under subalgebras and homomorphic images, and also have products in the natural sense of category theory.
4. Any two bases of $A = \mathcal{M}(P)$ have a common algebraic subdivision iff any two regular triangulations Δ and ∇ of P have a common Farey subdivision. Passing to homogeneous coordinates this is equivalent to saying that the corresponding regular fans of Δ and ∇ have a common Farey subdivision. This is a reformulation in the language of fans, of the strong form of Oda's conjecture [98, p. 59], [99, p. 183].

5. As shown in Example 5.5, the conjecture fails for $n = 1$.
6. As we have seen in Theorem 17.16, this is true for every one-dimensional $P \subseteq [0, 1]^n$. Gluschkof [100], characterized *injective* MV-algebras as complete MV-algebras whose associated unital ℓ -group is divisible.
7. In [101] it is proved that if $\dim(P) \leq 1$, the automorphism group of $\mathcal{M}(P)$ is finite.
8. If the answer is positive then for any input pair (m, ψ) , ($m = 1, 2, \dots, \psi \in \bigcup_n \text{FORM}_n$) we can decide if ψ is a presentation of the MV-algebra FREE_m . The procedure is as follows: (a) Check if $m_\psi = m$. (b) If this is the case, then $\text{LIND}_\psi \cong \text{FREE}_m$ iff the m -dimensional rational measure of the rational polyhedron $\hat{\psi}^{-1}(1)$ is equal to 1. To see this, one may argue as in the proof of Proposition 18.11, upon noting that the rational polyhedron $\hat{\psi}^{-1}(1)$ is effectively computable (by algorithm *Mod* in Lemma 18.1), and the satisfiability of ψ is decidable (by Lemma 18.1). Trivially, $\text{FREE}_p \not\cong \text{FREE}_q$ for $p \neq q$, as can be immediately verified by counting the number of maximal ideals of rank 2.
9. The problem is already open for one-generator subalgebras of FREE_1 .
10. By Theorem 2.20, the amalgamation property holds for the whole class of MV-algebras and for finitely presented MV-algebras (also see Corollary 6.7). In [102] it is proved that totally ordered MV-algebras have the amalgamation property.
11. This might pave the way to the construction of interesting classes of enriched Hilbert spaces.

Just as the strong Oda conjecture is equivalent to asserting the joint refinability of MV-algebraic bases, similarly the P/NP problem is equivalent to asking whether satisfiable $\mathbb{L}_\infty$ -formulas are recognized by some polynomial time algorithm. One may hope that progress on Problems (1–11) above leads to substantial developments of MV-algebra theory and Łukasiewicz logic, casting new light on the complexity of the satisfiability problem for interesting classes of formulas of $\mathbb{L}_\infty$, and on the $\mathbb{Z}$ -homeomorphism problem for n -dimensional polyhedra with $n > 1$.

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Chapter 21

Background Results

21.1 Appendix A: Background Results on MV-Algebras

To help the reader, we collect here several results on MV-algebras that have found use in earlier chapters. All proofs are given in [1]. We will write for example [1.1.5] instead of [1, 1.1.5]

Throughout this chapter, A and B denote MV-algebras

21.1. ([1.1.5]) For all $x, y \in A$, $x \vee y = (x \odot \neg y) \oplus y = (x \ominus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y) = x \odot (\neg x \oplus y)$.

21.2. ([1.1.7]) For all $x, y \in A$,

$$(x \odot \neg y) \wedge (y \odot \neg x) = 0.$$

21.3. ([1.1.8]) For all $x, y \in A$ with $x \wedge y = 0$ we have

$$n \cdot x \wedge n \cdot y = 0$$

for all $n = 1, 2, \dots$ (Notation of (2.8).)

21.4. ([1.2.3]) Let $\eta : A \rightarrow B$ a homomorphism. Then

- (i) For each (always proper) ideal j of B , the set $\eta^{-1}(j) = \{x \in A \mid \eta(x) \in j\}$ is an ideal of A . Thus in particular, $\ker(\eta)$ is an ideal of A .
- (ii) $\eta(x) \leq \eta(y)$ iff $x \ominus y \in \ker(\eta)$.
- (iii) η is injective iff $\ker(\eta) = \{0\}$.
- (iv) $\ker(\eta)$ is prime iff the image $\eta(A)$ is an MV-chain.

