Addis Ababa University
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School of Mechanical & Industrial Engineering

Engineering Mechanics II (Dynamics)
MEng 2052

Chapter Two

Kinematics of Particles

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Introduction

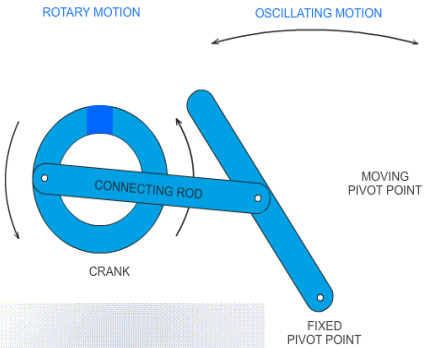
Kinematics: is the branch of dynamics which describes the motion of bodies *without* reference to the forces that either causes the motion or are generated as a result of the motion.

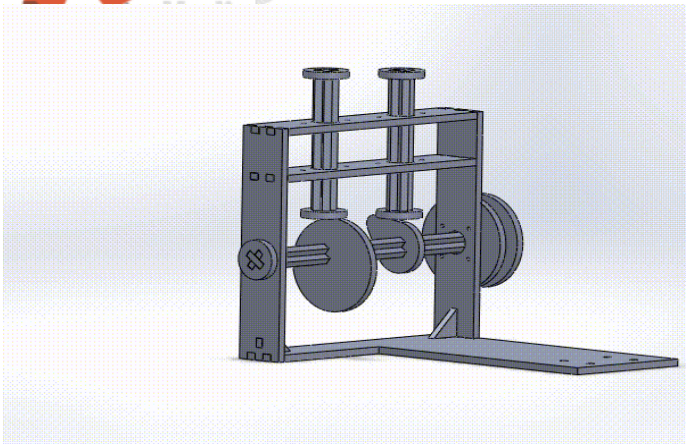
- Kinematics is often referred to as the "geometry of motion"

Examples of *kinematics problems* that engage the attention of engineers.

- The design of *cams*, *gears*, *linkages*, and other *machine elements* to control or produce certain desired motions, and
- The calculation of flight trajectory for aircraft, rockets and spacecraft.

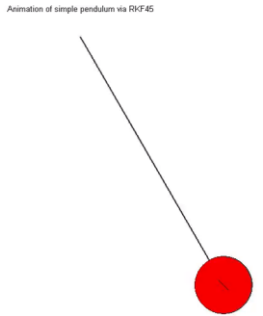


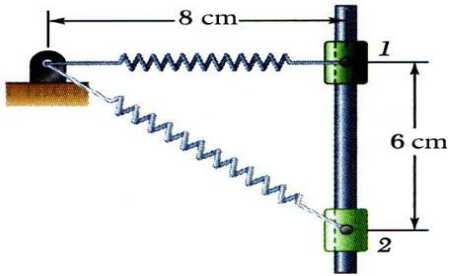


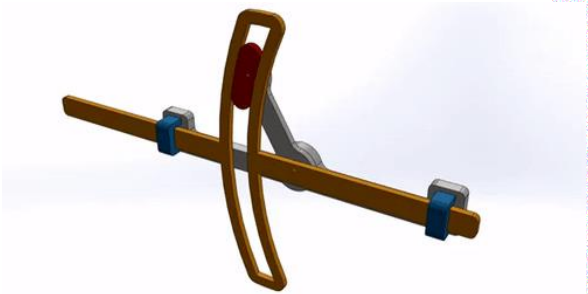


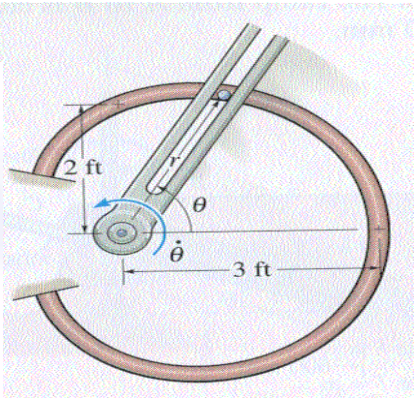
- If the particle is *confined* to a *specified path*, as with a bead sliding along a fixed wire, its motion is said to be *Constrained.*

Example 1. - A small rock tied to the end of a string and whirled in a circle undergoes constrained motion until the string breaks









- If there are *no* physical guides, the motion is said to be *unconstrained*.

Example 2. - Airplane, rocket





- The position of particle P at any time t can be described by specifying its:

  - *Rectangular* coordinates; X,Y,Z

  - *Cylindrical* coordinates; r,θ,z

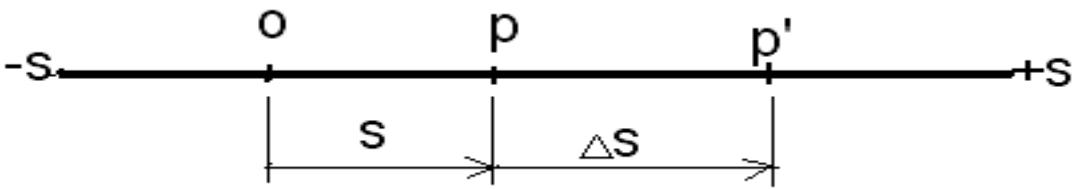
  - *Spherical* coordinates; R, θ,Φ

- Also described by measurements along the tangent *t* and normal *n* to the curve(path variable).

- The motion of particles(or rigid bodies) may be described by using coordinates measured from fixed reference axis *(absolute motion analysis)* or by using coordinates measured from moving reference axis *(relative motion analysis).*

Rectilinear motion

- Is a motion in which a particle moving along a *straight line(one-dimensional motion)*
- Consider a particle P moving along a straight line.



Average velocity: *f*or the time interval Δt, it is defined as the ratio of the displacement Δs to the time interval Δt.

$$V_{av} = \frac{\Delta s}{\Delta t} \qquad 2.1$$

- As Δt *becomes smaller and approaches zero in the limit, the average velocity approaches the instantaneous velocity of the particle.*

$$V = \lim_{\Delta t \to 0} V_{av} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \dot{S} \qquad 2.2$$

Average acceleration

For the time interval Δt, it is defined as the ratio of the change in velocity Δv to the time interval Δt.

$$a_{av} = \frac{\Delta v}{\Delta t} \qquad 2.3$$

Instantaneous acceleration

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \dot{v} \qquad 2.4(a)$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2} = \ddot{s} \qquad 2.4(b)$$

- Note:-The acceleration is positive or negative depending on whether the velocity increasing or decreasing.
- Considering equation *2.2* and *2.4(a)* , we have

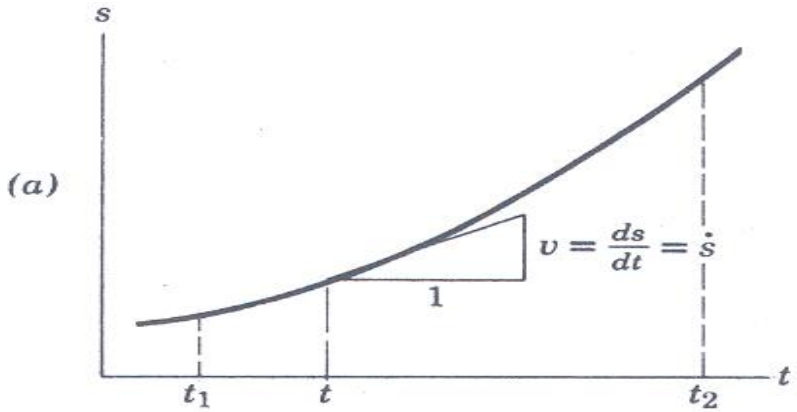
$$\mathrm{dt} = \frac{\mathrm{ds}}{\mathrm{v}} = \frac{dv}{a}$$

$$\Rightarrow vdv = ads$$

$$dt = \frac{ds}{\dot{s}} = \frac{d\dot{s}}{\ddot{s}} \Rightarrow \dot{s}\,d\dot{s} = \ddot{s}\,ds$$

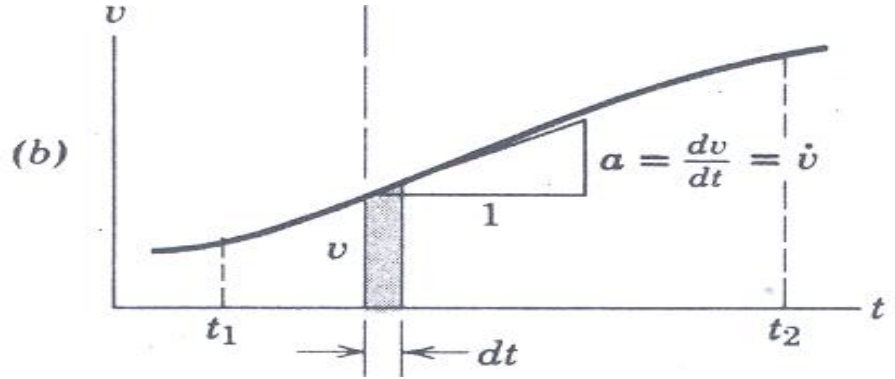
General Representation of Relationship among s, v, a & t.

1. Graph of *s* Vs *t*



- *By constructing tangent to the curve at any time t, we obtain the slope, which is the velocity v=ds/dt*

2. Graph of *v* Vs *t*



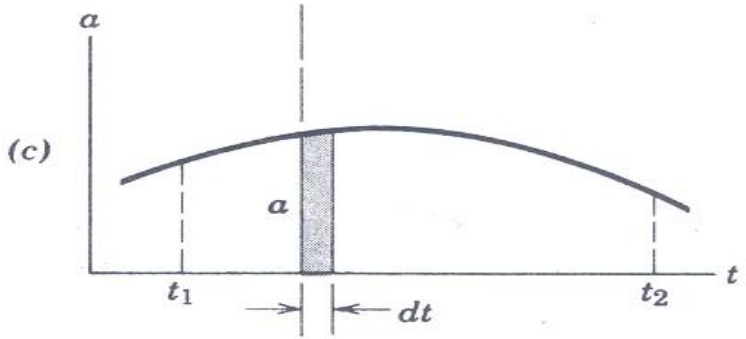
- *The slope dv/dt of the v-t curve at any instant gives the acceleration at that instant.*
- *The area under the v-t curve during time dt is vdt which is the displacement ds*

- The area under the v-t curve is the net displacement of the particle during the interval from t1 to t2.

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt$$

$$s_2 - s_2 = \text{(area under v-t curve)}$$

3. Graph of *a* vs *t*

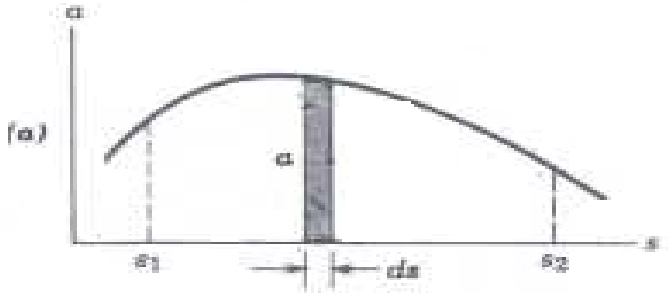


- The area under the a-t curve during time dt is the net change in velocity of the particle between t_1 and t_2.

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt$$

$v_2 - v_1$=(area under a-t curve)

4. Graph of *a* Vs *s*

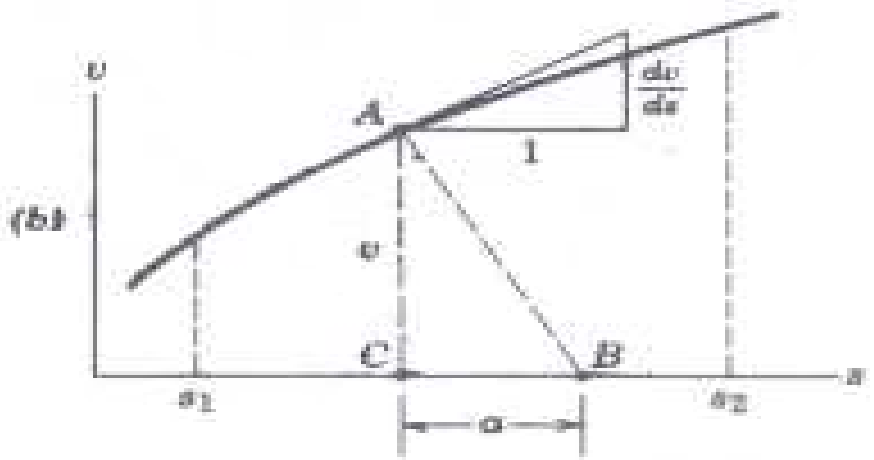


- The net area under the curve b/n position coordinates s_1 and s_2 is

$$\int_{v_1}^{v_2} v\,dv = \int_{s_1}^{s_2} a\,ds$$

$$\frac{1}{2}(v_2{}^2 - v_1{}^2) = \text{(areas under a-s curve)}$$

5. Graph of *v* vs. *s*



$$\tan\theta = \frac{dv/ds}{1} = \frac{\overline{CB}}{v}$$

$$\frac{dv}{ds} = \frac{\overline{CB}}{v} \Rightarrow vdv = \overline{CB}ds$$

- The graphical representations described are useful for:-
  - visualizing the relationships among the several motion quantities.
  - approximating results by graphical integration or differentiation when a lack of knowledge of the mathematical relationship prevents its expression as an explicit mathematical function .
  - experimental data and motions that involve discontinuous relationship b/n variables.

