



Calculus for Economists

Course Number- Econ 1011

CALCULUS AND ITS ECONOMIC APPLICATION

UNIT-ONE: INTRODUCTION

1.1 Mathematical vs. Non Mathematical Economics

Since mathematical economics is merely an approach to economic analysis it should not differ from the nonmathematical approach to economic analysis in any fundamental way. The purpose of any theoretical analysis regardless of the approach is always to derive a set of assumptions or postulates via a process of reasoning. The major difference between mathematical and theoretical economics lies principally in the fact that, in the former, the assumptions and conclusions are stated in mathematical symbols than words and in equations than sentences; moreover, in place of literary logic, use is made of mathematical theorems. The choice between mathematical logic and theoretical logic, again, is little import, but mathematics has the advantage of forcing analysis to make there assumptions explicit at every stages of reasoning.

Mathematical economics has claim to the following advantages

1. the language used is more concise and precise
2. there exists a wealthy of mathematical theorems at our services
3. In forcing us to state explicitly all assumptions as prerequisite to the uses of mathematical theorems and it allow us to treat the general n-variable cases.

In sum, we can link the mathematical approach to “a mode of transportation” that takes us from the set of postulates (assumptions) to set of conclusions (destination) at a good speed.

The term mathematical economics is sometimes confused with econometrics. As the metric mart of the second term implies econometrics is mainly concerned with the measurement of economic data. Hence it deals with the study of empirical observations using statistical methods of estimation and hypothesis testing.

Indeed, empirical studies and theoretical analysis are often complementary and mutually reinforcing. On one hand theories must be tested against empirical data for validity before they can be applied with confidence. On the other hand, statistical works needs economic theory as a guide, in order to determine the most relevant and fruitful direction of research.

1.2 Functions

The function concept is one of the most important concepts that frequently occur in many real world situations. The idea of corresponds plays a central role in its formulations. We have already had experiences with corresponds in everyday life. For example, to each person there corresponds an annual income; to each item in a super market there corresponds a price; for the manufacturer of x there corresponds a cost; to each square there corresponds an area.

One of the most important aspects of any science (managerial, economical, life, social, physical, etc) is the establishment of correspondence among various types of phenomena. Once a correspondence is known, predictions can be made. The application of calculus to practical problems depends on expressing physical quantities in terms of functions. Therefore, in order to analyze the relationship between different phenomena in general and to use calculus in particular we must understand the basic ideas of functions.

A function can be defined either in rule form or in set form

Definition in Rule form

A function is a rule (process or method) that produces a correspondence between one set of elements, called *the domain*, and a second set of elements, called *the range*, such that to each element in the domain there corresponds one and only one element in the range

Definition in Set form

A function is a set of ordered pairs with the property that no two ordered pairs have the same first element and different second element. The set of all first elements in a function is called the *domain* of the function, and the set of all second components is called the *range*.

Example: Consider the following three tables of the cube, square and square root

Table 1		Table 2		Table 3	
Domain (cube)	Range (Number)	Domain (Square)	Range (Number) (Square root)	Domain	Range (Number)
-2	-8	-2	4	0	0
-1	-1	-1	1	1	1
0	0	0	0		-1
1	1	1	1	4	2
2	8	2	4		-2
				9	3
					-3

Table 1 and 2 are functions, since to each domain values there correspond exactly one range value (for example, the cube of -2 is -8 and no other number). On the other hand, Table 3 is not a function, since to at least one domain value there corresponds more than one range value(for example to the domain value 9 there corresponds -3 and 3, both square roots of 9).

Since in a function element in the range are paired with elements in the domain by some rule or process, this correspondence (pairing) can be illustrated by pairs of ordered pairs. For example, if we list the ordered pairs in Table 1 and 2, we have the followings:

Function 1= $\{(-2,-8), (-1,-1), (0,0), (1, 1), (2, 8),\}$

Function 2= $\{(-2,-4), (-1,1), (0,0), (1,1), (2,4)\}$

In either cases, notice that no two ordered pairs have the same first component and different second components.

On the other hand, if we list the ordered pairs determined by Table 3, we have

$A= \{(0, 0), (1, 1), (1,-1), (4, 2), (4,-2) (9, 3), (9,-3)\}$

In this case, there are ordered pairs with the same first component and different second component (for example, (1, 1) and (1,-1)) which indicates once again that Table 3 is not a function.

