

Chapter 2: Solution of Non Linear Equations

Introduction

In this chapter we shall discuss some numerical methods for solving algebraic and transcendental equations. The equation $f(x) = 0$ is said to be algebraic if $f(x)$ is purely a polynomial in x . If $f(x)$ contains some other functions, namely, Trigonometric, Logarithmic, Exponential, etc., then the equation $f(x) = 0$ is called a Transcendental Equation.

One of the most common problem encountered in engineering analysis is that given a function $f(x)$, find the values of x for which $f(x) = 0$. The solution (values of x) are known as the *roots* of the equation $f(x) = 0$, or the *zeroes* of the function $f(x)$.

Theorem: If a function $f(x)$ assumes values of opposite sign at the end points of interval (a, b) , i.e., $f(a)f(b) < 0$ then the interval will contain at least one root of the equation $f(x) = 0$, in other words, there will be at least one number $c \in (a, b)$ such that $f(c) = 0$.

There are two types of methods available to find the roots of algebraic and transcendental equations of the form $f(x) = 0$.

1. **Direct Methods:** These methods give the exact value of the roots (in the absence of round off errors) in a finite number of steps and determine all the roots at the same time.
2. **Iterative methods:** These methods, also known as trial and error methods, are based on the idea of successive approximations, i.e., starting with one or more initial approximations to the value of the root, we obtain the sequence of approximations by repeating a fixed sequence of steps over and over again till we get the solution with reasonable accuracy and determine one or two roots at a time.

The indirect or iterative methods are further divided into two categories: bracketing and open methods. The bracketing methods require the limits between which the root lies, whereas the open methods require the initial estimation of the solution. The most popular method for solving a non-linear equation is the Newton-Raphson method and this method has a high rate of convergence to a solution.

Rate of Convergence

Definition: An iterate method is said to be of order p or has the rate of convergence p , if p is the largest positive number for which there exists a finite constant $c \neq 0$ such that $|e_{k+1}| \leq c|e_k|^p$ where $e_k = x_k - \xi$ is the error in the k^{th} iterate.

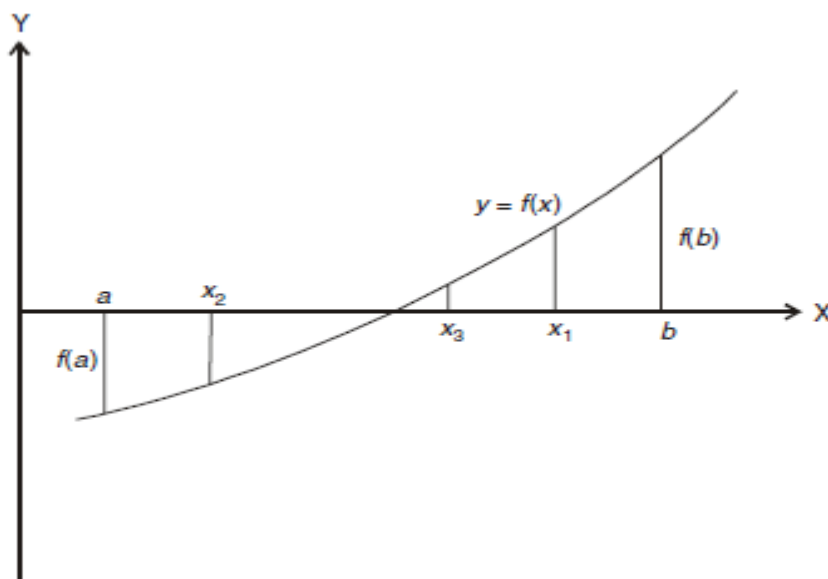
We present the following indirect or iterative methods with illustrative examples:

- | | |
|-------------------------|--------------------------|
| 1. Bisection Method | 4. Iteration Method and |
| 2. Regular Falsi Method | 5. Newton-Raphson Method |
| 3. Secant Method | (Newton's method) |

Bisection Method (Interval halving method)

This is one of the simplest iterative method and is strongly based on the property of intervals. To find a root using this method, let the function $f(x)$ be continuous between a and b and $f(a)f(b) < 0$. Then there is a root of $f(x) = 0$, lying between a and b . Let the first approximation be $x_1 = \frac{1}{2}(a + b)$ (i.e., average of the ends of the range).

