

Chapter 5: understanding Fixed-Income Risk and Return

1. Introduction

- Any analysis of fixed-rate securities starts with an understanding of their risk and return characteristics.
- The yield-to-maturity, or internal rate of return on future cash flows, is of particular focus.
- The return on a fixed-rate bond is affected by many factors, such as credit risk (potential default on payments) and interest rate risk (varying coupon reinvestment rate and sale price).
- The focus of this chapter is not on credit risk, but on interest rate risk & the impact of a change in interest rates.

2. Sources of Return

- A fixed-rate bond has 3 sources of return:
 - ⇒ Receipt of the promised coupon and principal payments on the scheduled dates
 - ⇒ Reinvestment of coupon payments
 - ⇒ Potential capital gains or losses on the sale of the bond prior to maturity
- Horizon yield:
 - ⇒ The internal rate of return between the investment horizon and the purchase price of the bond

a. Investment horizon and interest rate risk

- The *investment horizon* is at the heart of understanding interest rate risk and return for fixed-rate bonds
- If interest rates change, what happens to coupon reinvestment and bond price?
- There are 2 offsetting types of interest rate risk
 - ⇒ Coupon reinvestment risk
 - ⇒ Market price risk

b. Interest rate & return examples

- Investor A (the buy-and-hold investor) purchases a 10-year, 8% annual-pay bond at a price of 85.503075 (per 100 of par). The investor holds the bond until maturity and is able to reinvest all coupon payments received during the term of the bond at the YTM

at issuance. Compute the total amount of reinvestment income earned. Determine her realized rate of return on the investment.

⇒ To calculate the amount of reinvestment income earned, need to calculate YTM at issuance (= rate at which coupon payments are reinvested)

$$N=10; PMT = 8; PV = -\$85.503075; FV = \$100, CPT I/Y; I/Y = 10.40\%$$

⇒ Next, compute the future value of the ten \$8 coupon payments received over the bond's term if they're invested until maturity at the bond's YTM (10.40%)

$$I/Y = 10.40; PMT = 8; N = 10; PV = 0, CPT FV; FV = \$129.970678$$

⇒ Then isolate the reinvestment income as the difference between the FV of this annuity & the total amount of coupon income ($10 \times \$8 = \$80$) received

$$\text{Reinvestment income} = \$129.970678 - \$80 = \$49.970678$$

⇒ Investor's total return = reinvestment income + coupon payments + principal

$$\text{Total return} = \$49.970678 + \$80 + \$100 = \$229.970678$$

⇒ The realized rate of return: $\$85.503075 = \$229.970678 / (1+r)^{10}$; $r = 10.4\%$

⇒ **Takeaways:**

- Think of the reinvestment income as the “interest-on-interest” income from compounding
- This example illustrates that the YTM represents the investor's realized rate of return on a bond if the following 3 conditions hold:
 - The bond is held till maturity
 - The issuer does not default
 - All coupon payments can be reinvested through the term of the bond at a rate = the bond's YTM at issuance

- Investor B (the short-term investor) purchases a 10-year, 8% annual-pay bond at a price of 85.503075 (per 100 of par) implying a YTM of 10.40%. The investor holds the bond for 4 years, after which she sells the bond at a price that entails a YTM of 10.40%. She is also able to reinvest the coupons at 10.40%. Compute the total amount of reinvestment income earned by Investor B. Determine her realized rate of return on the investment.

- ⇒ The future value of the four \$8 coupon payments received over the investment horizon if they're invested until maturity at 10.40% (YTM at issuance) is:
 $I/Y = 10.4; PMT = \$8; N = 4; CPT FV; FV = \$37.347111$
- ⇒ Total reinvestment income = $\$37.347111 - (\$8 \times 4 = \$32) = \$5.347111$
- ⇒ Bond's selling price at the end of Year 4 is PV of expected future payments (e.g. for the next 6 years):
 $N = 6; I/Y = 10.40; PMT = \$8; FV = \$100; CPT PV; PV = \$89.66877$
- ⇒ Investor's total return = selling price + coupon receipts + reinvestment income
 $Total\ return = \$89.66877 + \$32 + \$5.547111 = \$127.015881$
- ⇒ The realized rate of return: $\$85.503075 = \$127.015881 / (1+r)^4; r = 10.4\%$
- ⇒ **Takeaways:**
 - The previous example illustrates that the horizon yield (or annualized holding period rate of return) = the YTM at issuance if the following 2 conditions hold:
 - o The bond is sold at its carrying value (i.e. at a price that lies on a constant-yield price trajectory)
 - o All coupon payments can be reinvested at a rate = the bond's YTM at issuance until the date of sale.

c. Interest rate & return examples when interest rates rise

- Same 2 investors:
 - ⇒ Investor A: a buy-and-hold investor holds bond until maturity
 - Purchases a 10-year, 8% annual-pay bond at a price of 85.503075 (per 100 of par)
 - ⇒ Investor B: a short-term investor who sells bond prior to maturity
 - Purchases same bond at same price and holds for 4 years
- Note: now assume interest rates change
 - ⇒ Examples #3 & #4 interest rates rise 100 bps
 - ⇒ Examples #5 & #6 interest rates fall 100 bps

d. Examples #3 & #4 – interest rates rise

- Investor A (buy-and-hold) purchases a 10-year, 8% annual-pay bond at a price of 85.503075 (per 100 of par). After she purchases bond, but before first coupon payment, interest rates rise to 11.40%.