General Methods for Determining the Velocity and Displacement Functions Representation of Relationship among s, v, a & t.

a) When the acceleration is constant, (a=const.)

- boundary conditions

at t=0 , s=s0 and v=v0

using integrating

$$a = \frac{dv}{dt} \Rightarrow adt = dv$$

$$\int_{v_o}^{v} dv = \int_{0}^{t} adt \Rightarrow v - v_o = at$$

$$\Rightarrow v = v_o + at$$

- Using

$$vdv = ads \Rightarrow \int_{v_o}^{v} vdv = \int_{s_o}^{s} ads$$

$$\Rightarrow \frac{v^2}{2}\Big|_{v_o}^{v} = as\Big|_{s_o}^{s} \Rightarrow \frac{v^2 - {v_o}^2}{2} = a(s - s_o)$$

$$\Rightarrow v^2 = {v_o}^2 + 2a(s - s_0)$$

- Using

$$v = \frac{ds}{dt} \Rightarrow ds = vdt$$

$$\Rightarrow \int_{s_0}^{s} ds = \int_{o}^{t} (v_o + at)dt$$

$$\Rightarrow s - s_o = v_o t + \frac{at^2}{2}$$

$$\Rightarrow s = s_o + v_o t + \frac{1}{2}at^2$$

- These relations are necessarily restricted to the special case where the acceleration is constant.
- The integration limits depend on the initial and final conditions and for a given problem may be different from those used here.
- Typically, conditions of motion are specified by the type of acceleration experienced by the particle. Determination of velocity and position requires two successive integrations.
- Three classes of motion may be defined for:

  - *acceleration given as a function of time, $a = f(t)$*

  - *acceleration given as a function of position, $a = f(x)$*

  - *acceleration given as a function of velocity, $a = f(v)$*

b) Acceleration given as a function of time, $a=f(t)$

$$\frac{dv}{dt}=a=f(t) \qquad dv=f(t)\,dt \qquad \int_{v_0}^{v(t)} dv=\int_0^t f(t)\,dt \qquad v(t)-v_0=\int_0^t f(t)\,dt$$

$$\frac{dx}{dt}=v(t) \qquad dx=v(t)\,dt \qquad \int_{x_0}^{x(t)} dx=\int_0^t v(t)\,dt \qquad x(t)-x_0=\int_0^t v(t)\,dt$$

c) Acceleration given as a function of *position*, $a = f(x)$

$$v=\frac{dx}{dt} \quad \text{or} \quad dt=\frac{dx}{v} \qquad a=\frac{dv}{dt} \quad \text{or} \quad a=v\frac{dv}{dx}=f(x)$$

$$v\,dv=f(x)\,dx \qquad \int_{v_0}^{v} v\,dv=\int_{x_0}^{x} f(x)\,dx \qquad \tfrac{1}{2}v^2-\tfrac{1}{2}v_0^2=\int_{x_0}^{x} f(x)\,dx$$

d) Acceleration given as a function of velocity, a = f(v)

$$\frac{dv}{dt} = a = f(v) \qquad \frac{dv}{f(v)} = dt \qquad \int_{v_o}^{v} \frac{dv}{f(v)} = \int_{0}^{t} dt$$

$$\int_{v_o}^{v} \frac{dv}{f(v)} = t$$

$$v\frac{dv}{dx} = a = f(v) \qquad dx = \frac{v\,dv}{f(v)} \qquad \int_{x_o}^{x} dx = \int_{v_o}^{v} \frac{v\,dv}{f(v)}$$

$$x - x_o = \int_{v_o}^{v} \frac{v\,dv}{f(v)}$$

Example 1

- Consider a particle moving in a straight line, and assuming that its position is defined by the equation

$$s = 6t^2 - t^3$$

- Where, t is express in seconds and s is in meters. Determine the velocity and acceleration of the particles at any time t

Example 2

- The acceleration of a particle is given by ,

  $a = 4t - 30$ where a is in meters per second squared and t is in seconds. Determine the velocity and displacement as function time. The initial displacement at t=0 is s_o=-5m, and the initial velocity is v_o=30m/s. solved

Example 3

- The position of a particle which moves along a straight line is defined by the relation $x = t^3 - 6t^2 - 15t + 40$, where x is expressed in m and t in second.

Determine:

a) The time at which the velocity will be zero.

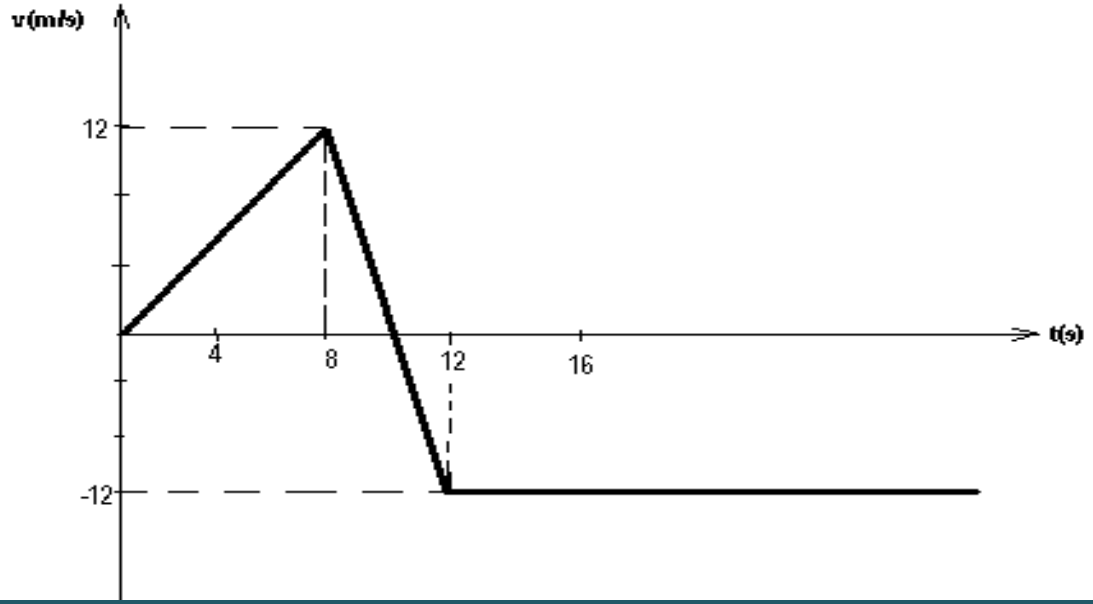
b) The position and distance traveled by the particle at that time.

c) The acceleration of the particle at that time.

d) The distance traveled by the particle between 4s and 6s

Example 4

- A particle moves in a straight line with velocity shown in the figure. Knowing that x=-12m at t=0



- *Draw the a-t and x-t graphs*, and
- *Determine:*

a) The total distance traveled by the particle when t=12s.

b) The two values of t for which the particle passes the origin.

c) The max. value of the position coordinate of the particle.

d) The value of t for which the particle is at a distance of 15m from the origin.

Plane Curvilinear Motion

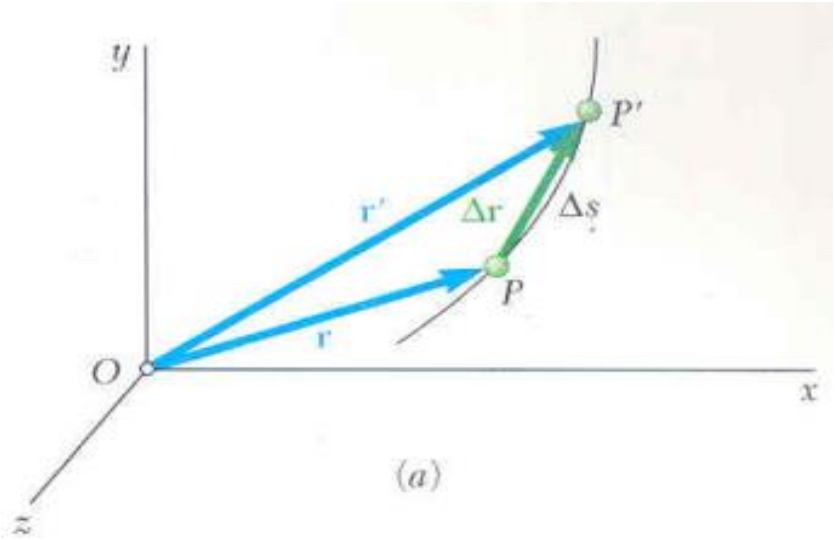
Curvilinear Motion of a Particle

- When a particle moves along a *curve* other than a straight line, we say that the particle is in *curvilinear motion*

The analysis of motion of a particle along a curved path that lies on a single plane.

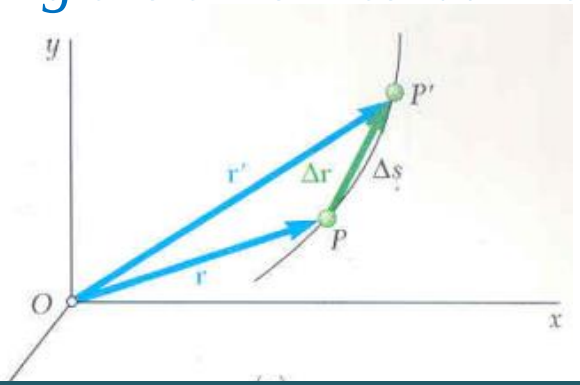
- Consider the continuous motion of a particle along a plane curve.

  - *At time t, the particle is at position P, which is located by the position vector* ***r*** *measured from some convenient fixed origin o.*



- *At time* $t + \Delta t$ *, the particle is at P' located by the position vector* $\vec{r} + \Delta\vec{r}$.

- *The vector Δr joining p and p' represents the change in the position vector during the time interval Δt (displacement) .*

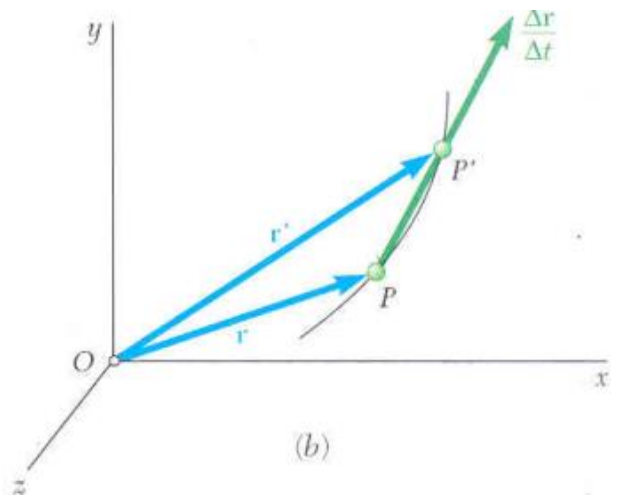


- The distance traveled by the particle as it moves along the path from P to P’ is the scalar length Δs measured along the path.
- The displacement of the particle, that represents the vector change of position and is clearly independent of the choice of origin.

- The average velocity of the particle between P and P'

  defined as: $\vec{V}_{av} = \dfrac{\overrightarrow{\Delta r}}{\Delta t}$

which is a vector whose direction is that of $\Delta\vec{r}$.



