



Chapter 3

LIMIT AND CONTINUITY

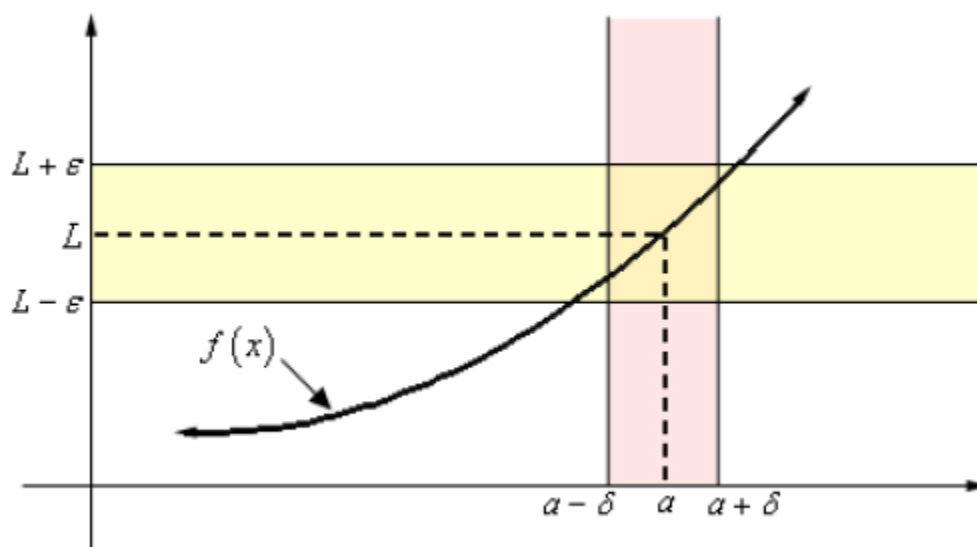
3.1 Definition of Limit

Definition : (Formal Definition of Limit)

Let f be a function defined on an open interval containing a , with the possible exception of a itself. Then the limit of $f(x)$ as x approaches a is the number L ,

written as $\lim_{x \rightarrow a} f(x) = L$ iff for every $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$

This idea can be represented graphically by the figure below



Note:

- The calculated δ should be given as a function of known value ϵ
- The calculated δ is not unique i.e. for δ value, Taking $\delta' < \delta$ is true



Example 1: Use the formal definition of the limit to prove the following.

$$\lim_{x \rightarrow 2} 5x - 4 = 6$$

Solution: Let us start by letting $\epsilon > 0$ be any number then we need to find a number $\delta > 0$ so that the following will be true.

$$|(5x-4)-6| < \epsilon \quad \text{whenever} \quad 0 < |x-2| < \delta$$

We'll start by simplifying the left inequality in an attempt to get a guess for δ .

$$\text{Doing this gives } |(5x-4)-6| < \epsilon \Rightarrow |(5x-10)| = 5|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{5}$$

So, if we do enough simplification on the left inequality we get something that looks like the right inequality and this leads us to choose $\delta = \frac{\epsilon}{5}$

Let's now verify this guess. So, again let $\epsilon > 0$ be any number and then choose

$$\delta = \frac{\epsilon}{5} \quad \text{Next, assume that, } 0 < |x-2| < \delta = \frac{\epsilon}{5} \text{ and we get the following,}$$

$$|(5x-4)-6| = |(5x-10)| = 5|x-2| < 5\left(\frac{\epsilon}{5}\right) = \epsilon$$

So, we've shown that $|(5x-4)-6| < \epsilon$ whenever $0 < |x-2| < \frac{\epsilon}{5}$ and so by our definition we have, $\lim_{x \rightarrow 2} 5x - 4 = 6$

Example 2: Use the definition of the limit to prove the following limit.

$$\lim_{x \rightarrow 4} x^2 + x - 11 = 9$$

Solution: Let $\epsilon > 0$ be any number then we need to find a number $\delta > 0$ so that the following will be true.

$$|(x^2 + x - 11) - 9| < \epsilon \quad \text{wherever} \quad 0 < |x-4| < \delta$$

$$|(x^2 + x - 11) - 9| = |x^2 + x - 20| = |(x+5)(x-4)| = |x+5||x-4| < \epsilon$$

We have managed to show that $|(x^2 + x - 11) - 9| < \epsilon \Rightarrow |x+5||x-4| < \epsilon$

However, unlike the previous example, we have got an extra term in here that doesn't show up in the right inequality above. We are going to find some way to get rid of the $|x+5|$.



