

Chapter 3

Counting methods

3.1. Sum and product rules

A. addition principle (AP)

Assume that there are n_1 different ways for the event E_1 to occur

n_2 different ways for the event E_2 to occur

n_3 different ways for the event E_3 to occur

n_4 different ways for the event E_4 to occur

n_5 different ways for the event E_5 to occur

n_k different ways for the event E_k to occur

If the ways in which these events occur are pair-wise disjoint, then the number of ways for at least one of the events $E_1, E_2, \dots$ or E_k to occur is given by $n_1 + n_2 + \dots + n_k = \sum_{i=1}^k n_i$

An equivalent form of AP,

let $A_1, A_2, \dots, A_k$ be any finite sets, if the given sets are pair-wise disjoint, then

$|\bigcup_{i=1}^k A_i| = |A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k|$, where $| \quad |$ stands for number of elements in A_i (or cardinality of A_i).

E.g.1. if there are 14 boys and 12 girls in a class, then find the number of ways of selecting one student as class representative.

Solution:- using sum rule, there are $14+12=26$ ways of selecting one student (either a boy or girl) as class representative.

E.g.2 A student can choose a computer project from one of three lists. The three lists contain 23, 15, 19 possible projects, respectively.

How many possible projects are there to choose from?

Solution:- the student can choose a project from 1st list in 23 ways

2nd list in 15 ways

3rd list in 19 ways

E.g.3. one can reach city B from city A by bus, train, air or by foot. Suppose that there are 2 ways by bus, 3 ways by train, 4 ways by air and 5 ways by foot. What is the total no of going from city A to city B?

Solution:- by (AP) there are a total of $2+3+4+5 = 14$ ways to go from city A to city B by bus or by train or by air or by road.

E.g. 4. let F is the event of choosing prime nos less than 10 and E is the event of choosing even no less than 10. In how many ways can F or E occur?

Solution:- $E = \{2, 3, 5, 7\}$, $F = \{2, 4, 6, 8, \dots\}$. $(E \text{ or } F) = 4+4-1 = 7$ ways

B. Multiplication principle (MP)

Assume that an event E can be decomposed in to reordered event $E_1, E_2, \dots, E_r$ and that there are: n_1 different ways for the event E_1 to occur

n_2 different ways for the event E_2 to occur

n_3 different ways for the event E_3 to occur

.

n_r different ways for the event E_r to occur

Then the total number of ways for the event E to occur is given by

$$n_1 \times n_2 \times \dots \times n_r = \prod_{i=1}^r n_i$$

An equivalent form of (MP), Let $A_1, A_2, \dots, A_k$ be any finite sets and

Let $\prod_{i=1}^r A_i = A_1 \times A_2 \times \dots \times A_r = \{(a_1, a_2, a_3, \dots, a_r) : a_i \in A_i, i = 1, 2, \dots, r\}$

Then $|\prod_{i=1}^r A_i| = |A_1| \times |A_2| \times \dots \times |A_r| = \prod_{i=1}^r |A_i|$

E.g.1. In how many ways can a college having 20 staff members elect dean, vice dean and program manager assuming no person is elected more than one position?

Solution:- $20 \times 19 \times 18 = 6840$.

E.g. 2. Suppose a college has 3 different calculus, 4 different algebra and 2 different optimization courses. Then

a. The no of ways a student can choose one of each kind of the course is $3 \times 4 \times 2 = 24$ ways

b. The no of ways a student can choose one of the courses is $3+4+2=9$

Exercise. 1. To reach city D from city A, one has to pass through city B and then through city C. If there are 2 ways to travel from A to B, 5 ways from B to C and 3 ways from c to D, what is the no of ways to go from city A to city D.

Solution:- by (MP), $A = B \equiv C \equiv D$. The no of ways is $2 \times 5 \times 3 = 30$

2. If two die are thrown once, then find the no of ways in which we get even sum.

Solution:- Even, $E_1 = \{(1,1)\}$, to get 2

$E_2 = \{(1,3), (3,1), (2,2)\}$ to get 4

$E_3 = \{(1,5), (5,1), (2,4), (4,2), (3,3)\}$ to get 6

$E_4 = \{(2,6), (6,2), (3,5), (5,3), (4,4)\}$ to get 8

$E_5 = \{(4,6), (6,4), (5,5)\}$ to get 10

$E_6 = \{(6,6)\}$ to get 12

Number of ways which we get even sum is $|E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6| = 18$

Note:- since 1st die can fall (event E_1) in 6 ways and the 2nd die can fall (event E_2) in 6 ways.