- ⇒ FV of ten \$8 coupon payments received over investment horizon if invested until maturity at 11.40% is:
 $I/Y = 11.40; PMT = \$8; N = 10; PV = 0; CPT FV; FV = \$136.380195$
- ⇒ Realized rate of return: $\$85.503075 = (\$136.380195 + \$100)/(1+r)^{10} = 10.7\%$
- ⇒ Benefits from higher coupon reinvestment rate as her realized rate of return rises by 30 bps from 10.40% to 10.70%
- Investor B (short-term investor) purchases a 10-year, 8% annual-pay bond at a price of 85.503075 (per 100 of par). After she purchases bond, but before first coupon payment, interest rates rise to 11.40%. Investor sells the bond at the end of Year 4 (when it's priced to yield 11.40%).
 - ⇒ FV of four \$8 coupon payments over investment horizon if invested until end of Yr 4 at 11.40% is:
 $I/Y = 11.40; PMT = \$8; PV = 0; N = 4; CPT FV; FV = \$37.8997$
 - ⇒ Selling price of the bond (at the end of Year 4): $N = 6; I/Y = 11.40; FV = \$100;$
 $CPT PV; PV = \$85.78041$ (the PV of the remaining payments)
 - ⇒ Realized rate of return: $\$85.5031 = (\$37.8997 + \$85.78041)/(1 + r)^4; r = 9.67\%$
 - ⇒ Has lower rate of return (9.67% vs 10.40%) given interest rates have gone up. Even though FV of reinvested coupons goes up is more than offset by capital loss of \$3.89 ($\$89.67 - \$85.78$) as bond sold for < its carrying value

e. Example #5 & #6 – interest rates fall

- Investor A (buy-and-hold) purchases a 10-year, 8% annual-pay bond at a price of 85.503075 (per 100 of par). After she purchases bond, but before first coupon payment, interest rates fall to 9.40%.
 - ⇒ FV of ten \$8 coupon payments received over investment horizon if invested until maturity at 9.40% is:
 $I/Y = 9.40; PMT = \$8; N = 10; PV = 0; CPT FV; FV = \$123.888356$
 - ⇒ Realized rate of return: $\$85.503075 = (\$123.888356 + \$100)/(1+r)^{10} = 10.1\%$
 - ⇒ Suffers from lower coupon reinvestment rate as her realized rate of return falls by 30 bps from 10.40% to 10.1%
- Investor B (short-term investor) purchases a 10-year, 8% annual-pay bond at a price of 85.503075 (per 100 of par). After she purchases bond, but before first coupon payment,

interest rates fall to 9.40%. Investor sells the bond at the end of Year 4 (when it's priced to yield 9.40%).

⇒ FV of four \$8 coupon payments over investment horizon if invested until end of Yr 4 at 9.40% is:

$$I/Y = 9.40; PMT = \$8; PV = 0; N = 4; CPT FV; FV = \$36.801397$$

⇒ Selling price of the bond (at the end of Year 4): $N = 6$; $I/Y = 9.40$; $FV = \$100$; CPT PV; $PV = \$93.793912$ (the PV of the remaining payments)

⇒ Realized rate of return: $\$85.503075 = (\$36.801397 + \$93.793912)/(1 + r)^4$; $r = 11.17\%$

⇒ Has higher rate of return (11.17% vs 10.40%) given interest rates have gone up. Even though FV of reinvested coupons goes down is more than offset by capital gain of \$4.125 ($\$93.794 - \$89.669$) as bond sold for > its carrying value

- Which situation provides the best return?

⇒ Buy-and-hold investor who holds an 8% 10-year bond until maturity

- If interest rates remain unchanged ($r = 10.4\%$)
- If interest rates go up able to reinvest coupons at higher rate ($r = 10.7\%$);
- If interest rates go down ($r = 10.1\%$)

⇒ Short-term investor who sells an 8% 10-year bond after 4 years

- If interest rates remain unchanged ($r = 10.4\%$)
- If interest rates go up ($r = 9.67\%$)
- If interest rates go down; able to realize gain on bond sale ($r = 11.17\%$)

f. Bond carrying value

- Carrying value:

⇒ The value of a bond (at any time between the purchase date & maturity date) that entails the same YTM as when the bond was purchased

⇒ The purchase price plus (minus) the amortized amount of the discount (premium) if the bond is purchased at a price below (above) par value

- Looking at Example #2:

- ⇒ Carrying value of the bond at end of Year 1 = \$86.395394 ($N = 9$; $I/Y = 10.40$; $PMT = \$8$; $FV = \$100$; CPT PV), which means that discount amortization for Year 1 = \$0.892319 (= \$86.395394 – 85.503075)
- ⇒ By the end of Year 4 (time of sale) the carrying value of the bond has risen to \$89.66877. This increase in the value of the bond is a movement along the constant-yield price trajectory ($YTM = 10.40\%$)
- Capital gains/losses arise if a bond is sold at a price different from its carrying value
- g. Coupon reinvestment risk and market price risk**
- The future value of **reinvested coupon** payments (and in a portfolio, the principal on bonds that mature before the horizon date) *increases* when interest rates go up and *decreases* when rates go down.
- The sale price on a bond that matures after the horizon date (and thus needs to be sold) *decreases* when interest rates go up and *increases* when rates go down.
- **Coupon reinvestment risk** matters more when the investor has a long-term horizon relative to the time-to-maturity of the bond. **Market price risk** matters more to short-term investors