21.5. ([1.2.6]) Let $\mathbf{i}$ be an ideal of an MV-algebra A . Then the binary relation $\sim_{\mathbf{i}}$ on A defined by $x \sim_{\mathbf{i}} y$ iff $\mathbf{d}(x, y) \in \mathbf{i}$ is a congruence relation other than A^2 , and $\mathbf{i} = \{x \in A \mid x \sim_{\mathbf{i}} 0\}$. Conversely, if $\sim$ is a congruence on A other than A^2 , then $\{x \in A \mid x \sim 0\}$ is an ideal, and $x \sim y$ iff $\mathbf{d}(x, y) \sim 0$. Therefore, the correspondence $\mathbf{i} \mapsto \sim_{\mathbf{i}}$ is a bijection from the set of ideals of A onto the set of congruences on A other than A^2 .

- 21.6 ([1.2.8]) Let A and B be MV-algebras and $\eta : A \rightarrow B$ a surjective homomorphism. Then there is an isomorphism $\theta : A/\ker(\eta) \cong B$ such that $\theta(x/\ker(\eta)) = \eta(x)$ for all $x \in A$.
- 21.7 ([1.2.10]) For every ideal j of A there is a canonical inclusion preserving one-one correspondence between the ideals of A containing j , and the ideals of A/j .
- 21.8. ([1.2.11])
- (i) Every ideal of A that contains a prime ideal is prime.
 - (ii) For each prime ideal j of A , the set of ideals of A containing j is totally ordered by inclusion.
- 21.9. ([1.2.12]) Every prime ideal of A is contained in a unique maximal ideal.
- 21.10. ([1.2.14]) Every ideal of an MV-algebra is an intersection of prime ideals.
- 21.11. ([1.2.15]) Every MV-algebra has a maximal ideal.
- 21.12. ([1.2.16]) Let A and B be MV-algebras and $\mathbf{m}$ be a maximal ideal of B .
- (i) For any homomorphism $\eta : A \rightarrow B$, the inverse image $\eta^{-1}(\mathbf{m})$ is a maximal ideal of A ;
 - (ii) For any subalgebra S of B , $S \cap \mathbf{m}$ is a maximal ideal of S .
- 21.13. ([1.3.3]) [Chang subdirect representation theorem] Every MV-algebra is a subdirect product of MV-chains.
- 21.14. ([1.4.7]) An MV-equation is satisfied by all MV-algebras iff it is satisfied by all MV-chains.
- 21.15. ([1.6.2(1.15)]) The following equation holds in every MV-algebra A :

$$(x \ominus y) \oplus ((x \oplus \neg y) \odot y) = x.$$

- 21.16. ([2.1.3]) Let (G, u) be a unital ℓ -group and $A = \Gamma(G, u)$.
- (i) For all $a, b \in A$, $a + b = (a \oplus b) + (a \odot b)$.
 - (ii) For all $x_1, \dots, x_n \in A$, $x_1 \oplus \dots \oplus x_n = u \wedge (x_1 + \dots + x_n)$.
 - (iii) The natural order of the MV-algebra A coincides with the order of $[0, u]$ inherited from G by restriction.
- 21.17. ([2.5.3]) [Chang Completeness Theorem] An equation holds in $[0, 1]$ iff it holds in every MV-algebra.
- 21.18. ([3.1.9]) Let $g : [0, 1]^n \rightarrow \mathbb{R}$ be a linear polynomial with integer coefficients,

$$g(x) = m_0x_0 + \dots + m_{n-1}x_{n-1} + b, (m_0, \dots, m_{n-1}, b \in \mathbb{Z}).$$

Then $(g \vee 0) \wedge 1 \in \mathcal{M}([0, 1]^n)$.

- 21.19 ([3.4.2]) Let $X \neq \emptyset$ be a compact Hausdorff space, and B a subalgebra of the MV-algebra $C(X)$. The map $j \mapsto \mathcal{Z}j$ is an inclusion reversing map from the set of ideals of B into the family of nonempty closed subsets of X .
- 21.20. ([3.4.3]) Let X be a compact Hausdorff space and A a separating subalgebra of $C(X)$.