Thus, there are $6 \times 6 = 36$ out comes when 2 dice are rolled.

Note: - If a natural number 'n' has its prime factorization,

$n = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_r^{k_r}$, where p_i 's are distinct and the k_i 's are positive integers, then the no of positive divisors of n is given by $\prod_{i=1}^r (k_i + 1)$.

E.g. 3. Find the number of positive divisors of 600 .

Solution:- we know $600 = 2^2 \cdot 3^1 \cdot 5^2$. Thus the no of positive divisors of 600 is $4 \times 2 \times 3 = 24$

3.2. Permutations

Let $A = \{a_1, a_2, \dots, a_n\}$ be a given set of n distinct objects.

For $0 \leq r \leq n$, an r - permutation of A is a way of arranging any r of the objects of A in a row.

When $r = n$, an n – permutation is simple called a permutation of A

E. g. Let $= \{a, b, c\}$. Then all the 2- permutations of A are ab, ac, ba, bc, ca, cb . They are 6 in no

An r -permutation of A can be formed in r steps. There are n choices for the 1st position, $(n - 1)$ choices for the 2nd position $(n - 2)$ choice for the 3rd position and so on until the r^{th} position is filled in $n - (r - 1)$ ways.

Thus by (MP), we have $n (n - 1) (n - 2) \dots (n - r + 1)$

Definition: - for an integer $n \geq 0$, n factorial denoted by $n!$ is defined

By $0! = 1, 1! = 1$

$n! = n (n - 1) (n - 2) \dots (3) (2) (1), \text{ for } n \geq 2.$

Note:- $p_r^n = p(n, r) = \frac{n (n-1) (n-2) \dots (n-r+1) (n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$ (check exercise)

E.g.1. The number of permutation of the letters in the word computer is $8!$.

If only five of the letters are used. Then $p(8, 5) = \frac{8!}{(8-5)!} = \frac{8!}{(3)!} = 8.7.6.5.4 =$

2. How many Pairs of dance partners can be selected from a group of 12 women and 20 men?

Soul: -The 1st women can be paired with any of 20 men

The 2nd women can be paired with any of the remaining 19 men

The 3rd women can be paired with any of the remaining and so on

This are $20.19.18. \dots .9 = p(20, 12)$

Exercise1. What is the value of n if $p(n, 5) = 12p(n, 3)$?

Soul:- $p(n, 5) = 12p(n, 3)$

$$\Rightarrow \frac{n!}{(n-5)!} = 12 \frac{n!}{(n-3)!} \Rightarrow n = 0 \text{ or } n = 7. \text{ Thus } n = 7 \text{ is the only answer.}$$

Exercise2. How many three letter "words" can be formed from letters in the set $\{a, b, y, z\}$

i. If repeated letters are allowed

ii. If repeated letters are not allowed

Soul:-Here n is 4 and r is 3. So i) the no of such words is $4.4.4=4^3=64$ ii) $p(4, 3) = \frac{4!}{(4-3)!} = 24$

Exercise3. how many "words" of 3 distinct letters can be formed from the letters of the word MAST?

Soul:- the no is $p_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$

Exercise4. How many different 4 digit nos can be formed out of the digits 1, 2, 3, 7, 8 and 9 when

a. repetition is allowed

b. repetition is not allowed

Solution:- a. $6.6.6.6 = 6^4$

$$\text{b. } p(6.4) = \frac{6!}{(6-4)!}$$

Note:- If there are n objects with n_1 of the 1st type, n_2 of the 2nd type, - - -, and n_r of the r^{th} type, Where $n_1 + n_2 + \dots + n_r = n$, then there are $\frac{n!}{n_1! n_2! \dots n_r!}$ linear arrangements of the given n -objects.

e.g.1. Find the no of arrangement (permutation) can be made from.