3. Interest Rate Risk on Fixed-rate Bonds

- The **duration** of a bond measures the sensitivity of the bond's full price (including accrued interest) to changes in the bond's yield-to-maturity or, more generally, to changes in benchmark interest rates.
- a. Bond duration explained**
- Duration tells investors:
 - ⇒ The weighted average of the time (in years) that it would take to receive all the bond's promised cash flows, where the weights are calculated as the present value of each cash flow divided by the bond's full price
 - ⇒ How much a bond's price might change when interest rates change i.e. how much risk they face from interest rate changes
- Higher duration: means bond's price is more sensitive to interest rate changes
- Lower duration: means bond's price is less sensitive to interest rate changes
- Example: Duration of 5 means the price of the bond will increase/decrease by 5% when interest rates move by 1%

b. Calculating Macaulay duration for an annual pay bond: example #1

- Consider a newly-issued 10-year, 8% annual-pay bond priced at 85.503075 per 100 of par value to yield 10.40%. Calculate the bond's Macaulay duration

Period (1)	Cash Flow (2)	Present Value (3)	Weight (4)	Period × Weight (5)
1	8	7.246377	0.08475	0.0847
2	8	6.563747	0.07677	0.1535
3	8	5.945423	0.06953	0.2086
4	8	5.385347	0.06298	0.2519
5	8	4.878032	0.05705	0.2853
6	8	4.418507	0.05168	0.3101
7	8	4.002271	0.04681	0.3277
8	8	3.625245	0.04240	0.3392
9	8	3.283737	0.03840	0.3456
10	108	40.154389	0.46963	4.6963
		85.503075	1.00000	<u>7.0029</u>

- Column 2: Annual coupon payment = $8\% \times \$100 = \$8$. Principal payment of \$100 at maturity
- Column 3: PV of each cash flow, e.g. PV of Year 10 cash flow ($\$108$) = $108 / (1.104)^{10} = 40.154389$
- Column 4: shows the weight (share of total PV corresponding to each cash flow). The weight of the Year 10 payment = $40.154389 / 85.503075 = 0.46963$ (or 46.963% of the PV cash flows)
- Column 5: multiplies the # of periods to receipt of the cash flow (Column 1) by the weight (Column 4). The sum of Column 5 (7.0029) is the Macaulay duration of the bond

**b. Calculating Macaulay duration (for semiannual-pay bond between coupon dates):
example #2**

- Consider a 7-year, 5% semiannual-pay bond that matures on September 15, 2020, which is purchased on January 3, 2014. Coupon payments are made on March 15 & September 15 each year. YTM is 5.00% quoted on a street-convention semiannual bond basis. $t = 15 + 30 + 30 + 30 + 3$; $t/T = 108/180 = 0.6$; $1 - 0.6 = 0.4$

Period	Time to Receipt	Cash Flow	Present Value	Weight	Time × Weight
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(1)	(2)	(3)	(4)	(5)	(6)
1	0.4000	2.5	2.475429	0.02439	0.0097561
2	1.4000	2.5	2.415053	0.02380	0.0333135
3	2.4000	2.5	2.356149	0.02321	0.0557160
4	3.4000	2.5	2.298682	0.02265	0.0770058
5	4.4000	2.5	2.242616	0.02210	0.0972240
6	5.4000	2.5	2.187918	0.02156	0.1164101
7	6.4000	2.5	2.134555	0.02103	0.1346024
8	7.4000	2.5	2.082492	0.02052	0.1518381
9	8.4000	2.5	2.031700	0.02002	0.1681530
10	9.4000	2.5	1.982146	0.01953	0.1835816
11	10.4000	2.5	1.933801	0.01905	0.1981576
12	11.4000	2.5	1.886635	0.01859	0.2119134
13	12.4000	2.5	1.840620	0.01814	0.2248803
14	13.4000	102.5	73.624790	0.72542	9.7206330
			101.492586	1.00000	<u>11.383185</u>

- Column 1: 14 semiannual periods remaining until maturity (including the current period)
- Column 2: contains # of semiannual periods to receipt of cash flow. Payment 1 will be received in $1 - (108/180) = 0.4$ semiannual periods, Payment 2 will be received in $2 - (108/180) = 1.4$ semiannual periods, and so on
- Column 3: The coupon payment per semiannual period is \$2.50
- Column 4: The annual YTM is 5.00% so the yield per semiannual period is 2.50%.
When this yield per semiannual period is used to compute the PV of each cash flow, the full price of the bond = 101.492586 (sum of Column 4)
- Column 5: contains the weights (share of the full price corresponding to each cash flow)
- Column 6: The values are obtained by multiplying values in Column 2 by Column 5
- Macaulay duration = sum of the values in Column 6
- The value obtained here (11.383185) is in terms of # of semiannual periods
- Similar to coupon rates and yields-to-maturity, duration statistics are also annualized

- This is done by dividing the duration per period by the number of periods in the year
- Annualized Macaulay duration = $11.383185/2 = 5.691592$ years

c. Types of bond duration

- There are several types of bond duration. In general, these can be divided into **yield duration** and **curve duration**.
 - ⇒ **Yield duration** is the sensitivity of the bond price with respect to the bond's own yield-to-maturity.
 - ⇒ **Curve duration** is the sensitivity of the bond price (or more generally, the market value of a financial asset or liability) with respect to a benchmark yield curve.

d. Yield duration statistics

- Yield duration statistics used in fixed-income analysis include
 - ⇒ Macaulay duration
 - ⇒ Modified duration
 - ⇒ Money duration
 - ⇒ Price value of a basis point (PVBP)

