

Chapter Two

Application

2.1 Application on area and volume

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors and consider the parallelogram that the two vectors make. Then

$$\|\mathbf{u} \times \mathbf{v}\| = \text{Area of the Parallelogram}$$

The direction of $\mathbf{u} \times \mathbf{v}$ is a right angle to the parallelogram that follows the right hand rule.

To find the volume of the parallelepiped spanned by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, we find the triple product:

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \text{Volume}$$

This can be found by computing the determinant of the three vectors:

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = u_1(v_2w_3 - v_3w_2) - v_1(u_2w_3 - u_3w_2) + w_1(u_2v_3 - u_3v_2)$$

Example :

1. Find the area of the parallelogram which is formed by the two vectors $\mathbf{u} = (1, 3, 2)$ and $\mathbf{v} = (-2, 1, 3)$.

2. Find the volume of the parallelepiped spanned by the vectors

$$\mathbf{u} = (3, -2, -1), \mathbf{v} = (1, 3, 2), \text{ and } \mathbf{w} = (-2, 1, 3).$$

Solution: 1. The area of the parallelogram is given by:

$$\|\mathbf{u} \times \mathbf{v}\| = \|(1, 3, 2) \times (-2, 1, 3)\| = \|(7, -7, 7)\| = \sqrt{147}$$

2. The volume of the parallelepiped spanned by the three vectors is:

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |(3, -2, -1) \cdot ((1, 3, 2) \times (-2, 1, 3))| = |(3, -2, -1) \cdot (7, -7, 7)| = 28$$

Exercise: Find the area of the triangle having vertices at $\mathbf{u} = (3, -2, -1)$, $\mathbf{v} = (1, 3, 2)$, and $\mathbf{w} = (-2, 1, 3)$.

Lines and planes

Activity : Given two vectors $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{B} \neq \mathbf{0}$, consider vectors of the form $\mathbf{A} + t\mathbf{B}$, where t varies over all real numbers. The vector $\mathbf{A} + 2\mathbf{B}$, for example is as shown in the figure. Show the vectors $\mathbf{A}$

2.2 Equations of Lines and Planes

An equation of three variable $F(x, y, z) = 0$ is called an equation of a surface S if

$$(x_1, y_1, z_1) \in S \text{ if and only if } F(x_1, y_1, z_1) = 0.$$

For instance,

$$x^2 + y^2 + z^2 = 1$$

is the equation of the unit sphere centered at the origin. The graph of a system of two equations

$$F(x, y, z) = 0, G(x, y, z) = 0$$

represents the intersection of two surfaces represented by $F(x, y, z) = 0$ and by $G(x, y, z) = 0$, respectively, and is usually a curve.

A) Lines in R^3 .

A line l is determined by two elements: one point P_0 on the line l and a direction $\vec{v}$ of l , i.e., any vector that is parallel to l . The goal here is to describe the line using algebra so that one is able to digitize it. Suppose that the coordinate of the point P_0 on the line and a direction $\vec{v}$ are given as:

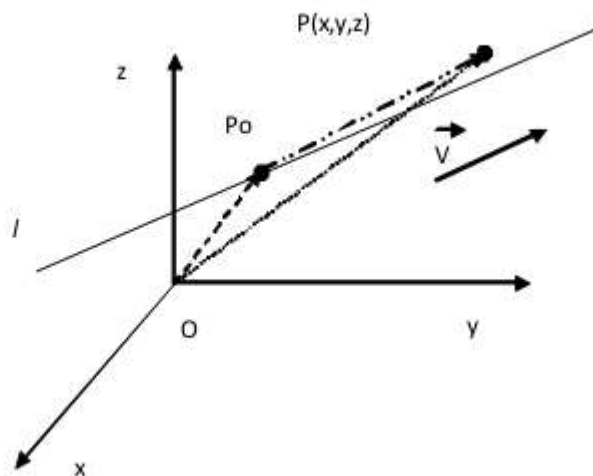
$$\begin{aligned} P_0(x_0, y_0, z_0) &\text{ is a given point on } l \\ \vec{v} = \langle a, b, c \rangle &\text{ is parallel to } l. \end{aligned}$$

Consider any point $P(x, y, z)$ in the space. Let $\overrightarrow{P_0P}$ be the vector connecting P_0 and P . If P is located exactly on the line, then $\overrightarrow{P_0P}$ is parallel to the line l , and thus it is parallel to $\vec{v}$. On the other hand, if P is off the line, then, since P_0 is on the line, $\overrightarrow{P_0P}$ cannot possibly be parallel to the line. Therefore, $\overrightarrow{P_0P}$ cannot possibly be parallel to $\vec{v}$. We just concluded that

$$P \text{ is on } l \text{ if and only if } \overrightarrow{P_0P} \text{ is parallel to } \vec{v}.$$