a) 4black, 3red, 2white

b) 2black, 2red, 2white

2. Find the no of arrangement (permutation) can be made out of the letters of the word

a) LAMP

b) FOOD

c) CALCULUS

3. a) Find the no of arrangement (permutation) that can be made out of the letters of the word "MATHEMATICS". In how many of these permutations

b) Do all the vowels occur together

c) Do the word start with I

d) Begins with the two A

e) Begins with E and ends with C

Solution :- 1. a) $n = 4 + 3 + 2 = 9 \therefore \frac{9!}{4!3!2!} = 630$

$$\text{b) } n = 2 + 2 + 2 = 6 \therefore \frac{6!}{2!2!2!} = 90$$

$$\text{2. a) } p(4, 4) = \frac{4!}{(4-4)!} = 24 \quad \text{b) there are 1F, 2'O', 1D. Thus } \frac{4!}{1!2!1!} = 12 \text{ different ways.}$$

$$\text{C) there are 2C, 1A, 2L, 2U, 1S. Thus } \frac{8!}{2!1!2!2!1!} = 5,040$$

3. a) The word "MATHEMATICS" consists of 11 letters of which 2 are A_s, 2 are M_s, 2 are T_s, and the rest all different. Therefore the no of arrangements are $\frac{11!}{2!2!2!}$

b) there are 2 are T_s, 2 are M_s, 1 is H, 1 is C, and 1 is S. Total no of arrangements are $\left(\frac{7!}{2!2!}\right)\left(\frac{4!}{2!}\right)$

$$\text{c) } \frac{10!}{2!2!2!}$$

$$\text{d) } \frac{9!}{2!2!}$$

$$\text{e) } \frac{9!}{2!2!2!}$$

Circular permutation:-

Definition: - The circular arrangement of n out on n distinct objects is $p(n-1, n-1) = (n-1)!$

e.g. The arrangement of 5 students in a linear is $p(5, 5) = 5! = 120$ ways. But in a circular is

$$p(5-1, 5-1) = (5-1)! = 4! = \frac{p(5, 5)}{5}$$

e.g. for A, B, C on circular chair, we have $\frac{p(3, 3)}{3} = p(2, 2) = 2$ ways of arrangements.

Restricted permutation

Permutation of n object in r places when x particular objects will

- i. always occur is $p(n-x, r-x) = \frac{(n-x)!}{(n-r)!}$
- ii. never occur is $p(n-x, r) = \frac{(n-x)!}{(n-x-r)!}$

e.g. In how many ways can we choose 11 out of 18 players?

- i. If 2 papers are choose
- ii. If 2 players are not choose

Solution:- i. $p(18-2, 11-2) = p(16, 9)$ ii. $p(18-2, 11) = p(16, 11)$

3.3. Combinations

In how many ways can we choose two letters from A, B, C?

Solution:- 3 ways. i.e, AB, Bc and AC.

Definition:- combination is the no of ways of selecting r objects from a set of n - objects without regarding to the order of selection. It is denoted by $c(n, r)$ or $\binom{n}{r}$ and is $c(n, r) = \frac{p(n, r)}{r!} = \frac{n!}{(n-r)! r!}$ and is called the no of combination of objects taken r at a time.

E.g. 1 how many different 5 cards can be selected out of 52 different cards

Solution:- $c(52, 5) = \frac{52!}{(52-5)! 5!}$

Exercise1. How many different seven person committees can be formed each containing three women from an available set of 20 women and 4 men an available set of 30 men?

Solution:- we have two tasks

Task1: choose 3 women from the set of 20 women

Task2: choose 4 men from the set of 30 men

Thus task1 can be performed in $c(20, 3)=1140$ ways and

Task2 can be performed in $c(30, 4)=27,405$ ways

By multiplication principle (MP), there are $(1140) (27,405) = 31,241,700$ different committees.

Exercise2. A men is going to toss a coin eight times. In how many ways can get five heads and three fails.

Soultion: $-c(8, 5) = c(8, 3) = 56$.

Restricted combination

Combination of n object taking r at a time when x particular will

i) always occur is $c(n - x, r - x)$

ii) never occur is $c(n - x, r)$

e.g. how many committees of five people can be chosen from 20 men and 12 women

a. if exactly three men must be on each committee

b. if at least four women must be on each committee

Solution:- a. $c(20, 3) c(12, 2) = (1140) (66) = 75,240$

b. by addition principle (AP), $c(12, 4) c(20, 1) + c(12, 5) c(20, 0) = 495(20) + 792 = 10,692$.

