Chapter Three

Transportation and Assignment problems

One important application of linear programming is in the area of physical distribution (transportation) of goods and services from several supply origins to several demand destinations. It is possible to express a transportation problem mathematically in terms of an LP model, which can be solved using simplex method. But since it involves a large number of variables and constraints, it takes a long time to solve it. However, transportation algorithm called stepping stone method and MODI methods have been developed to handle it successfully.

The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations. The objective is to determine the number of units which should be shipped from an origin to destinations. The objective is to determine the number of units which should be shipped from an origin to a destination in order to satisfy the required quantity of goods or services at each demand destinations, with in the limited quantity of goods and services available at each supply origin, at minimum transportation cost and or time.

Formulating the model

A transportation model typically involves a set of sending locations called origins and a set of receiving locations called destinations. In order to develop a model of transportation problem, it is necessary to have the following information.

- Supply quantity
- Demand quantity
- Unit transportation cost for each origin destination route

The transportation algorithm requires the assumption that all goods be homogenous, so that any origin is capable of supplying any destination and the assumption that transportation costs are a direct linear function of the quantity shipped over any route.

Example

	Project #1	Project #2	Project #3	Total Supply
Farm A	4	2	8	100
Farm B	5	1	9	200

Farm C	7	6	3	200
Total Demand	50	150	300	500

Finding initial feasible solution

A transportation model is similar to simplex technique because both involve an initial feasible solution that is evaluated to determine if it can be improved. A solution to a transportation problem consists of quantities that are assigned to the various routes (cells in the table). These values can range from zero which implies that no units will be shipped over that route to the maximum that equals the smaller of the two quantities: the row (supply) or the column (demand). Such a solution is a feasible solution.

A feasible solution is one in which assignments are made in such a way that all supply and demand requirements are satisfied. The number of non-zero (occupied cells) should equal one less than the sum of number of rows and columns in transportation table.

Sometimes, fewer occupied cells appear in a solution. When that happens, the solution is referred to as degenerate solution.

Remarks

- When the total supply equals total demand the problem is called balanced transportation problem, otherwise an unbalanced transportation problem. The unbalanced transportation problem can be made balanced by adding dummy supply centre or dummy demand centre.
- Cells in transportation table having positive allocations will be called occupied cells and otherwise empty or non-occupied cells.

Finding solution

Finding solution for transportation algorithm may be summarized as follows.

Steps

1. Formulate the problem and set up in matrix form
2. Obtain an initial basic feasible solution
   - ✓ Northwest corner point approach
   - ✓ Least cost method
   - ✓ Vogel’s approximation (Penalty) method

The initial solution obtained by any of the three methods must satisfy the following conditions

- ✓ The solution must be feasible- it must satisfy all the demand and supply constraints
- ✓ The number of positive allocations must be m+n-1 , where m is number of rows and n is number of columns

Any solution that satisfies the above conditions is called non degenerate basic feasible solution.

3. test the initial solution for optimality
    - ✓ Stepping stone method
    - ✓ MODI (modified Distribution)
4. update the solution

Methods for finding initial solutions

There are methods available to obtain an initial basic feasible solution. Some of the methods will be discussed here under.

Northwest corner point approach

It is a simple and efficient method to obtain an initial solution. This method does not take in to account the cost of transportation on any route. The method can be summarized as:

- ✓ Start with the cell at the upper left (northwest) corner of the transportation matrix and allocate as much as possible to the minimum of the rim values for the first row and first column
- ✓ If allocation made is equal to the supply available at the first source, move vertically down and apply the same in the second row and first column.
- ✓ If allocation made in first step is equal to demand at the first destination, move horizontally to the first row second column and apply the same rule
- ✓ If demand and supply for the first row and first column are equal allocate to the first row first column and move diagonally.

Least cost method

Under least cost method of allocation, we try to allocate as much as possible through those routes where unit transportation cost is lowest. This method takes in to account the minimum unit cost of transportation for obtaining initial solution and can be summarized as follows.

- ✓ Select the cell with the smallest unit cost in the entire transportation table and allocate as much as possible to this cell and eliminate that row or column in which either supply or demand is exhausted. In case the smallest unit cost cell is not unique, then select the cell where maximum allocation can be made
- ✓ After adjusting the supply and demand for all uncrossed out rows and columns repeat the procedure with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell and eliminate the row in which either a demand or supply is exhausted.
- ✓ Repeat the procedure until the entire available supply at various sources and demand destinations are satisfied.

Vogel's Approximation

It is alternatively called penalty or regret method

It is heuristic method and is preferred to the other two. Each allocation is made on the basis of opportunity cost that would have been incurred if allocation in certain cells with minimum unit transportation cost were missed. Allocations are made to minimize penalty cost or regret. The method gives an initial solution which is nearer to the optimal solution or is optimal by itself. The steps involved are summarized as follows:

- ✓ Calculate penalties for row and column by taking the difference between the smallest and next smallest unit cost in the same row or column.
- ✓ Select the row or column with the largest penalty cost and allocate as much as possible in the cell having the least cost in the selected row or column satisfying the rim condition. When there is a tie in the penalties it can broken by selecting the cell where maximum selection can be made.
- ✓ Adjust the supply and demand and cross out the satisfied row or column.

