Chapter 5

Numerical Methods for Ordinary Differential Equations

INTRODUCTION

Numerical methods are becoming more and more important in engineering applications, simply because of the difficulties encountered in finding exact analytical solutions but also, because of the ease with which numerical techniques can be used in conjunction with modern high-speed digital computers. Several numerical procedures for solving initial value problems involving first-order ordinary differential equations are discussed in this chapter.

Definition: Solutions (particular) to differential equations may require satisfied certain defined conditions and such conditions are called initial conditions if they are given at only one point of the independent variable.

- ✓ The problem of solving an n^{th} order ordinary differential equation together with initial conditions is called an initial value problem (IVP).
- ✓ A first order initial value problem can be written as:

$$y'(x) = \frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad \text{…….. (1)}$$

The single step methods for the solution of the IVP in (1):

1. **Taylor series method**

The Taylor series method is the fundamental numerical method for the solution of the IVP given in (1).

Expanding $y(x)$ in Taylor series about a point x_i,

$$y(x) = y(x_i) + y'(x_i)(x - x_i) + y''(x_i)\frac{(x-x_i)^2}{2!} + \cdots + y^{(n)}(x_i)\frac{(x-x_i)^n}{n!} + error \qquad \text{…….. (2)}$$

We denote the numerical solution and the exact solution at x_i by y_i and $y(x_i)$, respectively.

Now, consider the interval $[x_i, x_{i+1}]$. Substituting the length of the interval $h = x_{i+1} - x_i$ and $x = x_{i+1}$ in (2), we obtain

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \cdots + \frac{h^n}{n!}y^{(n)}(x_i) + error$$

Neglecting the error term, we obtain the Taylor method as:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \cdots + \frac{h^n}{n!}y_i^{(n)} \qquad \text{…….. (3)}$$

is called an explicit single step Taylor series method of order n.

Examples: Solve the following IVPs by using Taylor series method of 1st, 2nd and 3rd order.

a. $yy' = x, y(0) = 1$ in $0 \leq x \leq 0.4$ with $h = 0.2$.

b. $y' = x(y + 1), y(0) = 1$ in $0 \leq x \leq 0.4$ with $h = 0.1$.

c. $y' = x - y^2, y(0) = 1$ at $x = 0.1$ and 0.2.

d. $\frac{dy}{dx} = 3x+y^2$, $x \in [0, 2]$ with $h = 0.5$ and $y = 1$ when $x = 0$.

Exercise:

i. Use the 2nd-order Taylor series method on $(2, 3)$ for the IVP, $\frac{dy}{dx} = -xy^2, y(2) = 1$ Take $h = 0.1$.

ii. Use the fourth order Taylor series method with a single integration step to determine $y(0.2)$ given that $\frac{dy}{dx} + 4y = x^2 \quad y(0) = 1$.

2. **Euler's Method**

Euler's explicit method (also called the *forward Euler method*) is a single-step, explicit method for solving a first-order ordinary differential equation.

Let the differential equation is given by:

$$y' = \frac{dy}{dx} = f(x, y), \text{ with } y(x_0) = y_0 \quad (4)$$

To solve the equation given in (4) for the values of y at $x = x_i = x_0 + ih, i = 1,2, \ldots . n$ integrate equation (4) with one step size. $y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx$ (5)

Take $f(x, y) = f(x_0, y_0)$ in the range x_0 to x_1, then we get

$$y_1 = y_0 + (x_1 - x_0)f(x_0, y_0) = y_0 + hf(x_0, y_0)$$

Similarly, for the range x_1 to $x_2, f(x, y) = f(x_1, y_1)$, then

$$y_2 = y_1 + (x_2 - x_1)f(x_1, y_1) = y_1 + hf(x_1, y_1)$$

In general form we have

$y_{n+1} = y_n + hf(x_n, y_n)$, where $n = 0,1,2, \ldots$

This is called **Euler's** method.

Example: Using Euler's method solve for $x = 0.1$ of $\frac{dy}{dx} = \frac{x-y}{x+y}$ given that

$x_0 = 0, y_0 = 1$ and $h = 0.02$.

