

Axiomatic Systems and Incidence Geometry

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Axiomatic System

An **axiomatic system**, or **axiom system**, includes:

- Undefined terms

Such terms are necessary to avoid circularity:

- *fast*—adj. 'swift; quick; speedy'
- *swift*—adj. 'rapid; fast'
- *rapid*—adj. 'fast; quick'

...and so on.

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Axiomatic System

An **axiomatic system**, or **axiom system**, includes:

- Undefined terms
- Axioms, or statements about those terms, taken to be true without proof. Also called "postulates."
- Theorems, or statements proved from the axioms (and previously proved theorems)
- (Definitions, which can make things more concise.)

A **model** for an axiom system is an already understood (usually mathematical) system in which:

- every undefined term has a specific meaning in that system, and
- all the axioms are true.

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Example 1

Axiom System:

- Undefined terms: member, committee, on.
- Axiom 1: Every committee has exactly two members on it.
- Axiom 2: Every member is on at least two committees.

Model:

- A **Member** is one of: Joan, Anne, Blair
- A **Committee** is one of: {Joan, Anne}, {Joan, Blair}, {Anne, Blair}
- **On**: Belonging to the set.

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Example 2

Axiom System:

- Undefined terms: member, committee, on.
- Axiom 1: Every committee has exactly two members on it.
- Axiom 2: Every member is on at least two committees.

Model:

- A **Member** is one of: Joan, Anne, Blair, Lacey
- A **Committee** is one of: {Joan, Anne}, {Joan, Blair}, {Joan, Lacey}, {Anne, Blair}, {Anne, Lacey}, {Blair, Lacey}
- **On**: Belonging to the set.

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Example 3 – Axiom System

- Undefined terms: vector, +, $\cdot$

• Axioms:

- For all vectors x and y , $x+y$ is another vector.
- For all real numbers r and all vectors x , $r \cdot x$ is a vector
- For all vectors x and y , $x+y = y+x$.
- For all vectors x , y , and z , $(x+y)+z = x+(y+z)$.
- There is a vector 0 such that for every vector x , $0+x = x+0 = x$.
- For every vector x there is a vector $-x$ such that $(-x) + x = x + (-x) = 0$.
- For every vector x and all real numbers c, d , we have
 - $0 \cdot x = 0$
 - $1 \cdot x = x$
 - $(cd) \cdot x = c \cdot (d \cdot x)$
- For all vectors x and y and real numbers c, d , we have
 - $c \cdot (x+y) = c \cdot x + c \cdot y$
 - $(c+d) \cdot x = c \cdot x + d \cdot x$

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Example 3 – Model 1

- A vector is an ordered pair of real numbers (x, y) .
- $(x, y) + (z, w) = (x+z, y+w)$
- $c \cdot (x, y) = (cx, cy)$

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Example 3 – Model 2

- Vectors are real-valued functions $f: \mathbb{R} \rightarrow \mathbb{R}$.
- $(f + g)(x) = f(x) + g(x)$
- $(c \cdot f)(x) = c \cdot (f(x))$

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Axiom Systems and Models

- Notice that a given axiom system can have more than one model. These models can be quite different. Thus some statements can be true in one model and false in another model.
- However, there is a relationship between **theorems** and models.

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Some facts about statements, axioms, and models:

- Every *theorem* of an axiom system is true in all models of the axiom system (**soundness**).
- A statement that is true in one model could be false in another model and so would not be a theorem. But:
- If a statement is true in *every* model of an axiom system, then it is a theorem (**completeness**).