Now, if $f(x_1) = 0$ then x_1 is a root of $f(x) = 0$. Otherwise, the root will lie between a and x_1 or x_1 and b depending upon whether $f(x_1)f(a) < 0$ or $f(x_1)f(b) < 0$ respectively.



Then, we bisect the interval and continue the process till the root is found to be desired accuracy. In the above figure, $f(x_1)f(a) < 0$; therefore, the root lies in between a and x_1 . The second approximation to the root now is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is negative as shown in the figure then the root lies in between x_2 and x_1 , and the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on. This method is simple but slowly convergent. It is also called as Bolzano method.

Procedure for the Bisection Method to Find the Root of the Equation $f(x) = 0$

- Step 1: Choose two initial guess values (approximation) a and b such that $f(a)f(b) < 0$.
- Step 2: Evaluate the mid point x_1 of a and b given by $x_1 = \frac{1}{2}(a + b)$ and also evaluate $f(x_1)$.
- Step 3: If $f(a)f(x_1) < 0$, then set $b = x_1$ else set $a = x_1$. Then apply the formula of step 2.
- Step 4: Stop evaluation when the difference of two successive values of x_1 obtained from step 2, is numerically less than the prescribed accuracy.

Order of Convergence of Bisection Method

As the error decreases with each step by a factor of $\frac{1}{2}$ (i.e. $\frac{E_{k+1}}{E_k} = \frac{1}{2}$), then the convergence in the bisection method is linear.

Example: Find the root of equation $x^3 - x - 1 = 0$ for the root lying between 1 and 2, correct to two decimal places by bisection method.

Sol. Let $f(x) = x^3 - x - 1 = 0$

Since $f(1) = 1^3 - 1 - 1 = -1$, which is negative

and $f(2) = 2^3 - 2 - 1 = 5$, which is positive

Therefore, $f(1)$ is negative and $f(2)$ is positive, so at least one real root will lie between 1 and 2.

First iteration: Now using Bisection Method, we can take first approximation

$$x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Then,
$$\begin{aligned} f(1.5) &= (1.5)^3 - 1.5 - 1 \\ &= 3.375 - 1.5 - 1 = 0.875 \end{aligned}$$

$\therefore f(1.5) > 0$ that is, positive

So root will now lie between 1 and 1.5.

Second iteration: The Second approximation is given by $x_2 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$.

Then,
$$\begin{aligned} f(1.25) &= (1.25)^3 - 1.25 - 1 \\ &= 1.953 - 2.25 = -0.297 < 0 \end{aligned}$$

$\therefore f(1.25)$ is negative.

Therefore, $f(1.5)$ is positive and $f(1.25)$ is negative, so that root will lie between 1.25 and 1.5.

Third iteration: The third approximation is given by

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$

$$x_3 = 1.375$$

Now
$$f(1.375) = (1.375)^3 - 1.375 - 1$$

$$f(1.375) = 0.2246$$

$\therefore f(1.375)$ is positive.

$\therefore$ The required root lies between 1.25 and 1.375.

Fourth iteration: The fourth approximation is given by

$$x_4 = \frac{1.25 + 1.375}{2} = 1.313$$

Now
$$f(1.313) = (1.313)^3 - 1.313 - 1$$

$$f(1.313) = -0.0494$$

Therefore, $f(1.313)$ is negative and $f(1.375)$ is positive. Thus root lies between 1.313 and 1.375.

Fifth iteration: The fifth approximation is given by

$$x_5 = \frac{1.313 + 1.375}{2} = 1.344$$

$$\therefore f(1.344) = (1.344)^3 - 1.344 - 1 = 0.0837$$

$$\therefore f(1.313) > 0$$

$\therefore f(1.313)$ is negative and $f(1.344)$ is positive, so root lies between 1.313 and 1.344.