- (i) The map $x \mapsto h_x$ is a one-one correspondence between X and $\mu(A)$;
 - (ii) For each closed set $S \subseteq X$, $\mathcal{Z}h_S = S$;
 - (iii) For each ideal j of A , $h(\mathcal{Z}(j))$ is the intersection of all maximal ideals in A containing j .
- 21.21. ([3.4.5]) Let X be a nonempty compact Hausdorff space and A a separating subalgebra of $C(X)$. For each ideal j of A the map $f/j \mapsto f \upharpoonright \mathcal{Z}j$ is an isomorphism from A/j onto $A \upharpoonright \mathcal{Z}j$ iff j is an intersection of maximal ideals of A .
- 21.22. ([3.4.6]) For each cardinal $\kappa \geq 1$, the MV-algebra $\mathbf{FREE}_\kappa$ is a separating subalgebra of $C([0, 1]^\kappa)$.
- 21.23. ([3.4.7]) The map $x \mapsto h_x$ is a one-one correspondence between $[0, 1]^\kappa$ and $\mu(\mathbf{FREE}_\kappa)$. The inverse is given by $j \in \mu(\mathbf{FREE}_\kappa) \mapsto$ the only point of $\mathcal{Z}j$.
- 21.24. ([3.4.8]) Let $f, g \in \mathbf{FREE}_\kappa$. Then $g \in \langle f \rangle \Leftrightarrow \mathcal{Z}g \supseteq \mathcal{Z}f$.
- 21.25. ([3.4.9]) Each principal ideal of $\mathbf{FREE}_\kappa$ is an intersection of maximal ideals.
- 21.26. ([3.5.1]) For every MV-algebra A the following conditions are equivalent:
- (i) A is simple.
 - (ii) for every nonzero element $x \in A$ there is an integer $n > 0$ such that $1 = x \oplus \cdots \oplus x$ (n times).
 - (iii) A is isomorphic to a subalgebra of $[0, 1]$.
- 21.27. ([3.5.3]) Let A be a subalgebra of $[0, 1]$, and $a = \inf\{x \in A \mid x > 0\}$. If $a = 0$ then A is a dense subchain of $[0, 1]$. If $a > 0$ then A coincides with the n -element MV-chain, for a uniquely determined $n \geq 2$.
- 21.28. ([3.5.4]) An MV-algebra is finite and simple iff it is isomorphic to a finite MV-chain.
- 21.29. ([3.6.4]) An element $a \in A$ belongs to the radical of A iff $a = 0$ or a is infinitesimal.
- 21.30. ([3.6.5]) A is finite iff it is a finite product of finite chains.
- 21.31. ([3.6.6]) Let j be an ideal of A . Then A/j is semisimple iff j is an intersection of maximal ideals of A .
- 21.32. ([3.6.7]) Let $\kappa \geq 1$ be a cardinal. An MV-algebra A with κ generators is semisimple iff for some nonempty closed subset $X \subseteq [0, 1]^\kappa$, A is isomorphic to the MV-algebra of restrictions to X of all functions in $\mathcal{M}([0, 1]^\kappa)$.
- 21.33. ([3.6.8]) A is semisimple iff it is isomorphic to a separating MV-algebra of $[0, 1]$ -valued continuous functions on some nonempty compact Hausdorff space.
- 21.34. ([4.5.3]) The tautology problem in the infinite-valued calculus of Łukasiewicz is decidable.
- 21.35. ([6.2.4]) For every element $x \in A$, the following conditions are equivalent:
- (i) There is an integer $n \geq 1$ such that $\neg x \vee n \cdot x = 1$;
 - (ii) There is an integer $n \geq 1$ such that $n \cdot x = (n + 1) \cdot x$;
 - (iii) There is an integer $n \geq 1$ such that $n \cdot x$ belongs to the center $B(A)$.

21.36. ([6.3.2]) For any MV-algebra A the following conditions are equivalent:

- (i) A is hyperarchimedean.
- (ii) Every prime ideal of A is maximal.
- (iii) Every ideal of A is an intersection of the maximal ideals containing it.
- (iv) For every ideal j of A , A/j is semisimple.

21.37. ([6.3.5]) Let $X \neq \emptyset$ be a compact Hausdorff space. Then the MV-algebra $C(X)$ has a hyperarchimedean separating subalgebra A iff X is a boolean space. In this case, X is homeomorphic to $\mu(B(A))$.

21.38. ([6.6.2]) Every σ -complete MV-algebra A (whence, a fortiori, every complete MV-algebra) is semisimple.

