

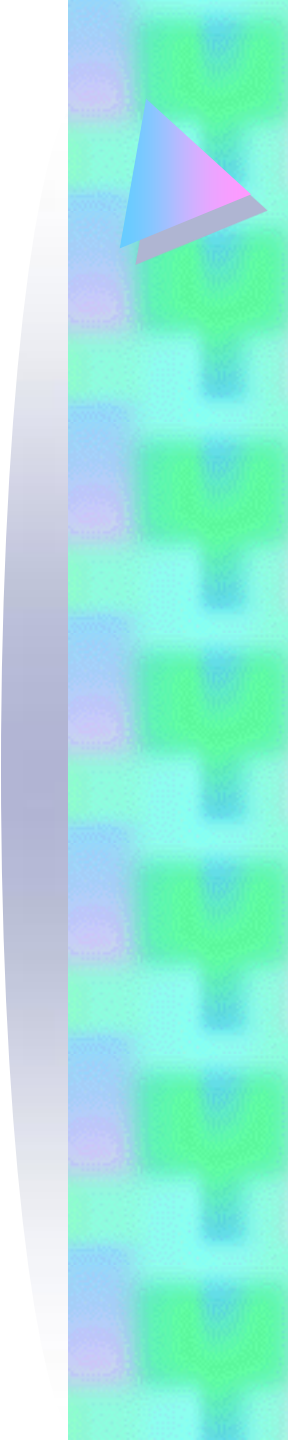


CHAPTER V INTEGER PROGRAMMING: TRANSPORTATION ALGORITHM

Prepared by:

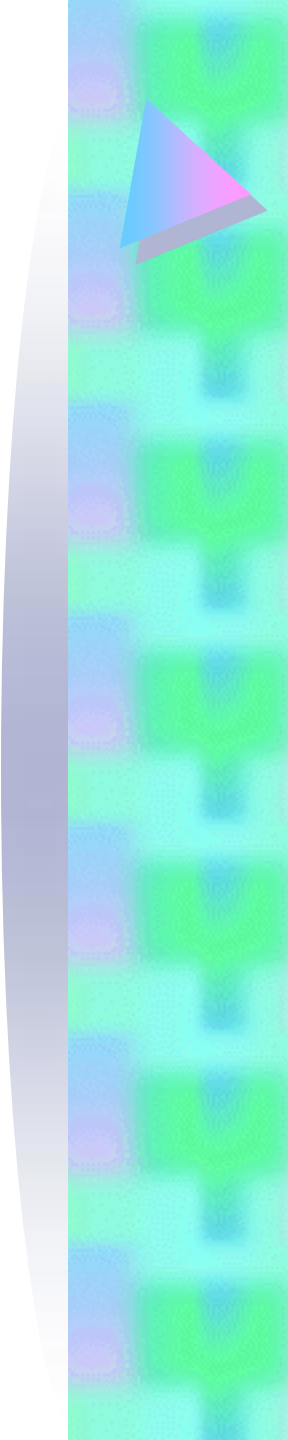
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Transportation Algorithm

- A transportation algorithm involves m sources, each of which requires a_i ($i=1,2,\dots,m$) units of a homogeneous product, and n destinations, each of which requires b_j ($j=1,2,\dots,n$) units of this product.
- The cost c_{ij} of transporting one unit of product from the i th source to the j th destination is given for each i and j .
- The objective is to develop an integral transportation schedule that meets all demands from current inventory at a minimum total shipping cost.



Standard Mathematical Model

- It is assumed that total supply and total demand are equal; that is:

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

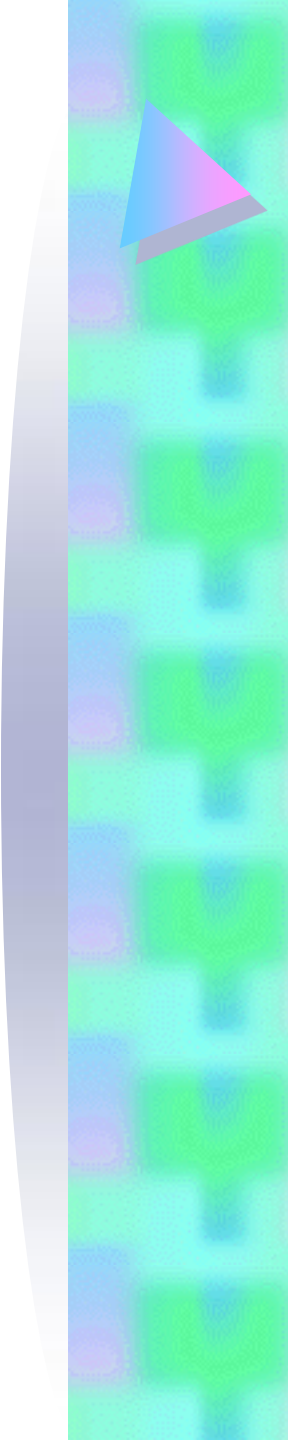
- Let x_{ij} represent the (unknown) number of units to be shipped from source i to destination j . The standard mathematical model for this problem is:

$$\text{minimize : } z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\text{subject to : } \sum_{j=1}^n x_{ij} = a_i \quad (i = 1, 2, \dots, m)$$

$$\sum_{i=1}^m x_{ij} = b_j \quad (j = 1, 2, \dots, n)$$

with : all x_{ij} nonnegative and integral.

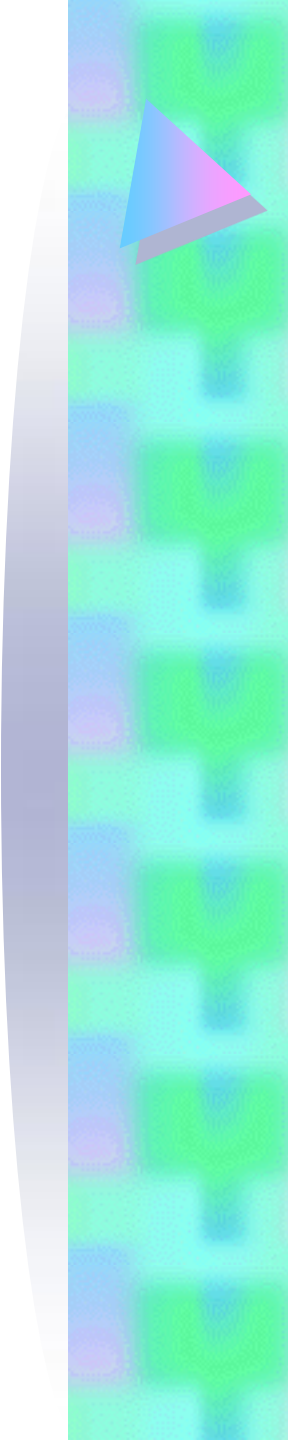


The Transportation Algorithm

- **The transportation algorithm is the simplex method specialized to the format of the Transportation Tableau, it involves:**
 - 1. Finding an initial, basic feasible solution**
 - 2. Testing the solution for optimality**
 - 3. Improving the solution when it is not optimal; and**
 - 4. Repeating steps 2 and 3 until the optimal solution is obtained.**