To do this let's just note that if, by some chance, we can show that $|x + 5| < K$ for some number K then, we'll have the following,

$$|x + 5| |x - 4| < K |x - 4|$$

If we now assume that what we really want to show is $|x - 4| < \epsilon$ instead of

$$|x + 5| |x - 4| < \epsilon \quad \text{we get the following,} \quad |x - 4| < \frac{\epsilon}{K}$$

All this work however, is based on the assumption that we can show that $x + 5 < K$ for some K . We are only concerned with what is happening around the point in question, $x = 4$ in this case. So, it is safe to assume that whatever x is, it must be close to $x = 4$. Or in terms of an inequality, we can assume that, $|x - 4| < 1$

Why choose 1 here? There is no reason other than it is a nice number to work with. We could just have easily chosen 2, or 5, or fraction number.. The only difference our choice will make is on the actual value of K that we end up with.

So, let's start with $|x - 4| < 1$ then, $-1 < x - 4 < 1 \Rightarrow 3 < x < 5$

If we now add 5 to all parts of this inequality we get, $8 < x + 5 < 10$

So, provided $|x - 4| < 1$ we can see that $x + 5 < 10$ which in turn gives us,

$$|x - 4| < \frac{\epsilon}{K} = \frac{\epsilon}{10}$$

So, to this point we make two assumptions about $|x - 4|$. We've assumed that,

$$|x - 4| < \frac{\epsilon}{K} \quad \text{and} \quad |x - 4| < 1$$

It may not seem like it, but we're now ready to choose a δ . In the previous examples we had only a single assumption and we used that to give us δ . In this case we've got two and they both need to be true. So, will let δ be the smaller of the two assumptions, 1 and $\frac{\epsilon}{10}$. Mathematically, this is written as, $\delta = \min\{1, \frac{\epsilon}{10}\}$

By doing this we can guarantee that, $\delta \leq \frac{\epsilon}{10}$ and $\delta \leq 1$

Now, we've made our choice for δ we need to verify it. So, $\epsilon > 0$ be any number and then choose $\delta = \min\{1, \frac{\epsilon}{10}\}$

Assume that, $0 < |x - 4| < \delta = \min\{1, \frac{\epsilon}{10}\}$. First, we get that,

$$0 < |x - 4| < \delta \leq \frac{\epsilon}{10} \Rightarrow |x - 4| < \frac{\epsilon}{10}$$

We also get, $0 < |x - 4| < \delta \leq 1 \Rightarrow |x - 4| < 1 \Rightarrow |x + 5| < 10$

Finally, all we need to do is,



$$|(x^2 + x - 11) - 9| = |x^2 + x - 20| = |(x+5)(x-4)| < 10|x-4| < 10 \cdot \frac{\epsilon}{10} < \epsilon$$

We have now managed to show that,

$|(x^2 + x - 11) - 9| < \epsilon$ whenever $0 < |x - 4| < \min\{1, \frac{\epsilon}{10}\}$ and so by our definition we

have, $\lim_{x \rightarrow 4} x^2 + x - 11 = 9$

Example 3: Using the definition of limit prove that

$$\lim_{x \rightarrow 3} \frac{2}{x+3} = \frac{1}{3}$$

Solution: Let $\epsilon > 0$ be given. We must find δ such that $0 < |x - 3| < \delta \Rightarrow |f(x) - L| < \epsilon$. Consider, $|\frac{2}{x+3} - \frac{1}{3}| = \frac{1}{3} |\frac{x-3}{x+3}| < \epsilon$

Take $\delta = 1$, then

$$|x - 3| < \delta \Rightarrow |x - 3| < 1$$

$$\Rightarrow -1 < x - 3 < 1$$

$$\Rightarrow 2 < x < 4 \Rightarrow 5 < |x+3| < 7$$

$$\Rightarrow \frac{1}{7} < \frac{1}{|x+3|} < \frac{1}{5} \text{ thus } \frac{1}{3} \frac{|x-3|}{|x+3|} < \frac{1}{15} |x-3| < \epsilon \text{ iff } |x-3| < 15\epsilon$$

