

Class 11 Maths Chapter 8 Binomial Theorem

Binomial Theorem for Positive Integer

If n is any positive integer, then

$$(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_n a^n.$$

$$(x + a)^n = \sum_{r=0}^n {}^nC_r x^{n-r} a^r$$

This is called binomial theorem.

Here, ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are called binomial coefficients and

${}^nC_r = n! / r!(n-r)!$ for $0 \leq r \leq n$.

Properties of Binomial Theorem for Positive Integer

(i) Total number of terms in the expansion of $(x + a)^n$ is $(n + 1)$.

(ii) The sum of the indices of x and a in each term is n .

(iii) The above expansion is also true when x and a are complex numbers.

(iv) The coefficient of terms equidistant from the beginning and the end are equal. These coefficients are known as the binomial coefficients and

${}^nC_r = {}^nC_{n-r}, r = 0, 1, 2, \dots, n$.

(v) General term in the expansion of $(x + c)^n$ is given by

$$T_{r+1} = {}^nC_r x^{n-r} a^r.$$

(vi) The values of the binomial coefficients steadily increase to maximum and then steadily decrease .

(vii)

$$(x - a)^n = {}^nC_0 - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \dots + (-1)^n {}^nC_n a^n$$

$$\text{i.e., } (x - a)^n = \sum_{r=0}^n (-1)^r {}^nC_r \cdot x^{n-r} \cdot a^r$$

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$$

$$\text{i.e., } (1 + x)^n = \sum_{r=0}^n {}^nC_r \cdot x^r$$

(viii)

(ix) The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

(x)

$$(xi) (a) \quad (x + a)^n + (x - a)^n = 2 ({}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \dots)$$

$$(b) \quad (x + a)^n - (x - a)^n = 2 ({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \dots)$$

(xii) (a) If n is odd, then $(x + a)^n + (x - a)^n$ and $(x + a)^n - (x - a)^n$ both have the same number of terms equal to $(n + 1) / 2$.(b) If n is even, then $(x + a)^n + (x - a)^n$ has $(n + 1) / 2$ terms. and $(x + a)^n - (x - a)^n$ has $(n / 2)$ terms.(xiii) In the binomial expansion of $(x + a)^n$, the r th term from the end is $(n - r + 2)$ th term

$$(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \dots + (-1)^r {}^nC_r x^r + \dots + (-1)^n {}^nC_n x^n$$

$$\text{i.e., } (1 - x)^n = \sum_{r=0}^n (-1)^r {}^nC_r \cdot x^r$$

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(xiv) If n is a positive integer, then number of terms in $(x + y + z)^n$ is $(n + 1)(n + 2) / 2$.

Middle term in the Expansion of $(1 + x)^n$

- (i) If n is even, then in the expansion of $(x + a)^n$, the middle term is $(n/2 + 1)^{\text{th}}$ terms.
- (ii) If n is odd, then in the expansion of $(x + a)^n$, the middle terms are $(n + 1) / 2$ th term and $(n + 3) / 2$ th term.

Greatest Coefficient

- (i) If n is even, then in $(x + a)^n$, the greatest coefficient is ${}^nC_{n/2}$
- (ii) If n is odd, then in $(x + a)^n$, the greatest coefficient is ${}^nC_{n-1/2}$ or ${}^nC_{n+1/2}$ both being equal.

Greatest Term

In the expansion of $(x + a)^n$

- (i) If $n + 1 / x/a + 1$ is an integer = p (say), then greatest term is $T_p == T_{p+1}$.
- (ii) If $n + 1 / x/a + 1$ is not an integer with m as integral part of $n + 1 / x/a + 1$, then T_{m+1} is the greatest term.

