



First Order And Second Order Response Of RL And RC Circuit

Topic 5

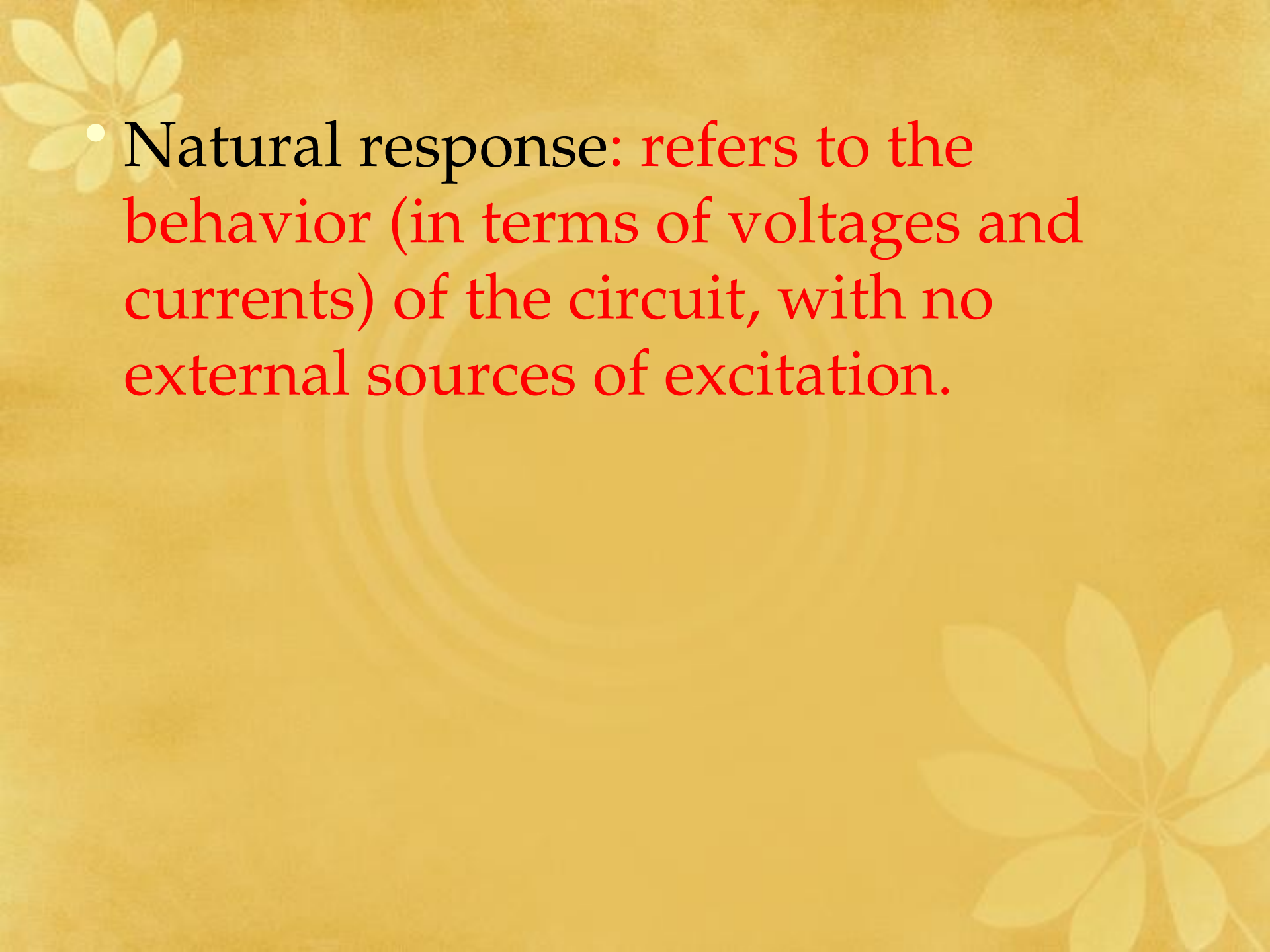
First-Order and Second-Order Response of RL and RC Circuit

- Natural response of RL and RC Circuit
- Step Response of RL and RC Circuit
- General solutions for natural and step response
- Sequential switching
- Introduction to the natural and step response of RLC circuit
- Natural response of series and parallel RLC circuit
- Step response of series and parallel RLC circuit

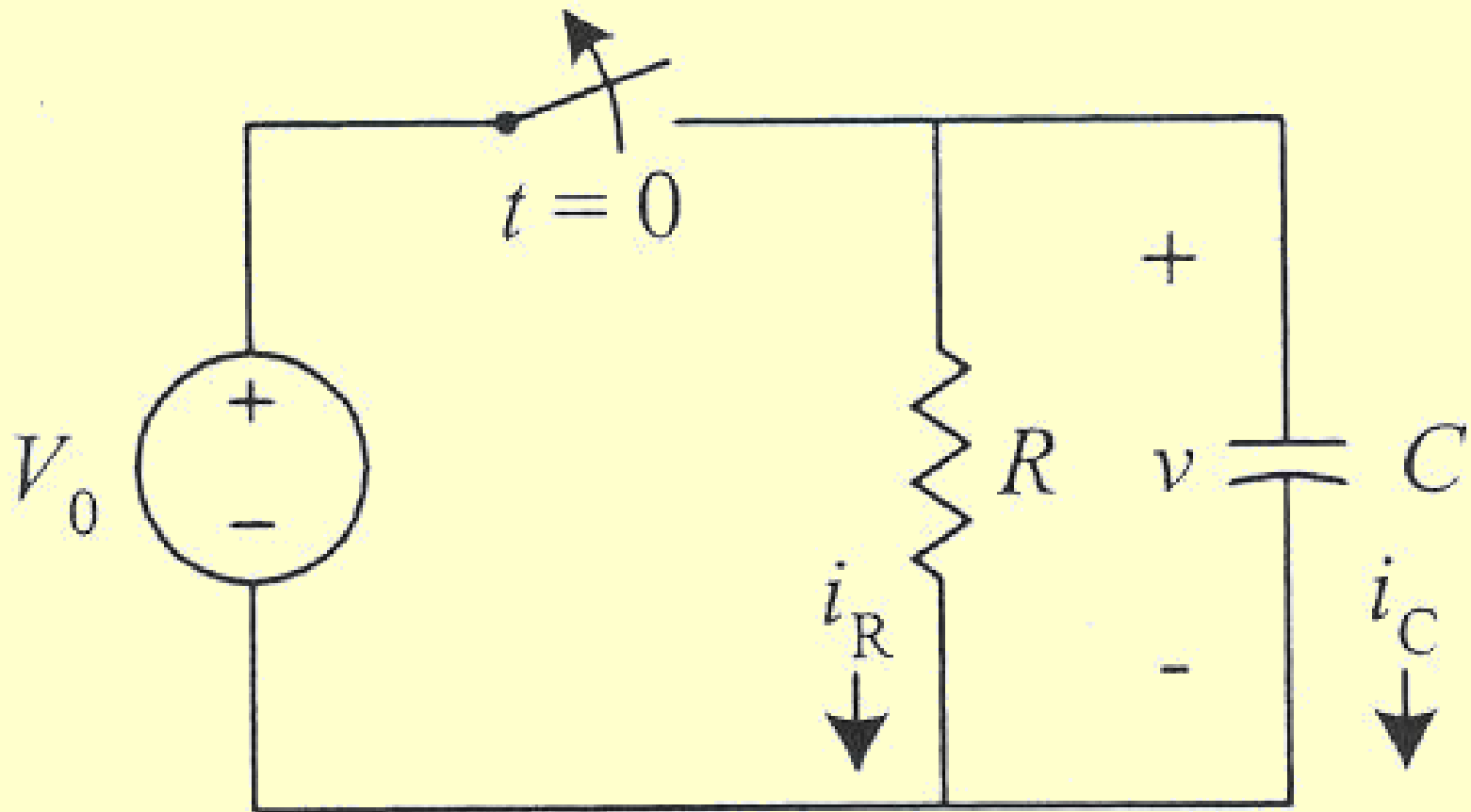


Natural response of RL and RC Circuit

- RL- resistor-inductor
 - RC-resistor-capacitor
 - First-order circuit: RL or RC circuit because their voltages and currents are described by first-order differential equation.
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- Natural response: refers to the behavior (in terms of voltages and currents) of the circuit, with no external sources of excitation.

Natural response of RC circuit





Consider the conditions below:

1. At $t < 0$, switch is in a closed position for along time.
 2. At $t=0$, the instant when the switch is opened
 3. At $t > 0$, switch is not close for along time
- 

- For $t \leq 0$, $v(t) = V_0$.

□ For $t \geq 0$:

$$i_c + i_R = 0$$

$$C \frac{dv(t)}{dt} + \frac{v(t)}{R} = 0$$

$$\frac{dv(t)}{dt} + \frac{v(t)}{RC} = 0$$

$$\frac{dv(t)}{dt} = - \frac{v(t)}{RC}$$

$$\frac{dv(t)}{v(t)} = - \frac{1}{RC} dt$$

$$\frac{du}{u} = - \frac{1}{RC} dv$$

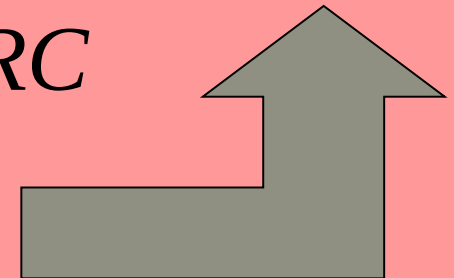
$$\int_{V_0}^{v(t)} \frac{1}{u} du = - \frac{1}{RC} \int_0^t dv$$