Sixth iteration: The sixth approximation is given by

$$x_6 = \frac{1.313 + 1.344}{2} = 1.329$$

$$\therefore f(1.329) = (1.329)^3 - 1.329 - 1 = 0.0183$$

$$\therefore f(1.329) > 0$$

$\therefore f(1.313)$ is negative and $f(1.329)$ is positive, so that the required root lies between 1.313 and 1.329.

Seventh iteration: The seventh approximation is given by

$$x_7 = \frac{1.313 + 1.329}{2} = 1.321$$

From above iterations, the root of $f(x) = x^3 - x - 1 = 0$ up to two places of decimals is 1.32, which is of desired accuracy.

Exercises 1: Using the bisection method solves the following equations.

1. Approximate $\sqrt[3]{2}$, correct to three decimal places.
2. $x^3 - 9x + 1 = 0$ for the root lying between 2 and 3, correct to three significant figures.
3. Perform five iterations to obtain the smallest positive root of $f(x) = x^3 - 5x + 1 = 0$.
4. $3x - e^x = 0$ for $1 \leq x \leq 2$, accurate to within 10^{-4} .

Exercises 2: Use the bisection method find the drag coefficient c needed for a parachutist of mass $m = 68.1$ kg to have a velocity of $v = 40$ m/s after free-falling for time $t = 10$ s. Note: The acceleration due to gravity is $g = 9.8$ m/s² and $f(c) = \frac{mg}{c}(1 - e^{-t(c/m)}) - v$.

False position method

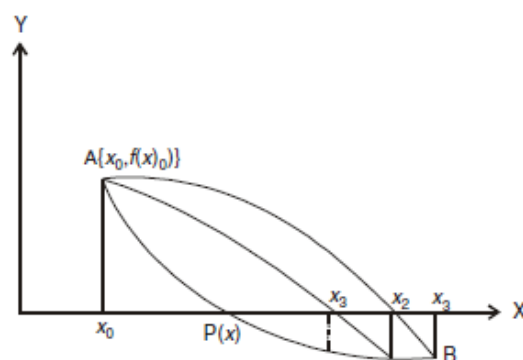
This method is essentially same as the bisection method except that instead of bisecting the interval.

In this method, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of $y = f(x)$ crosses the X-axis between these two points, a root must lie in between these points.

Consequently, $f(x_0)f(x_1) < 0$. Equation of the chord joining points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point



of intersection of the chord with X-axis as an approximation to the root.

So the abscissa of the point where chord cuts $y = 0$ is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

The value of x_2 can also be put in the following form:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

In general, the $(i + 1)$ th approximation to the root is given by

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Procedure for the False Position Method to Find the Root of the Equation $f(x) = 0$

Step 1: Choose two initial guess values (approximations) x_0 and x_1 (where $x_1 > x_0$) such that $f(x_0)f(x_1) < 0$.

Step 2: Find the next approximation x_2 using the formula

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

and also evaluate $f(x_2)$.

Step 3: If $f(x_2)f(x_1) < 0$, then go to the next step. If not, rename x_0 as x_1 and then go to the next step.

Step 4: Evaluate successive approximations using the formula

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}, \text{ where } i = 2, 3, 4, \dots$$

But before applying the formula for x_{i+1} , ensure whether $f(x_{i-1}) \cdot f(x_i) < 0$; if not, rename x_{i-2} as x_{i-1} and proceed.

Step 5: Stop the evaluation when $|x_i - x_{i-1}| < \epsilon$, where ϵ is the prescribed accuracy.

Note: The order (or rate) of convergence of false position method is 1.618.

Example: Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by the method of false position up to three places of decimal.

Sol. Given that $f(x) = x^3 - 2x - 5 = 0$
So that $f(2) = (2)^3 - 2(2) - 5 = -1$

and $f(3) = (3)^3 - 2(3) - 5 = 16$

Therefore, a root lies between 2 and 3.