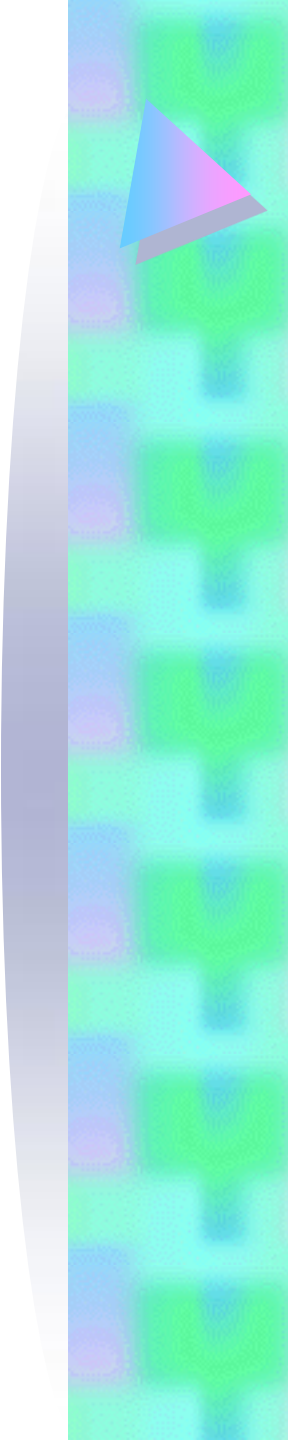
The Transportation Tableau

		Destination							
		1	2	3	...	n	Supply	u _i	
Sources	1	c ₁₁ x ₁₁	c ₁₂ x ₁₂	c ₁₃ x ₁₃	...	c _{1n} x _{1n}	D ₁	u ₁	
	2	c ₂₁ x ₂₁	c ₂₂ x ₂₂	c ₂₃ x ₂₃	...	c _{2n} x _{2n}	D ₂	u ₂	
	3	c ₃₁ x ₃₁	c ₃₂ x ₃₂	c ₃₃ x ₃₃	...	c _{3n} x _{3n}	D ₃	u ₃	
	...	...	...	...	...	...	...	...	
	m	c _{m1} x _{m1}	c _{m2} x _{m2}	c _{m3} x _{m3}	...	c _{mn} x _{mn}	D _m	u _m	
	Demand	b ₁	b ₂	b ₃	...	B _n			
	v _j	v ₁	v ₂	v ₃	...	V _n			



Example No. 1

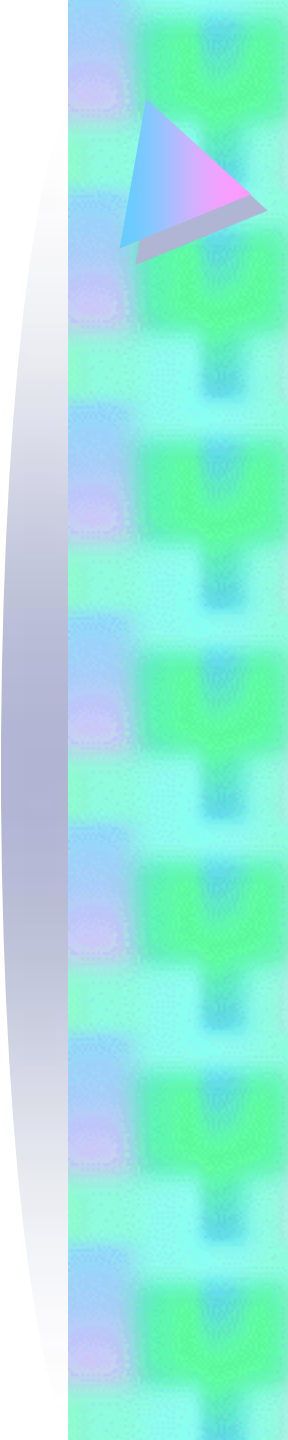
- A car rental company is faced with an allocation problem resulting from rental agreements that allow cars to be returned to locations other than those at which they were originally rented. At the present time, there are two locations (sources) with 15 and 13 surplus cars, respectively, and four locations (destinations) requiring 9, 6, 7, and 9 cars, respectively.



**Unit transportation costs (in dollars)
between the locations are as follows:**

	Dest. 1	Dest. 2	Dest. 3	Dest. 4
Source 1	45	17	21	30
Source 2	14	18	19	31

**Set-up the initial transportation
tableau for the minimum-cost schedule.**



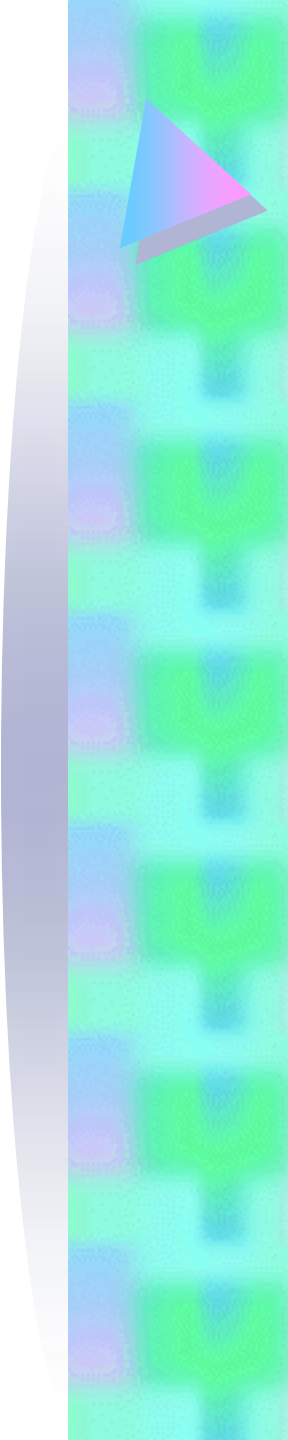
Solution

- Since the total demand ($9 + 6 + 7 + 9 = 31$) exceeds the total supply ($15 + 13 = 28$), a dummy source is created having the supply equal to the 3-unit shortage.
- In reality, shipments from this fictitious source are never made, so the associated shipping costs are taken as zero.
- Positive allocations from this source to a destination represents cars that cannot be delivered due to a shortage of supply, they are shortages a destination will experience under an optimal shipping schedule.