- The instantaneous velocity $\vec{v}$,

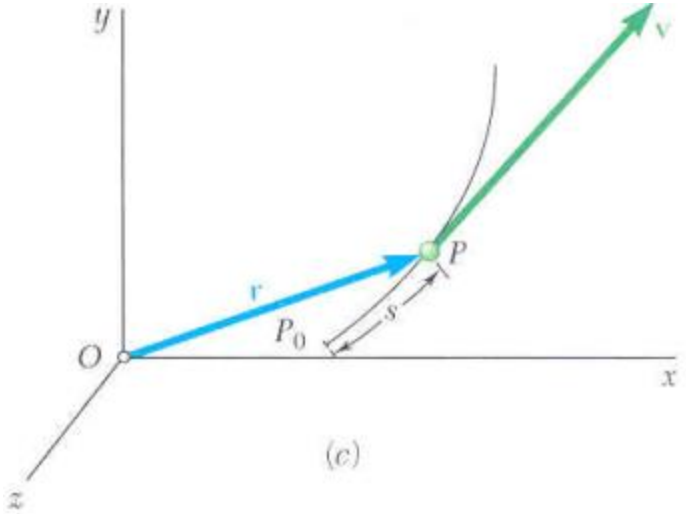
$$\vec{v} = \lim_{\Delta t \to 0} \overrightarrow{v_{av}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{r}$$

Note: As Δt approaches zero, the direction of $\Delta\vec{r}$ approaches to the tangent of the path.

Hence the velocity V is always a vector tangent to the path.

$$v = |\vec{v}| = \frac{ds}{dt} = \dot{s}$$

- The derivative of a vector is itself a vector having both a magnitude and a direction.



Note: there is a clear distinction between the magnitude of the derivative and the derivative of the magnitude.

- The magnitude of the derivative.

$$\left|\frac{d\vec{r}}{dt}\right| = \left|\dot{\vec{r}}\right| = \left|\vec{v}\right| = v \Rightarrow speed$$

- The derivative of the magnitude

$$\frac{d\left|\vec{r}\right|}{dt} = \frac{dr}{dt} = \dot{r}$$

The rate at which the length of the position $\vec{r}$ vector is changing.

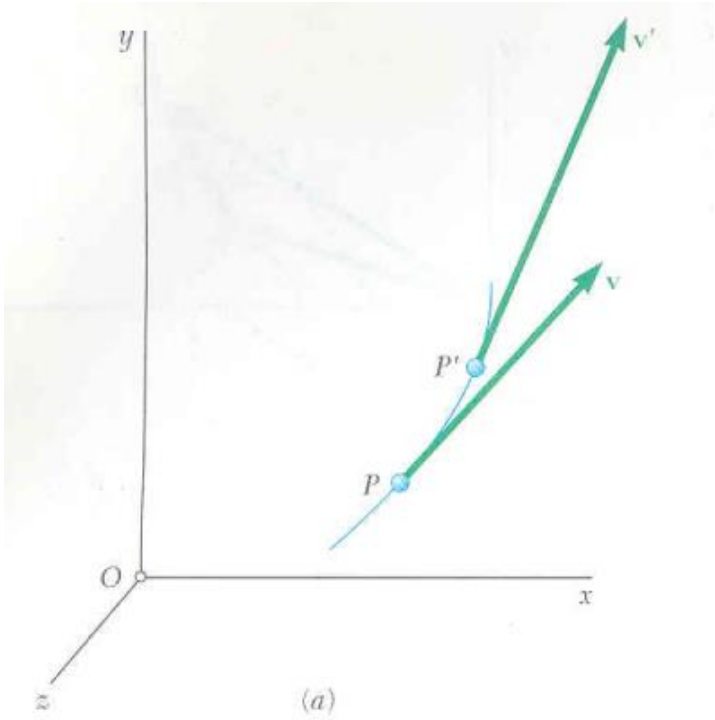
- The magnitude of the vector v is called the speed of the particle.

$$v = \lim_{\Delta t \to 0} \frac{pp'}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

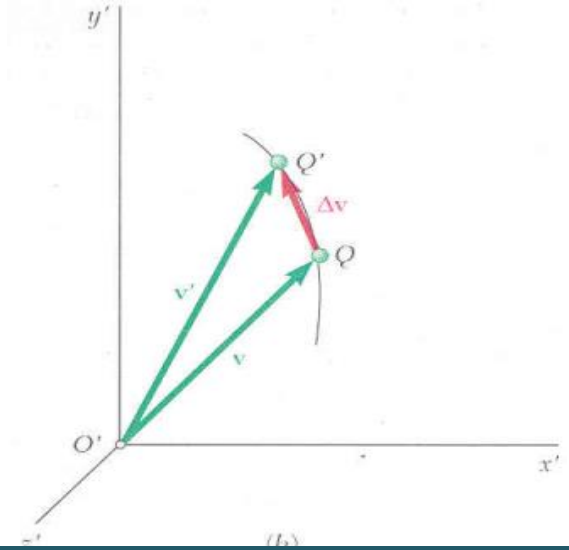
$$v = \frac{ds}{dt}$$

Consider the following figure

- let the velocity at p be $\vec{v}$
- let the velocity at p' be $\vec{v}'$



- Let us draw both vectors v and v’ from the same origin o’. The vector Δv joining Q and Q’ represents the change in the velocity of the particle during the time interval Δt.



$$v' = \Delta v + v$$

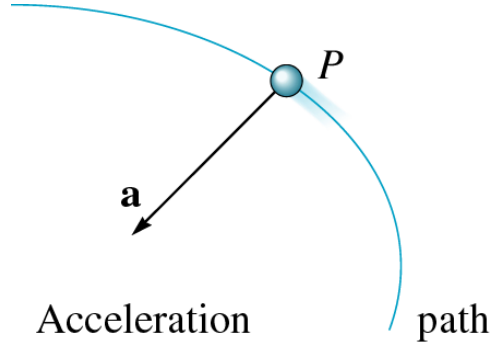
- *Average acceleration*, of the particle between P and P' is defined as $\frac{\Delta \vec{v}}{\Delta t}$, which is a vector and whose direction is that of Δv.

$$\overrightarrow{a_{av}} = \frac{\Delta \vec{v}}{\Delta t}$$

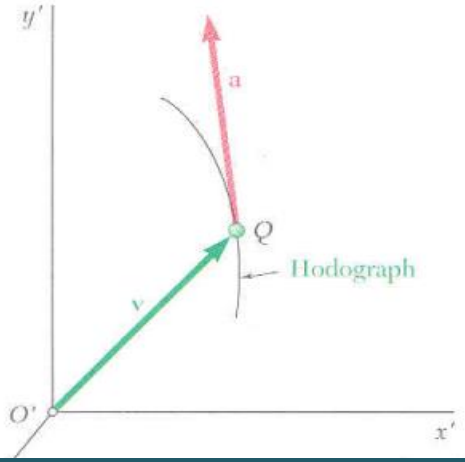
- *Instantaneous acceleration* $\vec{a}$,

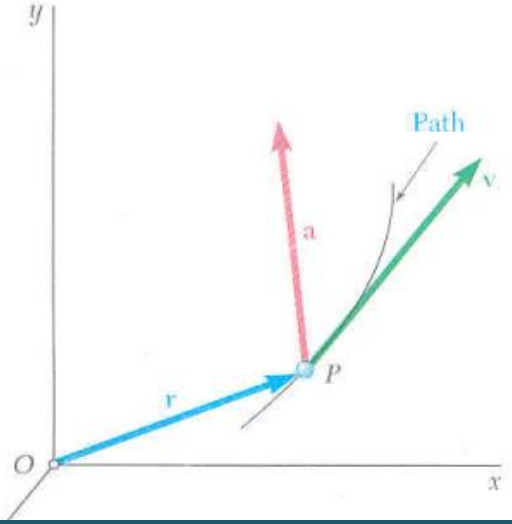
$$a = \lim \overrightarrow{a_{av}} = \lim \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}}$$

Note: The direction of the acceleration of a particle in curvilinear motion is neither tangent to the path nor normal to the path.



- Suppose we take the set of velocity vectors and trace out a continuous curve; such a curve is called a *hodograph*.
- The acceleration vector is tangent to the *hodograph*, but this does *not* produce vectors tangent to the *path* of the particle.





Rectangular co-ordinates (x-y-z)

- This is particularly useful for describing motions where the *x,y and z-components* of acceleration are *independently generated*.

- When the position of a particle P is defined at any instant by its rectangular coordinate x,y and z, it is convenient to resolve the *velocity v* and the *acceleration a* of the particle into rectangular components.

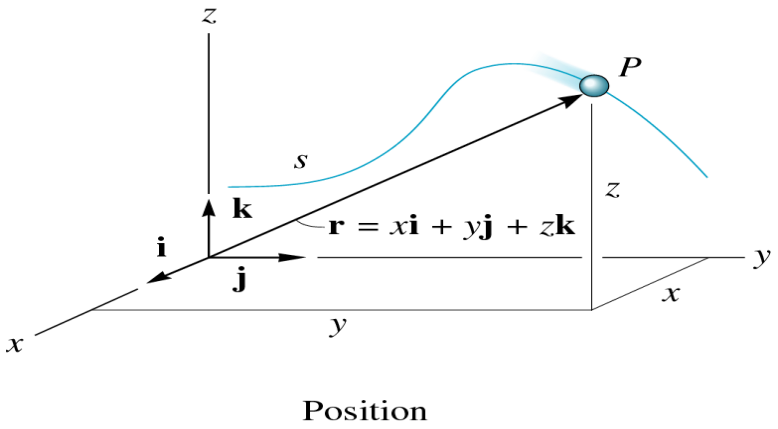
- Resolving the position vector r of the particle into rectangular components,

$$r = xi + yj + zk$$

- Differentiating

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$v = \dot{x}\,i + \dot{y}\,j + \dot{z}\,k$$



Position

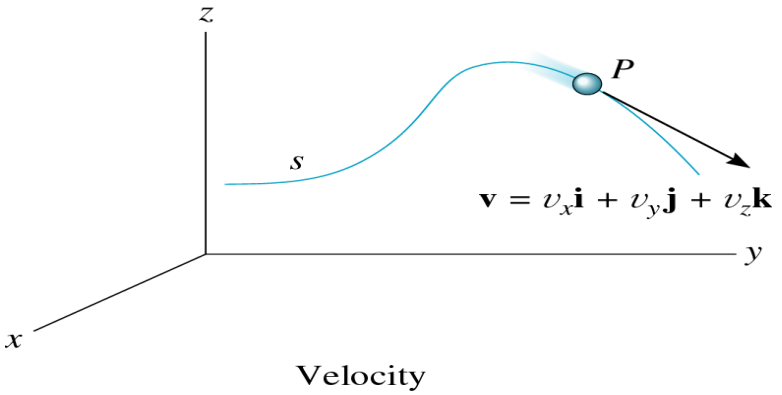
• All of the following are equivalent:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$= \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$= v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$



Velocity

• Since the *speed* is defined as the magnitude of the velocity, we have:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

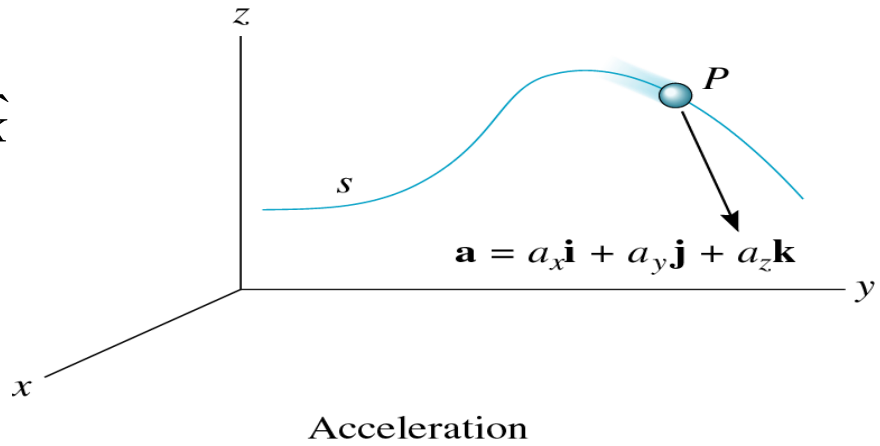
Similarly,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$= \dot{v}_x\hat{i} + \dot{v}_y\hat{j} + \dot{v}_z\hat{k}$$

$$= \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$



Acceleration

- The magnitude of the acceleration vector is:

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

- From the above equations that the scalar components of the velocity and acceleration are

$$v_x = \dot{x} \quad v_y = \dot{y} \quad v_z = \dot{z}$$

$$a_x = \ddot{x} \quad a_y = \ddot{y} \quad a_z = \ddot{z}$$

- The use of *rectangular components* to describe the *position*, the *velocity* and the *acceleration* of a particle is particularly *effective* when the component a_x of the acceleration depends only upon t,x and/or $v_{x,}$ similarly for a_y and $a_{z.}$

- The motion of the particle in the *x direction*, its motion in the *y direction*, and its motion in the *z direction* can be considered *separately*.