First approximation: Therefore taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -1$, $f(x_1) = 16$, then by Regula-Falsi method, we get

$$\begin{aligned}x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\&= 2 - \frac{3-2}{16+1}(-1) = 2 + \frac{1}{17} = 2.0588\end{aligned}$$

Now,
$$\begin{aligned}f(x_2) &= f(2.0588) \\&= (2.0588)^3 - 2(2.0588) - 5 = -0.3911\end{aligned}$$

Therefore, root lies between 2.0588 and 3.

Second approximation: Now, taking $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -0.3911$, $f(x_1) = 16$, then by Regula-Falsi method, we get

$$\begin{aligned}x_3 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\&= 2.0588 - \frac{3-2.0588}{16+0.3911}(-0.3911) = 2.0588 + 0.0225 = 2.0813\end{aligned}$$

Now,
$$\begin{aligned}f(x_3) &= f(2.0813) \\&= (2.0813)^3 - 2(2.0813) - 5 = -0.1468\end{aligned}$$

Therefore, root lies between 2.0813 and 3.

Third approximation: Taking $x_0 = 2.0813$ and $x_1 = 3$, $f(x_0) = -0.1468$, $f(x_1) = 16$. Then by Regula-Falsi method, we get

$$\begin{aligned}x_4 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\&= 2.0813 - \frac{3-2.0813}{16+0.1468}(-0.1468) = 2.0813 + 0.0084 = 2.0897\end{aligned}$$

Now,
$$\begin{aligned}f(x_4) &= f(2.0897) \\&= (2.0897)^3 - 2(2.0897) - 5 \\&= 9.1254 - 9.1794 = -0.054\end{aligned}$$

Therefore, root lies between 2.0897 and 3.

Fourth approximation: Now, taking $x_0 = 2.0897$, $x_1 = 3$, $f(x_0) = -0.054$, $f(x_1) = 16$, then by Regula-Falsi method, we get

$$\begin{aligned}x_5 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\&= 2.0897 - \frac{3-2.0897}{16+0.054}(-0.054) \\&= 2.0897 + 0.0031 = 2.0928\end{aligned}$$

$$\begin{aligned}
\text{Now,} \quad f(x_5) &= f(2.0928) \\
&= (2.0928)^3 - 2(2.0928) - 5 \\
&= 9.1661 - 9.1856 = -0.0195
\end{aligned}$$

Therefore, root lies between 2.0928 and 3.

Fifth approximation: Now, taking $x_0 = 2.0928$, $x_1 = 3$, $f(x_0) = -0.0195$, $f(x_1) = 16$, then we get

$$\begin{aligned}
x_6 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 2.0928 - \frac{3 - 2.0928}{16 + 0.0195} (-0.0195) \\
&= 2.0928 + 0.0011 = 2.0939
\end{aligned}$$

$$\begin{aligned}
\text{Now,} \quad f(x_6) &= f(2.0939) \\
&= (2.0939)^3 - 2(2.0939) - 5 \\
&= 9.1805 - 9.1879 = -0.0074
\end{aligned}$$

Thus the root lies between 2.0939 and 3.

Sixth approximation: Now, taking $x_0 = 2.0939$, $x_1 = 3$, $f(x_0) = -0.0074$, $f(x_1) = 16$, then we get

$$\begin{aligned}
x_7 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 2.0939 - \frac{3 - 2.0939}{16 + 0.0074} (-0.0074) \\
&= 2.0939 + 0.00042 = 2.0943
\end{aligned}$$

$$\begin{aligned}
\text{Now,} \quad f(x_7) &= f(2.0943) \\
&= (2.0943)^3 - 2(2.0943) - 5 \\
&= 9.1858 - 9.1886 = -0.0028
\end{aligned}$$

Therefore, root lies between 2.0943 and 3.

Seventh approximation: Taking $x_0 = 2.0943$, $x_1 = 3$, $f(x_0) = -0.0028$, $f(x_1) = 16$, then by False position method, we get

$$\begin{aligned}
x_8 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 2.0943 - \frac{3 - 2.0943}{16 + 0.0028} (-0.0028) \\
&= 2.0943 + 0.00016 = 2.0945
\end{aligned}$$

Hence, the root is 2.094 correct to three decimal places.