Transportation Tableau

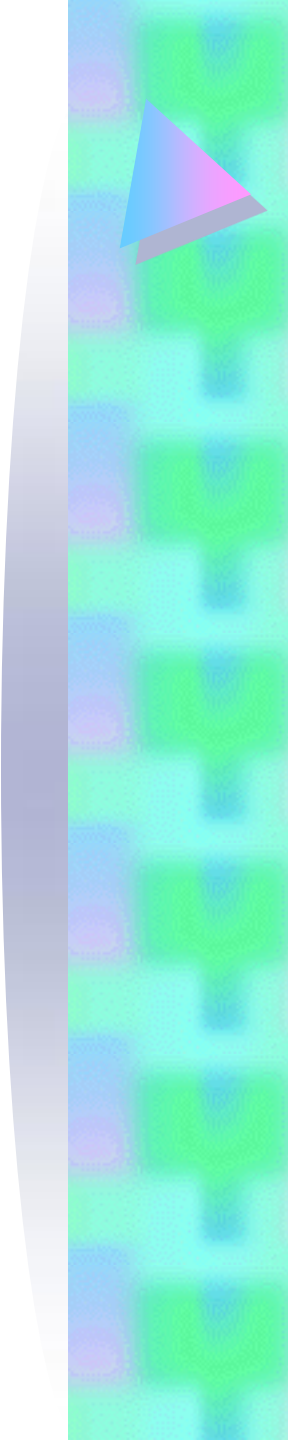
- The x_{ij} , u_i , v_j are not yet entered since they are unknown for the moment.

		Destination				Supply	u_i
		1	2	3	4		
Sources	1	45	17	21	30	15	
	2	14	18	19	31	13	
	Dummy 3	0	0	0	0	3	
	Demand	9	6	7	9	Tableau 1A	
	v_j						



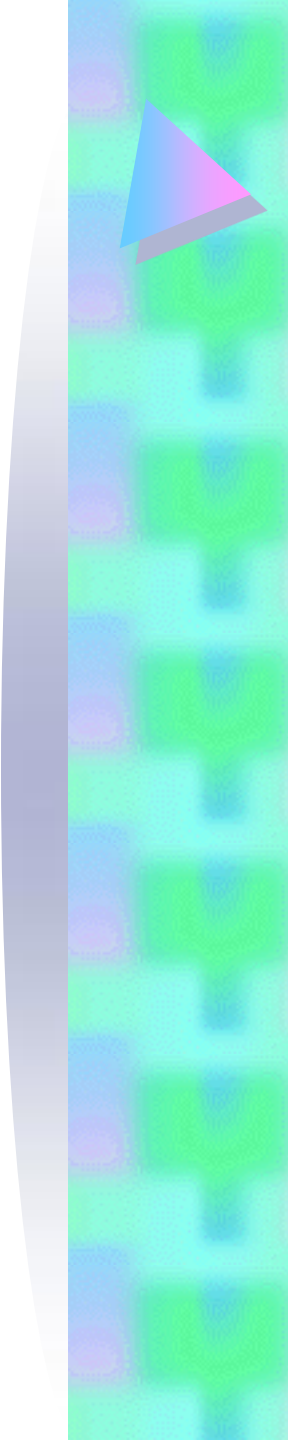
Northwest Corner Rule

- Beginning with cell (1,1) in the Transportation Tableau, allocate to x_{11} , as many units as possible without violating the constraints. This will be the smaller of a_1 and b_1 . Thereafter, continue by moving one cell to the right, if some supply remains, or if not, one cell down.
- At each step, allocate as much as possible to the cell (variable) under considerations without violating the constraints. The sum of the i th row allocations cannot exceed a_i , the sum of the j th column allocations cannot exceed b_j , and no allocation can be negative. The allocation may be zero.



1. An Initial Basic Solution

- Variables that are assigned values by either one of these starting procedures (becomes the basic variables in the initial solution. The unassigned variables are non-basic and, therefore, zero.
- The northwest corner rule is simpler than other rules to apply. However, another method, the Vogel's Method (which will be discussed later), which takes into account the unit shipping costs, usually results in a closer-to-optimal starting solution.



Example No. 2

- Use the Northwest Corner Rule to obtain an initial allocation to Example No. 1
- **Solution**
 1. Begin with x_{11} , and assign it the minimum of $a_1 = 15$ and $b_1 = 9$. Thus $x_{11} = 9$, leaving 6 surplus cars at the first source.
 2. Move one cell to the right and assign $x_{12} = 6$. These allocations together exhaust the supply at the first source, move one cell down.

		Destination					
		1	2	3	4	Supply	ui
1	45	9	17	21	30	15	
2	14	0	18	19	31	13	
Dummy 3	0		0		0	3	
Demand	9	6	7	9			
vj							

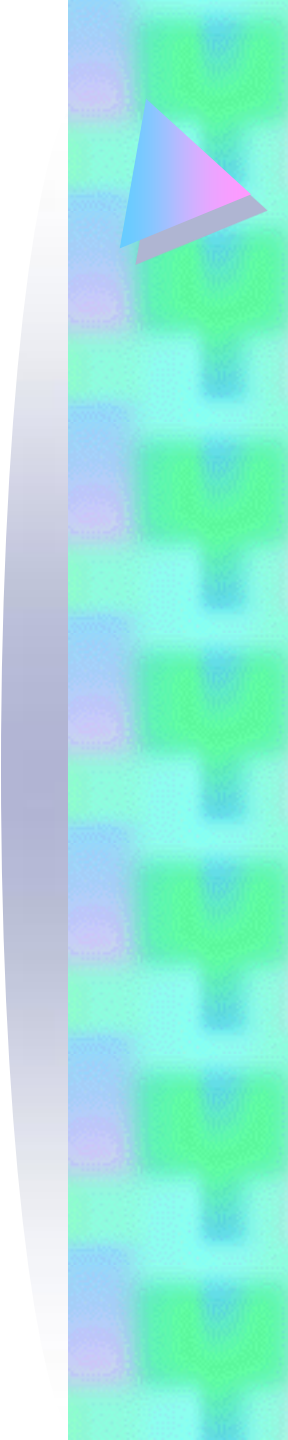
- Consider x_{22} , observe however, that the demand at the second destination has been satisfied by the x_{12} allocation.
- Since we cannot deliver additional cars to it without exceeding its demand, we must assign $x_{22} = 0$ and move one cell to the right.