Now

$$\begin{aligned} \overrightarrow{P_0P} &= \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = \langle x - x_0, y - y_0, z - z_0 \rangle \\ \vec{v} &= \langle a, b, c \rangle. \end{aligned}$$



So

$$\overrightarrow{P_0P} \parallel \vec{v} \iff \overrightarrow{P_0P} = t\vec{v} \text{ (for a constant } t)$$

which is equivalent to

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We called these three equation *symmetric form* of the system of equations for line l .

If we set

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t,$$

which is equivalent to

$$\begin{aligned} \frac{x - x_0}{a} &= t \\ \frac{y - y_0}{b} &= t \\ \frac{z - z_0}{c} &= t, \end{aligned}$$

Or

$$\begin{aligned}x &= x_0 + at \\y &= y_0 + bt \\z &= z_0 + ct,\end{aligned}$$

We call it the *parametric form* of the system of equations for line l . This system can be written in the form of vector equation:

$$\vec{r} = \vec{r}_0 + t\vec{v}, \quad \vec{r} = \langle x, y, z \rangle, \quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle.$$

Example 4.1. (a) Find the equation of the line passing through $(5, -1, 3)$, having direction $\vec{v} = \langle 1, 0, -2 \rangle$. Express answer in (i) symmetric form, (ii) vector form, and (iii) parametric form.

(b) Find two other points on the line.

Solution: (a) (i)

$$\frac{x-5}{1} = \frac{y+1}{0} = \frac{z-3}{-2}$$

(ii)

$$\vec{r} = \langle 5, -1, 3 \rangle + t \langle 1, 0, -2 \rangle$$

(iii)

$$\begin{aligned}x &= 5 + t \\y &= -1 \\z &= 3 - 2t\end{aligned}$$

(b) Take $t = 1$, $(x, y, z) = (6, -1, 1)$. Take $t = -1$, $(x, y, z) = (4, -1, 5)$.

Example 4.2. (a) Find the equation, in symmetric form, of the line l passing through $A(2, 4, -3)$ and $B(3, -1, 1)$. (b) Determine where the line l intersects xy -plane.

Solution. (a) The line is parallel to vector $\overrightarrow{AB}$. So we choose this vector as the direction of l ,

$$\vec{v} = \overrightarrow{AB} = \langle 3, -1, 1 \rangle - \langle 2, 4, -3 \rangle = \langle 1, -5, 4 \rangle,$$

and $A(2, 4, -3)$ as the point on l . Thus, the equation is

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}.$$

(b) If this line crosses xy -plane somewhere at (x, y, z) , then $z = 0$. So this point $(x, y, 0)$ satisfies the line equation, i.e.,

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{0+3}{4}.$$

We solve this system to obtain

$$\begin{aligned}x &= 2 + \frac{3}{4} = \frac{11}{4} \\y &= 4 - 5\left(\frac{3}{4}\right) = \frac{1}{4} \\z &= 0.\end{aligned}$$

Example 4.3. Given two lines

$$\begin{aligned}l_1 : x &= 1 + t, \quad y = -2 + 3t, \quad z = 4 - t \\l_2 : x &= 2t, \quad y = 3 + t, \quad z = -3 + 4t.\end{aligned}$$

Determine whether they intersect each other, or they are parallel, or neither (skew lines).

Solution: First of all, in each line equation, " t " is a parameter (or free variable) that can be chosen arbitrarily. Therefore, the parameter " t " in the equations for line l_1 is DIFFERENT from the parameter " t " in the equations for line l_2 . To clarify this issue, we rewrite as

$$\begin{aligned}l_1 : x &= 1 + t, \quad y = -2 + 3t, \quad z = 4 - t \\l_2 : x &= 2s, \quad y = 3 + s, \quad z = -3 + 4s,\end{aligned}$$

and intersection of these two lines consists of solutions of the following system of six equations,

$$\begin{aligned}x &= 1 + t, \quad y = -2 + 3t, \quad z = 4 - t \\x &= 2s, \quad y = 3 + s, \quad z = -3 + 4s,\end{aligned}$$

for five variables: x, y, z, t, s . Two lines intersect each other if and only if this system has a solution. If, for instance, $(x_0, y_0, z_0, t_0, s_0)$ is a solution, then the first three components, (x_0, y_0, z_0) is a point of intersection.