$$\ln v(t) - \ln V_0 = - \frac{1}{RC} (t - 0)$$

$$\ln \left(\frac{v(t)}{V_0} \right) = - \frac{t}{RC}$$

$$v(t) = V_0 e^{-t/RC}$$

voltage



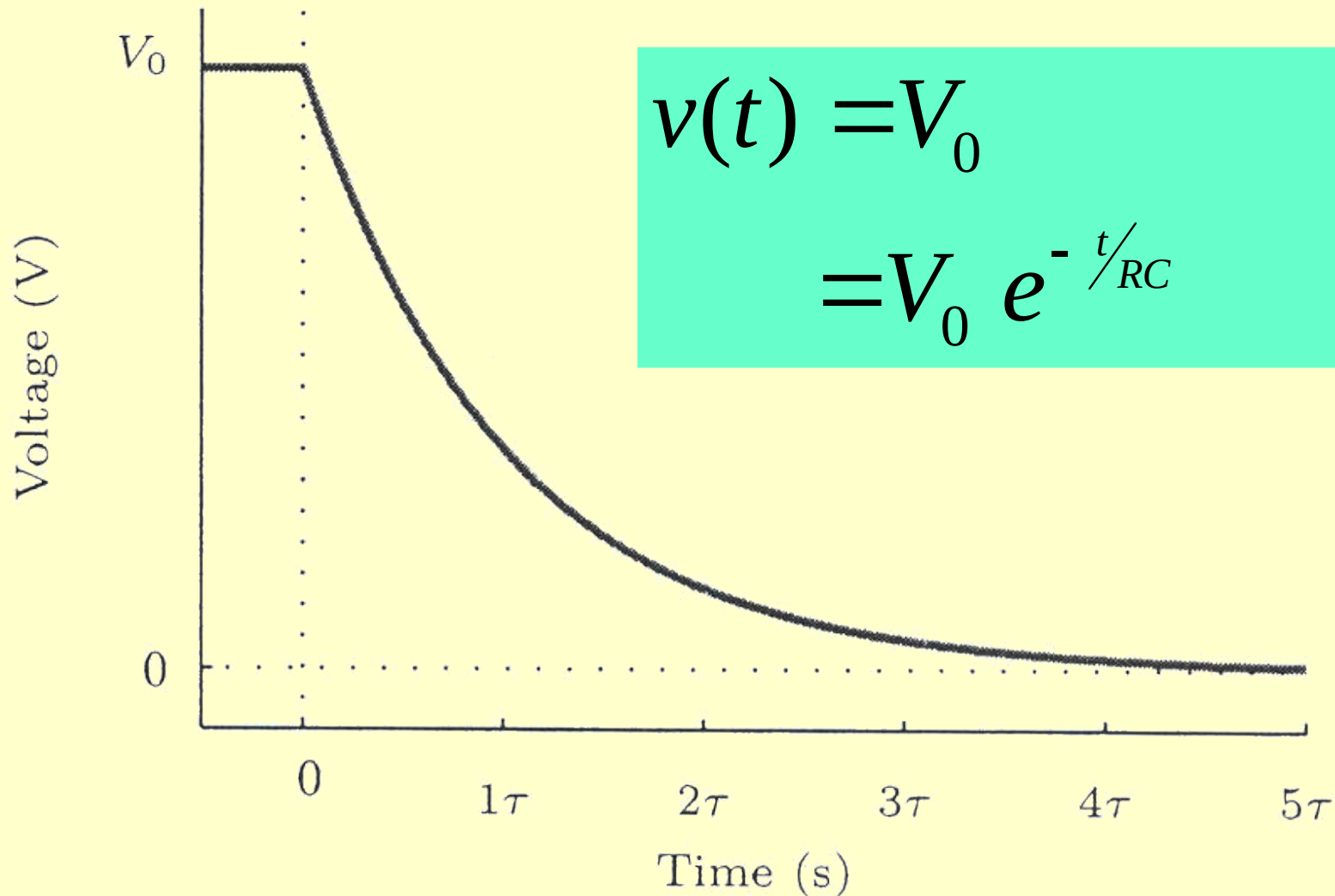
- Thus for $t > 0$,

$$v(t) = V_0 e^{-t/RC}$$

$$i_c(t) = - \frac{v(t)}{R} = - \frac{V_0}{R} e^{-t/RC}$$

$$W(t) = \frac{1}{2} C (v(t))^2 = \frac{1}{2} C V_0^2 e^{-2t/RC}$$

The graph of the natural response of RC circuit



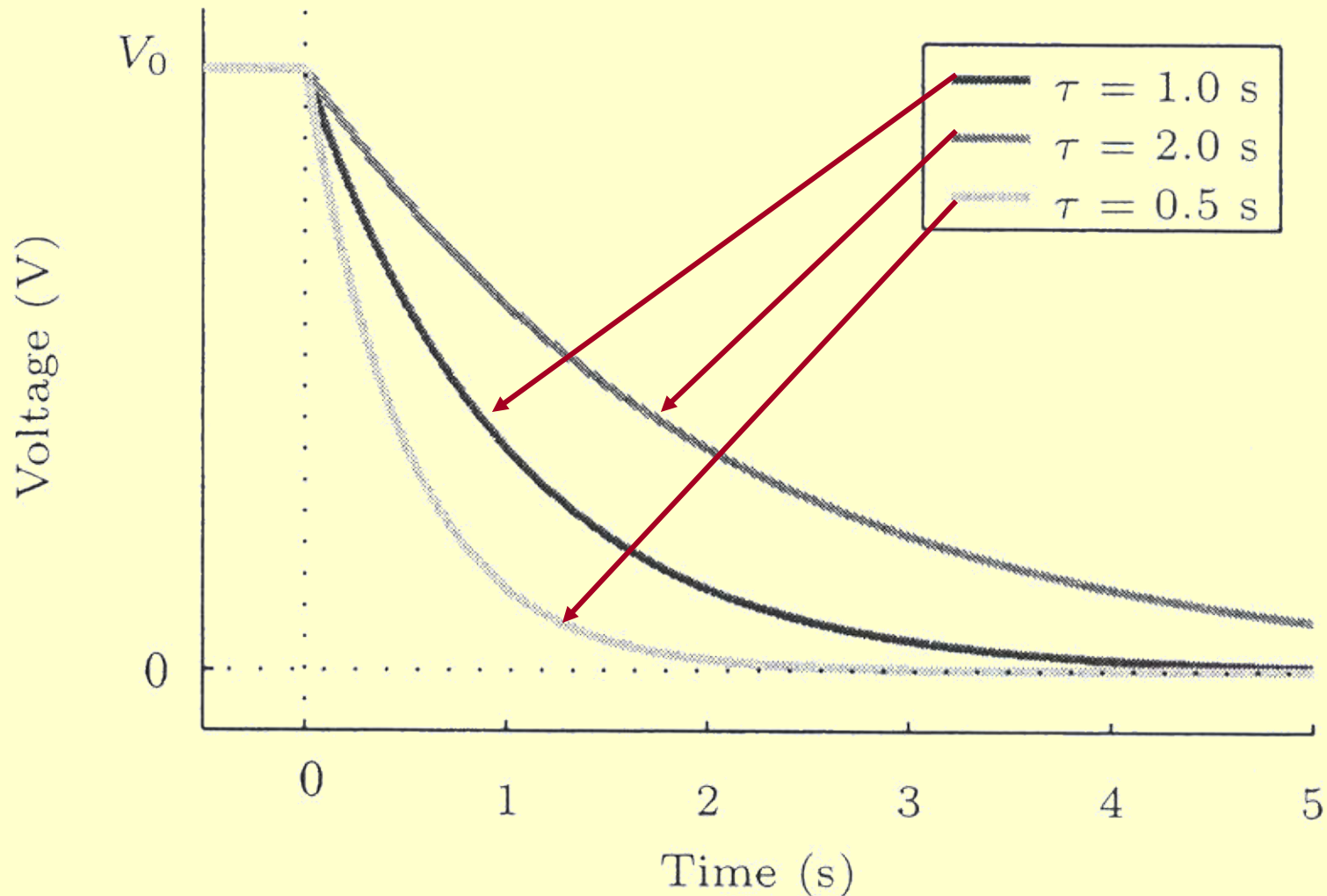
$$v(t) = V_0 \quad t \leq 0$$

$$= V_0 e^{-t/\tau_{RC}} \quad t \geq 0$$

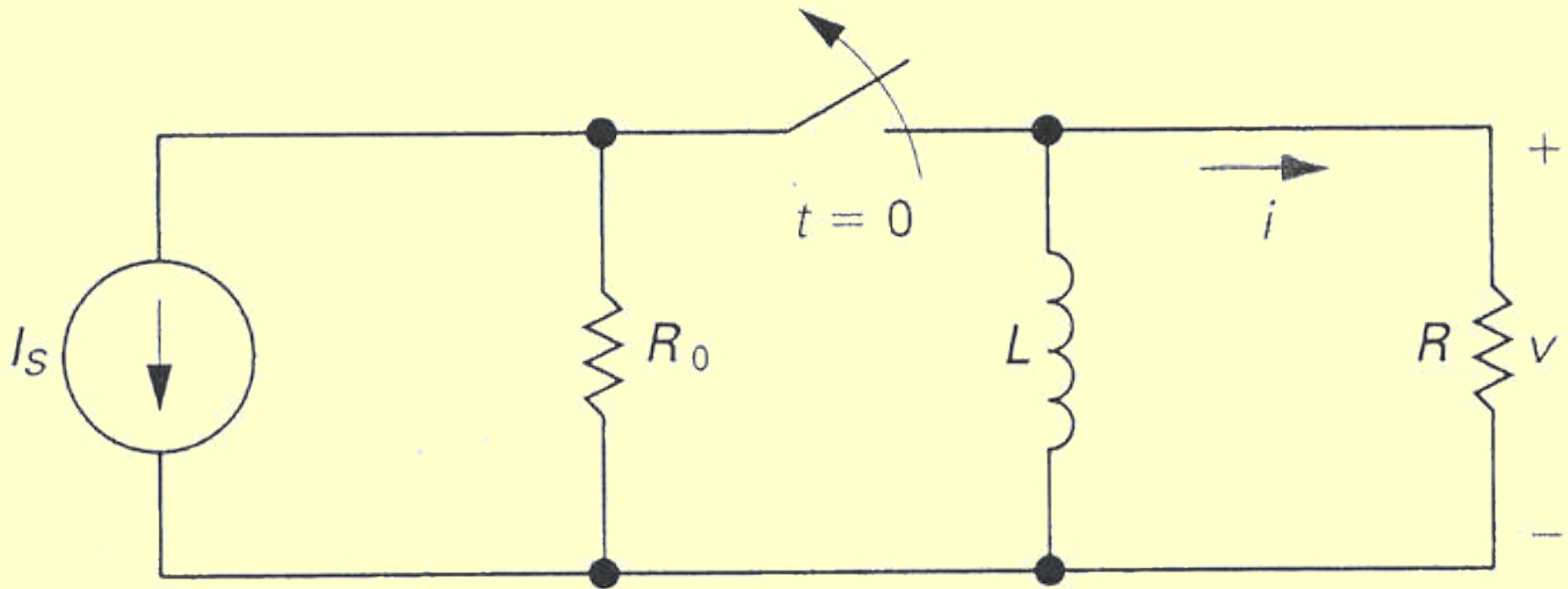
- The time constant, $\tau = RC$ and thus,

$$v(t) = V_0 e^{-t/\tau}$$

- The time constant, τ determine how fast the voltage reach the steady state:



Natural response of RL circuit





Consider the conditions below:

1. At $t < 0$, switch is in a closed position for along time.
 2. At $t=0$, the instant when the switch is opened
 3. At $t > 0$, switch is not close for along time
- 

- For $t \leq 0$, $i(t) = I_0$

□ For $t > 0$,

$$v(t) + Ri(t) = 0$$

$$L \frac{di(t)}{dt} + Ri(t) = 0$$

$$L \frac{di(t)}{dt} = - Ri(t)$$

$$\frac{di(t)}{i(t)} = - \frac{R}{L} dt$$

$$\frac{du}{u} = - \frac{R}{L} dv$$

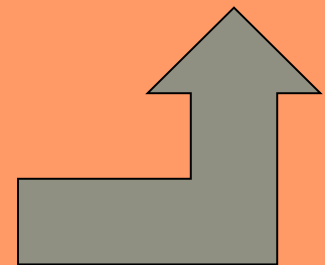
$$\int_{i(0)}^{i(t)} \frac{1}{u} du = - \frac{R}{L} \int dv$$

$$\ln i(t) - \ln i(0) = - \frac{R}{L} (t - 0)$$

$$\ln \left(\frac{i(t)}{i(0)} \right) = - t \frac{R}{L}$$

$$i(t) = i(0) e^{-tR/L}$$

current



- Thus for $t > 0$,

$$i(t) = I_0 e^{-tR/L}$$

$$\begin{aligned} v(t) &= -i(t)R \\ &= -RI_0 e^{-tR/L} \end{aligned}$$

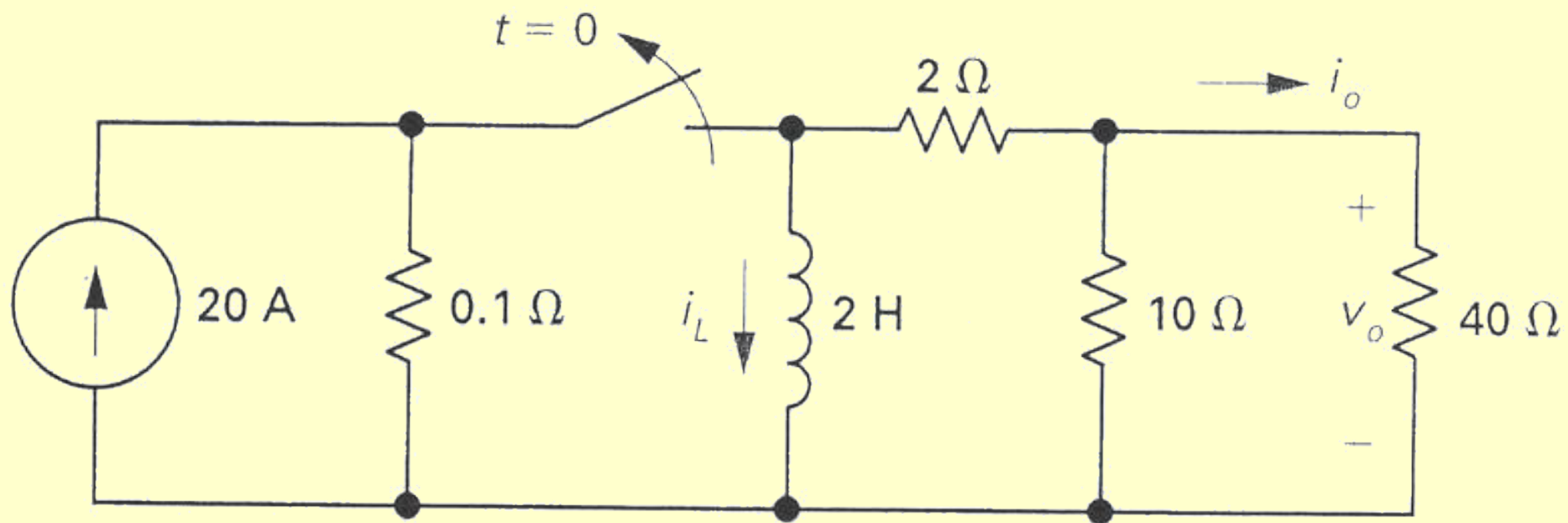
$$\begin{aligned} w(t) &= \frac{1}{2} L (i(t))^2 \\ &= \frac{1}{2} L I_0^2 e^{-2tR/L} \end{aligned}$$

Example...

