

Econ 2012: Linear Algebra for Economists

Chapter-2: Systems of Linear Equations

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Introduction

- In a static-equilibrium model, the standard problem is that of finding the set of values of the endogenous variables which will satisfy the equilibrium condition of the model.
- This involves solving two or more equations simultaneously.
- Consider for example the market model. In this model we need to solve the demand functions and the supply functions of several commodities simultaneously to find the equilibrium prices and the equilibrium quantities.
- For the two commodity case this is relatively easy. One can use graphical methods, the method of substitution or elimination to solve for the equilibrium price and quantity.
- However, as more and more commodities are incorporated to the model, such methods become cumbersome and thus we resort to matrix algebra.

- Matrix algebra can enable us to do many things:
 - It permits the concise expression of a system of linear equations.
 - It provides a shorthand method to determine whether a solution exists before it is attempted.
 - It gives a method of finding that solution (if it exists).
- In this chapter, we will apply the tools of matrix algebra to analyze and solve systems of linear equations.
- We will also introduce and discuss vectors, linear independence and rank of a matrix.

Matrix Representation of Linear Equations

- A system of m linear equations in n variables can be arranged into the following format.

[illegible]

- There are essentially three types of ingredients in the equation system (1).
- The first is the set of coefficients a_{ij} ; the second is the set of variables $x_1, \dots, x_n$; and the last is the set of constant terms $b_1, \dots, b_m$.

Matrix Representation of Linear Equations

- If we arrange the three sets as three matrices and label them, respectively, as A , x , and b , then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad (2)$$

- With the matrices defined in (2), we can express the equation system (1) simply as

$$A\mathbf{x} = \mathbf{b}$$

- The matrix A in this equation is called the *coefficient matrix* of the system, $\mathbf{x}$ is the vector of unknowns, and $\mathbf{b}$ is the vector of known elements.

Solving Systems of Linear Equations

- A solution of system (1) is an ordered set or list of numbers $s_1, s_2, \dots, s_n$ that satisfies all the equations simultaneously when we put $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$.
- Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.
- If system (1) has at least one solution, it is said to be *consistent*. When the system has no solution, it is said to be *inconsistent*.
- In this section, we will discuss Gaussian Elimination, Inverse Method, and Cramer's rule to solve systems of linear equations.

Solving Systems of Linear Equations

Gaussian Elimination

- Consider the system (1). If we mentally keep track of the location of the +’s, the x ’s, and the =’s, (1) can be abbreviated by writing only the matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

- This is called the *augmented matrix* for the system.
- The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but is easier to solve.

Gaussian Elimination

- This new system is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically:
 - 1 Multiply an equation through by a nonzero constant.
 - 2 Interchange two equations.
 - 3 Add a multiple of one equation to another.
- Since the rows of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:
 - 1 Multiply a row through by a nonzero constant.
 - 2 Interchange two rows.
 - 3 Add a multiple of one row to another row.
- These are called *elementary row operations*.

A matrix is in a ***reduced row-echelon form*** if it has the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a *leading 1*.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row-echelon form***.

Example

The following matrices are in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Echelon Forms

Example

The following matrices are in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row-echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Echelon Forms

Thus, with any real numbers substituted for the *'s, all matrices of the following types are in row-echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon Forms

Thus, with any real numbers substituted for the *'s, all matrices of the following types are in row-echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moreover, all matrices of the following types are in reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian Elimination

- If, by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in reduced row-echelon form, then the solution set of the system will be evident by inspection or after a few simple steps. The next example illustrates this situation.

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Example

Solve the system of equations given below by using elementary row operations.

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

Gaussian Elimination

Solution

The augmented matrix of the system is

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array} \right]$$

We then apply successive row operations to get the reduced row-echelon form.

$$(1) \begin{array}{l} R_2 \rightarrow -2R_1 + R_2 \\ R_3 \rightarrow -3R_1 + R_3 \end{array} \quad \left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right] \quad \begin{array}{l} \text{A zero is obtained in} \\ \text{the } a_{21} \text{ and } a_{31} \\ \text{positions.} \end{array}$$

Gaussian Elimination

Solution (*continued*)

$$(2) R_2 \rightarrow \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & \color{red}{1} & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

A one is obtained in the a_{22} position.

$$(3) R_2 \rightarrow -3R_2 + R_3$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & \color{red}{0} & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

A zero is obtained in the a_{32} position.

$$(4) R_3 \rightarrow -2R_3$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & \color{red}{1} & 3 \end{bmatrix}$$

A one is obtained in the a_{33} position.

Solution (*continued*)

$$(5) \quad R_1 \rightarrow -1R_2 + R_1 \quad \left[\begin{array}{cccc} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array} \right]$$

A zero is obtained in the a_{12} position.

$$(6) \quad \begin{array}{l} R_2 \rightarrow \frac{7}{2}R_3 + R_2 \\ R_1 \rightarrow -\frac{11}{2}R_3 + R_1 \end{array} \quad \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

A zero is obtained in the a_{13} and a_{23} positions.

The solution $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ is now evident.

Gaussian Elimination and Gauss-Jordan Elimination

- If we use only the first four steps, the above procedure produces a row-echelon form and is called *Gaussian elimination*.
- Carrying the procedure through to the sixth step and producing a matrix in reduced row-echelon form is called *Gauss-Jordan elimination*.

Back-Substitution

- It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form without continuing all the way to the reduced row-echelon form.
- When this is done, the corresponding system of equations can be solved by a technique called *back-substitution*.

