

Grade 11 Unit 3 CONTRIBUTERS

Entrance Hub Team Experts

Coordinate Geometry

Online Educational Centers

Ellipse

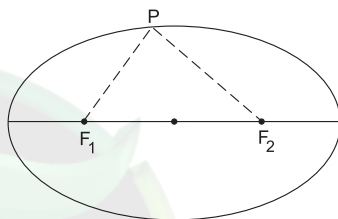
ELLIPSE It is the path traced by a point which moves in a plane in such a way that the sum of its distance from two fixed points in the plane is a constant.

The two fixed points are called the *foci* of the ellipse.

NOTE The plural of focus is foci.

In the given figure, F_1 and F_2 are two fixed points and P is a point which moves in such a way that $PF_1 + PF_2 = \text{constant}$.

The path traced by the point P is called an *ellipse*, and the points F_1 and F_2 are called its *foci*.

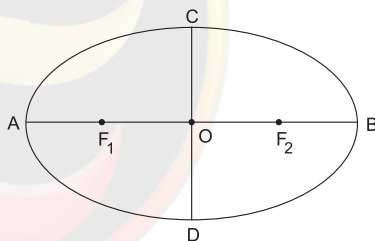


SOME MORE TERMS RELATED TO AN ELLIPSE

(I) CENTRE OF THE ELLIPSE

The midpoint of the line segment joining the foci, is called the *centre* of the ellipse.

In the given figure, F_1 and F_2 are the foci of the ellipse and O is its *centre*, where $OF_1 = OF_2$.



(II) AXES OF THE ELLIPSE

MAJOR AXIS: The line segment through the foci of the ellipse with its end points on the ellipse, is called its *major axis*.

In the given figure, AB is the major axis of the ellipse.

MINOR AXIS: The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its *minor axis*.

In the given figure, CD is the minor axis of the ellipse.

(III) VERTICES OF AN ELLIPSE

The end points of the major axis of an ellipse are called its *vertices*.

In the given figure, A and B are the vertices of the ellipse.

AN IMPORTANT NOTE In an ellipse, we take:

$$\text{Length of the major axis} = AB = 2a.$$

$$\text{Length of the minor axis} = CD = 2b.$$

Distance between the foci = $F_1F_2 = 2c$.

Length of the semi-major axis = a .

Length of the semi-minor axis = b .

(IV) ECCENTRICITY OF AN ELLIPSE

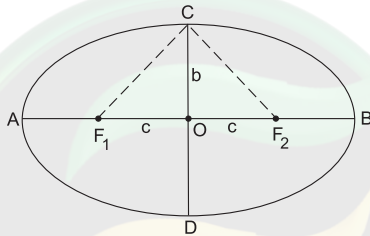
The ratio $\frac{c}{a}$ is always constant and it is denoted by e , called the *eccentricity* of the ellipse.

For an ellipse, we have $0 < e < 1$ $\left[\because c < a \Leftrightarrow e = \frac{c}{a} < 1 \right]$.

AN IMPORTANT RESULT

In the given ellipse, it is being given that $AB = 2a$, $CD = 2b$ and $F_1F_2 = 2c$.

Prove that: $(a^2 - c^2) = b^2$.



PROOF Clearly, COD is the perpendicular bisector of F_1F_2 .

Join F_1C and F_2C .

Now, $F_1O = c$, $OC = b \Rightarrow F_1C = \sqrt{c^2 + b^2}$.

And, $OF_2 = c$, $OC = b \Rightarrow F_2C = \sqrt{c^2 + b^2}$.

Since B and C both lie on the given ellipse, we have

$F_1B + F_2B = F_1C + F_2C = \text{constant}$ [by the definition of an ellipse]

$\Rightarrow (F_1O + OB) + (OB - OF_2) = F_1C + F_2C$

$\Rightarrow (c + a) + (a - c) = 2\sqrt{c^2 + b^2}$

[$\because F_1O = c$, $OF_2 = c$, $OB = a$ and $F_1C = F_2C = \sqrt{c^2 + b^2}$]

$\Rightarrow a = \sqrt{c^2 + b^2} \Rightarrow a^2 = (c^2 + b^2) \Rightarrow (a^2 - c^2) = b^2$.

Hence, $(a^2 - c^2) = b^2$.

STANDARD EQUATION OF AN ELLIPSE

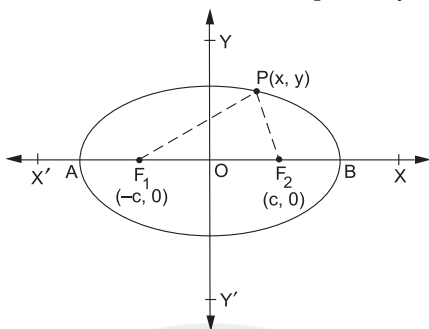
THEOREM Prove that the standard equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are the lengths of the semi-major axis and the semi-minor axis respectively and $a > b$.