Projectile motion

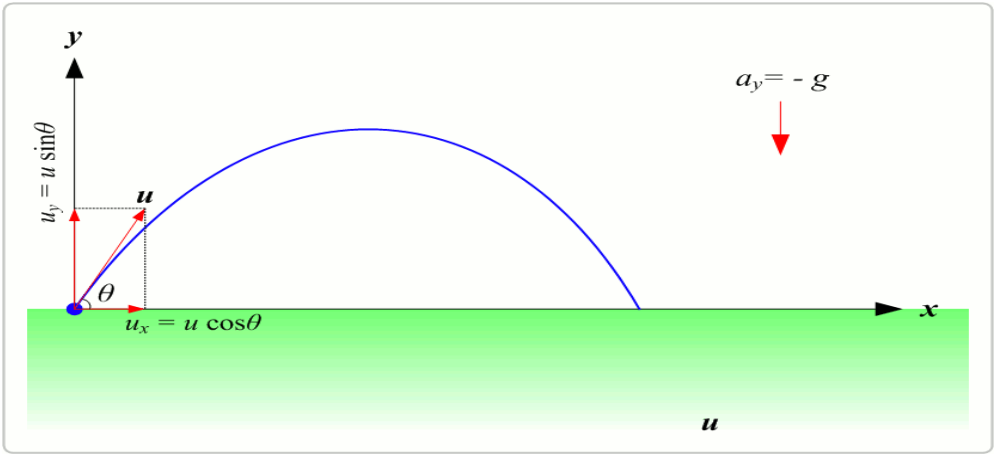
- An important application of two – dimensional kinematic theory is the problem of projectile motion.

Assumptions

- Neglect the aerodynamic drag, the earth curvature and rotation,
- The altitude range is so small enough so that the acceleration due to gravity can be considered constant, therefore;

- Rectangular coordinates are useful for the *trajectory analysis*.
- In the case of the motion of a projectile, it can be shown that the components of the acceleration are

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$



Boundary conditions

at $t=0$; $x=x_o$,$y=y_o$; $v_x=v_{xo}$ and $v_y=v_{y0}$

Position

$$x = x_0 + v_{x0}t$$

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2$$

$$z = z_o + v_{zo}t$$

Velocity

$$v_x = \dot{x} = v_{x0}$$

$$v_y = \dot{y} = v_{y0} - gt$$

$$v_z = \dot{z} = v_{zo}$$

$$v_y^2 = v_{yo}^2 - 2g(y - y_o)$$

- *In all these expressions, the subscript zero denotes initial conditions*

- *But for two dimensional motion of the projectile,*

$$x = x_0 + v_{x0}t$$

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2$$

$$v_x = \dot{x} = v_{x0}$$

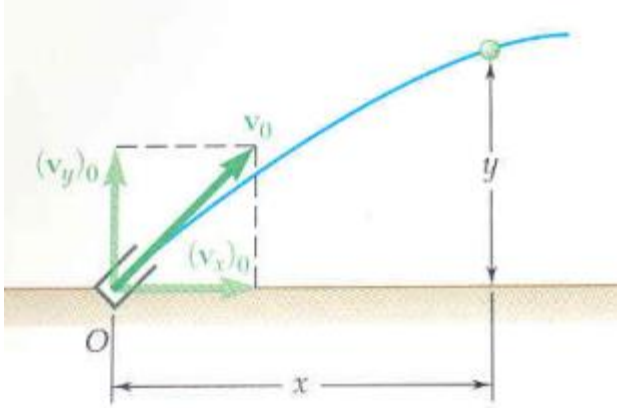
$$v_y = \dot{y} = v_{y0} - gt$$

$$v_y{}^2 = v_{yo}{}^2 - 2g(y - y_o)$$

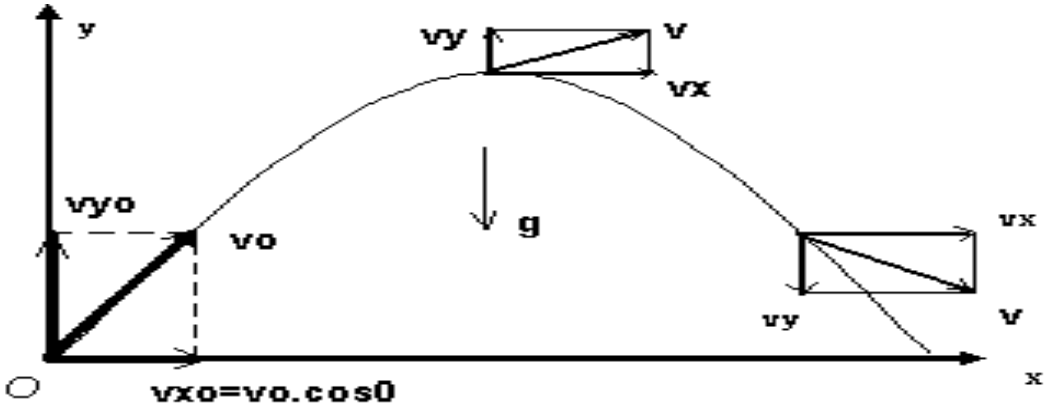
- If the projectile is fired from the origin O, we have $x_o=y_o=0$ and the equation of motion reduced to

$$x = v_{x0}t \qquad v_x = v_{x0}$$

$$y = v_{y0}t - \frac{1}{2}gt^2 \qquad v_y = v_{y0} - gt$$



(a) Motion of a projectile

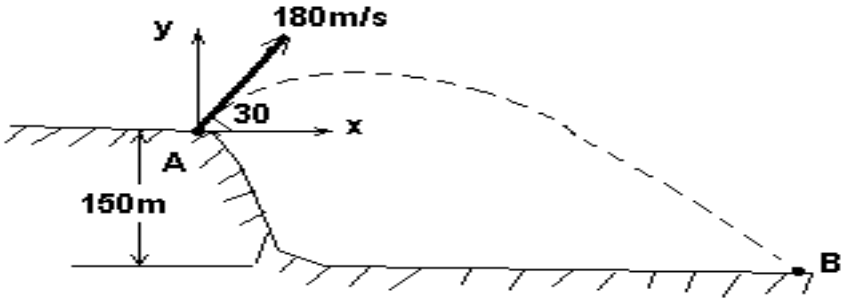

y
vy
v
vx
vyo
vo
g
vx
vy
v
x
O
vxo=vo.cos0

Example 1

A projectile is fired from the edge of a 150m cliff with an initial velocity of 180m/s at angle of 30^0 with the horizontal. Neglect air resistance, find

a) the horizontal distance from gun to the point where the projectile strikes the ground.

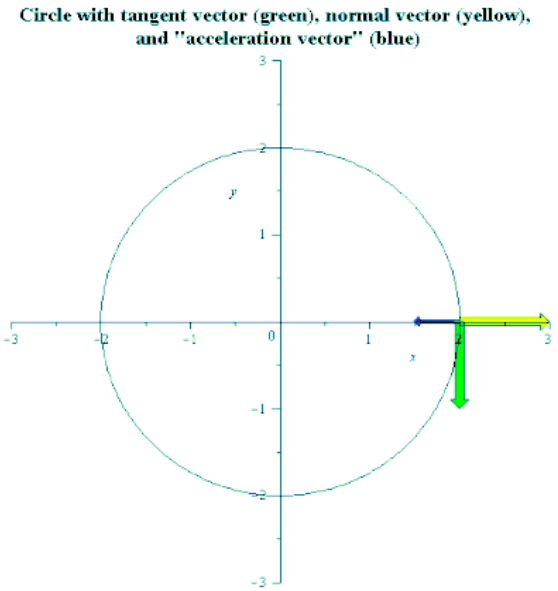
b) the greatest elevation above the ground reached by the projectile.

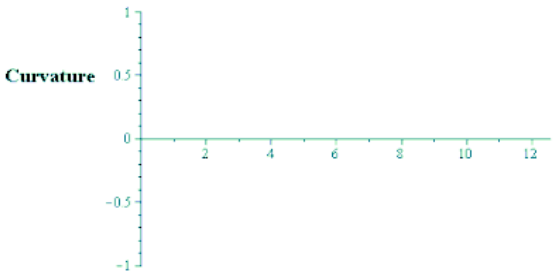


Curvilinear Motion

Normal and Tangential Coordinates



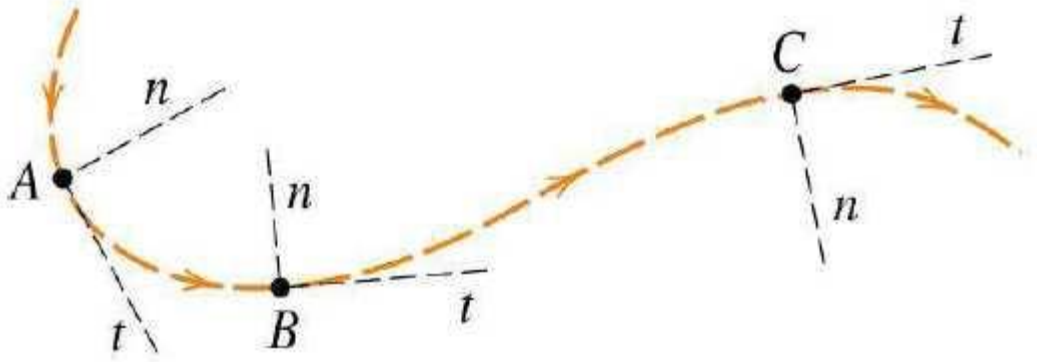




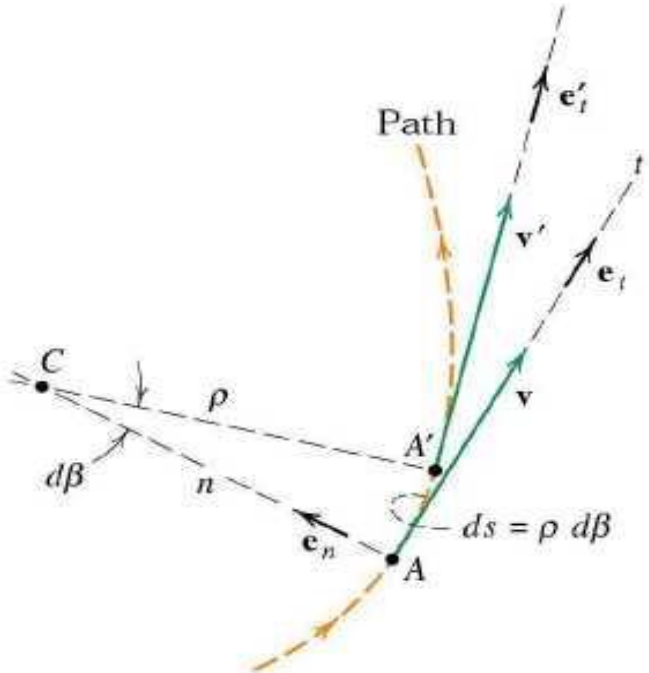
- When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian.
- When the path of motion is known, *normal* (n) and *tangential* (t) coordinates are often used.
- They are *path variables*, which are measurements made along the tangent *t* and normal *n* to the path of the particle.
- They are considered to move along the path with the particle.
- In the n-t coordinate system, the origin is located on the particle (the origin moves with the particle).

- The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion.

- The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.



- The coordinate *n* and *t* will now be used to describe the velocity *v* and acceleration *a*.

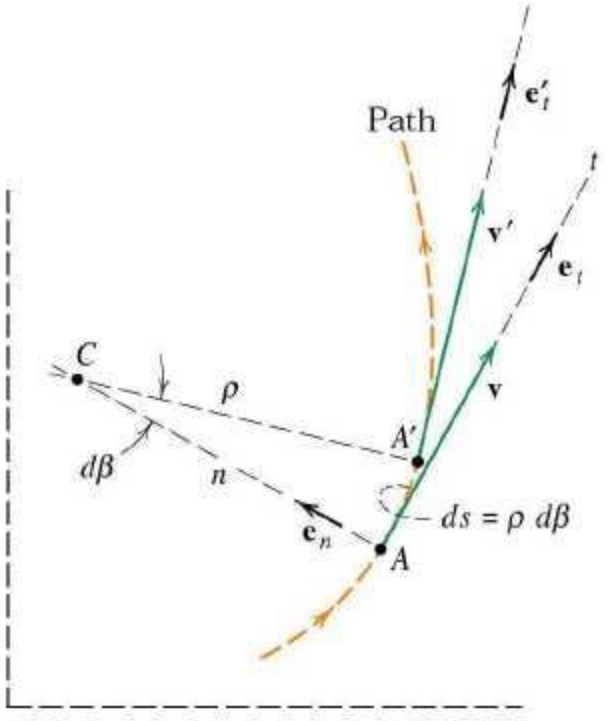


- Similarly to the unit vectors ***i*** and ***j*** introduced for rectangular coordinate system, unit vectors for *t-n* coordinate system can be used.

