

Chapter 5

INTEGRATIONS

5.1. Indefinite Integrals

Definition: Let f be a function whose domain is an open interval I . Then any function F such that $F'(x) = f(x)$ for each x in I is called an **anti-derivative** of f . (Note that f is the derivative of F on I .)

Example: 1. $F(x) = x^3 + c$ for constant c is the anti-derivative of $f(x) = 3x^2$.

Theorem: Let f be a function whose domain is an open interval I .

- i) If $F(x)$ is an anti-derivative of $f(x)$, then $F(x) + C$ is an anti-derivative of $f(x)$ for every real number C .
- ii) If $F(x)$ and $G(x)$ are two anti-derivatives of $f(x)$, then $F(x) - G(x)$ is constant for all x in I . That is, $G(x) = F(x) + C$ for some real number C

Definition: Let the domain of f be an open interval I and suppose f has anti-derivative. The family of all anti-derivatives of f is called the indefinite integral of f and is denoted by.

$$\int f(x) dx$$

Example: 1. $\int x dx = \frac{1}{2} x^2 + C$ 2. $\int x^2 dx = \frac{1}{3} x^3 + C$

Basic Rules of Indefinite Integrals

Theorem: Let f and g be a function of x whose domains are an open interval I and suppose f and g are differentiable for every x in I .

- 1) $\int dx = x + C$
- 2) $\int c dx = c \int dx$.
- 3) $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$.
- 4) $\int x^r dx = \frac{x^{r+1}}{r+1} + C$, where r is rational, $r \neq -1$, and $x > 0$ on I
- 5) $\int \frac{1}{x} dx = \ln|x| + C$ ($x \neq 0$)

$$6) \int \sin x \, dx = -\cos x + C, \int \cos x \, dx = \sin x + C \text{ and } \int \sec^2 x \, dx = \tan x + C$$

$$7) \int e^x \, dx = e^x + C \quad \text{and} \quad \int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$8) \int \sinh x \, dx = \cosh x + C, \int \cosh x \, dx = \sinh x + C \text{ and } \int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$9) \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C, \int \frac{dx}{1+x^2} = \tan^{-1} x + C \text{ and } \int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$$

Example: 1. $\int (2x^{-1} + 3\sin x) \, dx = 2\ln|x| - 3\cos x + C$

2. $\int (4x^3 - 6x^2 + 2x + 1) \, dx = x^4 - 2x^3 + x^2 + x + C.$

3. $\int \left(\frac{3}{x^2} + \sqrt{x} \right) \, dx = -\frac{3}{x} + \frac{2}{3} x^{\frac{3}{2}} + C$

5.2 Techniques (or rules) of Integration

Unlike differentiation there is no set of rules which you can follow to find the indefinite integral of any given function. These are integrals of a certain form for which we have an answer. If we are given an integral which doesn't correspond to one of the "standard integrals" then there are certain techniques we can use to try to convert its form into one or more of the standard integrals. Hence we discuss different Techniques of Integration like

- ✓ Integration by substitution.
- ✓ Integration by parts
- ✓ Integration by partial Fractions
- ✓ Integration by trigonometric substitution

A. Integration by substitution

A smart idea consists in "cleaning" them through an algebraic substitution which transforms the given integrals into easier ones. Let us first explain how the substitution technique works.

Theorem: Suppose I and J are open intervals, f has domain I , g maps J into I , and g is differentiable on J . Assume that we take u as independent variable.

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du = F(u) + C, \text{ where } x \text{ is the independent variable and } u = g(x)$$

Example: Evaluate the following integrals

$$\text{a) } \int \frac{x}{x^2+1} dx \quad \text{b) } \int 3x^2(x^3+1)^{\frac{1}{2}} dx$$

Solution a) $\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \int \frac{du}{u}, \quad \begin{matrix} u = x^2 + 1 \\ du = 2x dx \end{matrix}$

$$= \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln(x^2 + 1) + C$$

$$\text{b) } \int 3x^2(x^3+1)^{\frac{1}{2}} dx = \int u^{\frac{1}{2}} du, \quad \begin{matrix} u = x^3 + 1 \\ du = 3x^2 dx \end{matrix}$$

$$= \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3} (x^3 + 1)^{\frac{3}{2}} + C$$