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Theorems and Models

The World of Theorems and Proofs

- P is a theorem (you can prove P)
- P is **not** a theorem (you can't prove P)
- $\neg P$ is a theorem (you can prove $\neg P$)
- $\neg P$ is not a theorem (you can't prove $\neg P$)

The World of Models

- P is true in every model
- P is false in **some** model
- $\neg P$ is true in every model,
- $\neg P$ is false in **some** model

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Theorems and Models

The World of Theorems and Proofs

- P is a theorem (you can prove P)
- P is **not** a theorem (you can't prove P)
- $\neg P$ is a theorem (you can prove $\neg P$)
- $\neg P$ is not a theorem (you can't prove $\neg P$)

The World of Models

- P is true in every model
- P is false in **some** model
- $\neg P$ is true in every model,
- $\neg P$ is false in **some** model so P is true in **some** model

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Theorems and Models

The World of Theorems and Proofs

-
- You can't prove P
-
- You can't prove $\neg P$

The World of Models

-
- P is false in **some** model
-
- P is true in **some** model

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Theorems and Models

The World of Theorems and Proofs

- You can't prove P
- You can't prove $\neg P$
- Then you can't prove P or its negation. In other words,
- P is **independent** (or **undecidable**)

The World of Models

- P is false in **some** model
- P is true in **some** model
- There is a model where P is false, and another model where P is true.

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Example

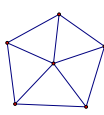
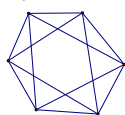
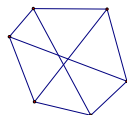
Axioms:

1. There exist exactly six points.
2. Each line is a set of exactly two points.
3. Each point lies on at least three lines.

Which, if any, of the following two statements are theorems in this axioms system? What about their negations?

Statement 1: Each point lies on exactly three lines.

Statement 2: There is a point which lies on more than three lines.



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Example

Axioms:

1. There exist exactly six points.
2. Each line is a set of exactly two points.
3. Each point lies on at least three lines.

Each of the following statements is independent of the axiom system.
We showed that by building models.

Statement 1: Each point lies on exactly three lines.

Statement 2: There is a point which lies on more than three lines.

Statement 3: Five of the six points lie on exactly three lines, and the sixth lies on more than three lines.

Statement 4: Each point lies on exactly four lines.

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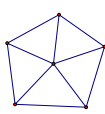
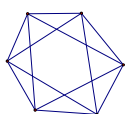
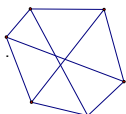
Example

Axioms:

1. There exist exactly six points.
2. Each line is a set of exactly two points.
3. Each point lies on at least three lines.

So these are all possible "sharper" alternatives to Axiom 3:

- 3'. Each point lies on exactly three lines.
- 3". Each point lies on exactly four lines.
- 3'''. Five of the six points lie on exactly three lines, and the sixth lies on more than three lines.



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Some facts about statements, axioms, and models:

Axiom systems ought to be:

- **Consistent**, that is, free from contradictions. This is true provided there is a model for the system. If so, we know we cannot prove a contradiction through logical reasoning from the axioms.
- **Independent**, so that every axiom is independent of the others. Thus, each axiom is essential and cannot be proved from the others. This can be demonstrated using a series of models.

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Example - Inconsistent

Axioms:

1. There exist exactly six points.
2. Each line is a set of exactly two points.
3. Each point lies on at least three lines.
4. There exist at least 47 lines.

Note that each line is exactly two points, and if each possible pair of points made a line, the *largest* number of lines we could get would be $\binom{6}{2} = 15$ possible lines. We have a contradiction: "There are at most 15 lines, and there are at least 47 lines." You can't find a model of that.

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Example - Dependent

Axioms:

1. There exist exactly six points.
2. Each line is a set of exactly two points.
3. Each point lies on at least three lines.
4. No point lies on more than 5 distinct lines.

Note that, given any point A, there are at most 5 other points that could be used to form a distinct line that includes A. Hence there are only five lines that A could lie on. So no point could lie on more than five lines.

Thus Axiom 4 is actually a theorem; it can be proved from Axioms 1, 2, and 3. This axiom system is dependent.

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Some facts about statements, axioms, and models:

In addition, axiom systems can be:

- **Complete**, so that any additional statement appended as an axiom to the system is either redundant (already provable from the axioms) or inconsistent (so its negation is provable). That is, there are no undecidable statements.
- **Categorical**, so that all models for the system are isomorphic, i.e., exactly the same except for renaming.

(Note: If an axiom system is categorical, it is complete. The proof is a fun exercise for you.)

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By the way. . . .

One of the great results of modern mathematical logic is that **any** consistent mathematical system rich enough to develop regular old arithmetic will have undecidable statements. Thus, no such axiom system can be **complete**. This was proved by Kurt Gödel in 1931.