Degenerate Solution

- Continuing in this manner, we obtain the degenerate solution (fewer than $4 + 3 - 1 = 6$ positive entries) below:

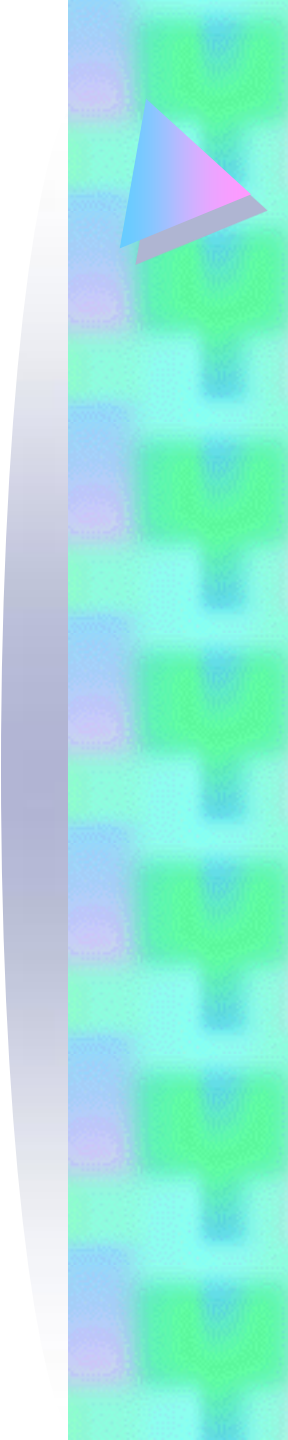
		Destination				Supply	ui
		1	2	3	4		
Sources	1	45 9	17 6	21	30	15	
	2	14	18 0	19 7	31 6	13	
	Dummy 3	0	0	0	0 3	3	
	Demand	9	6	7	9		
vj							

Tableau 1B



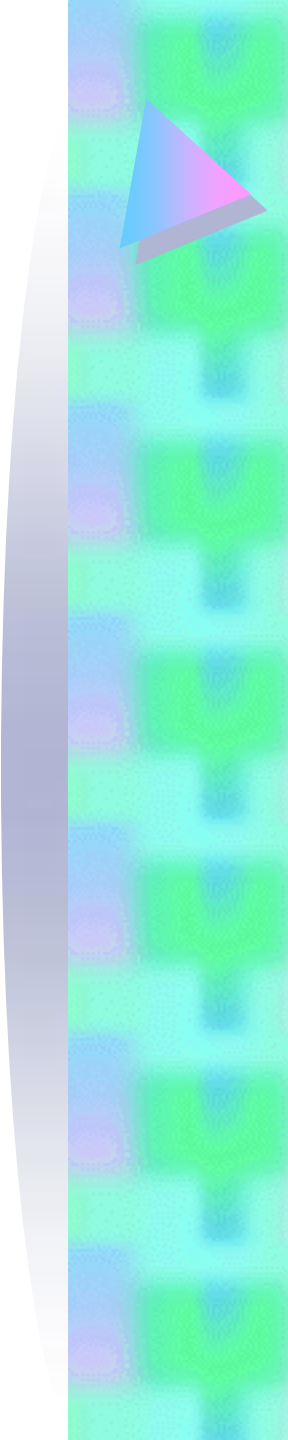
DEGENERACY

- The northwest corner rule always generates an initial basic solution, but it may fail to provide values $n + m - 1$ positive values, thus yielding a degenerate solution.
- Improving a degenerate solution may result in replacing one basic variable having a zero value by another such. Although the two degenerate solutions are effectively the same-only the designation of the basic variables has changed, not their values.



2. Test for Optimality

- Assign one (any one) of the u_i or v_j in the tableau the value zero and calculate the remaining u_i and v_j so that for each basic variable $u_i + v_j = c_{ij}$
- Then, for each nonbasic variable, calculate the quantity $c_{ij} - u_i - v_j$. If all these latter quantities are nonnegative, the current solution is optimal, otherwise, the current solution is not optimal and needs to be improved.

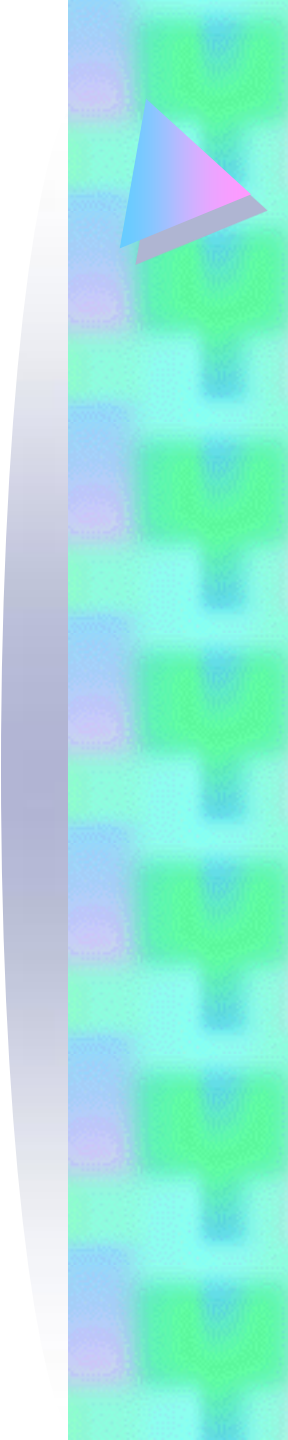


Example No. 3

- Solve the transportation problem in Example No. 1.

- Solution:

To determine whether the initial allocation found in Tableau 1B is optimal, first calculate the terms u_i and v_j with respect to the basic-variable cells of the tableau. Arbitrarily choosing $u_2 = 0$ (since the second row contains more basic variables than any other row or column.

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- This choice will simplify the computations giving:

$$(2, 2) \text{ cell: } u_2 + v_2 = c_{22} \quad 0 + v_2 = 18 \text{ or } v_2 = 18$$

$$(2, 3) \text{ cell: } u_2 + v_3 = c_{23} \quad 0 + v_3 = 19 \text{ or } v_3 = 19$$

$$(2, 4) \text{ cell: } u_2 + v_4 = c_{24} \quad 0 + v_4 = 31 \text{ or } v_4 = 31$$

$$(1, 2) \text{ cell: } u_1 + v_2 = c_{12} \quad u_1 + 18 = 17 \text{ or } u_1 = -1$$

$$(1, 1) \text{ cell: } u_1 + v_1 = c_{11} \quad -1 + v_1 = 45 \text{ or } v_1 = 46$$

$$(3, 4) \text{ cell: } u_3 + v_4 = c_{34} \quad u_3 + 31 = 0 \text{ or } u_3 = -31$$

- Place these values in Tableau 1C and calculate the quantities $c_{ij} - u_i - v_j$ for each nonbasic-variable cells in Tableau 1B

Tableau 1C

Sources	Destination				Supply	ui
	1	2	3	4		
	45	17	21	30		
	9	6				
	15					
	-1					
2	14	18	19	31	13	0
		0	7	6		
Dummy 3	0	0	0	0	3	-31
				3		
Demand	9	6	7	9		
vj	46	18	19	31		