A. The Macaulay duration (MACDUR)

$$D = \frac{\left[\frac{(1 - \frac{t}{T}) \times PMT}{(1+r)^{1-t/T}} + \frac{(2 - \frac{t}{T}) \times PMT}{(1+r)^{2-t/T}} + \dots + \frac{(N - \frac{t}{T}) \times (PMT + FV)}{(1+r)^{N-t/T}} \right]}{\frac{PMT}{(1+r)^{1-t/T}} + \frac{PMT}{(1+r)^{2-t/T}} + \dots + \frac{PMT + FV}{(1+r)^{N-t/T}}}$$

- ⇒ t = number of days from the last coupon payment to the settlement date
 - ⇒ T = number of days in the coupon period
 - ⇒ PMT = coupon payment per period
 - ⇒ FV = par value
 - ⇒ r = YTM (discount rate per period)
 - ⇒ N = number of coupon periods to maturity
- **Macaulay duration alternative formula**

$$D = \left[\frac{1+r}{r} - \frac{1+r + [N \times (c-r)]}{c \times [(1+r)^N - 1] + r} \right] \times \left(\frac{t}{T} \right)$$

- ⇒ c = coupon rate per period (as a decimal)
 - ⇒ t = number of days from the last coupon payment to the settlement date

- ⇒ T = number of days in the coupon period
- ⇒ **PMT** = coupon payment per period
- ⇒ **FV** = par value
- ⇒ r = YTM (market discount rate) per period
- ⇒ N = number of coupon periods to maturity
- ⇒ The formula calculates the Macaulay duration in terms of periods. This needs to be annualized (by dividing the Macaulay duration by the number of coupon payment periods per year)

- **Calculating the Macaulay duration**

- ⇒ **Example:** A 6% annual payment bond matures on 14 February 2025 and is purchased for settlement on 11 July 2019. The YTM is 4.00%. Calculate the bond's Macaulay duration (actual/actual convention):

Period	Time to Receipt	Cash Flow	Present Value	Weight	Time x Weight
1	0.5973	6	5.8611	0.0522	0.0312
2	1.5973	6	5.6357	0.0502	0.0802
3	2.5973	6	5.4189	0.0483	0.1254
4	3.5973	6	5.2105	0.0464	0.1670
5	4.5973	6	5.0101	0.0446	0.2052
6	5.5973	106	<u>85.1071</u>	<u>0.7582</u>	<u>4.2441</u>
			112.2433	1.0000	4.8530

- ⇒ The Macaulay duration is 4.8530 years

- **Calculating the Macaulay duration with the alternative formula:**

- ⇒ Using the alternative formula, the calculation is as follows:

$$MacDur = \frac{1+0.04}{0.04} - \frac{1+0.04 + [6 \times (0.06 - 0.04)]}{0.06 \times [(1+0.04)^6 - 1] + 0.04} - \frac{147}{365}$$

↳ 4.8530 years

B. Modified duration

- **Modified duration (ModDur)** is a measure of the interest rate sensitivity of a bond.

$$ModDur = \frac{MacDur}{1+r}$$

⇒ where r is the yield per period.

- **ModDur and MacDur** are expressed in periodic terms. They can be annualized by dividing by the number of periods in the year (the periodicity) to get **AnnMacDur** and **AnnModDur**.

- **Calculating modified duration: problem**

⇒ Determine the annual modified duration of a 7-year, 5% semiannual-pay bond that matures on September 15, 2020. The bond's yield-to-maturity is 5.00% quoted on a semiannual bond basis. Its Macaulay duration is 11.383185 semiannual periods

$$MD = \frac{D}{1+r}$$

- $MD = 11.383185 / (1 + .025) = 11.1055$ semiannual periods
- Annualized MD = $11.1055 / 2 = 5.552773$ years

- **Factors that impact a bond's duration (and therefore interest rate risk)**

⇒ The longer the term-to-maturity, the higher the duration
⇒ The lower the coupon rate, the higher the duration
⇒ The lower the yield to maturity, the higher the duration

- **Estimating bond price changes**

⇒ **Modified duration** provides a linear estimate of the % price change for a bond given a change in its YTM (formula provides good estimates for small changes in yields, but its accuracy fades with larger changes given the convexity effect)

$$\Rightarrow \% \Delta PV^{Full} \approx -MD \times \Delta Yield (\%)$$

- MD is expressed in annual terms
- Bond prices are inversely related to changes in yields, which explains the minus sign in the formula
- To get the change in bond price, the % change must be multiplied by the original bond price

- **Using modified duration to estimate the change in a bond's price: problem**

⇒ Given an annualized modified duration of 5.552773, estimate the change in value of the bond in response to:

(a) a 100 bp increase in yields and

(b) a 100 bp decrease in yields

$$\Rightarrow \% \Delta PV^{Full} \approx -MD \times \Delta Yield (\%)$$

- (a) a 100 bp increase in yields (to 6.00%), the estimated loss in value:

$$\begin{aligned}\% \Delta PV^{Full} &\approx -MD \times \Delta Yield (\%) \\ &= -5.552773 \times 0.01 \\ &= -.05552773 \\ &= -5.55\%\end{aligned}$$

- (b) a 100 bp decrease in yields (to 4.00%), the estimated increase in value:

$$\begin{aligned}&= -5.552773 \times -0.01 \\ &= .05552773 \\ &= 5.55\%\end{aligned}$$

⇒ MD gives the same % change for either an increase or decrease in yields

- **Approximate modified duration**

⇒ An alternative approach is to estimate the **approximate modified duration (AMD)** directly (esp if the Macaulay duration is not already known):

$$AMD = \frac{\frac{-\frac{\partial PV}{\partial Y}}{PV}}{\frac{\partial Y}{Y}}$$

- **PV₀** = price of the bond at the current yield (its original price)
- **PV₊** = price of the bond if the yield increases (by ΔYield)
- **PV₋** = the price of the bond if the yield decreases (by ΔYield)
- **ΔYield** = the change in yield

⇒ **Example:** Consider a 6% semiannual coupon paying bond with 4 years to maturity currently priced at par (YTM = 6%)

- If the YTM increases/decreases by annualized 20 bps, the price raises/decreases to 99.301 and 100.705, respectively:

$$AMD = \frac{(100.705) - (99.301)}{2 \times (0.002) \times (100)} = 3.51 \text{ years}$$

⇒ The approximate modified duration is a linear approximation of the slope of the price/yield curve. The difference is due to the convexity of the price/yield curve.