Back-Substitution

- It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form without continuing all the way to the reduced row-echelon form.
- When this is done, the corresponding system of equations can be solved by a technique called *back-substitution*.

Example

From the computations in the last example, a row-echelon form of the augmented matrix is

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Example (*continued*)

The system corresponding to this matrix is

$$x_1 + x_2 + 2x_3 = 9$$

$$x_2 - \frac{7}{2}x_3 = -\frac{17}{2}$$

$$x_3 = 3$$

Solving for the leading variables yields

$$x_1 = 9 - x_2 - 2x_3$$

$$x_2 = -\frac{17}{2} + \frac{7}{2}x_3$$

$$x_3 = 3$$

Substituting the bottom equation into those above and then substituting the second equation into the top yields $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Solving Linear Equations with the Inverse

- An inverse matrix can be used to solve systems of equations. If

$$A\mathbf{x} = \mathbf{b}$$

and the inverse A^{-1} exists, premultiplying both sides of the equation by A^{-1} yields the solution for $\mathbf{x}$, given by

$$\mathbf{x} = A^{-1}\mathbf{b} \tag{3}$$

Solving Linear Equations with the Inverse

- Using the general formula for the inverse discussed in chapter 1, the solution can be written as

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad (4)$$

Solving Linear Equations with the Inverse

Example

Use the inverse method to solve the following system of equations

$$4x_1 + x_2 - 5x_3 = 8$$

$$-2x_1 + 3x_2 + x_3 = 12$$

$$3x_1 - x_2 + 4x_3 = 5$$

Solving Linear Equations with the Inverse

Example

Use the inverse method to solve the following system of equations

$$4x_1 + x_2 - 5x_3 = 8$$

$$-2x_1 + 3x_2 + x_3 = 12$$

$$3x_1 - x_2 + 4x_3 = 5$$

Solution

First, express the system of equations in matrix form,

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

Solving Linear Equations with the Inverse

Solution (*continued*)

Using (3)

$$\begin{aligned}\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= \frac{1}{98} \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix} \\ &= \frac{1}{98} \begin{bmatrix} 196 \\ 490 \\ 98 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}\end{aligned}$$

Thus, $x_1 = 2$, $x_2 = 5$, and $x_3 = 1$.

Cramer's Rule

- *Cramer's rule* provides a simplified method of solving a system of linear equations through the use of determinants.
- From the expression in (4) the solution for the first unknown, x_1 , is seen to be

$$x_1 = \frac{1}{|A|} (b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1})$$

- However, the expression in parentheses is nothing but the evaluation of the determinant of a matrix derived from A by replacing its first column by the column vector $\mathbf{b}$.
- More precisely

$$(b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}) = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Cramer's Rule

- Similarly, for x_2 , the solution is obtained by evaluating the determinant of the matrix A where its second column has been replaced by $\mathbf{b}$, to the determinant of A .
- In general, Cramer's rule states x_j , the j th unknown, is calculated as

$$x_j = \frac{|A_j|}{|A|}, \quad (5)$$

where $|A_j|$ is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of x_j with the column vector of constants $\mathbf{b}$.

Cramer's Rule

Example

Use Cramer's rule to solve the system of equations

$$6x_1 + 5x_2 = 49$$

$$3x_1 + 4x_2 = 32$$

Cramer's Rule

Example

Use Cramer's rule to solve the system of equations

$$6x_1 + 5x_2 = 49$$

$$3x_1 + 4x_2 = 32$$

Solution

Rewrite the system as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 49 \\ 32 \end{bmatrix}$$

Cramer's Rule

Solution (*continued*)

To solve for x_1 , replace column 1 of A , with the vector of constants $\mathbf{b}$, forming a new matrix A_1 .

$$A_1 = \begin{bmatrix} 49 & 5 \\ 32 & 4 \end{bmatrix}$$

Then using (4)

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 49 & 5 \\ 32 & 4 \end{vmatrix}}{\begin{vmatrix} 6 & 5 \\ 3 & 4 \end{vmatrix}} = \frac{36}{9} = 4$$

Cramer's Rule

Solution (*continued*)

To solve for x_2 , replace column 2, the coefficients of x_2 , from the *original* matrix, with the column vector of constants $\mathbf{b}$, forming a new matrix A_2 . Then using (4)

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 6 & 49 \\ 3 & 32 \end{vmatrix}}{\begin{vmatrix} 6 & 5 \\ 3 & 4 \end{vmatrix}} = \frac{45}{9} = 5$$

Thus $x_1 = 4$, and $x_2 = 5$.

Homogeneous Systems of Linear Equations

- A system of linear equations is said to be *homogeneous* if the constant terms are all zero; that is, the system has the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

- A homogeneous system will always have the so-called *trivial* solution $x_1 = x_2 = \cdots = x_n = 0$.
- If there are other solutions, they are called *nontrivial* solutions.

Homogeneous Systems of Linear Equations

Theorem 2.1

A homogeneous linear system with n equations and n unknowns has nontrivial solutions if and only if the coefficient matrix $A = (a)_{ij}$ is singular (that is, if and only if $|A| = 0$).

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A homogeneous linear system with n equations and n unknowns has nontrivial solutions if and only if the coefficient matrix $A = (a)_{ij}$ is singular (that is, if and only if $|A| = 0$).