PROOF Let $X'OX$ and YOY' be the coordinate axes.

Let us consider an ellipse in which a and b are the lengths of the semi-major axis and semi-minor axis respectively.



Choose a real number c such that $c^2 = (a^2 - b^2)$.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the two fixed points and let $P(x, y)$ be an arbitrary point on the given ellipse, moving on it in such a way that $PF_1 + PF_2 = 2a$. Then,

$$\begin{aligned}
 & PF_1 + PF_2 = 2a \\
 \Rightarrow & \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \\
 \Rightarrow & \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} \\
 \Rightarrow & (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2} \quad [\text{on squaring both sides}] \\
 \Rightarrow & (x+c)^2 - (x-c)^2 - 4a^2 = -4a\sqrt{(x-c)^2 + y^2} \\
 \Rightarrow & 4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2} \\
 \Rightarrow & \sqrt{(x-c)^2 + y^2} = a - \frac{cx}{a} \quad [\text{on dividing both sides by } -4a] \\
 \Rightarrow & (x-c)^2 + y^2 = a^2 - 2cx + \frac{c^2x^2}{a^2} \quad [\text{on squaring both sides}] \\
 \Rightarrow & x^2 - \frac{c^2x^2}{a^2} + y^2 = (a^2 - c^2) \\
 \Rightarrow & x^2 \left(1 - \frac{c^2}{a^2} \right) + y^2 = (a^2 - c^2) \\
 \Rightarrow & \frac{x^2(a^2 - c^2)}{a^2} + y^2 = (a^2 - c^2) \\
 \Rightarrow & \frac{x^2}{a^2} + \frac{y^2}{(a^2 - c^2)} = 1 \quad [\text{on dividing both sides by } (a^2 - c^2)] \\
 \Rightarrow & \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad [\because (a^2 - c^2) = b^2].
 \end{aligned}$$

Thus, every point on the ellipse satisfies the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Conversely, let the equation of the given curve be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let $P(x, y)$ be an arbitrary point on this curve.

$$\begin{aligned} \text{Then, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 &\Rightarrow y^2 = b^2 \left(1 - \frac{x^2}{a^2} \right) \\ &\Rightarrow y^2 = \frac{b^2(a^2 - x^2)}{a^2}. \end{aligned} \quad \dots (i)$$

Also, let $(a^2 - b^2) = c^2$ (ii)

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be two fixed points on the x -axis. Then,

$$\begin{aligned} PF_1 &= \sqrt{(x + c)^2 + y^2} \\ &= \sqrt{(x + c)^2 + \frac{b^2(a^2 - x^2)}{a^2}} && \text{[using (i)]} \\ &= \sqrt{(x + c)^2 + \frac{(a^2 - c^2)(a^2 - x^2)}{a^2}} && \text{[using (ii)]} \\ &= \sqrt{a^2 + 2cx + \frac{c^2 x^2}{a^2}} = \sqrt{\left(a + \frac{cx}{a}\right)^2} = \left(a + \frac{cx}{a}\right) \end{aligned}$$

$$\text{Similarly, } PF_2 = \left(a - \frac{cx}{a}\right).$$

$$\therefore PF_1 + PF_2 = \left(a + \frac{cx}{a} + a - \frac{cx}{a}\right)$$

$$\Leftrightarrow PF_1 + PF_2 = 2a.$$

This shows that the given curve is an ellipse.

Hence, the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

AN IMPORTANT NOTE The foci of an ellipse always lie on the major axis.

HORIZONTAL ELLIPSE

In the given equation of an ellipse, if the coefficient of x^2 has the larger denominator then its major axis lies along the x -axis.

Such an ellipse is called a *horizontal ellipse*.

Thus, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an horizontal ellipse, if $a^2 > b^2$.

EXAMPLE $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is an horizontal ellipse.

LATUS RECTUM OF A HORIZONTAL ELLIPSE

The *latus rectum* of an ellipse is a line segment perpendicular to the major axis, passing through any of the foci with end points lying on the ellipse.



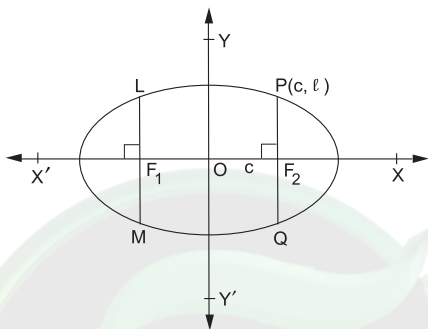
To Find the Length of Latus Rectum of the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$.