- For this purpose we introduce unit vector
  - $\mathbf{e}_t$ in the t-direction
  - $\mathbf{e}_n$ in the n-direction.
- $\mathbf{e}_t$ - *directed toward the direction of motion.*
- $\mathbf{e}_n$-*directed toward the center of curvature of the path.*

- During the differential increment of time *dt*, the particle moves a differential distance *ds* along the curve from A to A'.
- With the radius of curvature of the path at this position designated by ρ, we see that

$$ds = \rho d\beta$$



velocity

- The magnitude of the velocity is:-

$$v = |\vec{v}| = \frac{ds}{dt} = \frac{\rho d\beta}{dt} = \rho \frac{d\beta}{dt}$$

- Since it is unnecessary to consider the differential change in ρ between A and A',

$$\vec{v} = v\vec{e}_t = \rho\dot{\beta}\vec{e}_t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Acceleration

- The *acceleration* a of the particle was defined by:

$$\vec{a} = \frac{dv}{dt} = \frac{d}{dt}(ve_t)$$

- Now differentiate the velocity by applying the ordinary rule (chain rule) for the differentiation of the product of a *scalar* and a *vector*.

$$\vec{a} = \frac{dv}{dt} = \frac{d}{dt}(ve_t) = \frac{dv}{dt}e_t + v\frac{de_t}{dt}$$

$$\vec{a} = \dot{v}e_t + v\dot{e}_t$$

- Where the unit vector e_t now has a derivative because its direction changes.

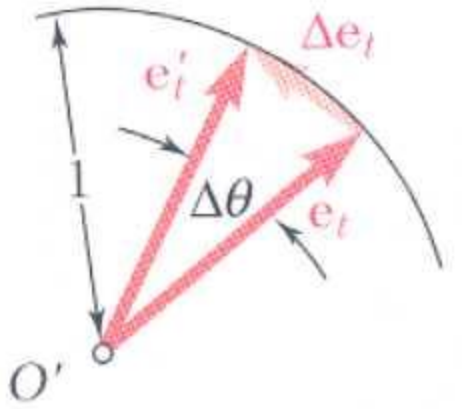
$$\vec{a} = \frac{dv}{dt} e_t + v \frac{de_t}{dt} \ldots\ldots\ldots (1)$$

- To find the derivative of $\frac{de_t}{dt}$ consider the following figure
- Using vector addition

$$\mathbf{e'_t = e_t + \Delta e_t}$$

- Since the magnitude

$$|\mathbf{e'_t}| = |\mathbf{e_t}| = 1$$

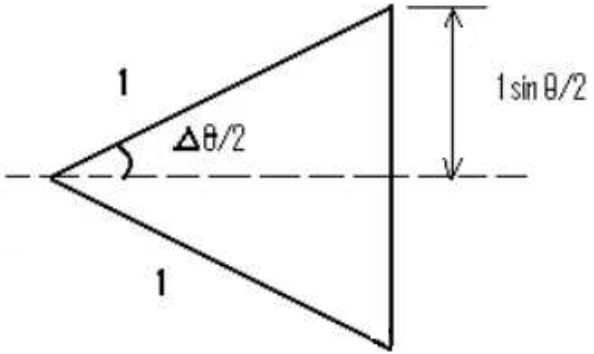


- The magnitude of Δe_t

  $| \Delta \mathbf{e_t} | = 2 \sin \Delta\theta/2$

- Dividing both sides by $\Delta\theta$

$$\left|\frac{\Delta e_t}{\Delta\theta}\right| = \frac{2\sin \Delta\theta/2}{\Delta\theta}$$



- As $\Delta\theta \rightarrow 0$, $\frac{\Delta e_t}{\Delta\theta}$ is tangent to the path;i.e,

perpendicular to $\mathbf{e_t}$.

- Taking the limit as $\Delta\theta \to 0$

$$\lim_{\Delta\theta\to 0}\left|\frac{\Delta e_t}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\frac{\sin \Delta\theta/2}{\Delta\theta/2} = 1$$

- The vector obtained in the limit is a unit vector along the normal to the path of the particle.

$$\lim_{\Delta\theta\to 0}\left|\frac{\Delta e_t}{\Delta\theta}\right| = \left|\frac{de_t}{d\theta}\right| = 1$$

- But $\frac{de_t}{d\theta} = 1.e_n$

$$\therefore e_n = \frac{de_t}{d\theta} \Rightarrow de_t = d\theta.e_n$$

- Dividing both sides by dt

$$\frac{de_t}{dt} = \frac{d\theta}{dt}.e_n$$ But $d\theta = ds/\rho$

- Then $$\frac{de_t}{dt} = \frac{1}{\rho}\frac{ds}{dt}.e_n \Rightarrow \frac{de_t}{dt} = \frac{v}{\rho}e_n$$

- Equation (1) becomes

$$a = v\frac{de_t}{dt} + \frac{dv}{dt}e_t \Rightarrow a = \frac{v^2}{\rho}.e_n + \frac{dv}{dt}.e_t$$

- We can write $a = a_n e_n + a_t e_t$

where, $a_n = \frac{v^2}{\rho} = \rho\dot{\theta}^2$ and $a_t = \dot{v} = \rho\ddot{\theta}$

$$|a| = a = \sqrt{a_n^2 + a_t^2}$$

Note:

- a_n is always directed towards the center of curvature of the path.
- a_t is directed towards the positive t-direction of the motion if the speed v is increasing and towards the negative t-direction if the speed *v* is decreasing.
- At the inflection point in the curve, the normal acceleration, $\frac{v^2}{\rho}$ goes to zero since ρ becomes infinity.

Special Case of Motion

- Circular motion

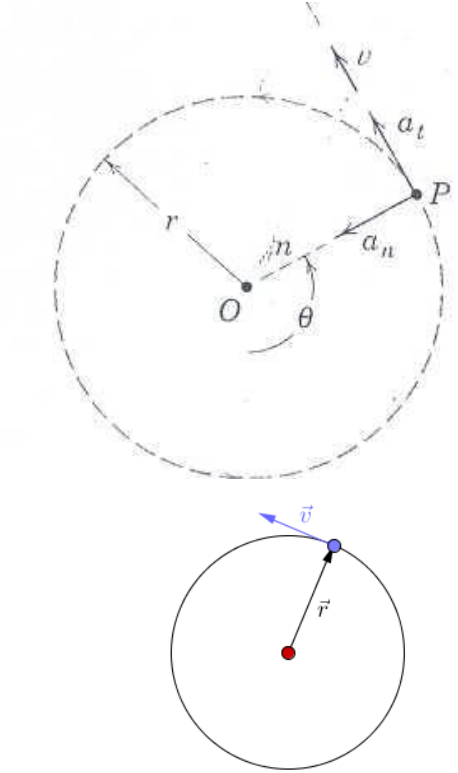
$$a_n = \frac{v^2}{\rho} \quad \text{but } \rho = r \text{ and } \quad v = r\dot{\theta}$$

$$a_n = r\dot{\theta}^2$$

$$a_t = \frac{dv}{dt} = \frac{d}{dt}\left(r\dot{\theta}\right) = r\frac{d\dot{\theta}}{dt}$$

$$a_t = r\ddot{\theta}$$

$$a = r\ddot{\theta}e_t + r\dot{\theta}^2 e_n$$



- The particle moves along a path expressed as y = f(x). The radius of curvature, ρ, at any point on the path can be calculated from

$$\rho_{xy} = \left| \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{d^2y/dx^2} \right|$$

Applications

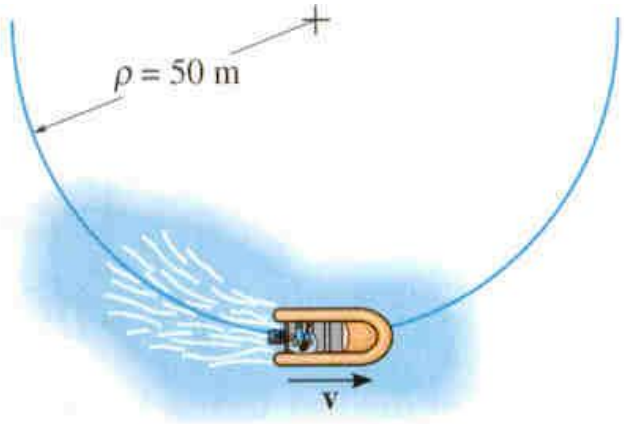


Cars traveling along a clover-leaf interchange experience an acceleration due to a change in speed as well as due to a change in direction of the velocity.

Example 1

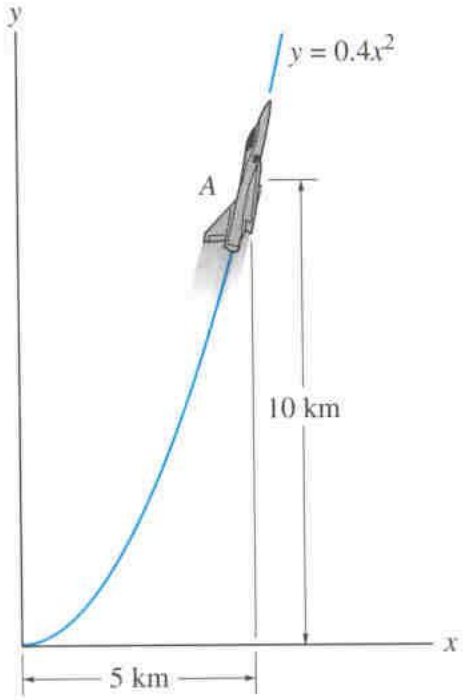
- Starting from rest, a motorboat travels around a circular path of $r = 50$ m at a speed that increases with time, $v = (0.2\ t^2)$ m/s.

Find the magnitudes of the boat's *velocity* and *acceleration* at the instant $t = 3$ *s*.



Example 2

- A jet plane travels along a vertical parabolic path defined by the equation $y = 0.4x^2$. At point A, the jet has a speed of *200 m/s*, which is increasing at the rate of $0.8\ m/s^2$. Find the *magnitude* of the plane's *acceleration* when it is at point A.



Example 3

- A race traveling at a speed of *250km/h* on the straightway applies his brakes at point A and reduce his speed at a uniform rate to *200km/h* at C in a distance of *300m*.
- Calculate the magnitude of the *total acceleration* of the race car an instant after it passes *point B*.

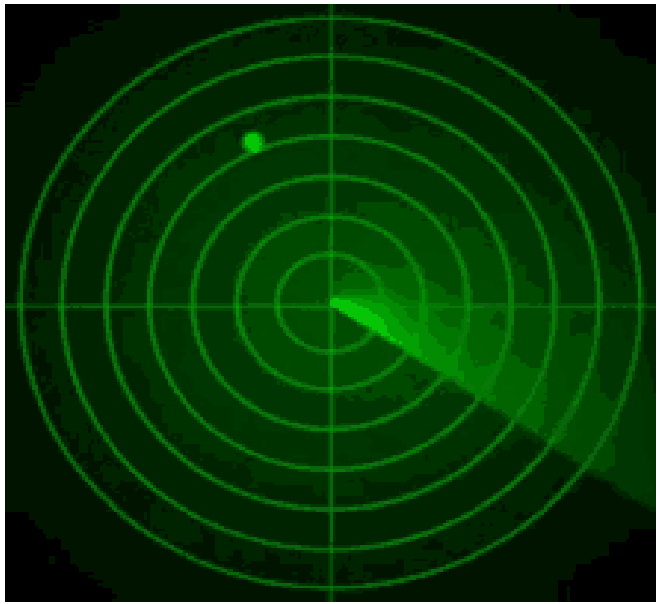
Example 4

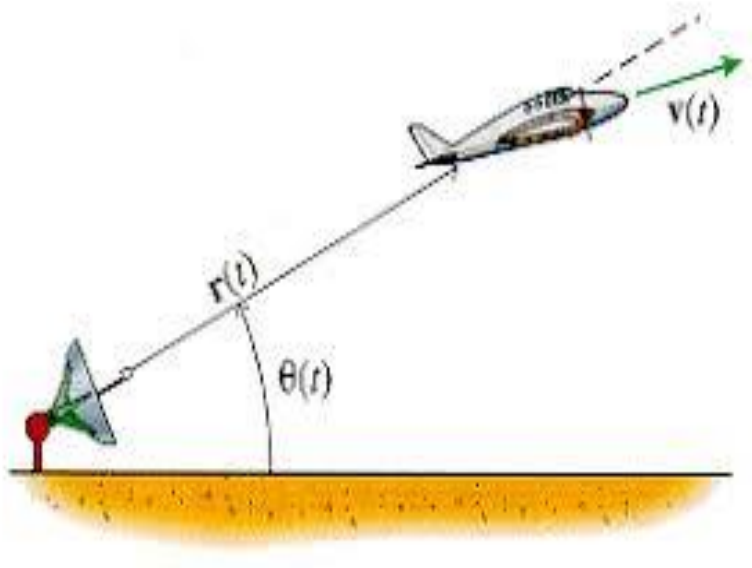
- The motion of pin A in the fixed circular slot is controlled by a guide B, which is being elevated by its lead screw with a constant upward velocity v_o=2m/s for the interval of its motion.
- Calculate both the *normal* and *tangential components* of *acceleration* of *pin A* as it passes the position for which

.