The switch in the circuit has been closed for a long time before it is opened at $t=0$.

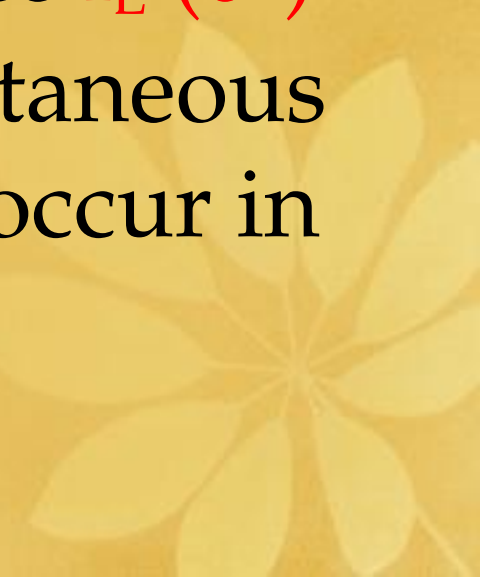
Find

- a) $I_L(t)$ for $t \geq 0$
- b) $I_0(t)$ for $t \geq 0+$
- c) $V_0(t)$ for $t \geq 0+$
- d) The percentage of the total energy stored in the 2H inductor that is dissipated in the 10Ω resistor.





Solution

- a) The switch has been closed for along time prior to $t=0$, so voltage across the inductor must be zero at $t = 0^-$. Therefore the initial current in the inductor is $20A$ at $t = 0^-$. Hence $i_L(0^+)$ also is $20A$, because an instantaneous change in the current cannot occur in an inductor.
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- The equivalent resistance and time constant:

$$R_{eq} = 2 + (40 \parallel 10) = 10\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{2}{10} = 0.2 \text{ sec}$$

- The expression of inductor current, $i_L(t)$ as,

$$\begin{aligned} i_L(t) &= i(0^+) e^{-t/\tau} \\ &= 20 e^{-5t} \text{ A} \quad t \geq 0 \end{aligned}$$

b) The current in the 40Ω resistor can be determine using current division,

$$i_0 = -i_L \left(\frac{10}{10 + 40} \right)$$

- Note that this expression is valid for $t \geq 0^+$ because $i_0 = 0$ at $t = 0^-$.
- The inductor behaves as a short circuit prior to the switch being opened, producing an instantaneous change in the current i_0 . Then,

$$i_0(t) = -4e^{-5t} \text{ A} \quad t \geq 0^+$$

- c) The voltage V_0 directly obtain using Ohm's law

$$\begin{aligned} V_0(t) &= 40i_0 \\ &= -160e^{-5t} \text{ V} \quad t \geq 0^+ \end{aligned}$$

- d) The power dissipated in the 10Ω resistor is

$$\begin{aligned} p_{10\Omega}(t) &= \frac{V_0^2}{10} \\ &= 2560e^{-10t} \text{ W} \quad t \geq 0^+ \end{aligned}$$

- The total energy dissipated in the 10Ω resistor is

$$W_{10\Omega}(t) = \int_0^{\infty} 2560e^{-10t} dt$$
$$= 256J$$

- The initial energy stored in the 2H inductor is

$$\begin{aligned} W(0) &= \frac{1}{2} L i^2(0) \\ &= \frac{1}{2} (2)(400) = 400 J \end{aligned}$$

- 
- Therefore the percentage of energy dissipated in the 10Ω resistor is,

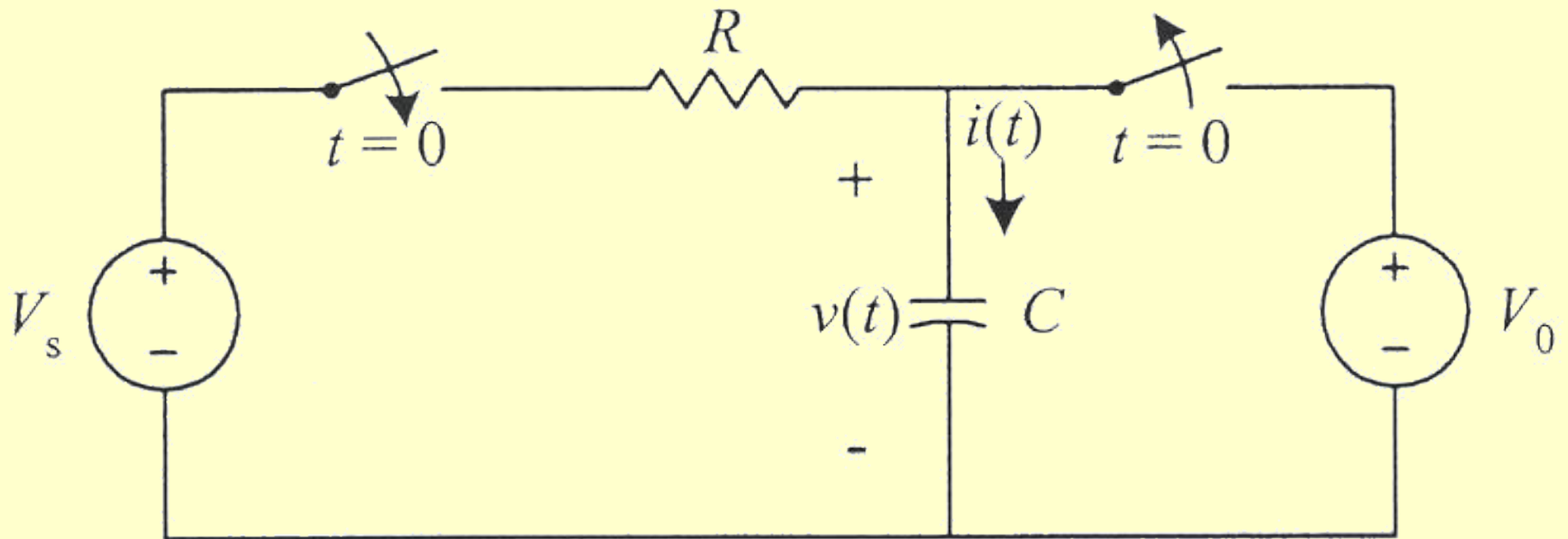
$$\frac{256}{400}(100) = 64\%$$

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Step response of RC circuit

- The step response of a circuit is its behavior when the excitation is the step function, which maybe a voltage or a current source.





Consider the conditions below:

1. At $t < 0$, switch is in a closed and opened position for along time.
 2. At $t=0$, the instant when the switch is opened and closed
 3. At $t > 0$, switch is not close and opened for along time
- 

• For $t \leq 0$, $v(t) = V_0$

■ For $t > 0$,

$$V_s - v(t) = Ri(t)$$

$$V_s - v(t) = RC \frac{dv(t)}{dt}$$

$$\frac{1}{RC} dt = \frac{dv(t)}{V_s - v(t)}$$

$$-\frac{1}{RC} dt = \frac{dv(t)}{v(t) - V_s}$$

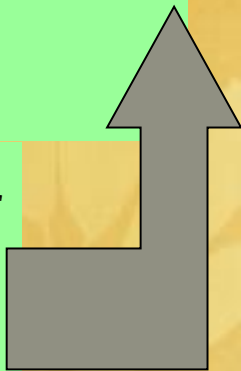
$$-\frac{1}{RC} dv = \frac{du}{u - V_s}$$

$$-\frac{t}{RC} = \ln(v(t) - V_s) - \ln(V_0 - V_s)$$

$$-\frac{t}{RC} = \ln\left(\frac{v(t) - V_s}{V_0 - V_s}\right)$$

voltage

$$\begin{aligned} v(t) &= V_s + (V_0 - V_s)e^{-t/RC} \\ &= V_s + (V_0 - V_s)e^{-t/\tau} \end{aligned}$$



- Thus for $t > 0$

$$V = V_s + (V_0 - V_s)e^{-t/\tau}$$

$$= V_f + V_n$$

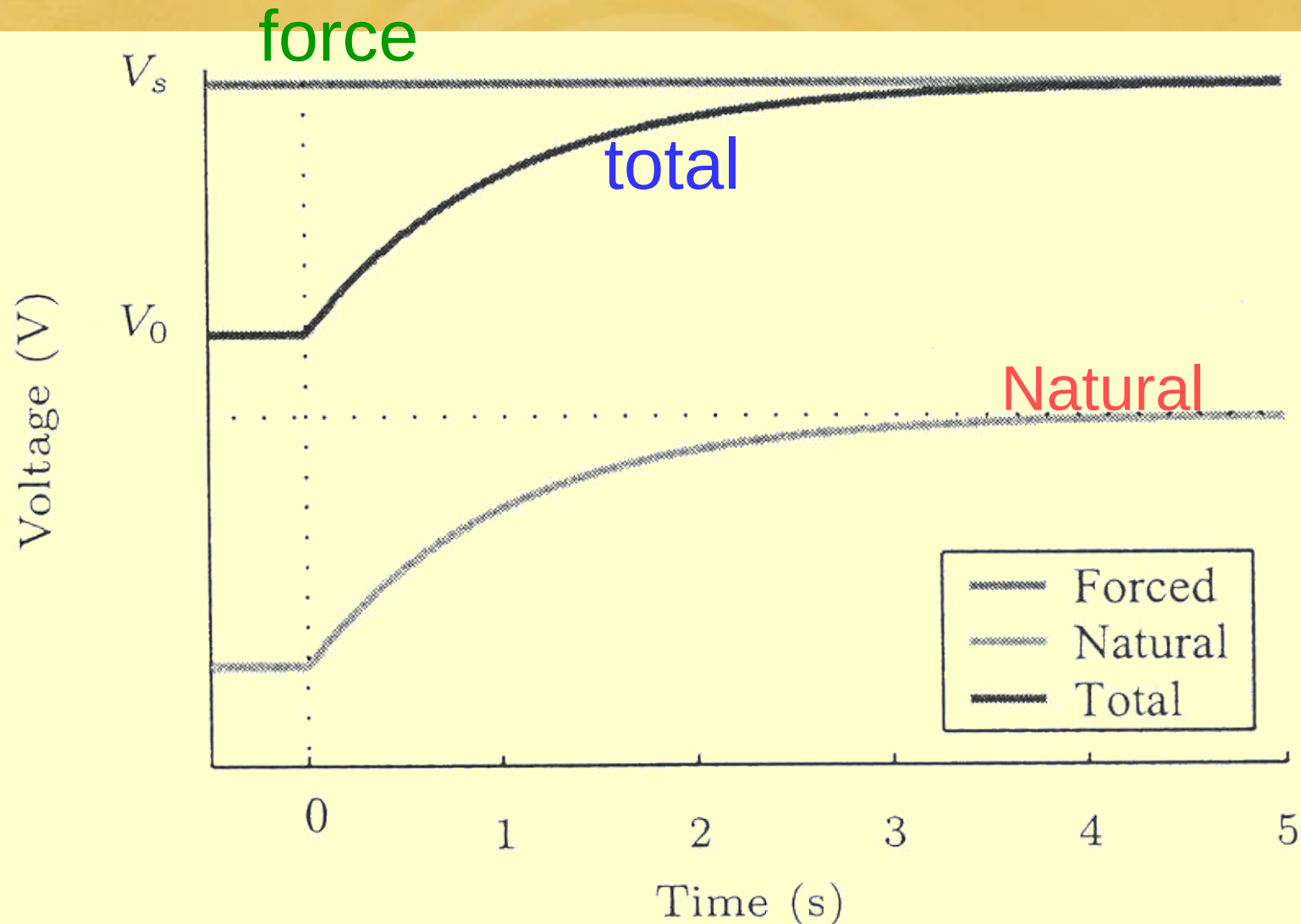
Where

$$V_f = V_s$$

$$V_n = (V_0 - V_s)e^{-t/\tau}$$

- 
- V_f = force voltage or also known as steady state response
 - V_n = transient voltage is the circuit's temporary response that will die out with time
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Graf Sambutan Langkah Litar RC



