

Chapter Two

Functions

Our everyday lives are filled with situations in which we encounter relationships between two sets. For example,

- To each automobile, there corresponds a license plate number
- To each circle, there corresponds a circumference
- To each number, there corresponds its square

In order to apply mathematics to a variety of disciplines, we must make the idea of a “relationship” between two sets mathematically precise.

On completion of this chapter students will be able to:

- understand the concept of real numbers
- use properties of real numbers to solve problems
- determine whether a given real number is rational number or not
- solve linear equations and inequalities
- solve quadratic equations and inequalities
- understand the notion of relation and function
- determine the domain and range of relations and functions
- find the inverse of a relation
- define polynomial and rational functions
- perform the fundamental operations on polynomials
- find the inverse of an invertible function
- apply the theorems on polynomials to find the zeros of polynomial functions
- apply theorems on polynomials to solve related problems
- sketch and analyze the graphs of rational functions
- define exponential, logarithmic, and trigonometric functions
- sketch the graph of exponential, logarithmic, and trigonometric functions
- use basic properties of logarithmic, exponential and trigonometric functions to solve problems

In this chapter, before discussing the idea of relations and functions we first review the system of real numbers, linear and quadratic equations and inequalities.

1.1 The real number systems

At the end of this section, students will be able to:

- understand the concept of real numbers
- use properties of real numbers to solve problems
- determine whether a given real number is rational number or not

In this section we will define what the real numbers are and what are their properties? To answer, we start with some simpler number systems.

• The integers and the rational numbers

The simplest numbers of all are the natural numbers,

$$1, 2, 3, 4, 5, 6, \dots$$

With them we can count: our books, our friends, and our money. If we adjoin their negatives and zero, we obtain the integers;

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

When we try to measure length, weight, or voltage, the integers are inadequate. They are spaced too far apart to give sufficient precision. Thus, we are led to consider quotients (ratios) of integers, numbers such as:

$$\frac{3}{4}, \frac{-7}{8}, \frac{21}{5}, \frac{19}{-2}, \frac{16}{2} \text{ and } \frac{-17}{1}$$

Note that we included $\frac{16}{2}$ and $\frac{-17}{1}$, though we would normally write them as 8 and - 17, since they are equal to the latter by the ordinary meaning of division. We did not include $\frac{5}{0}$ or $\frac{-9}{0}$, since it is impossible to make sense out of these symbols. In fact, let us agree once and for all to banish division by zero from this section. Numbers which can be written in the form $\frac{m}{n}$, where m and n are integers with $n \neq 0$, are called rational numbers.

Do the rational numbers serve to measure all lengths? No. This surprising

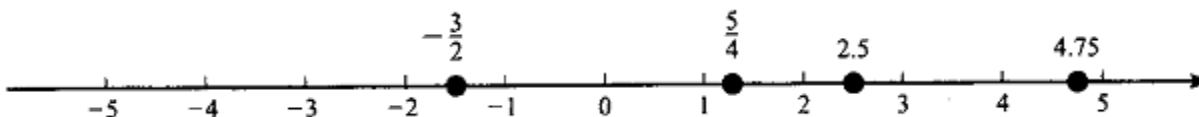
fact was discovered by the ancient Greeks long ago. They showed that while $\sqrt{2}$ measures the hypotenuse of a right triangle with sides of length 1, it cannot be written as a quotient of two integers (see exercise...). Thus, $\sqrt{2}$ is an irrational (not rational) number. So are $\sqrt{3}, \sqrt{5}, \sqrt[3]{7}, \pi$ and a host of other numbers.

- **The real numbers**

Consider the set of all numbers (rational and irrational) that can measure lengths, together with their negatives and zero. We call these numbers the real numbers.

The set of real numbers denoted by $\mathfrak{R}$ can be described as the union of the set of rational and irrational numbers. i.e $\mathfrak{R} = \{x : x \text{ is a rational number or an irrational number}\}$.

The real numbers may be viewed as labels for points along a horizontal line. There they measure the distance to the right or left (the directed distance) from a fixed point called the origin and labeled 0. Each point on the number line corresponds a unique real number and vice-versa.



Most students will remember that the number system can be enlarged still more to the so-called complex numbers. These are numbers of the form $a+b\sqrt{-1}$, where a and b are real numbers.

- **The four arithmetic operations**

Give two real numbers x and y , we may add or multiply them to obtain two new real numbers $x+y$ and $x \cdot y$ (also written simply as xy). The real numbers along with the operations of addition (+) and multiplication ($\cdot$), obey the 11 properties listed below. Most of these properties are straightforward and may seem trivial. Nevertheless, we shall see that these

11 basic properties are quite powerful in that they are the basis for simplifying algebraic expressions.

The commutative Properties

1. *For addition:* $a+b=b+a$
2. *For multiplication:* $ab=ba$

The associative properties

3. *For addition:* $a+(b+c)=(a+b)+c$
4. *For multiplication:* $a(bc)=(ab)c$

The distributive property

5. $a(b+c)=ab+ac$ or $(b+c)a=ba+ca$

Identities

6. *For addition:* There is a unique number called the additive identity, represented by 0, which has the property that $a+0=a=0+a$ for all real numbers a .
7. *For multiplication:* There is a unique real number called the multiplicative identity, represented by 1, which has the property that $a \cdot 1 = a = 1 \cdot a$ for all real numbers a .

Inverses

8. *For addition:* Each real number a has a unique additive inverse, represented by $-a$, which has the property that $a+(-a)=0=(-a)+a$
9. *For multiplication:* Each real number a , except 0, has a unique multiplicative inverse, represented by $\frac{1}{a}$, which has the property that $a \cdot (\frac{1}{a}) = 1 = (\frac{1}{a})a$.

Closure properties

10. *For addition:* The sum of two real numbers is a real number.
11. *For multiplication:* The product of two real numbers is a real number.

Subtraction and division are defined by:

$$x - y = x + (-y) \text{ and } x \div y = x \cdot \frac{1}{y}, \text{ where } y \neq 0.$$

In the product ab , a and b are called factors, in the sum $a+b$, a and b are called terms.

Example 2.1: The set of irrational numbers is not closed under addition and multiplication, because $\sqrt{2} + (-\sqrt{2}) = 0$ and $\sqrt{2}\sqrt{8} = \sqrt{16} = 4$, which are rational numbers.

- **The order relation on the set of real numbers**

The nonzero real numbers separate nicely into two disjoint sets - the positive real numbers and the negative real numbers. This fact allows us to introduce the order relation $<$ (read "is less than") by

$$x < y \Leftrightarrow y - x \text{ is positive}$$

We agree that $x < y$ and $y > x$ will mean the same thing. The order relation $\leq$ (read "is less than or equal to") is a first cousin of $<$. It is defined by

$$x \leq y \Leftrightarrow y - x \text{ is positive or zero}$$

The order relation $<$ has the following properties:

The order property

1. **Trichotomy:** If x and y are numbers, exactly one of the following holds:

$$x < y \text{ or } x = y \text{ or } x > y$$

2. **Transitivity:** $x < y$ and $y < z \Rightarrow x < z$

3. **Addition:** $x < y \Leftrightarrow x + z < y + z$

4. **Multiplication:** When z is positive, $x < y \Leftrightarrow xz < yz$,

When z is negative, $x < y \Leftrightarrow xz > yz$

- **Intervals**

Let a and b be two real numbers such that $a < b$, then the intervals which are subsets of $\mathbf{R}$ with end points a and b are denoted and defined as below:

- i) $(a, b) = \{x: a < x < b\}$ open interval from a to b .
- ii) $[a, b] = \{x: a \leq x \leq b\}$ closed interval from a to b .
- iii) $(a, b] = \{x: a < x \leq b\}$ open-closed interval from a to b .
- iv) $[a, b) = \{x: a \leq x < b\}$ closed-open interval from a to b .

Exercise 2.1

1. Simplify as much as possible:

a) $4 - 3(8 - 12) - 6$ c) $\frac{5}{6} - (\frac{1}{4} + \frac{2}{3})$

b) $2[3 - 2(4 - 8)]$ d) $\frac{\frac{1}{2} - \frac{3}{4} + \frac{7}{8}}{\frac{1}{2} + \frac{3}{4} - \frac{7}{8}}$

2. Which of the following statements are true and which of them are false?

- a) The sum of any two rational numbers is rational.
 b) The sum of any two irrational numbers is irrational.
 c) The product of any two rational numbers is rational.
 d) The product of any two irrational numbers is irrational.

3. Find the value of each of the following, if undefined, say so.

a) $0 \cdot 0$ c) $\frac{0}{0}$ e) $\frac{0}{8}$

b) $\frac{8}{0}$ d) 8^0 f) 0^8

4. Show that division by 0 is meaningless as follows: Suppose $a \neq 0$. If $\frac{a}{0} = b$, then $a = 0 \cdot b = 0$, which is a contradiction. Now find a reason why $\frac{0}{0}$ is also meaningless.

5. Prove each if $a > 0$ $b > 0$,

a) $a < b \Leftrightarrow a^2 < b^2$ b) $a < b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$

6. Which of the following are always correct if $a \leq b$?

a) $a - 4 \leq b - 4$ b) $-a \leq -b$ c) $a^2 \leq ab$ d) $a^2 \leq a^2 b$

2.2 Equations and Inequalities: Linear and Quadratic

At the end of this section, students will be able to:

- solve linear equations and inequalities
- solve quadratic equations and inequalities identify the notions of the common sets of numbers

• Linear Equations and inequalities

An equation is a symbolic statement of equality. That is, rather than writing “twice a number is four less than the number,” we write $2x = x - 4$. Our goal is to find the solution to a given equation. By solution we mean the value or values of the variable that make the algebraic statement true.

Definition 2.1: (Linear Equation)

A linear equation in one variable is an equation that can be put in the form $ax + b = 0$, where a and b are constants, and $a \neq 0$.

Equations that have the same solutions are called equivalent equations. For example, $3x - 1 = 5$ and $3x = 6$ are equivalent equations because the solution set of both equations is $\{2\}$. Our goal here is to take an equation and with the help of a few properties, gradually, change the given equation into an equivalent equation of the form $x = a$, where x is the variable for which we are solving. These properties are:

1. The addition property

If $a = b$, then $a + c = b + c$. That is, adding the same quantity to both sides of an equation will produce an equivalent equation.

2. The multiplication property

If $a = b$, then $ac = bc$. That is, multiplying both sides of an equation by the same nonzero quantity will produce an equivalent equation.

Example 2.2:

1. Solve for x

a) $820x = 10x + 30(50 - x)$

b) $3(2x + 1) = 2(1 - 5x) + 6x + 11$

Solution:

$$820x = 10x + 30(50 - x)$$

a)

Simplify the right hand side

$$820x = 10x + 1500 - 30x$$

$$820x = 1500 - 20x$$

Applying the addition property (add $20x$ to both sides)

$$840x = 1500$$

$$\text{Thus, } x = \frac{1500}{840} = \frac{25}{14}$$

Remember to check by substituting $\frac{25}{14}$ for x in the original equation.

$$3(2x + 1) = 2(1 - 5x) + 6x + 11$$

b)

(The given equation)

$$6x + 3 = 2 - 10x + 6x + 11$$

(Removing parentheses by distribution)

$$6x + 10x - 6x = 2 + 11 - 3$$

(Collecting like terms: 'variables to the left and

the left and

numbers to the right')

$$10x = 10$$

$$x = 1$$

(Dividing both sides by 10)

Therefore, the solution set (S.S) is $\{1\}$.

$$\frac{8x+3}{2} - 5(x+2) = -3(x + \frac{5}{6})$$

2. Find the solution set of

$$\text{Solution: } \frac{8x+3}{2} - 5(x+2) = -3(x + \frac{5}{6}) \quad (\text{The given equation})$$

This gives us:

$$4x + \frac{3}{2} - 5x - 10 = -3x - \frac{5}{2}$$

$$4x - 5x + 3x = -\frac{5}{2} - \frac{3}{2} + 10$$

Using addition property

$$2x = 6$$

Hence, $x = 3$. That is, the solution set is $\{3\}$.

3. A computer discount store held an end of summer sale on two types of computers. They collected Birr 41,800 on the sale of 58 computers. If

one type sold for Birr 600 and the other type sold for Birr 850, how many of each type were sold?

Solution: If we let x to be the number of Birr 600 computers sold, then $58-x$ = the number of computers that are sold for Birr 850 (since 58 were sold all together).

Our equation involves the amount of money collected on the sale of each type of computer that is, the value of computers sold). Thus we have:

$$600x + 850(58-x) = 41,800, \text{ which yields}$$

$$x = 30$$

Hence, there were 30 computers sold at Birr 600 and 28 computers sold at 850.

Remark: The solution set of some equation can be the set of all rational numbers. This is the case when the equation is satisfied by every rational number.

Example 2.3: Find the solution set of $5x - 2(x-1) + 4 = 3(x+2)$

Solution: $5x - 2(x-1) + 4 = 3(x+2)$ (The given equation)

$5x - 2x + 2 + 4 = 3x + 6$ (Removing parentheses by distribution)

$$3x + 6 = 3x + 6 \quad \text{(Combining like terms)}$$

This is always true whatever the value of x is. In fact, subtracting $3x$ from both sides of the last equation we get $6=6$ which is always true. This means the given equation is satisfied if you take any number for x as you wish. Thus, $S.S = \mathcal{R}$.

Remark: There are also some equations which cannot be satisfied by any number. For example, the equation $x+10 = x$ says 'If you increase a number x by 10, the result is x itself (unchanged)'. Obviously, there is no such a number. The solution set of such equation is **empty set**. If you try to solve such equation, you end up with a false statement (false equality). For example, an attempt to solve $x+10 = x$ leads to the following:

$10 + x - x = x - x$ (Subtracting x from both sides of the equation)

$$10 = 0, \text{ which is false.}$$

Hence, the solution set of $x+10 = x$ is $\emptyset$ (empty set).

Example 2.4: Find the solution set of $6 + 3(1 - x) = 2(1 - 5x) + 7x$

Solution: $6 + 3(1 - x) = 2(1 - 5x) + 7x$ (The given equation)

$6 + 3 - 3x = 2 - 10x + 7x$ (Removing parentheses by distribution)

$9 - 3x = 2 - 3x$ (Combining like terms)

$9 - 3x + 3x = 2 - 3x + 3x$ (Adding $3x$ to both sides)

$9 = 2$, which is false.

This means the solution set of the given equation is empty, $\emptyset$.

Example 2.5: A man has a daughter and a son. The man is five times older than his daughter. Moreover, his age is twice of the sum of the ages of his daughter and son. His daughter is 3 years younger than his son. How old is the man and his children?

Solution: The unknowns in the problem are age of the man, age of his daughter, and age of his son. So, let $m = \text{Age of the man}$; $d = \text{Age of the daughter}$; and $s = \text{Age of the son}$. Then, 'The man is 5 times older than his daughter' means $m = 5d$. Moreover, 'Age of the man is twice the sum of the ages of his daughter and son' means $m = 2(d + s)$. 'His daughter is 3 years younger than his son' means $d = s - 3$.

Now, from the last (3rd) equation you can get $s = d + 3$. Substitute this in the 2nd equation to get $m = 2(d + d + 3) = 2(2d + 3)$. This is, $m = 4d + 6$. Next substitute this in the 1st equation to get $4d + 6 = 5d$ or $6 = 5d - 4d = d$. Hence, $d = 6$. From this, $s = d + 3 = 6 + 3 = 9$, and $m = 5d = 5 \times 6 = 30$.

Therefore, the age of the man is 30, age of his daughter is 6 and age of his son is 9.

Definition 2.2: (Linear Inequalities)

A linear inequality is an inequality that can be put in the form $ax + b < 0$, where a and b are constants with $a \neq 0$. (The $<$ symbol can be replaced with $>$, $\leq$ or $\geq$)

To solve inequalities, we will need the following properties of inequalities.

For $a, b, c \in \mathbb{R}$, if $a < b$, then

$$1) a + c < b + c$$

$$c < 0$$

$$2) ac < bc, \text{ when } c > 0$$

$$3) ac > bc, \text{ when } c < 0$$

Thus, to produce an equivalent inequality, we may add (subtract) the same quantity to (from) both sides of an inequality, or multiply (divide) both sides by the same positive quantity. On the other hand, we must reverse the inequality symbol to produce an equivalent inequality if we multiply (divide) both sides by the same negative quantity.

Example 2.6:

$$5x + 8(20 - x) \geq 2(x - 5)$$

1. Solve the linear inequality .

Solution: $5x + 8(20 - x) \geq 2(x - 5)$

Simplify each side

$$5x + 160 - 8x \geq 2x - 10$$

$$160 - 3x \geq 2x - 10$$

Now apply the inequality property

$$-5x \geq -170$$

Divide both sides by - 5

$$x \leq 34$$

Note that the inequality symbol is

reversed

Thus, the solution set is $\{x \in \mathbb{R} : x \leq 34\} = (-\infty, 34]$.

Example 2.7: Find the solution set of the inequality $3x - 5(x + 2) \geq 0$.

Solution: $3x - 5(x + 2) > 0$ (The given inequality)

$$3x - 5x + 10 > 0$$
 (Removing the parentheses by distribution)

$$-2x + 10 > 0$$
 (Combining like terms)

$$-2x > -10$$
 (Subtracting 10 from both sides)

$$x < \frac{-10}{-2}$$

(Dividing both sides by -2 reverse the inequality)

That is, $x < 5$. Therefore, S.S = $\{x : x < 5\}$, the set of all real numbers less 5.

The solution of an inequality is sometimes required to be only in a given **domain** (set). If so, a solution set should contain only those solutions that belong to the specified domain.

Example 2.8: Find the solution set of $x - 4(x + 1) \geq -13 - (x - 2)$ in the set of natural numbers, $\mathbb{N}$.

Solution: $x - 4(x + 1) \geq -13 - (x - 2)$ (The given inequality)

$$\begin{aligned}
 x - 4x - 4 &\geq -13 - x + 2 && \text{(Removing parentheses by distribution)} \\
 -3x - 4 &\geq -11 - x && \text{(Combining like terms; i.e., } x - 4x = -3x \text{ and} \\
 -13 + 2 &= -11) \\
 -3x + x &\geq -11 + 4 && \text{(Collecting like terms)} \\
 -2x &\geq -7 && \text{(Next, division of both sides of this by -2} \\
 &&& \text{reverses the} \\
 &&& \text{inequality)}
 \end{aligned}$$

$$x \leq \frac{7}{2}; \quad \text{i.e., } x \leq 3.5$$

Thus, the solution of the given inequality in $\mathbb{N}$ is $\{1, 2, 3\}$. (Recall: $\mathbb{N} = \{1, 2, 3, \dots\}$)

Some inequalities may have no solution in the specified domain as in the following example.

