Chapter 3: Solving System of Equations

3.1 System of Linear Equations

In this section we present the solution of $n$linear simultaneous algebraic equations in $n$unknowns. Linearsystems of equations are associated with many problems in engineering and science, as well as withapplications of mathematics to the social sciences and quantitative study of business and economic problems.

A system of algebraic equations has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Where the coefficients $a_{ij}$and the constants $b_i$are known and $x_i$represent the unknowns. In matrix notation, the equations are written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Or simply$\boldsymbol{Ax = b}$

Note 1:A system of linear equations in $n$unknowns has a unique solution, provided that the determinant of the coefficient matrix is non-singular.

Note 2: If $b_i = 0, \forall i = 1,2,...n$ then the above linear system is called Homogenous otherwise, non-Homogenous.

Methods of Solution

1. **Direct methods**: The common characteristics of direct methods are that they transform the original equation into equivalent equations (equations that have the same solution) that can be solved more easily. The solution does not contain any truncation errors but the round off errors is introduced due to floating point operations.

i) **Gauss Elimination Method:** is a popular technique for solving system of linear equations.

Consider the following system of linear simultaneous equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1(1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2(2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3(3) \end{aligned}$$

In this method, the unknowns are eliminated such that the elimination process leads to an upper triangular system and the unknowns are obtained by back substitution. It is assumed$a_{11} \neq 0$.

The method can be described by the following steps:

Step 1: Eliminate x_1 from the second and third equations.

Using the first equation (1), the following operations are performed:

$(2) - \frac{a_{21}}{a_{11}}(1)$ and $(3) - \frac{a_{31}}{a_{11}}(1)$

gives$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1(4)$

$$a'_{22}x_2 + a'_{23}x_3 = b'_2(5)$$
$$a'_{32}x_2 + a'_{33}x_3 = b'_3(6)$$

Equation (4) is called the pivotal equationand the coefficient a_{11} is the pivot element.

Step 2: Eliminate $x_2$from the (6) using (5) by assuming$a'_{22} \neq 0$.

i.e.,$(6) - \frac{a'_{32}}{a'_{22}}(5)$

we get $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1(7)$

$$a'_{22}x_2 + a'_{23}x_3 = b'_2(8)$$
$$a''_{33}x_3 = b''_3(9)$$

Here (8) is called the pivotal equationand the coefficient a''_{22} is the pivot element.

Step 3: To find x_1, x_2 and x_3, we apply back substitution starting from Eq. (9) giving x_3, then x_2 from Eq. (8) and $x_1$from Eq. (7).

Pivoting: Gauss elimination method fails if any one of the pivots in the above equations (1) to (9) becomeszero.To overcome this difficulty, the equations are to be rewritten in a slightly different order such that the pivotsare not zero.

Partial pivoting method:

Step 1: The numerically largest coefficient of x_1 is selected from all the equations are pivot and thecorresponding equation becomes the first equation (1).

Step 2: The numerically largest coefficient of x_2 is selected from all the remaining equations as *pivot* and thecorresponding equation becomes the second equation (5). This process is repeated till an equation intoa simple variable is obtained.

Complete pivoting method: In this method, we select at each stage the numerically largest coefficient of the complete matrix of coefficients.This procedure leads to an interchange of the equations as well as interchange of the position of variables.

Example: Solve the following equations by Gauss elimination method:

$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$
$$x + 3y + 2z = 5$$

Solution:

Solution:

$2x + 4y - 6z = -4$ (E.1)

$x + 5y + 3z = 10$ (E.2)

$x + 3y + 2z = 5$ (E.3)

To eliminate *x* from (E.2) and (E.3) using (E.1):

i.e., (E.2)$-\frac{1}{2}$(E.1) and (E.3)$-\frac{1}{2}$(E.1), gives

$$2x + 4y - 6z = -4$$

$3y + 6z = 12$ (E.4)

$y + 5z = 7$(E.5)

To eliminate $y$from (E.5) using (E.4): i.e., i.e., (E.5)$-\frac{1}{3}$ (E.4), we obtain

$2x + 4y - 6z = -4$ (E.6)

$3y + 6z = 12$ (E.7)

$3z = 3$ (E.8)

Now, Evaluation of the unknowns by back substitution:

from (E.8) $3z = 3 \Rightarrow z = 1$, from (E.7) $3y + 6(1) = 12$

$\Rightarrow 3y = 12 - 6 = 6$

$\Rightarrow y = 2$ and

from (E.6), $2x + 4(2) - 6(1) = -4$

$\Rightarrow 2x = -4 - 8 + 6 = -6$

$\Rightarrow x = -3.$

Exercise: Use the method of Gaussian elimination to solve the following system of linear equations:

a) $\begin{cases} 2x + 4y + z = 1 \\ x - 2y - 3z = 2 \\ x + y - z = -1 \end{cases}$

b) $\begin{cases} x - 2y + z = 1 \\ -x + y - z = 4 \\ -x + 2y - 4z = 8 \end{cases}$

c) $\begin{cases} x_1 + x_2 + x_3 - x_4 = 2 \\ 4x_1 + 4x_2 + x_3 + x_4 = 11 \\ x_1 - x_2 - x_3 + 2x_4 = 0 \\ 2x_1 + x_2 + 2x_3 - 2x_4 = 2 \end{cases}$