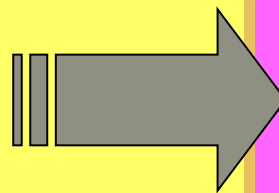
- The current for step response of RC circuit

$$i(t) = C \frac{dv}{dt}$$

$$= C \left(- \frac{1}{\tau} (V_0 - V_s) e^{-t/\tau} \right)$$

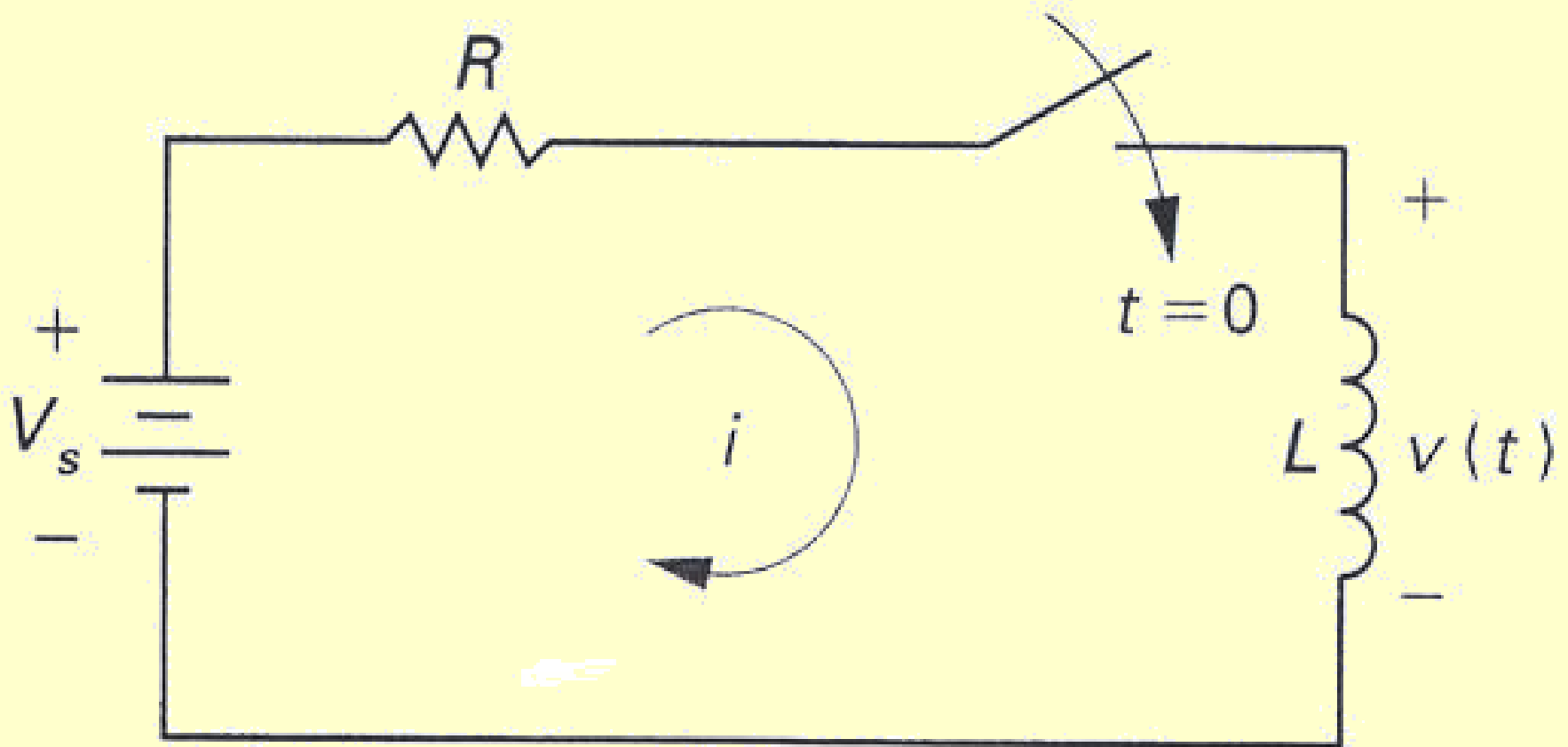
$$= - \frac{1}{R} (V_0 - V_s) e^{-t/\tau}$$

$$= \left(\frac{V_s}{R} - \frac{V_0}{R} \right) e^{-t/\tau}$$



$$i(t) = i(0^+) e^{-t/\tau}$$

Step response of RL circuit





Consider the conditions below:

1. At $t < 0$, switch is in a opened position for along time.
 2. At $t=0$, the instant when the switch is closed
 3. At $t > 0$, switch is not open for along time
- 

- $i(t) = I_0$ for $t \leq 0$.

□ For $t > 0$,

$$V_s - Ri(t) = v(t)$$

$$V_s - Ri(t) = L \frac{di(t)}{dt}$$

$$\frac{V_s}{R} - i(t) = \frac{L}{R} \frac{di(t)}{dt}$$

$$\frac{R}{L} dt = \frac{di(t)}{V_s/R - i(t)}$$

$$- \frac{R}{L} dt = \frac{di}{i - V_s/R}$$

$$- \frac{R}{L} dv = \frac{du}{u - V_s/R}$$

$$- \frac{R}{L} \int_0^t dv = \int_0^{i(t)} \frac{du}{u - V_s/R}$$

$$- t \frac{R}{L} = \ln(i(t) - V_s/R) - \ln(I_0 - V_s/R)$$

$$- t \frac{R}{L} = \ln \left(\frac{i(t) - V_s/R}{I_0 - V_s/R} \right)$$

$$i(t) = V_s/R + (I_0 - V_s/R)e^{-tR/L}$$

Current



• Thus,

$$\begin{aligned} i(t) &= I_0 & t \leq 0 \\ &= V_s / R + (I_0 - V_s / R) e^{-tR/L} & t \geq 0 \end{aligned}$$

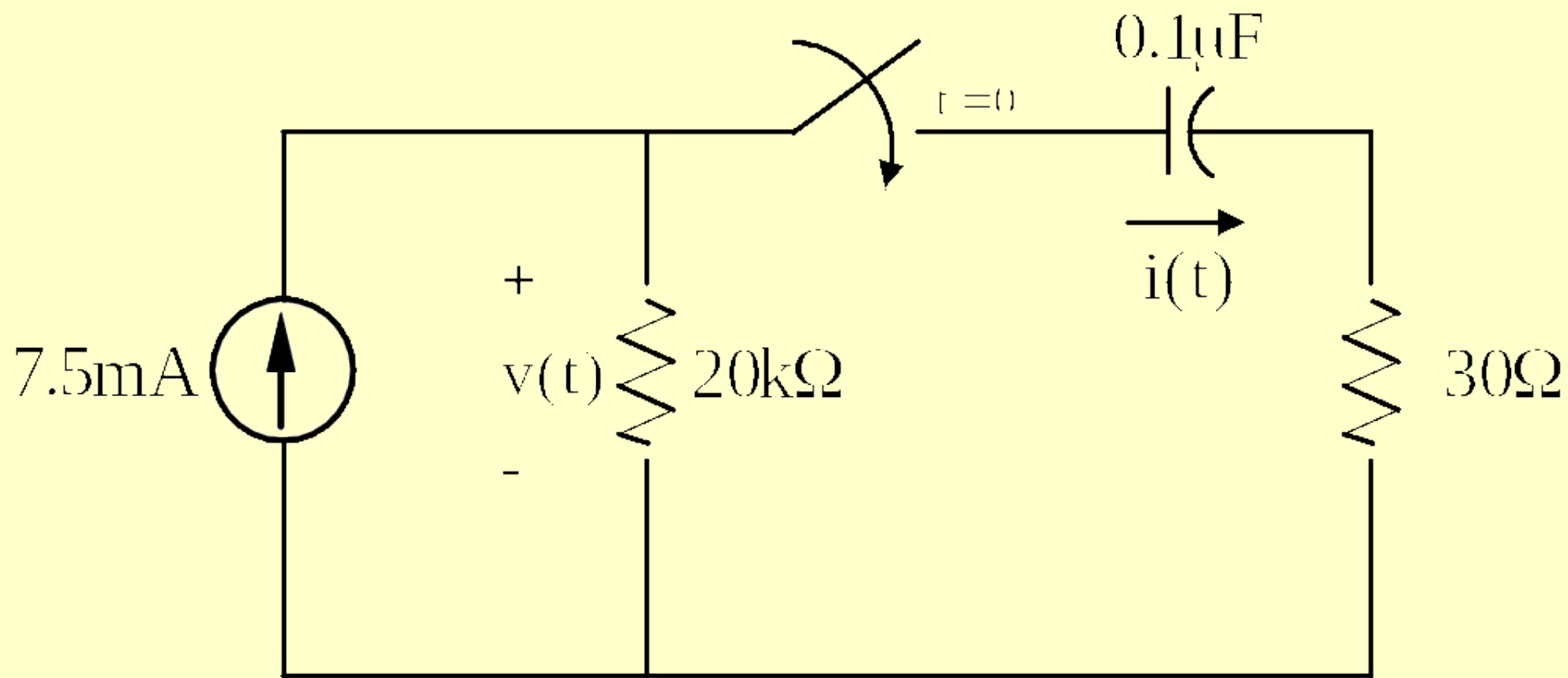
$$\begin{aligned} v(t) &= L \frac{di(t)}{dt} & t \leq 0 \\ &= (V_s - RI_0) e^{-tR/L} & t \geq 0 \end{aligned}$$



Question

The switch in the circuit has been open for along time. The initial charge on the capacitor is zero. At $t = 0$, the switch is closed. Find the expression for

- a) $i(t)$ for $t \geq 0$
 - b) $v(t)$ when $t \geq 0+$
- 



Answer (a)

- Initial voltage on the capacitor is zero. The current in the $30\text{k}\Omega$ resistor is

$$i(0^+) = \frac{(7.5)(20)}{50} = 3\text{mA}$$

- The final value of the capacitor current will be zero because the capacitor eventually will appear as an open circuit in terms of dc current. Thus $i_f = 0$.
- The time constant, τ is

$$\begin{aligned}\tau &= (20 + 30)10^3 (0.1) \times 10^{-6} \\ &= 5\text{ms}\end{aligned}$$

- Thus, the expression of the current $i(t)$ for $t \geq 0$ is

$$i(t) = i(0^+) e^{-t/\tau}$$

$$= 3 e^{-t/5 \times 10^{-3}}$$

$$= 3 e^{-200t} \text{ mA} \quad t \geq 0^+$$

Answer (b)

- The initial value of voltage is zero and the final value is

$$V_f = (7.5)(20) = 150V$$

- The capacitor $v_C(t)$ is

$$\begin{aligned}v_C(t) &= V_f + (V_0 - V_f)e^{-t/\tau} \\&= 150 + (0 - 150)e^{-200t} \\&= 150 - 150e^{-200t} \text{ V} \quad t \geq 0\end{aligned}$$

- Thus, the expression of $v(t)$ is

$$\begin{aligned}v(t) &= 150 - 150e^{-200t} + (30)(3)e^{-200t} \\&= (150 - 60e^{-200t}) \text{ V} \quad t \geq 0^+\end{aligned}$$

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General solutions for natural and step response

- There is common pattern for voltages, currents and energies:

$$v(t) = V_f + (V_0 - V_f)e^{-t/\tau}$$

$$i(t) = I_f + (I_0 - I_f)e^{-t/\tau}$$

$$W(t) = W_f + (W_0 - W_f)e^{-2t/\tau}$$



The general solution can be
compute as:

$$x(t) = x_f + (x_0 - x_f) e^{-t/\tau}$$

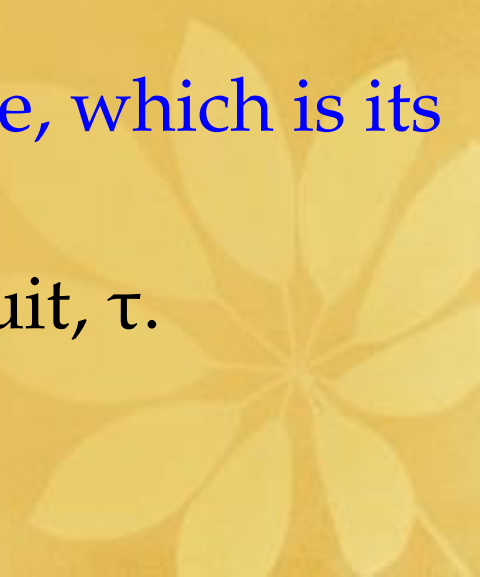


Write out in words:

$$\left[\begin{array}{l} \text{the unknown} \\ \text{variable as a} \\ \text{function of time} \end{array} \right] = \left[\begin{array}{l} \text{the final} \\ \text{value of the} \\ \text{variable} \end{array} \right] + \left[\left(\begin{array}{l} \text{the initial} \\ \text{value of the} \\ \text{variable} \end{array} \right) - \left(\begin{array}{l} \text{the final} \\ \text{value of the} \\ \text{variable} \end{array} \right) \right] e^{-\frac{t}{\text{time constant}}}$$