Let $X'OX$ and YOY' be the coordinate axes.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci of the given ellipse, where $c^2 = (a^2 - b^2)$.

Let LF_1M and PF_2Q be the latus rectums.

Let $PF_2 = l$. Then, the coordinates of P are $P(c, l)$.



Since $P(c, l)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$\begin{aligned} \frac{c^2}{a^2} + \frac{l^2}{b^2} &= 1 \Rightarrow \frac{l^2}{b^2} = \left(1 - \frac{c^2}{a^2}\right) \\ \Rightarrow l^2 &= b^2 \left(1 - \frac{c^2}{a^2}\right) = b^2 \left[1 - \frac{(a^2 - b^2)}{a^2}\right] \quad [\because c^2 = (a^2 - b^2)] \\ \Rightarrow l^2 &= \frac{b^4}{a^2} \Rightarrow l = \frac{b^2}{a} \end{aligned}$$

$$\therefore PF_2 = l = \frac{b^2}{a} \Leftrightarrow PF_2Q = 2l = \frac{2b^2}{a}.$$

$$\text{Similarly, } LF_1M = 2l = \frac{2b^2}{a}.$$

Hence, the length of the latus rectum is $\frac{2b^2}{a}$.

RESULTS ON HORIZONTAL ELLIPSE

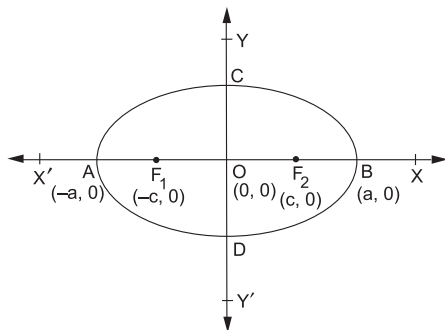
(i) The standard form of equation of a horizontal ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < b < a.$$

(ii) Its centre is $O(0, 0)$.

(iii) Its vertices are $A(-a, 0)$ and $B(a, 0)$.

(iv) Its foci are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c^2 = (a^2 - b^2)$.

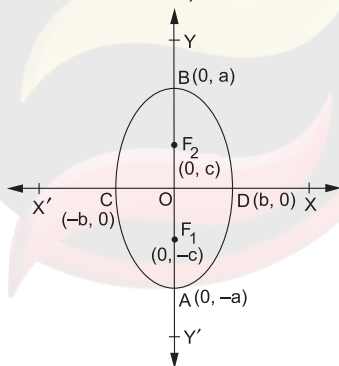


- (v) Length of the major axis, $AB = 2a$ and length of the minor axis, $CD = 2b$.
- (vi) Equation of the major axis is $y = 0$ and that of the minor axis is $x = 0$.
- (vii) Length of the latus rectum $= \frac{2b^2}{a}$.
- (viii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$.

VERTICAL ELLIPSE

In the given equation of an ellipse, if the coefficient of x^2 has the smaller denominator, then its major axis lies along the y -axis.

Such an ellipse is called a *vertical ellipse*.



RESULTS ON VERTICAL ELLIPSE

- (i) The standard form of the equation of a vertical ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $0 < b < a$.
- (ii) Its centre is $O(0, 0)$.
- (iii) Its vertices are $A(0, -a)$ and $B(0, a)$.
- (iv) Its foci are $F_1(0, -c)$ and $F_2(0, c)$, where $a^2 - b^2 = c^2$,
i.e., $F_1(0, -ae)$ and $F_2(0, ae)$.
- (v) Length of the major axis $= 2a$ and length of the minor axis $= 2b$.

(vi) Equation of the major axis is $x = 0$ and that of the minor axis is $y = 0$.

(vii) Length of latus rectum = $\frac{2b^2}{a}$.

(viii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$.

SUMMARY		
Properties	Horizontal Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $0 < b < a$ and $c^2 = (a^2 - b^2)$	Vertical Ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $0 < b < a$ and $c^2 = (a^2 - b^2)$
(i) Centre	(0, 0)	(0, 0)
(ii) Vertices	$A(-a, 0)$, $B(a, 0)$	$A(0, -a)$ and $B(0, a)$
(iii) Foci	$F_1(-c, 0)$ and $F_2(c, 0)$ or $(-ae, 0)$ and $(ae, 0)$	$F_1(0, -c)$ and $F_2(0, c)$ or $(0, -ae)$ and $(0, ae)$
(iv) Length of the major axis	$2a$	$2a$
(v) Length of the minor axis	$2b$	$2b$
(vi) Equation of the major axis	$y = 0$	$x = 0$
(vii) Equation of the minor axis	$x = 0$	$y = 0$
(viii) Length of the latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
(ix) Eccentricity	$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$	$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$

SOLVED EXAMPLES

EXAMPLE 1 Find the lengths of the major and minor axes; coordinates of the vertices and the foci, the eccentricity and length of the latus rectum of the ellipse:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

SOLUTION Given equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse*.