(ii) **LU decomposition method**

It is possible to show that any square matrix Acan be expressed as a productof a lower triangular matrix $L$and an upper triangular matrix U.

$$A = LU$$

For instance

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

The process of computing $L$and $U$for a given $A$is known as LU-Decomposition or LU-Factorization. LU-decomposition is not unique (the combinations of $L$and $U$for a prescribed A are endless), unless certain constraints are placed on Lor U. These constraints distinguish onetype of decomposition from another.

The three decomposition methods are given below:

1. Doolittle's decomposition: Constraints are $L_{ii} = 1$,
2. Crout's decomposition: Constrains are $U_{ii} = 1$ and
3. Cholesky's decomposition: Constraints are $L = U^T$ for $i = 1, 2, 3, \dots, n$.

After decomposing the matrix A, it is easier to solve the equations $Ax = b$.We can rewrite the equations as $LUx = b$

or denoting $Ux = v$, the above equation becomes

$$Lv = b$$

This equation $Lv = b$can be solved for $v$by forward substitution. Then $Ux = v$will yield $x$by thebackward substitution process. The advantage of LU decomposition method over the Gauss eliminationmethod is that once $A$is decomposed, we can solve $Ax = b$for as many constant vectors $b$as we please.Also, the forward and backward substitutions operations are much less time consuming than thedecomposition process.

Doolittle's Method: This method is based on the fact that every square matrix A can be expressed as the product of a lowertriangular matrix and an upper triangular matrix, provided all the principle minors of A are non-singular. Also,such a factorization, if exists, is unique.

Method: Decomposes$A$ to L and U

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- U is the same as the coefficient matrix at the end of the forward elimination step.
- $L$is obtained using the *multipliers* that were used in the forward elimination process.

Consider the system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Finding the U matrix: Using the Forward Elimination Procedure of Gauss Elimination, we get

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a''_{33}x_3 = b''_3$$

$$\text{Thus, } U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

Finding the L matrix: Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad L_{31} = \frac{a_{31}}{a_{11}}$$

$\Rightarrow L_{21} = \frac{a_{21}}{a_{11}}$ and

From the second step of forward elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{pmatrix} \Rightarrow L_{32} = \frac{a'_{32}}{a'_{22}}$$

Thus, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a'_{32}}{a'_{22}} & 1 \end{pmatrix}$

Or, To obtain the elements of $L$and U, we first find the first row of $U$and the first column of L. Then, wedetermine the second row of $U$and the second column of L. Finally, we compute the third row of U.

Example:Solve the following equations by Doolittle's method.

$$\begin{aligned} 2x + y + 4z &= 12 \\ 8x - 3y + 2z &= 20 \\ 4x + 11y - z &= 33 \end{aligned}$$

Solution:

Solution:

We have, $A = \begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $b = \begin{pmatrix} 12 \\ 20 \\ 33 \end{pmatrix}$

Let $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix} \quad \Rightarrow \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \quad \begin{pmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 9 & -9 \end{pmatrix}$$

$\Rightarrow L_{21} = 4$ and $L_{31} = 2$

$$R_3 \to R_3 - \frac{-9}{7} R_2 \begin{pmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{pmatrix} \Rightarrow L_{32} = \frac{-9}{7}.$$

Thus, $L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -9/7 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{pmatrix}$

Writing: $Lv = b$, we get $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -9/7 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \\ 33 \end{pmatrix}$ by forward elimination which gives

$v_1 = 12, \Rightarrow 4v_1 + v_2 = 20 \Rightarrow v_2 = 20 - 4(12) = -28$ and $2v_1 - \frac{9}{7}v_2 + v_3 = 33$

$\Rightarrow v_3 = 33 - 2(12) + \frac{9}{7}(-28) \Rightarrow v_3 = 33 - 24 - 36 = -27.$

The solution to the original system is given by: $Ux = v$

$$\begin{pmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ -28 \\ -27 \end{pmatrix}$$

$\Rightarrow \left.\begin{array}{r} 2x + y + 4z = 12 \\ -7y - 14z = -28 \\ -27z = -27 \end{array}\right\}$, then by back substitutation $z = 1, y = 2$ and $x = 3$.