When computing the step and natural responses of circuits, it may help to follow these steps:

1. Identify the variable of interest for the circuit. For RC circuits, it is most convenient to choose the capacitive voltage, for RL circuits, it is best to choose the inductive current.
 2. Determine the initial value of the variable, which is its value at t_0 .
 3. Calculate the final value of the variable, which is its value as $t \rightarrow \infty$.
 4. Calculate the time constant of the circuit, τ .
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First-Order and Second-Order Response of RL and RC Circuit

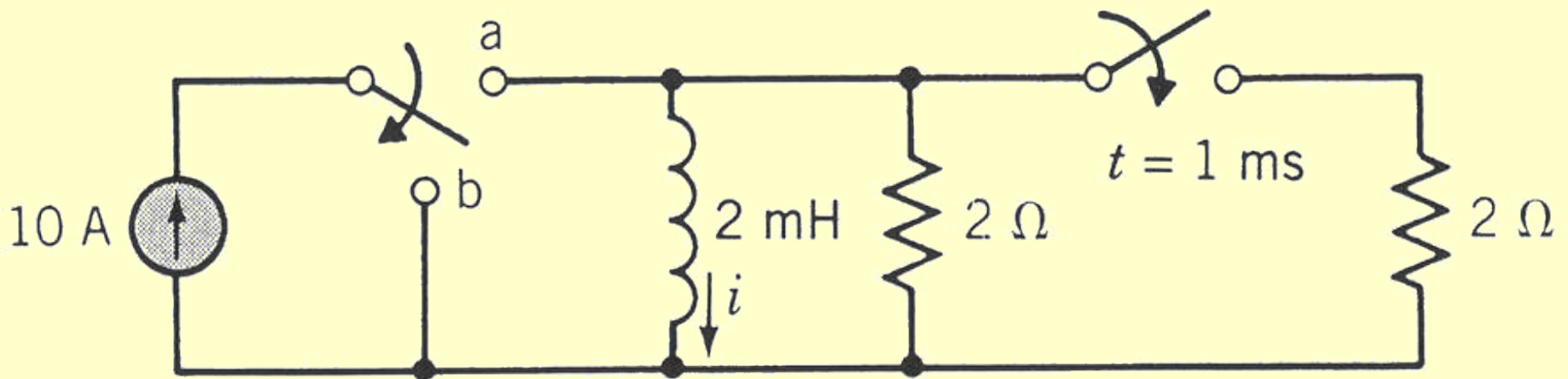
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Sequential switching

- Sequential switching is whenever switching occurs more than once in a circuit.
 - The time reference for all switchings cannot be $t = 0$.
- 

Example...



First switch move from a to b at $t=0$ and second switch closed at $t=1\text{ms}$. Find the current, i for $t \geq 0$.

- Step 1: current value at $t=0^-$ is determine as assume that the first switch at point a and second switch opened for along time. Therefore, the current, $i(0^-)=10A$.
- When $t=0$, an RL circuit is obtain as

$$\tau = \frac{L}{R} = 1ms.$$

- Thus the current for $0 \leq t \leq 1ms$ is,

$$i = 10e^{-1000t} A$$

- At $t=t_1=1\text{ms}$,

$$i(t_1) = 10e^{-1} \\ = 3.68\text{A}$$

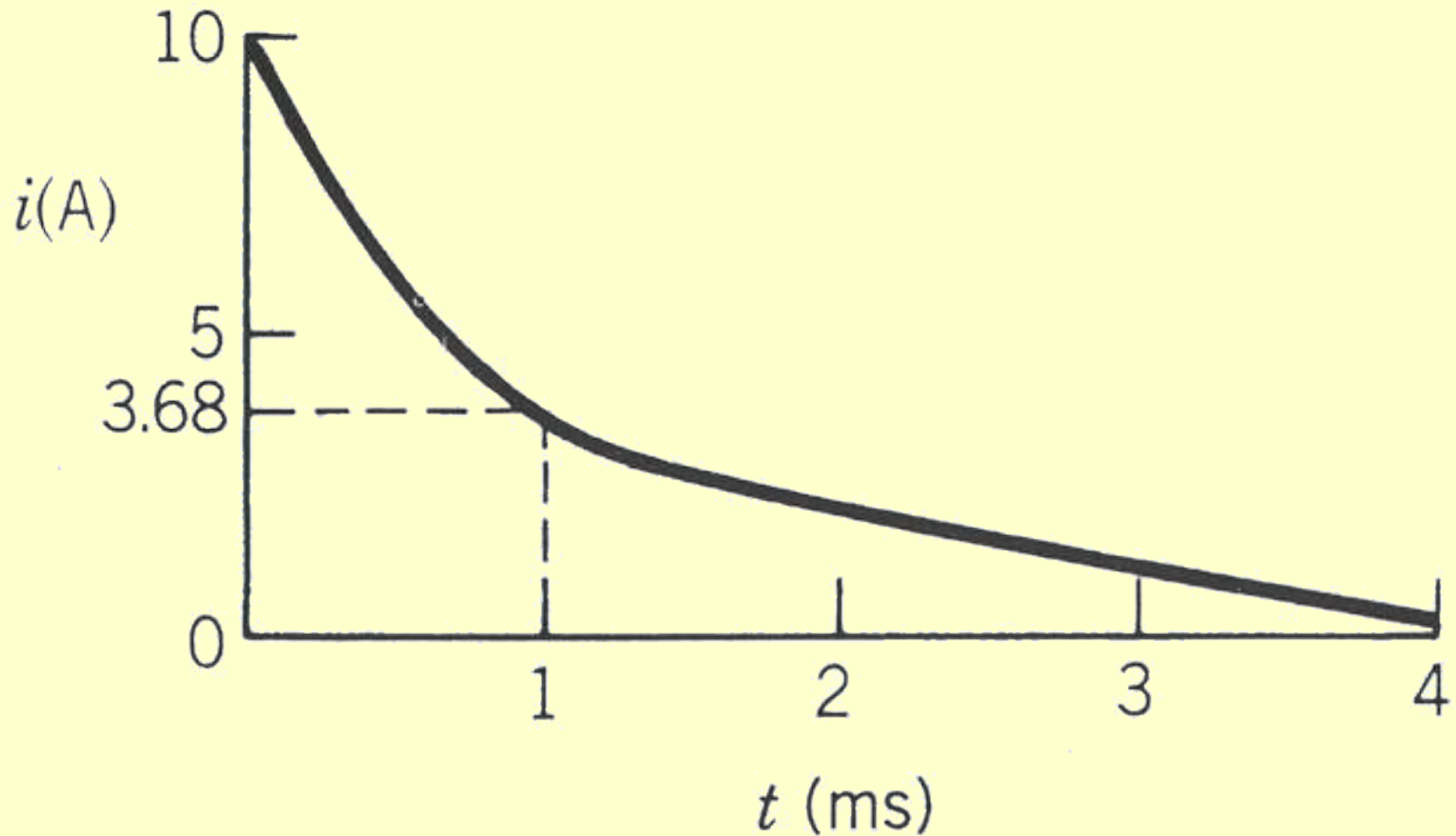
When switch is closed at $t=1\text{ms}$, the equivalent resistance is 1Ω . Then,

$$\tau_1 = \frac{L}{R} = \frac{2}{1} = 2\text{ms}$$

- Thus i for $t \geq 1\text{ms}$ is

$$\begin{aligned} i &= i(t_1) e^{-\frac{(t-t_1)}{\tau_1}} \\ &= 3.68 e^{-\frac{(t-t_1)}{\tau_1}} \text{ A} \end{aligned}$$

The graph of current for $t \geq 0$



First-Order and Second-Order Response of RL and RC Circuit

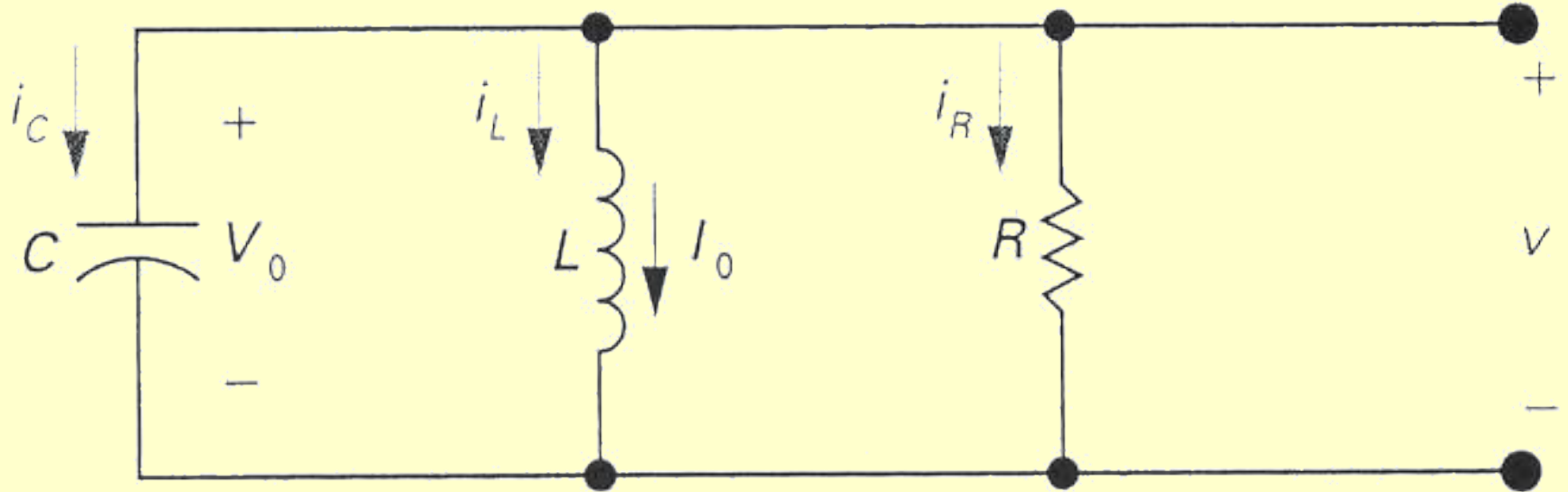
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Second order response for RLC c

- **RLC circuit:** consist of resistor, inductor and capacitor
 - **Second order response :** response from RLC circuit
 - **Type of RLC circuit:**
 1. Series RLC
 2. Parallel RLC
- 

Natural response of parallel RLC



- Summing all the currents away from node,

$$\frac{V}{R} + \frac{1}{L} \int_0^t v d\tau + I_0 + C \frac{dv}{dt} = 0$$

- Differentiating once with respect to t ,

$$\frac{1}{R} \frac{dv}{dt} + \frac{v}{L} + C \frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{v}{LC} = 0$$

- Assume that $v = Ae^{st}$

$$As^2 e^{st} + \frac{As}{RC} e^{st} + \frac{A}{LC} e^{st} = 0$$

$$Ae^{st} \left(s^2 + \frac{s}{RC} + \frac{1}{LC} \right) = 0$$

characteristic equation

- Characteristic equation is zero:

$$\left(s^2 + \frac{s}{RC} + \frac{1}{LC} \right) = 0$$

- The two roots:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- The natural response of series RLC:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- The two roots:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

- where:

$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

• Summary

Parameter	Terminology	Value in natural response
s_1, s_2	Characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
α	Neper frequency	$\alpha = \frac{1}{2RC}$
ω_0	Resonant radian frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$

- 
- The two roots s_1 and s_2 are depend on α and ω_o value.
 - 3 possible condition is:
 1. If $\omega_o < \alpha^2$, the voltage response is overdamped
 2. If $\omega_o > \alpha^2$, the voltage response is underdamped
 3. If $\omega_o = \alpha^2$, the voltage response is critically damped
- 

Overdamped voltage response

- overdamped voltage solution

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- The constant of A_1 dan A_2 can be obtain from,

$$v(0^+) = A_1 + A_2$$

$$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2$$

- The value of $v(0^+) = V_0$ and initial value of dv/dt is

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C}$$



The process for finding the overdamped response, $v(t)$:

1. Find the roots of the characteristic equation, s_1 dan s_2 , using the value of R , L and C .
 2. Find $v(0+)$ and $dv(0+)/dt$ using circuit analysis.
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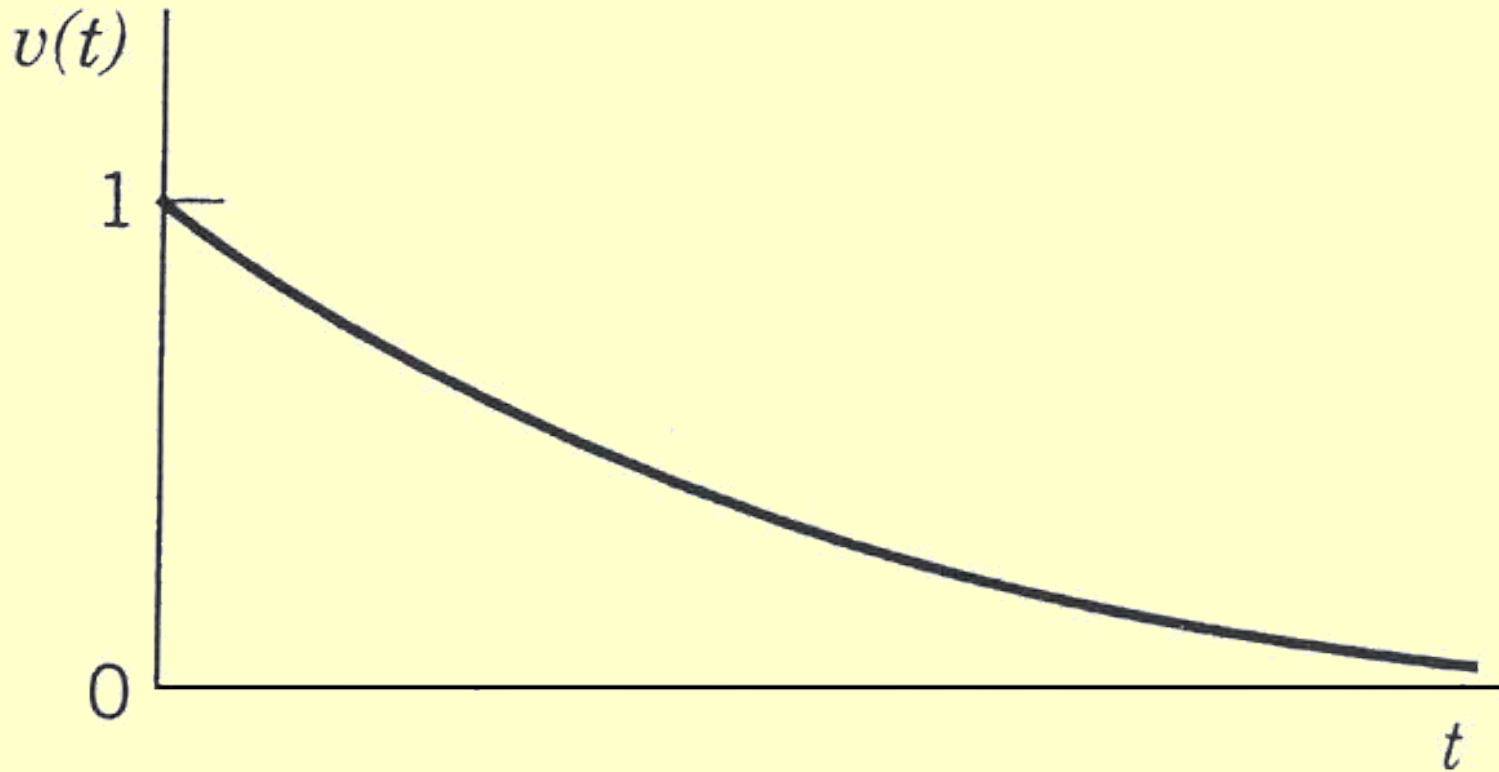
3. Find the values of A_1 and A_2 by solving equation below simultaneously:

$$v(0^+) = A_1 + A_2$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = s_1 A_1 + s_2 A_2$$

4. Substitute the value for s_1 , s_2 , A_1 dan A_2 to determine the expression for $v(t)$ for $t \geq 0$.

- Example of overdamped voltage response for $v(0) = 1V$ and $i(0) =$





Underdamped voltage response

- When $\omega_o^2 > \alpha^2$, the roots of the characteristic equation are complex and the response is underdamped.
- 

- The roots s_1 and s_2 as,

$$\begin{aligned} s_1 &= -\alpha + \sqrt{-\left(\omega_0^2 - \alpha^2\right)} \\ &= -\alpha + j\sqrt{\omega_0^2 - \alpha^2} \\ &= -\alpha + j\omega_d \end{aligned}$$

$$s_2 = -\alpha - j\omega_d$$

- ω_d : damped radian frequency

- The underdamped voltage response of a parallel RLC circuit is

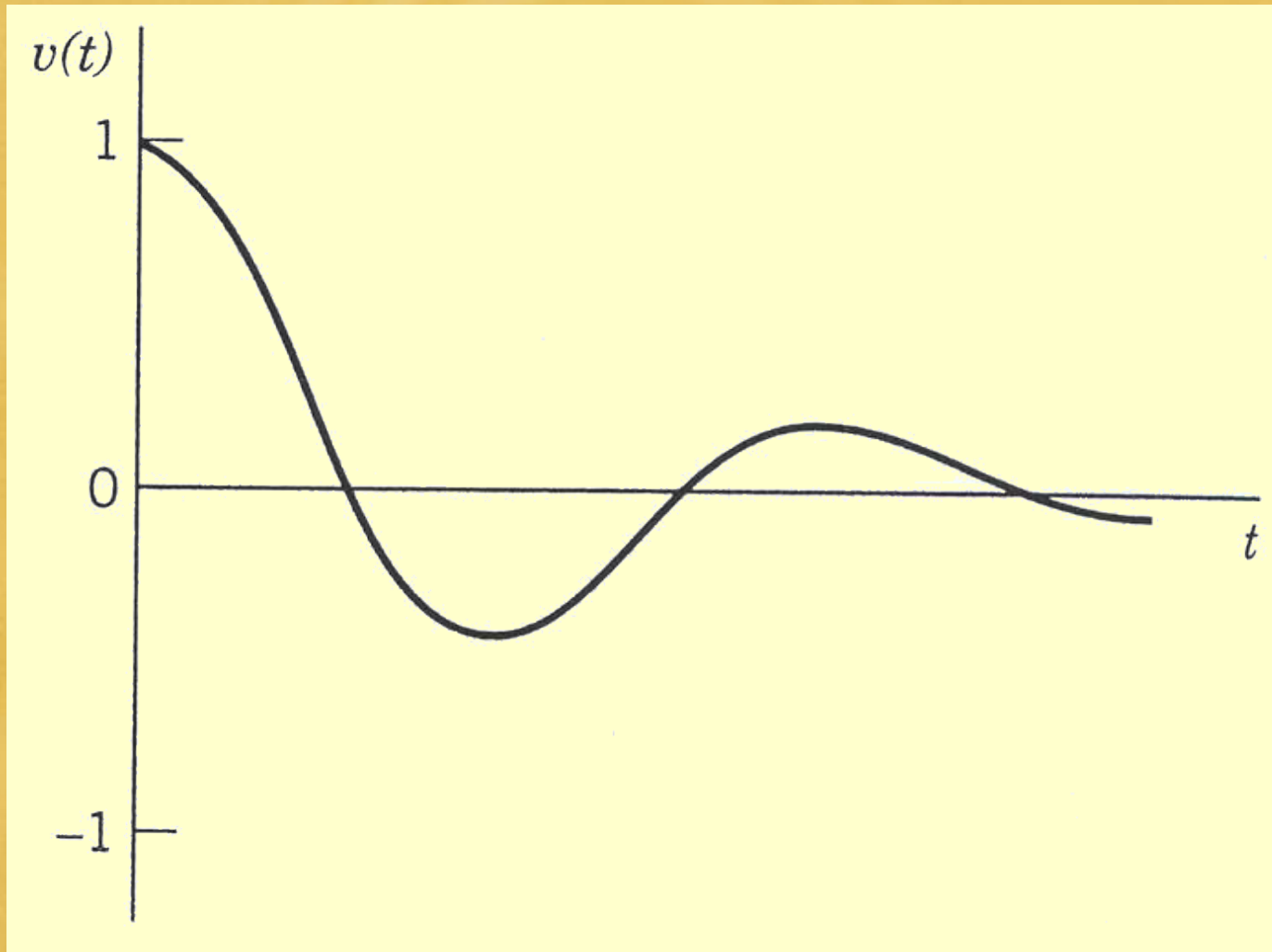
$$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

- The constants B_1 dan B_2 are real not complex number.
- The two simultaneous equation that determine B_1 and B_2 are:

$$v(0^+) = V_0 = B_1$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -\alpha_1 B_1 + \omega_d B_2$$

Example of underdamped voltage response for $v(0) = 1\text{V}$ and $i(0) = 0$





Critically Damped voltage response

- A circuit is critically damped when $\omega_o^2 = \alpha^2$ ($\omega_o = \alpha$). The two roots of the characteristic equation are real and equal that is,

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

- The solution for the voltage is

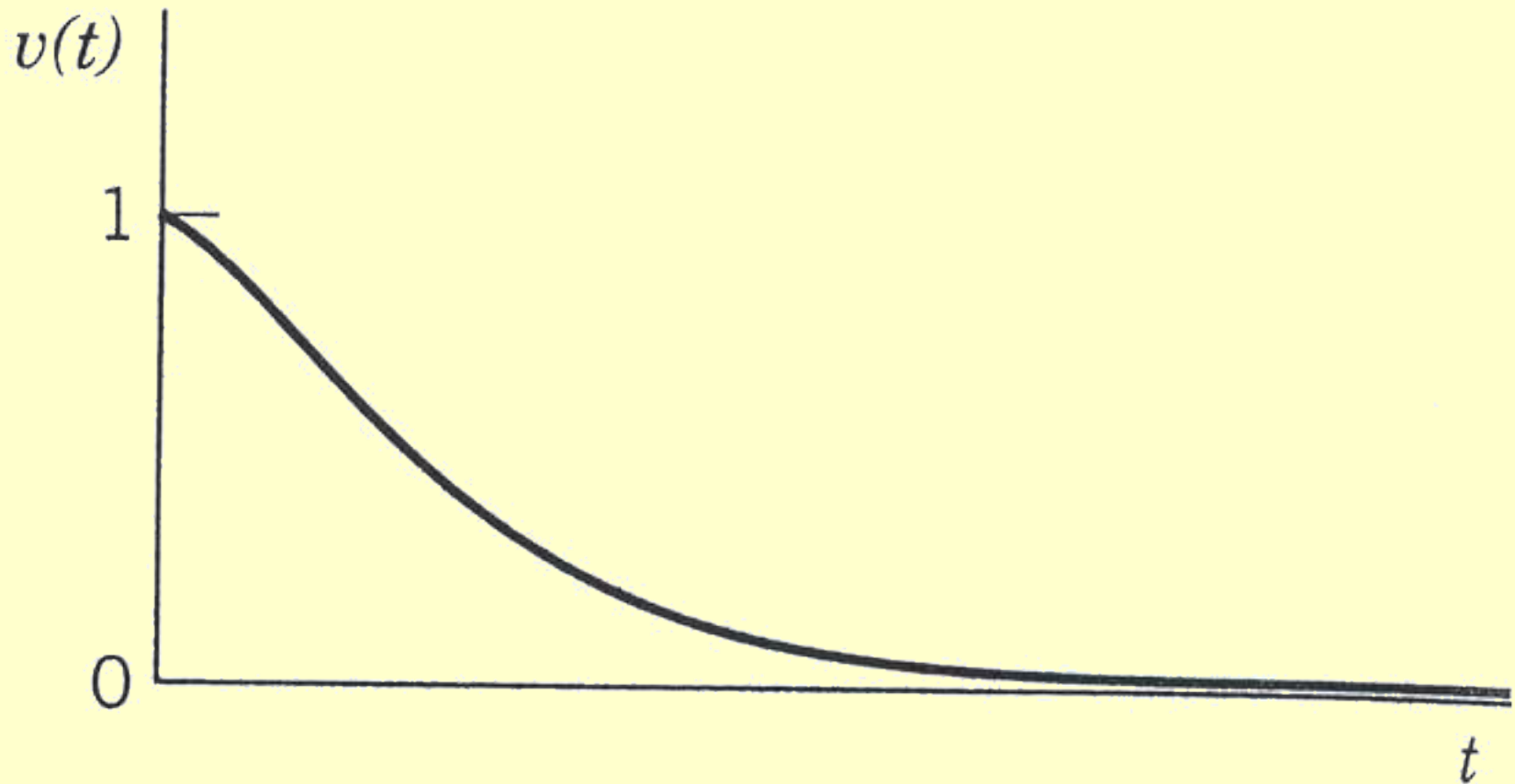
$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