This demonstrates an inherent limitation of mathematical axiom systems; there is no set of axioms for mathematics from which everything can be deduced. There are a number of philosophical debates surrounding this issue.

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What does this have to do with us?

First, in our work in geometry, we will establish an axiom system a little at a time. Occasionally, we will stop to consider whether the axiom we are about to add is in fact independent of the axioms we have established so far. That will mean the new axiom is really adding something – that it couldn't have been proved as a theorem. Thus, we will have to have some facility in creating models.

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What does this have to do with us?

Second, for centuries it was suspected that Euclid's Fifth Postulate was dependent on Postulates I through IV, and one of the great stories in the history of mathematics is how we gradually came to understand its independence from those postulate. This was shown by developing one model that satisfied the first four postulates but not the fifth, and another model that satisfied all five postulates. We will look at these models later in the class.

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Incidence Geometry

And some models

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Incidence Geometry

- Undefined terms: *point*, *line*, *lie on*.
- Axioms:
 1. For every pair of distinct points P and Q there exists exactly one line l such that both P and Q lie on l .
 2. For every line l there exist two distinct points P and Q such that both P and Q lie on l .
 3. There exist three points that do not all lie on any one line.

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Some terminology

- Definition: Three points P , Q , and R are said to be *collinear* provided there is one line l such that P , Q , and R all lie on l . The points are *noncollinear* provided no such line exists.
- (Axiom 3 can now be stated as: "There exist three noncollinear points.")

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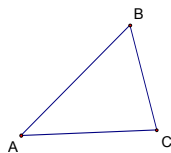
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Example ?

Three point plane:

- Points: Symbols A, B, and C.
- Lines: Pairs of points; $\{A, B\}$, $\{B, C\}$, $\{A, C\}$
- Lie on: "is an element of"



- YES

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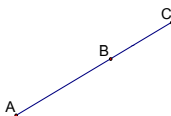
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Example ?

Three point line:

- Points: Symbols A, B, and C.
- Lines: The set of all points: $\{A, B, C\}$
- Lie on: "is an element of"



- NO

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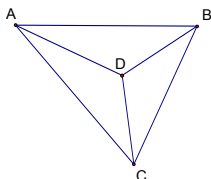
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Example?

Four-point geometry

- Points: Symbols A, B, C and D.
- Lines: Pairs of points: $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$
- Lie on: "is an element of"



- YES

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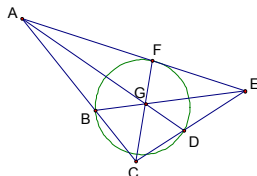
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Example

Fano's Geometry

- Points: Symbols A, B, C, D, E, F, and G.
- Lines: Any of the following: $\{A, B, C\}$, $\{C, D, E\}$, $\{E, F, A\}$, $\{A, G, D\}$, $\{C, G, F\}$, $\{E, G, B\}$, $\{B, D, F\}$
- Lie on: "is an element of"

• YES



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Example?

Cartesian Plane

- Points: All ordered pairs (x, y) of real numbers
- Lines: Nonempty sets of all points satisfying the equation $ax + by + c = 0$ for real numbers a, b, c , with a and b not both zero.
- Lie on: A point lies on a line if the point makes the equation of the line true.

• YES

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Example?

Spherical Geometry:

- Points: $\{(x, y, z) | x^2 + y^2 + z^2 = 1\}$
(In other words, points in the geometry are any regular Cartesian points on the sphere of radius 1 centered at the origin.)
- Lines: Points simultaneously satisfying the equation above and the equation of a plane passing through the origin; in other words, the intersections of any plane containing the origin with the unit sphere. Lines in this model are the "great circles" on the sphere. Great circles, like lines of longitude on the earth, always have their center at the center of the sphere.
- Lie on: A point lies on a line if it satisfies the equation of the plane that forms the line.

• NO

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Example?