21.39. ([6.6.4]) Let A be a complete MV-algebra. Let $\{x_i \mid i \in I\} \subseteq A$. Then for each $x \in A$, the following generalized distributive laws hold:

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i) \quad \text{and} \quad x \vee \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (x \vee x_i).$$

21.40. ([6.6.5]) If A is a complete MV-algebra then $B(A)$ is a complete boolean algebra, and for every set $\{b_i \mid i \in I\} \subseteq B(A)$ we have

$$\bigvee_{i \in I} b_i \in B(A) \quad \text{and} \quad \bigwedge_{i \in I} b_i \in B(A).$$

21.41. ([7.1.8]) The Γ functor is a natural equivalence between the category of unital ℓ -groups and the category of MV-algebras.

21.42. ([7.2.2]) Let (G, u) be a unital ℓ -group and $A = \Gamma(G, u)$. Then the map ϕ given by $\phi: j \mapsto \phi(j) = \{x \in G \mid |x| \wedge u \in j\}$ is an order-isomorphism from the set of ideals of A , ordered by inclusion, onto the set of ℓ -ideals of G . The inverse map is given by $i \mapsto i \cap [0, u]$.

21.43. ([7.2.3]) The map $j \mapsto j \cap [0, u]$ defines an isomorphism between the partially ordered set of prime ℓ -ideals of G and the partially ordered set of prime ideals of $\Gamma(G, u)$, both sets being equipped with the inclusion ordering.

21.44. ([7.2.4]) Let (G, u) be a unital ℓ -group. Then for every ℓ -ideal j of G , $\Gamma(G/j, u/j) \cong \Gamma(G, u)/(j \cap [0, u])$.

21.45. ([7.2.6]) Two MV-subalgebras of $[0, 1]$ are isomorphic iff they are equal. The identity map is the only automorphism of any subalgebra of $[0, 1]$.

21.46. ([7.2.7]) There are uncountably many nonisomorphic simple subalgebras of $[0, 1]$ with one generator.

21.47. ([9.1.4]) Let Δ be a regular triangulation of $[0, 1]^n$, v a vertex in Δ , and h_v its Schauder hat.

- (i) If $S \in \Delta$ has v among its vertices then there are integers $a_0, \dots, a_{n-1}, b$ such that, for all $x = (x_0, \dots, x_{n-1}) \in S$,

$$h_v(x_0, \dots, x_{n-1}) = a_0 x_0 + \dots + a_{n-1} x_{n-1} + b.$$

- (ii) h_v belongs to FREE_n .

- 21.48. ([9.1.5]) [McNaughton theorem] For each cardinal $\kappa \geq 1$, FREE_κ is isomorphic to the MV-algebra of McNaughton functions over $[0, 1]^\kappa$ with pointwise operations.
- 21.49. ([9.3.8]) The tautology problem for $\mathbb{L}_\infty$ is coNP-complete.

21.2 Appendix B: Miscellaneous Results

We collect here several results used in the course of the book, together with references for their proofs.

21.2.1 Polyhedral Geometry and Topology

- 21.50. The convex hull of a compact subset of $\mathbb{R}^n$ is compact [2, I 2.8].
- 21.51. Any two compact convex disjoint polyhedra in $\mathbb{R}^n$ are strongly separated by a hyperplane [3, I 3.5], [2, p. 29].
- 21.52. Every polyhedron P is the support of a simplicial complex $\mathcal{C}$. If P is rational then $\mathcal{C}$ can be assumed to be rational [4, 2.9–2.11], [5, p. 14].
- 21.53. Every complex can be subdivided into a triangulation without adding new vertices [3, III 2.6].
- 21.54. Every convex polyhedron P can be equivalently represented (i) as the convex hull of its vertices, and (ii) as the intersection of its finitely many *supporting half-spaces*, i.e., half-spaces bounded by supporting hyperplanes [3, II 1.2–1.5], [2, II 9.2].
- 21.55. [Carathéodory Theorem] For every $X \subseteq \mathbb{R}^n$ with $\dim(\text{aff}(X)) = m$, the convex hull $\text{conv}(X)$ is the set of all convex combinations of at most $m + 1$ points of X [2, I 2.4], [3, I 2.3].
- 21.56. The half-open parallelepiped determined by the primitive generating vectors of a non-regular cone contains a non-zero integer point [6, Theorems 446, 447].
- 21.57. Let Σ be a regular fan whose support is contained in $\mathbb{R}^n$. Then there is a regular fan Σ' with support coinciding with $\mathbb{R}^n$ such that Σ is a subset of Σ' . In other words, all cones of Σ are left unchanged in the extension Σ' [3, III 2.8, VI 9.3(b)].
- 21.58. [Solution of weak Oda Conjecture] Any two regular triangulations having the same support are connected by a path of Farey blow-ups and blow-downs [7, 13.3], [8].
- 21.59. Let Δ be a regular triangulation and ∇ a subset of Δ which is a simplicial complex. Then the first barycentric subdivision ∇' of ∇ is *full* in the first barycentric subdivision Δ' of Δ , in the sense that whenever S is a simplex of Δ' having all its vertices in ∇' , then S is a simplex of ∇' [9, Remark, p. 55], [4, 3.3].