Exercise: Applying (*a*) Crouts' method (*b*) triangularization method solves the ff equations.

a) $2x_1 + x_2 + x_3 = 7$
$x_1 + 2x_2 + x_3 = 8$
$x_1 + x_2 + 2x_3 = 9$

b) $3x + 2y + 7z = 42x + 3y + z = 53x + 4y + z = 7$

c) $10x + y + z = 122x + 10y + z = 132x + 2y + 10z = 14$

d) $2x - 6y + 8z = 245x + 4y - 3z = 23x + y + 2z = 16$

Exercise: An electrical engineer supervises the production of three types of electrical components. Three kinds of material-metal, plastic, and rubber—are required for production. The amounts needed to produce each component are

Component	Metal (g/component)	Plastic (g/component)	Rubber (g/component)
1	15	0.25	1.0
2	17	0.33	1.2
3	19	0.42	1.6

If totals of $2.12, 0.0434$, and $0.164\ kg$ of metal, plastic, andrubber, respectively, are available each day, how many components can be produced per day?

2. **Iterative or indirect methods**: An iterative technique to solve the $n\ x\ n$linear system $Ax = b$ starts with an initial approximation $x^{(0)}$ to the solution x and generates a sequence ofvectors $\{x^{(k)}\}_{k=1}^{\infty}$thatconverges to x. Iterative techniques involve a process that converts the system $Ax = b$ intoan equivalent system of the form $x = Cx + d$for some fixed matrix$C$and

vector d.After the initial vector $x^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing $x^{(k)} = Cx^{(k-1)} + d$, for each $k = 1,2,3,\ldots$ and $C_{ii} = 0$.

Convergence Criterion

1. $\varepsilon_{a,i} = \left|\frac{x_i^k - x_i^{k-1}}{x_i^k}\right| \times 100\% < \varepsilon$ for all x_i
2. Norm of the residual vector $|Ax - b| < \varepsilon$

Convergence and Diagonal Dominant

- Sufficient condition -A is diagonally dominant. In other words, the solution will exist (iterating will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system $Ax = b$ are greater than thesum of absolute values of the other coefficients of that row.
- Necessary and sufficient condition-The magnitude of the largest eigenvalue of C(*not* A) is less than 1.

(i) **Jacobi's Method**

The Jacobi iterative methodis obtained by solving the i^{th} equation in $Ax = b$for $x_i$to obtain (provided$a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(-\frac{a_{ij}x_j}{a_{ii}}\right) + \frac{b_i}{a_{ii}}, \text{ for } i = 1, 2, \ldots, n.$$

For each $k \geq 1$, generate the components $x_i^{(k)}$of $\boldsymbol{x}^{(k)}$from the components of$\boldsymbol{x}^{(k-1)}$by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \text{ for } i = 1, 2, \ldots, n. \quad \ldots\ldots(*)$$

(ii) **Gauss Seidel Method**

A possible improvement in Jacobi's Method can be seen by reconsidering Eq. (*). The components of $x^{(k-1)}$) are used to compute all the components$x_i^{(k)}$of$\boldsymbol{x}^{(k)}$. But, for $i > 1$, the components$x_1^{(k)}, \ldots, x_{i-1}^{(k)}$of $\boldsymbol{x}^{(k)}$have already been computed and are expected to be better approximations to the actual solutions$x_1, \ldots, x_{i-1}$than are$x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$. It seemsreasonable, then, to compute$x_i^{(k)}$using these most recently calculated values. That is, to use $x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}\left(a_{ij}x_j^{(k)}\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{(k-1)}\right) + b_i\right]$, for $i = 1, 2, \ldots, n,$ instead of Eq. (*).

This modification is called the Gauss-Seideliterative technique.

(iii) **SOR method**

Starting from Gauss-Seidel method, we introduce the successive over-relaxation method (or SOR method)

$x_i^{(k)} = \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1}\left(a_{ij}x_j^{(k)}\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{(k-1)}\right)\right] + (1-\omega)x_i^{(k-1)}$, for $i = 1, 2, \ldots, n.$

It is consistent for any$0 < \omega < 2$ and for$\omega = 1$ it coincides with Gauss-Seidelmethod. In particular, if $\omega \in (0, 1)$ the method is called under-relaxation,while if $\omega > 1$ it is called over-relaxation.The idea is to choose avalue for ωthat will accelerate the rate of convergence of the iterates to the solution.

Example:Solve the following equations by Gauss-Seidel and SOR (for $\omega = 1.2, 1.5, 1.8$)method correct to two decimal places.

a) $\begin{cases} 15x + 3y - 2z = 85 \\ 2x + 10y + z = 51 \\ x - 2y + 8z = 5 \end{cases}$

b) $\begin{cases} x + 2y + z = 0 \\ 3x + y - z = 0 \\ x - y + 4z = 3 \end{cases}$

c) $\begin{cases} 10x_1 - 2x_2 - x_3 - x_4 = 3 \\ -2x_1 + 10x_2 - x_3 - x_4 = 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 = 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 = -9 \end{cases}$

Solution:

Note 1: Iterative methods are generally less efficient than direct methods due to the large number of operations or iterations required.

Note 2: Iterative procedures are self-correcting, meaning that round off errors (or even arithmetic mistakes) in one iteration cycle is corrected in subsequent cycles and the solution contains truncation error.

Note 3: A serious drawback of iterative methods is that they do not always converge to the solution.

Note 4: The initial guess affects only the number of iterations that are required for convergence.