Example 2.9: Find the solution set of $7x + 6 \leq 3x + 2$ in the set of **whole numbers, $\mathbf{W}$** .

$$\begin{aligned}
 \textbf{Solution:} \quad 7x - 2 &\leq 3x - 6 && \text{(The given inequality)} \\
 7x - 3x &\leq -6 + 2 && \text{(Collecting like terms)} \\
 4x &\leq -4 \\
 \frac{4x}{4} &\leq \frac{-4}{4} && \text{or } x \leq -1
 \end{aligned}$$

However, there is no negative whole number. Therefore, the solution set of the given inequality in $\mathbf{W}$ is $\emptyset$, empty set. (Recall: $\mathbf{W} = \{0, 1, 2, 3, \dots\}$)

Find the solution set of the inequality
Example 2.10:

$$\frac{1}{6}(x+3) + \frac{1}{2}x - \frac{3}{2} \leq \frac{3}{2}(x+1) \quad \text{in } \mathbb{Q}.$$

Solution: The inequality involves fractional numbers. Thus, like for the case of linear equations, clear the denominators by multiplying both sides of the inequality by the LCM of the denominators. The denominators in this equation are 6 and 2; and their LCM is 6. Thus, multiply every term in both sides of the given inequality by 6. That is,

$$6\left[\frac{1}{6}(x+3)\right] + 6\left[\frac{1}{2}x\right] - 6\left[\frac{3}{2}\right] \leq 6\left[\frac{3}{2}(x+1)\right] \quad \text{(The inequality is not reversed because } 6 > 0)$$

$$x + 3 + 3x - 9 \leq 9(x + 1) \quad \text{(Simplifying/clear$$

denominators)

$$4x - 6 \leq 9x + 9$$

$$4x - 9x \leq 9 + 6$$

(Collecting like terms)

$$-5x \leq 15$$

(Next, division of both sides

by -5)

$$x \geq \frac{15}{-5} \quad \text{or} \quad x \geq -3.$$

Therefore, S.S = $\{ x \in \mathbb{Q} \mid x \geq -3 \}$.

• Quadratic Equations and Inequalities

A quadratic equation is a polynomial equation in which the highest degree of the variable is 2. We define the standard form of a quadratic equation as $Ax^2 + Bx + C = 0$, where $A \neq 0$.

As with linear equations, the solutions of quadratic equations are values of the variable that make the equation a true statement. The solutions of $Ax^2 + Bx + C = 0$ are also called the roots of the polynomial equation $Ax^2 + Bx + C = 0$.

In solving the equation $Ax^2 + Bx + C = 0$, if the polynomial $Ax^2 + Bx + C$ can be factored, then we can use the zero product rule (which is stated below) to reduce the problem to that of solving two linear equations. For example, to solve the equation $x^2 + x - 6 = 0$, we can factor the left hand side to get $(x - 2)(x + 3) = 0$. Hence, we can conclude that $x - 2 = 0$ or $x + 3 = 0$, which yields $x = 2$ or $x = -3$.

The Zero-Product Rule: If $a \cdot b = 0$, then $a = 0$ or $b = 0$

Another method is to apply the Square Root Theorem.

The Square Root Theorem: If $x^2 = d$, then

$$x = \pm \sqrt{d}.$$

Example 2.11: Solve the following

a) $4x^2 + 10x = 6$

b) $5x^2 - 6 = 8$

c) $(x-2)^2 = 6$

Solution: a) $4x^2 + 10x = 6$

Put into standard form

$$4x^2 + 10x - 6 = 0$$

Factor the left hand side

$$2(2x-1)(x+3) = 0$$

Hence we have

$$2x-1=0 \text{ or } x+3=0$$

Solving each linear equation, we get

$$x = \frac{1}{2} \text{ or } x = -3$$

b) We note that there is no first-degree term, so our approach will be to apply the Square Root Theorem.

$$5x^2 - 6 = 8$$

Isolate x^2 on the left-hand side before applying the

square root theorem

$$5x^2 = 14$$

$$x^2 = \frac{14}{5}$$

Applying the square root theorem we get

$$x = \pm \sqrt{\frac{14}{5}}$$

c) Since it is in the form of a squared quantity equal to a number, we

$$x = 2 \pm \sqrt{6}$$

will apply the Square Root Theorem to get

Part (c) of the above example illustrates that if we can construct a perfect square binomial from a quadratic equation (i.e., get the equation in the form $(x+p)^2 = d$), then we can apply the Square Root Theorem and solve for x to get $x = -p \pm \sqrt{d}$.

The method of constructing a perfect square is called completing the square. It is based on the fact that in multiplying out the perfect square $(x+p)^2$, with p a constant, we get

$$(x+p)^2 = x^2 + 2px + p^2$$

Notice the relationship between the constant term, p^2 , and the coefficient of the middle term, $2p$: The constant term is the square of half the coefficient of the middle term.

Example 2.12: Solve by completing the square: $2x^2 - 8x + 4 = 6$.

Solution: $2x^2 - 8x + 4 = 6$ Divide both sides by 2, the coefficient of x^2

$x^2 - 4x + 2 = 3$ Isolate the constant term on the right-hand side

$x^2 - 4x = 1$ Take half the middle term coefficient, square it

$\left(\frac{1}{2}(-4)\right)^2 = 4$, we add 4 to both sides of the equation

$x^2 - 4x + 4 = 1 + 4$ Factor the left hand side
 $(x - 2)^2 = 5$ Solve for x using the Square Root Theorem

$$x = 2 \pm \sqrt{5}.$$

Unlike the factoring method, all quadratic equations can be solved by completing the square. If we were to complete the square for the general quadratic equation $Ax^2 + Bx + C = 0$, $A \neq 0$, we would arrive at the formula given below.

The Quadratic Formula: If $Ax^2 + Bx + C = 0$ and $A \neq 0$, then

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Example 2.13: Solve the following using the quadratic formula: $x^2 - 8 = -6x$.

Solution: Writing the equation in standard form we get, $x^2 + 6x - 8 = 0$. By the quadratic formula we have:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-8)}}{2(1)} = \frac{-6 \pm \sqrt{68}}{2} = \frac{-6 \pm 2\sqrt{17}}{2} = -3 \pm \sqrt{17}$$

Thus, the solution set is $[-3 - \sqrt{17}, -3 + \sqrt{17}]$.

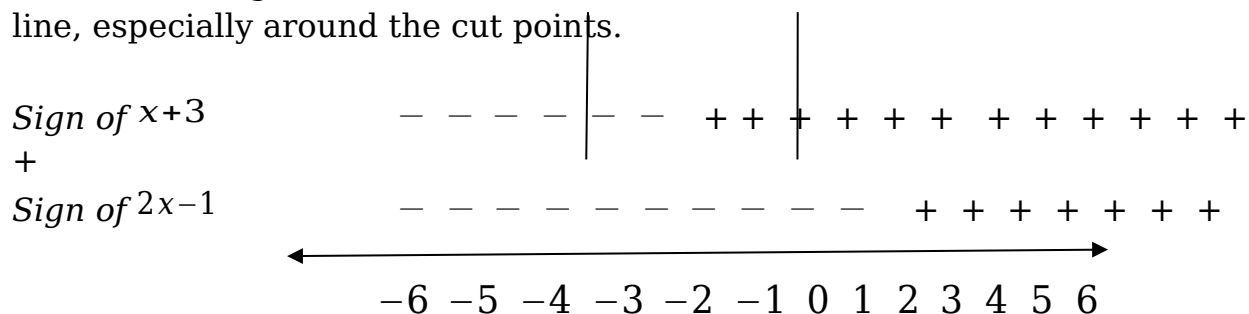
A **quadratic inequality** is in standard form if it is in the form $Ax^2 + Bx + C < 0$. (We can replace $<$ with $>$, $\leq$, or $\geq$.)

If we keep in mind that $u > 0$ means u is positive, then solving an inequality

such as $2x^2+5x-3>0$ means we are interested in finding the values of x that will make $2x^2+5x-3$ positive. Or, since $2x^2+5x-3=(2x-1)(x+3)$, we are looking for values of x that make $(2x-1)(x+3)$ positive. For $(2x-1)(x+3)$ to be positive, the factors must be either both positive or both negative. To determine when this happens, we first find the values of x for which $(2x-1)(x+3)$ is equal to 0; we call these the cut points of $(2x-1)(x+3)$. The cut points are $\frac{1}{2}$ and -3 .

Thus, our approach in solving quadratic inequalities will be primarily algebraic. After putting the inequality in standard form, we will determine the sign of each factor of the expression for various values of x . Then, we determine the solution by examining the sign of the product. This process is called a **sign analysis**.

Returning to the problem $2x^2+5x-3>0$, we draw a number line and examine the sign of each factor as x takes on various values on the number line, especially around the cut points.



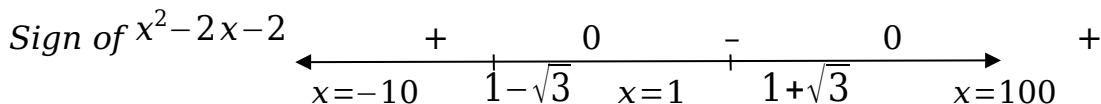
The above figure illustrates that the factor $x+3$ is negative when $x<-3$ and positive when $x>-3$. It is also shown that $2x-1$ is negative when $x<\frac{1}{2}$ and positive when $x>\frac{1}{2}$. Thus the product of the two factors is positive when $x<-3$ and $x>\frac{1}{2}$. Therefore, the solution set is $(-\infty, -3) \cup (\frac{1}{2}, \infty)$.

Remark: 1. The cut points of the inequalities will break up the number line into intervals.

2. The sign of the product does not change within an interval, i.e., if the expression is positive (or negative) for one value within the interval, it is positive (or negative) for all values within the interval.

Example 2.14: Solve the quadratic inequality $x^2-2x-2<0$.

Solution: Since we cannot factor $x^2 - 2x - 2$, we use the quadratic formula to find that its roots are $1 \pm \sqrt{3}$. This gives the cut points for the polynomial $x^2 - 2x - 2$. We use the sign analysis (see the figure below) with the test points given. Note: $1 + \sqrt{3} \approx 2.7$ and $1 - \sqrt{3} \approx -0.7$.



Substituting the test values -10 , 1 , and 100 for x in the expression $x^2 - 2x - 2$, we find that $x^2 - 2x - 2$ is negative only when x is in the interval $(1 - \sqrt{3}, 1 + \sqrt{3})$.

Exercise 2.2

1. Solve the linear equations

a)	$2 - 3(x - 4) = 2(x - 1)$	d)	$\frac{2}{x+3} + 4 = \frac{5-x}{x+3}$
b)	$3x - [2 + 3(2 - x)] = 5 - (3 - x)$	e)	$\frac{6}{x^2 - 3x} = \frac{12}{x} + \frac{1}{x-3}$
c)	$\frac{3}{4}(2x - 3) = \frac{2}{3}x + 5$		

2. Solve the linear inequalities

a)	$4x + \frac{2}{3} \leq 2x - (3x + 1)$	b)	$5x - 2 > 3x - (x - \frac{1}{5})$	c)	$\frac{5x-2}{3} \geq \frac{x+3}{4}$
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3. A truck carries a load of 50 boxes; some are 20 kg boxes and the rest are 25 kg boxes. If the total weight of all boxes is 1175 kg, how many of each type are there?

4. The product of two numbers is 5. If their sum $\frac{9}{2}$, find the numbers.

5. Solve

a)	$2x^2 - 7x = 15$	c)	$x^2 + 2x - 4 = 0$	e)	$3x^2 - 6x + 5 = 0$
b)	$x - 3 = \frac{1}{x+3}$	d)	$\frac{1}{x-5} + \frac{3}{x+2} = 4$		

6. Solve the quadratic inequalities

- a) $x^2 + 2x - 24 > 0$ d) $2x^2 - x - 2 \geq 0$
 b) $x^2 - 5x \leq 24$ e) $x^2 \leq 16$
 c) $x^2 - 3x - 3 < 0$

7. A student was given the inequality: $\frac{3}{x-2} > 4$. The first step the student took in solving this inequality was to transform it into $3 > 4(x-2)$. Explain what the student did wrong.

2.3. Review of relations and functions

After completing this section, the student should be able to:

- define Cartesian product of two sets
- understand the notion of relation and function
- know the difference between relation and function
- determine the domain and range of relations and functions
- find the inverse of a relation

The student is familiar with the phrase ordered pair. In the ordered pair $(2,3)$, $(-2,4)$ and (a,b) ; 2 , -2 and a are the first coordinates while 3 , 4 and b are the second coordinates.

• Cartesian Product

Given sets $A = \{3, 4\}$ and $B = \{4, 5, 9\}$. Then, the set $\{(3,4), (3,5), (3,9), (4,4), (4,5), (4,9)\}$ is the Cartesian product of A and B , and it is denoted by $A \times B$.

Definition 2.3: Suppose A and B are sets. The Cartesian product of A and B , denoted by $A \times B$, is the set which contains every ordered pair whose first coordinate is an element of A and second coordinate is an

element of B , i.e.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 2.15: For $A = \{2, 4\}$ and $B = \{-1, 3\}$, we have

$$\begin{aligned} A \times B &= \{(2, -1), (2, 3), (4, -1), (4, 3)\} \\ \text{a) } & \quad \quad \quad , \text{ and} \\ B \times A &= \{(-1, 2), (-1, 4), (3, 2), (3, 4)\} \\ \text{b) } & \quad \quad \quad . \end{aligned}$$

From this example, we can see that $A \times B$ and $B \times A$ are not equal. Recall that two sets are equal if one is a subset of the other and vice versa. To check equality of Cartesian products we need to define equality of ordered pairs.

Definition 2.4: (Equality of ordered Pairs)

Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Example 2.16: Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Then,

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}.$$

Definition 2.5: (Relation)

If A and B are sets, any subset of $A \times B$ is called a relation from A into B .

Suppose R is a relation from a set A to a set B . Then, $R \subseteq A \times B$ and hence for each $(a, b) \in A \times B$, we have either $(a, b) \in R$ or $(a, b) \notin R$. If $(a, b) \in R$, we say “ a is R -related (or simply related) to b ”, and write aRb . If $(a, b) \notin R$, we say that “ a is not related to b ”.

In particular if R is a relation from a set A to itself, then we say that R is a relation on A .

Example 2.17:

$$\begin{aligned} & A = \{1, 3, 5, 7\} \quad B = \{6, 8\} \\ 1. \text{ Let } & \quad \quad \text{and } \quad \quad . \text{ Let } R \text{ be the relation “less than” from } A \\ & B \quad R = \{(1, 6), (1, 8), (3, 6), (3, 8), (5, 6), (5, 8), (7, 8)\} \\ \text{to } & \quad . \text{ Then, } \end{aligned}$$

2. Let $A=\{1,2,3,4,5\}$ and $B=\{a,b,c\}$.

a) The following are relations from A into B ;

$$R_1=\{(1,a)\}$$

i)

$$R_2=\{(2,b),(3,b),(4,c),(5,a)\}$$

ii)

$$R_3=\{(1,a),((2,b),(3,c)\}$$

iii)

b) The following are relations from B to A ;

$$R_4=\{(a,3),(b,1)\}$$

i)

$$R_5=\{(b,2),(c,4),(a,2),(b,3)\}$$

ii)

$$R_6=\{(b,5)\}$$

iii)

Definition 2.6: Let R be a relation from A into B . Then,

a) the domain of R , denoted by $Dom(R)$, is the set of first coordinates of the elements of R , i.e

$$Dom(R)=\{a \in A : (a,b) \in R\}$$

b) the range of R , denoted by $Range(R)$, is the set of second coordinates of elements of R , i.e

$$Range(R)=\{b \in B : (a,b) \in R\}$$

Remark: If R is a relation from the set A to the set B , then the set B is called the codomain of the relation R . The range of relation is always a subset of the codomain.

Example 2.18:

1. The set $R=\{(4,7),(5,8),(6,10)\}$ is a relation from the set $A=\{1,2,3,4,5,6\}$ to the set $B=\{7,8,9,10\}$. The domain of R is $\{4,5,6\}$, the range of R is $\{7,8,10\}$ and the codomain of R is $\{6,7,8,9,10\}$.

2. The set of ordered pairs $R = \{(8,2), (6,-3), (5,7), (5,-3)\}$ is a relation between the sets $\{5,6,8\}$ and $\{2,-3,7\}$, where $\{5,6,7\}$ is the domain and $\{2,-3,7\}$ is the range.

Remark:

- If $(a,b) \in R$ for a relation R , we say a is related to (or paired with) b .
Note that a may also be paired with an element different from b . In any case, b is called the image of a while a is called the pre-image of b .
- If the domain and/or range of a relation is infinite, we cannot list each element assignment, so instead we use set builder notation to describe the relation. The situation we will encounter most frequently is that of a relation defined by an equation or formula. For example,

$$R = \{(x,y) : y = 2x - 3, x, y \in \mathbb{R}\}$$
is a relation for which the range value is 3 less than twice the domain value. Hence, $(0,-3), (0.5,-2)$ and $(-2,-7)$ are examples of ordered pairs that are of the assignment.

Example 2.19:

1. Let $A = \{1, 2, 3, 4, 6\}$
- Let R be the relation on A defined by $R = \{(a,b) : a \text{ is a factor of } b\}$. Find the domain and range of R .

Solution: We have

$$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\}.$$

$$\text{Then, } \text{Dom}(R) = \{1, 2, 3, 4, 6\} \text{ and } \text{Range}(R) = \{1, 2, 3, 4, 6\}.$$

2. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, \dots, 67\}$.

Let $R = \{(x, y) : x \text{ is cube root of } y\}$. Find a) R b) $Dom(R)$ c) $Range(R)$

Solution: We have $1 = \sqrt[3]{1}, 2 = \sqrt[3]{8}, 3 = \sqrt[3]{27}, 4 = \sqrt[3]{64}, 5 = \sqrt[3]{125}$ and 1, 8, 27 and 64 are in B whereas 125 is not in B . Thus, $R = \{(1, 1), (2, 8), (3, 27), (4, 64)\}$, $Dom(R) = \{1, 2, 3, 4\}$ and $Range(R) = \{1, 8, 27, 64\}$.

Remark:

- A relation R on a set A is called $R = A \times A$
 - a universal relation if $R = \{(a, a) : a \in A\}$
 - identity relation if $R = \emptyset$
 - void or empty relation if
- If R is a relation from A to B , then the inverse relation of R , denoted by R^{-1} , is a relation from B to A and is defined as:

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$

Observe that $Dom(R) = Range(R^{-1})$ and $Range(R) = Dom(R^{-1})$. For instance, if $R = \{(1, 4), (9, 15), (10, 2)\}$ is a relation on a set $A = \{1, 2, 3, \dots, 20\}$, then $R^{-1} = \{(4, 1), (15, 9), (2, 10)\}$

Example 2.20: Let R be a relation defined on IN by $R = \{(a, b) : a, b \in IN, a + 2b = 11\}$.

Find a) R b) $Dom(R)$ c) $Range(R)$ d) R^{-1}

Solution: The smallest natural number is 1.

$$b=1 \Rightarrow a+2(1)=11 \Rightarrow a=9$$

$$b=2 \Rightarrow a+2(2)=11 \Rightarrow a=7$$

$$b=3 \Rightarrow a+2(3)=11 \Rightarrow a=5$$

$$b=4 \Rightarrow a+2(4)=11 \Rightarrow a=3$$

$$b=5 \Rightarrow a+2(5)=11 \Rightarrow a=1$$

$$b=6 \Rightarrow a+2(6)=11 \Rightarrow a=-1 \notin \mathbb{N}$$

Therefore, $R=\{(9,1),(7,2),(5,3),(3,4),(1,5)\}$, $\text{Dom}(R)=\{1,3,5,7,9\}$, $\text{Range}(R)=\{1,2,3,4,5\}$
and $R^{-1}=\{(1,9),(2,7),(3,5),(4,3),(5,1)\}$.