The Klein Disk

- Points are all ordered pairs of real numbers which lie strictly inside the unit circle: $\{(x,y) \mid x^2 + y^2 < 1\}$.
- Lines are nonempty sets of all points satisfying the equation $ax + by + c = 0$ for real numbers a, b, c , with not both a and b zero.
- Lies on: Like the previous models; points satisfy the equation of the line.
- Thus, the model is the interior of the unit circle, and lines are whatever is left of regular lines when they intersect that interior.

• YES

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Parallelism and Incidence Geometry

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Definition

- Two lines l and m are said to be *parallel* if there is no point P such that P lies on both l and m . When l and m are parallel, we write $l \parallel m$.

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Parallel Postulates

- For historical reasons, three different possible axioms about parallel lines play an important role in our study of geometry. They are the *Euclidean Parallel Postulate*, the *Elliptic Parallel Postulate*, and the *Hyperbolic Parallel Postulate*.

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Parallel Postulates

- Euclidean*: For every line l and for every point P that does not lie on l , there is **exactly one** line m such that P lies on m and m is parallel to l .
- Elliptic*: For every line l and for every point P that does not lie on l , there is **no** line m such that P lies on m and m is parallel to l .
- Hyperbolic*: For every line l and for every point P that does not lie on l , there are **at least two** lines m and n such that P lies on both m and n and both m and n are parallel to l .

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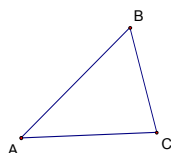
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Which Parallel Postulate is Satisfied?

Three point plane:

- Points: Symbols A , B , and C .
- Lines: Pairs of points; $\{A, B\}$, $\{B, C\}$, $\{A, C\}$
- Lie on: "is an element of"
- ELLIPTIC**



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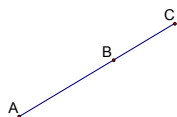
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Which Parallel Postulate is Satisfied?

Three point line:

- Points: Symbols A, B, and C.
- Lines: The set of all points: $\{A, B, C\}$
- Lie on: "is an element of"
- **ALL, but vacuously!**
(Very dull.)



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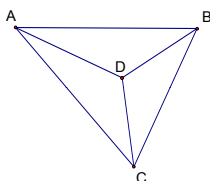
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Which Parallel Postulate is Satisfied?

Four-point geometry

- Points: Symbols A, B, C and D.
- Lines: Pairs of points: $\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{B,C\}$, $\{B,D\}$, $\{C,D\}$
- Lie on: "is an element of"
- **EUCLIDEAN**



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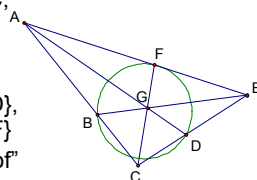
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Which Parallel Postulate is Satisfied?

Fano's Geometry

- Points: Symbols A, B, C, D, E, F, and G.
- Lines: Any of the following: $\{A,B,C\}$, $\{C,D,E\}$, $\{E,F,A\}$, $\{A,G,D\}$, $\{C,G,F\}$, $\{E,G,B\}$, $\{B,D,F\}$
- Lie on: "is an element of"
- **ELLIPTIC**



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Which Parallel Postulate is Satisfied?

Cartesian Plane

- Points: All ordered pairs (x,y) of real numbers
- Lines: Nonempty sets of all points satisfying the equation $ax + by + c = 0$ for real numbers a, b, c , with not both a and b zero.
- Lie on: A point lies on a line if the point makes the equation of the line true.
- **EUCLIDEAN**

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Which Parallel Postulate is Satisfied?

Spherical Geometry:

- Points: $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$
(In other words, points in the geometry are any regular Cartesian points on the sphere of radius 1 centered at the origin.)
- Lines: Points simultaneously satisfying the equation above and the equation of a plane passing through the origin; in other words, the intersections of any plane containing the origin with the unit sphere. Lines in this model are the "great circles" on the sphere. Great circles, like lines of longitude on the earth, always have their center at the center of the sphere.
- Lie on: A point lies on a line if it satisfies the equation of the plane that forms the line.
- **ELLIPTIC**

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Which Parallel Postulate is Satisfied?

