Econ 2012: Linear Algebra for Economists
Chapter-1: Matrix Algebra

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July 27, 2021

Overview

1. Introduction
2. Matrices and Matrix Operations
   - Addition and Subtraction of Matrices
   - Scalar Multiplication
   - Multiplication of Matrices
3. Powers and Trace of a Square Matrix
4. The Transpose
5. Special Matrices
6. Determinants and Inverse Matrices
   - Determinants
   - The Inverse of a Matrix

Introduction

- Linear Algebra has many useful applications in economics.
- Most importantly we can use the tools of linear algebra to analyze and solve systems of linear equations:
- The methods of linear algebra are also used extensively in the theory of differential and difference equations, in linear and nonlinear optimization theory, in statistics and econometrics etc.

Matrices

- A *matrix* is simply a rectangular array of numbers enclosed in parenthesis. It is conventionally denoted by a capital letter.
- When there are m *rows* and n *columns* in the array, we have an m-by-n matrix (written as $m \times n$).
- In general, an $m \times n$ matrix is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

Matrices

- The matrix A in (1) is said to have *order* $m \times n$.
- The mn numbers that constitute A are called its *elements* or *entries*.
- In particular, a_{ij} denotes the element in the ith row and the jth column.
- For brevity, the $m \times n$ matrix in (1) is often expressed as $(a_{ij})_{m \times n}$, or more simply as (a_{ij}), if the order $m \times n$ is either obvious or unimportant.

Matrix Operations

- As a preliminary, let us first define the word *equality*

Definition 1.1

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if and only if they have the same dimension and have identical elements in the corresponding locations in the array. In other words, $A = B$ if and only if $a_{ij} = b_{ij}$ for all values of i and j.

Example

When is $\begin{bmatrix} 3 & t-1 \\ 2t & u \end{bmatrix} = \begin{bmatrix} t & 2v \\ u+1 & t+w \end{bmatrix}$?

Solution

Both are 2×2 matrices. Since both have four elements, equality requires the four equations $3 = t, t-1 = 2v, 2t = u+1$ and $u = t+w$ to be satisfied. By solving these simultaneous equations, it follows that the two matrices are equal if and only if $t = 3, v = 1, u = 5$, and $w = 2$. Then both matrices are equal to $\begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix}$.

Addition and Subtraction of Matrices

- For two matrices A and B, the addition/subtraction operation is defined if and only if A and B have the same dimension.
- In that case, the addition of $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined as the addition of each pair of corresponding elements. The same is true for subtraction.

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{bmatrix}$$

- In general, we may state the rule thus:

$$[a_{ij}] \pm [b_{ij}] = [c_{ij}] \quad \text{where } c_{ij} = a_{ij} \pm b_{ij}$$

Scalar Multiplication

- To multiply a matrix by a number (scalar) is to multiply every element of that matrix by the given scalar.
- If α is a real number, we define αA by

$$\alpha A = \alpha \left(a_{ij}\right) = \left(\alpha a_{ij}\right).$$

Example

Compute $A + B, 3A$, and $\left(-\frac{1}{2}\right) B$, if

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -3 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}.$$

Solution

$$A + B = \begin{bmatrix} 1 & 3 & 2 \\ 5 & -3 & 1 \end{bmatrix}, \quad 3A = \begin{bmatrix} 3 & 6 & 0 \\ 12 & -9 & -3 \end{bmatrix},$$

$$\left(-\frac{1}{2}\right) B = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & -1 \end{bmatrix}.$$

Rules

Rules for Matrix Addition and Multiplication by Scalars

Let A, B, and C be arbitrary $m \times n$ matrices, and let α and β be real numbers. Also, let $\mathbf{0}$ denote the $m \times n$ matrix consisting only of zeros, called the *zero matrix*. Then:

(a) $(A + B) + C = A + (B + C)$

(b) $A + B = B + A$

(c) $A + \mathbf{0} = A$

(d) $A + (-A) = \mathbf{0}$

(e) $(\alpha + \beta)A = \alpha A + \beta A$

(f) $\alpha(A + B) = \alpha A + \alpha B$

Multiplication of Matrices

- Suppose that, given two matrices A and B, we want to find the product AB. The conformability condition for multiplication is that the column dimension of A must be equal to the row dimension of B.