Exercise 1: Using False position method, find a root of the function

1. $f(x) = e^x - 3x^2$ to an accuracy of four decimal places in $[0.5, 1]$
2. $f(x) = \cos x - 3x + 5$ correct to four decimal places.
3. $f(x) = x^2 - \log_e x - 12$ correct to three decimal places.

Exercise 2: The velocity v of a falling parachutist is given by

$$v = \frac{mg}{c} (1 - e^{-t(c/m)}) \quad \text{where } g = 9.8 \text{ m/s}^2. \text{ For a parachutist with a drag coefficient } c = 15$$

kg/s , compute the mass m so that the velocity is $v = 35 \frac{m}{s}$ at $t = 9s$. Use the false-position method to determine m to a level of accuracy 0.1%.

Exercise 3: Water is flowing in a trapezoidal channel at a rate of $Q = 20 m^3/s$. The critical depth y for such a channel must satisfy the equation $0 = 1 - \frac{Q^2}{gA_c^3} B$ where $g = 9.8 m/s^2$, A_c = the cross-sectional area (m^2), and B = the width of the channel at the surface (m). For this case, the width and the cross-sectional area can be related to depth y by $B = 3 + y$ and $A_c = 3y + \frac{y^2}{2}$. Solve for the critical depth using **(a)** bisection, and **(b)** false position. Use initial guesses of $x_l = 0.5$ and $x_u = 2.5$, and iterate until the approximate error falls below 1% or the number of iterations exceeds 10. Discuss your results.

Secant Method

The Secant method is similar to the Regula-Falsi method, except for the fact that we drop the condition that $f(x)$ should have opposite signs at the two points used to generate the next approximation. Instead, we always retain the last two points to generate the next. Thus, if x_{n-1} and x_n are two approximations to the root, then the next approximation x_{n+1} to root is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, \dots$$

Geometrically, in Secant method we replace the function $f(x)$ by a straight line passing through the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$ and take the point of intersection of the straight line with the x -axis as the next approximation to the root. In contrast to the Regula-Falsi method, the Secant iteration does not bracket the root and it is not even necessary to bracket the root to start the iteration. Hence, it is obvious that the iteration may not always coverage on the other hand, it generally converges faster. Thus, by dropping the necessity of bracketing the root, we improve the rate of convergence, however, in some cases, the iteration may not converge at all.

Procedure for Secant Method to Find the Root of $f(x)=0$

Step 1: Choose the interval $[x_0, x_1]$ in which $f(x)=0$ has a root, where $x_1 > x_0$.

Step 2: Find the next approximation x_2 of the required root using the formula

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1).$$

Step 3: Find the successive approximations of the required root using the formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, \dots$$

Step 4: Stop the process when the prescribed accuracy is obtained.

Note: The rate of convergence for the Secant method as $p = 1.618$ is superlinear.

Example: Compute root of the equation $x^2 e^{-x/2} = 1$ in the interval $[0, 2]$ using Secant method. The root should be correct to three decimal places.

Sol. We have, $x_0 = 1.42$, $x_1 = 1.43$, $f(x_0) = -0.0086$, $f(x_1) = 0.00034$

By Secant method,

First approximation:

First approximation is given by

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right] f(x_1)$$

$$x_2 = 1.43 - \left(\frac{1.43 - 1.42}{0.00034 + 0.0086} \right) (0.00034) = 1.4296$$

$$f(x_2) = -0.000011$$

Second approximation: Second approximation is given by

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)} \right] f(x_2)$$

$$x_3 = 1.4296 - \left(\frac{1.4296 - 1.42}{-0.000011 - 0.00034} \right) (-0.000011) = 1.4292$$

Since x_2 and x_3 agree up to three decimal places hence the required root is 1.429.