- **Approximate Macaulay duration**

⇒ The approximate Macaulay duration (ApproxMacDur) is calculated from the approximate modified duration (ApproxModDur).

- $ApproxMacDur = ApproxModDur \times (1+r)$

⇒ In the example from a previous slide:

$$ApproxMacDur = 3.51 \times (1+0.03) = 3.615 \text{ years}$$

- The given yield-to-maturity of 6% is quoted on a semiannual bond basis. The yield per period is 3%.

- **Curve duration statistics**

⇒ Curve duration statistics used in fixed-income analysis include

- Effective duration
- Key rate duration

⇒ Assess the interest rate risk of a bond given a change in a benchmark yield curve

- **Effective duration**

⇒ Another approach to assess the interest rate risk of a bond is to estimate the % change in price given a change in a benchmark yield curve—for example, the government par curve

⇒ This estimate is very similar to the formula for approximate modified duration. It's called the “**effective duration**”:

$$EffDur = \frac{-\frac{\Delta PV}{PV}}{\Delta Curve}$$

- where $\Delta Curve$ is a parallel shift in the benchmark curve
- Note that:
 - Effective duration is a curve duration statistic that measures interest rate risk in terms of a change in the benchmark yield curve ($\Delta Curve$)
 - Modified duration is a yield duration statistic that measures interest rate risk in terms of a change in the bond's own YTM ($\Delta Yield$)

- **When use effective duration:**

⇒ Effective duration is essential to the measurement of the interest rate risk of a complex bond, such as a bond with an embedded option.

⇒ A callable/puttable bond does not have a well-defined internal rate of return (yield-to-maturity). Therefore, yield duration statistics, such as modified and Macaulay durations, do not apply. Effective duration is the appropriate measure of interest rate risk.

e. Bonds with embedded options

- The presence of an embedded option reduces the duration of a bond & reduces the sensitivity of the bond price to changes in the benchmark yield curve, assuming no change in credit risk
- ⇒ When interest rates are high vs the coupon rate it's highly unlikely the issuer will call the bond, so the value of the embedded call option is low, so effective durations of the callable & noncallable bonds are similar
- ⇒ When interest rates are low vs the coupon rate it's more likely the issuer will call the bond, so the embedded call option gains value for the issuer, so the effective duration of the callable bond is lower than that of an otherwise identical noncallable bond (expected life shortens as the weighted average time to receipt of cash is reduced)

f. Computing effective duration: problem

- A callable bond is currently trading at 101.05 per 100 of par. When the government par curve is raised by 25 bps, the value of the bond falls to 99.04, and when it is lowered by 25 bps the value of the bond rises to 102.87. Compute the effective duration of this bond

$$EffDur = \frac{-\frac{1}{PV_0} \frac{dPV_0}{d\Delta}}{\frac{1}{PV_0} \frac{dPV_0}{d\Delta}}$$

$$PV_0 = 101.05; PV_- = 102.87; PV_+ = 99.04; \Delta \text{ Curve} = 0.0025$$

$$EffDur = (102.87 - 99.04) / (2 \times 0.0025 \times 101.05) = 7.5804$$

g. Key rate duration

- A **key rate duration** (or partial duration) is a measure of a bond's sensitivity to a change in the benchmark yield curve at a specific maturity segment.

- In contrast to effective duration, key rate durations help identify “shaping risk” for a bond—that is, a bond’s sensitivity to changes in the shape of the benchmark yield curve (e.g., the yield curve becoming steeper or flatter).

⇒ Recall that effective duration measures a bond’s sensitivity to a parallel shift in the yield curve

. **Calculating the duration of a bond portfolio**

- Bonds are typically held in a portfolio.
- There are two ways to calculate the duration of a bond portfolio.
 - ⇒ The weighted average of time to receipt of the aggregate cash flows
 - This method is the theoretically correct approach, but it is difficult to use in practice.
 - ⇒ The weighted average of the individual bond durations that comprise the portfolio
 - This method is commonly used by fixed-income portfolio managers, but it has its own limitations.

. **Portfolio duration equation**

- Portfolio duration = $w_1D_1 + w_2D_2 + \dots + w_ND_N$
- **N** = Number of bonds in portfolio
- **D₁** = Duration of Bond 1
- **w₁** = Weight of bond, e.g. Market value of Bond 1 divided by the market value of portfolio
 - ⇒ 2 advantages to this approach: can be used for portfolios holding bonds with embedded options or floating rate notes (by using effective durations for these securities) and duration calculated can be used as a measure of interest rate risk. e.g. can assert that if the YTM for bonds in the portfolio change by 100 bps the estimated change in portfolio value will be x% (in the other direction)
 - ⇒ A disadvantage is that this measure of duration assumes a parallel change in the yield curve (i.e., yields across all maturities change by the same amount and in the same direction). Portfolios of bonds are composed of a variety of bonds that may have different maturities, credit risks, and embedded options, and realistically speaking, there is no reason to expect the change in yields on all these bonds to be identical