• Functions

Mathematically, it is important for us to distinguish among the relations that assign a unique range element to each domain element and those that do not.

Definition 2.7: (Function)

A function is a relation in which each element of the domain corresponds to exactly one element of the range.

Example 2.21: Determine whether the following relations are functions.

$$R=\{(5,-2),(3,5),(3,7)\}$$

a)

$$\text{b) } \{(2,4),(3,4),(6,-4)\}$$

Solution:

a) Since the domain element 3 is assigned to two different values in the range, 5 and 7, it is not a function.

$$\{2,3,6\}$$

b) Each element in the domain, $\{2,3,6\}$, is assigned no more than one value in the range, 2 is assigned only 4, 3 is assigned only 4, and 6 is assigned only -4. Therefore, it is a function.

Remark: Map or mapping, transformation and correspondence are synonyms for the word function. If f is a function and $(x,y) \in f$, we say x is mapped to y .

Definition 2.8: A relation f from A into B is called a function from A into B , denoted by

$$f: A \rightarrow B \quad \text{or} \quad A \xrightarrow{f} B$$

if and only if

$$(i) \quad \text{Dom}(f)=A$$

- (ii) No element of A is mapped by f to more than one element in B ,
i.e. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Remark: 1. If to the element x of A corresponds $y (\in B)$ under the function f , then we write $f(x) = y$ and y is called the image of x under f and x is called a pre-image of y under f .

2. The symbol $f(x)$ is read as “ f of x ” but not “ f times x ”.

3. In order to show that a relation f from A into B is a function, we first show that the domain of f is A and next we show that f is well defined or single-valued, i.e. if $x = y$ in A , then $f(x) = f(y)$ in B for all $x, y \in A$.

Example 2.22:

$$A = \{1, 2, 3, 4\} \quad B = \{1, 6, 8, 11, 15\}$$

1. Let f from A to B and g from A to B . Which of the following are functions

from A to B .

a) f defined by $f(1) = 1, f(2) = 6, f(3) = 8, f(4) = 8$

b) f defined by $f(1) = 1, f(2) = 6, f(3) = 15$

c) f defined by $f(1) = 6, f(2) = 6, f(3) = 6, f(4) = 6$

d) f defined by $f(1) = 1, f(2) = 6, f(2) = 8, f(3) = 8, f(4) = 11$

e) f defined by $f(1) = 1, f(2) = 8, f(3) = 11, f(4) = 15$

Solution:

a) f is a function because to each element of A there corresponds exactly one element of B .

b) f is not a function because there is no element of B which correspond to 4 ($\in A$).

c) f is a function because to each element of A there corresponds exactly one element

of B . In the given function, the images of all element of A are the same.

d) f is not a function because there are two elements of B which are corresponding to 2.

In other words, the image of 2 is not unique.

e) f is a function because to each element of A there corresponds exactly one element of B .

As with relations, we can describe a function with an equation. For example, $y=2x+1$ is a function, since each x will produce only one y .

2. Let $f = \{(x, y) : y = x^2\}$. Then, f maps:

1 to 1	-1 to 1
2 to 4	-2 to 4
3 to 9	-3 to 9

More generally any real number x is mapped to its square. As the square of a number is unique, f maps every real number to a unique number. Thus, f is a function from $\mathbb{R}$ into $\mathbb{R}$.

We will find it useful to use the following vocabulary: The independent variable refers to the variable representing possible values in the domain, and the dependent variable refers to the variable representing possible values in the range. Thus, in our usual ordered pair notation (x, y) , x is the independent variable and y is the dependent variable.

- **Domain, Codomain and range of a function**

For the function $f: A \rightarrow B$

- (i) The set A is called the domain of f
- (ii) The set B is called the codomain of f
- (iii) The set $\{f(x): x \in A\}$ of all image of elements of A is called the range of f

Example 2.23:

- Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, \dots, 10\}$. Let $f: A \rightarrow B$ be the correspondence which assigns to each element in A , its square. Thus, we have $f(1) = 1, f(2) = 4, f(3) = 9$. Therefore, f is a function and $Dom(f) = \{1, 2, 3\}$, $Range(f) = \{1, 4, 9\}$ and codomain of f is $\{1, 2, 3, \dots, 10\}$.

- Let $A = \{2, 4, 6, 7, 9\}$, $B = IN$. Let x and y represent the elements in the sets A and B , respectively. Let $f: A \rightarrow B$ be a function defined by $f(x) = 15x + 17, x \in A$.

The variable x can take values 2, 4, 6, 7, 9. Thus, we have

$$f(2) = 15(2) + 17 = 47, f(4) = 77, f(6) = 107, f(7) = 122, f(9) = 152$$

This implies that $Dom(f) = \{2, 4, 6, 7, 9\}$, $Range(f) = \{47, 77, 107, 122, 152\}$ and codomain of f is IN .

- Let f be the subset of $Q \times Z$ defined by $f = \left\{ \left(\frac{p}{q}, p \right) : p, q \in Z, q \neq 0 \right\}$. Is f a function?

Solution: First we note that $\text{Dom}(f)=Q$. Then, f satisfies condition (i) in the definition of a function. Now, $(\frac{2}{3}, 2) \in f$, $(\frac{4}{6}, 4) \in f$ and $\frac{2}{3} = \frac{4}{6}$ but $f(\frac{2}{3}) = 2 \neq 4 = f(\frac{4}{6})$. Thus f is not well defined. Hence, f is not a function from Q to Z .

4. Let f be the subset of $Z \times Z$ defined by $f = \{(mn, m+n) : m, n \in Z\}$. Is f a function?

Solution: First we show that f satisfies condition (i) in the definition. Let x be any element of Z . Then, $x = x \cdot 1$. Hence, $(x, x+1) = (x \cdot 1, x+1) \in f$. This implies that $x \in \text{Dom}(f)$. Thus, $Z \subseteq \text{Dom}(f)$. However, $\text{Dom}(f) \subseteq Z$ and so $\text{Dom}(f) = Z$. Now, $4 \in Z$ and $4 = 4 \cdot 1 = 2 \cdot 2$. Thus, $(4 \cdot 1, 4+1)$ and $(2 \cdot 2, 2+2)$ are in f . Hence we find that $4 \cdot 1 = 2 \cdot 2$ and $f(4 \cdot 1) = 5 \neq 4 = f(2 \cdot 2)$. This implies that f is not well defined, i.e., f does not satisfy condition (ii). Hence, f is not a function from Z to Z .

5. Determine whether the following equations determine y as a function of x , if so, find the domain.

$$\begin{array}{lll} \text{a)} & y = -3x + 5 & \text{b)} & y = \frac{2x}{3x-5} & \text{c)} & y^2 = x \end{array}$$

Solution:

- a) To determine whether $y = -3x + 5$ gives y as a function of x , we need to know whether each x -value uniquely determines a y -value.

Looking at the equation $y = -3x + 5$, we can see that once x is chosen we multiply it by -3 and then add 5 . Thus, for each x there is a unique y . Therefore, $y = -3x + 5$ is a function.

$$y = \frac{2x}{3x-5}$$

b) Looking at the equation carefully, we can see that each x -value uniquely determines a y -value (one x -value can not produce

$$y = \frac{2x}{3x-5}$$

two different y -values). Therefore, is a function.

As for its domain, we ask ourselves. Are there any values of x that

must be excluded? Since $y = \frac{2x}{3x-5}$ is a fractional expression, we must exclude any value of x that makes the denominator equal to zero. We must have

$$3x-5 \neq 0 \Leftrightarrow x \neq \frac{5}{3}$$

Therefore, the domain consists of all real numbers except for $\frac{5}{3}$.

Thus, $Dom(f) = \{x : x \neq \frac{5}{3}\}$.

$$y^2 = x$$

$$x = 9$$

$$y^2 = 9$$

c) For the equation , if we choose we get , which

$$y = \pm 3$$

gives . In other words, there are two values associated

$$x = 9$$

$$y^2 = x$$

with . Therefore, is not a function.

$$y = \sqrt{3x-x^2}$$

6. Find the domain of the function .

Solution: Since $\mathcal{Y}$ is defined and real when the expression under the radical is non-negative, we need x to satisfy the inequality

$$3x-x^2 \geq 0 \Leftrightarrow x(3-x) \geq 0$$

This is a quadratic inequality, which can be solved by analyzing signs:

$$\begin{array}{c} \text{Sign of } 3x-x^2 \\ \hline - \quad - \quad - \quad | \quad + \quad + \quad + \quad | \quad - \quad - \quad - \\ \quad \quad \quad 0 \quad \quad 3 \end{array}$$

Since we want $3x - x^2 = x(3 - x)$ to be non-negative, the sign analysis shows us that the domain is $\{x: 0 \leq x \leq 3\}$ or $[0, 3]$.

Exercise 2.3

1. Let R be a relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ defined by $R = \{(a, b) : a + b \leq 9\}$.

- i) List the elements of R
- ii) Is $R = R^{-1}$

2. Let R be a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ defined by $R = \{(a, b) : a - b \mid b\}$.
divides

- i) List the elements of R
- ii) Find $Dom(R) \wedge Range(R)$
- iii) Find the elements of R^{-1}
- iv) Find $Dom(R^{-1}) \wedge Range(R^{-1})$

3. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation on A by $R = \{(x, y) : y = x + 1\}$. Write down the domain, codomain and range of R . Find R^{-1} .

4. Find the domain and range of the relation $\{(x, y) : |x| + y \geq 2\}$.

5. Let $A = \{1, 2, 3\}$ and $B = \{3, 5, 6, 8\}$. Which of the following are functions from A to B ?

$$f = \{(1,3), (2,3), (3,3)\}$$

a)

$$f = \{(1,8), (2,5)\}$$

c)

$$f = \{(1,3), (2,5), (1,6)\}$$

b)

$$f = \{(1,6), (2,5), (3,3)\}$$

d)

6. Determine the domain and range of the given relation. Is the relation a function?

$$\{(-4,-3), (2,-5), (4,6), (2,0)\}$$

a)

$$\{(-\frac{1}{2}, \frac{1}{6}), (-1,1), (\frac{1}{3}, \frac{1}{8})\}$$

d)

$$\{(8,-2), (6, -\frac{3}{2}), (-1,5)\}$$

b)

$$\{(0,5), (1,5), (2,5), (3,5), (4,5), (5,5)\}$$

e)

$$\{(-\sqrt{3},3), (-1,1), (0,0), (1,1), (\sqrt{3},3)\}$$

c)

$$f) \quad \{(5,0), (5,1), (5,2), (5,3), (5,4),$$

$$(5,5)\}$$

7. Find the domain and range of the following functions.

$$f(x) = 1 + 8x - 2x^2$$

a)

$$f(x) = \sqrt{x^2 - 6x + 8}$$

c)

$$f(x) = \frac{1}{x^2 - 5x + 6}$$

b)

$$f(x) = \begin{cases} 3x+4, & -1 \leq x < 2 \end{cases}$$

d)

$$f(x) = \begin{cases} 3x-5, & x < 1 \end{cases}$$

8. Given

$$\text{Find } a) f(-3)$$

$$b) f(1)$$

$$c) f(6)$$

2.4 Real Valued functions and their properties

After completing this section, the student should be able to:

- perform the four fundamental operations on polynomials
- compose functions to get a new function
- determine the domain of the sum, difference, product and quotient of two functions
- define equality of two functions

Let f be a function from set A to set B . If B is a subset of real number system $\mathfrak{R}$, then f is called a real valued function, and in particular if A is also a subset of $\mathfrak{R}$, then $f: A \rightarrow B$ is called a real function.

Example 2.24: 1. The function $f: \mathfrak{R} \rightarrow \mathfrak{R}$ defined by $f(x) = x^2 + 3x + 7$, $x \in \mathfrak{R}$ is a real function.

2. The function $f: \mathfrak{R} \rightarrow \mathfrak{R}$ defined as $f(x) = |x|$ is also a real valued function.

• Operations on functions

Functions are not numbers. But just as two numbers a and b can be added to produce a new number $a+b$, so two functions f and g can be added to produce a new function $f+g$. This is just one of the several operations on functions that we will describe in this section.

Consider functions f and g with formulas $f(x) = \frac{x-3}{2}$, $g(x) = \sqrt{x}$. We can make a new function $f+g$ by having it assign to x the value $\frac{x-3}{2} + \sqrt{x}$, that is,

$$(f+g)(x) = f(x) + g(x) = \frac{x-3}{2} + \sqrt{x}.$$

Definition 2.9: Sum, Difference, Product and Quotient of two functions

Let $f(x)$ and $g(x)$ be two functions. We define the following four functions:

- | | |
|---|-------------------------------------|
| 1. $(f+g)(x) = f(x) + g(x)$ | The sum of the two functions |
| 2. $(f-g)(x) = f(x) - g(x)$ | The difference of the two functions |
| 3. $(f \cdot g)(x) = f(x)g(x)$ | The product of the two functions |
| 4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
(provided $g(x) \neq 0$) | The quotient of the two functions |

Since an x -value must be an input into both f and g , the domain of $(f+g)(x)$ is the set of all x common to the domain of f and g . This is usually written as $Dom(f+g)=Dom(f)\cap Dom(g)$. Similar statements hold for the domains of the difference and product of two functions. In the case of the quotient, we must impose the additional restriction that all elements in the domain of g for which $g(x)=0$ are excluded.

Example 2.25:

1. Let $f(x)=3x^2+2$ and $g(x)=5x-4$. Find each of the following and its domain

a) $(f+g)(x)$ b) $(f-g)(x)$ c) $(f \cdot g)(x)$ d) $\left(\frac{f}{g}\right)(x)$

Solution:

a) $(f+g)(x)=f(x)+g(x)=(3x^2+2)+(5x-4)=3x^2+5x-2$

b) $(f-g)(x)=f(x)-g(x)=(3x^2+2)-(5x-4)=3x^2-5x+6$

c) $(f \cdot g)(x)=(3x^2+2)(5x-4)=15x^3-12x^2+10x-8$

d) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{3x^2+2}{5x-4}$

We have

$$Dom(f+g)=Dom(f-g)=Dom(fg)=Dom(f)\cap Dom(g)=\mathbb{R}\cap\mathbb{R}=\mathbb{R}$$

$$Dom\left(\frac{f}{g}\right)=[Dom(f)\cap Dom(g)]\setminus\{x:g(x)=0\}=\mathbb{R}\setminus\left\{\frac{5}{4}\right\}$$

2. Let $f(x)=\sqrt[4]{x+1}$ and $g(x)=\sqrt{9-x^2}$, with respective domains $[-1,\infty)$ and $[-3,3]$. Find formulas for $f+g$, $f-g$, $f \cdot g$, $\frac{f}{g}$ and f^3 and give their domains.

Solution:

Formula	Domain
$(f+g)(x) = f(x) + g(x) = \sqrt[4]{x+1} + \sqrt{9-x^2}$	$[-1, 3]$
$(f-g)(x) = f(x) - g(x) = \sqrt[4]{x+1} - \sqrt{9-x^2}$	$[-1, 3]$
$(f \cdot g)(x) = f(x) \cdot g(x) = \sqrt[4]{x+1} \cdot \sqrt{9-x^2}$	$[-1, 3]$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt[4]{x+1}}{\sqrt{9-x^2}}$	$[-1, 3]$
$f^3(x) = (f(x))^3 = (\sqrt[4]{x+1})^3 = (x+1)^{\frac{3}{4}}$	$[-1, \infty)$

There is yet another way of producing a new function from two given functions.

Definition 2.10: (Composition of functions)

Given two functions $f(x)$ and $g(x)$, the composition of the two functions is denoted by $f \circ g$ and is defined by:

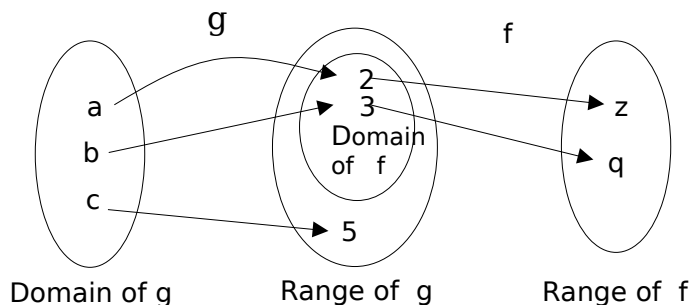
$$(f \circ g)(x) = f[g(x)].$$

$(f \circ g)(x)$ is read as f composed with g of x . The domain of $f \circ g$ consists of those x 's in the domain of g whose range values are in the domain of f , i.e. those x 's for which $g(x)$ is in the domain of f .

Example 2.26:

1. Suppose $f = \{(2, z), (3, q)\}$ and $g = \{(a, 2), (b, 3), (c, 5)\}$. The function $(f \circ g)(x) = f(g(x))$ is found by taking elements in the domain of g and evaluating as follows:
- $$(f \circ g)(a) = f(g(a)) = f(2) = z, \quad (f \circ g)(b) = f(g(b)) = f(3) = q$$

If we attempt to find $f(g(c))$ we get $f(5)$, but 5 is not in the domain of $f(x)$ and so we cannot find $(f \circ g)(c)$. Hence, $f \circ g = \{(a, z), (b, q)\}$. The figure below illustrates this situation.



2. Given $f(x) = 5x^2 - 3x + 2$ and $g(x) = 4x + 3$, find
- a) $(f \circ g)(-2)$ b) $(g \circ f)(2)$ c) $(f \circ g)(x)$ d) $(g \circ f)(x)$

Solution:

- a) $(f \circ g)(-2) = f(g(-2))$ $g(-2) = 4(-2) + 3 = -5$
 First evaluate
 $= f(-5)$
 $= 5(-5)^2 - 3(-5) + 2 = 142$
- b) $(g \circ f)(2) = g(f(2))$ $f(2) = 5(2)^2 - 3(2) + 2 = 16$
 First evaluate
 $= g(16)$
 $= 4(16) + 3 = 67$
- c) $(f \circ g)(x) = f(g(x))$ $g(x) = 4x + 3$
 But
 $= f(4x + 3)$
 $= 5(4x + 3)^2 - 3(4x + 3) + 2$
 $= 80x^2 + 108x + 38$
- d) $(g \circ f)(x) = g(f(x))$ $f(x) = 5x^2 - 3x + 2$
 But
 $= g(5x^2 - 3x + 2)$
 $= 4(5x^2 - 3x + 2) + 3$
 $= 20x^2 - 12x + 11$

3. Given $f(x) = \frac{x}{x+1}$ and $g(x) = \frac{2}{x-1}$, find
- a) $(f \circ g)(x)$ and its domain b) $(g \circ f)(x)$ and its domain

$$(f \circ g)(x) = f\left(\frac{2}{x-1}\right) = \frac{\frac{2}{x-1}}{\frac{2}{x-1} + 1} = \frac{2}{x+1}$$

Solution: a) . Thus, $Dom(f \circ g) = \{x : x \neq \pm 1\}$.

$$(g \circ f)(x) = g(f(x)) = \frac{2}{\frac{x}{x+1} - 1} = -2x - 2$$

b) . Since x must first be an input

into $f(x)$ and so must be in the domain of f , we see that $Dom(g \circ f) = \{x : x \neq -1\}$.