Iteration Method (Method of Successive Approximation)

This method is also known as the direct substitution method or method of fixed iterations. To find the root of the equation $f(x) = 0$ by successive approximations, we rewrite the given equation in the form

$$x = g(x)$$

Now, first we assume the approximate value of root (let x_0), then substitute it in $g(x)$ to have a first approximation x_1 given by

$$x_1 = g(x_0)$$

Similarly, the second approximation x_2 is given by

$$x_2 = g(x_1)$$

In general,

$$x_{i+1} = g(x_i)$$

Procedure For Iteration Method To Find The Root of The Equation $f(x) = 0$

Step 1: Take an initial approximation as x_0 .

Step 2: Find the next (first) approximation x_1 by using $x_1 = g(x_0)$.

Step 3: Follow the above procedure to find the successive approximations x_{i+1} by using

$$x_{i+1} = g(x_i), i = 1, 2, 3, \dots$$

Step 4: Stop the evaluation where relative error $\leq \varepsilon$, where ε is the prescribed accuracy.

Note 1: The iteration method $x = g(x)$ is convergent if $|g'(x)| < 1$.

Note 2: When $|g'(x)| > 1 \Rightarrow g'(x) > 1$ or $g'(x) < -1$, the iterative process is divergent.

Note 3: The rate of convergence of iteration method is 1. In other words, iteration method converges linearly.

Example: Find a real root correct up to four decimal places of the equation $2x - \log_{10} x - 7 = 0$ using iteration method.

Sol. Here, we have $f(x) = 2x - \log_{10} x - 7 = 0$

Now, we find that $f(3) = -1.447 = -ve$ and $f(4) = 0.398 = +ve$

Therefore, at least one real root of $f(x) = 0$ lies between $x = 3$ and $x = 4$.

Now, the given equation can be re-written as

$$x = \frac{1}{2} [\log_{10} x + 7] = g(x), \text{ say.}$$

Now, $g'(x) = \frac{1}{2x}$, from which we clearly note that $|g'(x)| < 1$ for all $x \in (3, 4)$.

Again since $|f(4)| < |f(3)|$, therefore, root is nearer to $x = 4$. Let the initial approximation be $x_0 = 3.6$ because $f(3.6)$ tends to zero. Then from the iterative formula $x_{i+1} = g(x_i)$, we obtain

$$x_1 = g(x_0) = \frac{1}{2} [\log_{10} x_0 + 7] = \frac{1}{2} [\log_{10} 3.6 + 7] = 3.77815$$

$$x_2 = g(x_1) = g(3.77815) = 3.78863$$

$$x_3 = g(x_2) = g(3.78863) = 3.78924$$

$$x_4 = g(x_3) = g(3.78924) = 3.78927$$

Hence, the root of the equation correct to the four places of decimals is 3.7893.

Newton-Raphson Method

This method can be derived from Taylor's series as follows:

Let $f(x) = 0$ be the equation for which we are assuming x_0 be the initial approximation and h be a small corrections to x_0 , so that $f(x_0 + h) = 0$.

Expanding it by Taylor's series, we get

$$f(x) = f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0.$$

Since h is small, we can neglect second and higher degree terms in h and therefore, we get $f(x_0) + hf'(x_0) = 0$

From which we have, $h = -\frac{f(x_0)}{f'(x_0)}$ where $f'(x_0) \neq 0$

Hence, if x_0 be the initial approximation, then next (or first) approximation x_1 is given by