Important Results on Binomial Coefficients

$$(i) {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$(ii) \frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$$

$$(iii) \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

$$(iv) C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$(v) C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

$$(vi) C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0$$

$$(vii) C_0 C_r + C_1 C_{r+1} + \dots + C_{n-r} C_n = {}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!(n+r)!}$$

$$(viii) C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{(n!)^2}$$

$$(ix) C_0 - C_2 + C_4 - C_6 + \dots = (\sqrt{2})^n \cos \frac{n\pi}{4}$$

$$(x) C_1 - C_3 + C_5 - C_7 + \dots = (\sqrt{2})^n \sin \frac{n\pi}{4}$$

$$(xi) C_0 - C_1 + C_2 - C_3 + \dots + (-1)^r C_r = (-1)^r {}^{n-1}C_r, r < n$$

$$(xii) C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots = \begin{cases} 0, & \text{if } n \text{ is odd.} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

$$(xiii) C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{(n+1)}$$

$$(xiv) C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$$

$$(xv) C_0 + \frac{C_1}{2} + \frac{C_2}{2^2} + \frac{C_3}{2^3} + \dots + \frac{C_n}{2^n} = \left(\frac{3}{2}\right)^n$$

$$(xvi) \sum_{r=0}^n (-1)^r {}^nC_r \left\{ \frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \dots \text{ upto } m \text{ terms} \right\} \\ = \frac{2^{mn} - 1}{2^{mn} (2^n - 1)}$$

Divisibility Problems

From the expansion, $(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$

We can conclude that,

(i) $(1+x)^n - 1 = {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$ is divisible by x i.e., it is multiple of x .

$$(1+x)^n - 1 = M(x)$$

$$(ii) (1+x)^n - 1 - nx = {}^nC_2x^2 + {}^nC_3x^3 + \dots + {}^nC_nx^n = M(x^2)$$

$$(1+x)^n - 1 - nx - \frac{n(n-1)}{2}x^2 = {}^nC_3x^3 + {}^nC_4x^4 + \dots + {}^nC_nx^n$$

$$= M(x^3)$$

(iii)

Multinomial theorem

For any $n \in \mathbb{N}$,

$$(x_1 + x_2)^n = \sum_{r_1 + r_2 = n} \frac{n!}{r_1!r_2!} x_1^{r_1} x_2^{r_2}$$

(i)

$$(x_1 + x_2 + \dots + x_n)^n = \sum_{r_1 + r_2 + \dots + r_k = n} \frac{n!}{r_1!r_2!\dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$$

(ii)

(iii) The general term in the above expansion is

$$\frac{n!}{r_1!r_2!\dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$$

(iv) The greatest coefficient in the expansion of $(x_1 + x_2 + \dots +$

$x_m)^n$ is $\frac{n!}{(q!)^{m-r} [(q+1)!]^r}$ where q and r are the quotient and remainder respectively, when n is divided by m .

(v) Number of non-negative integral solutions of $x_1 + x_2 + \dots + x_n = n$ is ${}^{n+r-1}C_{r-1}$

R-f Factor Relations

Here, we are going to discuss problem involving $(\sqrt[n]{A+B})^{\sup > n} = I + f$, Where I and n are positive integers.

$$0 \leq f \leq 1, |A - B^2| = k \text{ and } |\sqrt[n]{A - B}| < 1$$

Binomial Theorem for any Index

If n is any rational number, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots, |x| < 1$$

(i) If in the above expansion, n is any positive integer, then the series in RHS is finite otherwise infinite.

(ii) General term in the expansion of $(1+x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2)\dots[n-(r-1)]}{r!} x^r$

(iii) Expansion of $(x+a)^n$ for any rational index

Case I. When $x > a$ i.e., $\frac{a}{x} < 1$

$$\begin{aligned} \text{In this case, } (x+a)^n &= \left\{ x \left(1 + \frac{a}{x} \right) \right\}^n = x^n \left(1 + \frac{a}{x} \right)^n \\ &= x^n \left\{ 1 + n \cdot \frac{a}{x} + \frac{n(n-1)}{2!} \left(\frac{a}{x} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{a}{x} \right)^3 + \dots \right\} \end{aligned}$$