4. Let $f(x) = \frac{6x}{x^2-9}$ and $g(x) = \sqrt{3x}$. Find $(f \circ g)(12)$ and $(g \circ f)(x)$ and its domain.

Solution: We have $(f \circ g)(12) = f(g(12)) = f(\sqrt{36}) = f(6) = \frac{36}{27} = \frac{4}{3}$.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{3x}) = \frac{6\sqrt{3x}}{(\sqrt{3x})^2 - 9} = \frac{6\sqrt{3x}}{3x - 9} = \frac{2\sqrt{3x}}{x - 3}$$

The domain of $f \circ g$ is $[0, 3) \cup (3, \infty)$.

We now explore the meaning of equality of two functions. Let $f: A \rightarrow B$ and $g: A \rightarrow B$ be two functions. Then, f and g are subsets of $A \times B$. Suppose $f = g$. Let x be any element of A . Then, $(x, f(x)) \in f = g$ and thus $(x, f(x)) \in g$. Since g is a function and $(x, f(x)), (x, g(x)) \in g$, we must have $f(x) = g(x)$. Conversely, assume that $g(x) = f(x)$ for all $x \in A$. Let $(x, y) \in f$. Then, $y = f(x) = g(x)$. Thus, $(x, y) \in g$, which implies that $f \subseteq g$. Similarly, we can show that $g \subseteq f$. It now follows that $f = g$. Thus two functions $f: A \rightarrow B$ and $g: A \rightarrow B$ are equal if and only if $f(x) = g(x)$ for all $x \in A$. In general we have the following definition.

Definition 2.11: (Equality of functions)

Two functions are said to be equal if and only if the following two conditions hold:

- i) The functions have the same domain;
- ii) Their functional values are equal at each element of the domain.

Example 2.27:

1. Let $f: \mathbb{Z} \rightarrow \mathbb{Z}^+ \cup \{0\}$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}^+ \cup \{0\}$ be defined by $f = \{(n, n^2) : n \in \mathbb{Z}\}$ and $g = \{(n, |n|^2) : n \in \mathbb{Z}\}$. Now, for all $n \in \mathbb{Z}$, $f(n) = n^2 = |n|^2 = g(n)$. Thus, $f = g$.

2. Let $f(x) = \frac{x^2 - 25}{x - 5}$, $x \in \mathbb{R} \setminus \{5\}$, and $g(x) = x + 5$, $x \in \mathbb{R}$. The function f and g are not equal because $\text{Dom}(f) \neq \text{Dom}(g)$.

Exercise 2.4

1. For $f(x) = x^2 + x$ and $g(x) = \frac{2}{x+3}$, find each value:
- $(f - g)(2)$
 - $\left(\frac{f}{g}\right)(1)$
 - $g^2(3)$
 - $(f \circ g)(1)$
 - $(g \circ f)(1)$
 - $(g \circ g)(3)$
2. If $f(x) = x^3 + 2$ and $g(x) = \frac{2}{x-1}$, find a formula for each of the following and state its domain.
- $(f + g)(x)$
 - $(f \circ g)(x)$
 - $\left(\frac{g}{f}\right)(x)$
 - $(g \circ f)(x)$
3. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$.
- Find $(f \circ g)(x)$ and its domain.
 - Find $(g \circ f)(x)$ and its domain.
 - Are $(f \circ g)(x)$ and $(g \circ f)(x)$ the same functions? Explain.

4. Let $f(x)=5x-3$. Find $g(x)$ so that $(f \circ g)(x)=2x+7$.
5. Let $f(x)=2x+1$. Find $g(x)$ so that $(f \circ g)(x)=3x-1$.
6. If f is a real function defined by $f(x)=\frac{x-1}{x+1}$. Show that $f(2x)=\frac{3f(x)+1}{f(x)+3}$.
7. Find two functions f and g so that the given function $h(x)=(f \circ g)(x)$, where
- a) $h(x)=(x+3)^3$ c) $h(x)=\frac{1}{x}+6$
- b) $h(x)=\sqrt{5x-3}$ d) $h(x)=\frac{1}{x+6}$
8. Let $f(x)=4x-3, g(x)=\frac{1}{x}$ and $h(x)=x^2-x$. Find
- a) $f(5x+7)$ c) $f(g(h(3)))$ e) $f(x+a)$
- b) $5f(x)+7$ d) $f(1) \cdot g(2) \cdot h(3)$ f) $f(x)+a$

2.5 Types of functions

After completing this section, the student should be able to:

- define one to oneness and ontoness of a function
- check invertibility of a function
- find the inverse of an invertible function

In this section we shall study some important types of functions.

• One to One functions

Definition 2.12: A function $f:A \rightarrow B$ is called **one to one**, often written $1-1$, if and only if for all $x_1, x_2 \in A$, $f(x_1)=f(x_2)$ implies $x_1=x_2$. In words, no two elements of A are mapped to one element of B .

Example 2.28:

1. If we consider the sets $A = \{1, 2, 3, \dots, 6\}$ and $B = \{7, a, b, c, d, 8, e\}$ and if $f = \{(2, a), (3, b), (4, b), (5, c), (6, 8)\}$ and $g = \{(1, 7), (2, a), (3, b), (4, c), (5, 8), (6, d)\}$, then both f and g are functions from A into B . Observe that f is not a 1 - 1 function because $f(3) = f(4)$ but $3 \neq 4$. However, g is a 1 - 1 function.

2. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 7, 8\}$. Consider the functions
- i) $f: A \rightarrow B$ defined as $f(1) = 1, f(2) = 4, f(3) = 4, f(4) = 8$
- ii) $g: A \rightarrow B$ defined as $f(1) = 4, f(2) = 7, f(3) = 1, f(4) = 8$

Then, f is not 1 - 1, but g is a 1 - 1 function.

• **Onto functions**

Definition 2.13: Let f be a function from a set A into a set B . Then f is called an **onto function (or f maps onto B)** if every element of B is image of some element in A , i.e, $Range(f) = B$.

Example 2.29:

1. Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 5\}$. The function $f: A \rightarrow B$ defined as $f(1) = 1, f(2) = 5, f(3) = 1$, is not onto because there is no element in B , whose image under f is 4. The function $g: A \rightarrow B$ given by $g = \{(1, 4), (2, 5), (3, 1)\}$ is onto because each element of B is the image of at least one element of A .

Note that if A is a non-empty set, the function $i_A: A \rightarrow A$ defined by $i_A(x) = x$ for all $x \in A$ is a 1 - 1 function from A onto A . i_A is called the **identity map** on A .

2. Consider the relation f from $\mathbb{Z}$ into $\mathbb{Z}$ defined by $f(n) = n^2$ for all $n \in \mathbb{Z}$. Now, domain of f is $\mathbb{Z}$. Also, if $n = n'$, then $n^2 = (n')^2$, $f(n) = f(n')$, i.e. $f(1) = 1 = f(-1)$. Hence, f is well defined and a function. However, $1 \neq -1$, which implies that f is not 1 - 1. For all $n \in \mathbb{Z}$, $f(n)$ is a non-negative integer. This shows that a negative integer has no preimage. Hence, f is not onto. Note that f is onto $\{0, 1, 4, 9, \dots\}$.
3. Consider the relation f from $\mathbb{Z}$ into $\mathbb{Z}$ defined by $f(n) = 2n$ for all $n \in \mathbb{Z}$. As in the previous example, we can show that f is a function. Let $n, n' \in \mathbb{Z}$ and suppose that $f(n) = f(n')$. Then $2n = 2n'$ and thus $n = n'$. Hence, f is 1 - 1. Since for all $n \in \mathbb{Z}$, $f(n)$ is an even integer; we see that an odd integer has no preimage. Therefore, f is not onto.

• 1 - 1 Correspondence

Definition 2.14: A function $f: A \rightarrow B$ is said to be a 1 - 1 correspondence if f is both 1 - 1 and onto.

Example 2.30:

1. Let $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{0, 5, 10, 15, 20, 25\}$. Suppose $f: A \rightarrow B$ given by $f(x) = 5x$ for all $x \in A$. One can easily see that every element of B has a preimage in A and hence f is onto. Moreover, if $f(x) = f(y)$, then

$5x=5y$, i.e. $x=y$. Hence, f is 1 - 1. Therefore, f is a 1 - 1 correspondence between A and B .

2. Let A be a finite set. If $f: A \rightarrow A$ is onto, then it is one to one.

Solution: Let $A = \{a_1, a_2, \dots, a_n\}$. Then $\text{Range}(f) = \{f(a_1), f(a_2), \dots, f(a_n)\}$. Since f is onto we have $\text{Range}(f) = A$. Thus, $A = \{f(a_1), f(a_2), \dots, f(a_n)\}$, which implies that $f(a_1), f(a_2), \dots, f(a_n)$ are all distinct. Hence, $a_i \neq a_j$ implies $f(a_i) \neq f(a_j)$ for all $1 \leq i, j \leq n$. Therefore, f is 1 - 1.

• Inverse of a function

Since a function is a relation, the inverse of a function f is denoted by f^{-1} and is defined by:

$$f^{-1} = \{(y, x) : (x, y) \in f\}$$

For instance, if $f = \{(2, 4), (3, 6), (1, 7)\}$, then $f^{-1} = \{(4, 2), (6, 3), (7, 1)\}$. Note that the inverse of a function is not always a function. To see this consider the function $f = \{(5, 4), (6, 3), (4, 5)\}$. Then, $f^{-1} = \{(4, 2), (6, 3), (4, 5)\}$, which is not a function.

As we have seen above not all functions have an inverse, so it is important to determine whether or not a function has an inverse before we try to find the inverse. If the function does not have an inverse, then we need to realize that it does not have an inverse so that we do not waste our time trying to find something that does not exist.

A one to one function is special because only one to one functions have inverse. If a function is one to one, to find the inverse we will follow the steps below:

1. Interchange x and y in the equation $y = f(x)$
2. Solving the resulting equation for y , we will obtain the inverse function.

Note that the domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original function.

Example 2.31:

1. Given $y = f(x) = x^3$. Find f^{-1} and its domain.

Solution: We begin by interchanging x and y , and we solve for y .

$$\begin{array}{ll} y = x^3 & \text{Interchange } x \text{ and } y \\ x = y^3 & \text{Take the cube root of both sides} \\ \sqrt[3]{x} = y & \text{This is the inverse of the function} \end{array}$$

Thus, $f^{-1}(x) = \sqrt[3]{x}$. The domain of f^{-1} is the set of all real numbers.

2. Let $y = f(x) = \frac{x}{x+2}$. Find $f^{-1}(x)$.

Solution: Again we begin by interchanging x and y , and then we solve for y .

$$y = \frac{x}{x+2} \quad \text{Interchange } x \text{ and } y$$

$$x = \frac{y}{y+2} \quad \text{Solving for } y$$

$$x(y+2) = y \Leftrightarrow xy + 2x = y \Leftrightarrow 2x = y(1-x) \Leftrightarrow y = \frac{2x}{1-x}$$

Thus, $f^{-1}(x) = \frac{2x}{1-x}$.

Remark: Even though, in general, we use an exponent of -1 to indicate a reciprocal, inverse function notation is an exception to this rule. Please be aware that $f^{-1}(x)$ is not the reciprocal of f . That is,

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

If we want to write the reciprocal of the function $f(x)$ by using a negative exponent, we must write

$$\frac{1}{f(x)} = [f(x)]^{-1}.$$

Exercise 2.5

1. Consider the function $f = \{(x, x^2) : x \in S\}$ from $S = \{-3, -2, -1, 0, 1, 2, 3\}$ into $\mathbb{Z}$.
Is f one to one? Is it onto?

2. Let $A = \{1, 2, 3\}$. List all one to one functions from A onto A .

3. Let $f: A \rightarrow B$. Let f^{-1} be the inverse relation, i.e.
 $f^{-1} = \{(y, x) \in B \times A : f(x) = y\}$.

a) Show by an example that f^{-1} need not be a function.

b) Show that f^{-1} is a function from $\text{Range}(f)$ into A if and only if f is 1 - 1.

c) Show that f^{-1} is a function from B into A if and only if f is 1 - 1 and onto.

d) Show that if f^{-1} is a function from B into A , then $f^{-1} = f^{-1}$.

4. Let $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and $B = \{x \in \mathbb{R} : 5 \leq x \leq 8\}$. Show that $f: A \rightarrow B$ defined by $f(x) = 5 + (8 - 5)x$ is a 1 - 1 function from A onto B .

5. Which of the following functions are one to one?

a) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 4, x \in \mathbb{R}$

b) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 6x - 1, x \in \mathbb{R}$

c) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2 + 7, x \in \mathbb{R}$

d) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3, x \in \mathbb{R}$

e) $f: \mathbb{R} \setminus \{7\} \rightarrow \mathbb{R} \setminus \{7\}$ defined by $f(x) = \frac{2x+1}{x-7}, x \in \mathbb{R} \setminus \{7\}$

6. Which of the following functions are onto?

a) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 115x + 49, x \in \mathbb{R}$

b) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = |x|, x \in \mathbb{R}$

c) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x^2}, x \in \mathbb{R}$

d) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2 + 4, x \in \mathbb{R}$

7. Find $f^{-1}(x)$ if

a) $f(x) = 7x - 6$

d) $f(x) = \frac{4-x}{3x}$

g) $f(x) = -(x+2)^2 - 1$

b) $f(x) = \frac{2x-9}{4}$

e) $f(x) = \frac{5x+3}{1-2x}$

h) $f(x) = \frac{2x}{1+x}$

c) $f(x) = 1 - \frac{3}{x}$

f) $f(x) = \sqrt[3]{x+1}$

2.6 Polynomials, zeros of polynomials, rational functions and their graphs

After completing this section, the student should be able to:

- define polynomial and rational functions
- apply the theorems on polynomials to find the zeros of polynomial functions
- use the division algorithm to find quotient and remainder
- apply theorems on polynomials to solve related problems
- sketch and analyze the graphs of rational functions

The functions described in this section frequently occur as mathematical models of real-life situations. For instance, in business the demand function gives the price per item, P , in terms of the number of items sold, x . Suppose a company finds that the price P (in Birr) for its model GC-5

calculator is related to the number of calculators sold, x (in millions), and is given by the demand function $p=80-x^2$.

The manufacturer's revenue is determined by multiplying the number of items sold (x) by the price per item (p). Thus, the revenue function is

$$R=xp=x(80-x^2)=80x-x^3$$

These demand and revenue functions are examples of polynomial functions. The major aim of this section is to better understand the significance of applied functions (such as this demand function). In order to do this, we need to analyze the domain, range, and behavior of such functions.

- **Polynomial functions**

Definition 2.15: A polynomial function is a function of the form

$$y=a_n x^n+a_{n-1} x^{n-1}+\cdots+a_1 x+a_0, \quad a_n \neq 0.$$

Each a_i is assumed to be a real number, and n is a non-negative integer, a_n is called the leading coefficient. Such a polynomial is said to be of degree n .

Remark:

1. The domain of a polynomial function is always the set of real numbers.
2. (Types of polynomials)
 - A polynomial of degree 1 is called a linear function.
 - A polynomial of degree 2 is called quadratic function.
 - A polynomial of degree 3 is called a cubic function.

$$\text{i.e } p(x)=a_3 x^3+a_2 x^2+a_1 x+a_0, \quad a_3 \neq 0.$$

Example 2.32: $p(x)=2x^2+1$, $q(x)=\sqrt{3}x^4+2x-\pi$ and $f(x)=2x^3$ are examples of polynomial functions.

- **Properties of polynomial functions**

1. The graph of a polynomial is a smooth unbroken curve. The word smooth means that the graph does not have any sharp corners as turning points.
2. If P is a polynomial of degree n , then it has at most n zeros. Thus, a quadratic polynomial has at most 2 zeros.
3. The graph of a polynomial function of degree n can have at most $n-1$

turning points. Thus, the graph of a polynomial of degree 5 can have at most 4 turning points.

4. The graph of a polynomial always exhibits the characteristic that as $|x|$ gets very large, $|y|$ gets very large.

- **Zeros of a polynomial**

The zeros of a polynomial function provide valuable information that can be helpful in sketching its graph. One can find the zeros by factorizing the polynomial. However, we have no general method for factorizing polynomials of degree greater than 2. In this subsection, we turn our attention to methods that will allow us to find zeros of higher degree polynomials. To do this, we first need to discuss about the division algorithm.

Division Algorithm
Let $p(x)$ and $d(x)$ be polynomials with $d(x) \neq 0$, and with the degree of $d(x)$ less than or equal to the degree of $p(x)$. Then there are polynomials $q(x)$ and $R(x)$ such that $\underbrace{p(x)}_{\text{dividend}} = \underbrace{d(x)}_{\text{divisor}} \cdot \underbrace{q(x)}_{\text{quotient}} + \underbrace{R(x)}_{\text{remainder}},$ where either $R(x) = 0$ or the degree of $R(x)$ is less than degree of $d(x)$.

Example 2.33: Divide $\frac{x^4 - 1}{x^4 + 2x}$.

Solution: Using long division we have

$$\begin{array}{r}
 x^2+2x \overline{) x^4+0x^3+0x^2+0x+1} \\
 \underline{-(x^4+2x^3)} \\
 -2x^3+0x^2 \\
 \underline{-(-2x^3-4x^2)} \\
 4x^2+0x \\
 \underline{-(4x^2+8x)} \\
 -8x-1
 \end{array}$$

This long division means $\underbrace{x^4-1}_{\text{dividend}} = \underbrace{(x^2+2x)}_{\text{divisor}} \cdot \underbrace{(x^2-2x+4)}_{\text{quotient}} + \underbrace{(-8x-1)}_{\text{remainder}} .$

With the aid of the division algorithm, we can derive two important theorems that will allow us to recognize the zeros of polynomials.

If we apply the division algorithm where the divisor, $d(x)$, is linear (that is of the form $x-r$), we get

$$p(x) = (x-r)q(x) + R$$

Note that since the divisor is of the first degree, the remainder R , must be a constant. If we now substitute $x=r$, into this equation, we get

$$P(r) = (r-r)q(r) + R = 0 \cdot q(r) + R$$

Therefore, $p(r) = R$.

The result we just proved is called the remainder theorem.

The Remainder Theorem

When a polynomial $p(x)$ of degree at least 1 is divided by $x-r$, then the remainder is $p(r)$.

Example 2.34: The remainder when $P(x) = x^3 - x^2 + 3x - 1$ is divided by $x-2$ is $p(2) = 9$.

As a consequence of the remainder theorem, if $x-r$ is a factor of $p(x)$, then the remainder must be 0. Conversely, if the remainder is 0, then $x-r$, is a factor of $p(x)$. This is known as the Factor Theorem.

The Factor Theorem

$x-r$ is a factor of $p(x)$ if and only if $p(r)=0$.

The next theorem, called location theorem, allows us to verify that a zero exists somewhere within an interval of numbers, and can also be used to zoom in closer on a value.

Location theorem

Let f be a polynomial function and a and b be real numbers such that $a < b$. If $f(a)f(b) < 0$, then there is at least one zero of f between a and b .

The Factor and Remainder theorems establish the intimate relationship between the factors of a polynomial $p(x)$ and its zeros. Recall that a polynomial of degree n can have at most n zeros.

Does every polynomial have a zero? Our answer depends on the number system in which we are working. If we restrict ourselves to the set of real number system, then we are already familiar with the fact that the polynomial $p(x)=x^2+1$ has no real zeros. However, this polynomial does have two zeros in the complex number system. (The zeros are i and $-i$). Carl Friedrich Gauss (1777-1855), in his doctoral dissertation, proved that within the complex number system, every polynomial of degree ≥ 1 has at least one zero. This fact is usually referred to as the Fundamental theorem of Algebra.