- The two simultaneous equations needed to determine D_1 and D_2 are,

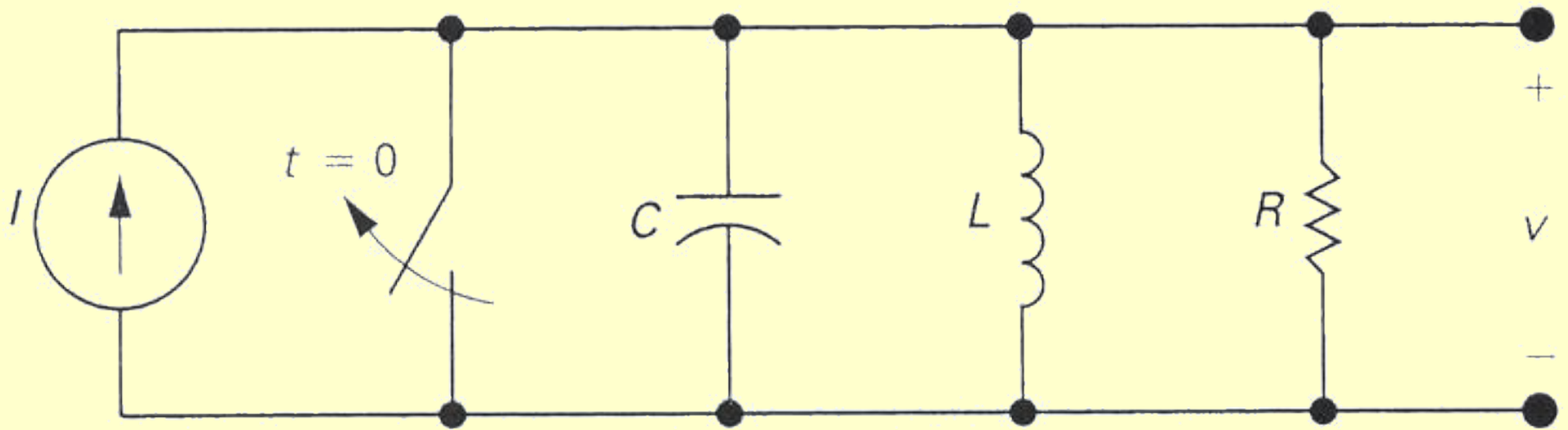
$$v(0^+) = V_0 = D_2$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2$$

Example of the critically damped voltage response for $v(0) = 1\text{V}$ and $i(0) = 0$



The step response of a parallel RLC circuit



- From the KCL,

$$\dot{i}_L + \dot{i}_R + \dot{i}_C = I$$

$$\dot{i}_L + \frac{v}{R} + C \frac{dv}{dt} = I$$

- Because

$$v = L \frac{di}{dt}$$

- We get

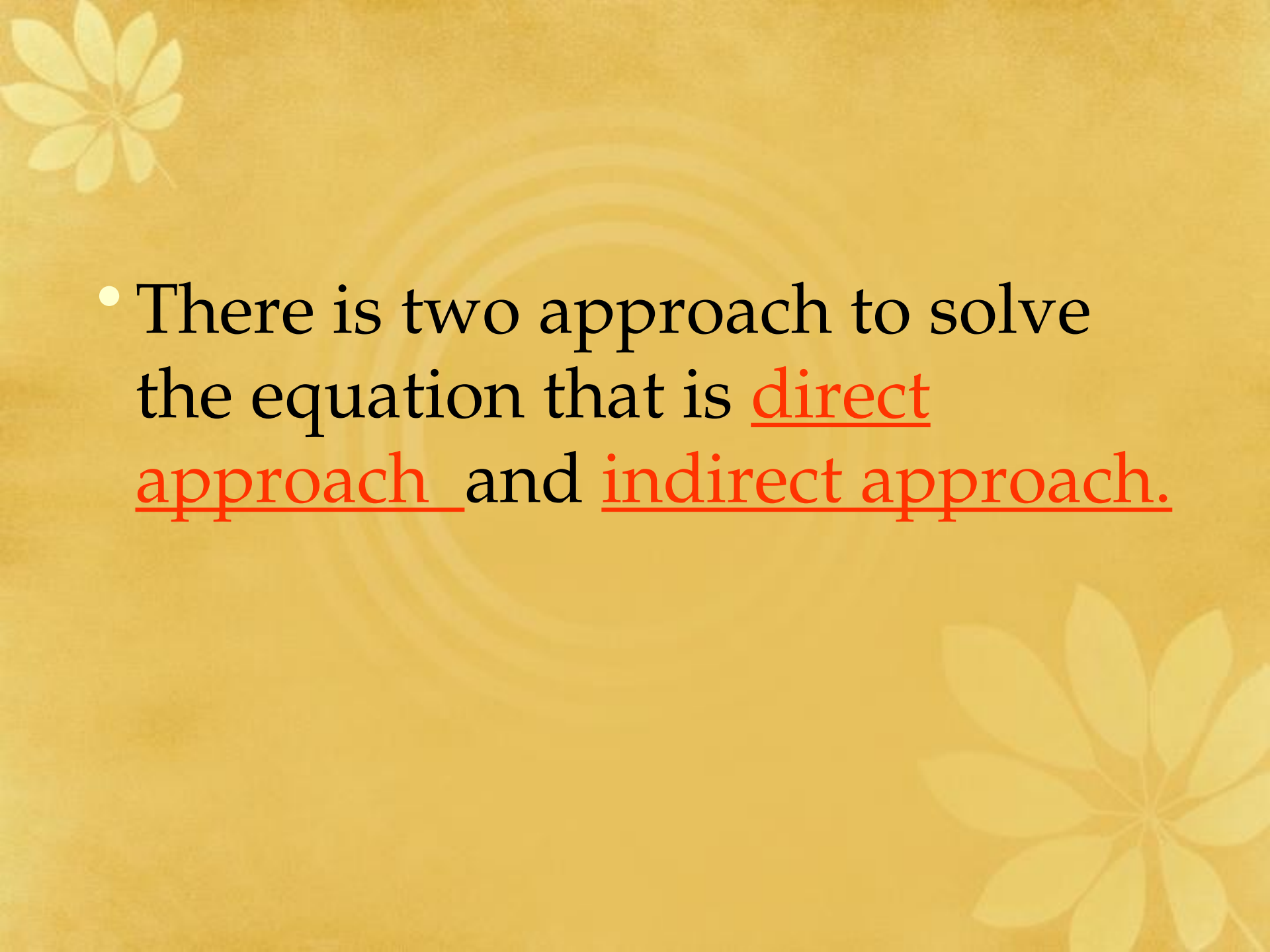
$$\frac{dv}{dt} = L \frac{d^2 i_L}{dt^2}$$



- Thus,

$$i_L + \frac{L}{R} \frac{di_L}{dt} + LC \frac{d^2 i_L}{dt^2} = I$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC} \frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

- 
- There is two approach to solve the equation that is direct approach and indirect approach.

Indirect approach

- From the KCL:

$$\frac{1}{L} \int_0^t v d\tau + \frac{v}{R} + C \frac{dv}{dt} = I$$

- Differentiate once with respect to t :

$$\frac{v}{L} + \frac{1}{R} \frac{dv}{dt} + C \frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{v}{LC} = 0$$

- The solution for v depends on the roots of the characteristic equation:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$v = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

- Substitute into KCL equation :

$$i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$i_L = I + B_1' e^{-\alpha t} \cos \omega_d t \\ + B_2' e^{-\alpha t} \sin \omega_d t$$

$$i_L = I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$



Direct approach

- It is much easier to find the primed constants directly in terms of the initial values of the response function.

$$A_1', A_2', B_1', B_2', D_1', D_2'$$

- The primed constants could be find from

$$i_L(0)$$

and

$$\frac{di_L(0)}{dt}$$

- 
- 
- The solution for a second-order differential equation equals the forced response plus a response function identical in form to natural response.

- If I_f and V_f is the final value of the response function, the solution for the step function can be write in the form,

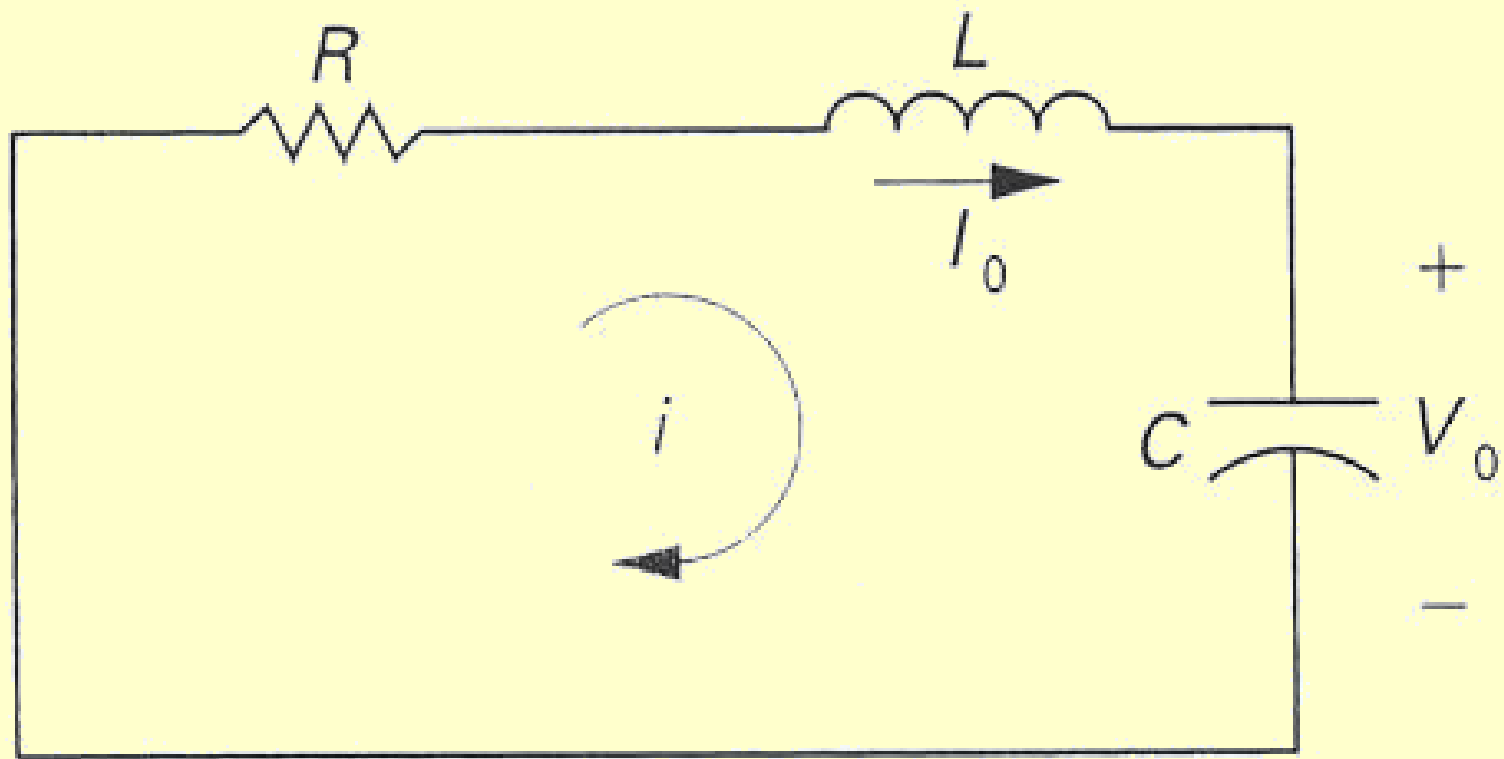
$$i = I_f + \left\{ \begin{array}{l} \text{Function of the same form} \\ \text{as the natural response} \end{array} \right\}$$

$$v = V_f + \left\{ \begin{array}{l} \text{function of the same form} \\ \text{as the natural response} \end{array} \right\}$$

Natural response of a series RLC

- The procedures for finding the natural response of a series RLC circuit is the same as those to find the natural response of a parallel RLC circuit because both circuits are described by differential equations that have same form.