Evaluating solution for optimality

The test for optimality for feasible solution involves a cost evaluation of empty cells to see if an improved solution is possible. There are two method of checking for optimality.

- ❖ The stepping stone method

- ❖ Modified distribution method (MODI)

Evaluation using stepping stone method

It involves tracing a series of closed paths in the transportation table, using one such path for all empty cells. The path represents a shift of one unit in to an empty cell and it enables the manager or the analyst to answer what if questions. The stepping stone path can also be used to determine the maximum number of units that can be shifted in to the empty cells as well as the modifications to other completed cells needed to compensate for the shift in to the previously unused cell.

The name stepping stone relates to crossing a pond or stream by moving from stone to stone; in the case of transportation problem stones are completed cells.

Rules for stepping stone

- All unoccupied cells must be evaluated. Evaluate the cells one at a time.
- Except for the cells being evaluated only add or subtract in occupied cells. It permissible to skip over unoccupied or occupied cell to find occupied cell from which a path can continue)
- A path will consist of only horizontal and vertical moves, starting and ending with the empty cell being evaluated.
- Alternate + and – sign, beginning with the + sign in the cell being evaluated.

Modified Distribution (MODI)

- ✓ Involves the use of index numbers that are established for the rows and columns.
- ✓ These are based on the unit costs of the occupied cells of the occupied cells.
- ✓ The index numbers can be used for the evaluation of empty cells
- ✓ There is one index number for each row and one for each column which can be conveniently displayed on the upper part of the table and left edge.

- ✓ The index numbers are determined in such a way that for any occupied cell the sum of the row index and column index equals the cells unit transportation cost.
- ✓ The index numbers are determined sequentially in a manner dictated by position of occupied cells. The process always begins by assigning a value of zero as the index of row 1.

The cells evaluation (improvement potential) can be for each of the occupied cells can be determined using:

Cell evaluation= cell cost-row index- column index

When cell evaluations are positive or zero, an optimal solution has been found. If one or more is negative, the cell with the largest negative should be brought in to the solution because that route has the largest potential for improvement per unit.

Developing an improved solution

Developing an improved solution to a transportation problem requires focusing on the unoccupied cell that has the largest negative cell evaluation. Improving the solution involves reallocating quantities in the transportation table. The stepping stone path is necessary for determining how many units can be reallocated while retaining the balance of supply and demand for the table. It also reveals which cells must have quantity changes and both the magnitude and direction of changes. The limit on subtraction is the smallest quantity in negative position along the cell path. After developing a new and improved solution, the combination of routes needs to be checked for feasibility and optimality. If optimal solution is arrived on it needs to be adopted as transportation route for the organization; if not the steps need to be repeated until optimal transportation table is found.

Some special issues

There are some special issues that deserve special attention before winding the discussion on transportation algorithm.

- ✓ Alternate optimal solution
- ✓ Degeneracy
- ✓ Avoiding unacceptable or prohibited route assignments
- ✓ Unbalanced supply and demand

- ✓ Solving maximization problems

Alternate optimal solutions

Transportation problems may have multiple optimal solutions. This can be useful for the managers since it gives the manager an option of bringing non quantitative considerations in to account. The existence of an alternate solution is evidenced by an empty cell evaluation of zero.

Degeneracy

A transportation problem is degenerate when there are too few occupied cells to enable all the empty cells to be evaluated. In the case of the stepping stone method, there will be at least one empty cell for which an evaluation path can not be constructed. For the MODI method it means that it will be impossible to determine all the row and column index numbers. It is relatively simple to determine if a given solution is degenerate. A solution is degenerate if the number of occupied cells is less than the number of rows plus number of columns minus one.

Obviously, some modification has to be made in order to determine id a given solution is optimal or not. The modification is to treat some of the empty cells as occupied cells. This is accomplished by placing delta (Δ) in one of the empty cells. The delta represents an extremely small quantity; it is so small that supply and demand for the row and column involved will be unaffected even without modifying other quantities in the row or column and so small that total cost will not change. The purpose of delta is to enable the evaluation of the remaining empty cells. The choice of location for delta can be somewhat tricky. Some empty cells may unsuitable if they do not enable the evaluation of the remaining empty cells. Moreover, the delta can not be placed in a cell which later turns out to be in a negative position of a cell path involved in reallocation.

Degeneracy may happen at two stages;

- ✓ When obtaining an initial basic feasible solution
- ✓ At any stage while moving towards optimal solution.