3.2 **System of Non-Linear Equations Using Newton's method**

A system of nonlinear equations has the form

$$\left.\begin{array}{l} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \quad\vdots \\ f_n(x_1, x_2, \dots, x_n) = 0 \end{array}\right\} \quad \dots\dots\dots (1)$$

where at least one f_i is non linear equation.

Now, starting with the initial approximations $\boldsymbol{x^{(0)}} = \left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)^T$, we obtain the sequence of iterates using Newton Raphson method as

$$\boldsymbol{x^{(k+1)}} = \boldsymbol{x^{(k)}} - J^{-1}\boldsymbol{f^{(k)}}, \qquad k = 0, 1, 2, \dots$$

where $\boldsymbol{x^{(k)}} = \left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)^T$, $\boldsymbol{f^{(k)}} = (f_1^{(k)}, f_2^{(k)}, \dots, f_n^{(k)})^T$, $\boldsymbol{f_i^{(k)}} = f_i(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$,

and $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}) & \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}) \\ \frac{\partial f_2}{\partial x_1}(\boldsymbol{x}) & \frac{\partial f_2}{\partial x_2}(\boldsymbol{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\boldsymbol{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\boldsymbol{x}) & \frac{\partial f_n}{\partial x_2}(\boldsymbol{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\boldsymbol{x}) \end{bmatrix}$.

Example: Solve the system $\begin{cases} x^2 + y^2 = 1.12 \\ xy = 0.23 \end{cases}$ correct to three decimal places.

Solution: From the system

$f_1(x, y) = x^2 + y^2 - 1.12 \Rightarrow \frac{\partial f_1}{\partial x} = 2x, \quad \frac{\partial f_1}{\partial y} = 2y$

$f_2(x, y) = xy - 0.23 \Rightarrow \frac{\partial f_2}{\partial x} = y, \quad \frac{\partial f_1}{\partial y} = x$

⇒we have the Jacobian matrix $J(x,y)=\begin{bmatrix}2x & 2y\\ y & x\end{bmatrix}\Rightarrow J^{-1}(x,y)=\frac{1}{2(x^2-y^2)}\begin{bmatrix}x & -2y\\ -y & 2x\end{bmatrix}$

Now using the method $\boldsymbol{x^{(k+1)}=x^{(k)}-J^{-1}f^{(k)}}$ $y_{(2)}=1\ 10\ 51-2x_1-z_1=3.904$ $z_{(2)}=1\ 8\ 5-x_1+2y_1=1.192$ Iteration 3: when $k=3$ $x_{(3)}=1\ 15\ 85-3y_2+2z_2=5.045$ $y_{(3)}=1\ 10\ 51-2x_2-z_2=4.035$ $z_{(3)}=1\ 8\ 5-x_2+2y_2=1.010$

$\begin{pmatrix}x^{(k+1)}\\ y^{(k+1)}\end{pmatrix}=\begin{pmatrix}x^{(k)}\\ y^{(k)}\end{pmatrix}-J^{-1}(x^{(k)},y^{(k)})\begin{pmatrix}f_1(x^{(k)},y^{(k)})\\ f_2(x^{(k)},y^{(k)})\end{pmatrix}$, for $k=0,1,2,$ ...and starting with $x^{(0)}=1, y^{(0)}=0.23$ we have the following sequence of iterates.

First iteration, when $k=0$

$$\begin{pmatrix}x^{(1)}\\ y^{(1)}\end{pmatrix}=\begin{pmatrix}x^{(0)}\\ y^{(0)}\end{pmatrix}-J^{-1}(x^{(0)},y^{(0)})\begin{pmatrix}f_1(x^{(0)},y^{(0)})\\ f_2(x^{(0)},y^{(0)})\end{pmatrix}=\begin{pmatrix}1.0354\\ 0.2218\end{pmatrix}$$

Second iteration, when $k=1$

$$\begin{pmatrix}x^{(2)}\\ y^{(2)}\end{pmatrix}=\begin{pmatrix}x^{(1)}\\ y^{(1)}\end{pmatrix}-J^{-1}(x^{(1)},y^{(1)})\begin{pmatrix}f_1(x^{(1)},y^{(1)})\\ f_2(x^{(1)},y^{(1)})\end{pmatrix}=\begin{pmatrix}1.0347\\ 0.2223\end{pmatrix}$$

Third iteration, when $k=2$

$$\begin{pmatrix}x^{(3)}\\ y^{(3)}\end{pmatrix}=\begin{pmatrix}x^{(2)}\\ y^{(2)}\end{pmatrix}-J^{-1}(x^{(2)},y^{(2)})\begin{pmatrix}f_1(x^{(2)},y^{(2)})\\ f_2(x^{(2)},y^{(2)})\end{pmatrix}=\begin{pmatrix}1.0347\\ 0.2223\end{pmatrix}$$

Hence the solution correct to three decimal places is obtained as $x=1.035$ and $y=0.222$.