Fundamental Theorem of Algebra

If $p(x)$ is a polynomial of degree $n > 0$ whose coefficients are complex numbers, then $p(x)$ has at least one zero in the complex number system.

Note that since all real numbers are complex numbers, a polynomial with real coefficients also satisfies the Fundamental theorem of Algebra. As an immediate consequence of the Fundamental theorem of Algebra, we have

The linear Factorization Theorem

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 1$ and $a_n \neq 0$, then

$p(x) = a_n (x - r_1)(x - r_2) \cdots (x - r_n)$, where the r_i are complex numbers (possible real and not necessarily distinct).

From the linear factorization theorem, it follows that every polynomial of degree $n \geq 1$ has exactly n zeros in the complex number system, where a root of multiplicity k counted k times.

Example 2.35: Express each of the polynomials in the form described by the Linear Factorization Theorem. List each zero and its multiplicity.

a) $p(x) = x^3 - 6x^2 - 16x$

b) $q(x) = 3x^2 - 10x + 8$

c) $f(x) = 2x^4 + 8x^3 + 10x^2$

Solution:

a) We may factorize $p(x)$ as follows:

$$\begin{aligned} p(x) &= x^3 - 6x^2 - 16x = x(x^2 - 6x - 16) \\ &= x(x - 8)(x + 2) \\ &= x(x - 8)(x - (-2)) \end{aligned}$$

The zeros of $p(x)$ are 0, 8, and -2 each of multiplicity one.

b) We may factorize $q(x)$ as follows:

$$\begin{aligned} q(x) &= 3x^2 - 10x + 8 = (3x - 4)(x - 2) \\ &= 3\left(x - \frac{4}{3}\right)(x - 2) \end{aligned}$$

Thus, the zeros of $q(x)$ are $\frac{4}{3}$ and 2, each of multiplicity one.

c) We may factorize $f(x)$ as follows:

$$\begin{aligned} f(x) &= 2x^4 + 8x^3 + 10x^2 = 2x^2(x^2 + 4x + 5) \\ &= 2x^2(x - (-2 + i))(x - (-2 - i)) \end{aligned}$$

Thus, the zeros of $f(x)$ are 0 with multiplicity two and $-2 + i$ and $-2 - i$ each with multiplicity one.

Example 2.36:

1. Find a polynomial $p(x)$ with exactly the following zeros and multiplicity.

zeros	multiplicity
-1	3
2	4
5	2

Are there any other polynomials that give the same roots and multiplicity?

- Find a polynomial $f(x)$ having the zeros described in part (a) such that $f(1) = 32$.

Solution:

- Based on the Factor Theorem, we may write the polynomial as:

$$p(x) = (x - (-1))^3 (x - 2)^4 (x - 5)^2 = (x + 1)^3 (x - 2)^4 (x - 5)^2$$

which gives the required roots and multiplicities.

Any polynomial of the form $kp(x)$, where k is a non-zero constant will give the same roots and multiplicities.

- Based on part (1), we know that $f(x) = k(x + 1)^3 (x - 2)^4 (x - 5)^2$. Since we want $f(1) = 32$, we have

$$f(1) = k(1 + 1)^3 (1 - 2)^4 (1 - 5)^2$$

$$32 = k(8)(1)(16) \Rightarrow k = \frac{1}{4}$$

Thus, $f(x) = \frac{1}{4}(x + 1)^3 (x - 2)^4 (x - 5)^2$.

Our experience in using the quadratic formula on quadratic equations with real coefficients has shown us that complex roots always appear in conjugate pairs. For example, the roots of $x^2 - 2x + 5 = 0$ are $1 + 2i$ and $1 - 2i$. In fact, this property extends to all polynomial equations with real coefficients.

Conjugate Roots Theorem

Let $p(x)$ be a polynomial with real coefficients. If complex number $a + bi$ (where a and b are real numbers) is a zero of $p(x)$, then so is its conjugate $a - bi$.

Example 2.37: Let $r(x) = x^4 + 2x^3 - 9x^2 + 26x - 20$. Given that $1 - \sqrt{3}i$ is a zero, find the other zero of $r(x)$.

Solution: According to the Conjugate Roots Theorem, if $1-\sqrt{3}i$ is a zero, then its conjugate, $1+\sqrt{3}i$ must also be a zero. Therefore, $x-(1-\sqrt{3}i)$ and $x-(1+\sqrt{3}i)$ are both factors of $r(x)$, and so their product must be a factor of $r(x)$. That is, $[x-(1-\sqrt{3}i)][x-(1+\sqrt{3}i)]=x^2-2x+4$ is a factor of $r(x)$. Dividing $r(x)$ by x^2-2x+4 , we obtain

$$r(x)=(x^2-2x+4)(x^2+4x-5)=(x^2-2x+4)(x+5)(x-1).$$

Thus, the zeros of $r(x)$ are $1-\sqrt{3}i$, $1+\sqrt{3}i$, -5 and 1 .

The theorems we have discussed so far are called existence theorems because they ensure the existence of zeros and linear factors of polynomials. These theorems do not tell us how to find the zeros or the linear factors. The Linear Factorization Theorem guarantees that we can factor a polynomial of degree at least one into linear factors, but it does not tell us how.

We know from experience that if $p(x)$ happens to be a quadratic function, then we may find the zeros of $p(x)=Ax^2+Bx+C$ by using the quadratic formula to obtain the zeros

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

The rest of this subsection is devoted to developing some special methods for finding the zeros of a polynomial function.

As we have seen, even though we have no general techniques for factorizing polynomials of degree greater than 2, if we happen to know a root, say r , we can use long division to divide $p(x)$ by $x-r$ and obtain a quotient polynomial of lower degree. If we can get the quotient polynomial down to a quadratic, then we are able to determine all the roots. But how do we find a root to start the process? The following theorem can be most helpful.

The Rational Root Theorem

Suppose that $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $n\geq 1$, $a_n\neq 0$ is an n^{th} degree polynomial with integer coefficients. If $\frac{p}{q}$ is a rational root of

$f(x)=0$, where p and q have no common factor other than ± 1 , then p is a factor of a_0 and q is a factor of a_n .

To get a feeling as to why this theorem is true, suppose $\frac{3}{2}$ is a root of $a_3x^3+a_2x^2+a_1x+a_0=0$.

Then, $a_3\left(\frac{3}{2}\right)^3+a_2\left(\frac{3}{2}\right)^2+a_1\left(\frac{3}{2}\right)+a_0=0$ which implies that

$$\frac{27a_3}{8}+\frac{9a_2}{4}+\frac{3a_1}{2}+a_0=0 \quad \text{multiplying both sides by 8}$$

$$27a_3+18a_2+12a_1=-8a_0 \dots\dots\dots (1)$$

$$27a_3=-18a_2-12a_1-8a_0 \dots\dots\dots (2)$$

If we look at equation (1), the left hand side is divisible by 3, and therefore the right hand side must also be divisible by 3. Since 8 is not divisible by 3, a_0 must be divisible by 3. From equation (2), a_3 must be divisible by 2.

Example 2.38: Find all the zeros of the function $p(x)=2x^3+3x^2-23x-12$.

p

Solution: According to the Rational Root Theorem, if q is a rational root of the given equation, then p must be a factor of -12 and q must be a factor of 2. Thus, we have

possible values of p : $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

possible values of q : $\pm 1, \pm 2$

possible rational roots $\frac{p}{q}$: $\pm 1, \pm \frac{1}{2}, \pm 2, \pm 3, \pm \frac{3}{2}, \pm 4, \pm 6, \pm 12$

We may check these possible roots by substituting the value in $p(x)$. Now $p(1)=-30$ and $p(-1)=12$. Since $p(1)$ is negative and $p(-1)$ is positive, by intermediate value theorem, $p(x)$ has a zero between -1 and 1 . Since $p(-\frac{1}{2})=0$, then $(x+\frac{1}{2})$ is a factor of $p(x)$. Using long division, we obtain

$$p(x)=2x^3+3x^2-23x-12=(x+\frac{1}{2})(2x^2+2x-24)$$

$$=2(x+\frac{1}{2})(x+4)(x-3)$$

Therefore, the zeros of $p(x)$ are $-\frac{1}{2}$, -4 and 3 .

• Rational Functions and their Graphs

A rational function is a function of the form $f(x) = \frac{n(x)}{d(x)}$ where both $n(x)$ and $d(x)$ are polynomials and $d(x) \neq 0$.

Example 2.39: The functions $f(x) = \frac{3}{x+5}$, $f(x) = \frac{x-1}{x^2-4}$ and $f(x) = \frac{x^5+2x^3-x+1}{x+5x}$ are examples of rational function.

Note that the domain of the rational function $f(x) = \frac{n(x)}{d(x)}$ is $\{x : d(x) \neq 0\}$

Example 2.40: Find the domain and zeros of the function $f(x) = \frac{3x-5}{x^2-x-12}$.

Solution: The values of x for which $x^2-x-12=0$ are excluded from the domain of f . Since $x^2-x-12=(x-4)(x+3)$, we have $Dom(f)=\{x : x \neq -3, 4\}$. To find the zeros of $f(x)$, we solve the equation

$$\frac{n(x)}{d(x)} = 0 \Leftrightarrow n(x) = 0 \wedge d(x) \neq 0$$

Therefore, to find the zeros of $f(x)$, we solve $3x-5=0$, giving $x = \frac{5}{3}$. Since $\frac{5}{3}$ does not make the denominator zero, it is the only zero of $f(x)$.

The following terms and notations are useful in our next discussion.

Given a number a ,

- x approaches a from the right means x takes any value near and near to a but $x > a$. This is denoted by: $x \rightarrow a^+$ (read: ' x approaches a from the right').

For instance, $x \rightarrow 1^+$ means x can be 1.001 , 1.0001 , 1.00001 , 1.000001 , etc.

- x approaches a from the left means x takes any value near and near to a but $x < a$.

This is denoted by: $x \rightarrow a^-$ (read: ' x approaches a from the left').

For instance, $x \rightarrow 1^-$ means x can be 0.99, 0.999, 0.9999, 0.9999, etc.

- $x \rightarrow \infty$ (read: ' x approaches or tends to *infinity*') means the value of x gets indefinitely larger and larger in magnitude (keep increasing without bound). For instance, x can be 10^6 , 10^{10} , 10^{12} , etc.
- $x \rightarrow -\infty$ (read: ' x approaches or tends to negative *infinity*') means the value of x is negative and gets indefinitely larger and larger negative in magnitude (keep decreasing without bound). For instance, x can be -10^6 , -10^{10} , -10^{12} , etc.

The same meanings apply also for the values of a function f if we wrote $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$. The following figure illustrates these notion and notations.

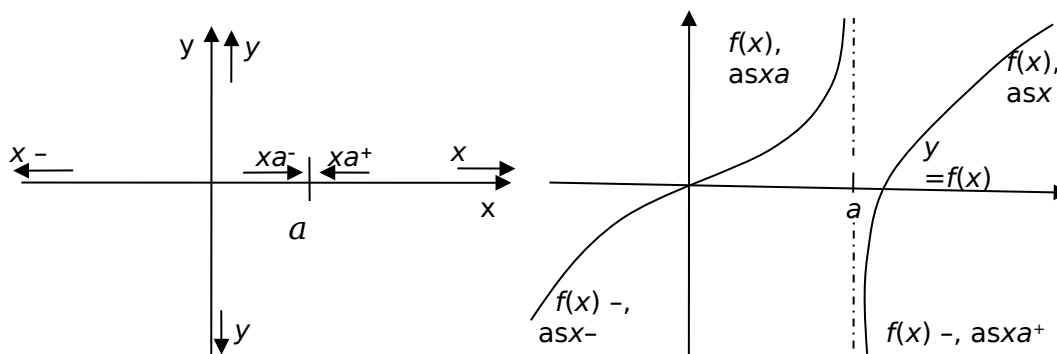


Fig. 2.1. Graphical illustration of the idea of $x \rightarrow a^+$, $f(x) \rightarrow \infty$, etc.

We may also write $f(x) \rightarrow b$ (read: ' $f(x)$ approaches b ') to mean the function values, $f(x)$, becomes arbitrarily closer and closer to b (i.e., approximately

b) but not exactly equal to b . For instance, if $f(x) = \frac{1}{x}$, then $f(x) \rightarrow 0$ as $x \rightarrow \infty$;

i.e., $\frac{1}{x}$ is approximately 0 when x is arbitrarily large.

The following steps are usually used to sketch (or draw) the graph of a rational function $f(x)$.

1. Identify the domain and simplify it.
2. Find the intercepts of the graph whenever possible. Recall the following:
 - y-intercept is the point on y-axis where the graph of $y = f(x)$ intersects with the y-axis. At this point $x=0$. Thus, $y = f(0)$, or $(0, f(0))$ is the y-intercept if $0 \in \text{Dom}(f)$.
 - x-intercept is the point on x-axis where the graph of $y = f(x)$ intersects with the x-axis. At this point $y=0$. Thus, $x=a$ or $(a, 0)$ is x-intercept if $f(a)=0$.

3. Determine the asymptotes of the graph. Here, remember the following.

- **Vertical Asymptote:** The vertical line $x=a$ is called a vertical asymptote(VA) of $f(x)$ if
 - i) $a \notin \text{dom}(f)$, i.e., f is not defined at $x=a$; and
 - ii) $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$ when $x \rightarrow a^+$ or $x \rightarrow a^-$. In this case, the graph of f is almost vertically rising upward (if $f(x) \rightarrow \infty$) or sinking downward (if $f(x) \rightarrow -\infty$) along with the vertical line $x=a$ when x approaches a either from the right or from the left.

Example 2.41: Consider $f(x) = \frac{1}{(x-a)^n}$, where $a \geq 0$ and n is a positive integer.

Obviously $a \notin \text{Dom}(f)$. Next, we investigate the trend of the values of $f(x)$ near a . To do this, we consider two cases, when n is even or odd:

Suppose n is even: In this case $(x-a)^n > 0$ for all $x \in \mathbb{R} \setminus \{a\}$; and since $(x-a)^n \rightarrow 0$ as $x \rightarrow a^+$ or $x \rightarrow a^-$. Hence,

$f(x) = \frac{1}{(x-a)^n} \rightarrow \infty$ as $x \rightarrow a^+$ or $x \rightarrow a^-$. Therefore, $x=a$ is a VA of $f(x)$. Moreover, $y = 1/a^n$ or $(0, 1/a^n)$ is its y-intercept since $f(0) = 1/a^n$. However, it has no x-intercept since $f(x) > 0$ for all x in its domain (See, Fig. 2.2 (A)).

Suppose n is odd: In this case $(x-a)^n > 0$ for all $x > a$ and $1/(x-a)^n \rightarrow \infty$ when $x \rightarrow a^+$ as in the above case. Thus, $x=a$ is its VA. However, $1/(x-a)^n \rightarrow -\infty$ when $x \rightarrow a^-$ since $(x-a)^n < 0$ for $x < a$. Moreover, $y = -1/a^n$ or $(0, -1/a^n)$ is its y-intercept since $f(0) = -1/a^n$. However, it has no x-intercept also in this case. (See, Fig. 2.2 (B)).

Note that in both cases, $f(x) = \frac{1}{(x-a)^n} \rightarrow 0$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

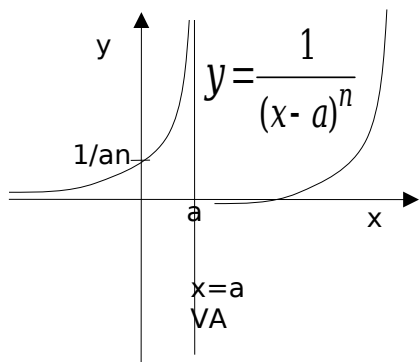


Fig. 2.2 (A)

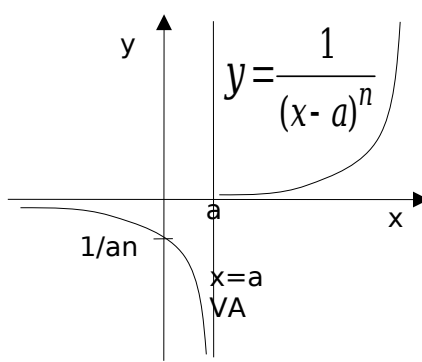


Fig. 2.2 (B)

Remark: Let $f(x) = \frac{n(x)}{d(x)}$ be a rational function. Then,

1. if $d(a) = 0$ and $n(a) \neq 0$, then $x=a$ is a VA of f .
2. if $d(a) = 0 = n(a)$, then $x=a$ may or may **not** be a VA of f . In this case, simplify $f(x)$ and look for VA of the simplest form of f .

- Horizontal Asymptote: A horizontal line $y=b$ is called horizontal asymptote (HA) of $f(x)$ if the value of the function becomes closer and closer to b (i.e., $f(x) \rightarrow b$) as $x \rightarrow \infty$ or as $x \rightarrow -\infty$.

In this case, the graph of f becomes almost a horizontal line along with (or near) the line $y=b$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$. For instance, from the above

example, the HA of $f(x) = \frac{1}{(x-a)^n}$ is $y=0$ (the x -axis), for any positive integer n (See, Fig. 2.2).

Remark: A rational function $f(x) = \frac{n(x)}{d(x)}$ has a HA only when $\text{degree}(n(x)) \leq \text{degree}(d(x))$.

In this case, (i) If $\text{degree}(n(x)) < \text{degree}(d(x))$, then $y = 0$ (the x -axis) is the HA of f .

(ii) If $\text{degree}(n(x)) = \text{degree}(d(x)) = n$, i.e.,

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0},$$

then $y = \frac{a_n}{b_n}$ is the HA of f .

- Oblique Asymptote: The oblique line $y=ax+b$, $a \neq 0$, is called an oblique asymptote (OA) of f if the value of the function, $f(x)$, becomes closer and closer to $ax+b$ (i.e., $f(x)$ becomes approximately $ax+b$) as either $x \rightarrow \infty$ or $x \rightarrow -\infty$. In this case, the graph of f becomes almost a straight line along with (or near) the oblique line $y=ax+b$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

Note: A rational function $f(x) = \frac{n(x)}{d(x)}$ has an OA only when $\text{degree}(n(x)) = \text{degree}(d(x)) + 1$. In this case, using long division, if the quotient of $n(x) \div d(x)$ is $ax+b$, then $y=ax+b$ is the OA of f .

Example 2.42: Sketch the graphs of (a) $f(x) = \frac{x+2}{x-1}$ (b) $g(x) = \frac{x^2+3x+2}{x^2-1}$

Solution: (a) Since $x-1=0$ at $x=1$, $\text{dom}(f) = \mathbb{R} \setminus \{1\}$.

- Intercepts: y-intercept: $x=0 \Rightarrow y=f(0) = -2$. Hence, $(0, -2)$ is y-intercept.

x-intercept: $y=0 \Rightarrow x+2=0 \Rightarrow x=-2$. Hence, $(-2, 0)$ is x-intercept.

- Asymptotes:

- VA: Since $x-1=0$ at $x=1$ and $x+2 \neq 0$ at $x=1$, $x=1$ is VA of f . In fact, if $x \rightarrow 1^+$, then $x+2 \approx 3$ but the denominator $x-1$ is almost 0 (but positive).

Consequently, $f(x) \rightarrow \infty$ as $x \rightarrow 1^+$.