Curvilinear Motion

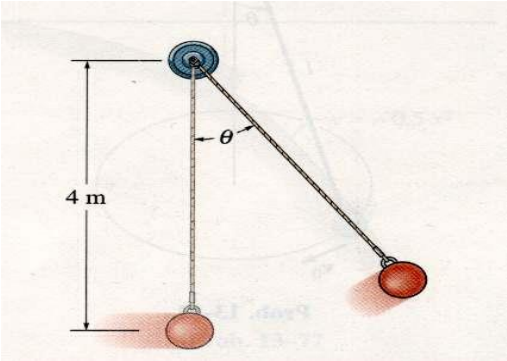
Polar Coordinate System (r- ɵ)

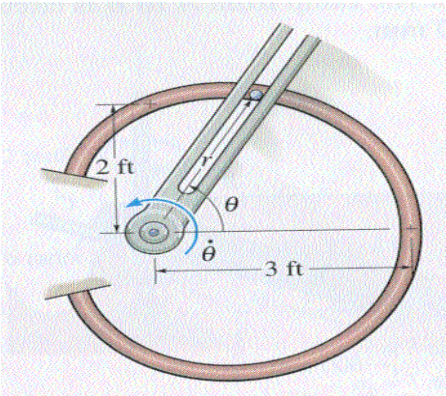


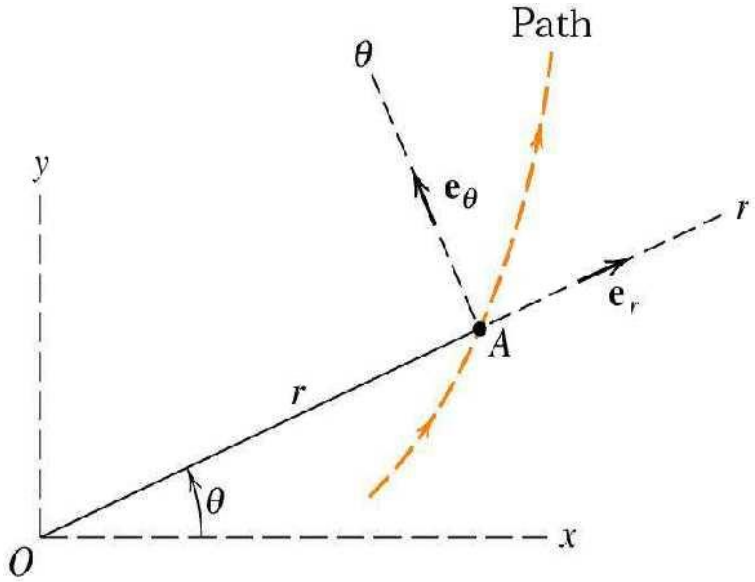


Polar Coordinate System (r- ɵ)

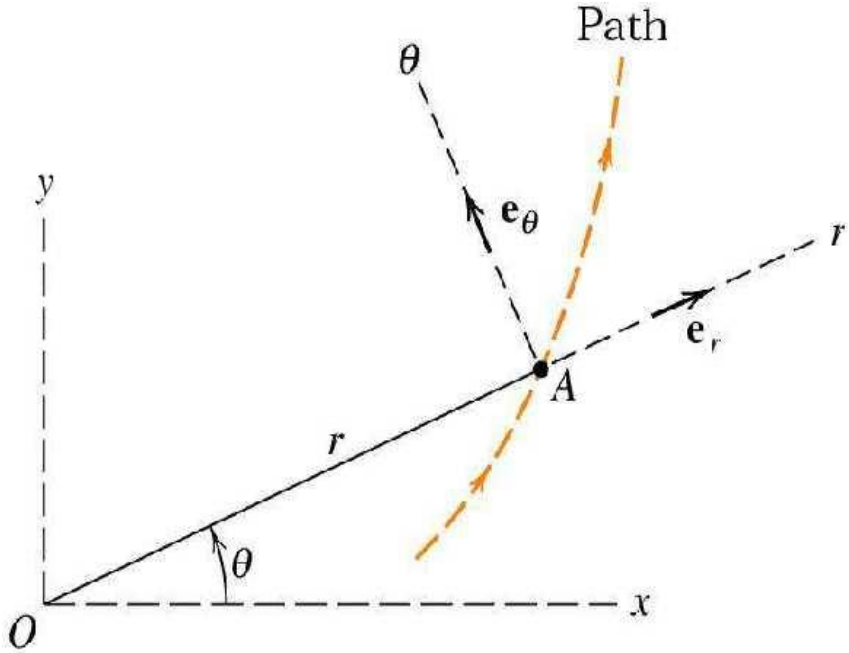
- The third description for *plane curvilinear motion.*
- Where the particle is located by the *radial distance r* from a fixed pole and by an *angular measurement ɵ* to the radial line.
- Polar coordinates are particularly useful when a motion is *constrained* through the control of a radial distance and an angular position, or when an unconstrained motion is observed by measurements of a radial distance and an angular position.


4 m
θ


2 ft
θ
θ̇
3 ft


Path
θ
y
e_θ
r
e_r
A
r
θ
O
x

- An arbitrary fixed line, such as the x-axis, is used as a reference for the measurement ɵ.
- Unit vectors e_r and e_θ are established in the *positive r* and *ɵ* directions, respectively.



- The position vector $\vec{r}$ to the particle at A has a magnitude equal to the radial distance r and a direction specified by the unit vector e_r.
- We express the location of the particle at A by the vector

$$\vec{r} = r.\vec{e}_r$$

Velocity

- The velocity is obtained by differentiating the vector **r.**

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr.\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt}$$

$$\vec{v} = \dot{r}.\vec{e}_r + r.\dot{\vec{e}}_r$$

- Where the unit vector e_r now has a derivative because its direction changes.

- We obtain the derivation in exactly the same way that we derived for e_t.

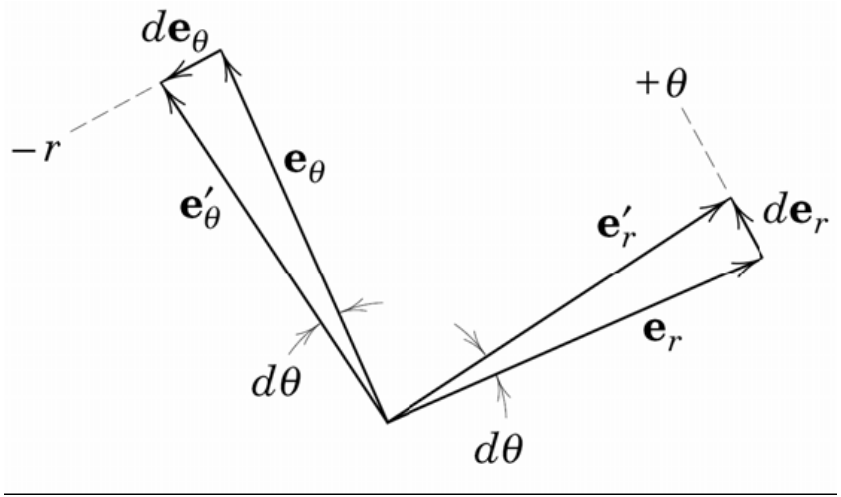
- To find the derivative of $\frac{de_r}{dt}$ consider the following figure
- Using vector addition

$$\mathbf{e'_r} = \mathbf{e_r} + \Delta\mathbf{e_r}$$

$$\mathbf{e'_\theta} = \mathbf{e_\theta} + \Delta\mathbf{e_\theta}$$

- Since the magnitude

$$|\mathbf{e'_r}| = |\mathbf{e_r}| = |\mathbf{e'_\theta}| = |\mathbf{e_\theta}| = 1$$

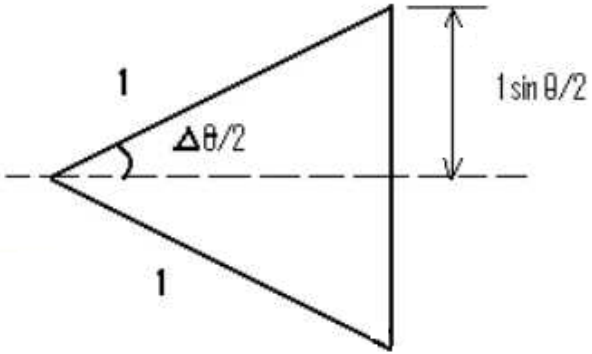


- The magnitude of Δe_r and $\Delta \mathbf{e}_\theta$

$$|\Delta \mathbf{e}_r| = |\Delta \mathbf{e}_\theta| = 2 \sin \Delta\theta/2$$

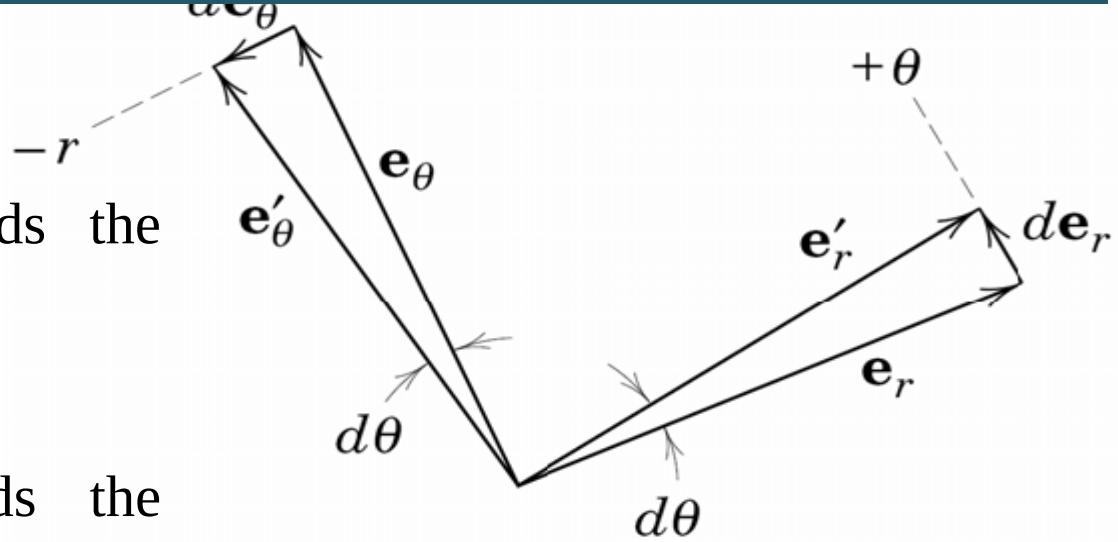
- Dividing both sides by $\Delta\theta$

$$\left|\frac{\Delta e_r}{\Delta\theta}\right| = \left|\frac{\Delta e_\theta}{\Delta\theta}\right| = \frac{2\sin \Delta\theta/2}{\Delta\theta}$$



- As $\Delta\theta \rightarrow 0$, $\frac{\Delta e_r}{\Delta\theta}$ is perpendicular to $\mathbf{e}_r$.

Note: As $\Delta\theta \to 0$,



1. $\frac{\Delta e_r}{\Delta\theta}$ is directed towards the positive $\mathbf{e}_\theta$ direction.

2. $\frac{\Delta e_\theta}{\Delta\theta}$ is directed towards the negative $\mathbf{e}_r$ direction.