(1, 3) cell: $c_{13} - u_1 - v_3 = 21 - (-1) - 19 = 3$

(1, 4) cell: $c_{14} - u_1 - v_4 = 30 - (-1) - 31 = 0$

(2, 1) cell: $c_{21} - u_2 - v_1 = 14 - 0 - 46 = -32$

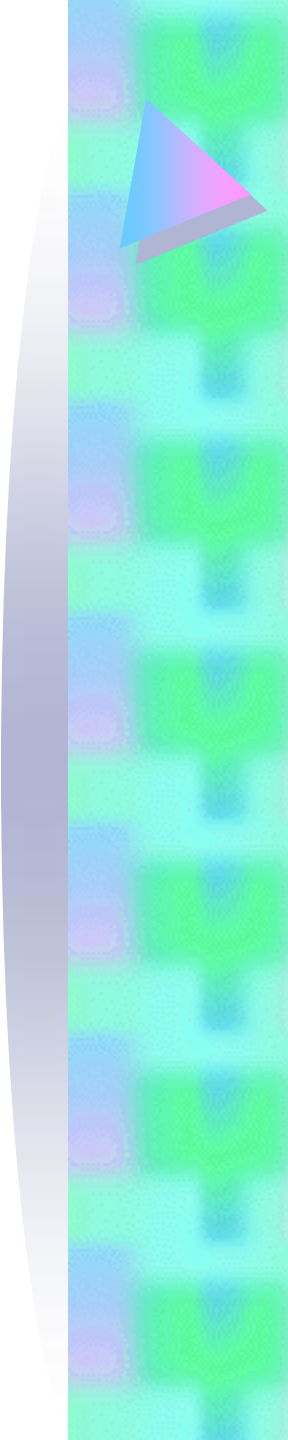
(3, 1) cell: $c_{31} - u_3 - v_1 = 0 - (-31) - 46 = -15$

(3, 2) cell: $c_{32} - u_3 - v_2 = 0 - (-31) - 18 = 13$

(3, 3) cell: $c_{33} - u_3 - v_3 = 0 - (-31) - 19 = 12$

- The results are recorded in Tableau 1C below enclosed in parentheses:

		Destination					
		1	2	3	4	Supply	ui
1	45	17	21	30	15	-1	
	9	6	(3)	(0)			
2	14	18	19	31	13	0	
	(-32)	0	7	6			
Dummy 3	0	0	0	0	3	-31	
	(-15)	(13)	(12)	3			
Demand		9	6	7	9		
vj		46	18	19	31		

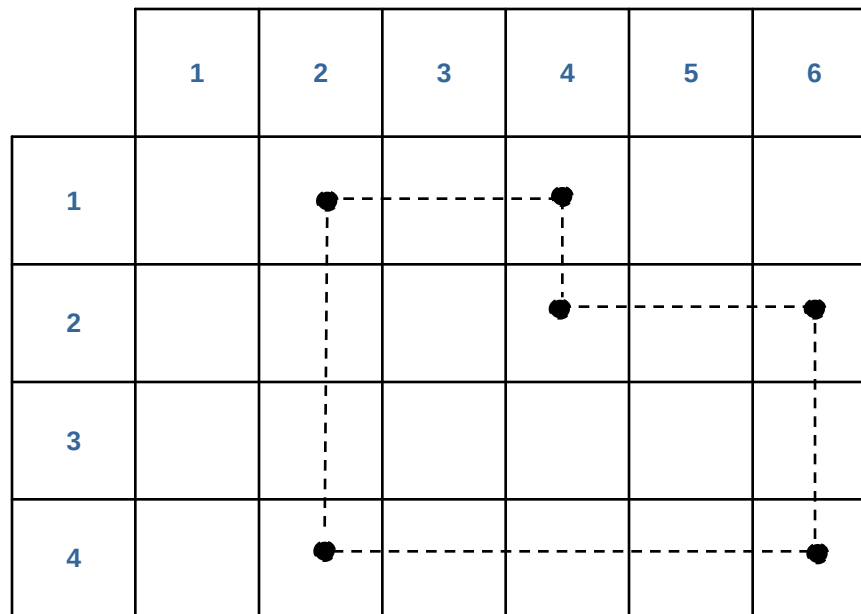


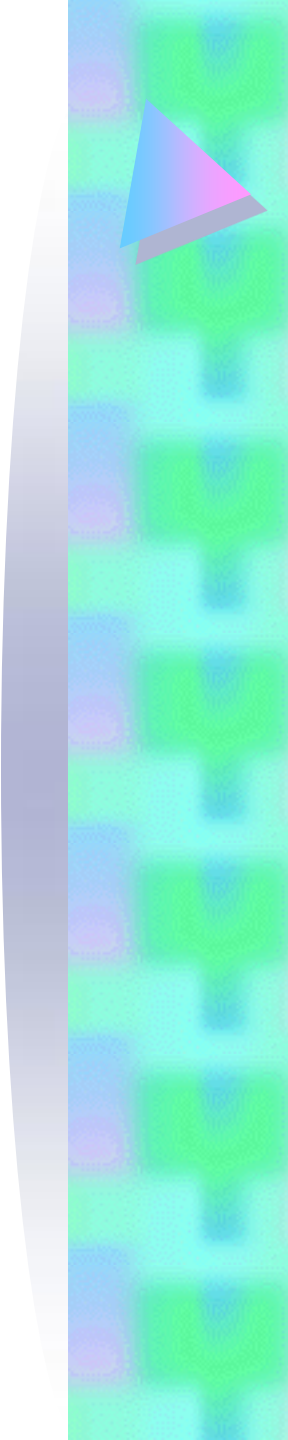
3. Improving the Solution by Looping

- Since at least one of these $(c_{ij} - u_i - v_j)$ -values is negative, the current solution is not optimal.
- Loop – a sequence of cells in the Transportation Tableau such that:
 - i. Each pair of consecutive cells lie in either the same row or the same column
 - ii. No three consecutive cells lie in the same row or column
 - iii. The first and last cells of the sequence lie in the same row or column
 - iv. No cell appears more than once in the sequence.