B. Integration by parts

The General Method

Suppose we have two functions multiplied by each other $h(x) = f(x)g(x)$ and differentiate according to the product rule:

$$h'(x) = [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Then integrating both sides $\int [f(x)g(x)]' dx = \int [f'(x)g(x) + f(x)g'(x)] dx$

and rearranging gives $\int f(x)g'(x) dx = \int [f(x)g(x)]' dx - \int f'(x)g(x) dx$

or as the formula is better known $\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$

This relationship is what we call Integration by Parts. Let us look at it more carefully and then see some examples. First, it asks us to break the integrand up into two pieces and identify one of the pieces, $g'(x)$, as the derivative of another function. In other words, we anti-differentiate a "part" of the integrand. The other part(x) is differentiated. The hope is that when you apply this, the new integral $\int f'(x)g(x) dx$ is simpler to anti-differentiate.

Theorem: Let f and g be differentiable functions, then

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

The u - v Method:

The u - v Method is given by $\int u dv = uv - \int v du$ which is identical to the canonical method.

Set $u = f(x), du = f'(x)dx$
 $v = g(x), dv = g'(x)dx$

Many find this a lot easier to remember: $\int u dv = uv - \int v du$

Note: Sometimes it is necessary to integrate by parts more than once.

Example Integrate $\int x e^{3x} dx$

Solution: We use integration by parts. Notice that we need to use substitution to find the integral of e^x . Now let $u = x$ and $dv = e^{3x} dx$, so that $du = dx$ and $v = 1/3 e^{3x}$

Hence we have $\int x e^{3x} dx = x \left(\frac{1}{3} e^{3x} \right) - \int \frac{1}{3} e^{3x} dx = \frac{x e^{3x}}{3} - \frac{1}{9} e^{3x} + C$

Reduction formulas:

a) $\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$ for $n \geq 2$

b) $\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$ for $n \geq 2$

Example Evaluate the following integrals.

$$\begin{aligned} \int \sin^3 x dx &= -\frac{1}{3} \sin^{-2} x \cos x + \frac{2}{3} \int \sin x dx \\ &= -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C \end{aligned}$$

Note: Integrals of the form $\int \sin^m x \cos^n x dx$

- a) If n is odd, substitute $u = \sin x$ and use the identity $\cos^2 x = 1 - \sin^2 x$
- b) if m is odd, substitute $u = \cos x$ and use the identity $\sin^2 x = 1 - \cos^2 x$
- c) if m and n are even, reduce to smaller power of m or n and use the identities

$$\sin x \cos x = \frac{1}{2} \sin 2x, \quad \sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example Evaluate $\int \sin^5 x \cos^4 x dx$

$$\begin{aligned} \int \sin^5 x \cos^4 x dx &= \int \sin^4 x \cos^4 x \sin x dx = \int (\sin^2 x)^2 \cos^4 x \sin x dx \\ &= \int (1 - \cos^2 x)^2 \cos^4 x \sin x dx, \quad \begin{array}{l} \text{Let } u = \cos x \\ du = -\sin x dx \end{array} \end{aligned}$$

$$\begin{aligned}
&= -\int (1-u^2)^2 u^4 du = -\int (u^4 - 2u^6 + u^8) du \\
&= -\frac{1}{5} u^5 + \frac{2}{7} u^7 - \frac{1}{9} u^9 + C \\
&= -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C
\end{aligned}$$

C. Integration by Partial Fractions

In algebra we learn how to find the common denominator of two expressions and then combine like terms. Partial fractions are all about going backwards.

Objective: To integrate rational functions $f(x) = \frac{N(x)}{D(x)}$

1. If the degree of the numerator is greater than the degree of the denominator perform long division.
2. Factor the numerator and the denominator of the remainder in to linear and quadratic factors.
3. Rewrite the transformed rational functions as a sum of expressions of the types.