Example

Determine whether the following system of equations has nontrivial solutions.

$$2x_1 + 2x_2 - x_3 = 0$$

$$4x_1 \quad \quad + 2x_3 = 0$$

$$6x_2 - 3x_3 = 0$$

Homogeneous Systems of Linear Equations

Solution

This system of equations can be written in matrix form as

$$\begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix is

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{vmatrix} = -24$$

Since the determinant of the coefficient matrix is different from zero, the system of equations has only the trivial solution, i.e.,

$$x_1 = 0, x_2 = 0, x_3 = 0.$$

Homogeneous Systems of Linear Equations

Example

Examine for what values of λ the following system of equations has nontrivial solutions.

$$5x + 2y + z = \lambda x$$

$$2x + y = \lambda y$$

$$x + \quad + z = \lambda z$$

Homogeneous Systems of Linear Equations

Example

Examine for what values of λ the following system of equations has nontrivial solutions.

$$5x + 2y + z = \lambda x$$

$$2x + y = \lambda y$$

$$x + z = \lambda z$$

Solution

The variables x, y , and z appear on both sides of the equations, so we start by putting the system into standard form:

$$(5 - \lambda)x + 2y + z = 0$$

$$2x + (1 - \lambda)y = 0$$

$$x + (1 - \lambda)z = 0$$

Homogeneous Systems of Linear Equations

Solution (*continued*)

According to Theorem 2.1, this system has a nontrivial solution if and only if the coefficient matrix is singular:

$$\begin{vmatrix} 5 - \lambda & 2 & 1 \\ 2 & 1 - \lambda & 0 \\ 1 & 0 & 1 - \lambda \end{vmatrix} = 0$$

The value of the determinant is found to be $\lambda(1 - \lambda)(\lambda - 6)$. Hence, the system has nontrivial solutions if and only if $\lambda = 0, 1$ or 6 .

Note Actually, using terminology to be explained in Chapter 3, this

example asks us to find the eigenvalues of the matrix $\begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

- A vector can be thought of as a special kind of matrix, with n rows and just one column.
- Thus we can write the vector $\mathbf{v}$ as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- We refer to this as a *column vector* and its $1 \times n$ transpose

$$\mathbf{v}' = [v_1 \quad v_2 \quad \cdots \quad v_n]$$

as a *row vector*.

- The rules for adding, subtracting, and multiplying vectors then follow from those defined in chapter 1 for matrices in general.
- Thus addition (subtraction) is defined for two vectors with the same dimension,

$$\mathbf{w} \pm \mathbf{v} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \pm \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \pm v_1 \\ w_2 \pm v_2 \\ \vdots \\ w_n \pm v_n \end{bmatrix}.$$

Vectors

The Inner Product

- The product of two vectors is now referred to as the *inner product*, denoted $\mathbf{w} \cdot \mathbf{v}$, and is defined for two vectors of the same dimension

$$\mathbf{w} \cdot \mathbf{v} \equiv \mathbf{w}'\mathbf{v} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$
$$= \sum_{i=1}^n w_i v_i$$

- Note that $\mathbf{w}'\mathbf{v} = \mathbf{v}'\mathbf{w}$.

Geometric Interpretation of Vectors

- Vectors, in contrast to general matrices, are easily interpreted geometrically.
- Define a (real) vector as an array of (real) numbers and define the operations of addition, subtraction, and multiplication as above.
- The link with geometry is established by noting that we can associate with any two-component vector, $\mathbf{v}$, a point in $\mathbb{R}^2$, with v_1 for the x -coordinate and v_2 the y -coordinate, as Figure 1 illustrates for

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Geometric Interpretation of Vectors

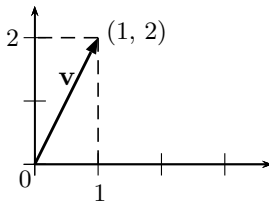


Figure 1: Vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- The vector is pictured as an arrow from the origin to a point with coordinates given by the two elements of $\mathbf{v}$.
- The first element gives the coordinate on the horizontal axis and the second gives the coordinate on the vertical axis.

Geometric Interpretation of Vectors

The length of a vector

- Using Pythagoras's theorem, we can define the length of this vector as the square root of the sum of squares of the coordinates.
- So the length of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\|\mathbf{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$.
- In general, for an n -dimensional vector, we have

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- The length of a vector is also often called its *Euclidean norm*.
- Note that the length of the vector $\mathbf{v}$ may be defined equivalently as the square root of the inner product of $\mathbf{v}$ with itself:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}'\mathbf{v}}.$$

Geometric Interpretation of Vectors

Example

$$\text{If } \mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix},$$

- a) compute $\mathbf{a} \cdot \mathbf{b}$;
- b) find the lengths of $\mathbf{a}$ and $\mathbf{b}$.

Geometric Interpretation of Vectors

Example

$$\text{If } \mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix},$$

- a) compute $\mathbf{a} \cdot \mathbf{b}$;
- b) find the lengths of $\mathbf{a}$ and $\mathbf{b}$.

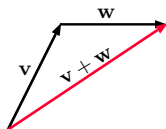
Solution

- a) We get $\mathbf{a} \cdot \mathbf{b} = 1 \cdot (-3) + (-2) \cdot 2 + 3 \cdot 5 = 8$.
- b) For $\mathbf{a}$ we have $\|\mathbf{a}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$.
For $\mathbf{b}$ we have $\|\mathbf{b}\| = \sqrt{(-3)^2 + 2^2 + 5^2} = \sqrt{38}$.

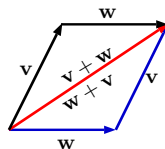
Geometric Interpretation of Vectors

Vector Addition

- We now look at the graphical interpretation of the operation of *vector addition*.
- If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then the sum $\mathbf{v} + \mathbf{w}$ is the vector determined as follows: Position the vector $\mathbf{w}$ so that its initial point coincides with the terminal point of $\mathbf{v}$. The vector is represented by the arrow from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$ (Figure 2a).