Example No. 4

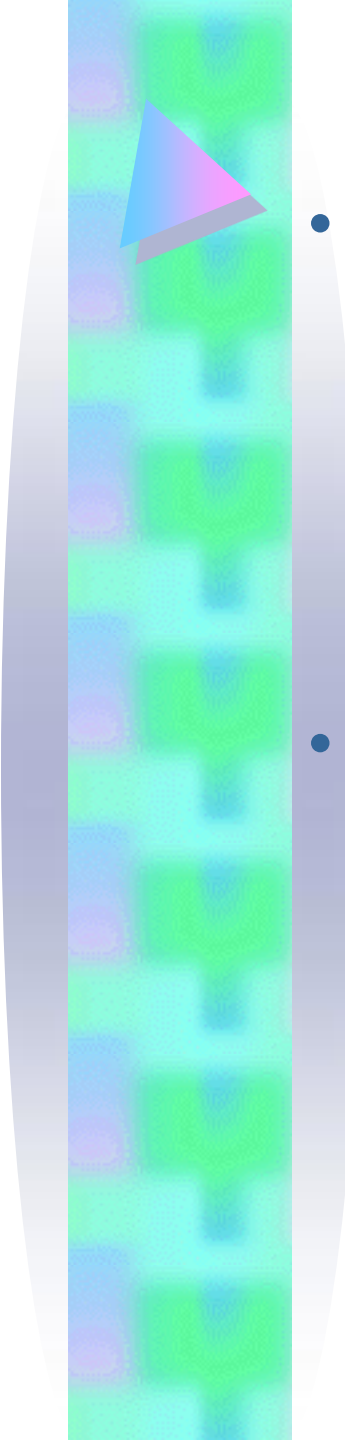
- The sequences $\{(1,2), (1,4), (2,4), (2,6), (4,6), (4,2)\}$ illustrated below is a loop.





Improving The Tableau

- A better solution can be obtained by increasing the allocation to the variable (cell) having the largest negative entry, here (2,1) cell of Tableau 1C.
- Do it by placing a boldface plus sign (signaling an increase) in the (2,1) cell and identify a loop containing, besides this cell, only basic-variable cells.

- 
- Increase the allocation to (2,1) cell as much as possible, simultaneously adjusting the other cell allocations in the loop so as not to violate the supply, demand or nonnegativity constraints.
 - Any positive allocation to the (2, 1) cell would force x_{22} to become negative. To avoid this, but still make x_{21} basic, assign $x_{21} = 0$ and remove x_{22} from the set of basic variables.

Looping

		Destination				Supply	ui
		1	2	3	4		
1		45	17	21	30	15	-1
			9	6	(3)		
2		14	18	19	31	13	0
		(-32)	+	0	7		
Dummy 3		0	0	0	0	3	-31
		(-15)	(13)	(12)	3		
Demand		9	6	7	9		
vj		46	18	19	31		

Sources

Tableau 1D

Destination

	1	2	3	4	Supply	ui
1	45 9	17 6	21	30	15	
2	14 0	18	19 7	31 6	13	
Dummy 3	0	0	0	0 3	3	
Demand	9	6	7	9		
vj						

Sources

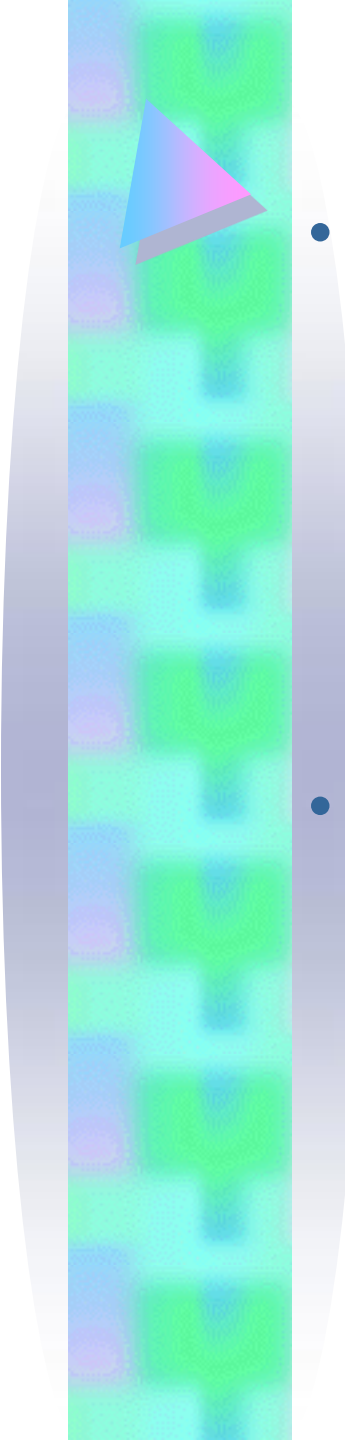
- 
- Now, check whether Tableau 1D is optimal. Calculate the new u_i and v_j with respect to the new basic variables, and then compute $c_{ij} - u_i - v_j$. Again we arbitrarily choose $u_2 = 0$.
 - The results is shown in Tableau 1E. Since two entries are negative, the current solution is not yet optimal, and a better solution can be obtained by increasing the allocation to the (1,4) cell.

Tableau 1E

		Destination				Supply	ui
		1	2	3	4		
1	45	17	21	30	15	31	
	9	6	(-29)	(-32)			
2	14	18	19	31	13	0	
	0	(32)	7	6			
Dummy 3	0	0	0	0	3	-31	
	(17)	(45)	(12)	3			
Demand		9	6	7	9		
vj		14	-14	19	31		

Sources

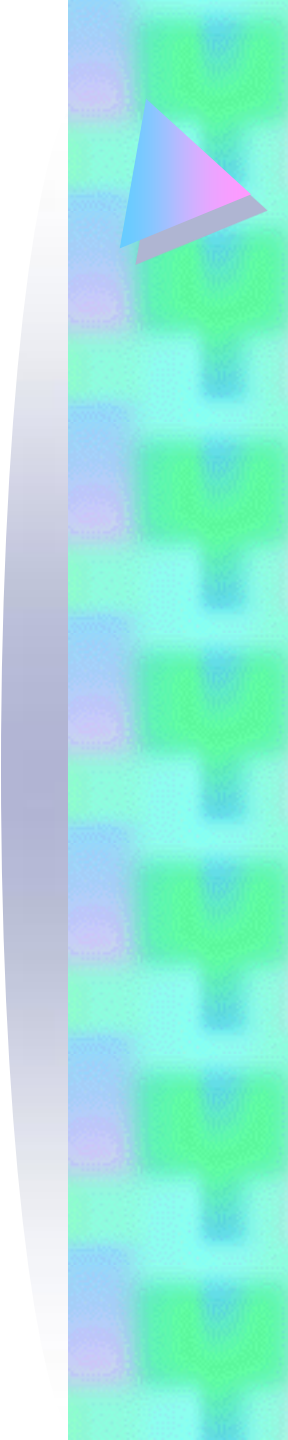
- 
- The loop whereby this is accomplished is indicated in Tableau 1E. Any amount added to cell (1,4) must be simultaneously subtracted from cells (1,1) and (2,4) and then added to cell (2,1), so as not to violate the supply-demand constraints.
 - The new nondegenerate solution is shown in Tableau 1F