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_r}{(ax+b)^r}$$

Example Evaluate the integral $\int \frac{4x+2}{x^2+x-6} dx$

Solution: First we find the partial fraction decomposition of the integrand.

We factor to get $\frac{4x+2}{x^2+x-6} = \frac{4x+2}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

Multiply by the common denominator to get: $A(x+3) + B(x-2) = 4x+2$

Let $x = 2$, then, $5A = 10$ and $A = 2$, Let $x = -3$, then, $-5B = -10$ and $B = 2$

Now we can integrate $\int \frac{4x+2}{x^2+x-6} dx = \int \left[\frac{2}{x-2} + \frac{2}{x+3} \right] dx$

$$= 2 \ln|x-2| + 2 \ln|x+3| + C$$

D. Integration by trigonometric substitution

An integrand which contains one of the forms $\sqrt{a^2 - b^2 x^2}$, $\sqrt{a^2 + b^2 x^2}$, or $\sqrt{b^2 x^2 - a^2}$ may be transformed into another involving a trigonometric function of a new variable as follows:

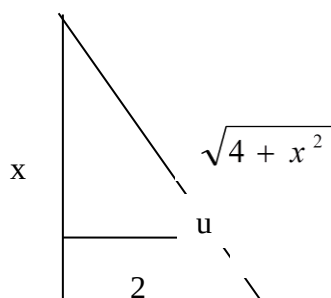
Form in Integrand	Substitution	New Form in Integrand
$\sqrt{a^2 - b^2 x^2}$	$x = \frac{a}{b} \sin u$	$a \sqrt{1 - \sin^2 u} = a \cos u$
$\sqrt{a^2 + b^2 x^2}$	$x = \frac{a}{b} \tan u$	$a \sqrt{1 + \tan^2 u} = a \sec u$
$\sqrt{b^2 x^2 - a^2}$	$x = \frac{a}{b} \sec u$	$a \sqrt{\sec^2 u - 1} = a \tan u$

Example Evaluate $\int \frac{dx}{x^2 \sqrt{4 + x^2}}$

Solution: Let $x = 2 \tan u$ so that $dx = 2 \sec^2 u du$ and hence $\sqrt{4 + x^2} = 2 \sec u$.

$$\begin{aligned} \text{Hence, } \int \frac{dx}{x^2 \sqrt{4 + x^2}} &= \int \frac{2 \sec^2 u du}{4 \tan^2 u \times 2 \sec u} = \frac{1}{4} \int \frac{\sec u du}{\tan^2 u} = \frac{1}{4} \int \frac{\cos u}{\sin^2 u} du \\ &= \frac{-1}{4 \sin u} + C \end{aligned}$$

Note that $\tan u = \frac{x}{2}$ and thus $\sin u = \frac{x}{\sqrt{4 + x^2}}$. (See the following figure)



$$\text{Therefore, } \int \frac{dx}{x^2 \sqrt{4 + x^2}} = \frac{-\sqrt{4 + x^2}}{4x} + C$$

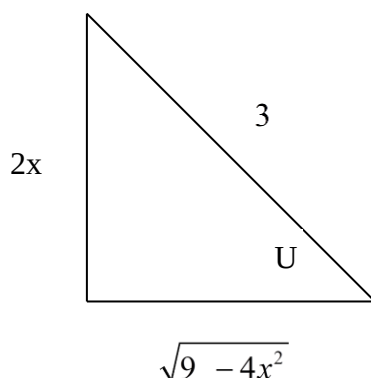
Example Evaluate $\int \frac{\sqrt{9-4x^2}}{x} dx$

Solution: Let $x = \frac{3}{2} \sin u$, so that $dx = \frac{3}{2} \cos u du$ and $\sqrt{9-4x^2} = 3 \cos u$.