Moreover, $f(x) \rightarrow -\infty$ as $x \rightarrow 1^-$ (since, if $x \rightarrow 1^-$ then $x-1$ is almost 0 but negative).

(So, the graph of f rises up to $+\infty$ at the right side of $x=1$, and sink down to $-\infty$ at the left side of $x=1$)

- HA: Note that if you divide $x+2$ by $x-1$, the quotient is 1 and remainder is 3. Thus,

$$f(x) = \frac{x+2}{x-1} = 1 + \frac{3}{x-1}. \text{ Thus, if } x \rightarrow \infty \text{ (or } x \rightarrow -\infty), \text{ then } \frac{3}{x-1} \rightarrow 0 \text{ so that } f(x) \rightarrow 1. \text{ Hence, } y=1 \text{ is the HA of } f.$$

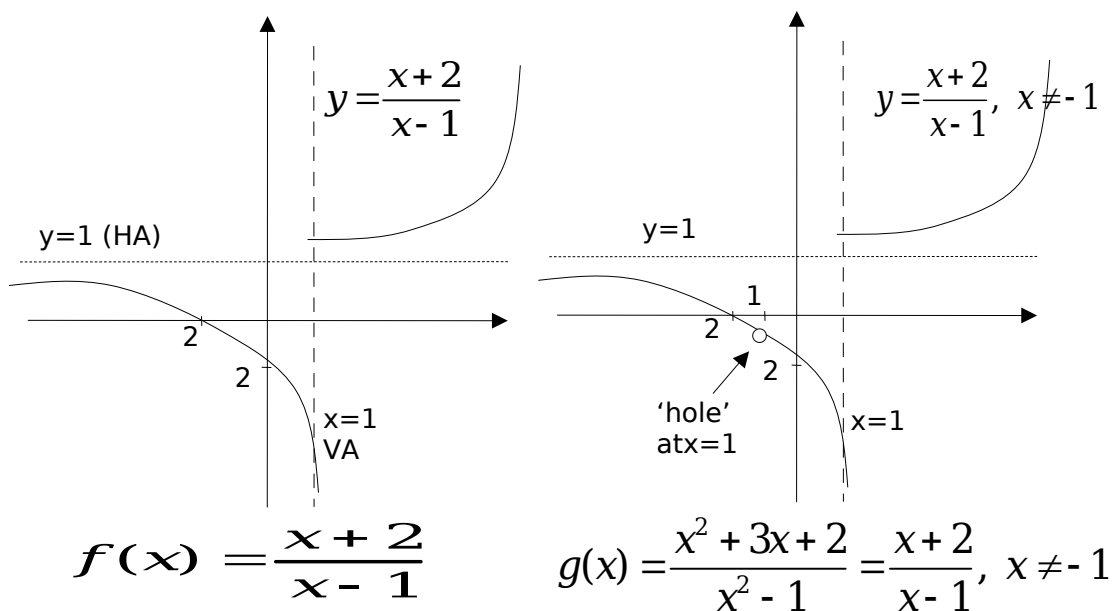
Using these information, you can sketch the graph of f as displayed below in Fig. 2.3 (A).

(b) Both the denominator and numerator are 0 at $x=1$. So, first factorize and simplify them:

$$\begin{aligned} x^2+3x+2 &= (x+2)(x+1) \quad \text{and} \quad x^2-1 = (x-1)(x+1). \text{ Therefore,} \\ g(x) &= \frac{x^2+3x+2}{x^2-1} = \frac{(x+2)(x+1)}{(x-1)(x+1)}, \quad x \neq -1 \\ &= \frac{x+2}{x-1}. \end{aligned} \quad (\text{So, } \text{dom}(g) = \mathbb{R} \setminus \{1, -1\})$$

This implies that only $x=1$ is VA.

Hence, the graph of $g(x) = \frac{x+2}{x-1}, x \neq -1$ is exactly the same as that of $f(x) = \frac{x+2}{x-1}$ except that $g(x)$ is not defined at $x=-1$. Therefore, the graph of g and its VA are the same as that of f except that there should be a 'hole' at the point corresponding to $x=-1$ on the graph of g as shown on Fig. 2.3(B) below.



Exercise 2.6

- Perform the requested divisions. Find the quotient and remainder and verify the Remainder Theorem by computing $p(a)$.
 - Divide $p(x)=x^2-5x+8$ by $x+4$
 - Divide $p(x)=2x^3-7x^2+x+4$ by $x-4$
 - Divide $p(x)=1-x^4$ by $x-1$
 - Divide $p(x)=x^5-2x^2-3$ by $x+1$
- Given that $p(4)=0$, factor $p(x)=2x^3-11x^2+10x+8$ as completely as possible.
- Given that $r(x)=4x^3-x^2-36x+9$ and $r(\frac{1}{4})=0$, find the remaining zeros of $r(x)$.
- Given that 3 is a double zero of $p(x)=x^4-3x^3-19x^2+87x-90$, find all the zeros of $p(x)$.
- Write the general polynomial $p(x)$ whose only zeros are 1, 2 and 3, with multiplicity 3, 2 and 1 respectively. What is its degree?
 - Find $p(x)$ described in part (a) if $p(0)=6$.
- If $2-3i$ is a root of $p(x)=2x^3-5x^2+14x+39$, find the remaining zeros of $p(x)$.
- Determine the rational zeros of the polynomials

a) $p(x) = x^3 - 4x^2 - 7x + 10$

b) $p(x) = 2x^3 - 5x^2 - 28x + 15$

c) $p(x) = 6x^3 + x^2 - 4x + 1$

8. Find the domain and the real zeros of the given function.

a) $f(x) = \frac{3}{x^2 - 25}$ b) $g(x) = \frac{x-3}{x^2 + 4x - 12}$ c) $f(x) = \frac{(x-3)^2}{x^3 - 3x^2 + 2x}$ d) $f(x) = \frac{x^2 - 16}{x^2 + 4}$

9. Sketch the graph of

a) $f(x) = \frac{1-x}{x-3}$ b) $f(x) = \frac{x^2+1}{x}$ c) $f(x) = \frac{1}{x} + 2$ d) $f(x) = \frac{x^2}{x^2-4}$

10. Determine the behavior of $f(x) = \frac{x^3 - 8x - 3}{x-3}$ when x is near 3.

11. The graph of any rational function in which the degree of the numerator is exactly one more than the degree of the denominator will have an oblique (or slant) asymptote.

a) Use long division to show that

$$y = f(x) = \frac{x^2 - x + 6}{x - 2} = x + 1 + \frac{8}{x - 2}$$

b) Show that this means that the line $y = x + 1$ is a slant asymptote for the graph and sketch the graph of $y = f(x)$.

2.7 Definition and basic properties of logarithmic, exponential, and trigonometric functions and their graphs

After completing this section, the student should be able to:

- define exponential, logarithmic and trigonometric functions
- understand the relationship between exponential and logarithmic functions
- sketch the graph of exponential, logarithmic, and trigonometric functions
- use basic properties of logarithmic, exponential and trigonometric functions to solve problems

- **Exponents and radicals**

Definition 2.16: For a natural number n and a real number x , the power x^n , read “the n^{th} power of x ” or “ x raised to n ”, is defined as follows:

$$x^n = \underbrace{x \cdot x \cdots x}_{n \text{ factors each equal to } x}$$

In the symbol x^n , x is called the base and n is called the exponent.

For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.

Based on the definition of x^n , n must be a natural number. It does not make sense for n to be negative or zero. However, we can extend the definition of exponents to include 0 and negative exponents.

Definition 2.17: (Zero and Negative Exponents)

Definition of zero Exponent

$$x^0 = 1 \quad (x \neq 0)$$

Note: 0^0 is undefined.

Definition of Negative

$$x^{-n} = \frac{1}{x^n} \quad (x \neq 0)$$

As a result of the above definition, we have $\frac{1}{x^{-n}} = x^n$. We have the following rules of exponents for integer exponents:

Rules for Integer Exponents

$$1. \quad x^n \cdot x^m = x^{n+m}$$

$$4. \quad (xy)^n = x^n y^n$$

$$2. \quad (x^n)^m = x^{nm}$$

$$5. \quad \frac{x^n}{x^m} = x^{n-m}$$

$$3. \quad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \quad (y \neq 0)$$

Next we extend the definition of exponents even further to include rational number exponents. To do this, we assume that we want the rules for integer exponents also to apply to rational exponents and then use the rules to show us to define a rational exponent. For example, how do we define $a^{\frac{1}{2}}$?

Consider $9^{\frac{1}{2}}$.

If we apply rule 2 and square $9^{\frac{1}{2}}$, we get $\left(9^{\frac{1}{2}}\right)^2 = 9^{\frac{1}{2} \cdot 2} = 9^1 = 9$. Thus, $9^{\frac{1}{2}}$ is a number that, when squared, yields 9. There are two possible answers: 3 and -3, since squaring either number will yield 9. To avoid ambiguity, we define $a^{\frac{1}{2}}$ (called the principal square root of a) as the non-negative quantity that, when squared, yield a . Thus, $9^{\frac{1}{2}} = 3$.

We will arrive at the definition of $a^{\frac{1}{3}}$ in the same way as we did for $a^{\frac{1}{2}}$. For example, if we cube $8^{\frac{1}{3}}$, we get $\left(8^{\frac{1}{3}}\right)^3 = 8^{\frac{1}{3} \cdot 3} = 8^1 = 8$. Thus, $8^{\frac{1}{3}}$ is the number that, when cubed, yields 8. Since $2^3 = 8$ we have $8^{\frac{1}{3}} = 2$. Similarly, $(-27)^{\frac{1}{3}} = -3$. Thus, we define $a^{\frac{1}{3}}$ (called the cube root of a) as the quantity that, when cubed yields a .

Definition 2.18: (Rational Exponent $a^{\frac{1}{n}}$)

If n is an odd positive integer, then $a^{\frac{1}{n}} = b$ if and only if $b^n = a$

If n is an even positive integer and $a \geq 0$, then $a^{\frac{1}{n}} = |b|$ if and only if $b^n = a$

We call $a^{\frac{1}{n}}$ the principal n^{th} root of a . Hence, $a^{\frac{1}{n}}$ is the real number (nonnegative when n is even) that, when raised to the n^{th} power, yields a . Therefore,

$$\begin{aligned} (16)^{\frac{1}{2}} &= 4 \quad \text{since } 4^2 = 16 \\ (-125)^{\frac{1}{3}} &= -5 \quad \text{since } (-5)^3 = -125 \\ \left(\frac{1}{81}\right)^{\frac{1}{4}} &= \frac{1}{3} \quad \text{since } \left(\frac{1}{3}\right)^4 = \frac{1}{81} \\ 27^{\frac{1}{3}} &= 3 \quad \text{since } 3^3 = 27 \\ (-16)^{\frac{1}{4}} &\text{ is not a real number} \end{aligned}$$

Thus far, we have defined $a^{\frac{1}{n}}$, where n is a natural number. With the help of the second rule for exponent, we can define the expression $a^{\frac{m}{n}}$, where m

and n are natural numbers and $\frac{m}{n}$ is reduced to lowest terms.

Definition 2.19: (Rational Exponent $a^{\frac{m}{n}}$)

If $a^{\frac{1}{n}}$ is a real number, then $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$ (i.e. the n^{th} root of a raised to the m^{th} power)

We can also define negative rational exponents:

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} \quad (a \neq 0)$$

Example 2.43: Evaluate the following

a) $27^{\frac{2}{3}}$ b) $36^{-\frac{1}{2}}$ c) $(-32)^{-\frac{3}{5}}$

Solution: We have

a) $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = 3^2 = 9$

b) $36^{-\frac{1}{2}} = \frac{1}{36^{\frac{1}{2}}} = \frac{1}{6}$

c) $(-32)^{-\frac{3}{5}} = \frac{1}{(-32)^{\frac{3}{5}}} = \frac{1}{\left((-32)^{\frac{1}{5}}\right)^3} = \frac{1}{(-2)^3} = -\frac{1}{8}$

Radical notation is an alternative way of writing an expression with rational exponents. We define for real number a , the n^{th} root of a as follows:

Definition 2.20 (n^{th} root of a): $\sqrt[n]{a} = a^{\frac{1}{n}}$, where n is a positive integer.

The number $\sqrt[n]{a}$ is also called the principal n^{th} root of a . If the n^{th} root of a exists, we have:

For a a real number and n a positive integer,

$$\sqrt[n]{a^n} = |a|, \text{ if } n \text{ is even}$$

For example, $\sqrt[3]{5^3} = 5$ and $\sqrt[4]{(-3)^4} = 3$.

• Exponential Functions

In the previous sections we examined functions of the form $f(x) = x^n$, where n is a constant. How is this function different from $f(x) = n^x$.

Definition 2.21: A function of the form $y = f(x) = b^x$, where $b > 0$ and $b \neq 1$, is called an exponential function.

Example 2.44: The functions $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = \left(\frac{1}{2}\right)^x$ are examples of exponential functions.

As usual the first question raised when we encounter a new function is its domain. Since rational exponents are well defined, we know that any rational number will be in the domain of an exponential function. For example, let $f(x) = 3^x$. Then as x takes on the rational values $x = 4, -2, \frac{1}{2}$ and $\frac{4}{5}$, we have

$$\begin{aligned} f(4) &= 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81 & f(-2) &= 3^{-2} = \frac{1}{3^2} = \frac{1}{9} \\ f\left(\frac{1}{2}\right) &= 3^{\frac{1}{2}} = \sqrt{3} & f\left(\frac{4}{5}\right) &= 3^{\frac{4}{5}} = \sqrt[5]{3^4} = \sqrt[5]{81} \end{aligned}$$

Note that even though we do not know the exact values of $\sqrt{3}$ and $\sqrt[5]{81}$, we do know exactly what they mean. However, what about $f(x)$ for irrational values of x ? For instance, $f(\sqrt{2}) = 3^{\sqrt{2}} = ?$

We have not defined the meaning of irrational exponents. In fact, a precise formal definition of b^x where x is irrational requires the ideas of calculus. However, we can get an idea of what $3^{\sqrt{2}}$ should be by using successive

rational approximations to $\sqrt{2}$. For example, we have

$$1.414 < \sqrt{2} < 1.415$$

Thus, it would seem reasonable to expect that $3^{1.414} < 3^{\sqrt{2}} < 3^{1.415}$. Since 1.414 and 1.415 are rational numbers, $3^{1.414}$ and $3^{1.415}$ are well defined, even though we cannot compute their values by hand. Using a calculator, we get $4.7276950 < 3^{\sqrt{2}} < 4.7328918$. If we use better approximations to $\sqrt{2}$, we get $3^{1.4142} < 3^{\sqrt{2}} < 3^{1.4143}$. Using a calculator again, we get $4.7287339 < 3^{\sqrt{2}} < 4.7292535$. Computing $3^{\sqrt{2}}$ directly on a calculator gives $3^{\sqrt{2}} \approx 4.7288044$. This numerical evidence suggests that as x approaches $\sqrt{2}$, the values of 3^x approach a unique real number that we designate by $3^{\sqrt{2}}$, and so we will accept without proof, the fact that the domain of the exponential function is the set of real numbers.

The exponential function $y=b^x$, where $b>0$ and $b\neq 1$, is defined for all real values of x . In addition all the rules for rational exponents hold for real number exponents as well.

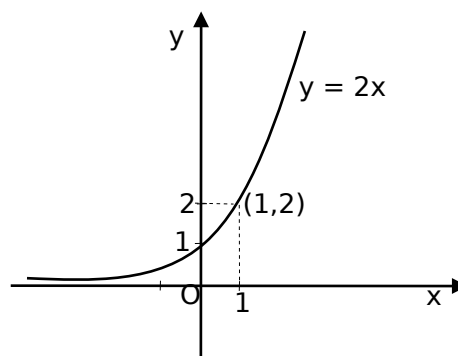
Before we state some general facts about exponential functions, let's see if we can determine what the graph of an exponential function will look like.

Example 2.45:

1. Sketch the graph of the function $y=2^x$ and identify its domain and range.

Solution: To aid in our analysis, we set up a short table of values to give us a frame of reference.

x	y
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



With these points in hand, we draw a smooth curve through the points obtaining the graph appearing above. Observe that the domain of $y=2^x$ is $\mathbb{R}$, the graph has no x -intercepts, as $x \rightarrow +\infty$, the y values are increasing very rapidly, whereas as $x \rightarrow -\infty$, the y values are getting closer and closer to 0. Thus, x -axis is a horizontal asymptote, the y -intercept is 1 and the range of $y=2^x$ is the set of positive real numbers.

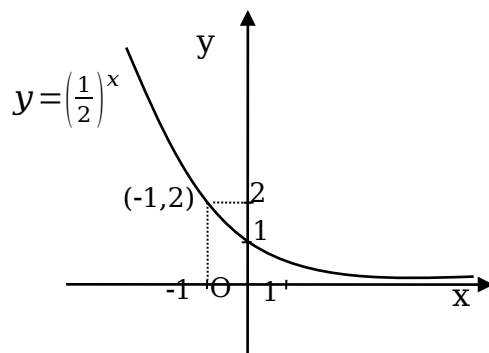
$$y = f(x) = \left(\frac{1}{2}\right)^x$$

2. Sketch the graph of .

Solution: It would be instructive to compute a table of values as we did in example 1 above (you are urged to do so). However, we will take a different

approach. We note that $y = f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$. If $f(x) = 2^x$, then $f(-x) = 2^{-x}$.

Thus by the graphing principle for $f(-x)$, we can obtain the graph of $y = 2^{-x}$ by reflecting the graph of $y = 2^x$ about the y -axis.



Here again the x -axis is a horizontal asymptote, there is no x -intercept, 1 is y -intercept and the range is the set of positive real numbers. However, the graph is now decreasing rather than increasing.

The following box summarizes the important facts about exponential functions and their graphs.

The Exponential function $y=f(x)=b^x$

1. The domain of the exponential function is the set of real numbers
2. The range of the exponential function is the set of positive real numbers
3. The graph of $y=b^x$ exhibits exponential growth if $b>1$ or exponential decay if $0<b<1$.
4. The y -intercept is 1.
5. The x -intercept is a horizontal asymptote
6. The exponential function is 1 - 1. Algebraically if $b^x=b^y$, then $x=y$

Example 2.46: Sketch the graph of each of the following. Find the domain, range, intercepts, and asymptotes.

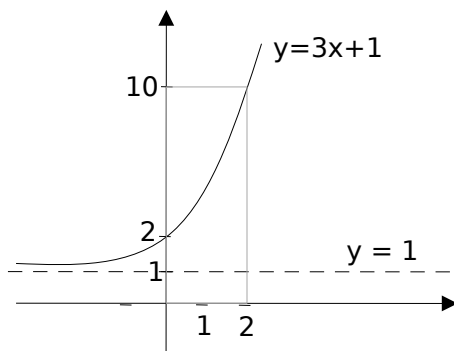
a) $y=3^x+1$

b) $y=3^{x+1}$

c) $y=-9^{-x}+3$

Solution:

- a) To get the graph of $y=3^x+1$. We start with the graph of $y=3^x$, which is the basic exponential growth graph, and shift it up 1 unit.