Now, ($a^2 = 16$ and $b^2 = 9 \Rightarrow (a = 4$ and $b = 3)$).

$$\therefore c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}.$$

Thus, $a = 4$, $b = 3$ and $c = \sqrt{7}$.

- (i) Length of the major axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the minor axis $= 2b = (2 \times 3)$ units $= 6$ units.
- (ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.
- (iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-\sqrt{7}, 0)$ and $F_2(\sqrt{7}, 0)$.
- (iv) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{7}}{4}$.
- (v) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 9)}{3}$ units $= \frac{9}{2}$ units.

EXAMPLE 2 Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:

$$4x^2 + 9y^2 = 144$$

SOLUTION The given equation may be written as

$$\frac{x^2}{36} + \frac{y^2}{16} = 1.$$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse*.

Now, $(a^2 = 36 \text{ and } b^2 = 16) \Rightarrow (a = 6 \text{ and } b = 4)$.

$$\therefore c = \sqrt{a^2 - b^2} = \sqrt{36 - 16} = \sqrt{20} = 2\sqrt{5}.$$

Thus, $a = 6$, $b = 4$ and $c = 2\sqrt{5}$.

- (i) Length of the major axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the minor axis $= 2b = (2 \times 4)$ units $= 8$ units.
- (ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.
- (iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-2\sqrt{5}, 0)$ and $F_2(2\sqrt{5}, 0)$.
- (iv) Eccentricity, $e = \frac{c}{a} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$.
- (v) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 16)}{6}$ units $= \frac{16}{3}$ units.

EXAMPLE 3 Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:

$$\frac{x^2}{4} + \frac{y^2}{36} = 1.$$

SOLUTION Given equation is $\frac{x^2}{4} + \frac{y^2}{36} = 1$.

This is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *vertical ellipse*.

Now, $(b^2 = 4 \text{ and } a^2 = 36) \Rightarrow (b = 2 \text{ and } a = 6)$.

$$\therefore c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = \sqrt{32} = 4\sqrt{2}.$$

Thus, $a = 6$, $b = 2$ and $c = 4\sqrt{2}$.

- (i) Length of the major axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the minor axis $= 2b = (2 \times 2)$ units $= 4$ units.
- (ii) Coordinates of its vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -6)$ and $B(0, 6)$.
- (iii) Coordinates of its foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$.
- (iv) Eccentricity, $e = \frac{c}{a} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$.
- (v) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 4)}{6}$ units $= \frac{4}{3}$ units.

EXAMPLE 4 Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:

$$4x^2 + y^2 = 100.$$

SOLUTION The given equation of the ellipse may be written as

$$\frac{x^2}{25} + \frac{y^2}{100} = 1.$$

This is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *vertical ellipse*.

Now, $(b^2 = 25 \text{ and } a^2 = 100) \Rightarrow (b = 5 \text{ and } a = 10)$.

$$\therefore c = \sqrt{a^2 - b^2} = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}.$$

Thus, $a = 10$, $b = 5$ and $c = 5\sqrt{3}$.

- (i) Length of the major axis $= 2a = (2 \times 10)$ units $= 20$ units.
Length of the minor axis $= 2b = (2 \times 5)$ units $= 10$ units.
- (ii) Coordinates of the vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -10)$ and $B(0, 10)$.
- (iii) Coordinates of the foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -5\sqrt{3})$ and $F_2(0, 5\sqrt{3})$.
- (iv) Eccentricity, $e = \frac{c}{a} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$.
- (v) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 25)}{10}$ units $= 5$ units.

EXAMPLE 5 Find the equation of an ellipse whose vertices are at $(\pm 5, 0)$ and foci at $(\pm 4, 0)$.

SOLUTION Since the vertices of the given ellipse are on the x -axis, so it is a *horizontal ellipse*.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Its vertices are $(\pm a, 0)$ and therefore, $a = 5$.

Its foci are $(\pm c, 0)$ and therefore, $c = 4$.

$$\therefore c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (25 - 16) = 9.$$

Thus, $a^2 = 5^2 = 25$ and $b^2 = 9$.

Hence, the required equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

EXAMPLE 6 Find the equation of an ellipse whose foci are $(\pm 4, 0)$ and the eccentricity is $\frac{1}{3}$.

SOLUTION Since the foci of the given ellipse are on the x -axis, so it is a *horizontal ellipse*.