We now proceed to solution the system by eliminating x,y,z:

$$1 + t = 2s \quad (1)$$

$$-2 + 3t = 3 + s \quad (2)$$

$$4 - t = -3 + 4s. \quad (3)$$

There are three equations with *two* unknowns. We start with two equations, for instance, the first and the second equation:

$$1 + t = 2s$$

$$-2 + 3t = 3 + s.$$

This can be easily solved as, by subtracting 2 times the second equation from the first equation, i.e.,

$$7 - 5t = -6 \implies t = \frac{11}{5},$$

$$s = \frac{1+t}{2} = \frac{8}{5}.$$

We need to verify that the solution, $t = \frac{11}{5}$, $s = \frac{8}{5}$, from the first two equations (1) & (2), satisfies the third equation (3). So

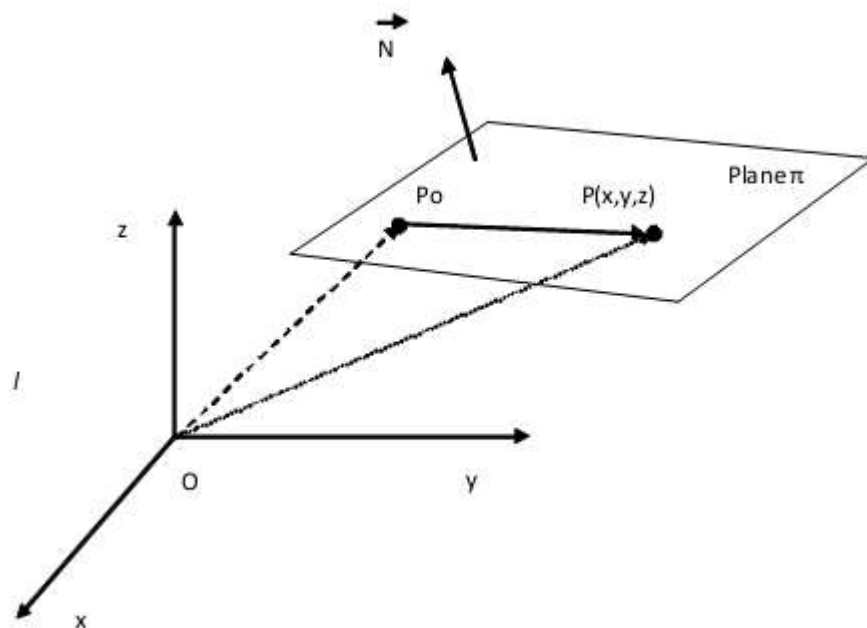
$$LHS \text{ of } (3) = 4 - t = 4 - \frac{11}{5} = \frac{9}{5}$$

$$RHS \text{ of } (3) = -3 + 4s = -3 + 4\left(\frac{8}{5}\right) = \frac{17}{5}.$$

Apparently,

$$t = \frac{11}{5}, s = \frac{8}{5}$$

is not a solution of the entire system (1)-(3). We thus conclude that these two line cannot possibly intersect. Answer: skew lines



B) Equations of Plane.

Definition. Any vector that is perpendicular to a plane is called a normal vector to the plane.

There are two normal directions (opposite to each other) to a given plane. For any given vector $\vec{n}$, there are infinite many parallel planes that are all having $\vec{n}$ as their normal vector. If we also know a point on the plane, then, this plane is uniquely determined. In other words, a plane π can be determined by a point $P_0(x_0, y_0, z_0)$ on the plane and a vector as its normal vector $\vec{n} = \langle A, B, C \rangle$.

For any point $P(x, y, z)$, if this point P is on the plane π , then the line segment P_0P entirely lies on the plane. Consequently, vector

$$\overrightarrow{P_0P} = \vec{r} - \vec{r_0} = \langle x - x_0, y - y_0, z - z_0 \rangle, \text{ where } \vec{r} = \langle x, y, z \rangle, \vec{r_0} = \langle x_0, y_0, z_0 \rangle,$$

is perpendicular to the normal vector $\vec{n}$. On the other hand, if P is off the

plane π , then $\vec{r} = \overrightarrow{P_0P}$ is not perpendicular to $\vec{n}$. We conclude that

$$P \in \pi \text{ (} P \text{ belongs to } \pi \text{)} \iff \overrightarrow{P_0P} \cdot \vec{n} = 0,$$

or

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0. \quad (\text{Vector Equation})$$

We call it vector equation of the plane π . In terms of components,

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle A, B, C \rangle = 0.$$

We obtain scalar form of equation of plane π :

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \quad (\text{Scalar Equation})$$

or

$$Ax + By + Cz + D = 0. \quad (\text{Linear Equation})$$

In 3D spaces, any linear equation as above represents a plane with a normal vector $\vec{n} = \langle A, B, C \rangle$. (In 2D, any linear equation is a straight line.)