$$\text{Hence, } \int \frac{\sqrt{9-4x^2}}{x} dx = \int \frac{3 \cos u}{\left(\frac{3}{2}\right) \sin u} \left(\frac{3}{2}\right) \cos u du$$

$$\begin{aligned} \int \frac{\sqrt{9-4x^2}}{x} dx &= \int \frac{3 \cos u}{\frac{3}{2} \sin u} \left(\frac{3}{2} \cos u\right) du = 3 \int \frac{\cos^2 u}{\sin u} du = 3 \int \frac{1 - \sin^2 u}{\sin u} du \\ &= 3 \int \csc u du - 3 \int \sin u du = 3 \ln |\csc u - \cot u| + 3 \cos u + c \end{aligned}$$

Note that $\cos u = \frac{\sqrt{9-4x^2}}{3}$, $\cot u = \frac{\sqrt{9-4x^2}}{2x}$ and $\csc u = \frac{3}{2x}$



$$\text{Hence } \int \frac{\sqrt{9-4x^2}}{x} dx = 3 \ln \left| \frac{3 - \sqrt{9-4x^2}}{x} \right| + \sqrt{9-4x^2} + c$$

5.3. Integration of Inverse Trigonometric Functions

1. $\int \sin^{-1} x dx$

Using integration by parts, let $u = \sin^{-1} x$ and $dv = dx$.

$$\text{Then } du = \frac{1}{\sqrt{1-x^2}} dx \text{ and } v = x.$$

$$\text{So, } \int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C \text{ -----by substitution}$$

$$2. \int \tan^{-1} x dx$$

Using integration by parts, let $u = \tan^{-1} x$ and $dv = dx$.

$$\text{Then } du = \frac{1}{1+x^2} dx \text{ and } v = x.$$

$$\text{So, } \int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln|x^2 + 1| + C \text{ -----by substitution}$$

Exercise:

$$1. \int \cos^{-1} x dx$$

$$2. \int \cot^{-1} x dx$$

$$3. \int \sec^{-1} x dx$$

$$4. \int \csc^{-1} x dx$$

5.4 The Definite Integral and its Properties

Theorem

Let f continuous on $[a, b]$. For any $\varepsilon > 0$, there is a number $\delta > 0$ such that the following statements holds:

If $P = \{x_0, x_1, \dots, x_n\}$ is any partition of $[a, b]$ each of whose subintervals has length less than δ and if $x_{k-1} \leq t_k \leq x_k$ for each $k = 1, 2, \dots, n$, then the associated Riemann Sum

$$\sum_{k=1}^n f(t_k) \Delta x_k \text{ satisfies } \left| \int_a^b f(x) dx - \sum_{k=1}^n f(t_k) \Delta x_k \right| < \varepsilon \text{ i.e.}$$

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(t_k) \Delta x_k$$

$$\text{i.e. } \int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(t_k) \Delta x_k \text{ where } \|P\| \text{ denotes the largest of the lengths}$$

of the subintervals associated with P and is called the normal of P .

5.4.2 Properties of the Definite Integral

Definition Let f be continuous on $[a, b]$. Then

$$\int_b^a f(x)dx = 0 \quad \text{and} \quad \int_b^a f(x)dx = -\int_a^b f(x)dx \quad [\text{A line segment has area zero}]$$

Example : $\int_4^1 x^2 dx = -\int_1^4 x^2 dx = -\frac{1}{3}(4^3 - 1^3) = -21$

Theorem (Rectangle Property) : For any number a, b and k , $\int_a^b k dx = k(b - a)$

Theorem (Addition Property)

Let f be continuous on an interval containing a, b and c .

$$\text{Then, } \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Example .Evaluate

$$\int_1^5 f(x)dx, \text{ where } f(x) = \begin{cases} 2x - 1, & \text{for } 1 \leq x \leq 3 \\ 5, & \text{for } 3 \leq x \leq 5 \end{cases}$$

Solutions: $\int_1^5 f(x)dx = \int_1^3 f(x)dx + \int_3^5 f(x)dx = \int_1^3 (2x - 1)dx + \int_3^5 5dx = -4$

Corollary Let f be non – negative and continuous on $[a, b]$. Then $\int_a^b f(x)dx \geq 0$

5.4.3 Fundamental Theorem of Calculus

Theorem 5.5.1 (Fundamental theorem of calculus I)

The first fundamental theorem of calculus states that, if f is continuous on the closed interval $[a, b]$ and F is the anti-derivative (indefinite integral) of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \quad (1)$$

Remark. The number $F(b) - F(a)$ is also denoted by $[F(x)]_a^b$ (or $F(x) \Big|_a^b$).