Exercise. In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 teachers and 7 students if

i. there is no restriction in the selection

ii. The committee must include exactly 2 teachers

iii. the committee must include at least 3 teachers

iv. A particular teacher and particular student can not be both in the committee.

Soul: -the no of ways is

i. $c(11, 5) = \frac{11!}{(11-5)!5!} = 462$

ii. $c(4, 2)c(7, 3) = 210$

iii. $c(4, 3)c(7, 2) + c(4, 4)c(7, 1) = 91$

iv. $c(3, 3)c(6, 2) + c(3, 2)c(6, 3) + c(3, 1)c(6, 4) + c(3, 0)c(6, 5) = 378$

3.4. Binomial theorem

The sum of two distinct terms say " $x+y$ " is called a binomial then $(x + y)^n$ for $n \in N$ is binomial expansion.

Note:- (binomial theorem)

If x and y are variables and new, then

$$(x + y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n} x^0 y^n$$

$$= x^n + \frac{n!}{1!(n-1)!} x^{n-1} y^1 + \frac{n!}{2!(n-2)!} x^{n-2} y^2 + \dots + \frac{n!}{r!(n-r)!} x^{n-r} y^r + \dots + \frac{n!}{n!(n-n)!} y^n$$

e.g. 1. Expand $(x + y)^3$ and $(x - y)^4$

Solution:- $(x + y)^3 = x^3 + \frac{3!}{1!(3-1)!}x^2y^1 + \frac{3!}{2!(3-2)!}x^1y^2 + y^3 = x^3 + 3x^2y^1 + 3xy^2 + y^3$

$$(x - y)^4 = x^4 + \frac{4!}{1!(4-1)!}x^3(-y)^1 + \frac{4!}{2!2!}x^2(-y)^2 + \frac{4!}{3!1}x^1(-y)^3 + (-y)^4$$

$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$$

e.g.2. Find the fourth term in the expansion of $(x + y)^9$

Solution:- since y^1 appears in the 2nd term

y^2 Appears in the 3rd term

y^3 Appears in the 4th term

Then the variables of the fourth term are x^6y^3 . And the coefficient must be $\frac{n!}{r!(n-r)!} = \frac{5!}{3!(9-3)!} = 84$

Thus fourth term is $\frac{5!}{3!(9-3)!}x^6y^3 = 84x^6y^3$

e.g.3. Find coefficient of a^3b^4 in the expansion of $(3a - 4b)^7$

Solution:- $-\binom{7}{4}(3)^3(-4)^4 = 76,545$

Note: - 1. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$

2. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} = 0$

e.g. 4. using the Binomial theorem, expand $(x - \frac{4}{x})^6$ and find the constant term

Soul:- $(x - \frac{4}{x})^6 = x^6 + \binom{6}{1}x^5\left(\frac{-4}{x}\right)^1 + \binom{6}{2}x^4\left(\frac{-4}{x}\right)^2 + \dots + \left(\frac{-4}{x}\right)^6$

Theorem (Multinomial them)

For positive integer n , the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3} \dots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \dots + x_t)^n$ is $\frac{n!}{n_1!n_2! \dots n_t!}$, Where $0 \leq n_i \leq n, \forall i, 1 \leq i \leq t$ and

$$n_1 + n_2 + \dots + n_t = n.$$

e.g.1. In the expansion of $(x + y + z)^7$, find the coefficient of a) $x^2y^2z^3$ b) xy^4z^2 c) y^4z^3

e.g. 2. In the expansion of $(a + 3b - 2c + 3)^{12}$, find the coefficient of

a. $a^2b^6c^4$ b. $a^2b^6c^4$ c. b^5c^3

Solution:- 2. a. $3^6(-2)^4 \frac{12!}{2!6!4!0!}$

b. $3^3(-2)^4 3^3 \frac{12!}{2!3!4!3!}$

c. $3^5(-2)^3 3^4 \frac{12!}{0!5!3!4!}$

3.5 The pigeonhole principle

The pigeonhole principle states that if $n + 1$ objects (pigeons) are placed into n boxes (pigeonholes), then at least one box contains more than one object.

e.g. 1. In any given set of 14 people at least two of them have their birth day during the same month.