(a) The sum $\mathbf{v} + \mathbf{w}$



(b) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

Figure 2

Geometric Interpretation of Vectors

Vector Addition

- In Figure 2b we have constructed two sums, $\mathbf{v} + \mathbf{w}$ and $\mathbf{w} + \mathbf{v}$. It is evident that

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

and that the sum coincides with the diagonal of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ when these vectors are positioned so that they have the same initial point.

- The next example shows this numerically.

Geometric Interpretation of Vectors

Vector Addition

Example

If we add the two vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, we obtain

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Graphically, this is shown in Figure 3.

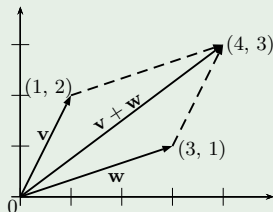


Figure 3

Geometric Interpretation of Vectors

Scalar Multiplication

- If $\mathbf{v}$ is a nonzero vector and k is a nonzero real number (scalar), then the product $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if $k > 0$ and opposite to that of $\mathbf{v}$ if $k < 0$.
- Figure 4 illustrates the relation between a vector $\mathbf{v}$ and the vectors $\frac{1}{2}\mathbf{v}$, $(-1)\mathbf{v}$, $2\mathbf{v}$, and $(-3)\mathbf{v}$.

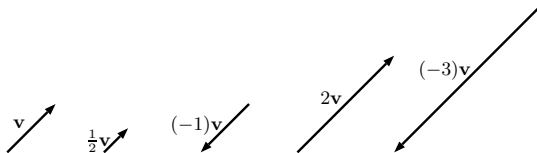


Figure 4

Geometric Interpretation of Vectors

Scalar Multiplication

- Note that the vector $(-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but is oppositely directed.
- A vector of the form $k\mathbf{v}$ is called a *scalar multiple* of $\mathbf{v}$.
- As it can be seen from Figure 4, vectors that are scalar multiples of each other are parallel.

Vector Orthogonality

- When the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$ is 90° , they are said to be *orthogonal*, and we write $\mathbf{v} \perp \mathbf{w}$.
- For two orthogonal vectors we state the following without proof.

Two vectors, $\mathbf{v}$ and $\mathbf{w}$, are *orthogonal* if and only if their inner product is 0. In symbols:

$$\mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$$

Vector Orthogonality

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Two vectors, $\mathbf{v}$ and $\mathbf{w}$, are *orthogonal* if and only if their inner product is 0. In symbols:

$$\mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$$

Example

a) The vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are orthogonal since
 $\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-2) + 2 \cdot 1 = 0$.

b) The vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ are not orthogonal since
 $\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-1) + (-1) \cdot 1 + (1) \cdot (-1) = -3 \neq 0$.

Linear Independence

- A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is said to be *linearly dependent* iff any one of them can be expressed as a linear combination of the remaining vectors; otherwise they are *linearly independent*.

Example

The three vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ are linearly dependent because $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$3\mathbf{v}_1 - 2\mathbf{v}_2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \mathbf{v}_3$$

Note that this last equation is alternatively expressible as

$$3\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$$

where $\mathbf{0}$ is a null vector (also called the *zero vector*).

Linear Independence

- In $\mathbb{R}^2$, the two vectors $\mathbf{v}$ and $\mathbf{w}$ are *linearly dependent* iff one of the vectors, say $\mathbf{v}$, is a scalar multiple of the other, so that $\mathbf{v} = \lambda\mathbf{w}$. Geometrically, their arrows lie on a single straight line (see Figure 5).
- If, however, it is impossible to express one as a multiple of the other, then the two vectors $\mathbf{v}$ and $\mathbf{w}$ are *linearly independent*. Geometrically, their arrows do not lie on a single straight line (see Figure 6).
- For a set of n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, linear dependence requires that at least one of the vectors can be written as a *linear combination* of the others; for instance,

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n \quad (6)$$

where $c_2, \dots, c_n$ are constants.

Linear Independence

- If, on the other hand, none of them can be written as a linear combination of the others, then the vectors are said to be *linearly independent*.
- Formally, we have the following definition.

Definition 2.1

The n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in $\mathbb{R}^m$ are ***linearly dependent*** if there exists numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, not all zero, such that

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \dots + \lambda_n \mathbf{v}_n = \mathbf{0}$$

If this equation holds only when $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$, then the vectors are ***linearly independent***.

Linear Independence

- The formal definitions of linear dependence and independence given above may seem rather odd. However, we can easily show that it is similar to our earlier statements about linear dependence and independence.
- First, suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent. Then the equation $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \dots + \lambda_n \mathbf{v}_n = \mathbf{0}$ holds with at least *one* of the coefficients λ_i different from 0. After reordering the vectors $\mathbf{v}_i$ and the corresponding scalars λ_i , if necessary, we can assume that $\lambda_1 \neq 0$. Solving the equation for $\mathbf{v}_1$ yields

$$\mathbf{v}_1 = -(\lambda_2/\lambda_1)\mathbf{v}_2 - \dots - (\lambda_n/\lambda_1)\mathbf{v}_n$$

This is similar to (5) where $\mathbf{v}_1$ is a linear combination of the other vectors.

Linear Independence

- Using Definition 2.1, two vectors in $\mathbb{R}^2$, $\mathbf{v}$ and $\mathbf{w}$, are *linearly independent* if the scalars λ_1 and λ_2 satisfying

$$\lambda_1 \mathbf{v} + \lambda_2 \mathbf{w} = \mathbf{0}$$

are zero.