Tableau 1F

Sources	Destination				Supply	ui
	1	2	3	4		
	1	2	3	4		
	1	2	3	4		
	1	2	3	4		
1	45	17	21	30	15	
2	14	18	19	31	13	
Dummy 3	0	0	0	0	3	
Demand	9	6	7	9		
vj						

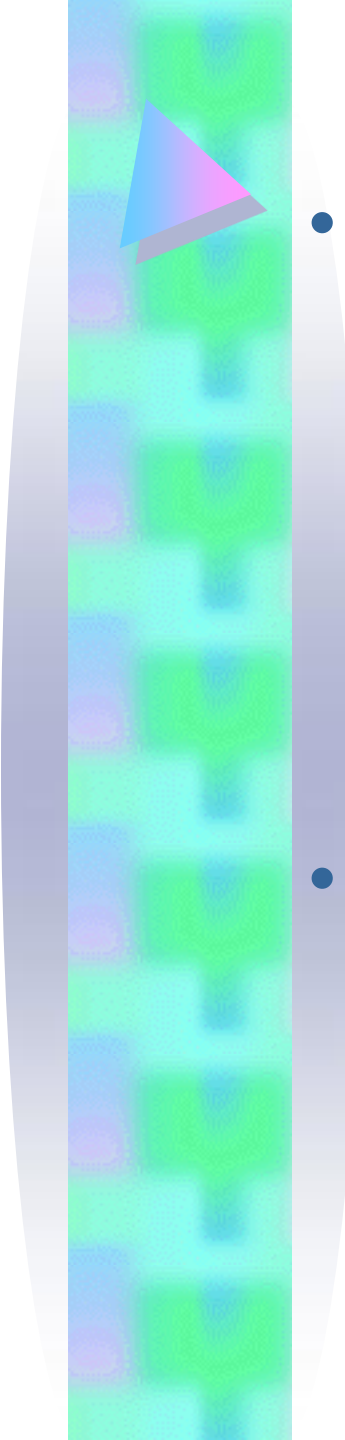
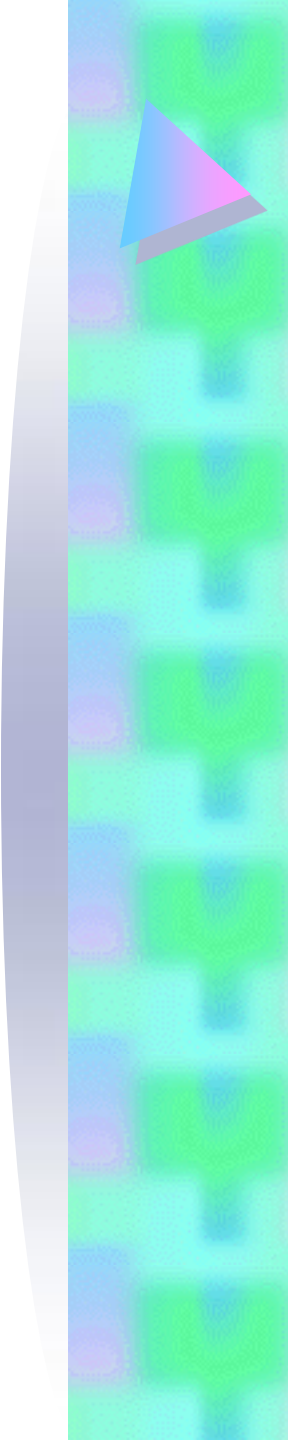
- 
- After one further optimality test and consequent change of basis, we obtain Tableau 1H, which also shows the results of the optimality test of the new basic solution.
 - It is seen that each $c_{ij} - u_i - v_j$ is nonnegative; hence the new solution is optimal.

Tableau 1H

	Destination				Supply	ui
	1	2	3	4		
1	45	17	21	30	15	0
	(29)	6	3	6		
2	14	18	19	31	13	-2
	9	(3)	4	(3)		
Dummy 3	0	0	0	0	3	-30
	(14)	(13)	(9)	3		
Demand	9	6	7	9		
vj	16	17	21	30		

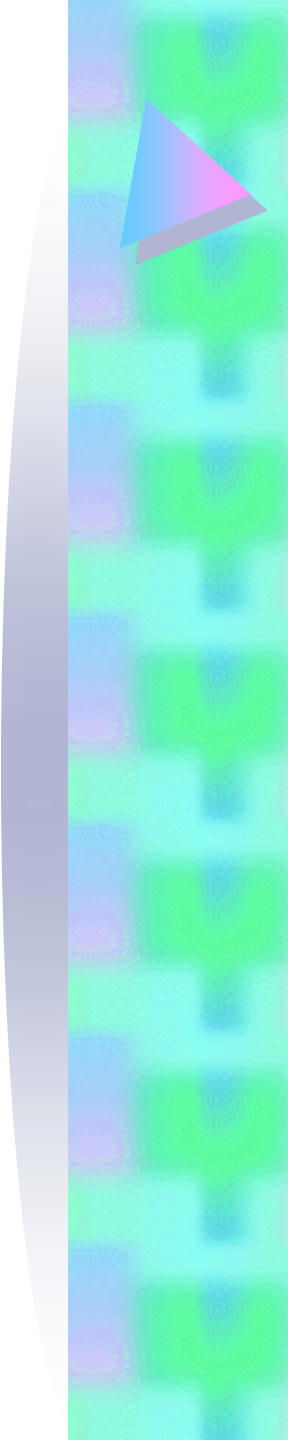
That is, $x_{12}^* = 6$, $x_{13}^* = 3$, $x_{14}^* = 6$, $x_{21}^* = 9$, $x_{23}^* = 4$, $x_{34}^* = 3$, with all other variables nonbasic and, therefore, zero. Furthermore:

$$z^* = 6(17) + 3(21) + 6(30) + 9(14) + 4(19) + 3(0) = \$547$$



Vogel's Method

- For each row and each column having some supply or some demand remaining, calculate its difference, which is the nonnegative difference between the two smallest shipping costs c_{ij} associated with unassigned variables in that row or column.
- Consider the row having the largest difference; in case of a tie, arbitrarily choose one. In this row or column, locate that unassigned variable (cell) having the smallest unit shipping cost and allocate to it as many units as possible without violating constraints. Recalculate the new differences and repeat the above procedure until all demands are satisfied.



Example No. 5

- **Use Vogel's Method to determine an initial basic solution to the transportation problem in Example No. 1**
- **Solution:**

The two smallest costs in row 1 of Transportation Tableau are 17 and 21; their difference is 4. The two smallest costs in row 2 are 14 and 18; their difference is also 4. The two smallest costs in row 3 both 0; so their difference is 0. repeating this analysis on the columns, we generate the differences shown in Tableau 2.