2. Among any group of 8 people at least two were born on the same day of the week.

The generalized pigeonhole principle:-

If N objects are placed into k boxes then there is at least one box containing $\left\lceil \frac{N}{k} \right\rceil$ objects. i.e., the number of objects exceeds a multiple of the number of boxes.

Note:- $\lceil x \rceil$ is the smallest integer l , such that $l \geq x$.

e.g. 1. Among 100 people there are at least $\left\lceil \frac{100}{13} \right\rceil = 8$ who were born in the same month.

2. What is the minimum number of students in a discrete mathematics class to be sure that at least 6 will receive the same grade if the possible grades are A, B, C, D and F?

Solution: - let N be the smallest integer than $\left\lceil \frac{N}{5} \right\rceil = 6$ Thus $N = 5.5 + 1 = 26$

3.6 generating functions

Definition:- given a sequence $\{a_k\}$ with $k \geq 0$, its generating function $G(x)$ is defined by

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{k \geq 0} a_k x^k$$

e.g. 1. For any $n \in \mathbb{Z}^+$,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n. \text{ Thus } (1+x)^n \text{ is the generating function for}$$

the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots, \binom{n}{n}, 0, 0, 0, \dots$.

e.g.2. a) For $n \in \mathbb{Z}^+$, From $y^{n+1} - x^{n+1} = (y-x)(y^n + y^{n-1}x + y^{n-2}x^2 + \dots + x^n)$,

we have, $1 - x^{n+1} = (1-x)(1+x+x^2+\dots+x^n)$

$$\Rightarrow \frac{1-x^{n+1}}{1-x} = 1+x+x^2+\dots+x^n,$$

Thus $\frac{1-x^{n+1}}{1-x}$ is a generating function for the sequence 1, 1, 1, ..., 1, 0, 0, 0, ...

Where the first $n+1$ terms are 1.

b) Extending the idea in part (a) we find that

$\frac{1}{1-x} = 1 + x + x^2 + \dots$. where $|x| < 1$ as $n \rightarrow \infty$, $x^{n+1} \rightarrow 0$ from 2a and so, $\frac{1}{1-x}$ is generating function for the sequence 1, 1, 1, $\dots$.

c. from $\frac{1}{1-x} = 1 + x + x^2 + \dots$.

we can taking the derivative for both sides

$$\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} (1 + x + x^2 + x^3 + \dots)$$

$$\Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\Rightarrow \frac{1}{(1-x)^2} \text{ is the generating function for the sequence } 1, 2, 3, 4, \dots$$

d. multiply both sides by x , we have

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \dots \text{ is also the generating function for the sequence } 0, 1, 2, 3, 4, \dots$$

e. continuing from part (d).

Example:- 3. $f(x) = \frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \dots$ is the generating function for the sequence 1, 2^1 , 2^2 , 2^3 , $\dots$

Exercise:-1. Let $f(x) = \frac{1}{1-x}$. Then find the sequence for

a. $F(x) = f(x) - x^2$

b. $H(x) = f(x) + 2x^3$

2. Find the generating function for the sequence 0, 2, 6, 12, 20, 30, 42, $\dots$

Solution:- $G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

$$G(x) = 0 + 2x + 6x^2 + 12x^3 + \dots$$

$$G(x) = (0 + 0^2) + (1 + 1^2)x + (2 + 2^2)x^2 + (3 + 3^2)x^3 + \dots$$

$$= (0 + 1x + 2x^2 + 3x^3 + \dots) + (0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \dots)$$

$$= \frac{x}{(1-x)^2} + \frac{x(x+1)}{(1-x)^3}$$

$$= \frac{2x}{(1-x)^3} \text{ is generating function for the given sequence } 0, 2, 6, 12, 20, 30, 42, \dots$$

3. find the generating function for the sequence

a. $1, 2, 3, 3, 3, - - -$

b. $1, 2, 3, 4, 4, 4, - - -$

c. $1, 4, 7, 10, 13, - - -$

d. $a_n = (-1)^n n$