Definition 1.2

Suppose that $A = (a_{ij})_{m\times n}$ and that $B = (b_{ij})_{n\times p}$. Then the product $C = AB$ is the $m \times p$ matrix C , whose element in the ith row and the jth column is the inner product

$$c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ik}b_{kj} + ... + a_{in}b_{nj}$$

of the ith row of A and the jth column of B.

- Note that to get c_{ij} we multiply each component a_{ir} in the ith row of A by the corresponding component b_{rj} in the jth column of B, then add all the products.

Multiplication of Matrices

- One way of visualizing matrix multiplication is this:

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

Multiplication of Matrices

Example

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$. Compute the matrix product AB. Is the product BA defined?

Solution

A is 3×3 and B is 3×2 matrix:

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \cdot 3 + 1 \cdot 1 + 2 \cdot (-1) & 0 \cdot 2 + 1 \cdot 0 + 2 \cdot 1 \\ 2 \cdot 3 + 3 \cdot 1 + 1 \cdot (-1) & 2 \cdot 2 + 3 \cdot 0 + 1 \cdot 1 \\ 4 \cdot 3 + (-1) \cdot 1 + 6 \cdot (-1) & 4 \cdot 2 + (-1) \cdot 0 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 8 & 5 \\ 5 & 14 \end{bmatrix}.$$

The matrix product BA is not defined because the number of columns in $B(= 2)$ is not equal to the number of rows in $A(= 3)$.

Multiplication of Matrices

Rules for Matrix Multiplication

If A, B, and C are matrices whose dimensions are such that the given operations are defined, and α is an arbitrary scalar, then:

$$(AB)\,C = A\,(BC) \qquad \text{(associative law)}$$
$$A\,(B+C) = AB + AC \qquad \text{(left distributive law)}$$
$$(A+B)\,C = AC + BC \qquad \text{(right distributive law)}$$
$$(\alpha A)\,B = A\,(\alpha B) = \alpha\,(AB)$$

- Note that both the left and right distributive laws are stated here because, unlike for numbers, matrix multiplication is not commutative, and so $A\,(B+C) \neq (B+C)\,A$ in general.

Powers of Matrices

- If A is a square matrix, the associative law allows us to write AA as A^2, and AAA as A^3, and so on.
- In general,

$$A^n = AA \cdots A \qquad (A \text{ is repeated } n \text{ times})$$

Example

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Compute A^2, A^3, and A^4. Then guess the general form A^n.

Powers of Matrices

Solution

We find that $A^2 = AA = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, $A^3 = AAA = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$, $A^4 = A^3A = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$.

A reasonable guess, therefore, is that for all natural numbers n,

$$A^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$$

The Trace of a Matrix

- The trace of a square matrix A is given by the sum of the elements of the main diagonal.
- In other words, if A is $n \times n$, then the trace is defined as

$$\text{trace}(A_n) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

- The trace is defined only for square matrices.

Theorem 1.1

For two matrices $\underset{(p\times q)}{A}$ and $\underset{(q\times p)}{B}$, we have

$$\text{trace}(AB) = \text{trace}(BA)$$

The Trace of a Matrix

Proof.

Let $c_{ii} = (AB)_{ii}$. Then $c_{ii} = \sum_{j=1}^{q} a_{ij} b_{ji}$ for $i = 1, \ldots, p$ and

$$\text{trace}(AB) = \sum_{i=1}^{p} c_{ii} = \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} b_{ji}.$$

On the other hand, let $d_{jj} = (BA)_{jj}$. Then $d_{jj} = \sum_{i=1}^{p} b_{ji} a_{ij}$ for $j = 1, \ldots, q$ and

$$\text{trace}(BA) = \sum_{j=1}^{q} d_{jj} = \sum_{j=1}^{q} \sum_{i=1}^{p} b_{ji} a_{ij}.$$

Thus we have

$$\text{trace}(AB) = \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} b_{ji} = \sum_{j=1}^{q} \sum_{i=1}^{p} b_{ji} a_{ij} = \text{trace}(BA),$$

as desired. □

The Trace of a Matrix

Example

1. Find the traces of the following matrices

(a) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(b) $B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$

2. For the matrices A and B below, verify that $\text{trace}(AB) = \text{trace}(BA)$:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$$

The Transpose

- Consider any $m \times n$ matrix A. The transpose of A, denoted by $A^{'}$, is defined as the $n \times m$ matrix whose first column is the first row of A, whose second column is the second row of A, and so on. Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Longrightarrow A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- So we can write $A' = (a'_{ij})$, where $a'_{ij} = a_{ji}$. The subscripts i and j have to be interchanged because the jth row of A becomes the jth column of A', whereas the ith column of A becomes the ith row of A'.