The Klein Disk

- Points are all ordered pairs of real numbers which lie strictly inside the unit circle: $\{(x,y) \mid x^2 + y^2 < 1\}$.
- Lines are nonempty sets of all points satisfying the equation $ax + by + c = 0$ for real numbers a, b, c , with not both a and b zero.
- Lies on: Like the previous models; points satisfy the equation of the line.
- Thus, the model is the interior of the unit circle, and lines are whatever is left of regular lines when they intersect that interior.
- **HYPERBOLIC**

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Finite Geometries

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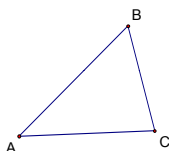
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Three Point Geometry

Axioms:

1. There exist exactly three distinct points.
2. Each two distinct points lie on exactly one line.
3. Each two distinct lines intersect in at least one point.
4. Not all the points are on the same line.



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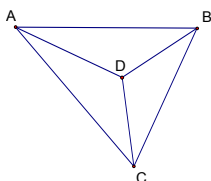
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Four Point Geometry

Axioms:

1. There exist exactly four points.
2. Each pair of points are together on exactly one line.
3. Each line consists of exactly two points.



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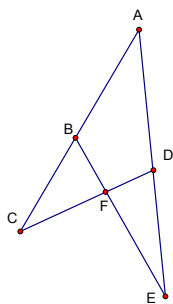
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Four Line Geometry

Axioms:

1. There exist exactly four lines.
2. Each pair of lines has exactly one point in common.
3. Each point is on exactly two lines.



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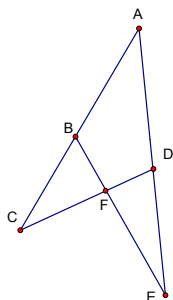
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Four Line Geometry

By the way, is this an Incidence Geometry?

NO.



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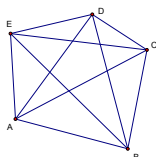
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Five Point Geometry

Axioms:

1. There exist exactly five points.
2. Each two distinct points have exactly one line on both of them.
3. Each line has exactly two points.



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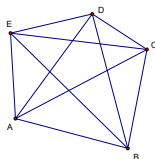
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Five Point Geometry

By the way, which
parallel postulate
does the Five Point
Geometry satisfy?

HYPERBOLIC



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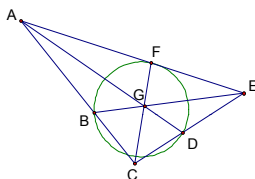
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Fano's Geometry

Axioms:

1. Every line of the geometry has exactly three points on it.
2. Not all points of the geometry are on the same line.
3. There exists at least one line.
4. For each two distinct points, there exists exactly one line on both of them.
5. Each two lines have at least one point in common.



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Young's Geometry

Axioms:

1. Every line of the geometry has exactly three points on it.
2. Not all points of the geometry are on the same line.
3. There exists at least one line.
4. For each two distinct points, there exists exactly one line on both of them.
5. For each line l and each point P not on l , there exists exactly one line on P that does not contain any points on l .

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Model for Young's Geometry

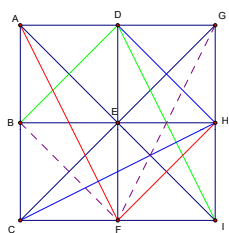
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A	A	A	B	B	B	C	C	D	D	G	H
B	D	E	E	D	F	F	E	E	H	H	F
C	G	I	H	I	G	I	G	F	C	I	A

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Model for Young's Geometry



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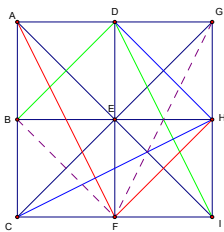
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Model for Young's Geometry

- By the way, which Parallel Postulate is satisfied by Young's Geometry?

- **EUCLIDEAN**



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Interesting Fact:

- The axiom sets for these finite geometries are categorical, and are thus complete. You can decide if a statement is a theorem simply by deciding if it is true of “the” model.

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Interesting Fact:

- An Incidence Geometry that satisfies the Euclidean parallel postulate is called an **affine geometry**. These include:
 - Four point geometry (the Four Point Affine Plane)
 - Young’s geometry (the Nine Point Affine Plane)
 - Cartesian Plane

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