Series RLC circuit



- Summing the voltage around the loop,

$$Ri + L \frac{di}{dt} + \frac{1}{C} \int_0^t i d\tau + V_0 = 0$$

- Differentiate once with respect to t ,

$$R \frac{di}{dt} + L \frac{d^2 i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0$$

- The characteristic equation for the series RLC circuit is,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

- The roots of the characteristic equation are,

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

@

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

- Neper frequency (α) for series RLC,

$$\alpha = \frac{R}{2L} \text{ rad / s}$$

And the resonant radian frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad / s}$$



The current response will be overdamped, underdamped or critically damped according to,

$$\omega_0^2 < \alpha^2$$

$$\omega_0^2 > \alpha^2$$

$$\omega_0^2 = \alpha^2$$


- Thus the three possible solutions for the currents are,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t \\ + B_2 e^{-\alpha t} \sin \omega_d t$$

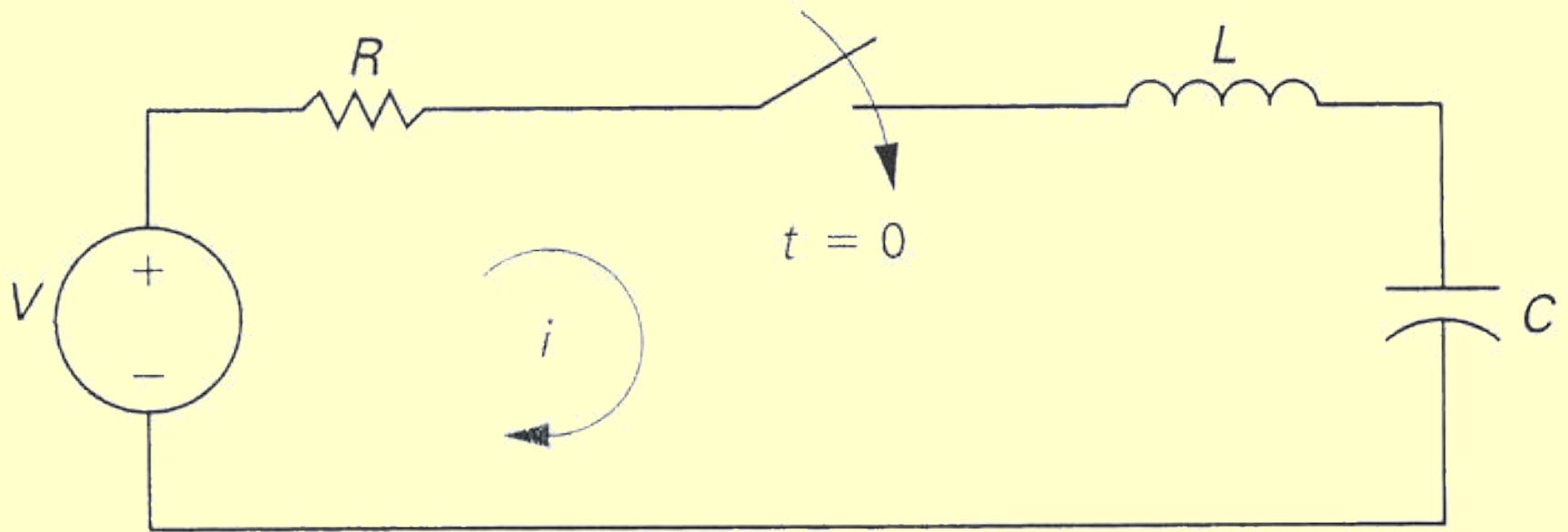
$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$



Step response of series RLC

- The procedures is the same as the parallel circuit.
- 

Series RLC circuit



- 
- Use KVL,

$$v = Ri + L \frac{di}{dt} + v_C$$

- The current, i is related to the capacitor voltage (v_C) by expression,

$$i = C \frac{dv_C}{dt}$$

- Differentiate once i with respect to t

$$\frac{di}{dt} = C \frac{d^2 v_C}{dt^2}$$

- Substitute into KVL equation,

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}$$

- Three possible solution for v_C are,

$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

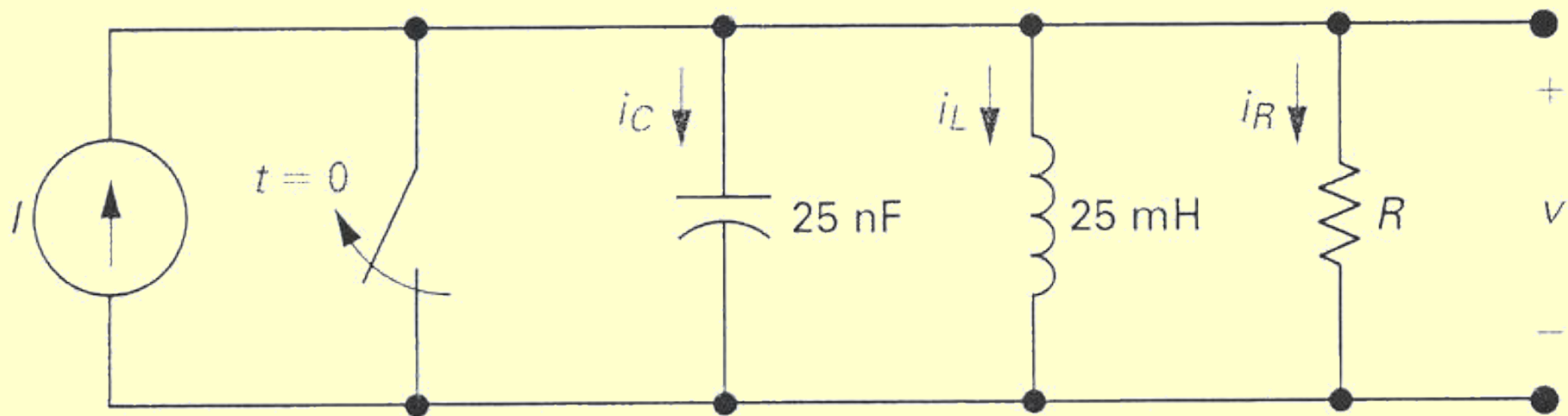
$$v_C = V_f + B_1' e^{-\alpha t} \cos \omega_d t \\ + B_2' e^{-\alpha t} \sin \omega_d t$$

$$v_C = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

Example 1 (Step response of parallel RLC)

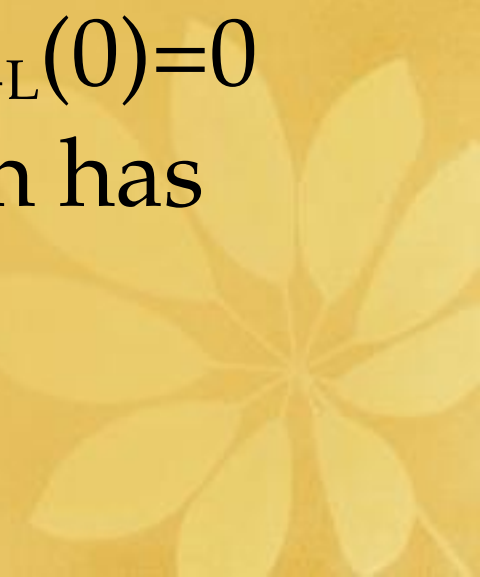
The initial energy stored in the circuit is zero. At $t = 0$, a DC current source of 24mA is applied to the circuit. The value of the resistor is 400Ω .

1. What is the initial value of i_L ?
2. What is the initial value of di_L/dt ?
3. What are the roots of the characteristic equation?
4. What is the numerical expression for $i_L(t)$ when $t \geq 0$?





Solution

1. No energy is stored in the circuit prior to the application of the DC source, so the initial current in the inductor is zero. The inductor prohibits an instantaneous change in inductor current, therefore $i_L(0)=0$ immediately after the switch has been opened.
- 

2. The initial voltage on the capacitor is zero before the switch has been opened, therefore it will be zero immediately after. Because

$$v = L \frac{di_L}{dt}$$

thus

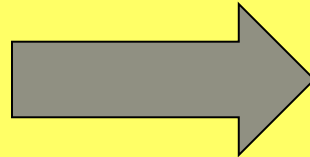
$$\frac{di_L(0^+)}{dt} = 0$$

3. From the circuit elements,

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(400)(25)}$$

$$= 5 \times 10^4 \text{ rad / s}$$



$$\alpha^2 = 25 \times 10^8$$

- Thus the roots of the characteristic equation are real,

$$s_1 = -5 \times 10^4 + 3 \times 10^4$$

$$= -20\,000 \text{ rad} / s$$

$$s_2 = -5 \times 10^4 - 3 \times 10^4$$

$$= -80\,000 \text{ rad} / s$$

4. The inductor current response will be overdamped.

$$i_L = I_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Two simultaneous equations:

$$i_L(0) = I_f + A_1' + A_2' = 0$$

$$\frac{di_L(0)}{dt} = s_1 A_1' + s_2 A_2' = 0$$

$$A_1' = -32mA \quad A_2' = 8mA$$

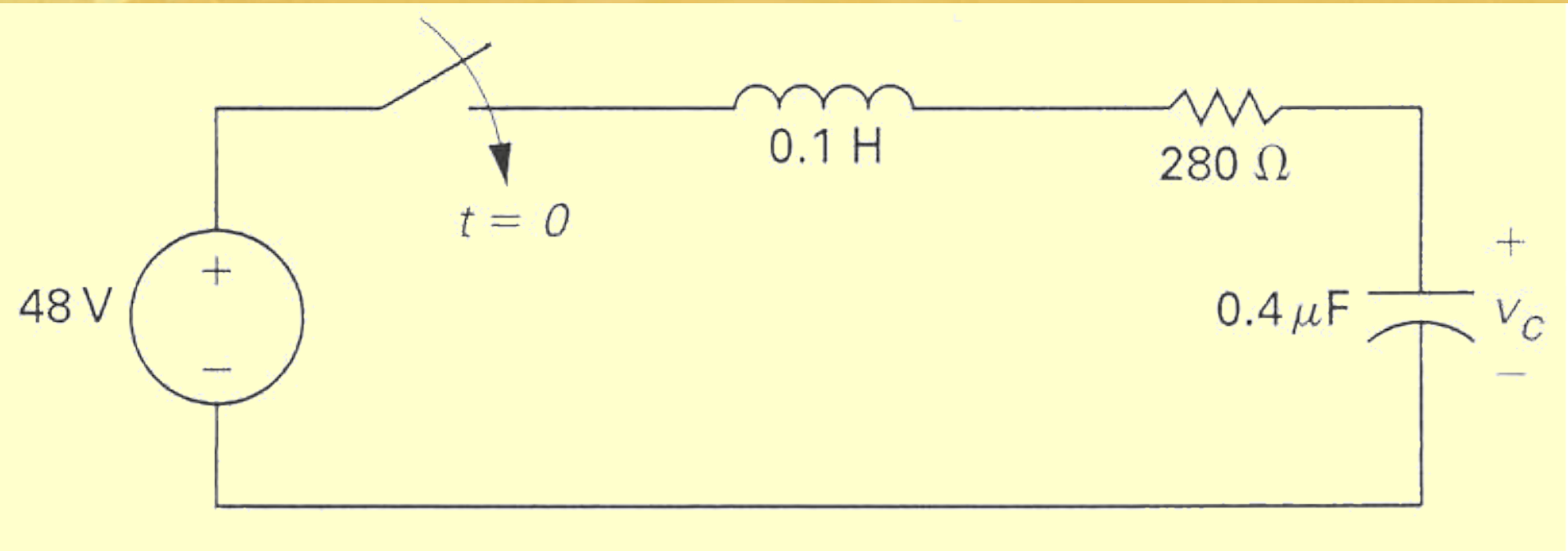
- Numerical solution:

$$i_L(t) = \left(\begin{array}{c} 24 - 32e^{-20000t} \\ + 8e^{-80000t} \end{array} \right) mA$$

for $t \geq 0$

Example 2 (step response of series RLC)

- No energy is stored in the 100mH inductor or 0.4 μ F capacitor when switch in the circuit is closed. Find $v_C(t)$ for $t \geq 0$.



Solution

- The roots of the characteristic equation:

$$\begin{aligned}s_1 &= -\frac{280}{0.2} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}} \\ &= (-1400 + j4800) \text{ rad / s} \\ s_2 &= (-1400 - j4800) \text{ rad / s}\end{aligned}$$

- The roots are complex, so the voltage response is underdamped. Thus:

$$v_C = 48 + B_1 e^{-1400t} \cos 4800t + B_2 e^{-1400t} \sin 4800t \quad t \geq 0$$

- No energy is stored in the circuit initially, so both $v_C(0)$ and $dv_C(0^+)/dt$ are zero. Then:

$$v_C(0) = 0 = 48 + B_1'$$

$$\frac{dv_C(0^+)}{dt} = 0 = 4800B_2' - 1400B_1'$$

- Solving for B_1' and B_2' yields,

$$B_1' = -48V$$

$$B_2' = -14V$$

- Thus, the solution for $v_C(t)$,

$$v_C(t) = \begin{pmatrix} 48 - 48e^{-1400t} \cos 4800t \\ -14e^{-1400t} \sin 4800t \end{pmatrix} V$$

for $t \geq 0$