Example 4.4. Find the equation of the plane passing through $P_0(2, 4, -1)$ having a normal vector $\vec{n} = \langle 2, 3, 4 \rangle$.

Solution: $A = 2, B = 3, C = 4$. The equation is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0,$$

or

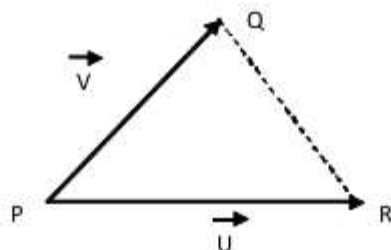
$$2x + 3y + 4z - 12 = 0.$$

Example 4.5. Find the equation of the plane passing through $P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$.

Solution: Let

$$\vec{u} = \overrightarrow{PR} = \langle 5, 2, 0 \rangle - \langle 1, 3, 2 \rangle = \langle 4, -1, -2 \rangle$$

$$\vec{v} = \overrightarrow{PQ} = \langle 3, -1, 6 \rangle - \langle 1, 3, 2 \rangle = \langle 2, -4, 4 \rangle$$



The vector

$$\begin{aligned}
 \vec{u} \times \vec{v} &= \begin{vmatrix} i & j & k \\ 4 & -1 & -2 \\ 2 & -4 & 4 \end{vmatrix} \\
 &= \begin{vmatrix} -1 & -2 \\ -4 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 4 & -2 \\ 2 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 4 & -1 \\ 2 & -4 \end{vmatrix} \vec{k} \\
 &= -12\vec{i} - 20\vec{j} - 14\vec{k} \\
 &= -2(6\vec{i} + 10\vec{j} + 7\vec{k})
 \end{aligned}$$

is perpendicular to both $\vec{u}$ and $\vec{v}$. Thus,

$$\vec{n} = (6\vec{i} + 10\vec{j} + 7\vec{k})$$

is perpendicular to π . Now, we take this normal vector and one point $P(1, 3, 2)$ (you may choose Q or R , instead), and the equation is

$$6(x - 1) + 10(y - 3) + 7(z - 2) = 0$$

or

$$6x + 10y + 7z - 50 = 0.$$

Note that if we chose $Q(3, -1, 6)$ as the known point, then the equation would be

$$6(x - 3) + 10(y + 1) + 7(z - 6) = 0$$

or

$$6x + 10y + 7z - 50 = 0.$$

Example 4.6. Find the intersection, if any, of the line

$$x = 2 + 3t, \quad y = -4t, \quad z = 5 + t$$

and the plane

$$4x + 5y - 2z = 18.$$

Solution: We need to solve the system of all four equations

$$x = 2 + 3t$$

$$y = -4t$$

$$z = 5 + t$$

$$4x + 5y - 2z = 18$$

for x, y, z, t . To this end, we substitute the first three equations into the last one. This leads to

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18,$$

which can be simplified to

$$-10t - 2 = 18.$$

So

$$t = -2,$$

and

$$x = 2 + 3t = -4$$

$$y = -4t = 8$$

$$z = 5 + t = 3.$$

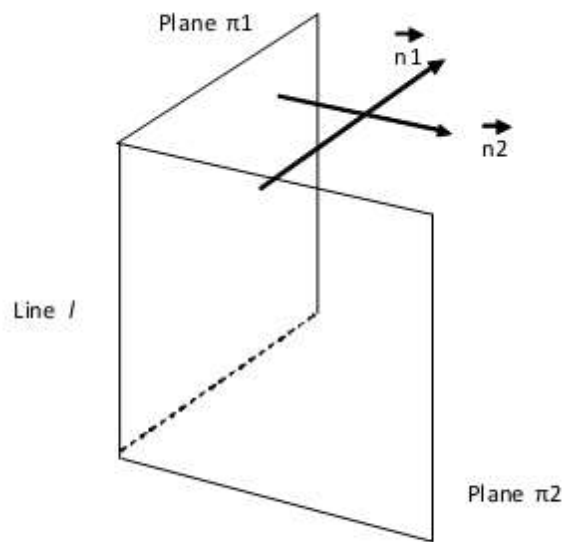
Answer: the intersection is $(-4, 8, 3)$.