Case II. When $x < a$ i.e., $\frac{x}{a} < 1$

$$\begin{aligned} \text{In this case, } (x+a)^n &= \left\{ a \left(1 + \frac{x}{a} \right) \right\}^n = a^n \left(1 + \frac{x}{a} \right)^n \\ &= a^n \left\{ 1 + n \cdot \frac{x}{a} + \frac{n(n-1)}{2!} \left(\frac{x}{a} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{x}{a} \right)^3 + \dots \right\} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad (1-x)^{-n} &= 1 + nx + \frac{n(n+1)}{1 \cdot 2} x^2 + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} x^3 + \dots \\ &= 1 + {}^nC_1 x + {}^{(n+1)}C_2 x^2 + {}^{(n+2)}C_3 x^3 + \dots \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad (1+x)^{-n} &= 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots \\ &\quad + (-1)^r \frac{n(n+1)(n+2) \dots (n+r-1)}{r!} x^r + \dots \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad (1-x)^n &= 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \dots \\ &\quad + (-1)^r \frac{n(n-1)(n-2) \dots (n-r+1)}{r!} x^r + \dots \end{aligned}$$

$$\text{(vii)} \quad (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \infty$$

$$\text{(viii)} \quad (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots \infty$$

$$\text{(ix)} \quad (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots \infty$$

$$\text{(x)} \quad (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \infty$$

$$\text{(xi)} \quad (1+x)^{-3} = 1 - 3x + 6x^2 - \dots \infty$$

$$\text{(xii)} \quad (1-x)^{-3} = 1 + 3x + 6x^2 - \dots \infty$$

$$\text{(xiii)} \quad (1+x)^n = 1 + nx, \text{ if } x^2, x^3, \dots \text{ are all very small as compared to } x.$$

Important Results

(i) Coefficient of x^m in the expansion of $(ax^p + b/x^q)^n$ is the coefficient of T_{r+1} where $r = np - m/p + q$

(ii) The term independent of x in the expansion of $ax^p + b/x^q$ is the coefficient of T_{r+1} where $r = np/p + q$

(iii) If the coefficient of r th, $(r+1)$ th and $(r+2)$ th term of $(1+x)^n$ are in AP, then $n^2 - (4r+1)n + 4r^2 = 2$

(iv) In the expansion of $(x+a)^n$

$$T_{r+1} / T_r = n - r + 1 / r * a / x$$

(v) (a) The coefficient of x^{n-1} in the expansion of

$$(x-1)(x-2) \dots (x-n) = -n(n+1)/2$$

(b) The coefficient of x^{n-1} in the expansion of

$$(x+1)(x+2) \dots (x+n) = n(n+1)/2$$

(vi) If the coefficient of p th and q th terms in the expansion of $(1+x)^n$ are equal, then $p+q = n+2$

(vii) If the coefficients of x^r and x^{r+1} in the expansion of $(a+x/b)^n$ are equal, then

$$n = (r+1)(ab+1) - 1$$

(viii) The number of term in the expansion of $(x_1 + x_2 + \dots + x_r)^n$ is $n+r-1C_{r-1}$.

(ix) If n is a positive integer and $a_1, a_2, \dots, a_m \in \mathbb{C}$, then the coefficient of x^r in the expansion of $(a_1 + a_2x + a_3x^2 + \dots + a_mx^{m-1})^n$ is

$$\sum \frac{n!}{n_1! n_2! \dots n_m!} a_1^{n_1} x a_2^{n_2} \dots a_m^{n_m}$$

(x) For $|x| < 1$,

$$(a) 1 + x + x^2 + x^3 + \dots + \infty = 1 / (1-x)$$

$$(b) 1 + 2x + 3x^2 + \dots + \infty = 1 / (1-x)^2$$

(xi) Total number of terms in the expansion of $(a+b+c+d)^n$ is $(n+1)(n+2)(n+3)/6$.

Important Points to be Remembered

(i) If n is a positive integer, then $(1+x)^n$ contains $(n+1)$ terms i.e., a finite number of terms. When n is general exponent, then the expansion of $(1+x)^n$ contains infinitely many terms.

(ii) When n is a positive integer, the expansion of $(1+x)^n$ is valid for all values of x . If n is general exponent, the expansion of $(1+x)^n$ is valid for the values of x satisfying the condition $|x| < 1$.