Now choose $\delta = \min(1, 15\epsilon)$

$$\text{Thus, } 0 < |x - 3| < \delta \Rightarrow \left| f(x) - \frac{1}{3} \right| = \left| \frac{2}{x+3} - \frac{1}{3} \right| = \frac{1}{3} \frac{|x-3|}{|x+3|} < \frac{1}{15} 15\epsilon = \epsilon$$

Therefore, by definition $\lim_{x \rightarrow 3} \frac{2}{x+3} = \frac{1}{3}$

Example 4. A machinist is required to manufacture a circular metal disk with area 1000cm^2

- What radius produces such a disk?
- If the machinist is allowed error tolerance of $\pm 5\text{cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius

Solution: a) We know that the area of a circle with radius r is given by πr^2

$$\Rightarrow \pi r^2 = 1000$$

$$\Rightarrow r = \sqrt{\frac{1000}{\pi}} \text{cm} = 17.84\text{cm}$$

- The error tolerance $= \pm 5 \Rightarrow \epsilon = 5$

Let $A(r)$ be the area of the circular metal disk for radius r

Consider $a = 17.84$ and $A(r) = f(r)$, $L = 1000$.



Hence applying the definition, there should be positive δ such that

$$0 < |r - 17.84| < \delta \Rightarrow |f(r) - 1000| < 5$$

$$|f(r) - 1000| = \left| \pi r^2 - 1000 \right| = \left| \pi \left(r^2 - \frac{1000}{\pi} \right) \right| = \pi \left| \left(r - \sqrt{\frac{1000}{\pi}} \right) \left(r + \sqrt{\frac{1000}{\pi}} \right) \right|$$

$$= \pi |r - 17.84| |r + 17.84| \text{ to get ride of } |r + 17.84|,$$

$$\text{Let } \delta_1 = 1$$

$$0 < |r - 17.84| < \delta \Rightarrow |r - 17.84| < 1$$

$$\Rightarrow -1 < r - 17.84 < 1$$

$$\Rightarrow 16.84 < r < 18.84$$

$$\Rightarrow 16.84 + 17.84 < r + 17.84 < 18.84 + 17.84 \Rightarrow 34.68 < r + 17.84 < 36.68$$

$$|f(r) - 1000| = \pi |r - 17.84| |r + 17.84| < 36.68\pi |r - 17.84| < \epsilon = 5$$

$$\Rightarrow |r - 17.84| < \frac{5}{36.68\pi} = 0.043$$

$$\Rightarrow \text{Now choose } \delta_2 = 0.043$$

$$\text{choose } \delta = \min \{ \delta_1, \delta_2 \} = \min \{ 1, 0.043 \} = 0.043$$

Thus to keep ± 5 the error tolerance, the machinist should take radius of ± 0.043 from the radius 17.84 value. That is the machinist should take the radius of the metal disk within the interval

$$(17.84 - 0.043, 17.84 + 0.043) = (17.797, 17.883)$$

3.2 Basic Limit Theorems

First we will assume that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist and that c is any constant.

Then,

$$1. \lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$$

$$2. \lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$4. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{Provided that } \lim_{x \rightarrow a} g(x) \neq 0$$

$$5. \lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n \text{ where } n \text{ is any real number. In this property } n \text{ can be any real number (positive, negative, integer, fraction, irrational, zero, etc.).}$$

In the case that n is an integer this rule can be thought of as an extended case of 3.

Example: Evaluate the following limits



a) $\lim_{x \rightarrow 5} (2x^2 - 3x + 4)$

b) $\lim_{h \rightarrow 0} \frac{(3+h)^2 - h}{h}$

c) $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

d) $\lim_{x \rightarrow -1} \frac{x^2 + 2x + 1}{x^4 - 1}$

Solution

a) $\lim_{x \rightarrow 5} (2x^2 - 3x + 4) = 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 = 2(25) - 3(5) + 4 = 39$

b) $\lim_{h \rightarrow 0} \frac{(3+h)^2 - h}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 6h}{h} = \lim_{h \rightarrow 0} h + \lim_{h \rightarrow 0} 6 = 6$

c) $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \lim_{h \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{6}$

d) $\lim_{x \rightarrow -1} \frac{x^2 + 2x + 1}{x^4 - 1} = \lim_{x \rightarrow -1} \frac{x+1}{(x^2+1)(x-1)} = 0$