- If the λ_i s are nonzero, then the pair of vectors $\mathbf{v}$ and $\mathbf{w}$ would be proportional and they would be linearly dependent.
- To see this, note that a nontrivial solution to the equation $\lambda_1 \mathbf{v} + \lambda_2 \mathbf{w} = \mathbf{0}$ would imply that

$$\mathbf{v} = -\frac{\lambda_2}{\lambda_1} \mathbf{w}$$

which states that $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$.

Linear Independence

Example

- a) Prove that $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ are linearly dependent. Illustrate.
- b) Prove that $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly independent. Illustrate.

Linear Independence

Example

- a) Prove that $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ are linearly dependent. Illustrate.
- b) Prove that $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly independent. Illustrate.

Solution

- a) Here $\mathbf{w} = 2\mathbf{v}$, so $2\mathbf{v} - \mathbf{w} = \mathbf{0}$. Choosing $\lambda_1 = 2$ and $\lambda_2 = -1$ yields $\lambda_1\mathbf{v} + \lambda_2\mathbf{w} = \mathbf{0}$, so according to Definition 2.1, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent. The vector $\mathbf{v}$ points in the same direction as $\mathbf{w}$, and is twice as long. See Figure 5 on the next slide.

Linear Independence

Solution (*continued*)

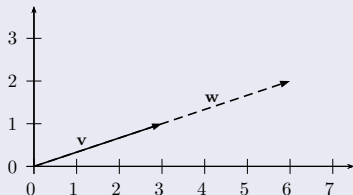


Figure 5 $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent

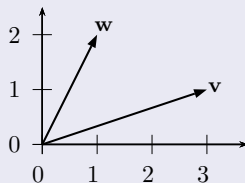


Figure 6 $\mathbf{v}$ and $\mathbf{w}$ are linearly independent

b) In this case the equation $\lambda_1 \mathbf{v} + \lambda_2 \mathbf{w} = \mathbf{0}$ reduces to

$$3\lambda_1 + \lambda_2 = 0$$

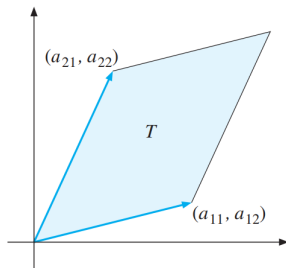
$$\lambda_1 + 2\lambda_2 = 0$$

The only solution is $\lambda_1 = \lambda_2 = 0$, so $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. See Figure 6.

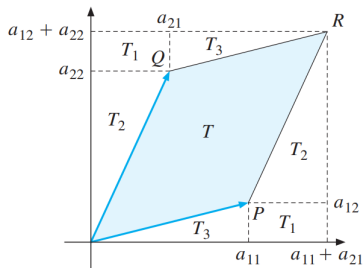
Digression: A Geometric Interpretation of Determinants

- Determinants of order 2 have a nice geometric interpretation.
- If the two rows of the matrix are represented as the vectors shown in Figure 7(a), then its determinant equals the shaded area of the parallelogram.
- If we interchange the two rows, however, the determinant becomes a negative number equal to minus this shaded area.
- Figure 7(b) illustrates why the result claimed in Figure 7(a) is true. We want to find area T .
- Note that $2T_1 + 2T_2 + 2T_3 + T = (a_{11} + a_{21})(a_{12} + a_{22})$, where $T_1 = a_{12}a_{21}$, $T_2 = \frac{1}{2}a_{21}a_{22}$, and $T_3 = \frac{1}{2}a_{11}a_{12}$.
- Hence, $T = a_{11}a_{22} - a_{21}a_{12}$, by elementary algebra.

Digression: A Geometric Interpretation of Determinants



(a)



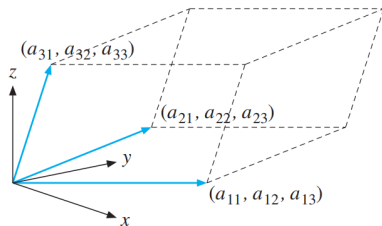
(b)

Figure 7

Digression: A Geometric Interpretation of Determinants

- Like determinants of order 2, those of order 3 also have a geometric interpretation which is shown and explained in Figure 8.
- The rows of the determinant correspond to three different 3-vectors represented in the diagram.
- These vectors determine a box which is not rectangular with right-angles at each of its six corners, but a distorted "parallelepiped" which has six faces that are all parallelograms whose opposite edges are parallel.

Digression: A Geometric Interpretation of Determinants



the volume of the
“box” spanned by
the three vectors $= \pm \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Figure 8

Linear Independence and Systems of Linear Equations

- Any system of linear equations can be written as a vector equation.
- For instance, the system

$$2x_1 + 2x_2 - x_3 = -3$$

$$4x_1 \quad \quad + 2x_3 = 8$$

$$6x_2 - 3x_3 = -12$$

can be written as the vector equation

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = \mathbf{b} \tag{7}$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ -12 \end{bmatrix}$$

Linear Independence and Systems of Linear Equations

- In (6) we say that $\mathbf{b}$ is expressed as a linear combination of the column vectors of the coefficient matrix A .
- Solving system (6) we get $x_1 = 1/2$, $x_2 = -1/2$, and $x_3 = 3$. Thus

$$\mathbf{b} = (1/2)\mathbf{v}_1 + (-1/2)\mathbf{v}_2 + 3\mathbf{v}_3.$$

- In this case, we say that $\mathbf{b}$ is *linearly dependent* on the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Linear Independence and Systems of Linear Equations

- Consider the general system of m equations in n unknowns, written both in its usual form and also as a vector equation:

[illegible]

- Here $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the column vectors of the coefficients, and $\mathbf{b}$ is the column vector with components $b_1, \dots, b_m$.