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

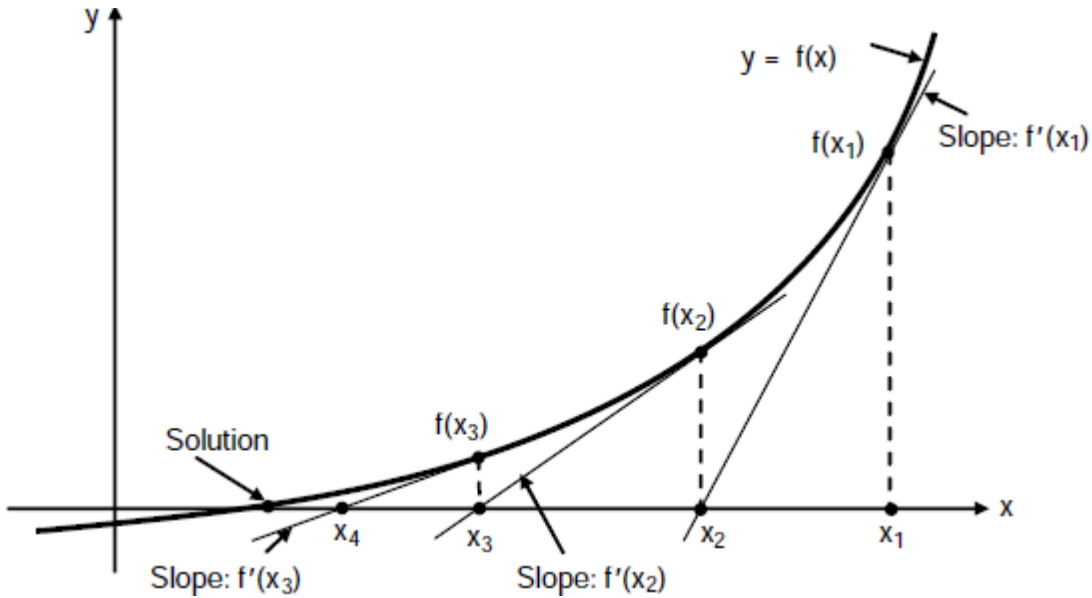
The next and second approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This formula is well known as Newton-Raphson formula or Newton's method.



Algorithm for Newton-Raphson Method to Find the Root of the Equation $f(x) = 0$

Step 1: Take a trial solution (initial approximation) as x_0 . Find $f(x_0)$ and $f'(x_0)$

Step 2: Follow the above procedure to find the successive approximations x_{n+1} using the

$$\text{formula } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } n = 0, 1, 2, 3, \dots$$

Step 3: Stop the process when $\left| \frac{x_{n+1} - x_n}{x_{n+1}} \right| \leq \varepsilon$, where ε is the prescribed accuracy.

Order (or Rate) of Convergence of Newton-Raphson Method

Let α be the actual root of equation $f(x) = 0$ i.e., $f(\alpha) = 0$. Let x_n and x_{n+1} be two successive approximations to the actual root α . If e_n and e_{n+1} are the corresponding errors we have, $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$. By Newton's-Raphson formula,

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

$$e_{n+1} = e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \dots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \dots} \quad (\text{By Taylor's expansion})$$

$$e_{n+1} = e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \dots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \dots} \quad [\because f(\alpha) = 0]$$

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{2[f'(\alpha) + e_n f''(\alpha)]} \quad (\text{On neglecting high powers of } e_n)$$

$$\begin{aligned}
&= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha) \left\{ 1 + e_n \frac{f''(\alpha)}{f'(\alpha)} \right\}} \\
&= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left\{ 1 + e_n \frac{f''(\alpha)}{f'(\alpha)} \right\}^{-1} \\
&= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left\{ 1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \dots \right\} \\
&= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^3}{2} \left\{ \frac{f''(\alpha)}{f'(\alpha)} \right\}^2 + \dots \\
\text{or} \quad \frac{e_{n+1}}{e_n^2} &= \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n}{2} \left\{ \frac{f''(\alpha)}{f'(\alpha)} \right\}^2 + \dots \\
&= \frac{f''(\alpha)}{2f'(\alpha)} \quad (\text{Neglecting terms containing powers of } e_n)
\end{aligned}$$

Hence by definition, the order of convergence of Newton-Raphson method is 2 i.e., Newton Raphson method is **quadratic convergent**. This also shows that subsequent error at each step is proportional to the square of the previous error and as such the convergence is quadratic.

Example 1: Find the real root of the equation $\log x - \cos x = 0$ correct to three places of decimal by Newton-Raphson's method.

Sol. Let $f(x) = \log x - \cos x = 0$...(1)

So that $f(1) = -0.5403$

$f(2) = 1.1092$

$\therefore$ The root lies between 1 and 2.