Let the required equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a^2 > b^2.$$

Let its foci be $(\pm c, 0)$. Then, $c = 4$.

$$\text{Also, } e = \frac{c}{a} \Leftrightarrow a = \frac{c}{e} = \frac{4}{(1/3)} = 12.$$

$$\text{Now, } c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (144 - 16) = 128.$$

$$\therefore a^2 = (12)^2 = 144 \text{ and } b^2 = 128.$$

Hence, the required equation is $\frac{x^2}{144} + \frac{y^2}{128} = 1$.

EXAMPLE 7 Find the equation of an ellipse whose major axis lies on the x -axis and which passes through the points $(4, 3)$ and $(6, 2)$.

SOLUTION Since the major axis of the ellipse lies on the x -axis, so it is a *horizontal ellipse*.

Let the required equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a^2 > b^2). \quad \dots (i)$$

$$\text{Since } (4, 3) \text{ lies on (i), we have } \frac{16}{a^2} + \frac{9}{b^2} = 1. \quad \dots (ii)$$

$$\text{Also, since } (6, 2) \text{ lies on (i), we have } \frac{36}{a^2} + \frac{4}{b^2} = 1. \quad \dots (iii)$$

Putting $\frac{1}{a^2} = u$ and $\frac{1}{b^2} = v$ in (ii) and (iii), we get

$$16u + 9v = 1 \quad \dots (iv)$$

$$36u + 4v = 1 \quad \dots (v)$$

On multiplying (iv) by 9 and (v) by 4, and subtracting, we get

$$65v = 5 \Leftrightarrow v = \frac{1}{13} \Leftrightarrow \frac{1}{b^2} = \frac{1}{13} \Leftrightarrow b^2 = 13.$$

Putting $v = \frac{1}{13}$ in (iv), we get

$$\begin{aligned} 16u &= \left(1 - \frac{9}{13}\right) \Leftrightarrow 16u = \frac{4}{13} \Leftrightarrow u = \left(\frac{4}{13} \times \frac{1}{16}\right) = \frac{1}{52} \\ &\Leftrightarrow \frac{1}{a^2} = \frac{1}{52} \Leftrightarrow a^2 = 52. \end{aligned}$$

Thus, $a^2 = 52$ and $b^2 = 13$.

Hence, the required equation is $\frac{x^2}{52} + \frac{y^2}{13} = 1$.

EXAMPLE 8 Find the equation of the ellipse, the ends of whose major axis are $(\pm 3, 0)$ and the ends of whose minor axis are $(0, \pm 2)$.

SOLUTION Clearly, the major axis of the given ellipse lies on the x -axis. So, it is a horizontal ellipse.

Let the required equation be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Its vertices are $(\pm a, 0)$ and, therefore, $a = 3$

[$\because$ ends of the major axis are the vertices].

Ends of the minor axis are $C(0, -2)$ and $D(0, 2)$.

$\therefore CD = 4$, i.e., length of the minor axis = 4 units.

$\therefore 2b = 4 \Leftrightarrow b = 2$

Now, $(a = 3 \text{ and } b = 2) \Rightarrow (a^2 = 9 \text{ and } b^2 = 4)$.

Hence, the required equation is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

EXAMPLE 9 Find the equation of the ellipse whose centre lies at the origin, major axis lies on the x -axis, the eccentricity is $\frac{2}{3}$ and the length of the latus rectum is 5 units.

SOLUTION Since the major axis of the ellipse lies on the x -axis, it is a horizontal ellipse.

Let the required equation be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Length of its latus rectum = $\frac{2b^2}{a}$.

$\therefore \frac{2b^2}{a} = 5 \Leftrightarrow \frac{2a^2(1 - e^2)}{a} = 5$ [$\because b^2 = a^2(1 - e^2)$]

$\Leftrightarrow a = \frac{5}{2(1 - e^2)} = \frac{5}{2\left(1 - \frac{4}{9}\right)} = \left(\frac{5}{2} \times \frac{9}{5}\right) = \frac{9}{2}$.

Also, $b^2 = a^2(1 - e^2) = \frac{81}{4} \times \left(1 - \frac{4}{9}\right) = \left(\frac{81}{4} \times \frac{5}{9}\right) = \frac{45}{4}$.

$\therefore a^2 = \left(\frac{9}{2}\right)^2 = \frac{81}{4}$ and $b^2 = \frac{45}{4}$.

Hence, the required equation is

$$\frac{x^2}{(81/4)} + \frac{y^2}{(45/4)} = 1 \Leftrightarrow \frac{4x^2}{81} + \frac{4y^2}{45} = 1 \Leftrightarrow 20x^2 + 36y^2 = 405.$$



EXAMPLE 10 Find the equation of an ellipse whose vertices are $(0, \pm 13)$ and the foci are $(0, \pm 5)$.

SOLUTION Since the vertices of the ellipse lie on the y -axis, it is a *vertical ellipse*.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a)$ and therefore, $a = 13$.