Exercise: Solve the system $\begin{cases} ycos(xy)+1=0\\ \sin(xy)+x-y=0\end{cases}$ correct to two decimal places, and has one solution close to $x=1$ and $y=2$.

Chapter 4: Interpolation and Curve Fitting

Introduction

Interpolation is the method of estimating unknown values with the help of given set ofobservations. According to *Theile* Interpolation is, "The art of reading between the lines of the table" and According to *W.M. Harper*:"Interpolation consists in reading a value which lies between twoextreme points".

Also interpolation means insertion or filling up intermediate terms of the series. It is the techniqueof obtaining the value of a function for any intermediate values of the independent variables *i.e.*,of argument when the values of the function corresponding to number of values of argument aregiven.

The process of computing the value of function outside the range of given values of the variable is called extrapolation.

There are three methods to interpolate the certain quantity.

1. **Graphical Method**: - According the Graphical Method, take a suitable scale for the values of $x$and y *i.e.*, for arguments and entries values, and plot the different points on theGraph for these values of $x$and y. Draw the Curve passes through the different plottedpoints and find a

point on the curve corresponding to argument $x$and obtain the corresponding entry value y. This method is not reliable most of the cases.

2. **Method of curve fitting**: -This method can be used only in those cases in which theform of the function in known. This method is not exact and becomes more complicatedwhen the number of observations is sufficiently large. The only merit of this method liesin the fact that it gives closer approximation than the graphical method.
3. **Use the finite differences**: -The study of finite differences for the purpose of interpolation can be divided into three parts.
    (a) The technique of interpolation with equal intervals.
    (b) The technique of interpolation with unequal intervals.
    (c) The technique of central differences.

These methods are less approximate than the others and do not assume the form of thefunction to be known. On using the method of calculus of finite difference, the calculationsremain simple even if some additional observations are included in the given data.

Finite differences: - Let $y = f(x)$ be a function tabulated for the equally spaced values or argument$a = x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh = b$, where $h$is the increment given to the independent variable offunction $y = f(x)$. To determine the values of function $y = f(x)$ for given intermediate or argumentvalues of x, three types of differences are useful:

Forward or Leading Differences

If we subtract from each value of $y$except y_0, the previous value of y, we have$y_1 - y_0, y_2 - y_1, y_3 - y_2 \dots, y_n - y_{n-1}$.These differences are called the first forward differencesof $y$and is denoted by Δy. The symbol Δ denotes the forward difference operator. That is,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

Also it can be written as,

$$\Delta f(x) = f(x + h) - f(x)$$

where $h$is the interval of differencing.

Similarly for second and higher order differences,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

Or

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$$
$$\dots\dots\dots\dots\dots\dots\dots\dots..$$
$$\Delta^3 y_{n-1} = \Delta^2 y_n - \Delta^2 y_{n-1}$$

In general,n^{th}forward difference are given by

$$\Delta^n y_{r-1} = \Delta^{n-1} y_r - \Delta^{n-1} y_{r-1}$$
$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$$

Forward difference table:

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

where$x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots., x_n = x_0 + nh.$

Example 1: Construct a forward difference table for the following values:

x	0	5	10	15	20	25
$f(x)$	7	11	14	18	24	32

Example 2: If $y = x^3 + x^2 - 2x + 1$, calculate values of y for $x = 0, 1, 2, 3, 4, 5$ and form the difference table. Also find the value of y at $x = 6$ by extending the table and verify that the same value is obtained by substitution.

Backward or Ascending Differences

If we subtract from each value of $y$except y_0, the previous value of y, we have$y_1 - y_0, y_2 - y_1, y_3 - y_2 \ldots, y_n - y_{n-1}$.These differences are called the first backward differences of $y$and are denoted by ∇y. The symbol ∇ denotes the backward difference operator. That is,

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\nabla y_3 = y_3 - y_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\nabla y_n = y_n - y_{n-1}$$

Also it can be written as,

$$f(x) = f(x) - f(x-h)$$

where $h$is the interval of differencing.

Similarly, second backward difference is given by,

$$\nabla^2 f(x) = \nabla f(x) - \nabla f(x-h)$$

In general,n^{th}backward difference are given by

$$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}$$
$$\nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h)$$

Backward difference table:

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

where, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, …., $x_n = x_0 + nh$.

Example: Construct the backward difference table for $y = \log x$ given that:

x	10	20	30	40	50
y	1	1.3010	1.4771	1.6021	1.6990

and find the values of $\nabla^3 \log 40$ and $\nabla^4 \log 50$.

The operator *E*: The operator E is called shift operator or displacement or translation operator. It shows the operation of increasing the argument value $x$by its interval of differencing$h$so that

$$Ef(x) = f(x + h) \text{ or } Ey_x = y_{x+h}$$

Similarly,$Ef(x + h) = f(x + 2h)$

.....

In general,$E^n y_x = y_{x+nh}$ or $E^n f(x) = f(x + nh)$

In the same manner, $E^{-1} f(x) = f(x - h)$

Also, $E^{-2} f(x) = f(x - 2h)$

$$E^{-n} f(x) = f(x - nh)$$

This is called inverse of shift operator.