Example

A) The set $S = \{(-2,4), (-1,3), (0,1), (1,3), (2,4)\}$ defines a function since no two ordered pairs have the same first component and different second components. The domain and range are

Domain = $\{-2, -1, 0, 1, 2\}$ and Range = $\{1, 3, 4\}$

B) The set $T = \{(2,1), (1,3), (2,3), (1,4)\}$ does not define a function since there are ordered pairs with the same first component and different second components for example (1,3) and (1,4)

Exercise

1. A) Define a function?

B) Give an example of a function both in a set form and in a rule form

2. Determine whether each set defines a function. If it does, then state the domain and range.

A). $S = \{(1, 2), (2, 4), (3, 6), (2, 8), (1, 10)\}$

B). $T = \{(-3, 1), (-1, 2), (3, 2), (5, 1)\}$

Functions specified by Equations

An equation is a mathematical statement which involves the symbol '='. It sets two algebraic expressions equal to each other. $3x+2 = \frac{x}{3}-5$ and $4y-5 = 8y$ are equations in one variable and $Y=2x-5$ is an equation in two variables, x and y .

Since most variables in applied science like economics are by their nature restricted to real numbers the rules associating range values with domain values usually expressed in equations forms. The input values are domain values, and the output values are range values. The equations (a rule) assign each domain value x a range value y .

For an equation in two variables, if there it correspond exactly one values of the output to each value of the input, then the equation specifies a function. If there is any value of the input which correspond more than one value of the output, then the equation does not specify a function.

Example

Determine which of the following equations specify functions with input x

a) $4y - 3x = 8$

b) $y^2 - x^2 = 9, x \in \mathbb{R}$

a) Solving for the output y , we have

$$4y - 3x = 8$$

$$4y = 8 + 3x$$

$$y = 2 + \frac{3}{4}x$$

Since each input value x corresponds to exactly one output value this equation specifies a function.

b) Solving for the output y we have

$$y^2 - x^2 = 9$$

$$y^2 = 9 + x^2$$

$$y = \pm \sqrt{9 + x^2}$$

Since $9 + x^2$ is always a positive real number for any real number x and since each positive real number has two square roots, to each input value x there corresponds two output values ($y = \sqrt{9 + x^2}$ and $y = -\sqrt{9 + x^2}$). For example, if $x = 4$, then equation in (b) is satisfied for $y = 5$ and $y = -5$. Thus, an equation in (b) does not specify function.

Agreement on Domain and Ranges

If a function is specified by an equation and the domain is not indicated, then we assumed that the domain is set of all input values x for which an equation can be evaluated and the range is the set of all output values y corresponding to input values x .

Example

Find the domain and range of the function specified by the equation $Y = \sqrt{x + 4}$. Since we cannot have a negative number inside the square root, the domain is all real number greater than or equal to -4 , i.e., $x \geq -4$. The range is all real number's greater than or equal to 0 , i.e., $X \geq 0$.

Example

The total cost C of a firm per day is expressed in terms of its daily output Q by the equation

$C = 200 + 4Q$. The firm has a capacity limit of 100 units of output per day. What are the domain and the range of the cost equation?

Since we cannot have a negative quantity of output and the maximum output limit is 100, the domain is the set of values $0 \leq Q \leq 100$ or more formally, $\text{Domain} = \{Q / 0 \leq Q \leq 100\}$

As for the range, the minimum cost C value is 200 (when $Q = 0$) and the maximum C value is 600 (when $Q = 100$). Thus, $\text{Range} = \{C / 200 \leq C \leq 600\}$

Exercise

1. Determine which of the following equations specify functions with input X

a) $y^2 - x^4 = 9, X \in \mathbb{R}$

b) $3y - 2x = 3, x \in \mathbb{R}$

2. Find the domain and range of the following equations (functions)

a) $Y = 5x + 2$

b) $x^2 - y = 1$

c) $Y = \sqrt{2 - x}$

d) $y = \frac{5x+1}{2x-3}$

1.2.1 Type of Functions

1. Polynomial is a continuous function with a general form of;

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

In which each term contains a coefficient as well as a non-negative integer power of the variable x . The superscript indicators of the power of x are called *exponents*. The highest power involved, i.e., the value of n (when $a_n \neq 0$), is often called the *degree* of the polynomial function. Depending on the value of the integer n (i.e., the degree), polynomial function is further divided into the following sub-classes.

1.1 Constant function is a polynomial with degree zero ($n = 0$).

Its graph is a horizontal line and its slope is zero. It has a general form of $Y = f(x) = k$, where k is any constant number

For instance, $f(x) = 5$ and $g(x) = \frac{1}{2}$ are constant functions

1.2 Linear function is a polynomial with degree one ($n=1$).

Its graph is a straight line and it has a constant slope. It has a general form of

$$F(x) = a_0 + a_1x, \quad a_1 \neq 0, \text{ where } a_0 \text{ and } a_1 \text{ are constants,}$$

For instance, $f(x) = 2+5x$ is a linear function.