The Transpose

Example

Let $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \\ 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \end{bmatrix}$. Find A' and B'.

Solution

$$A' = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}.$$

Rules for Transposition

- The following rules apply to matrix transposition:

Rules for Transposition

(a) $(A')' = A$

(b) $(A + B)' = A' + B'$

(c) $(\alpha A)' = \alpha A'$

(d) $(AB)' = B'A'$

- We will provide a proof for (b) and (d). (a) and (c) are left as exercises.

Rules for Transposition

Proof.

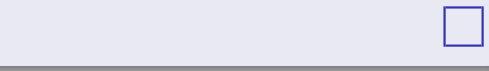
(b)

$$\begin{aligned}
[(A+B)']_{ij} &= [A+B]_{ji} && \text{Definition of transpose} \\
&= [A]_{ji} + [B]_{ji} && \text{Definition of matrix summation} \\
&= a_{ji} + b_{ji}
\end{aligned}$$

and

$$\begin{aligned}
[A'+B']_{ij} &= [A']_{ij} + [B']_{ij} && \text{Defintion of matrix summation} \\
&= [A]_{ji} + [B]_{ji} && \text{Matrix transposition} \\
&= a_{ji} + b_{ji}
\end{aligned}$$

We've shown that $[(A+B)']_{ij} = [A'+B']_{ij}$ for $i = 1, \ldots, q$ and $j = 1, \ldots, p$. Hence, $(A+B)' = A'+B'$.



Rules for Transposition

Proof.

(d) Let A be $(p \times q)$ and B be $(q \times r)$ matrices. First, note that both $(AB)'$ and $B'A'$ are $r \times p$ matrices. We will show that $[(AB)']_{ij} = [B'A']_{ij}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, p$.

$$[(AB)']_{ij} = [AB]_{ji} \qquad \text{Definition of transpose}$$

$$= A_{j\bullet} \cdot B_{\bullet i} = \sum_{k=1}^{q} a_{jk} b_{ki} \qquad \text{Matrix product}$$

and

$$[B'A']_{ij} = (B')_{i\bullet} \cdot (A')_{\bullet j} \qquad \text{Matrix product}$$

$$= B_{\bullet i} \cdot A_{j\bullet} = A_{j\bullet} \cdot B_{\bullet i} = \sum_{k=1}^{q} a_{jk} b_{ki}. \qquad \text{By transposition}$$

The proof is complete. □

Symmetric Matrices

- Square matrices with the property that they are symmetric about the main diagonal are called symmetric.

Example

$$\begin{bmatrix} -3 & 2 \\ 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 5 \\ -1 & -3 & 2 \\ 5 & 2 & 8 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

- Symmetric matrices are characterized by the fact that they are equal to their own transposes:

$$\text{The matrix } A \text{ is symmetric } \iff A = A'$$

The Identity Matrix

- The identity matrix of order n, denoted by I_n (or often just by I), is the $n \times n$ matrix having ones along the main diagonal and zeros elsewhere:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n} \qquad \text{(identity matrix)}$$

- If A is any $m \times n$ matrix, it is easy to verify that $AI_n = A$. Likewise, if B is any $n \times m$ matrix, then $I_n B = B$. In particular,

$$AI_n = I_n A = A \qquad \text{(for every } n \times n \text{ matrix A)}$$

- Thus, I_n is the matrix equivalent of 1 in the real number system. In fact, it is the only matrix with this property.

Null Matrices

- A null matrix is composed of all 0s and can be of any dimension; it is not necessarily square.

Example

$$\mathbf{0}_{(2\times 2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{0}_{(2\times 3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Addition or subtraction of the null matrix leaves the original matrix unchanged.
- Multiplication by a null matrix produces a null matrix.

Idempotent Matrices

Definition 1.3

A square matrix A of any order is ***idempotent*** if

$$A = A^2 = A^3 = \cdots$$

where $A^2 = AA$, $A^3 = AAA$, etc.

Example

Verify that $A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$ is idempotent.