Let its foci be $(0, \pm c)$. Then, $c = 5$.

$$\therefore b^2 = (a^2 - c^2) = (169 - 25) = 144.$$

Thus, $b^2 = 144$ and $a^2 = (13)^2 = 169$.

Hence, the required equation is $\frac{x^2}{144} + \frac{y^2}{169} = 1$.

EXAMPLE 11 Find the equation of the ellipse whose foci are $(0, \pm 6)$ and the length of whose minor axis is 16.

SOLUTION Since the foci of the given ellipse lie on the y -axis, it is a *vertical ellipse*.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Then, its foci are $(0, \pm c)$ and therefore, $c = 6$.

Also, $b =$ length of the semi-minor axis $= \left(\frac{1}{2} \times 16\right) = 8$.

$$\therefore a^2 = (c^2 + b^2) = (36 + 64) = 100.$$

Thus, $b^2 = 64$ and $a^2 = 100$.

Hence, the required equation is $\frac{x^2}{64} + \frac{y^2}{100} = 1$.

EXAMPLE 12 Find the equation of the ellipse whose foci are $(0, \pm 5)$ and the length of whose major axis is 20.

SOLUTION Since the foci of the ellipse lie on the y -axis, it is a *vertical ellipse*.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Its foci are $(0, \pm c)$ and therefore, $c = 5$.

Also, $a =$ length of the semi-major axis $= \left(\frac{1}{2} \times 20\right) = 10$.

$$\text{Now, } c^2 = (a^2 - b^2) \Leftrightarrow b^2 = (a^2 - c^2) = (100 - 25) = 75.$$

Thus, $a^2 = (10)^2 = 100$ and $b^2 = 75$.

Hence, the required equation is $\frac{x^2}{75} + \frac{y^2}{100} = 1$.



EXAMPLE 13 Find the equation of the ellipse for which $e = \frac{4}{5}$ and whose vertices are $(0, \pm 10)$.

SOLUTION Since the vertices of the ellipse lie on the y -axis, it is a *vertical ellipse*.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a)$ and therefore, $a = 10$.

Let $c^2 = (a^2 - b^2)$.

Then, $e = \frac{c}{a} \Rightarrow c = ae = \left(10 \times \frac{4}{5}\right) = 8$.

Now, $c^2 = (a^2 - b^2) \Leftrightarrow b^2 = (a^2 - c^2) = (100 - 64) = 36$.

$\therefore a^2 = (10)^2 = 100$ and $b^2 = 36$.

Hence, the required equation is $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

EXAMPLE 14 Find the equation of the ellipse with centre at the origin, major-axis on the y -axis and passing through the points $(3, 2)$ and $(1, 6)$.

SOLUTION Since the major axis of the ellipse lies on the y -axis, it is a *vertical ellipse*.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (where $a^2 > b^2$). ... (i)

Since $(3, 2)$ lies on (i), we have $\frac{9}{b^2} + \frac{4}{a^2} = 1$ (ii)

Also, since $(1, 6)$ lies on (i), we have $\frac{1}{b^2} + \frac{36}{a^2} = 1$ (iii)

Putting $\frac{1}{b^2} = u$ and $\frac{1}{a^2} = v$, these equations become:

$9u + 4v = 1$... (iv)

and $u + 36v = 1$... (v)

On multiplying (v) by 9 and subtracting (iv) from it, we get

$$320v = 8 \Leftrightarrow v = \frac{8}{320} = \frac{1}{40} \Leftrightarrow \frac{1}{a^2} = \frac{1}{40} \Leftrightarrow a^2 = 40.$$

Putting $v = \frac{1}{40}$ in (v), we get

$$u + \left(36 \times \frac{1}{40}\right) = 1 \Leftrightarrow u = \left(1 - \frac{9}{10}\right) = \frac{1}{10} \Leftrightarrow \frac{1}{b^2} = \frac{1}{10} \Leftrightarrow b^2 = 10.$$

Thus, $b^2 = 10$ and $a^2 = 40$.

Hence, the required equation is $\frac{x^2}{10} + \frac{y^2}{40} = 1$.



Entrance Hub Practice Questions

Find the (i) lengths of major and minor axes, (ii) coordinates of the vertices, (iii) coordinates of the foci, (iv) eccentricity, and (v) length of the latus rectum of each of the following ellipses.