. **Calculating portfolio duration: problem**

- An investment fund contains the following 3 fixed-rate government bonds:

	Bond A	Bond B	Bond C
Par value	20,000,000	20,000,000	40,000,000
Coupon rate	7.0%	9.0%	6.0%
Time-to-maturity	3 years	5 years	7 years
Yield-to-maturity	7.21%	7.45%	7.71%
Market value (USD)	19,888,485	21,274,578	36,352,583
Macauley duration	2.757	4.165	5.748

- ⇒ The total market value of the portfolio is US\$77,515,646
- ⇒ Each bond is on a coupon date so that there is no accrued interest
- ⇒ The market values are the full prices given the par value
- ⇒ Coupons on all 3 bonds are paid semiannually
- ⇒ The yields-to-maturity are stated on a semiannual bond basis
- ⇒ The Macaulay durations are annualized
 - Calculate the portfolio's (annual) modified duration using shares of market values as weights
 - o duration into an annual modified duration convert the Macaulay duration into the number of semiannual periods (by multiplying it by 2)
 - o Divide this number of semiannual periods by $1 + \text{yield per semiannual period}$
 - o Divide the resulting number by 2 to obtain annual modified duration

$$\text{AnnualModDur} = \text{Annual MacDur} / (1 + r); r = \text{yield per semiannual period}$$

$$\text{ModDurA} = 2.757 / (1 + 0.0721/2) = 2.6611$$

$$\text{ModDurB} = 4.165 / (1 + 0.0745/2) = 4.0154$$

$$\text{ModDurC} = 5.748 / (1 + 0.0771/2) = 5.5346$$

- o Compute the wtd avg of these ModDur's based on the share of each bond in the market value of the portfolio

$$\begin{aligned} \text{ModDur}_{\text{Portfolio}} = & 2.611 \times (19,888,485/77,515,646) + 4.0154 \times \\ & (21,274,578/77,515,646) + 5.5346 \times (36,352,583/77,515,646) = 4.3804 \end{aligned}$$

- Estimate the percentage loss in the portfolio's market value if the (annual) yield-to-maturity on each bond goes up by 30 bps. $\% \Delta PV^{Full} = -AnnModDur \times \Delta Yield$
 - o $-4.3804 \times 0.0030 = -0.0131$ or 1.31%

4. Convexity – but duration is not fully accurate

- BUT: duration provides a good approximation for changes in bond prices only for small changes in yields
- Duration estimates the change in a bond's price along a straight line (which is tangent to the curved price-yield profile)
- The real relationship between the price of a bond and its YTM is convex: the graph forms a curve rather than a straight-line

a. Why the bond price-yield relationship is convex

- A bond price's change is magnified the lower the YTM (“the shorter the man”) and is lessened the higher the YTM (“the taller the man”)

b. Convexity statistic (adjustment)

- The true relationship between bond price and yield-to-maturity is the curved (convex) line, which shows the actual bond price given its market discount rate
 - ⇒ The linear approximation of estimated price change offered by duration is good for small YTM changes. But for larger changes, the difference becomes significant
- The **convexity statistic** for the bond is used to improve the estimate of the % price change provided by modified duration alone

$$\% \Delta PV^{Full} \approx (-AMD \times \Delta Yield) + \left[\frac{1}{2} \times Conv \times (\Delta Yield)^2 \right]$$

- The 1st part of equation captures the 1st-order (duration) effect
- The 2nd part of equation captures the 2nd-order (convexity adjustment) effect
- The convexity adjustment has a positive impact on the bond price estimate for either an increase or a decrease in yields

c. Approximate and money convexity

- Like modified duration, convexity can be accurately approximated
- Approximate convexity is calculated as:

$$\begin{array}{c}
 PV \\
 \downarrow \\
 - \\
 PV \\
 \downarrow \\
 + \\
 \frac{PV^- - PV^+}{2 \times \Delta \text{Yield}} \\
 \downarrow \\
 \Delta \text{Yield} \\
 \downarrow \\
 AConv
 \end{array}$$

⇒ **PV-** = the bond price when the yield is decreased

⇒ **PV+** = the bond price when the yield is increased

⇒ **PV₀** = the original bond price

⇒ **ΔYield** = the change in yield

- The **money convexity** of the bond is the annual convexity multiplied by the full price of the bond

d. Calculating approximate convexity

- **Example:** Consider a 6% semiannual coupon payment bond with 4 years to maturity that is currently priced at par (YTM = 6.00%) and has an annual modified duration of 3.51. If the YTM increases/decreases by 20 bps, the price decreases/increases to 99.301 and 100.705, respectively. Calculate ApproxCon and the effect of a 50 bps increase in yield on the bond price:

$$A_{approx\ Con} = \frac{100.705 + 99.301 - (2 \times 100)}{(0.0020)^2 \times 100} = 14.81$$

$$\% \Delta PV^{Full} \approx -3.51 \times 0.0050 + \frac{1}{2} \times 14.81 \times (0.0050)^2 = -0.0174$$

⇒ Modified duration alone estimates the price change to be -1.76% , convexity adds 2 bps to give an estimate of -1.74% .