1.3 Quadratic function is a polynomial with degree two ($n = 2$).

Its graph is either upward or downward parabola. Its slope is different at different points. It has a general form of: $F(x) = a_0 + a_1x + a_2x^2$, $a_2 \neq 0$, where a_0 , a_1 and a_2 are constants

1.4 Cubic function is a polynomial with degree three ($n = 3$)

Polynomials of higher degree are known by the degree as polynomial of degree 4, 5 and so on.

2. Rational function is a function in which y or $f(x)$ is expressed as a ratio of two polynomial functions. It is a discontinuous function; it is undefined at points where the denominator is equal to zero. Notice that any polynomial function is by itself a rational function, because it can always be expressed as a ratio of one.

If $p(x)$ and $Q(x)$ are polynomials, then $R(x) = \frac{p(x)}{Q(x)}$ is a rational function

A special rational function that has interesting applications in economics is the function

$$Y = \frac{a}{x} \text{ or } xy = a, \text{ where } a \text{ is any constant}$$

The graph of this function is *a rectangular hyperbola*.

3. Exponential functions ;are functions which generally denoted by $b^n = b*b*b*...*b$ n times

4. Logarithmic Functions: are the inverses of exponential functions.

Given $f(x) = b^n$ then $\log_b f(x) = n$

1.2.2 Multivariate Functions

Uni-variate function is a function which has only single independent variable. The word “uni” means “one” and “variate” means “variables (independent)”. It is also known as function of one independent variable or univariable function. It is of the form $Y = f(x)$

The law of demand, for example, is a univariate functions. It can be writer as $Q_d = f(P)$, cetirus paribus, where Q_d stands for quantity demand of commodity and P for price of the commodity.

Multivariate Functions is a function which has two or more independent variables. It is also known as a function of several independent variables or multivariable function.

A function of the form $Z = f(x, y)$ is a function of two in dependent variables if a given pairs of x and y values uniquely determined a value of the dependent variable Z . We have many occasions to use functions of this type in economic models. For instance, if the output (Q) is determined by the amount of labor (L) and capital (K) employed; then we can write the production function in general form of $Q = f(L, K)$.

Similarly the function of the form $W = f(x, y, z)$ is a function of three independent variables. Generally a function with n independent variables can be written as.

$$Y = (x_1, x_2, \dots, x_n)$$

Note that for economic applications, most functions of several independent variables restricted to a function of one independent variable by using a common assumption of *cetris paribus*.

For example, the quantity demand for a good (Q_d) depends on not only on its own price (p_0) but also the price of related good (p_1), on consumers' income(y), on the consumers' taste and preferences (T), on the amount of advertising (A) and on other factors. This demand function for a particular good might be of the form

$$Q_d = f(P_0, P_1, Y, T, A, \dots)$$

To derive a law of demand, with the help of Cetrus paribus (i.e. all factors except own price kept constant), this function can be changed to a univariate function of the form

$Q_d = f(P_0/P_1, Y, A...)$ The vertical line (bar) means that all dependent variable after it are assumed to be constant. Usually this univariate function is written as $Q_d = f(P_0)$, Ceteris paribus

1.2.3 Explicit Versus Implicit Functions

From the point of view of economic applications, it is worth to differentiate function as explicit functions and implicit functions. The aim of this section is to illustrate the difference and the functional form of these two types of functions.

Explicit function is a function in which the independent and dependent variables are explicitly (clearly) defined. Whenever the cause and effect relationship is clear and when the objective is to show that relationship, it is imperative to use explicit functions. An explicit function of one independent variable can be represented as $Y = f(x)$.

The law of supply for example, is an explicit function of one independent variable. It can be written as $Q_s = f(P_0)$, where Q_s stands for quantity supplied of a commodity and P_0 price of the commodity. This law asserts that it is the price of the commodity that determines the quantity supplied, if all other factors kept constant. Note that $Q_s = f(P)$ and $P = f(Q_s)$ do not have the same interpretive meanings.

An explicit function of two independent variables can be represented by $Z = f(X, Y)$. The production function, $Q = f(L, K)$ is an explicit function. It tells that the output (Q) is determined by two independent variables, labor (L) and capital (K).

Implicit Function: is a function in which the independent and dependent variables are not explicitly (clearly) defined. Whenever the cause and effect relationship is not clear or when the objective is not to show that relationship, it is better to use implicit functions.

- An implicit function of one independent variable can be represented by $f(x, y) = 0$.
- An implicit function of two independent variable can be written as $f(x, y, z) = 0$

The budget formula $M = P_x X + P_y Y$, Ceteris paribus, is an implicit function.

Where M is the budget (money), P_x is price of commodity x and P_y is price of Y .

In this case neither X or Y is a cause or an effect; there may or may not be any cause and effect relationship between them.

Note that an explicit function can always be transformed in to an implicit function by simply transposing the left hand side expression to the right side of the equal sign, but the converse is not always possible.