- Then, $$\lim_{\Delta\theta\to 0}\left|\frac{\Delta e_r}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\left|\frac{\Delta e_\theta}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\frac{\sin \Delta\theta/2}{\Delta\theta/2} = 1$$

- Therefore;

$$\lim_{\Delta\theta\to 0}\left|\frac{\Delta e_r}{\Delta\theta}\right| = \left|\frac{de_r}{\Delta\theta}\right| = 1$$

$$\lim_{\Delta\theta\to 0}\left|\frac{\Delta e_\theta}{\Delta\theta}\right| = \left|\frac{de_\theta}{\Delta\theta}\right| = 1$$

$$\frac{de_r}{d\theta} = 1.\vec{e}_\theta = \vec{e}_\theta \Rightarrow de_r = d\theta.\vec{e}_\theta$$

$$\frac{de_\theta}{d\theta} = -1.\vec{e}_r = -\vec{e}_r \Rightarrow de_\theta = -d\theta.\vec{e}_r$$

- Dividing both sides by dt, we have

$$\frac{d\vec{e}_r}{dt} = \frac{d\theta}{dt}\vec{e}_\theta \Rightarrow \frac{d\vec{e}_r}{dt} = \dot{\theta}.\vec{e}_\theta$$

$$\frac{d\vec{e}_\theta}{dt} = -\frac{d\theta}{dt}\vec{e}_r \Rightarrow \frac{d\vec{e}_\theta}{dt} = -\dot{\theta}.\vec{e}_r$$

- Therefore the velocity equation becomes;

$$\vec{v} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \dot{r}.\vec{e}_r + r.\dot{\theta}\vec{e}_\theta$$

- Where, $v_r = \dot{r}$ and $v_\theta = r.\dot{\theta}$

$$v = \sqrt{{v_r}^2 + {v_\theta}^2}$$

$$\alpha = \tan^{-1}\left(\frac{v_\theta}{v_r}\right)$$

- The r-component of v is merely the rate at which the vector r stretches.
- The ɵ-component of v is due to the rotation of r.

Acceleration

- Differentiating the expression for v to obtain the acceleration a.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\dot{r}\,\vec{e}_r + r\,\dot{\theta}\,\vec{e}_\theta\right)$$

$$\vec{a} = \frac{d\,\dot{r}}{dt}\vec{e}_r + \dot{r}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\dot{\theta}\,\vec{e}_\theta + r\frac{d\,\dot{\theta}}{dt}\vec{e}_\theta + r\,\dot{\theta}\frac{d\vec{e}_\theta}{dt}$$

- But from the previous derivation

$$\frac{de_r}{dt} = \dot{\theta}.\vec{e}_\theta, and\ \frac{de_\theta}{dt} = -\dot{\theta}.\vec{e}_r$$

- Substituting the above and simplifying

$$\vec{a} = \ddot{r}\,\vec{e}_r + \dot{r}\,\dot{\theta}\,\vec{e}_\theta + \dot{r}\,\dot{\theta}\,\vec{e}_\theta + r\,\ddot{\theta}\,\vec{e}_\theta + r\,\dot{\theta}\left(-\dot{\theta}\right)\vec{e}_r$$

$$\therefore \vec{a} = \left(\ddot{r} - r\,\dot{\theta}^{\,2}\right)\vec{e}_r + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\vec{e}_\theta$$

- Where

$$a_r = \left(\ddot{r} - r\,\dot{\theta}^{\,2}\right)$$

$$a_\theta = \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)$$

$$a = \sqrt{a_r{}^2 + a_\theta{}^2}$$

$$\phi = \tan^{-1}\left(\frac{a_\theta}{a_r}\right)$$

- For motion in a circular path
  - Velocity

$$\vec{v} = \dot{r}.e_r + r.\dot{\theta}\,e_\theta$$

Where, $\dot{r} = 0$ because r=constant

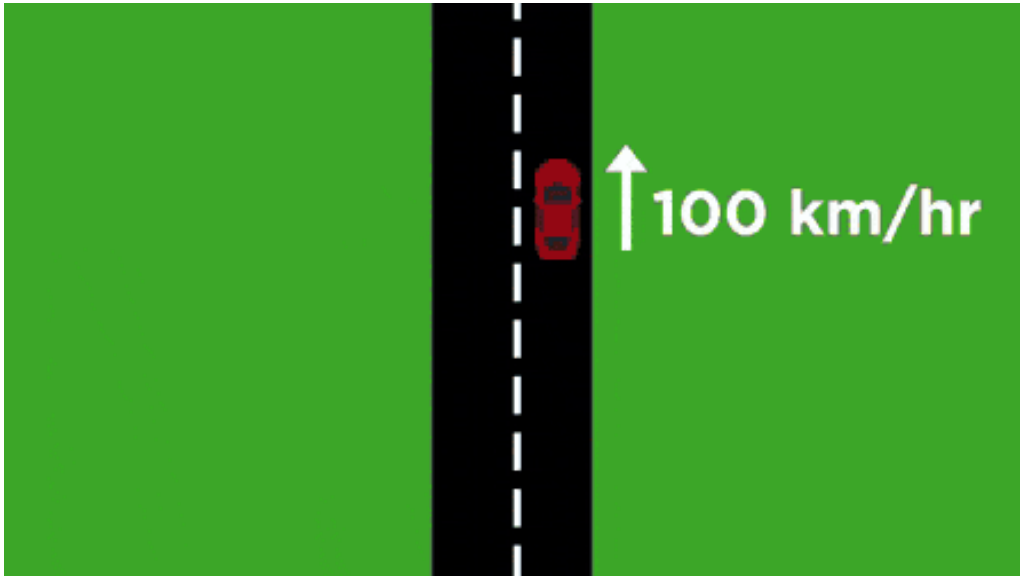
$$\therefore \vec{v} = r.\dot{\theta}\,e_\theta$$

  - Acceleration

where, $\dot{r} = \ddot{r} = 0$

$$\therefore a = \left(-r\dot{\theta}^{2}\right)e_r + \left(r\ddot{\theta}\right)e_\theta$$

Kinematics of Particles

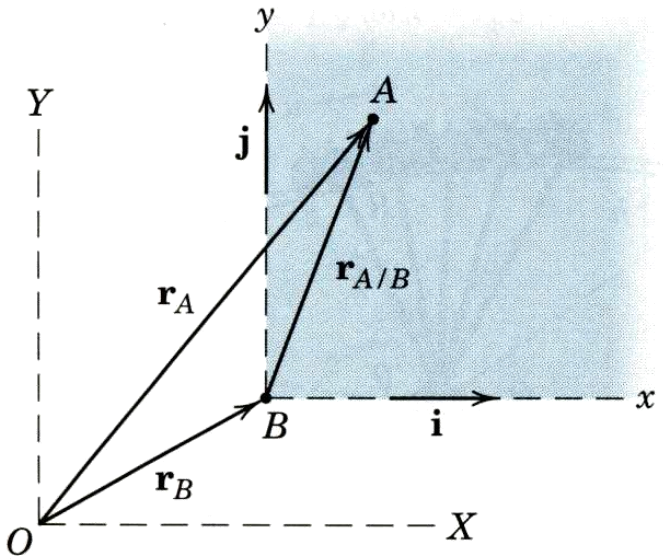


Relative motion

- In this portion *Relative motion analysis :* is the motion analysis of a particle using *moving reference system* coordinate in reference to fixed reference system.
- we will confine our attention to:-
  - moving reference systems that translate but do not rotate.
  - The relative motion analysis is limited to plane motion.
- Note: in this section we need

1. Inertial(fixed) frame of reference.
2. Translating(not rotating) frame of reference.

- Consider two particles *A* and *B* that may have separate curvilinear motion in a given plane or in parallel planes.
- X,Y : *inertial frame of reference*
- *X,y* : *translating coordinate system*



- Using vector addition:
  - *position vector* of particle B is

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

Where: r_A, r_B – absolute position vectors

$r_{B/A}$ – relative position vector of particle B (*B relative to A or B with respect to A*)

- Differentiating the above position vector once we obtain the *velocities* and twice to obtain *accelerations*. Thus,

- Velocity

$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt}$$

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

- Acceleration

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d\vec{v}_{B/A}}{dt}$$

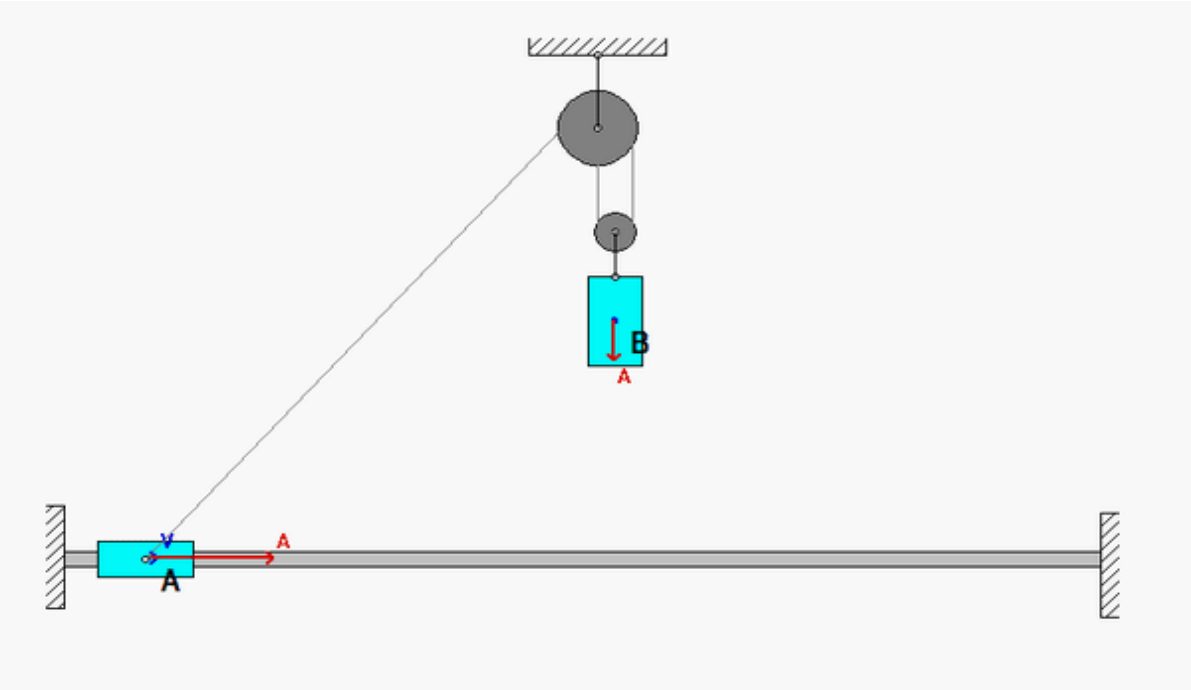
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

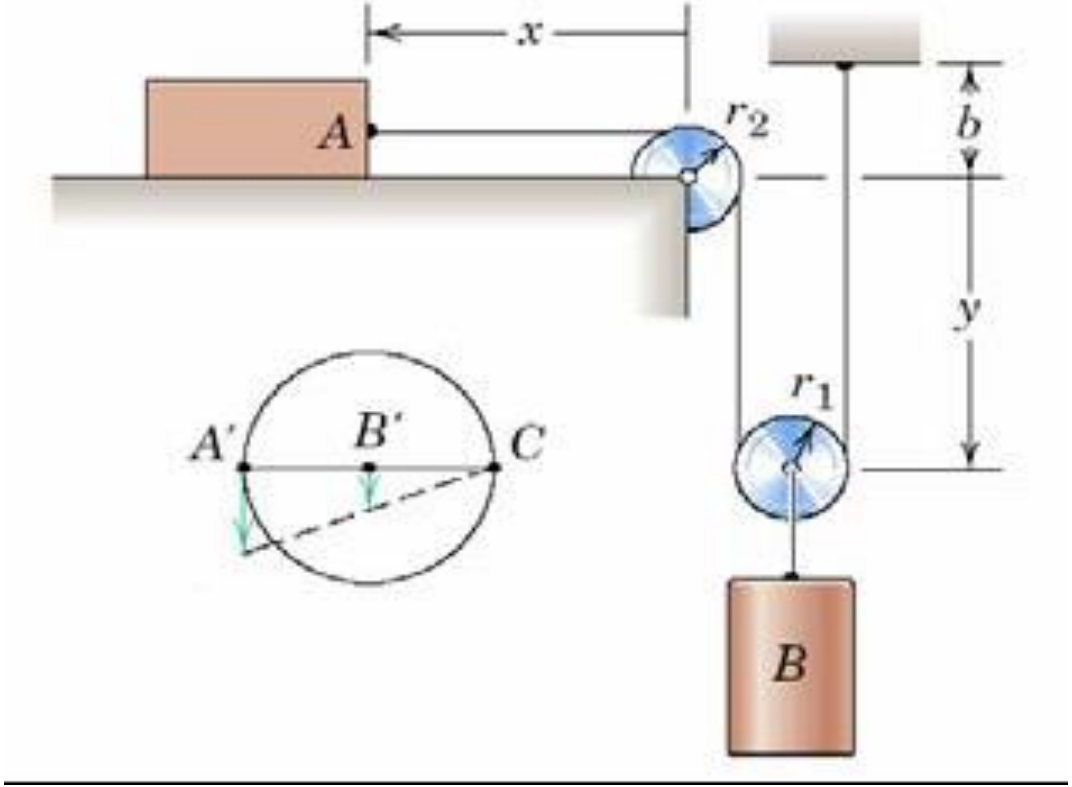
- *Note*: In relative motion analysis, we employed the following two methods,

  1. *Trigonometric(vector diagram)* – A sketch of the vector triangle is made to reveal the trigonometry

  2. *Vector algebra* – using unit vector i and j, express each of the vectors in vector form.