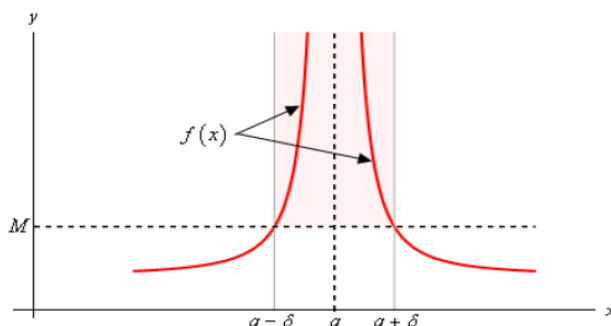
3.3 One sided Limits

Definition 1. For the right-hand limit we say that, $\lim_{x \rightarrow a^+} f(x) = L$ if for everynumber $\epsilon > 0$ there is some number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta$ (or $a < x < a + \delta$)**Definition 2.** For the left-hand limit we say that, $\lim_{x \rightarrow a^-} f(x) = L$ if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $-\delta < x - a < 0$ (or $a - \delta < x < a$)**Example.** Use the definition of the limit to prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ **Solution:** Let $\epsilon > 0$ be any number then we need to find a number $\delta > 0$ so that the following will be true. $|\sqrt{x} - 0| < \epsilon$ whenever $0 < x - 0 < \delta$ Or upon a little simplification we need to show, $\sqrt{x} < \epsilon \Rightarrow x < \epsilon^2$ So, it looks like we can choose $\delta = \epsilon^2$ Let's verify this. Let $\epsilon > 0$ be any number and choose $\delta = \epsilon^2$.Next assume that $0 < x < \epsilon^2$. This gives, $|\sqrt{x} - 0| = \sqrt{x} < \sqrt{\epsilon^2} = \epsilon$ We now have shown that, $|\sqrt{x} - 0| < \epsilon$ whenever $0 < x - 0 < \epsilon^2$ and so by the definition of the right-hand limit we have, $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$

**3.4 Infinite Limits, Limit at infinity and Asymptotes**

Definition1. Let $f(x)$ be a function defined on an interval that contains a , except possibly at a . Then we say that, $\lim_{x \rightarrow a} f(x) = \infty$ if for every number $M > 0$ there is some number $\delta > 0$ such that $f(x) > M$ Whenever $0 < x - a < \delta$

Definition2. Let $f(x)$ be a function defined on an interval that contains a , Except possibly at a . Then we say that, $\lim_{x \rightarrow a} f(x) = -\infty$ if for every number $N < 0$ there is some number $\delta > 0$ such that $f(x) < N$ whenever $0 < x - a < \delta$



No matter how large we choose M to be we can always find an interval around $x = a$, given by $0 < x - a < \delta$ for some number δ , so that as long as we stay within that interval the graph of the function will be above the line $y = M$ as shown in the graph.

Example Use the definition of the limit to prove the following limit. $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$

Solution: Let $M > 0$ be any number and we will need to choose a $\delta > 0$ so that,

$$\frac{1}{x^2} > M \text{ whenever } 0 < |x - 0| < \delta$$

Now we will start with the left inequality and try to get something in the end that looks like the right inequality. Recall that $\sqrt{x^2} = |x|$

$$\frac{1}{x^2} > M \Rightarrow x^2 < \frac{1}{M} \Rightarrow |x| < \frac{1}{\sqrt{M}}$$



So, it looks like we can choose $\delta = \frac{1}{\sqrt{M}}$. All we need to do now is verify this guess.

Let $M > 0$ be any number, choose $\delta = \frac{1}{\sqrt{M}}$ and assume that $0 < |x| < \frac{1}{\sqrt{M}}$

Doing this work gives, $|x| < \frac{1}{\sqrt{M}} \Rightarrow |x|^2 < \frac{1}{M} \Rightarrow x^2 < \frac{1}{M} \Rightarrow \frac{1}{x^2} > M$

So, we have managed to show that, $\frac{1}{x^2} > M$ whenever $0 < |x-0| < \frac{1}{\sqrt{M}}$

and so by the definition of the limit we have, $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$

Definition: The line $x = a$ is called a vertical asymptote of the graph of $y = f(x)$ if any one of the following limits holds true:

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty, \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty, \quad \lim_{x \rightarrow a} f(x) = \pm\infty$$

Example: $x = 0$ is the vertical asymptote of $f(x) = \frac{1}{x^2}$

Limit at infinity and Horizontal asymptotes

Definition:

- i. Let f be defined on an interval (a, ∞) , then $\lim_{x \rightarrow \infty} f(x) = L$ for every $\epsilon > 0$ there is a number $M > 0$ such that if $x > M$, then $|f(x) - L| < \epsilon$
- ii. Let f be defined on an interval $(-\infty, a)$, then $\lim_{x \rightarrow -\infty} f(x) = L$ for every $\epsilon > 0$ there is a number $M < 0$ such that if $x < M$, then $|f(x) - L| < \epsilon$

Definition: The line $y = L$ is called a horizontal asymptote if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

Example: The line $y = 0$ is the horizontal asymptote of $f(x) = \frac{1}{x}$

Example: Find the vertical and horizontal asymptotes of $f(x) = \frac{\sqrt{4x^2 + 1}}{2x - 4}$

Solution: Here the function is undefined for value where $2x - 4 = 0 \Rightarrow x = 2$

$$\lim_{x \rightarrow 2^+} \frac{\sqrt{4x^2 + 1}}{2x - 4} = \infty$$



Thus, the line $x = 2$ is the vertical asymptote of $f(x)$

To find the horizontal Asymptote

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{2x - 4} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2(4 + \frac{1}{x^2})}}{x(2 - \frac{4}{x})} = \lim_{x \rightarrow \infty} \frac{|x| \sqrt{(4 + \frac{1}{x^2})}}{x(2 - \frac{4}{x})} = 1$$

$$\text{Again, find } \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{2x - 4} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(4 + \frac{1}{x^2})}}{x(2 - \frac{4}{x})} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{(4 + \frac{1}{x^2})}}{x(2 - \frac{4}{x})} = -1$$

Thus, the lines $y = 1$ and $y = -1$ are the horizontal asymptote of $f(x)$.

Example: The increase in stress on a soil element located at a depth z vertically down due to a point load Q applied at a radial distance r is given by

$$\Delta\sigma_z = \frac{3Q}{2\pi z^2} \left[\frac{z^2}{r^2 + z^2} \right]^{5/2} \quad (\text{Muni Budhu page 136})$$

Discuss about the stress effect on a soil particle

- When the depth increase for constant radial distance
- When the radial distance increase at a constant depth

Solution:

$$\Delta\sigma_z = \frac{3Q}{2\pi z^2} \left[\frac{z^2}{r^2 + z^2} \right]^{5/2} = \frac{3Q}{2\pi z^2} \left[\frac{1}{1 + (\frac{r}{z})^2} \right]^{5/2}$$

a)

$$\lim_{z \rightarrow \infty} \Delta\sigma_z = \lim_{z \rightarrow \infty} \frac{3Q}{2\pi z^2} \left[\frac{1}{1 + (\frac{r}{z})^2} \right]^{5/2} = \lim_{z \rightarrow \infty} \frac{3Q}{2\pi z^2} = 0$$

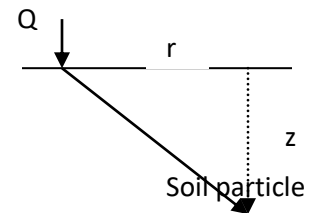
As it is known, as the depth increases the effect of the stress on a particle is none.

$$\Delta\sigma_z = \frac{3Q}{2\pi z^2} \left[\frac{z^2}{r^2 + z^2} \right]^{5/2} = \frac{3Q}{2\pi z^2} \left[\frac{(\frac{z}{r})^2}{1 + (\frac{z}{r})^2} \right]^{5/2} = \frac{3Q}{2\pi z^2} \left(\frac{z}{r} \right)^5 \left[\frac{1}{1 + (\frac{z}{r})^2} \right]^{5/2}$$

b)

$$\lim_{r \rightarrow \infty} \Delta\sigma_z = \lim_{r \rightarrow \infty} \frac{3Q}{2\pi z^2} \left(\frac{z}{r} \right)^5 \left[\frac{1}{1 + (\frac{z}{r})^2} \right]^{5/2} = \lim_{r \rightarrow \infty} \frac{3Qz^3}{2\pi r^5} = 0$$

As we are considering a particle farther from the point load keeping the depth constant, the effect of the stress is none



**3.5 Continuity of a function; One sided continuity; Intermediate Value Theorem****Continuity of function****Definition**

A function $f(x)$ is said to be **continuous** at $x = a$ if $\lim_{x \rightarrow a} f(x) = f(a)$.