So $\int_a^b f(x)dx = [F(x)] \Big|_a^b = F(b) - F(a)$

Theorem (Fundamental theorem of calculus II)

The second fundamental theorem of calculus holds for f a continuous function on an open interval I and a any point in I , and states that if F is defined by

$$F(x) = \int_a^x f(t)dt, \text{ then } F'(x) = f(x) \text{ at each point in } I. \quad (2)$$

Example Evaluate $\int_1^3 x dx$

Solution:
$$\int_1^3 x dx = \frac{x^2}{2} \Big|_1^3 = \frac{9}{2} - \frac{1}{2} = 4$$

Example If $F(x) = \int_3^{2x} \sqrt{t^2 + 1} dt$, find $F'(x)$

Solution:
$$F'(x) = \sqrt{(2x)^2 + 1} \cdot (2) = 2\sqrt{4x^2 + 1}$$

Example Evaluate $\int_0^{\frac{\pi}{2}} \cos t dt$

Solution: We have
$$\int_0^x \cos t dt = [\sin(t)]_0^x = \sin(x) - \sin(0) = \sin(x)$$

$$\text{So } \int_0^{\frac{\pi}{2}} \cos t dt = \sin \frac{\pi}{2} = 1$$

- ✓ Combining the Chain Rule with the Fundamental Theorem of Calculus, we can generate some nice results. Indeed, let $f(x)$ be continuous on $[a, b]$ and $u(x)$ be differentiable on

$$[a, b]. \text{ Define the function } F(x) = \int_a^{u(x)} f(t) dt$$

Then the Chain Rule implies that $F(x)$ is differentiable and $F'(x) = f(u(x))u'(x)$

- ✓ We can generalize this a little bit more to find the derivative of a function of the form

$$H(x) = \int_{u(x)}^{v(x)} f(t) dt \text{ where } u(x) \text{ and } v(x) \text{ are both differentiable on } [a, b].$$

$$\text{We have } H'(x) = f(v(x))v'(x) - f(u(x))u'(x)$$

Example Find the derivative of $\int_0^{x^2} \cos(t^2) dt$

Solution: Set $F(x) = \int_0^{x^2} \cos(t^2) dt$

From the Fundamental Theorem of Calculus, we know that $F(x)$ is an anti-derivative of $\cos(x^2)$.

We have $\int_0^{x^2} \cos(t^2) dt = F(x^2)$

The Chain Rule then implies that $\frac{d}{dx} \left(\int_0^{x^2} \cos(t^2) dt \right) = F'(x^2) 2x = 2x \cos((x^2)^2) = 2x \cos(x^4)$.

Note that $F(x)$ does not have an explicit form. So it is quite amazing that even if $F(x)$ is defined via some theoretical result, we are still able to find the derivative of the given function.

Example Find the derivative of $\int_x^{x^2} e^{-t^2} dt$

Solution let $F(x) = \int_0^x e^{-t^2} dt \Rightarrow \int_x^{x^2} e^{-t^2} dt = F(x^2) - F(x)$

$$\Rightarrow \frac{d}{dx} \left(\int_x^{x^2} e^{-t^2} dt \right) = F'(x^2) 2x - F'(x) = 2x e^{-x^4} - e^{-x^2}.$$

5.5 Improper Integrals

Up until now, all the integrals that we have performed have had finite limits and integrands with no asymptotes. Now we will investigate integrals with infinite limits and integrands with vertical asymptotes. Recall that the definition of an integral $\int_a^b f(x) dx$ requires the function $f(x)$ to be bounded on the bounded interval $[a, b]$ (where a and b are two real numbers). It is natural what happens if the function f is unbounded.