Solution:

It is only necessary to verify that $AA = A$. (Why?)

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 2 + (-2)\cdot 1 & 2\cdot(-2) + (-2)\cdot(-1) \\ 1\cdot 2 + (-1)\cdot 1 & 1\cdot(-2) + (-1)\cdot(-1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.$$

Diagonal Matrices

- A square matrix in which all the entries off the main diagonal are zero is called a diagonal matrix. Here are some examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

- A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Diagonal Matrices

- Powers of diagonal matrices are easy to compute; if D is the diagonal matrix and k is a positive integer, then

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Example

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$,

$A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-3)^{100} & 0 \\ 0 & 0 & 2^{100} \end{bmatrix}$, $A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$.

- Consider the pair of linear equations with its associated coefficient matrix:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}, \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{2}$$

- Solving the equation system (2) in the usual way yields

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \qquad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \tag{3}$$

- The two fractions have a common denominator D, equal to $a_{11}a_{22} - a_{21}a_{12}$.

Determinants of Order 2

- The number D must be nonzero for (3) to be valid, in which case system (2) has a unique solution specified by (3).
- In this sense, the value of the denominator determines whether system (2) has a unique solution.
- In fact, $D = a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the matrix A.
- The determinant of A is denoted by either $\det(A)$ or, $|A|$.
- Thus,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

for any 2×2 matrix A.

Determinants of Order 2

Example

a) $\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 4 \cdot 2 - 3 \cdot 1 = 5, \qquad \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - b_2 a_{12}$

b) Find the solutions to the equation $\begin{vmatrix} 2-x & 1 \\ 8 & -x \end{vmatrix} = 0.$

Solution:
Using the definition of the determinant for a 2×2 matrix, we have

$$(2-x)(-x) - 1 \cdot 8 = 0 \implies x^2 - 2x - 8 = 0$$

This is a quadratic equation; thus, using the quadratic formula

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-8)}}{2} = \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2} = -2 \text{ or } 4.$$

Therefore, the solution to the above equation is $x = -2$ or $x = 4$.

Determinants of Order 3

- Consider the system of three linear equations in three unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

- Here the coefficient matrix A is 3×3.
- This system can be solved eventually for x_1, x_2, and x_3 except in a degenerate case.
- The resulting expression for x_1 is

$$x_1 = \frac{b_1a_{22}a_{33} - b_1a_{23}a_{32} - b_2a_{12}a_{33} + b_2a_{13}a_{32} + b_3a_{12}a_{23} - b_3a_{22}a_{13}}{a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}}$$

Determinants of Order 3

- Although not presented here, the corresponding expressions for x_2 and x_3 share the same denominator as that given for x_1.
- This common denominator is called the determinant of A, denoted by $\det(A)$ or $|A|$.
- Thus, the determinant is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{cases} a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ -a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{cases} \tag{4}$$

Determinants of Order 3

Expansion by Cofactors

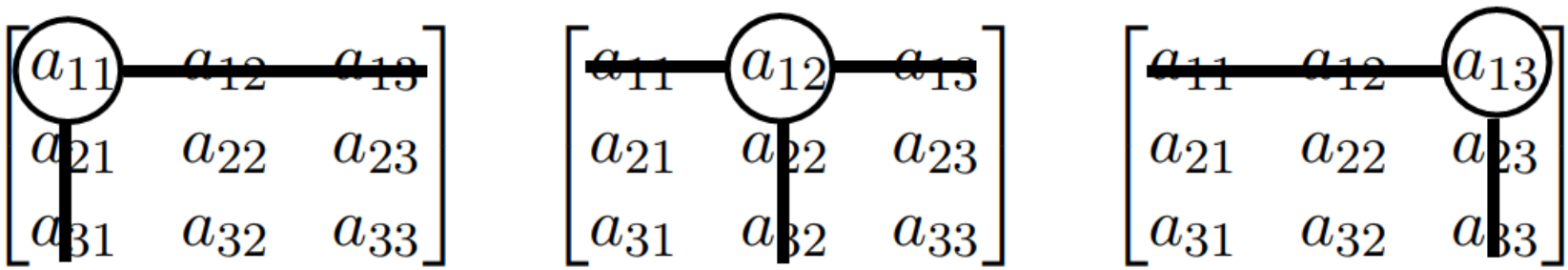
- Consider the sum of the six terms in (4). It looks quite messy, but the method of expansion by cofactors makes it easy to write down all the terms.
- First, note that each of the three elements a_{11}, a_{12}, and a_{13} in the first row of A appears in exactly two terms of (4).
- In fact, $|A|$ can be written as