Note: For the above definition one should check the following

- a) $f(a)$ is defined
- b) $\lim_{x \rightarrow a} f(x)$ exist
- c) $\lim_{x \rightarrow a} f(x) = f(a)$

If f is not continuous at a , then we say that f is discontinuous at a .

Example: $f(x) = \frac{x}{x+3}$ is continuous at 2 because $\lim_{x \rightarrow 2} \frac{x}{x+3} = \frac{2}{5} = f(2)$

Example: $g(x) = \begin{cases} 3\frac{\sin 2x}{6x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous at 0.

Because $\lim_{x \rightarrow 0} 3\frac{\sin 2x}{6x} = 1 = f(0)$

Example: $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ is discontinuous at 2

Because $\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x - 2} = 3 \neq f(2) = 1$

**One sided continuity****Definition:**

- (a) A function f is said to be continuous from the right at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$
- (b) A function f is said to be continuous from the left at a if $\lim_{x \rightarrow a^-} f(x) = f(a)$

Example: Discuss about the one sided continuity of $f(x) = \sqrt{1-x^2}$

Solution: $\lim_{x \rightarrow -1^+} \sqrt{1-x^2} = f(-1) = 0$ and $\lim_{x \rightarrow 1^-} \sqrt{1-x^2} = f(1) = 0$

Hence f is continuous to the right of -1 and to the left of 1

Continuity on Intervals**Definition:**

1. A function f is said to be continuous on (a,b) if it is continuous $\forall c \in (a,b)$
2. A function f is said to be continuous on $[a,b)$ if
 - a) $\forall c \in (a,b)$
 - b) $\lim_{x \rightarrow a^+} f(x) = f(a)$
3. A function f is said to be continuous on $(a,b]$ if
 - a) $\forall c \in (a,b)$
 - b) $\lim_{x \rightarrow b^-} f(x) = f(b)$
4. A function f is said to be continuous on $[a,b]$ if
 - a) $\forall c \in (a,b)$
 - b) $\lim_{x \rightarrow a^+} f(x) = f(a)$
 - c) $\lim_{x \rightarrow b^-} f(x) = f(b)$



Example: The function $f(x) = \sqrt{1-x^2}$ is continuous on $[-1,1]$

The function $f(x) = \sqrt{x-3}$ is continuous on $[3, \infty)$

Theorems on Continuity

1. If $f(x)$ is continuous at $x=b$ and $\lim_{x \rightarrow a} g(x) = b$ then,

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$$

Example: Evaluate $\lim_{x \rightarrow 0} e^{\sin x}$

Solution: Since we know that exponentials are continuous everywhere we can use the fact above. $\lim_{x \rightarrow 0} e^{\sin x} = e^{\lim_{x \rightarrow 0} \sin x} = e^0 = 1$

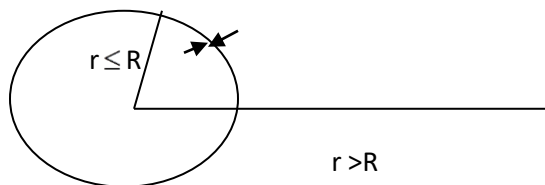
2. If f and g are continuous at a , and c is a constant then the following are also continuous at a .

$$f + g, f - g, cf, \text{ where } c \in \mathbb{R}, \frac{f}{g} \text{ where } g(a) \neq 0$$

3. Polynomial, Rationals, Root functions, Trigonometric functions Inverse trigonometric functions, exponential functions and Logarithmic functions are continuous on their domain.

Example: According to scientific finding, the gravitational force exerted by the earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \frac{GMr}{R^3}, & r < R \\ \frac{GM}{r^2}, & r \geq R \end{cases}$$



Where M and r are the mass and radius of the earth respectively and G is the gravitational constant. Is F a continuous function of r ?



Solution: The function f is clearly continuous for, $0 < r < R$ and $r > R$. The only place where discontinuity is suspected is at $r = R$. The function F is said to be continuous at R if $\lim_{r \rightarrow R^+} f(r) = f(R) = \lim_{r \rightarrow R^-} f(r)$

$$\text{Hence, } \lim_{r \rightarrow R^+} f(r) = \lim_{r \rightarrow R^+} \frac{GM}{r^2} = \frac{GM}{R^2} \quad \text{and} \quad \lim_{r \rightarrow R^-} f(r) = \lim_{r \rightarrow R^-} \frac{GMr}{r^3} = \lim_{r \rightarrow R^-} \frac{GM}{r^2} = \frac{GM}{R^2}$$

Thus the function is continuous. This implies that the gravitational force extremely closer to the surface beneath is the same as closer to the surface above.