Example 4.7. Given two planes

$$\pi_1 : x + y + z = 1$$

$$\pi_2 : x - 2y + 3z = 1.$$

Find (a) the line of intersection, and (b) the angle between two planes.



Solution: Plane π_1 and plane π_2 have normal vectors $\vec{n}_1$ and $\vec{n}_2$, respectively, as

$$\begin{aligned}\vec{n}_1 &= \langle 1, 1, 1 \rangle \\ \vec{n}_2 &= \langle 1, -2, 3 \rangle.\end{aligned}$$

The line is on both planes and thus is perpendicular to both normal vectors. The direction of the line is

$$\begin{aligned}\vec{v} = \vec{n}_1 \times \vec{n}_2 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \vec{k} \\ &= 5\vec{i} - 2\vec{j} - 3\vec{k}.\end{aligned}$$

To find the equation of the line, we also need a point on the line, i.e., on both planes. So we look for one solution to the system

$$\begin{aligned}x + y + z &= 1 \\ x - 2y + 3z &= 1.\end{aligned}$$

This system has infinite many solutions (why). Since we only need one solution, we set $z = 0$ to reduce the system to

$$\begin{aligned}x + y &= 1 \\x - 2y &= 1.\end{aligned}$$

Subtracting the second equation from the first, we find

$$\begin{aligned}3y &= 0 \implies y = 0 \\x &= 1.\end{aligned}$$

So $P(1, 0, 0) \in l$. The equation of the line, in parametric form, is

$$\begin{aligned}x &= 1 + 5t \\y &= -2t \\z &= -3t.\end{aligned}$$

Solution #2: Another way to find the equation of this line is to solve the system

$$\begin{aligned}x + y + z &= 1 \\x - 2y + 3z &= 1\end{aligned}$$

directly in terms of z . In other words, we choose z as parameter. To this end, we subtract the second equation from the first one to get

$$3y - 2z = 0 \implies y = \frac{2}{3}z.$$

Substituting this into plane π_1 :

$$x + \left(\frac{2}{3}z\right) + z = 1 \implies x = 1 - \frac{5}{3}z,$$

we obtain the equation of the line

$$\begin{aligned}x &= 1 - \frac{5}{3}z \\y &= \frac{2}{3}z \\z &= z.\end{aligned}$$

Note that z is basically a free variable. If we set $z = -3t$, this becomes

$$\begin{aligned}x &= 1 + 5t \\y &= -2t \\z &= -3t\end{aligned}$$

which is identical to what we got earlier.

(b) The angle between two planes is the same as the angle between their normal vectors. So

Homework:

1. Determine whether each statement is true or false.
 - (a) Two lines parallel to a third line are parallel.
 - (b) Two lines perpendicular to a third line are parallel.
 - (c) Two planes parallel to a third plane are parallel.
 - (d) Two planes perpendicular to a third plane are parallel.
 - (e) Two lines parallel to a plane are parallel.
 - (f) Two lines perpendicular to a plane are parallel.
 - (g) Two planes parallel to a line are parallel.
 - (h) Two planes perpendicular to a line are parallel.
 - (i) Two planes are either intersect or are parallel.
 - (j) Two lines are either intersect or are parallel.
 - (k) A plane and a line are either intersect or are parallel.
2. Find parametric equation and symmetric equation of line.
 - (a) The line through the point $(1, 0, -3)$ and parallel to $\langle 2, -4, 5 \rangle$.
 - (b) The line through the point the origin and parallel to the line $x = 2t$, $y = 1 - t$, $z = 4 + 3t$.
 - (c) The line through $(1, 1, 6)$ and perpendicular to the plane $x + 3y + z = 5$.
 - (d) The line through $(2, 1, 1)$ and perpendicular to $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{k}$.
 - (e) The line of intersection of the planes $x + 2y + z = 1$ and $x + y = 0$.
3. Find equation of plane.
 - (a) The plane through $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 0)$.
 - (b) The plane through $(5, 1, 0)$ and parallel to two lines $x = 2 + t$, $y = -2t$, $z = 1$ and $x = t$, $y = 2 - t$, $z = 2 - 3t$.
 - (c) The plane through $(-1, 2, 1)$ and contains the line of intersection of two planes $x + y - z = 2$ and $2x - y + 3z = 1$.
 - (d) The plane through $(-1, 2, 1)$ and perpendicular to the line of intersection of two planes $x + y - z = 2$ and $2x - y + 3z = 1$.
 - (e) The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$, and is perpendicular to the plane $x + y - 2z = 1$.