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

- Applying the rule for evaluating determinants of order 2, we see that this is the same as

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{5}$$

- Note that a_{11} is multiplied by the second-order determinant obtained by deleting the *first* row and the *first* column of $|A|$ (see next slide). This is called the *minor* of the element a_{11} and is denoted by $|M_{11}|$.
- Likewise, a_{12}, with a minus sign attached to it, is multiplied by the determinant obtained by deleting the *first* and the *second* column of $|A|$. Similarly this is the minor of a_{12} and is denoted by $|M_{12}|$ (see next slide).
- In general, the symbol $|M_{ij}|$ can be used to represent the minor obtained by deleting the ith row and jth column of a given determinant.



- Thus, the minors of a_{11}, a_{12}, and a_{13}, respectively, are

$$|M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad |M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad |M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

- A concept closely related to the minor is that of the *cofactor*.
- A cofactor, denoted by $|C_{ij}|$, is a minor with a prescribed sign. The rule for the sign of a cofactor is

$$|C_{ij}| = (-1)^{i+j} |M_{ij}|$$

- Thus, if the sum of the subscripts is an even number, $|C_{ij}| = |M_{ij}|$, since -1 raised to an even power is positive.
- If $i + j$ is equal to an odd number, $|C_{ij}| = -|M_{ij}|$, since -1 raised to an odd power is negative.

Example

Use (4) to calculate $|A| = \begin{vmatrix} 3 & 0 & 2 \\ -1 & 1 & 0 \\ 5 & 2 & 3 \end{vmatrix}$.

Solution

$$
\begin{aligned}
|A| &= 3 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} + 2 \cdot \begin{vmatrix} -1 & 1 \\ 5 & 2 \end{vmatrix} \\
&= 3 \cdot 3 - 0 + 2 \cdot (-2 - 5) \\
&= -5
\end{aligned}
$$

Laplace Expansion and Higher Order Determinants

- Laplace expansion is a method for evaluating determinants in terms of cofactors.
- In general, the Laplace expansion of an nth-order determinant will reduce the problem to one of evaluating n cofactors, each of which is of the $(n-1)$st order, and the repeated application of the process will lead to lower and lower orders of determinants.
- Laplace expansion permits evaluation of a determinant along any row or column.
- Selection of a row or column with more zeros than others simplifies evaluation of the determinant by eliminating terms.

Laplace Expansion and Higher Order Determinants

- The value of a determinant $|A|$ of order n can be found by the Laplace expansion of any row or any column as follows:

$$|A| = \sum_{j=1}^{n} a_{ij} |C_{ij}| \qquad \text{(expansion by the } i\text{th row)}$$

$$= \sum_{i=1}^{n} a_{ij} |C_{ij}| \qquad \text{(expansion by the } j\text{th column)}$$

Laplace Expansion and Higher Order Determinants

Example

Calculate $|A| = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix}$.

Solution

To find $|A|$ it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$|A| = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

Solution (*continued*)

For the 3×3 determinant, it will be easiest to use cofactor expansion along its second column, since it has the most zeros:

$$\begin{aligned} |A| &= 1 \cdot (-2) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ &= -2(1+2) \\ &= -6 \end{aligned}$$

We would have found the same answer if we had used any other row or column.

Basic Rules for Determinants

Rules for Determinants

Let A be an $n \times n$ matrix. Then:

A. If all the elements in a row (or column) of A are 0, then $|A| = 0$.

B. $|A'| = |A|$, where A' is the transpose of A.

C. If all the elements in a single row (or column) of A are multiplied by a number α, the determinant is multiplied by α.

D. If two rows (or two columns) of A are interchanged, the determinant changes sign, but the absolute value remains unchanged.

E. If two of the rows (or columns) of A are proportional, then $|A| = 0$.

F. The value of the determinant of A is unchanged if a multiple of one row (or one column) is added to a different row (or column) of A.