21.2.2 Measure Theory, Function Spaces and Functional Analysis

In what follows, for X a compact Hausdorff space, $\mathcal{C}(X)$ denotes the Banach algebra of all continuous real valued functions on X , with the sup norm.

- 21.60. [Baire category theorem] Let f_n be a sequence of real-valued continuous functions defined on a topological space X , and such that the finite limit

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

exists at each $x \in X$. Then the set of points at which f is discontinuous is meager [10, pp. 12–13].

- 21.61. [Lusin theorem]. Suppose μ is a regular Borel measure on a compact Hausdorff space X and f is a Borel measurable function on X . Then for every $\epsilon > 0$ there exists a compact subset Y of X such that f is continuous on Y and $\mu(X \setminus Y) < \epsilon$ [11, 2.23].

- 21.62. [Corollary of Lusin theorem]. If the distance between two real-valued continuous functions f and g on $[0, 1]^n$ is defined to be

$$\int_{[0,1]^n} |f(x) - g(x)| dx,$$

then the completion of the resulting metric space consists precisely of the Lebesgue integrable functions on $[0, 1]^n$, provided we identify any two that are equal almost everywhere [11, 68–69].

- 21.63. [Stone–Weierstrass Theorem] If X is a compact Hausdorff space and S is a separating subset of $\mathcal{C}(X)$ containing all the rational constant functions, closed under addition, multiplication by rational scalars, and pointwise max, then S is norm dense in $\mathcal{C}(X)$ [12, II 7.29], [13, p. 31].
- 21.64. [Riesz Representation Theorem] For any compact Hausdorff space X and regular Borel probability measure μ , let F_μ be defined by

$$F_\mu(f) = \int f d\mu.$$

Then the map $\mu \mapsto F_\mu$ is an affine isomorphism of the convex set of regular Borel probability measures on X and the convex set of normalized positive linear functionals on $\mathcal{C}(X)$ [12, 12.36–12.39], [14, C 18].

- 21.65. Let X be a compact Hausdorff space in which every open set is a countable union of compact sets. Then every Borel probability measure on X is regular [11, 2.18], [12, III 12.55].
- 21.66. [Krein–Milman Theorem] For any compact convex subset of a locally convex space X , $X = \text{cl conv ext } X$ [14, V 7.4].
- 21.67. [Cauchy–Bunyakowsky–Schwarz inequality] For any two elements x, y of a Hilbert space H we have the inequality $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$. The expression $\|x\| = \langle x, x \rangle^{1/2}$ defines a norm on H [14, 1.4–1.5].

21.2.3 Boolean Algebras, ℓ -Groups and Vector Lattices

- 21.68. Up to homeomorphism, nonempty basically disconnected compact Hausdorff spaces coincide with the Stone spaces of σ -complete boolean algebras [15, p. 73].
- 21.69. [Riesz decomposition] Suppose $0 \leq x_1, \dots, x_p$ and $0 \leq y_1, \dots, y_q$ are elements in an ℓ -group G and $\sum_{i=1}^p x_i = \sum_{j=1}^q y_j$. Then there is a double sequence

$$0 \leq z_{ij}, \quad 1 \leq i \leq p, \quad 1 \leq j \leq q$$

of elements of G such that $x_i = \sum_{j=1}^q z_{ij}$ for all i , and $y_j = \sum_{i=1}^p z_{ij}$ for all j [16, 1.2.16].