1. $\frac{x^2}{25} + \frac{y^2}{9} = 1$
2. $\frac{x^2}{49} + \frac{y^2}{36} = 1$
3. $16x^2 + 25y^2 = 400$
4. $x^2 + 4y^2 = 100$
5. $9x^2 + 16y^2 = 144$
6. $4x^2 + 9y^2 = 1$
7. $\frac{x^2}{4} + \frac{y^2}{25} = 1$
8. $\frac{x^2}{9} + \frac{y^2}{16} = 1$
9. $3x^2 + 2y^2 = 18$
10. $9x^2 + y^2 = 36$
11. $16x^2 + y^2 = 16$
12. $25x^2 + 4y^2 = 100$
13. Find the equation of the ellipse whose vertices are at $(\pm 6, 0)$ and foci at $(\pm 4, 0)$.
14. Find the equation of the ellipse whose vertices are at $(0, \pm 4)$ and foci at $(0, \pm \sqrt{7})$.
15. Find the equation of the ellipse the ends of whose major and minor axes are $(\pm 4, 0)$ and $(0, \pm 3)$ respectively.
16. The length of the major axis of an ellipse is 20 units and its foci are $(\pm 5\sqrt{3}, 0)$. Find the equation of the ellipse.
17. Find the equation of the ellipse whose foci are $(\pm 2, 0)$ and the eccentricity is $\frac{1}{2}$.
18. Find the equation of the ellipse whose foci are at $(\pm 1, 0)$ and $e = \frac{1}{2}$.
19. Find the equation of the ellipse whose foci are at $(0, \pm 4)$ and $e = \frac{4}{5}$.
20. Find the equation of the ellipse with centre at the origin, major axis on the x -axis and passing through the points $(4, 3)$ and $(-1, 4)$.
21. Find the equation of the ellipse with eccentricity $\frac{3}{4}$, foci on the y -axis, centre at the origin and passing through the point $(6, 4)$.
22. Find the equation of the ellipse which passes through the point $(4, 1)$ and having its foci at $(\pm 3, 0)$.
23. Find the equation of an ellipse, the lengths of whose major and minor axes are 10 and 8 units respectively.
24. Find the equation of an ellipse whose eccentricity is $\frac{2}{3}$, the latus rectum is 5 and the centre is at the origin.



25. Find the eccentricity of an ellipse whose latus rectum is one half of its minor axis.
26. Find the eccentricity of an ellipse whose latus rectum is one half of its major axis.

Entrance Hub Practice Questions

- (i) 10 units, 6 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-4, 0)$ and $F_2(4, 0)$
 (iv) $e = \frac{4}{5}$ (v) 3.6 units
- (i) 14 units, 12 units (ii) $A(-7, 0)$ and $B(7, 0)$
 (iii) $F_1(-\sqrt{13}, 0)$ and $F_2(\sqrt{13}, 0)$ (iv) $e = \frac{\sqrt{13}}{7}$ (v) $\frac{72}{7}$ units
- (i) 10 units, 8 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-3, 0)$ and $F_2(3, 0)$
 (iv) $e = \frac{3}{5}$ (v) $\frac{32}{5}$ units
- (i) 20 units, 10 units (ii) $A(-10, 0)$ and $B(10, 0)$
 (iii) $F_1(-5\sqrt{3}, 0)$ and $F_2(5\sqrt{3}, 0)$ (iv) $e = \frac{\sqrt{3}}{2}$ (v) 5 units
- (i) 8 units, 6 units (ii) $A(-4, 0)$ and $B(4, 0)$ (iii) $F_1(-\sqrt{7}, 0)$, $F_2(\sqrt{7}, 0)$
 (iv) $e = \frac{\sqrt{7}}{4}$ (v) $\frac{9}{2}$ units
- (i) 1 unit, $\frac{2}{3}$ unit (ii) $A\left(-\frac{1}{2}, 0\right)$ and $B\left(\frac{1}{2}, 0\right)$
 (iii) $F_1\left(\frac{-\sqrt{5}}{6}, 0\right)$ and $F_2\left(\frac{\sqrt{5}}{6}, 0\right)$ (iv) $e = \frac{\sqrt{5}}{3}$ (v) $\frac{4}{9}$ unit
- (i) 10 units, 4 units (ii) $A(0, -5)$ and $B(0, 5)$
 (iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$ (iv) $e = \frac{\sqrt{21}}{5}$ (v) $\frac{8}{5}$ units
- (i) 8 units, 6 units (ii) $A(0, -4)$ and $B(0, 4)$
 (iii) $F_1(0, -\sqrt{7})$ and $F_2(0, \sqrt{7})$ (iv) $e = \frac{\sqrt{7}}{4}$ (v) $4\frac{1}{2}$ units
- (i) 6 units, $2\sqrt{6}$ units (ii) $A(0, -3)$ and $B(0, 3)$
 (iii) $F_1(0, -\sqrt{3})$ and $F_2(0, \sqrt{3})$ (iv) $e = \frac{1}{\sqrt{3}}$ (v) 4 units