e. The convexity adjustment problem

- Consider the following information: 7 year, 5% semiannual coupon

Initial YTM = 5%

PV₀ = \$101.492586

PV+ = \$101.2110

PV- = \$101.7750

$$\Delta \text{Yield} = 5 \text{ bps}$$

$$\text{ApproxModDur} = 5.55278$$

$$\text{ApproxCon} = \frac{(\text{PV}_-)+(\text{PV}_+)-(2 \times \text{PV}_0)}{(\Delta \text{Yield})^2 \times (\text{PV}_0)}$$

$$\% \Delta \text{PV}^{\text{Full}} = (-\text{AnnModDur} \times \Delta \text{Yield}) + (\frac{1}{2} \times \text{AnnConvexity} \times (\Delta \text{Yield})^2)$$

⇒ Calculate approximate convexity.

$$\bullet \quad \text{ApproxCon} = \frac{101.775 + 101.211 - (2 \times 101.492586)}{(.0005)^2 \times 101.492586} = \frac{.000828}{.0000253731} = 32.637$$

⇒ Calculate the estimated convexity-adjusted % price change resulting from a 100 bp increase in the yield-to-maturity.

$$\bullet \quad \Delta \text{PV}^{\text{Full}} = (-5.55278 \times 0.01) + (\frac{1}{2} \times 32.637 \times (0.01)^2) = -.053896 \text{ or } -5.3896\%$$

⇒ Note: you can compare the estimated % price change with the actual change, by calculating the PV of the bond assuming the yield-to-maturity jumps to 6.00% on the settlement date.

$$\begin{aligned} \bullet \quad & N=14; \text{PMT} = -\$2.50; \text{FV} = -\$100; \text{I/Y} = 3\%; \text{CPT PV}; \text{PV} = 94.35196 \\ \bullet \quad & \text{PV}^{\text{Full}} = 94.35196 \times (1.03)^{(108/180)} = \$96.040249 \\ \bullet \quad & \Delta \text{PV}^{\text{Full}} = \frac{(\$96.040249 - \$101.492586)}{\$101.492586} = -0.0537215 \text{ or } -5.37215\% \end{aligned}$$

f. Ranking bonds in terms of interest rate risk: problem

- A fixed-income analyst has been assigned the task of ranking 3 bonds in terms of interest rate risk (measured in terms of the potential % price depreciation given forecasted worst-case scenario changes in the yields).

<u>Bond</u>	<u>Modified Duration</u>	<u>Convexity</u>	<u>ΔYield (bps)</u>
A	9.2	147.0	15
B	7.8	38.5	20
C	5.9	12.1	25

- The modified duration and convexity statistics are annualized.
- ΔYield is the projected increase in the annual yield-to-maturity.

- Calculate the approximate % price change for each bond given the projected increase in its yield using formula:
- $\% \Delta PV^{\text{Full}} = (-\text{AnnualModDur} \times \Delta \text{Yield}) + (1/2 \times \text{AnnConvexity} \times (\Delta \text{Yield})^2)$
- Rank the bonds in terms of interest rate risk.
 - ⇒ First, calculate the approximate % price change for each bond given the projected increase in its yield:
 - Bond A: $(-9.2 \times 0.0015) + [1/2 \times 147.0 \times (0.0015)^2] = 0.013635$ or -1.36%
 - Bond B: $(-7.8 \times 0.002) + [1/2 \times 38.5 \times (0.002)^2] = 0.015523$ or -1.55%
 - Bond C: $(-5.9 \times 0.0025) + [1/2 \times 12.1 \times (0.0025)^2] = 0.014712$ or -1.47%
 - ⇒ Based on these anticipated changes in yields, Bond B has the highest degree of interest rate risk (potential loss = 1.55%), followed by Bond C (potential loss = 1.47%) and Bond A (potential loss = 1.36%).

g. Effective convexity

- The **effective convexity** of a bond is a curve convexity statistic that measures the secondary effect of a change in a benchmark yield curve.

$$\begin{array}{c}
 PV \\
 \downarrow \\
 - \\
 PV \\
 \downarrow \\
 + \\
 \downarrow \\
 \downarrow \\
 \downarrow \\
 \downarrow \\
 \text{EffCon} =
 \end{array}$$

- For callable bonds (and other bonds with embedded options) the 1st-order effect of a change in the benchmark yield curve is measured by effective duration, and the 2nd-order effect of a change in the benchmark yield curve is measured by effective convexity
- The difference between this equation and the one that approximates yield convexity is that this one has the change in benchmark yield curve $(\Delta \text{Curve})^2$ whereas the other one has the change in YTM $(\Delta \text{Yield})^2$ in the denominator
- Effective convexity: problem
 - ⇒ Compute effective convexity for the following callable bond:

$$PV_0 = 101.05$$

$$PV_- = 102.87$$

$$PV+ = 99.04$$

$$\Delta\text{Curve} = 0.0025$$

$$\begin{aligned}\text{EffCon} &= \frac{(PV-)+(PV+)-(2 \times PV_0)}{(\Delta\text{Curve})^2 \times (PV_0)} \\ &= \frac{102.87 + 99.04 - (2 \times 101.05)}{(0.0025)^2 \times (101.05)} \\ &= -300.84\end{aligned}$$

⇒ Notice that the value of convexity for this callable bond is negative

h. Positive and negative convexity

- For the same decrease in YTM, the more convex bond appreciates more in price. For the same increase in YTM, the more convex bond depreciates less in price. The more convex bond outperforms the less convex bond in both bull (rising price) & bear (falling price) markets. A more convex bond will have a higher price than a less convex bond (other things being equal)
- Negative convexity (concavity) is possible for callable bonds. As benchmark yields decline, callable bonds can move into negative convexity where the embedded call option holds significant value to the issuer and is highly likely to be exercised
 - ⇒ Option-free bonds always have positive convexity
 - ⇒ Puttable bonds always have positive convexity