- 21.70. [The divisible hull] Any lattice-ordered abelian group G is embeddable into a rational vector lattice V in such a way that for all $v \in V$ there is an integer $n \geq 0$ such that $nv \in G$, and $G^+ = V^+ \cap G$. V is unique up to ℓ -isomorphism, and is called the *divisible hull* of G [16, 1.6.8–1.6.9].
- 21.71. [Hölder–Hion Theorem] For every unital ℓ -group (G, u) and maximal ideal $\mathfrak{m} \in \mu(G)$ there is exactly one pair $(\iota_{\mathfrak{m}}, R_{\mathfrak{m}})$ where $R_{\mathfrak{m}}$ is a unital ℓ -subgroup of $(\mathbb{R}, 1)$, and $\iota_{\mathfrak{m}}$ is a unital ℓ -isomorphism of the quotient $(G, u)/\mathfrak{m}$ onto $R_{\mathfrak{m}}$ [17, p. 45–47], [16, 2.6].
- 21.72. [Corollary of the Hölder–Hion Theorem] Let G and H be ℓ -subgroups of the additive group $\mathbb{R}$ of real numbers with natural order. Assume $1 \in G \cap H$. Then there is at most one ℓ -isomorphism f of G onto H such that $f(1) = 1$. Whenever such f exists, then G necessarily coincides with H , and f is the identity function on G [17, p. 45–47].
- 21.73. [Jordan decomposition] In every abelian ℓ -group G , (and hence, in every vector lattice), upon defining for all $a \in G$, $a^+ = 0 \vee a$ and $a^- = 0 \vee -a$, it follows that
- (i) $a = a^+ - a^-$ and $a^+ \wedge a^- = 0$.
 - (ii) The two equations $a = y - z$ and $y \wedge z = 0$ are simultaneously satisfied iff $y = a^+$ and $z = a^-$ [16, 1.3.3, 1.3.4].

21.2.4 Algebraic Topology

- 21.74. For any rational polyhedron $P \subseteq [0, 1]^n$ the following conditions are equivalent:
- (α) P is *contractible*, i.e., the identity map on P is nullhomotopic.
 - (β) P is *n -connected*, i.e., the homotopy group $\pi_i(P)$ is trivial for each $i = 0, \dots, n$.
 - (γ) P is a deformation retract of $[0, 1]^n$.
 - (δ) P is a retract of $[0, 1]^n$.

Proof $(\alpha) \Rightarrow (\beta)$ [18, p. 405].

$(\beta) \Rightarrow (\alpha)$ [19, p. 359].

$(\alpha) \Rightarrow (\gamma)$ is a consequence of Whitehead theorem [19, p. 346].

$(\gamma) \Rightarrow (\delta)$, trivial.

$(\delta) \Rightarrow (\alpha)$, because a retract of a contractible space (like $[0, 1]^n$) is contractible [19, 1.17]. $\square$

21.75. A one-dimensional polyhedron P is contractible iff it is connected and simply connected, i.e., P is a tree. [*Proof* By (B21.74) $\alpha \Leftrightarrow \beta$.]

21.76. Let $T \subseteq \mathbb{R}^n$ be a k -simplex and U a nonempty set in the relative interior of T . Then T and $T \setminus U$ are not homeomorphic.

Proof It is sufficient to prove that the singular homology groups of $T \setminus U$ and T are not isomorphic (we refer to [20] or [21] for notation and terminology). To this purpose let us first note that $H_{k-1}(T)$ is the trivial group $\{0\}$, because T is contractible. There remains to be proved that $H_{k-1}(T \setminus U)$ is different from $\{0\}$. As a matter of fact, let us consider the inclusion maps

$$S^{k-1} \rightarrow T \setminus U \rightarrow \mathbb{R}^n \setminus \{z\},$$

where z is any point in the nonempty set U . These continuous maps induce homomorphisms of homology groups

$$\mathbb{Z} = H_{k-1}(S^{k-1}) \rightarrow H_{k-1}(T \setminus U) \rightarrow H_{k-1}(\mathbb{R}^n \setminus \{z\}) = \mathbb{Z}. \quad (21.1)$$

Moreover, the composition of these two homomorphisms is the homomorphism induced by the inclusion

$$S^{k-1} \rightarrow \mathbb{R}^n \setminus \{z\}.$$

Because the latter map is a homotopy equivalence, it induces an isomorphism on homology groups. Thus, the above sequence (21.1) shows that $H_{k-1}(T \setminus U)$ contains $\mathbb{Z}$ as a summand, whence it cannot be equal to $\{0\}$. $\square$

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