Example: Find the constant a and b so that f can be continuous everywhere

$$f(x) = \begin{cases} ax + 3, & x < 1 \\ 4, & x = 1 \\ x^2 + b, & x > 1 \end{cases}$$

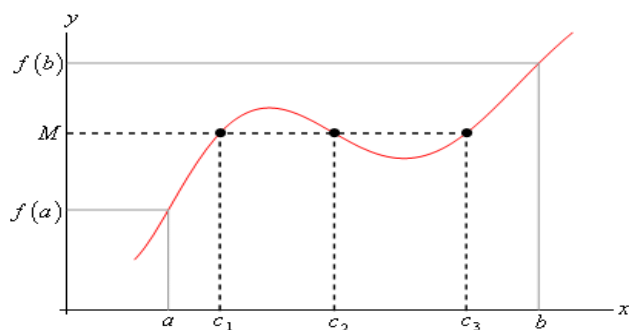
Solution: The function f is suspected to be discontinuous at 1; hence continuity should be assured at 1. Hence $\lim_{x \rightarrow 1^+} f(x) = f(1) = \lim_{x \rightarrow 1^-} f(x)$ Thus,

$$\lim_{x \rightarrow 1^-} ax + 3 = f(1) = a + 3 = 4 \Rightarrow a = 1$$

$$\lim_{x \rightarrow 1^+} x^2 + b = f(1) = 1 + b = 4 \Rightarrow b = 3$$

Intermediate Value Theorem (IVT)

Let f be continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$ then there exists at least one number c in (a, b) such that $f(c) = M$



Example: Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root in the interval $[-1, 2]$

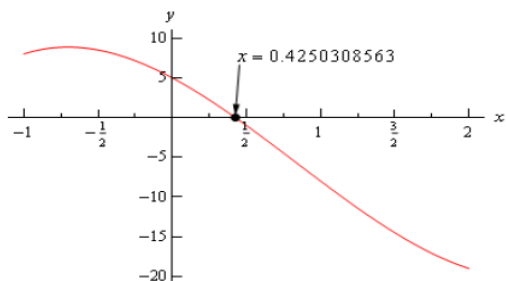


Solution: To do this all we need to do is to compute, $P(-1) = 8$ and $P(2) = -19$. Hence $-19 = P(2) < 0 < P(-1) = 8$

Therefore $M = 0$ is between $P(-1)$ and $P(2)$ and since $P(x)$ is a polynomial it is continuous everywhere and so in particular it is continuous on the interval $[-1, 2]$. So by the Intermediate Value Theorem there must be a number $-1 < c < 2$ so that, $P(c) = 0$. Therefore the polynomial does have a root between -1 and 2 .

For the sake of completeness here is a graph showing the root that we just proved existed.

Note that we used a computer program to actually find the root and that the Intermediate Value Theorem did not tell us what this value was.



Example:

When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by $Q(t) = Q_0(1 - e^{-\frac{t}{a}})$

(Maximum charge capacity = Q_0 in coulomb and time t in seconds)

Show that it takes between 3 and 5 seconds to recharge the capacitor to 90% of capacity if $a = 2$.

Solution:



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$$Q(t) = Q_0(1 - e^{-\frac{t}{a}}) \Rightarrow 90\% Q_0 = Q_0(1 - e^{-\frac{t}{2}})$$

$$\Rightarrow 0.9 = 1 - e^{-\frac{t}{2}} \Rightarrow 0.1 - e^{-\frac{t}{2}} = 0.$$

$$\text{Thus, take } f(t) = 0.1 - e^{-\frac{t}{2}}$$

$$f(3) = 0.1 - e^{-\frac{3}{2}} = -0.123 < 0 \quad \text{and} \quad f(5) = 0.1 - e^{-\frac{5}{2}} = 0.018 > 0$$

Thus, by intermediate value theorem the time to charge 90% of its maximum charge is between 3 and 5 seconds.

$$\text{Note: } 0.1 - e^{-\frac{t}{2}} = 0 \Rightarrow t = -2\ln(0.1) = 4.605 \text{ sec}$$