Definition

Let the function $f(x)$ becomes unbounded on the interval $[a, b]$ and the interval $[a, b]$ becomes unbounded (that is $a = -\infty$ or $b = \infty$), then we say that the integral $\int_a^b f(x) dx$ is **improper**

Type I: (Discontinuous Integrands)

Consider the function $f(x)$ defined on the interval $[a, b]$ (where a and b are real numbers). We have two cases $f(x)$ becomes unbounded around a or unbounded around b

$$\Rightarrow \int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx \quad \text{unbounded around } a$$

$$\Rightarrow \int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx \quad \text{unbounded around } b$$

Remark What happened if the function $f(x)$ is unbounded at more than one point on the interval $[a,b]$? Very easy, first you need to study $f(x)$ on $[a,b]$ and find out where the function is unbounded. Let us say that $f(x)$ is unbounded at c_1 and c_2 for example, with

$a \leq c_1 < c_2 \leq b$. Then you must choose a number α between c_1 and c_2 (that is $c_1 < \alpha < c_2$) and then write

$$\Rightarrow \int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{\alpha} f(x)dx + \int_{\alpha}^{c_2} f(x)dx + \int_{c_2}^b f(x)dx$$

Example evaluate the following improper integral

a) $\int_0^3 \frac{dx}{x-1}$

b. $\int_0^1 \ln x dx$

Type II: Consider the function $f(x)$ defined on the interval $[a, +\infty)$ or $(-\infty, a]$. In other words, the domain is unbounded not the function.

$$\Rightarrow \int_{-\infty}^a f(x)dx = \lim_{t \rightarrow -\infty} \int_t^a f(x)dx \quad \text{and}$$

$$\Rightarrow \int_a^{+\infty} f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx$$

Example Evaluate $\int_1^{\infty} \frac{1}{x^2} dx$

Solution We use the definition to find

$$\lim_{m \rightarrow \infty} \int_1^m \frac{1}{x^2} dx = \lim_{m \rightarrow \infty} \left[\frac{-1}{x} \right]_1^m = \lim_{m \rightarrow \infty} \left[\frac{-1}{m} + 1 \right]_1^m = 1$$

Example Evaluate $\int_0^1 \frac{1}{\sqrt{x}} dx$

Solution The integrand has a vertical asymptote at $x = 0$, so we must use part (ii) in the definition.

We have
$$\lim_{m \rightarrow 0^+} \int_m^1 x^{-\frac{1}{2}} dx = \lim_{m \rightarrow 0^+} \left[2x^{\frac{1}{2}} \right]_m^1 = \lim_{m \rightarrow 0^+} \left[2 - 2m^{\frac{1}{2}} \right] = 2$$

5.5.2 Convergence and Divergence of Improper Integrals

Consider a function $f(x)$ which exhibits a Type I or Type II behavior on the interval $[a, b]$ (in other words, the integral $\int_a^b f(x)dx$ is improper). We saw before that the integral is defined as a limit. Therefore we have two cases:

Definition Let $\int_a^b f(x)dx$ be improper integral.

1. If the limit exists (and is a number), we say that the improper integral is **Convergent**;
2. If the limit does not exist or it is infinite, then we say that the improper integral is **Divergent**.

If the improper integral $\int_a^b f(x)dx$ is split into a sum of improper integrals (because $f(x)$ presents more than one improper behavior on $[a, b]$), then the integral converges if and only if any single improper integral is convergent.

5.6 Applications of Integrals

5.6.1 Area

Definition if f is continuous and $f(x) \geq 0$ on $[a, b]$, the area of the region below the curve $y = f(x)$ from a to b is defined as the definite integral:

$$\text{Area} = \int_a^b f(x)dx$$

Theorem (Area of a region between two curves)

If f and g are continuous on $[a, b]$ and $f(x) \leq g(x) \forall x \in (a, b)$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is given by

$$A = \int_a^b [g(x) - f(x)]dx$$

Example

Find the area of the region that is enclosed between the curve $y = x^2$ and $y = x + 6$.