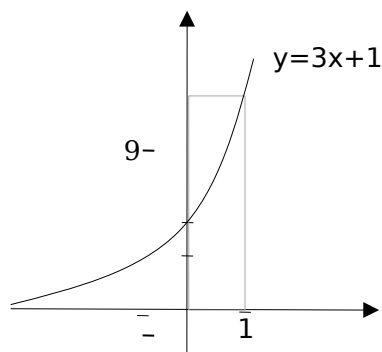


From the graph we see that

- $Dom(f)=\mathbb{R}$
- $Range(f)=(1, \infty)$
- The y -intercept is 2
- The line $y=1$ is a horizontal asymptote

- b) To get the graph of $y=3^{x+1}$, we start with the graph of $y=3^x$, and shift

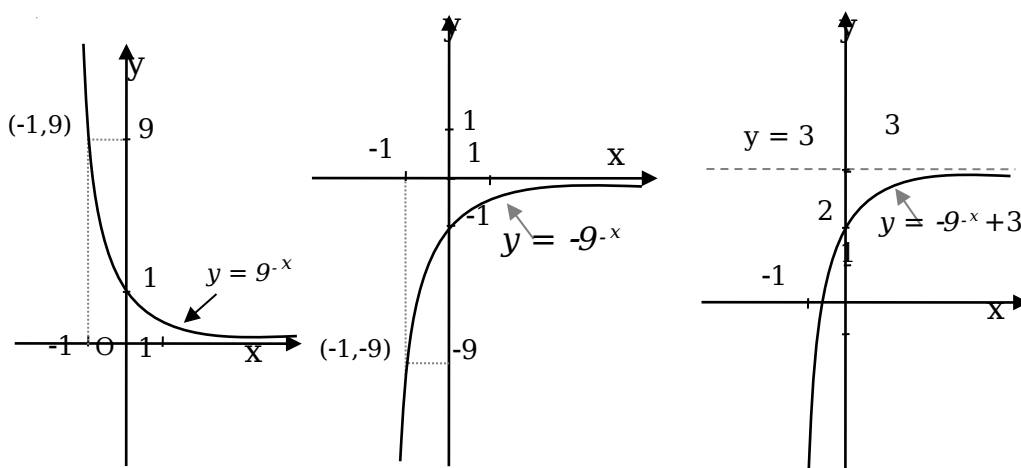
1 unit to the left.



From the graph we see that

- $Dom(f) = \mathbb{R}$
- $Range(f) = (0, \infty)$
- The y -intercept is 3
- The line $y = 0$ is a horizontal asymptote

c) To get the graph of $y = -9^{-x} + 3$, we start with the basic exponential decay $y = 9^{-x}$. We then reflect it with respect to the x -axis, which gives the graph of $y = -9^{-x}$. Finally, we shift this graph up 3 units to get the required graph of $y = -9^{-x} + 3$.



From the graph of $y = -9^{-x} + 3$, we can see that $Dom(h) = \mathbb{R}$, $Range(h) = (-\infty, 3)$, the line $y = 3$ is a horizontal asymptote, 2 is the y -intercept and $x = -\frac{1}{2}$ is the x -intercept.

Remark: When the base b of the exponential function $f(x) = b^x$ equals to the number e , where $e = 2.7182\dots$, we call the exponential function the natural exponential function.

• Logarithmic Functions

In the previous subsection we noted that the exponential function $f(x)=b^x$ (where $b>0$ and $b\neq 1$) is one to one. Thus, the exponential function has an inverse function. What is the inverse of $f(x)=b^x$?

To find the inverse of $f(x)=b^x$, let's review the process for finding an inverse function by comparing the process for the polynomial function $y=x^3$ and the exponential function $y=3^x$. Keep in mind that x is our independent variable and y is the dependent variable and so whenever possible we want a function solved explicitly for y .

To find the inverse of $y=x^3$		To find the inverse of $y=3^x$	
$y=x^3$	Interchange x and y	$y=3^x$	Interchange x and y
$x=y^3$	solve for y	$x=3^y$	solve for y
$y=\sqrt[3]{x}$		$y=??$	

There is no algebraic procedure we can use to solve $x=3^y$ for y . By introducing radical notations we could express the inverse of $y=x^3$ explicitly in the form $y=\sqrt[3]{x}$. In words, $y^3=x$ and $y=\sqrt[3]{x}$ both mean exactly the same thing: y is the number whose cube is x . Similarly, if we want to express $x=3^y$ explicitly as a function of x , we need to invent a special notation for this. The key idea is to take the equation $x=3^y$ and express it verbally.

$x=3^y$ means y is the exponent to which 3 must be raised to yield x

We introduce the following notation, which expresses this idea in a much more compact form.

Definition 2.22: For $b>0$ and $b\neq 1$, we write $y=\log_b x$ to mean y is the exponent to which b must be raised to yield x . In other words,

$$x=b^y \Leftrightarrow y=\log_b x$$

We read $y = \log_b x$ as “ y equals the logarithm of x to the base b ”.

REMEMBER: $y = \log_b x$ is an alternative way of writing $x = b^y$

When an expression is written in the form $x = b^y$, it is said to be in exponential form. When an expression is written in the form $y = \log_b x$, it is said to be in logarithmic form. The table below illustrates the equivalence of the exponential and logarithmic forms.

Exponential form	Logarithmic form
$4^2 = 16$	$\log_4 16 = 2$
$2^4 = 16$	$\log_2 16 = 4$
$5^{-3} = \frac{1}{125}$	$\log_5 \frac{1}{125} = -3$
$6^{\frac{1}{2}} = \sqrt{6}$	$\log_6 \sqrt{6} = \frac{1}{2}$
$7^0 = 1$	$\log_7 1 = 0$

Example 2.47:

1. Write each of the following in exponential form.

a) $\log_{\frac{1}{3^9}} = -2$ b) $\log_{16} 2 = \frac{1}{4}$

Solution: We have a) $\log_{\frac{1}{3^9}} = -2$ means $3^{-2} = \frac{1}{9}$.

b) $\log_{16} 2 = \frac{1}{4}$ means $16^{\frac{1}{4}} = 2$

2. Write each of the following in logarithmic form.

a) $10^{-3} = 0.001$ b) $27^{\frac{2}{3}} = 9$

Solution: We have a) $10^{-3} = 0.001$ means $\log_{10} 0.001 = -3$

b) $27^{\frac{2}{3}} = 9$ means $\log_{27} 9 = \frac{2}{3}$

3. Evaluate each of the following.

a) $\log_3 81$ b) $\log_8 \frac{1}{64}$

Solution:

a) To evaluate $\log_3 81$, we let $t = \log_3 81$, and then rewrite the equation in

exponential form, $3^t = 81$. Now, if we can express both sides in terms of the same base, we can solve the resulting exponential equation, as follows:

Let $t = \log_3 81$ Rewrite in exponential form
 $3^t = 81$ Express both sides in terms of the same
 base
 $3^t = 3^4$ Since the exponential function is 1 - 1
 $t = 4$

Therefore, $\log_3 81 = 4$.

b) We apply the same procedure as in part (a).

Let $t = \log_8 \frac{1}{64}$ Rewrite in exponential form
 $8^t = \frac{1}{64}$ Express both sides in terms of the same
 base
 $8^t = 8^{-2}$ Since the exponential function is 1 - 1
 $t = -2$

Therefore, $\log_8 \frac{1}{64} = -2$.

As was pointed out at the beginning of this subsection, logarithm notation was invented to express the inverse of the exponential function. Thus, $\log_b x$ is a function of x . We usually write $f(x) = \log_b x$ rather than writing $f(x) = \log_b(x)$ and use parenthesis only when needed to clarify the input to the log function. For example,

If $f(x) = \log_5(4-x)$, then $f(-1) = \log_5(4-(-1)) = \log_5 5 = 1$, whereas if $f(x) = 4 - \log_5 x$, then $f(-1) = 4 - \log_5(-1)$, which is undefined.

Example 2.48: Given $f(x) = \log_5 x$, find

- a) $f(25)$ b) $f(\frac{1}{25})$ c) $f(0)$ d) $f(-125)$

Solution:

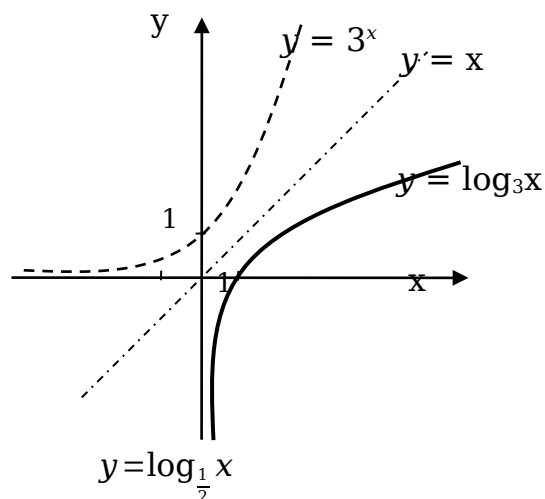
- a) $f(25) = \log_5 25 = 2$ (since $5^2 = 25$)
- b) $f\left(\frac{1}{25}\right) = \log_5 \frac{1}{25} = -2$ (since $5^{-2} = \frac{1}{25}$)
- c) $f(0) = \log_5 0$ is not defined (what power of 5 will yield 0?). We say that 0 is not in the domain of f .
- d) $f(-125) = \log_5(-125)$ is not defined (what power of 5 will yield -125?). We say that -125 is not in the domain of f .

Acknowledging that the logarithmic and exponential functions are inverses, we can derive a great deal of information about the logarithmic function and its graph from the exponential function and its graph.

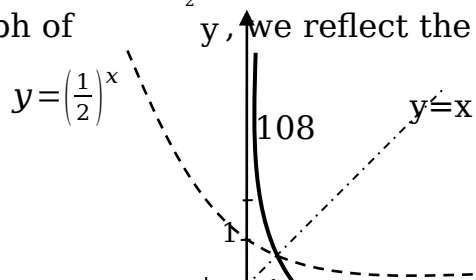
Example 2.49: Sketch the graph of the following functions. Find the domain and range of each.

- a) $y = \log_3 x$
- b) $y = \log_{\frac{1}{2}} x$

Solution: a) Since $y = \log_3 x$ is the inverse of $y = 3^x$, we can obtain the graph of $y = \log_3 x$ by reflecting the graph of $y = 3^x$ about the line $y = x$, as shown below.



- b) To get the graph of $y = \log_{\frac{1}{2}} x$, we reflect the graph of $y = \left(\frac{1}{2}\right)^x$ about the



line $y=x$ as shown below.

Taking note of the features of the two graphs we have the following important informations about the graph of the log function:

The Logarithmic Function $y=\log_b x$

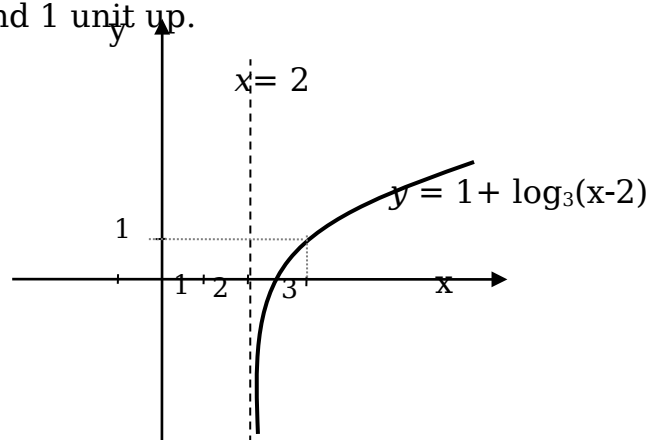
1. Its domain is the set of positive real numbers
2. Its range is the set of real numbers.
3. Its graph exhibits logarithmic growth if $b>1$ and logarithmic decay if $0<b<1$.
4. The x -intercept is 1. There is no y -intercept.
5. The y -axis is a vertical asymptote.

Example 2.50:

$$f(x)=1+\log_3(x-2)$$

1. Sketch the graph of $f(x)=1+\log_3(x-2)$. Find the domain, range, asymptote and intercepts.

Solution: We can obtain the graph of $y=1+\log_3(x-2)$ by applying the graphing principle to shift the basic logarithmic growth graph 2 units to the right and 1 unit up.



We have $Dom(f)=\{x:x>2\}$, $Range(f)=\mathbb{R}$ and the graph has the line $x=2$ as a vertical asymptote. To find the intercept, we set $y=0$ and solve for x . Setting $y=0$ and solving for x , we will obtain $x=\frac{7}{3}$. Thus, the x -intercept is $\frac{7}{3}$.

2. Find the inverse function for

$$\begin{array}{ll} y=f(x)=3^x+4 & y=g(x)=\log_3(x-2) \\ \text{a)} & \text{b)} \end{array}$$

Solution: Following the procedure for finding an inverse function, we have

$$\begin{array}{ll} \text{(a)} & y=3^x+4 \\ & \text{Interchange } x \text{ and } y \\ & x=3^y+4 \\ & \text{solve explicitly for } y \\ & x-4=3^y \\ & \text{Write in logarithmic form} \\ & y=\log_3(x-4) \\ & \text{Thus, } f^{-1}(x)=\log_3(x-4) \end{array} \quad \begin{array}{ll} \text{(b)} & y=\log_3(x-2) \\ & \text{Interchange } x \text{ and } y \\ & x=\log_3(y-2) \\ & \text{Write in logarithmic form} \\ & y-2=3^x \\ & \text{solve explicitly for } y \\ & y=3^x+2 \\ & \text{Thus, } g^{-1}(x)=3^x+2 \end{array}$$

The following table contains the basic properties of logarithm:

Properties of logarithm

Assume that b, u and v are positive and $b \neq 1$. Then

$$1. \log_b(uv)=\log_b u+\log_b v$$

In words, logarithm of a product is equal to the sum of the logs of the factors.

$$2. \log_b\left(\frac{u}{v}\right)=\log_b u-\log_b v$$

In words, the log of a quotient is the log of the numerator minus the log of the denominator.

$$3. \log_b(u^r)=r \log_b u$$

In words, the log of a power is the exponent times the log.

$$4. \log_b(b^x)=x \log_b b=x$$

$$5. b^{\log_b x}=x$$

Example 2.51:

1. Express in terms of simpler logarithms.

$$\log_b(x^3 y)$$

a)

$$\log_b(x^3 + y)$$

b)

$$\log_b\left(\frac{\sqrt{xy}}{z^3}\right)$$

c)

Solution:

$$\log_b(x^3 y) = \log_b x^3 + \log_b y = 3 \log_b x + \log_b y$$

a)

- b) Examining the properties of logarithms, we can see that they deal

$$\log_3(x^3 + y)$$

with log of a product, quotient and power. Thus, which is the log of a sum cannot be simplified using log properties.

- c) We have

$$\log_b\left(\frac{\sqrt{xy}}{z^3}\right) = \log_b \sqrt{xy} - \log_b(z^3) = \log_b(xy)^{\frac{1}{2}} - 3 \log_b z = \frac{1}{2}(\log_b x + \log_b y) - 3 \log_b z$$

$$\log_b \frac{1}{2} = -\log_b 2$$

2. Show that

$$\log_b \frac{1}{2} = \log_b 1 - \log_b 2 = 0 - \log_b 2 = -\log_b 2$$

The logarithmic function was introduced without stressing the particular base chosen. However, there are two bases of special importance in science and mathematics, namely, $b=10$ and $b=e$.

Definition 2.23: (Common Logarithm)

$f(x) = \log_{10} x$ is called the common logarithm function. We write $\log_{10} x = \log x$.

The inverse of the natural exponential function is called the natural logarithmic function and has its own special notation.

Definition 2.24: (Natural Logarithm)

$f(x) = \log_e x$ is called the natural logarithmic function. We write $\log_e x = \ln x$.

Example 2.52:

1. Evaluate $\log 1000$

$$\text{Solution: Let } a = \log 1000. \text{ Then, } a = \log_{10} 1000 = \log_{10}(10^3) = 3.$$

2. Find the inverse function of $f(x) = e^x + 1$.

Solution: Let $y = e^x + 1$ Interchange x and y
 $x = e^y + 1$ Solve for y
 $x - 1 = e^y$ Rewrite in logarithmic form
 $y = \ln(x - 1)$

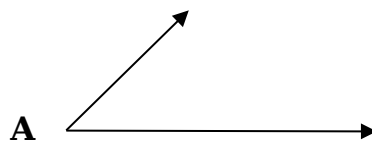
Thus, $f^{-1}(x) = \ln(x - 1)$.

• Trigonometric functions and their graphs

For the functions we have encountered so far, namely polynomial, rational and exponential functions, as the independent variable goes to infinity the graph of each of these three functions either goes to infinity (very quickly) for exponential functions or approaches a finite horizontal asymptote. None of these functions can model the regular periodic patterns that play an important role in the social, biological, and physical sciences: business cycles, agricultural seasons, heart rhythms, and hormone level fluctuations, and tides and planetary motions. The basic functions for studying regular periodic behaviour are the trigonometric functions. The domain of the trigonometric functions is more naturally the set of all geometric angles.

Angle Measurement

An angle is the figure formed by two half-lines or rays with a common end point. The common end point is called the vertex of the angle.



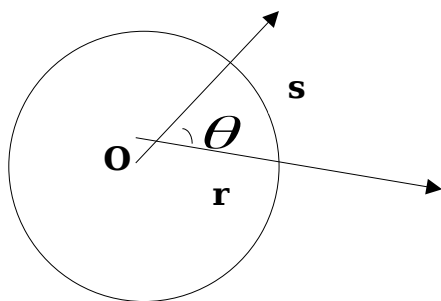
In forming the angle, one side remains fixed and the other side rotates. The fixed side is called the initial side and the side that rotates is called the terminal side. If the terminal side rotates in a counter clockwise direction, we call the angle positive angle, and if the terminal side rotates in a clockwise direction, we call the angle negative angle.

What we are trying to measure when we measure the size of an angle? A

moment of thought will lead us to the conclusion that when we measure an angle we are trying to answer the question: Through what part of a complete rotation has the terminal side rotated?

We will use degree ($^{\circ}$) as the unit of measurement for angles. Recall that the measure of a full round angle (full circle) is 360° , straight angle is 180° , and right angle is 90° .

An alternative unit of measure for angles which will indicate their size is the radian measure. To see the connection between the degree measure and radian measure of an angle, let us consider an angle θ and draw a circle of radius r with the vertex of θ at its center O . Let s represent the length of the arc of the circle intercepted by $\angle \theta$ (as shown below).



Basic geometry tells us that the central angle θ will be the same fractional part of one complete rotation as s will be of the circumference of the circle. For example, if θ is $\frac{1}{10}$ of a complete rotation, then s will be $\frac{1}{10}$ of the circumference of the circle. In other words, we can set up the following proportion:

$$\frac{\theta}{1 \text{ complete rotation}} = \frac{s}{\text{circumference of circle}} = \frac{s}{2\pi r}$$

Thus, we have the following conversion formula:

$$\frac{\theta \text{ in degrees}}{180^{\circ}} = \frac{\theta \text{ in radians}}{\pi}$$

Example 2.53:

1. Convert each of the following radian measures to degrees.

a) $\frac{\pi}{6}$

b) $\frac{3\pi}{5}$

Solution: a) By the conversion formula, we have $\frac{\theta}{180^\circ} = \frac{\frac{\pi}{6}}{\pi}$, which implies that $\theta = 30^\circ$.

b) Again using the conversion formula, we get $\frac{\theta}{180^\circ} = \frac{\frac{3\pi}{5}}{\pi}$, which implies that $\theta = 108^\circ$.