Basic Rules for Determinants

Rules for Determinants (*continued*)

G. The determinant of the product of two $n \times n$ matrices A and B is the product of the determinants of each of the factors:

$$|AB| = |A| \cdot |B|$$

H. If α is a real number,

$$|\alpha A| = \alpha^n |A|$$

Note The determinant of a sum is not the sum of the determinants:

$$|A + B| \neq |A| + |B| \qquad \text{(in general)}$$

Basic Rules for Determinants

Example

(a) We can use the above rules to evaluate the following determinants:

$$\begin{vmatrix} 3 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 0 & 5 \end{vmatrix} = 0 \qquad \text{(Rule A)}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 3 & -1 \\ -3 & -6 & -9 & -12 \end{vmatrix} = 0 \qquad \text{(Rule E, b/c } Row4 = -3 \cdot Row1)$$

(b) Find the determinant of B:

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{bmatrix} \quad \text{if} \quad |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Solution: By Rule C, $|B| = 3\,|A| = 21$.

Inverse Matrices

- An *inverse matrix* A^{-1}, which can be found only for a square matrix A, is a unique matrix satisfying the relationship

$$AA^{-1} = A^{-1}A = I$$

- Because $A^{-1}A = AA^{-1} = I$, the matrix A is also an inverse of A^{-1}—that is, A and A^{-1} are inverses of each other.
- Not every square matrix has an inverse.
- If a square matrix A has an inverse, A is said to be *nonsingular* ; if A possesses no inverse, it is called a *singular matrix*.

Inverse Matrices

Example

Show that $A = \begin{bmatrix} 5 & 6 \\ 5 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 1/2 & -3/10 \\ -1/4 & 1/4 \end{bmatrix}$ are inverses of each other.

Solution

$$\begin{bmatrix} 5 & 6 \\ 5 & 10 \end{bmatrix}\begin{bmatrix} 1/2 & -3/10 \\ -1/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 5/2-6/4 & -15/10+6/4 \\ 5/2-10/4 & -15/10+10/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and likewise we verify that $BA = I$.

Properties of the Inverse

Properties of the Inverse

Let A and B be invertible $n \times n$ matrices. Then:

(a) A^{-1} is invertible, and $(A^{-1})^{-1} = A$.

(b) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

(c) The transpose A' is invertible, and $(A')^{-1} = (A^{-1})'$.

(d) $(cA)^{-1} = c^{-1}A^{-1}$ whenever c is a number $\neq 0$.

Cofactor and Adjoint Matrices

- A *cofactor matrix* is a matrix in which every element a_{ij} is replaced with its cofactor $|C_{ij}|$.
- An *adjoint matrix* is the transpose of a cofactor matrix. Thus

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{bmatrix} \quad \text{adj } (A) = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & |C_{31}| \\ |C_{12}| & |C_{22}| & |C_{32}| \\ |C_{13}| & |C_{23}| & |C_{33}| \end{bmatrix}$$

Example

Find the cofactor matrix C and the adjoint matrix $\text{adj}(A)$ of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$$

Solution

Replacing the elements a_{ij} with their cofactors $|C_{ij}|$ according to the law of cofactors,

$$C = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & - \begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 5 & 3 \end{vmatrix} \\ - \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} & - \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & - \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & -6 & 7 \\ -9 & 3 & 9 \\ 5 & 0 & -10 \end{bmatrix}$$

Solution (continued)

The adjoint matrix $\mathrm{adj}(A)$ is the transpose of C,

$$\mathrm{adj}(A) = \begin{bmatrix} -2 & -9 & 5 \\ -6 & 3 & 0 \\ 7 & 9 & -10 \end{bmatrix}$$

A General Formula for the Inverse

General Formula for the Inverse

Any square matrix $A = (a_{ij})_{n\times n}$ with determinant $|A| \neq 0$ has a unique inverse A^{-1} satisfying $AA^{-1} = A^{-1}A = I$. This is given by

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$$

If $|A| = 0$, then there is no matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

A General Formula for the Inverse

Example

Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. Show that A has an inverse and find the inverse.

Solution

A has an inverse if and only if $|A| \neq 0$. Here we find that $|A| = -5$, so the inverse exists. The cofactors are

$$C_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10, \quad C_{12} = -\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = -15, \quad C_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4, \quad C_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 4, \quad C_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 14, \quad C_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6$$

Solution (*continued*)

Hence, the inverse of A is

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

(Check the result by showing that $AA^{-1} = I$.)

********** End of Chapter One **********