Also, $f(1.1) = \log 1.1 - \cos 1.1$
 $= 0.0953 - 0.4535 = -0.3582$

$f(1.2) = \log 1.2 - \cos 1.2 = -0.18$

$f(1.3) = \log 1.3 - \cos 1.3 = -0.0051$

$f(1.4) = \log 1.4 - \cos 1.4 = 0.1665$

Thus the root lies between 1.3 and 1.4.

From (1) $f'(x) = \frac{1}{x} + \sin x$...(2)

Then by Newton's-Raphson method, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Using (1) and (2) $x_{n+1} = x_n - \frac{\log x_n - \cos x_n}{\frac{1}{x_n} + \sin x_n}$

or $x_{n+1} = \frac{x_n + x_n^2 \sin x_n - x_n \log x_n + x_n \cos x_n}{1 + x_n \sin x_n}$...(3)

Let us take $x_0 = 1.3$

Now putting $n = 0$ into (3), we get first approximation

First approximation:

$$\begin{aligned}x_1 &= \frac{x_0 + x_0^2 \sin x_0 - x_0 \log x_0 + x_0 \cos x_0}{1 + x_0 \sin x_0} \\&= \frac{(1.3) + (1.3)^2 \sin (1.3) - (1.3) \log (1.3) + (1.3) \cos (1.3)}{1 + (1.3) \sin (1.3)} \\&= \frac{1.3 (1 + 1.2526 - 0.2623 + 0.2674)}{1 + 1.2526}\end{aligned}$$

or

$$x_1 = \frac{2.93501}{2.2526} = 1.3029$$

Now putting $n = 1$ into (3), we get second approximation

Second approximation:

$$\begin{aligned}x_2 &= \frac{x_1 + x_1^2 \sin x_1 - x_1 \log x_1 + x_1 \cos x_1}{1 + x_1 \sin x_1} \\&= \frac{1.3029 + (1.3029)^2 \sin (1.3029) - (1.3029) \log (1.3029) + (1.3029) \cos (1.3029)}{1 + (1.3029) \sin (1.3029)} \\&= \frac{1.3029 (1 + 1.2526 - 0.2645 + 0.2647)}{1 + 1.2564}\end{aligned}$$

or

$$x_2 = \frac{2.9401}{2.2564} = 1.3030$$

Hence the required root is 1.303 correct to three decimal places.

Example 2: Evaluate $\sqrt{12}$ to four decimal places by Newton's iterative method.

Sol. Let $x = \sqrt{12} \Rightarrow x^2 - 12 = 0$... (1)

Therefore Newton's Iterative formula gives,

$$\begin{aligned}x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\x_{n+1} &= x_n - \frac{x_n^2 - 12}{2x_n} = \frac{1}{2} \left(x_n + \frac{12}{x_n} \right) \quad \dots (2)\end{aligned}$$

Now since $f(3) = -3$ (-ve) and $f(4) = 4$ (+ve)

Therefore the root lies between 3 and 4.

Take $x_0 = 3.5$, equation (2) gives

$$\begin{aligned}x_1 &= \frac{1}{2} \left(x_0 + \frac{12}{x_0} \right) \\x_1 &= \frac{1}{2} \left(3.5 + \frac{12}{3.5} \right) = 3.4643 \\x_2 &= \frac{1}{2} \left(3.4643 + \frac{12}{3.4643} \right) = 3.4641\end{aligned}$$

$$x_3 = \frac{1}{2} \left(3.4641 + \frac{12}{3.4641} \right) = 3.4641$$

Since, $x_2 = x_3$ up to four decimal places. So we have $\sqrt{12} = 3.4641$.

Exercise: Find by Newton's method, the real root of the equation

- a. $x^4 - 11x + 8 = 0$, correct to five decimal places and the root near 2.
- b. $3x = \cos x + 1$, correct to four decimal places.
- c. $x \log_{10} x = 1.2$, correct to five decimal places.
- d. $x = e^{-x}$, correct to four decimal places.