Linear Independence and Systems of Linear Equations

- Suppose that (7) has two solutions $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$. Then

$$u_1 \mathbf{v}_1 + \dots + u_n \mathbf{v}_n = \mathbf{b} \quad \text{and} \quad v_1 \mathbf{v}_1 + \dots + v_n \mathbf{v}_n = \mathbf{b}$$

Subtracting the second equation from the first yields

$$(u_1 - v_1) \mathbf{v}_1 + \dots + (u_n - v_n) \mathbf{v}_n = \mathbf{0}$$

Let $\lambda_1 = u_1 - v_1, \dots, \lambda_n = u_n - v_n$. The two solutions are different if and only if $\lambda_1, \dots, \lambda_n$ are not all equal to 0. But this is exactly the definition of linear dependence (see Definition 2.1).

Linear Independence and Systems of Linear Equations

- Thus we conclude that if system (7) has more than one solution, then the column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent.
- Equivalently: *If the column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent, then system (7) has at most one solution.*
- Without saying more about the right hand side vector $\mathbf{b}$, however, we cannot know if there are any solutions at all, in general.
- But if the number of equations and the number of unknowns of the system are equal (hence the coefficient matrix is square), then the system has a unique solution iff the coefficient matrix has linearly independent columns (or rows).
- This is equivalent to saying that a system of n linear equations with n unknowns has a unique solution iff the determinant of the coefficient matrix is non-zero (hence nonsingular). The next theorem proves this result.

Linear Independence and Systems of Linear Equations

Theorem 2.2

The n column vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ of the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \text{where} \quad \mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

are linearly independent iff $|A| \neq 0$.

Theorem 2.2

The n column vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ of the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \text{where} \quad \mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

are linearly independent iff $|A| \neq 0$.

Proof: The vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are linearly independent iff the vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ has only the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. This vector equation is equivalent to a homogeneous system of equations, and according to Theorem 2.1, it has only the trivial solution iff $|A| \neq 0$.

Rank of a Matrix

- Associated with any matrix is an important number called its *rank*.
- An $m \times n$ matrix A has n column vectors, each with m components.
- The largest number of column vectors in A that form a linearly independent set is called the rank of A .
- We state without proof that for the $m \times n$ matrix A , the maximum number of linearly independent rows is equal to the number of linearly independent columns.

Definition 2.2

For any matrix A , the maximum number of linearly independent columns equals the number of linearly independent rows. This number is known as the *rank* of the matrix A and is written $\text{rank}(A)$.

An Efficient Way to find the Rank of a Matrix

- One method for finding the rank of a matrix A involves transforming A into its *row-echelon form* by using elementary row operations, which is discussed earlier in this chapter.
- The rank of A is simply the number of nonzero rows present in the row-echelon form of A .

Example

Find the rank of the matrix

$$A = \begin{bmatrix} 0 & -11 & -4 \\ 2 & 6 & 2 \\ 4 & 1 & 0 \end{bmatrix}$$

An Efficient Way to find the Rank of a Matrix

Solution

(1) $R_1 \leftrightarrow R_3$	$\begin{bmatrix} 4 & 1 & 0 \\ 2 & 6 & 2 \\ \textcolor{red}{0} & -11 & -4 \end{bmatrix}$	Interchange R_1 and R_3 to get 0 in the a_{31} position
(2) $R_1 \rightarrow \frac{1}{4}R_1$	$\begin{bmatrix} \textcolor{red}{1} & \frac{1}{4} & 0 \\ 2 & 6 & 2 \\ 0 & -11 & -4 \end{bmatrix}$	Multiply R_1 by $\frac{1}{4}$ to get 1 in the a_{11} position
(3) $R_2 \rightarrow -2R_1 + R_2$	$\begin{bmatrix} 1 & \frac{1}{4} & 0 \\ \textcolor{red}{0} & \frac{11}{2} & 2 \\ 0 & -11 & -4 \end{bmatrix}$	Multiply R_1 by -2 and add the result to R_2 to get 0 in the a_{21} position

An Efficient Way to find the Rank of a Matrix

Solution (*continued*)

$$(4) \quad R_2 \rightarrow \frac{2}{11}R_2 \quad \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \textcolor{red}{1} & \frac{4}{11} \\ 0 & -11 & -4 \end{bmatrix} \quad \begin{array}{l} \text{Multiply } R_2 \text{ by } \frac{2}{11} \text{ to get } 1 \\ \text{in the } a_{22} \text{ position} \end{array}$$

$$(5) \quad R_3 \rightarrow 11R_2 + R_3 \quad \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{4}{11} \\ 0 & \textcolor{red}{0} & 0 \end{bmatrix} \quad \begin{array}{l} \text{Multiply } R_2 \text{ by } 11 \text{ and add} \\ \text{the result to } R_3 \text{ to get } 0 \\ \text{in the } a_{32} \text{ position} \end{array}$$

The matrix in step 5 is the row-echelon form of A . Since this matrix contains two nonzero rows, $\text{rank}(A) = 2$.

Rank of a Matrix and nonsingularity

- By definition, for an $n \times n$ matrix A to be nonsingular, it must have n linearly independent rows (or columns); consequently, it must have rank n , and its row-echelon form must contain exactly n nonzero rows.
- Thus, in the case where A is a square matrix of order n , if the rank of A is n , then the matrix is nonsingular.
- In the last example, the matrix A is 3×3 , but $\text{rank}(A) = 2$; hence, A is singular (check this by computing the determinant).