Constrained Motion of Connected Particles

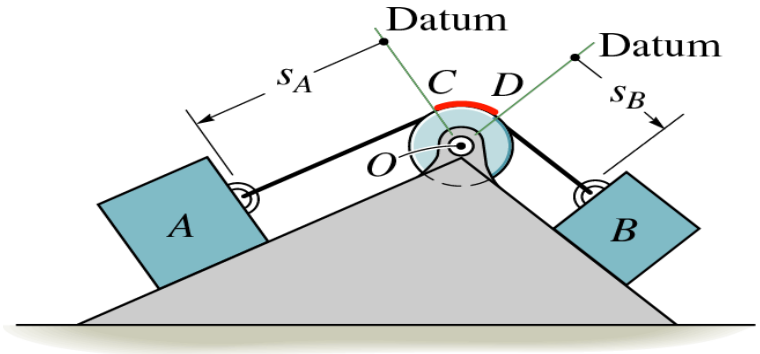



x
A
r_2
b
y
r_1
A'
B'
C
B

Constrained Motion (Dependent Motion)

- Sometimes the position of a particle will depend upon the position of another or of several particles.
- If the particles are connected together by an inextensible ropes, the resulting motion is called constrained motion

- Considering the figure, cable *AB* *is* subdivided into three segments:
  - the length in contact with the pulley, *CD*
  - the length *CA*
  - the length *DB*



- It is assumed that, no matter how A and B move, the length in contact with the pulley is constant.
- We could write:

$$s_A + l_{CD} + s_B = l_{AB} = \text{constant}$$

- Differentiating with respect to time,

$$\frac{ds_A}{dt} + \frac{ds_B}{dt} = 0$$

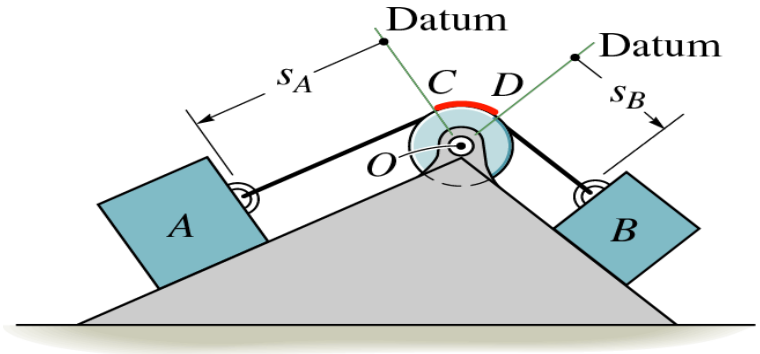
$$v_A + v_B = 0$$

- Differentiating the velocity equation

$$a_A + a_B = 0$$

Important points in this technique:

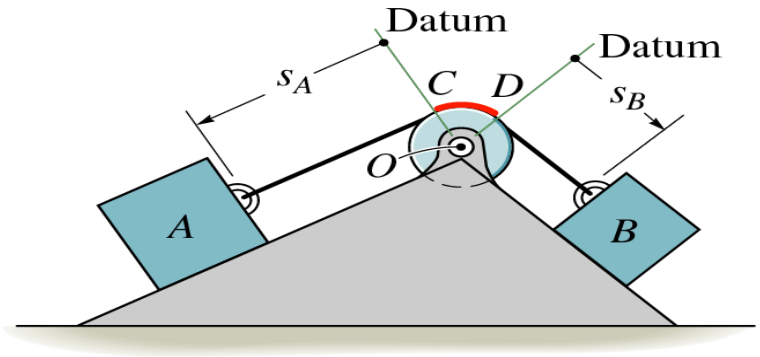
- Each datum must be defined from a *fixed position*.
- In many problems, there may be multiple lengths like l_{CD} that don't change as the system moves. Instead of giving each of these lengths a separate label, we may just incorporate them into an effective length:



- where it's understood that

l = cable length less the length in contact with the pulley = $l_{AB} - l_{CD}$.

$$s_A + s_B = l = \text{constant}$$



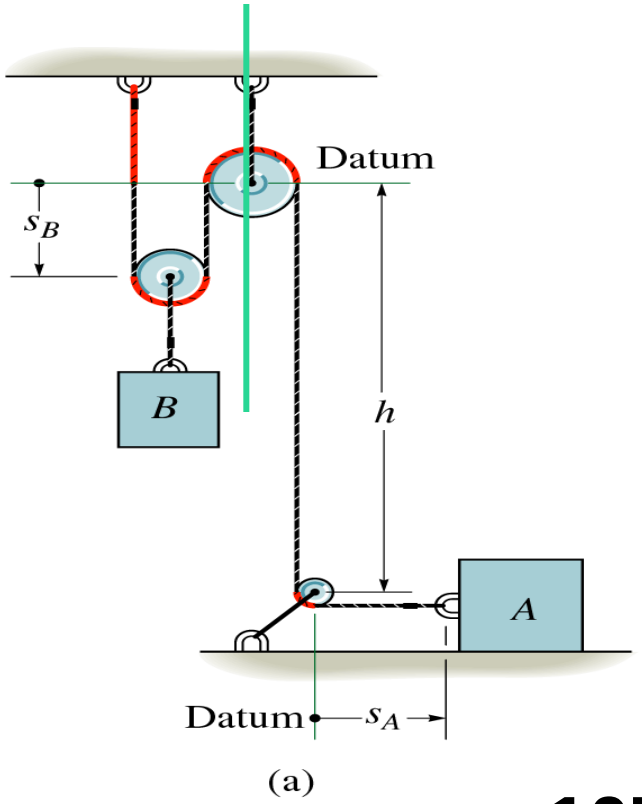
- considering the fig, we could write:

$$s_A + h + 2s_B = l = \text{constant}$$

- where l is the length of the cable less the red segments that remain unchanged in length as A and B move. Differentiating,

$$v_A + 2v_B = 0$$

$$a_A + 2a_B = 0$$



(a)

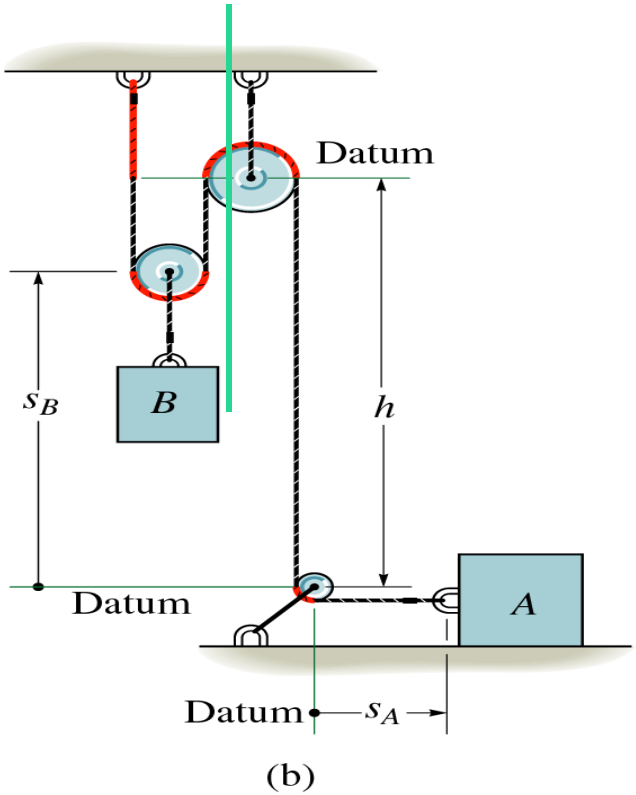
- we could also write the length of the cable by taking another datum:

$$s_A + h + 2(h - s_B) = l = \text{constant}$$

- Differentiating,

$$v_A + 2v_B = 0$$

$$a_A + 2a_B = 0$$



(b)

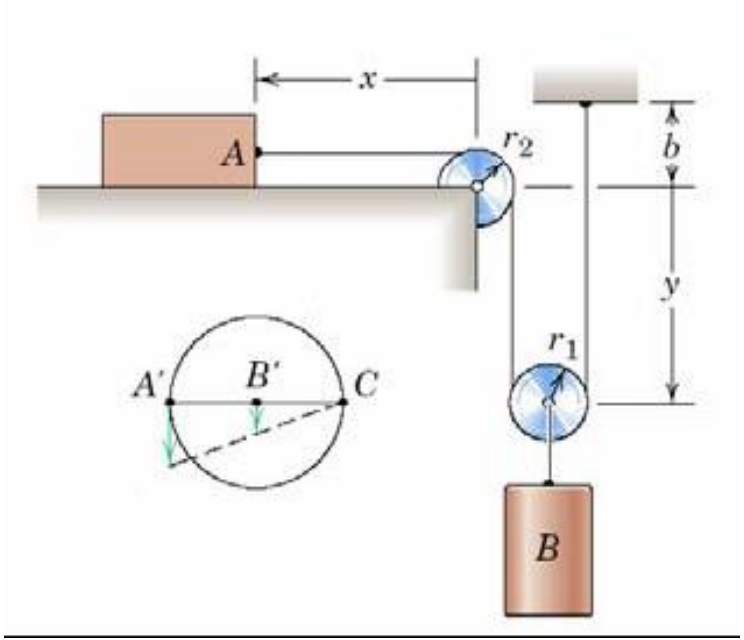
- Consider the fig.,

$$L = x + \frac{\pi r_2}{2} + 2y + \pi r_1 + b$$

- Since L, r_2, r_1 and b is constant, the first and second time derivatives are:-

$$0 = \dot{x} + 2\dot{y}$$

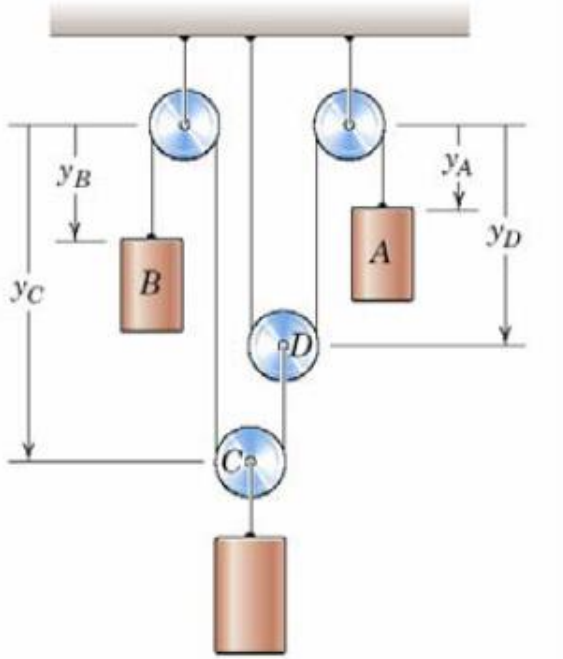
$$0 = \ddot{x} + 2\ddot{y}$$



- Consider the following figure

$$L_A = y_A + 2y_D + const.$$

$$L_B = y_B + y_C + (y_C - y_D) + const.$$



- NB. Clearly, it is impossible for the signs of all three terms to be positive simultaneously.

Suggested Problem

From 7th Edition , Merriam

Engineering Mechanics Dynamics

Chapter II Problem 2/……..

58,48,22,18,44,43,62,79,76,85,81,83,93,

107,120,128,129,114,111,136,140,143,145,

161,154,152,118,220,221,214,189,199,204,

192, and 184

Thank You

Any Question?