Solution To find the end point, we solve the equations

$$y = x^2 \text{ and } y = x + 6$$

$$\Rightarrow x^2 - x - 6 = (x - 3)(x + 2) = 0$$

$$\Rightarrow x = 3 \vee x = -2$$

$\therefore y = 9$ for $x = 3$ and $y = 4$ for $x = -2$

Hence, the two graphs intersect at the points (3; 9) and (-2, 4)

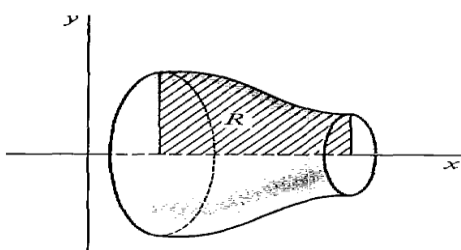
Moreover, if $f(x) = x + 6$ and $g(x) = x^2$, then

$$\therefore A = \int_{-2}^3 [x + 6 - x^2] dx = \left[\frac{1}{2} x^2 + 6x - \frac{1}{3} x^3 \right]_{-2}^3 = \frac{125}{6}$$

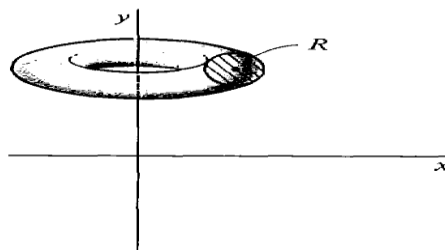
5.6.2 Volume

Integrals are used in this section to find the volume of solid revolution. A solid of revolution is generated by taking a region in the first quadrant of the plane and rotating it in space about the x-axis or y-axis.

Solids of revolution



Rotating about the x-axis



Rotating about the y-axis

Theorem Let S be a solid region obtained by revolving the graph of a continuous non – negative function f and has a cross – sectional area A .

i) If A is given by $A(x)$, which is perpendicular to the x – axis for $a \leq x \leq b$, then the volume V of

S is given by $V = \int_a^b A(x) dx = \pi \int_a^b [f(x)]^2 dx$ -----Horizontal axis of revolution

ii) If A is given by $A(y)$, which is perpendicular to the y – axis for $c \leq y \leq d$, then the volume V

of S is given by $V = \int_c^d A(y) dy = \pi \int_c^d [f(y)]^2 dy$ ---- Vertical axis of revolution

Theorem

If the solid is generated by revolving the region between the graphs of f and g

i) on $[a, b]$ with $g(x) \leq f(x) \forall x \in (a, b)$, Then

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx \quad \text{--- Horizontal axis of revolution}$$

ii) on $[c, d]$ with $g(y) \leq f(y) \forall y \in (c, d)$, then

$$V = \pi \int_c^d ([f(y)]^2 - [g(y)]^2) dy \quad \text{--- Vertical axis of revolution}$$

Example Let R be the region between the graphs of $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{x-1}$ on the interval $[1, 3]$. Find the volume V of the solid obtained by revolving R about the x -axis.

Solution: Clearly $f(x) \geq g(x) \forall x \in [1, 3]$, and

$$[f(x)]^2 = x+1 \text{ and } [g(x)]^2 = x-1. \text{ Moreover } [f(x)]^2 - [g(x)]^2 = x+1 - (x-1) = 2$$

$$\therefore V = \pi \int_1^3 2 dx = \pi(2x) \Big|_1^3 = 6\pi - 2\pi = 4\pi \text{ Cu. Units}$$

5.6.3 Arc length

Theorem Let f be continuous on $[a, b]$ and the curve of the graph of f on $[a, b]$ given by $y = f(x)$ in a plane. Then the length L of the curve between $(a, f(a))$ and $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Example: Find the length of the curve $y = 3x^{\frac{3}{2}} - 1$ from $x = 0$ to $x = 1$

Exercise

1. Find the length of the curve $y = 2x$

a) $(-1, -2)$ and $(4, 8)$ b) $(2, 6)$ and $(-2, 4)$

2. Find the length of the curve $x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}}$ from $y = 0$ to $y = 1$.