Other operators:

Operator name /symbol	**Definition**
Central difference operator δ	$\delta f(x) = f(x + h/2) - f(x - h/2)$
Average operator μ	$\mu f(x) = 0.5[f(x + h/2) - f(x - h/2)]$
Differential operator D	$Df(x) = f'(x)$

Properties of the operatorΔ:

i. If $c$is a constant then $\Delta c = 0$.

ii. Δ is distributive, i.e.,$\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$.

iii. If c is a constant then, $\Delta[cf(x)] = c\Delta f(x)$.

iv. If $m$and $n$are positive integers then, $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.

v. $\Delta[f(x)g(x)] = f(x)\,\Delta g(x) + g(x)\,\Delta f(x)$.

vi. $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$, where $g(x) \neq 0$.

Theorem:-If $f(x)$ be the n^{th} degree polynomial in x, then the n^{th} difference of $f(x)$ is constant and$\Delta^{n+1}f(x)$and all higher differences are zero when the values of the independent variables are atequal interval.

Interpolation with equal intervals

1. **Newton Forward Interpolation Formula**

Let $y = f(x)$ be a given function of $x$which takes the value $f(x_0), f(x_0 + h), f(x_0 + 2h), \ldots, f(x_0 + nh)$ for$(n + 1)$ equally spaced values $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$of theindependent variable x. Assume that$f(x)$ be a polynomial of n^{th} degree, given by

$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \ldots (1)$

and $f(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

Where $a_0, a_1, \ldots, a_n$are to be determined. Now to find the values of$a_0, a_1, \ldots, a_n$:

put $x = x_0,\ x_0 + h, x_0 + 2h, \ldots, x_0 + nh$in equation (1) successively.

Therefore for $x = x_0, f(x_0) = a_0$

for $x = x_0 + h, f(x_0 + h) = f(x_1) = a_0 + a_1(x_0 + h - x_0) + 0 = a_0 + ha_1$

$$\Rightarrow a_1 = \frac{f(x_0 + h) - a_0}{h} = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{\Delta f(x_0)}{h}$$

for $x = x_0 + 2h,\ f(x_0 + 2h) = f(x_2) = a_0 + a_1(x_0 + 2h - x_0) + a_2(x_0 + 2h - x_0)(x_0 + 2h - (x_0 + h)) + 0$

$$\Rightarrow f(x_0 + 2h) = a_0 + a_1(2h) + a_2(2h)(h)$$

$$\Rightarrow a_2 = \frac{f(x_0 + 2h) - a_0 - a_1(2h)}{2h^2} = \frac{f(x_0 + 2h) - f(x_0) - 2\Delta f(x_0)}{2h^2}$$

$$\Rightarrow a_2 = \frac{f(x_0 + 2h) - f(x_0) - 2(f(x_0 + h) - f(x_0))}{2h^2}$$

$$\Rightarrow a_2 = \frac{f(x_0 + 2h) - f(x_0) - 2f(x_0 + h) + 2f(x_0)}{2h^2} = \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{2h^2}$$

$$\Rightarrow a_2 = \frac{\Delta^2 f(x_0)}{2h^2} = \frac{\Delta^2 f(x_0)}{2!\,h^2}$$

Similarly, $\Rightarrow a_3 = \frac{\Delta^3 f(x_0)}{3!h^3}, \ldots, a_n = \frac{\Delta^n f(x_0)}{n!h^n}$

Put the values of $a_0, a_1, \ldots, a_n$, in the equation (1), we get

$$f(x) = f(x_0) + \frac{\Delta f(x_0)}{h}(x - x_0) + \frac{\Delta^2 f(x_0)}{2!\,h^2}(x - x_0)(x - x_1) + \cdots + \frac{\Delta^n f(x_0)}{n!\,h^n}(x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

Again, put $x = x_0 + hu \Rightarrow u = \frac{x - x_0}{h}$, we have

$$f(x) = f(x_0) + \Delta f(x_0)\frac{(x - x_0)}{h} + \frac{\Delta^2 f(x_0)}{2!}\frac{(x - x_0)}{h}\left(\frac{x - x_0}{h} - \frac{h}{h}\right) + \cdots + \frac{\Delta^n f(x_0)}{n!}\frac{(x - x_0)}{h}\left(\frac{x - x_0}{h} - \frac{h}{h}\right) \ldots \left(\frac{x - x_0}{h} - \frac{(n-1)h}{h}\right)$$

$$f(x) = f(x_0) + up\Delta f(x_0) + u(u-1)\frac{\Delta^2 f(x_0)}{2!} + \cdots + u(u-1) \ldots (u - (n-1)\frac{\Delta^n f(x_0)}{n!}$$

which is required formula for forward interpolation.

This formula is useful for interpolating the values of $f(x)$ near the starting of the set of values given and is applicablefor equally spaced argument only.