Rank of a Matrix

- From our discussion so far, we have the following result.

If A is an $n \times n$ matrix, then the following statements are equivalent.

- (a) A is invertible.
- (b) $|A| \neq 0$
- (c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- (d) The column vectors of A are linearly independent.
- (e) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
- (f) $\text{rank}(A) = n$

Rank of a Matrix and Linear Systems

- Consider the linear system of m simultaneous equations in n unknowns $A\mathbf{x} = \mathbf{b}$, where A is the $m \times n$ coefficient matrix.
- Let $[A \mid \mathbf{b}]$ be the augmented matrix of the system. Then we have the following theorem on linear systems.

Theorem 2.3

A necessary and sufficient condition for a linear system of equations to be consistent (that is, to have at least one solution) is that the rank of the coefficient matrix is equal to the rank of the augmented matrix. Briefly:

$$A\mathbf{x} = \mathbf{b} \quad \text{has a solution} \quad \Longleftrightarrow \quad \text{rank}(A) = \text{rank}([A \mid \mathbf{b}])$$

Rank of a Matrix and Linear Systems

Example

The augmented matrix for the system

$$\begin{array}{rcl} x_1 + x_2 + 2x_3 = 5 \\ -3x_1 + x_2 - 2x_3 = 10 \\ 2x_1 + x_2 + 3x_3 = 4 \end{array} \quad \text{is} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ -3 & 1 & -2 & 10 \\ 2 & 1 & 3 & 4 \end{array} \right]$$

which has the following row-echelon form (verify):

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & \frac{25}{4} \\ 0 & 0 & 0 & 1 \end{array} \right]$$

We see from the third row in this matrix that the system is inconsistent. It is also because of this row that the row-echelon form of the augmented

Rank of a Matrix and Linear Systems

Example (*continued*)

matrix has fewer zero rows than the row-echelon form of the coefficient matrix. This forces the coefficient matrix and the augmented matrix for the system to have different ranks.

Rank of a Matrix and Linear Systems

Example (*continued*)

matrix has fewer zero rows than the row-echelon form of the coefficient matrix. This forces the coefficient matrix and the augmented matrix for the system to have different ranks.

Example

What conditions must b_1 , b_2 , and b_3 satisfy in order for the system of equations

$$x_1 + x_2 + 2x_3 = b_1$$

$$x_1 + x_3 = b_2$$

$$2x_1 + x_2 + 3x_3 = b_3$$

to be consistent?

Rank of a Matrix and Linear Systems

Solution

The augmented matrix of the system is

$$\left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array} \right]$$

which can be reduced to row-echelon form as follows:

$$(1) \quad \begin{array}{l} R_2 \rightarrow -1R_1 + R_2 \\ R_3 \rightarrow -2R_1 + R_3 \end{array} \quad \left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array} \right] \quad \begin{array}{l} \text{A zero is obtained} \\ \text{in the } a_{21} \text{ and } a_{31} \\ \text{positions.} \end{array}$$

$$(2) \quad R_2 \rightarrow -1R_2 \quad \left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array} \right] \quad \begin{array}{l} \text{A one is obtained} \\ \text{in the } a_{22} \text{ position.} \end{array}$$

Rank of a Matrix and Linear Systems

Solution (*continued*)

$$(3) \ R_3 \rightarrow R_2 + R_3 \quad \left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right] \quad \begin{array}{l} \text{A zero is obtained} \\ \text{in the } a_{32} \text{ position.} \end{array}$$

It is now evident from the third row in the matrix that the system has a solution if and only if b_1 , b_2 , and b_3 satisfy the condition

$$b_3 - b_2 - b_1 = 0 \quad \text{or} \quad b_3 = b_1 + b_2$$

This is also the condition that needs to be satisfied for the coefficient matrix and the augmented matrix to have the same rank.

Economic Applications

Two-Commodity Market Model

- Let us consider a simple model in which only two commodities are related to each other.
- For simplicity, the demand and supply functions of both commodities are assumed to be linear. In parametric terms, such model can be written as

$$Q_{d1} - Q_{s1} = 0$$

$$Q_{d1} = a_0 + a_1P_1 + a_2P_2$$

$$Q_{s1} = b_0 + b_1P_1 + b_2P_2$$

$$Q_{d2} - Q_{s2} = 0$$

$$Q_{d2} = \alpha_0 + \alpha_1P_1 + \alpha_2P_2$$

$$Q_{s2} = \beta_0 + \beta_1P_1 + \beta_2P_2$$

Economic Applications

Two-Commodity Market Model

- By substituting the second and third equations into the first (for the first commodity) and the fifth and sixth equations into the fourth (for the second commodity), the model is reduced to two equations in two variables:

$$\begin{aligned}(a_0 - b_0) + (a_1 - b_1)P_1 + (a_2 - b_2)P_2 &= 0 \\ (\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)P_1 + (\alpha_2 - \beta_2)P_2 &= 0\end{aligned}$$

- Let us define the shorthand symbols

$$\begin{aligned}c_i &\equiv a_i - b_i \\ \gamma_i &\equiv \alpha_i - \beta_i\end{aligned}\quad (i = 0, 1, 2)$$