Exercise: Use Euler's method to solve the following differential equation of

i. $\frac{dy}{dx} = -xy^2, y(2) = 1$ and $2 < x < 3$ with step size $h = 0.1$.

ii. $\frac{dy}{dx} = \frac{1}{2}y, y(0) = 1$ and $0 < x < 1$ with step size $h = 0.1$.

3. **Modified Euler's method**

In case of Euler's method we approximate $f(x, y)$ by $f(x_0, y_0)$ in the range x_0 to x_1, but in this case we approximate $f(x, y)$ by means of Trapezoidal rule i.e.

$$f(x, y) = \frac{h}{2}(f(x_0, y_0) + f(x_1, {y_1}^{(0)}))$$

Then, $y_1 = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, {y_1}^{(0)}))$

Then, we get the iteration formula as:

$${y_1}^{(n+1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, {y_1}^{(n)}\right)\right), \ n = 0,1,2, \ldots$$

and ${y_1}^{(n)}$ is the n^{th} approximation of y_1.

The formula can be starting from ${y_1}^{(0)}$ by Euler's formula.

$$y_1^{(0)} = y_0 + hf(x_0, y_0).$$

$\therefore \ y_{k+1}^{(n+1)} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(n)})\right]$ is the $(n+1)^{th}$ a approximation of y_{k+1}.

Example: Using Modified Euler's method solves the following IVPs correct to 2 decimal places.

i. $y' = x - y^2, y(0) = 1$ and $h = 0.1$ at $x = 0.1$.

ii. $\frac{dy}{dx} = \frac{x-y}{x+y}$ given that $x_0 = 0, \ y_0 = 1$ and $h = 0.02$ at $x = 0.06$.

iii. $y' = x + y$ with $y(0) = 1, 0 < x < 0.06$ and $h = 0.02$.

iv. $\frac{dy}{dx} = \frac{1}{2}y, y(0) = 1$ and $0 < x < 1$ with step size $h = 0.2$.

4. **Runge-Kutta Methods**

Euler's method is less efficient in practical problems since it requires h to be small for obtaining reasonable accuracy. The Runge-Kutta methods are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points in the subinterval.

i. **First-order Runge-Kutta method**

The Euler's method is called First-order Runge-Kutta method

$$y_{n+1} = y_n + hf(x_n, y_n)$$

ii. **Second- order Runge-Kutta method**

The Euler's modified method is called Second-order Runge-Kutta method

$$y_{n+1} = y_n + \frac{1}{2}(k_0 + k_1)$$

where, $k_0 = hf(x_n, y_n)$ and $k_1 = hf(x_n + h, y_n + k)$

iii. **Third- order Runge-Kutta method**

$$y_{n+1} = y_n + \frac{1}{6}(k_0 + 4k_1 + k_2)$$

where, $k_0 = hf(x_n, y_n)$, $k_1 = hf(x_n + \frac{h}{2}, y_n + \frac{k_0}{2})$ and $k_2 = hf(x_n + h, y_n + k^*)$ such that $k^* = hf(x_n + h, y_n + k_0)$

iv. **Fourth- order Runge-Kutta method**: is the most commonly used.

$$y_{n+1} = y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

where, $k_0 = hf(x_n, y_n)$, $k_1 = hf(x_n + \frac{h}{2}, y_n + \frac{k_0}{2})$, $k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$ and $k_3 = hf(x_n + h, y_n + k_2)$

Example:

i. Use the second order Runge-Kutta method with $h = 0.1$, find y_1 and y_2 for $\frac{dy}{dx} = -xy^2, y(2) = 1.$

ii. Use Runge-Kutta method of order two to integrate $\frac{dy}{dx} = \sin(y)$ with $y(0) = 1$ from $x = 0$ to 0.5 in steps of $h = 0.1$. Keep four decimal places in the calculations.

iii. Use the Runge-Kutta method of order four with $h = 0.1$ to obtain an approximation to $y(1.5)$ for the solution of $\frac{dy}{dx} = 2xy$, $y(1) = 1$. The exact solution is given by $y = e^{x^2-1}$. Determine the relative error and the percentage relative error.

Solution:

Exercise: solve $y' = x - y^2$, with $h = 0.1, 0.05$ correct to four decimal places and find $y(0.1)$ and $y(0.2)$ using Runge-Kutta method of

a. First order
b. Second order
c. Third order
d. Fourth order