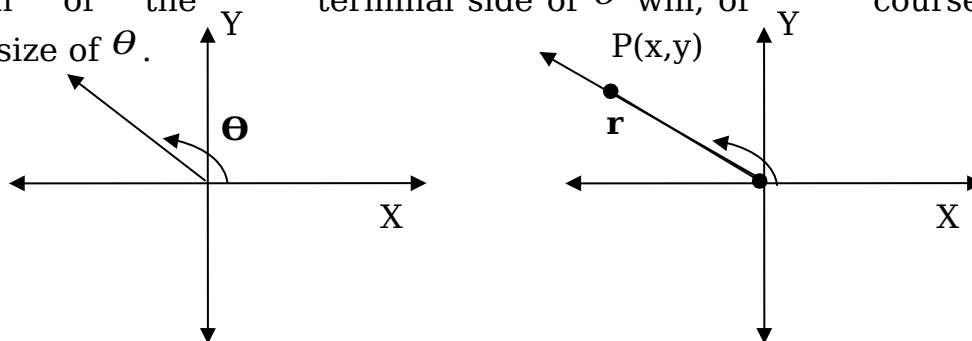
2. Convert to radian measures

a) 90° b) 270°

Solution: a) Let θ represent the radian measure of 90° . Using the conversion formula, we obtain: $\frac{\theta}{\pi} = \frac{90^\circ}{180^\circ}$, which implies that $\theta = \frac{\pi}{2}$.

b) Rather than using the conversion formula, we notice that $270^\circ = 3(90^\circ)$. In part (a) we found that $90^\circ = \frac{\pi}{2}$, and so we have $270^\circ = \frac{3\pi}{2}$.

To define the trigonometric functions, we will view all angles in the context of a Cartesian coordinate system: that is, given an angle θ , we begin by putting θ in standard position, meaning that the vertex of θ is placed at the origin and initial side of θ is placed along the positive x -axis. Thus the location of the terminal side of θ will, of course, depend on the size of θ .



We then locate a point (other than the origin) on the terminal side of θ and identify its coordinates (x, y) and its distance to the origin, denoted by r .

Then, r is positive.

With θ in standard position, we define the six trigonometric functions of θ as follows:

Definition 2.25	
<u>Name of function</u>	<u>Abbreviation</u>
<u>Definition</u>	
Sine θ	$\sin\theta$
$\sin\theta = \frac{y}{r}$	
Cosine θ	$\cos\theta$
$\cos\theta = \frac{x}{r}$	
Tangent θ	$\tan\theta$
$\tan\theta = \frac{y}{x}$	
Cosecant θ	$\csc\theta$
$\csc\theta = \frac{r}{y}$	
Secant θ	$\sec\theta$
$\sec\theta = \frac{r}{x}$	
Cotangent θ	$\cot\theta$
$\cot\theta = \frac{x}{y}$	

Recall that the radian measure of an angle is defined as $\theta = \frac{s}{r}$, where θ is angle in radians

s is the length of the arc intercepted by θ and r is the length of the radius.

Since s and r are both lengths, the quotient $\frac{s}{r}$ is a pure number without any units attached. Thus, any angle can be interpreted as a real number. Conversely, any real number can be interpreted as an angle. Thus, we can describe the domains of the trigonometric functions in the frame work of the real number systems. If we let $f(\theta) = \sin\theta$, the domain consists of all real

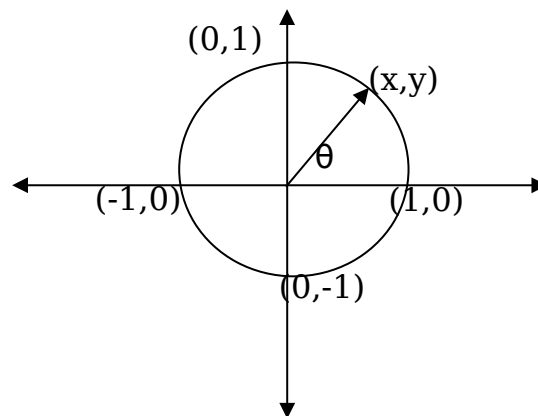
numbers θ for which $\sin\theta$ is defined. Since $\sin\theta = \frac{y}{r}$ and r is never equal to zero, the domain for $\sin\theta$ is the set of all real numbers. Similarly, the

domain of $f(\theta) = \cos\theta = \frac{x}{r}$ is also the set of all real numbers.

- **The graph of $y = \sin\theta$**

To analyze $f(\theta) = \sin\theta$, we keep in mind that once we choose a real number θ , we draw the angle, in standard position, that corresponds to θ . To simplify our analysis, we choose the point (x, y) on the terminal side so that $r=1$. That is, (x, y) is a point on the unit circle $x^2 + y^2 = 1$. Note that

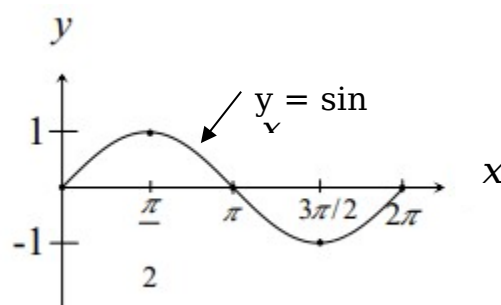
$$\sin\theta = \frac{y}{1} = y$$



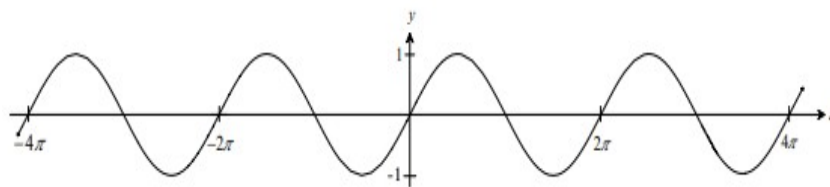
As the terminal side of θ moves through the first quadrant, y increases from 0 (when $\theta = 0$) to 1 (when $\theta = \frac{\pi}{2}$). Thus, as θ increases from 0 to $\frac{\pi}{2}$, $y = \sin\theta$ steadily increases from 0 to 1.

As θ increases from $\frac{\pi}{2}$ to π , $y = \sin\theta$ decreases from 1 to 0. A similar analysis reveals that as θ increases from π to $\frac{3\pi}{2}$, $\sin\theta$ decreases from 0 to -1; and as θ increases from $\frac{3\pi}{2}$ to 2π , $\sin\theta$ increases from -1 to 0.

Based on this analysis, we have the graph of $f(\theta) = \sin\theta$ in the interval $[0, 2\pi]$ as shown below.



Since the values of $f(\theta) = \sin\theta$ depend only on the position of the terminal side, adding or subtracting multiples of 2π to θ will leave the value of $f(\theta) = \sin\theta$ unchanged. Thus, the values of $f(\theta) = \sin\theta$ will repeat every 2π units. The complete graph of $f(\theta) = \sin\theta$ appears below.



The graph of $y = \sin x$, which is called the basic sine curve.

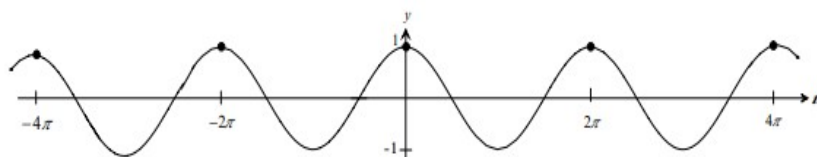
- **The graph of $y = \cos\theta$**

Applying the same type of analysis to $f(\theta) = \cos\theta$, we will be able to get a good idea of what its graph looks like. The figure below shows the angle corresponding to θ as it increases through quadrant I, II, III and IV.

Keeping in mind that $\cos\theta = \frac{x}{1} = x$, we have the following:

1. As θ increases from 0 to $\frac{\pi}{2}$, $x = \cos\theta$ decreases from 1 to 0.
2. As θ increases from $\frac{\pi}{2}$ to π , $x = \cos\theta$ decreases from 0 to -1.
3. As θ increases from π to $\frac{3\pi}{2}$, $x = \cos\theta$ increases from -1 to 0.
4. As θ increases from $\frac{3\pi}{2}$ to 2π , $x = \cos\theta$ increases from 0 to 1.

Based on this analysis, we have the graph of $f(\theta) = \cos\theta$ as shown below:

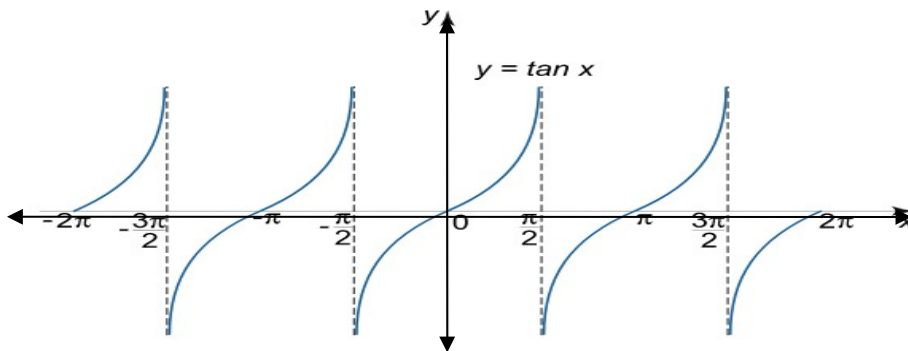


- **The graph of $y = \tan \theta$**

Since $\tan \theta = \frac{y}{x}$ is undefined whenever $x=0$, $\tan \theta$ is undefined whenever the terminal side of the angle corresponding to θ falls on the y -axis. This happens for $\theta = \frac{\pi}{2}$, to which we can add or subtract any multiple of π that will again bring the terminal side back to the y -axis. Thus, domain of $\tan \theta$ is $\{\theta : \theta \neq \frac{\pi}{2} + n\pi\}$, where n is an integer.

1. As θ increases from 0 to $\frac{\pi}{2}$, x decreases from 1 to 0 and y increases from 0 to 1; therefore, $\tan \theta = \frac{y}{x}$ increases from 0 to ∞ .
2. As θ increases from $\frac{\pi}{2}$ to π , x decreases from 0 to -1 and y decreases from 1 to 0; therefore, $\tan \theta = \frac{y}{x}$ increases from $-\infty$ to 0.
3. As θ increases from π to $\frac{3\pi}{2}$, x increases from -1 to 0 and y decreases from 0 to -1; therefore, $\tan \theta = \frac{y}{x}$ increases from 0 to ∞ .
4. As θ increases from $\frac{3\pi}{2}$ to 2π , x increases from 0 to 1 and y increases from -1 to 0; therefore, $\tan \theta = \frac{y}{x}$ increases from $-\infty$ to 0.

You may want to add some more specific values to this analysis. In any case, we get the following as the graph of the tangent function.



Definition 2.26: (Periodic function)

A function $y = f(x)$ is called periodic if there exists a number p such that $f(x + p) = f(x)$ for all x in the domain of f . The smallest such number p is called the period of the function.

A periodic function keeps repeating the same set of y -value over and over again. The graph of a periodic function shows the same basic segment of its graph being repeated. In the case of sine and cosine functions, the period is 2π . The period of the tangent function is π .

Definition 2.27: (Amplitude of a periodic function)

The amplitude of a periodic function $f(x)$ is

$$A = \frac{1}{2} [\text{maximum value of } f(x) - \text{minimum value of } f(x)]$$

Thus, the amplitude of the basic sine and cosine function is 1.

The portion of the graph of a sine or cosine function over one period is called a complete cycle of the graph. In other words, the minimal portion of a sine or cosine graph that keeps repeating itself is called a complete cycle of the graph.

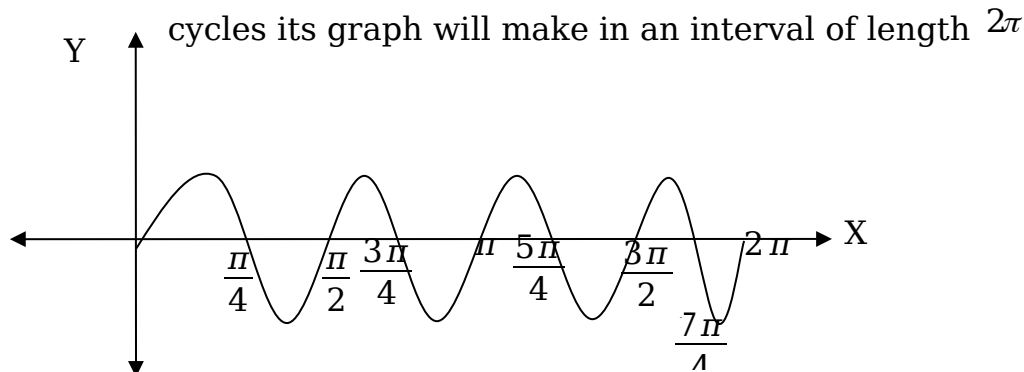
Definition 2.28: (Frequency of a periodic function)

The number of complete cycles a sine or cosine graph makes on an interval of length equal to 2π is called its frequency.

The frequency of the basic sine curve $y = \sin x$ and the basic cosine curve $y = \cos x$ is 1, because each graph makes 1 complete cycle in the interval $[0, 2\pi]$.

If a sine function has period of $\frac{\pi}{2}$ (see the figure below), then the number of

complete
cycles its graph will make in an interval of length 2π
is $\frac{2\pi}{\pi/2} = 4$.



Thus if a sine function has a period of $\frac{\pi}{2}$, its frequency is 4 and its graph will make 4 complete cycles in an interval of length 2π .

Example 2.54: Sketch the graph of $y = \sin 2x$ and find its amplitude, period and frequency.

Solution: We can obtain this graph by applying our knowledge of the basic sine graph. For the basic curve, we have

$$\sin 0 = 0 \quad \sin \frac{\pi}{2} = 1 \quad \sin \pi = 0 \quad \sin \frac{3\pi}{2} = -1 \quad \sin 2\pi = 0$$

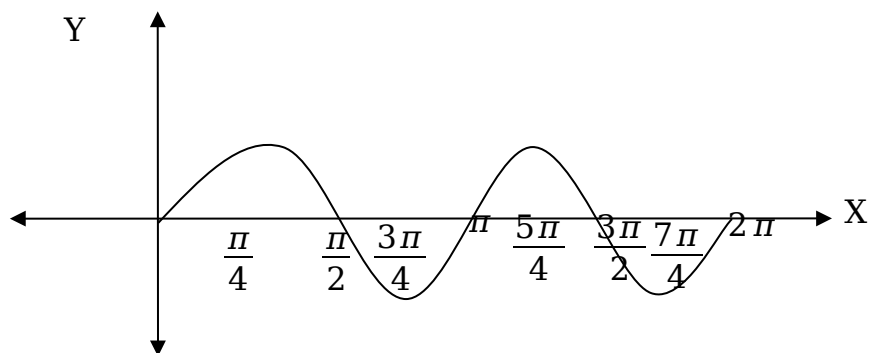
These quadrantal values serve as guide points, which help us draw the graph. To obtain similar guide points for $y = \sin 2x$, we ask for what values of x is

$$2x = 0 \quad 2x = \frac{\pi}{2} \quad 2x = \pi \quad 2x = \frac{3\pi}{2} \quad 2x = 2\pi$$

and we get

$$x = 0 \quad x = \frac{\pi}{4} \quad x = \frac{\pi}{2} \quad x = \frac{3\pi}{4} \quad x = \pi$$

Thus, $y = \sin 2x$ will have the values 0, 1, 0, -1, 0 at $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4},$ and π , respectively. The graph of $y = \sin 2x$ will thus complete one cycle in the interval $[0, \pi]$, and will repeat the same values in the interval $[\pi, 2\pi]$.



From this graph we see that $y = \sin 2x$ has an amplitude of 1, a period π , and a frequency of 2.

For convenience we summarize our discussion on the domains of the trigonometric functions in the table.

1. $f(x) = \sin x$	Domain = All real numbers
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2. $f(x) = \cos x$	Domain = All real numbers
3. $f(x) = \tan x$	Domain = $\{x: x \neq \frac{\pi}{2} + n\pi\}$
4. $f(x) = \csc x$	Domain = $\{x: x \neq n\pi\}$
5. $f(x) = \sec x$	Domain = $\{x: x \neq \frac{\pi}{2} + n\pi\}$
6. $f(x) = \cot x$	Domain = $\{x: x \neq n\pi\}$
	where n is an integer

We have the following trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$

Exercise 2.7

1. Find the domain of the given function.

a) $f(x) = \frac{1}{6^x}$ b) $g(x) = \sqrt{3^x + 1}$ c) $h(x) = \sqrt{2^x - 8}$ d) $f(x) = \frac{1}{2^{3x} - 2}$

2. Sketch the graph of the given function. Identify the domain, range, intercepts, and asymptotes.

a) $y = 5^{-x}$ b) $y = 9 - 3^x$ c) $y = 1 - e^{-x}$ d) $y = e^{x-2}$

3. Solve the given exponential equation.

a) $2^{x-1} = 8$ b) $3^{2x} = 243$ c) $8^x = \sqrt{2}$ d) $16^{3a-2} = \frac{1}{4}$

4. Let $f(x) = 2^x$. Show that $f(x+3) = 8f(x)$.

5. Let $g(x) = 5^x$. Show that $g(x-2) = \frac{1}{25}g(x)$.

6. Let $f(x) = 3^x$. Show that $\frac{f(x+2) - f(2)}{2} = 4(3^x)$.

7. Evaluate the given logarithmic expression (where it is defined).

a) $\log_2 32$ c) $\log_3(-9)$ e) $\log(\log 243)$
b) $\log_{\frac{1}{3}} 9$ d) $\log_6 \frac{1}{\sqrt{6}}$ f) $2^{\log_2 \sqrt{5}}$

8. If $f(x) = \log_2(x^2 - 4)$, find $f(6)$ and the domain of f .

9. If $g(x) = \log_3(x^2 - 4x + 3)$, find $f(4)$ and the domain of g .
10. Show that $\log_6 x = -\log_3 x$
11. Sketch the graph of the given function and identify the domain, range, intercepts and asymptotes.
- a) $f(x) = \log_2(x - 3)$ b) $f(x) = -3 + \log_2 x$ c) $f(x) = -\log_3(-x)$ d) $f(x) = 3\log_3 x$
12. Find the inverse of $f(x) = e^{(3x-1)}$.
13. Let $f(x) = e^{\sqrt{x}}$. Find a function so that $(f \circ g)(x) = (g \circ f)(x) = x$.
14. Convert the given angle from radians to degrees
- a) $\frac{\pi}{3}$ b) $-\frac{5\pi}{2}$ c) $-\frac{4\pi}{3}$
15. Convert the given angle from degrees to radians
- a) 315 b) -40 c) 330
16. Sketch the graph of
- a) $f(\theta) = \sec \theta$ c) $f(\theta) = \csc \theta$ e) $f(\theta) = \cot \theta$
b) $f(x) = 1 + \cos x$ d) $f(x) = \sin(x + \frac{\pi}{2})$ f) $f(x) = \tan 2x$
17. Verify the following identities:
- a) $(\sin x - \cos x)(\csc x + \sec x) = \tan x - \cot x$
b) $\sec^2 x - \csc^2 x = \tan^2 x - \cot^2 x$
18. Given $\tan \theta = \frac{1}{2}$ and $\sin \theta < 0$, find $\cos \theta$.