5. Interest Rate Risk and the Investment Horizon

- Recall Examples #1 to 6 last class where we worked with a 10-year, 8% annual-pay bond with 10.4% yield:
 - ⇒ Investor A (buy-and-hold investor) only faced coupon reinvestment risk (no market price risk). She experienced a higher return if interest rates increased & a lower return if rates decreased
 - ⇒ Investor B (4-year investment horizon investor) faced both market price risk & coupon reinvestment risk. She had a higher return if interest rates decreased than if rates increased; given her investment horizon and the change in yields, price risk > coupon reinvestment risk

a. Investment horizon, Macaulay duration and interest rate risk

- Now consider a 3rd investor, Investor C, with a 7-year investment horizon. The following is the total return calculations for this investor:

- ⇒ If interest rates remain at 10.40%, her horizon yield =10.40%
- ⇒ If the yield-to-maturity were to rise to 11.40%, her horizon yield =10.407%
- ⇒ If the yield-to-maturity were to fall to 9.40%, her horizon yield =10.408%
- For Investor C's horizon yields are almost the same in each of the 3 interest rate scenarios (approx 10.40%) regardless of whether interest rates rise, fall, or stay the same. This was not the same for Investors A & B
- This provides a very important point re Macaulay duration. Given a particular yield volatility assumption, Macaulay duration indicates the investment horizon for which coupon reinvestment risk & market price risk offset each other (are equal)
- The Macaulay duration of this bond is 7.003 years. This is an application where Macaulay duration is used rather than modified duration

b. Interest rate risk analysis

- An important aspect in understanding the interest rate risk and return characteristics of an investment in a fixed-rate bond is the **time horizon**.
- Bond duration is the primary measure of risk arising from a change in the yield-to-maturity; convexity is the secondary risk measure.
- The common assumption in interest rate risk analysis is a parallel shift in the yield curve. In reality, the shape of the yield curve changes based on factors affecting the supply and demand of shorter-term versus longer-term securities.

c. Yield volatility

- **The term structure of yield volatility** is the relationship between the volatility of bond yields-to-maturity and times-to-maturity (yield volatility for different maturities tends to be different).
- The importance of yield volatility in measuring interest rate risk is that bond price changes are products of 2 factors
 - ⇒ The impact *per* basis point change in the yield-to-maturity
 - The first factor is duration or the combination of duration and convexity.
 - ⇒ The number of basis points in the yield-to-maturity change
 - The second factor is the yield volatility.

d. General relationships among interest rate risk, the Macaulay duration, and the investment horizon

- When the investment horizon is greater than the Macaulay duration of a bond, coupon reinvestment risk dominates market price risk. The investor's risk is to lower interest rates.
- When the investment horizon is equal to the Macaulay duration of a bond, coupon reinvestment risk offsets market price risk.
- When the investment horizon is less than the Macaulay duration of a bond, market price risk dominates coupon reinvestment risk. The investor's risk is to higher interest rates.

e. Equation: duration gap

- The difference between the Macaulay duration of a bond and the investment horizon is called the “**duration gap**”

$$\text{Duration Gap} = \text{Macaulay Duration} - \text{Investment Horizon}$$

- ⇒ Investor A (with the 10-year horizon) has a negative duration gap and is currently at risk of lower interest rates
- ⇒ Investor B (with the 4-year horizon) has a positive duration gap and is currently at risk of higher interest rates
- ⇒ Investor C (with the 7-year horizon) has a duration gap of zero and is currently hedged against interest rate risk
- ⇒ As time passes, the investment horizon falls and the Macaulay duration of the bond also changes, so the duration gap changes as well

6. Credit and Liquidity Risk

- The yield-to-maturity on a corporate bond is composed of a government *benchmark* yield and a *spread* over that benchmark. A change in the bond's yield-to-maturity can originate in either component or a combination of the two.
- A change in the benchmark yield can arise from a change in either the expected inflation rate or the expected real rate of interest.
 - ⇒ The **inflation duration** would indicate the change in the bond price if expected inflation were to change by a certain amount.
 - ⇒ The **real rate duration** would indicate the bond price change if the real rate were to go up or down.

a. Impact of changes in yield-to-maturity

- A change in the spread can arise from a change in the credit risk of the issuer or in the liquidity of the bond.
- ⇒ For a bond with a given duration and convexity, the impact of changes in yield-to-maturity on the bond's price will be the same regardless of the source of the yield-to-maturity change.
- ⇒ The problem for a fixed-income analyst is that it is rare for the changes in the components of the overall yield-to-maturity to occur in isolation.

b. Price impact of a change in the credit spread: problem

- Consider an annual-pay bond that is trading at 93.75 per 100 of par. The duration of the bond is 7.25 and its convexity is 34.
- Compute the % change in the price of the bond if a credit rating downgrade results in its spread relative to the benchmark increasing by 25 basis points.
- $\% \Delta PV^{\text{Full}} = (-\text{AnnualModDur} \times \Delta \text{Yield}) + (\frac{1}{2} \times \text{AnnConvexity} \times (\Delta \text{Yield})^2)$
 - ⇒ The change in spread equals 25 bps, so compute the estimated change in the price of the bond given its duration and convexity and $\Delta \text{Yield} = 0.0025$.
 - ⇒ $\% \text{ change } PV^{\text{Full}} = (-7.25 \times 0.0025) + (\frac{1}{2} \times 34 \times (0.0025)^2) = -0.018 \text{ or } -1.8\%$
 - ⇒ The bond's value will fall by approximately 1.8% in response to the 25-bp increase in its credit spread.