Example 1: second degree polynomial passes through the points $(1,-1),(2,-2),(3,-1)$ and $(4,2)$. Find the polynomial and evaluate y at $x=0$.

Example 2:The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Use Newton's forward formula to find y when $x = 218\,ft$.

Example 3: Find e^{3x} for $x=0.05$ using the following table

x	*0*	*0.1*	*0.2*	*0.3*	*0.4*
e^{3x}	*1*	*1.3499*	*1.8221*	*2.4596*	*3.3201*

Example 4: Find the cubic polynomial which takes the following data:

x	0	1	2	3
$y=f(x)$	1	0	1	10

Example 5: Given that$\sqrt{12600}=112.24972, \sqrt{12610}=112.29426, \sqrt{12620}=112.33877$ and $\sqrt{12630}=112.38327$. Findthe value of 12616 .

2. **Newton Backward Interpolation Formula**

Let $y=f(x)$ be a given function of $x$which takes the value $f(x_0), f(x_0+h), f(x_0+2h),$ …, $f(x_0+nh)$ for $(n+1)$ equally spaced values $x_0, x_0+h, x_0+2h, \dots, x_0+nh$of the independent variable x. Assume that $f(x)$ be a polynomial of n^{th} degree, given by

$f(x)=a_0+a_1(x-x_n)+a_2(x-x_n)(x-x_{n-1})+\cdots+a_n(x-x_n)(x-x_{n-1})\dots(x-x_0)$(*)

and $f(x_i)=y_i$ for $i=0,1,\dots,n$.

Where $a_0, a_1, \dots, a_n$are to be determined. Now to find the values of $a_0, a_1, \dots, a_n$:

Put$x=x_0+nh, (n-1)h, x_0+(n-2)h, \dots, x_0$ in (*) respectively, we get

Therefore for $x=x_n$, $f(x_n)=a_0$

for $x=x_0+(n-1)h$, $f(x_0+(n-1)h)=f(x_{n-1})=a_0+a_1(x_0+(n-1)h-x_n)+0$

$$\Rightarrow a_1=\frac{f(x_0+(n-1)h)-f(x_n)}{-h}=\frac{f(x_n)-f(x_{n-1})}{h}=\frac{\nabla f(x_n)}{h}=\frac{\nabla y_n}{h}$$

Similarly,

$$a_2=\frac{\nabla^2 f(x_n)}{2!\,h^2}=\frac{\nabla^2 y_n}{2!\,h^2}, \dots, a_n=\frac{\nabla^n f(x_n)}{n!\,h^n}=\frac{\nabla^n y_n}{n!\,h^n}$$

Substituting the vaues in (*) and let $x=x_n+vh$, we obtain

$$f(x)=f(x_n)+v\nabla f(x_n)+v(v+1)\frac{\nabla^2 f(x_n)}{2!}+\cdots+v(v+1)\dots(v+n-1)\frac{\nabla^n f(x_n)}{n!}$$

This formula is called the Newton's backward interpolation formula and usedfor interpolating values of $y$near the end of the tabulated values, and also used for extrapolating values of$y$a little backward of y_n.

Example 1: The table below gives the value of $tanx$ for $0.10 \le x \le 0.30$

x	0.10	0.15	0.20	0.25	0.30
$tanx$	0.1003	0.1511	0.2027	0.2553	0.3093

Find

(a) $tan0.50$ (b) $tan0.26$ (c) $tan0.40$

Example 2: The area A of a circle of diameter d is given for the following values:

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find A for 105.

Example 3: Find the 4th degree polynomial which takes the following values using NBIF and evaluate at $x = 10$.

x	0	2	4	6	8
$f(x)$	1	2	1	10	9

Interpolation with unequal intervals

1. **Lagrange's Interpolation Formula**

Let$f(x_0), f(x_1), \dots, f(x_n)$be $(n + 1)$ entries of a function $y = f(x)$, where $f(x)$ is assumed to be a polynomialcorresponding to the arguments $x_0, x_1, \dots, x_n$. So that the polynomial $f(x)$ may be written as

$$f(x) = a_0(x - x_1)(x - x_2) \dots (x - x_n) + a_1(x - x_0)(x - x_2) \dots (x - x_n) + \cdots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1}) \quad \dots\dots (1)$$

where $a_0, a_1, \dots, a_n$are constants to be determined.

Putting $x = x_0, x_1, \dots, x_n$in (1) successively, we get

for $x = x_0$, $f(x_0) = a_0(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n) \Rightarrow a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}$

for $x = x_1$, $f(x_1) = a_1(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n) \Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}$

Similarly,

for $x = x_n$, $f(x_n) = a_n(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1}) \Rightarrow a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$

Substituting the values of $a_0, a_1, \dots, a_n$, in equation (1), we get

$$f(x) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}(x - x_1)(x - x_2) \dots (x - x_n) + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}(x - x_0)(x - x_2) \dots (x - x_n) + \cdots + \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$f(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} f(x_1) + \cdots + \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} f(x_n)$$

Or $y = f(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n)$, where the Lagrange basis functions are defined by

$$l_i(x) = \frac{\prod_{0\le k\le n; k\ne i}(x-x_k)}{\prod_{0\le k\le n; k\ne i}(x_i-x_k)} \text{ for } i = 0, 1, \dots, n.$$

This is called Lagrange's interpolation formula.