Economic Applications

Two-Commodity Market Model

- Then, after transposing the c_0 and γ_0 terms to the right-hand side, we get

$$c_1P_1 + c_2P_2 = -c_0$$

$$\gamma_1P_1 + \gamma_2P_2 = -\gamma_0$$

- We use Cramer's rule to solve this system. The three determinants needed— $|A|$, $|A_1|$, and $|A_2|$ —have the following values:

$$|A| = \begin{vmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} = c_1\gamma_2 - c_2\gamma_1$$

$$|A_1| = \begin{vmatrix} -c_0 & c_2 \\ -\gamma_0 & \gamma_2 \end{vmatrix} = -c_0\gamma_2 + c_2\gamma_0$$

$$|A_2| = \begin{vmatrix} c_1 & -c_0 \\ \gamma_1 & -\gamma_0 \end{vmatrix} = -c_1\gamma_0 + c_0\gamma_1$$

Economic Applications

Two-Commodity Market Model

- Therefore, the equilibrium prices must be

$$P_1^* = \frac{|A_1|}{|A|} = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1}$$

$$P_2^* = \frac{|A_2|}{|A|} = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}$$

- The equilibrium quantities can be found by setting $P_1 = P_1^*$ and $P_2 = P_2^*$ in the demand or supply functions.

Economic Applications

IS-LM Model: Closed Economy

- Let us look at the following IS-LM model. The goods market involves the following equations:

$$Y = C + I + G$$

$$C = a + b(1 - t)Y$$

$$I = d - ei$$

$$G = G_0$$

The endogenous variables are Y , C , I , and i (where i is the rate of interest). The exogenous variable is G_0 , while a , d , e , b , and t are structural parameters.

- The money market is described by

$$\text{Equilibrium condition:} \quad M_d = M_s$$

$$\text{Money demand:} \quad M_d = kY - li$$

$$\text{Money supply:} \quad M_s = M_0$$

where M_0 is the exogenous stock of money and k and l are parameters.

- These three equations can be condensed into:

$$M_0 = kY - li$$

- Together, the two sectors give us the following system of equations:

$$\begin{aligned}Y - C - I &= G_0 \\ b(1 - t)Y - C &= -a \\ I + ei &= d \\ kY - li &= M_0\end{aligned}$$

In matrix form, we have

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ b(1-t) & -1 & 0 & 0 \\ 0 & 0 & 1 & e \\ k & 0 & 0 & -l \end{bmatrix} \begin{bmatrix} Y \\ C \\ I \\ i \end{bmatrix} = \begin{bmatrix} G_0 \\ -a \\ d \\ M_0 \end{bmatrix}$$

Economic Applications

IS-LM Model: Closed Economy

- To find the determinant of the coefficient matrix, we can use Laplace expansion on one of the columns (preferably one with the most zeros). Expanding the fourth column, we find

$$\begin{aligned}|A| &= (-e) \begin{vmatrix} 1 & -1 & -1 \\ b(1-t) & -1 & 0 \\ k & 0 & 0 \end{vmatrix} - l \begin{vmatrix} 1 & -1 & -1 \\ b(1-t) & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\&= (-e)(k) \begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix} - l \begin{vmatrix} 1 & -1 \\ b(1-t) & -1 \end{vmatrix} \\&= ek - l[(-1) - (-1)b(1-t)] \\&= ek + l[1 - b(1-t)]\end{aligned}$$

Economic Applications

IS-LM Model: Closed Economy

- We can use Cramer's rule to find the equilibrium income Y^* .

$$Y^* = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} G_0 & -1 & -1 & 0 \\ -a & -1 & 0 & 0 \\ d & 0 & 1 & e \\ M_0 & 0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1 - t)]}$$

Using Laplace expansion on the second column of the numerator produces

$$Y^* = \frac{(-1)(-1)^3 \begin{vmatrix} -a & 0 & 0 \\ d & l & e \\ M_0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1 - t)]} + \frac{(-1)(-1)^4 \begin{vmatrix} G_0 & -1 & 0 \\ d & l & e \\ M_0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1 - t)]}$$

Economic Applications

IS-LM Model: Closed Economy

$$= \frac{\begin{vmatrix} -a & 0 & 0 \\ d & l & e \\ M_0 & 0 & -l \end{vmatrix} - \begin{vmatrix} G_0 & -1 & 0 \\ d & l & e \\ M_0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1 - t)]}$$

By further expansion, we obtain

$$\begin{aligned} Y^* &= \frac{(1) \begin{vmatrix} -a & 0 \\ M_0 & -l \end{vmatrix} - \left\{ (-1)(-1)^3 \begin{vmatrix} d & e \\ M_0 & -l \end{vmatrix} + (-1)^4 \begin{vmatrix} G_0 & 0 \\ M_0 & -l \end{vmatrix} \right\}}{ek + l[1 - b(1 - t)]} \\ &= \frac{al - [d(-l) - eM_0] - G_0(-l)}{ek + l[1 - b(1 - t)]} \\ &= \frac{l(a + d + G_0) + eM_0}{ek + l[1 - b(1 - t)]} \end{aligned}$$

Economic Applications

IS-LM Model: Closed Economy

- Since the solution to Y^* is linear with respect to the exogenous variables, we can rewrite Y^* as

$$Y^* = \left(\frac{e}{ek + l[1 - b(1 - t)]} \right) M_0 + \left(\frac{l}{ek + l[1 - b(1 - t)]} \right) (a + d + G_0)$$

- In this form, we can see that the Keynesian policy multipliers with respect to the money supply and government expenditure are the coefficients of M_0 and G_0 , that is,

Money-supply multiplier: $\frac{e}{ek + l[1 - b(1 - t)]}$

and

Government-expenditure multiplier: $\frac{l}{ek + l[1 - b(1 - t)]}$

***** End of Chapter Two *****