Degeneracy at the initial solution

To resolve degeneracy at the initial solution, we proceed by allocating very small quantity close to zero to one or more unoccupied cells so as to get m+n-1 number of occupied cells. In minimization transportation problems, it is better to allocate Δ to unoccupied cells that have the lowest

transportation costs whereas in maximization problems it should be allocated to a cell that has a high pay off value. In some cases Δ must be added in one of those occupied cells which make possible the determination of row and column indices.

Degeneracy at subsequent iteration

To resolve degeneracy which occurs during optimality test, the quantity can be allocated to one or more cells which have become unoccupied recently to have m+n-1 number of occupied cells in the new solution.

Unacceptable routes

In some cases certain origin-destination combination may be unacceptable. This may be due to weather factors, equipment breakdowns, labour problems, or skill requirements that either prohibit or make undesirable certain combinations of routes.

Such types of problems can be handled by placing large positive cost M (∞) to that route or cell.

Unequal supply and demand

For a feasible solution to exist, it is necessary that the total supply must equal total demand. But a situation may arise when the total available supply is not equal to the total demand.

- ✓ If supply exceeds total demand, an additional column (called dummy demand centre) can be added to the transportation table to absorb the excess supply. The unit transportation cost for the dummy column is zero because these represent product items that are not being made not being sent.
- ✓ If a total demand exceeds total supply, a dummy row (called dummy supply centre) can be added to the transportation table to account for the excess demand quantity. The unit transportation cost here also is zero.

Maximization problems

Some transportation type problems concern profits or revenues rather than costs. In such cases, the objective is to maximize rather than to minimize. Such problems can be handled by adding one additional step at the start: identify the cell with the largest profit and subtract all the other cell profits from the value. Then replace the cell profits with the resulting values. These values represent the opportunity costs that would be incurred by using routes with unit profits that are less than the largest unit

profit. Replace the original unit profits by the newly calculate opportunity costs and solve in the usual way for the minimum opportunity cost solution.

Assignment problems

Assignment problems are special cases of linear programming problems. They lend themselves to manual solution using special purpose algorithm. Assignment problems are characterized by a need to pair items in one group with items in the other group in one to one matching.

Typically the time or cost required to complete a project or a job will differ, depending on the machine used for the job or the staff member doing the project. The manager's goal in such a case is to develop an assignment schedule that minimizes the time or the cost of doing the project or the job. The goal can also be minimizing the distance travelled, scrap or some other measure of effectiveness.

Assignment models may also involve maximization of revenue or profit; in which case the role of the manager is finding for combination of activities and doers that will maximize the objective.

Example

Assume a manager as prepared a table that shows the cost of performing each of the four jobs by each of the four employees.

	Employees				
Jobs		A	B	C	D
	1	15	20	18	24
	2	12	17	16	15
	3	14	15	19	17
	4	11	14	12	13

The assignment method

Assignment method is based on minimization of opportunity cost that would result from potential pairings. These are additional costs incurred if the lowest cost assignment is not made either in terms of jobs or employees.

Steps in assignment algorithm

1. Row reduction

- ❖ Identify the minimum value in each row

- ❖ Subtract the minimum value in each row from all the values of the row.
- ❖ Use the resulting value to develop a new table.

2. Column reduction

- ❖ Identify the lowest cost value in each column
- ❖ Subtract the lowest cost value in each column from each of the values in that column.
- ❖ Use the resulting values to form new table

3. Check for optimality -Find the minimum number of lines needed to cover all the zero costs. If the minimum number of lines needed is equal to number of rows or columns then the assignment is optimal otherwise further reduction need to be done.
4. Further reduction: identify the smallest uncovered element and subtract it from all uncovered elements; and add it to elements covered two times.
5. check if optimal assignment is possible by drawing lines through all the zeros. If the number of lines needed is the same as the number of rows or columns then, the assignment is optimal and allocation can be made.

Some special situations

Sometimes, special conditions may arise. The following are some special conditions;

- ❖ The number of rows does not equal the number of columns
- ❖ The problem involves maximization rather than minimization
- ❖ Certain matches are undesirable or not allowed
- ❖ Multiple optimal solution exist

Unequal rows and columns

If the number of jobs is greater than the number of employees then an extra employee must be added to the table. The addition of employee however, does not mean real addition. Rather it is to handle the problem. The employee is nonexistent employee. Hence, the job assigned to non existent employee will not be processed immediately. The costs associated with the additional employee shall be zero.

If the goal is to maximize rather than to minimize, one extra step must be added to the start of the start of the process. Identify the largest value in each column and then subtract all numbers in each column

from the column maximum. Having done that repeat the steps that would be required if the problem were minimization.

Maximization

Identify the maximum value in each column and then subtract every value in a given column from the column maximum.

Undesirable allocations

When there is a restriction on employees or machines letter M is placed in the table at the position which is restricted. Analysis is done in the usual way except the letter M is not used in the reduction, nor any value added to it or subtracted from it.

Multiple optimal solutions

This is the case where there is no unique zero; resulting in more than one choice for assignment, and hence more than one optimal solution.