Note: - if $l_i(x)$ is n^{th} degree polynomial, then the interpolation is called n^{th} degree polynomial.

Example 1: Apply Lagrange's interpolation formula to find a polynomial which passes through the points $(0, -20), (1, -12), (3, -20)$ and $(4, -24)$.

Example 2: Determine by Lagrange's formula, the percentage number of criminals under 35 years:

Age	**% no. criminals**
Under 25 year	52
Under 30 years	67.3
Under 40 years	84.1
Under 50 years	94.4

Example 3: Find a Lagrange's interpolating polynomial for the given data:

$x_0 = 1, x_1 = 2.5, x_2 = 4$ and $x_3 = 5.5$

$f(x_0) = 4, f(x_1) = 7.5, f(x_2) = 13$ and $f(x_3) = 17.5$also, find the value of $f(5)$.

Note: In Lagrange's Interpolationformula $y$is expressed as a function of $x$as given below

$$y = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}y_1 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}y_n.$$

By interchanging $x$and $y$we can express $x$as a function of $y$as follows

$$x = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)}x_1 + \dots + \frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})}x_n.$$

The above formula may be used for inverse interpolation.

Example: The following table gives the values of $y$corresponding to certain values of x. Find the value of $x$when$y = 167.59789$ by applying Lagrange's inverse interpolation formula.

x	1	2	5	7
y	1	12	117	317

Ans. $x = 5.65238$.

2. **Newton divided difference Interpolation Formula**

The concept of divided differences: Let the function $y = f(x)$ be given at the point $x_0, x_1, \dots, x_n$(which need not be equally spaced) $f(x_0), f(x_1), \dots, f(x_n)$denote the $(n+1)$ values thefunction at the points $x_0, x_1, \dots, x_n$. Then the first divided differences of $f(x)$ for the arguments $x_0, x_1$is defined as

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

It is denoted by $f(x_0, x_1)$, or by$[x_0, x_1]$

$$\therefore\ f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

Similarly we can define

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

$$f(x_2, x_3) = \frac{f(x_2) - f(x_3)}{x_2 - x_3}$$

The second divided differences for the arguments$x_0, x_1, x_2, \ldots$ is defined as

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

Similarly the third divided differences for the arguments $x_0, x_1, x_2, x_3, \ldots$is defined as

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_0, x_1, x_2) - f(x_1, x_2, x_3)}{x_0 - x_3}$$

The first divided differences are called the divided differences of order one; the second divided differences are called the divided differences of order two, etc.

The divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_3)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Derivation of NDDIF: Let a function $f(x)$ be given for the $(n + 1)$ values $x_0, x_1, \ldots, x_n$as $f(x_0), f(x_1), \ldots, f(x_n)$where $x_0, x_1, \ldots, x_n$are not necessarily equispaced. From the definition of divided difference

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x, x_0)$ ……. (1)

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$\therefore\ f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$

Substituting in (1) we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1)$$

Proceeding in this way we get

$f(x) = f(x_0) + (x - x_0) f(x, x_0) + (x - x_0)(x - x_1) f(x, x_0, x_1) + \cdots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) f(x, x_0, x_1, x_2, \ldots, x_{n-1}) + (x - x_0)(x - x_1) \ldots (x - x_n) f(x, x_0, x_1, x_2, \ldots, x_n)$ .... (2)

If $f(x)$ is a polynomial of degree *n*, then the$(n + 1)^{th}$divided differences of $f(x)$ will be zero.

$$\therefore\ f(x, x_0, x_1, x_2, \ldots, x_n) = 0.$$

$\therefore$ Equation (2) can be written as

$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \cdots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) f(x_0, x_1, x_2, \ldots, x_n)$ ….. (3)

$\therefore$ Equation (3) is called Newton's General divided difference interpolation formula.

Example 1:Apply Newton's divided difference formula to find the value of $f(8)$ if $f(1) = 3, f(3) = 31, f(6) = 223, f(10) = 1011, f(11) = 1343.$

Example 2: Find the third divided difference with arguments $2, 4, 9, 10$ of the function $f(x) = x^3 - 2x.$

Example 3: The mode of a certain frequency Curve $y = f(x)$ is very near to $x = 9$ and the valuesof frequency density $f(x)$ for $x = 8.9, 9.0$ and 9.3 are respectively equal to $0.30, 0.35$ and 0.25. Calculatethe approximate value of mode.

Note: The divided differences are symmetrical in their arguments that are independent of the order of the arguments.

$$\text{i.e. } [x_0, x_1] = \frac{y_0 - y_1}{x_0 - x_1} = \frac{y_1 - y_0}{x_1 - x_0} = [x_1, x_0]$$