10. (i) 12 units, 4 units (ii) $A(0, -6)$ and $B(0, 6)$
 (iii) $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$ (iv) $e = \frac{2\sqrt{2}}{3}$ (v) $1\frac{1}{3}$ units
11. (i) 8 units, 2 units (ii) $A(0, -4)$ and $B(0, 4)$
 (iii) $F_1(0, -\sqrt{15})$ and $F_2(0, \sqrt{15})$ (iv) $e = \frac{\sqrt{15}}{4}$ (v) $\frac{1}{2}$ unit
12. (i) 10 units, 4 units (ii) $A(0, -5)$ and $B(0, 5)$
 (iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$ (iv) $e = \frac{\sqrt{21}}{5}$ (v) $1\frac{3}{5}$ units
13. $\frac{x^2}{36} + \frac{y^2}{20} = 1$ 14. $\frac{x^2}{9} + \frac{y^2}{16} = 1$ 15. $\frac{x^2}{16} + \frac{y^2}{9} = 1$
 16. $\frac{x^2}{100} + \frac{y^2}{25} = 1$ 17. $\frac{x^2}{16} + \frac{y^2}{12} = 1$ 18. $\frac{x^2}{4} + \frac{y^2}{3} = 1$
 19. $\frac{x^2}{9} + \frac{y^2}{25} = 1$ 20. $7x^2 + 15y^2 = 247$ 21. $16x^2 + 7y^2 = 688$
 22. $\frac{x^2}{18} + \frac{y^2}{9} = 1$ 23. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ 24. $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$
 25. $e = \frac{\sqrt{3}}{2}$ 26. $e = \frac{1}{\sqrt{2}}$

HINTS TO SOME SELECTED QUESTIONS

6. Given equation is $\frac{x^2}{(1/4)} + \frac{y^2}{(1/9)} = 1$.
 $\therefore a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$ and $c^2 = (a^2 - b^2) = \left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36}$.
 $\therefore a = \frac{1}{2}, b = \frac{1}{3}$ and $c = \frac{\sqrt{5}}{6}$.
15. The ends of the major axis are $(\pm a, 0)$ and the ends of the minor axis are $(0, \pm b)$.
 $\therefore a = 4$ and $b = 3$.
16. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
 Length of the major axis = 20 $\Leftrightarrow 2a = 20 \Leftrightarrow a = 10$.
 Foci are $(\pm c, 0) \Rightarrow c = 5\sqrt{3}$.
 Now, $c^2 = (a^2 - b^2) \Leftrightarrow b^2 = (a^2 - c^2) = (100 - 75) = 25$.
 $\therefore$ the required equation is $\frac{x^2}{100} + \frac{y^2}{25} = 1$.
19. Clearly, the major axis lies along the y -axis.
 Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Here, $c = 4$ and $e = \frac{4}{5}$. So, $a = \frac{c}{e} = 5$.

$$\therefore b^2 = (a^2 - c^2) = (25 - 16) = 9.$$

21. Since the centre of the ellipse lies at the origin and its foci lie on the y -axis, it is a vertical ellipse.

Let its equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (i)

Let $c^2 = a^2 - b^2$.

Now, $e = \frac{3}{4} \Leftrightarrow \frac{c}{a} = \frac{3}{4} \Leftrightarrow c = \frac{3}{4}a$.

$$\therefore b^2 = (a^2 - c^2) = \left(a^2 - \frac{9}{16}a^2\right) = \frac{7a^2}{16}.$$

So, the equation of the ellipse is

$$\frac{x^2}{\frac{7a^2}{16}} + \frac{y^2}{a^2} = 1 \Leftrightarrow 16x^2 + 7y^2 = 7a^2. \quad \dots \text{(ii)}$$

Since it passes through $(6, 4)$, putting $x = 6$ and $y = 4$ in (ii), we get

$$7a^2 = 688.$$

Hence, the required equation is $16x^2 + 7y^2 = 688$.

23. $2a = 10$ and $2b = 8$.

24. $\frac{2b^2}{a} = 5 \Rightarrow \frac{2a^2(1 - e^2)}{a} = 5 \Rightarrow a = \frac{9}{2}$.

Also, $b^2 = \frac{5a}{2} = \left(\frac{5}{2} \times \frac{9}{2}\right) = \frac{45}{4}$.

25. $\left\{ \frac{2b^2}{a} = \frac{1}{2} \times 2b \Rightarrow b = \frac{a}{2} \right\}$.

So, $b^2 = a^2(1 - e^2) \Rightarrow \frac{a^2}{4} = a^2(1 - e^2)$.

26. $\frac{2b^2}{a} = \frac{1}{2} \times 2a \Rightarrow a^2 = 2b^2 \Rightarrow a = \sqrt{2} b$. So, $e = \frac{b}{a} = \frac{1}{\sqrt{2}}$.

□