Tableau 2

						Difference	
		1	2	3	4	Supply	ui
Sources	1	45	17	21	30	15	4
	2	14	18	19	31	13	4
	Dummy 3	0	0	0	0	3	0
					3		
	Demand	9	6	7	9		
						Difference	
		14	17	19	30		

Since, the largest of these difference is 30, occurs in column 4, we locate the variable in this column having the unit shipping cost and allocate to it as many units as possible. Thus, $x_{34} = 3$, exhausting the supply of source 3 and eliminating row 3 from further consideration.

Tableau 2B

		1	2	3	4	Supply	ui	Difference	
Sources	1	45	17	21	30	15		4	4
	2	14	18	19	31	13		4	4
	Dummy 3	0	0	0	0	3	-	0	X
					3				
	Demand	9	6	7	9				
	vj								
Difference		14	17	19	30				
		31	1	2	1				

The largest difference is in column 1, and the variable in this column having the smallest cost is x_{21} . We assign $x_{21} = 9$, thereby meeting the demand of destination 2 and removing column 2 from further calculations.

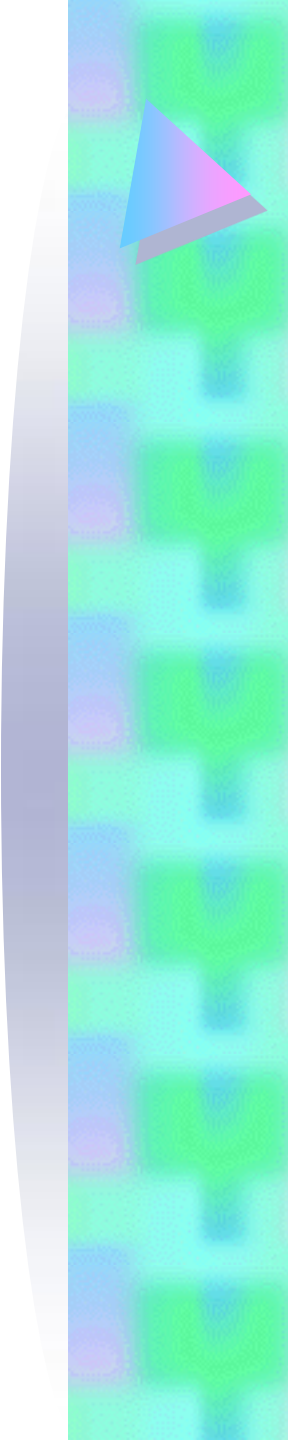
Final Optimal Solution

Difference
↓

	1	2	3	4	Supply	ui
1	45	17	21	30	15	
2	14	18	19	31	13	
Dummy 3	0	0	0	0	3	-
Demand	9	6	7	9		
vj						
Difference	14	17	19	30		
	31	1	2	1		
	X	1	2	1		
	X	X	2	1		

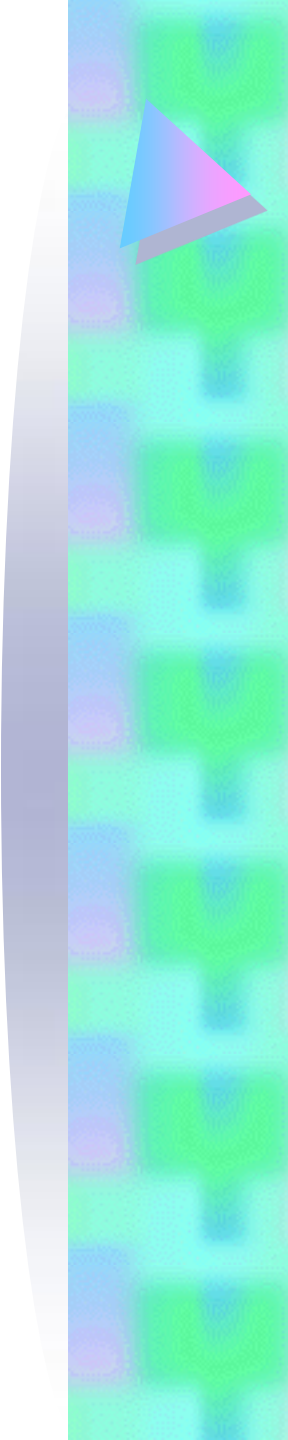
4	4	4	9
4	4	1	12
0	X	X	X

The result is the allocation shown in Tableau 1H



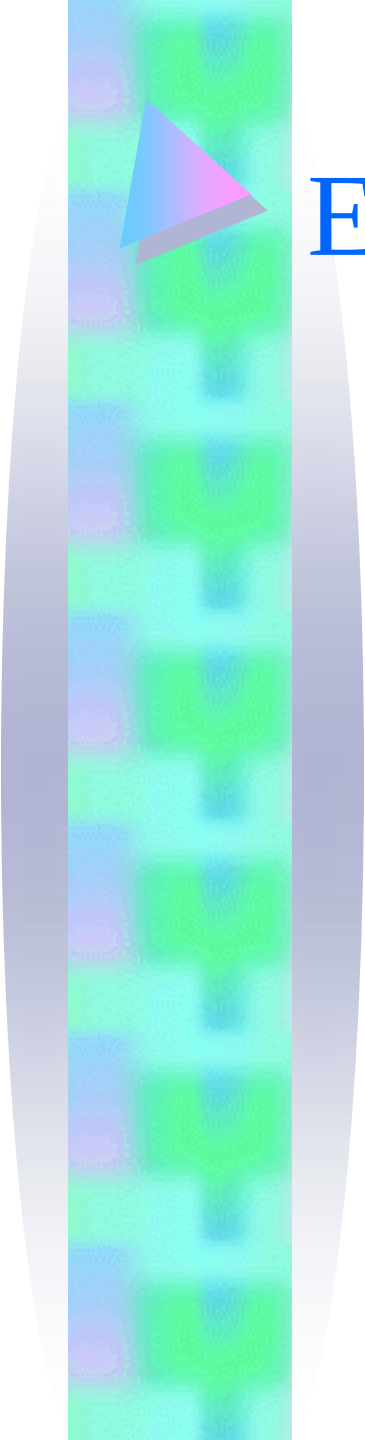
Assignment No. 4

- **A plastics manufacturer has 1200 boxes of transparent wrap in stock at one factory and another 1000 boxes at its second factory. The manufacturer has orders for this product from three different retailers, in quantities of 1000, 700, and 500 boxes, respectively.**

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- The unit shipping costs (in cents per box) from the factories to the retailers are as follows:

	Retailer 1	Retailer 2	Retailer 3
Factory 1	14	13	11
Factory 2	13	13	12

- Determine a minimum-cost shipping schedule for satisfying all demands from current inventory using BOTH the TRANSPORTATION ALGORITHM and VOGEL'S METHOD.



End of Chapter 5