Example

$Y = f(x) = 3x - 2$ is an explicit function. Its equivalent implicit function (form) will be

$Y - 3x + 2 = 0$ and it is the general form of $f(x, y) = 0$

Example

Write the implicit form of the function $Y = \frac{x^2 - 1}{x - 3}$.

By transposing the $\frac{x^2 - 1}{x - 3}$ to the right hand side, we have $Y - \frac{x^2 - 1}{x - 3} = 0$ $\frac{Y(x - 3) - (x^2 - 1)}{x - 3} = 0$

$$Y(x - 3) - (x^2 - 1) = 0 \quad Yx - 3y - x^2 + 1 = 0$$

Distinguishing between explicit and implicit functions is sometimes very important in managerial and economics model. For example, the following two demand functions may be taken as equivalent form from mathematical point of view, but they do not have the same economic meaning.

$$1. Q_d = 20 - 2p \qquad 2. P = 10 - \frac{1}{2} Q_d$$

Exercise

1. Explain the difference between explicit functions and implicit functions.
2. Write the implicit form of the following explicit functions

$$a). Y = f(x) = \frac{2}{3}x - 5 \quad b). Y = f(x) = \sqrt{x - 4}$$

1.2.4 Exponential and Logarithmic Functions

Reading assignments (take lecture note and read reference books)

1.2.5 Monotonic and Homogeneous functions

A. Monotonicity

The basic property of a function which has significant importance in economic relationships is whether it is increasing or decreasing. A **monotonic function** is a function which is either strictly increasing or strictly decreasing in its domain. In your elementary mathematics courses, you learned non-calculus (i.e, graphical and algebraic) approaches to determine whether a function is monotonic or not. In this subsection, we will see how the sign of the first derivative can easily tell us whether a function is strictly (monotonic) increasing or strictly (monotonic) decreasing on a given interval.

Definitions

1. A function $y = f(x)$ is **strictly (or monotonic) increasing** on the interval (a, b) if its graph moves up word from left to right on that interval. More precisely, a function f is strictly increasing on (a, b) if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ on (a, b) look figure a.

2. A function $y = f(x)$ is **strictly (or monotonic) decreasing** on the interval (a, b) if its graph moves down word from left to right on that interval. More precisely, a function f is strictly decreasing on (a, b) if $f(x_1) < f(x_2)$ whenever $x_1 > x_2$ on (a, b) look figure b.

A function $y = f(x)$ is neither monotonic increasing nor monotonic decreasing if it is strictly increasing on one interval and strictly decreasing on other interval in its domain, example look figure 1.1(c) it is strictly increasing on interval $(0, b)$ and strictly decreasing on interval (b, ∞) .

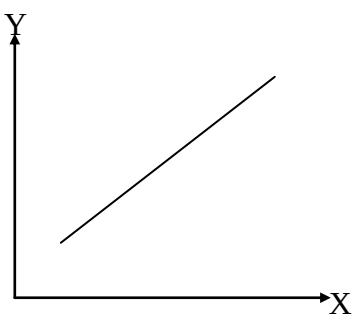


Figure (a)

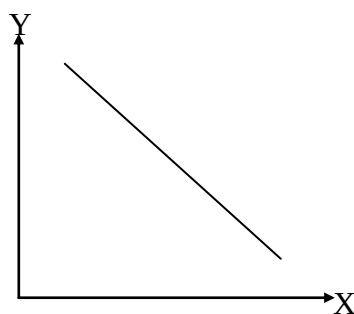


Figure (b)

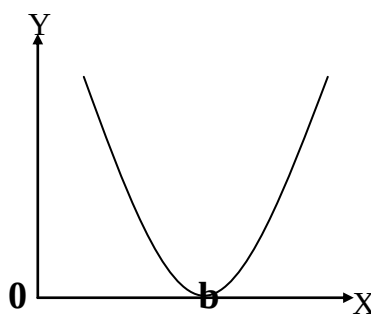


Figure (c)

B. Homogeneous Functions

Homogeneous functions arise naturally throughout economics. For instance, profit functions and cost functions that are derived from production functions, and demand functions that are derived from utility functions are homogeneous.

Now, let us start from non-calculus definition of homogeneous function and we will develop the differential calculus approach.

Definition:

A function $y = f(x_1, x_2, \dots, x_n)$ is **homogeneous with degree K** if

$$f(Sx_1, Sx_2, \dots, Sx_n) = S^K f(x_1, x_2, \dots, x_n) = S^K y$$

A function f is homogeneous with degree K if it satisfies the property that when all the independent variables are changes by same proportion or amount S , the dependent variable y changes by S^K .

Example Show that the following are homogeneous or not. If so, find the degree

$$a) y = f(x_1, x_2) = x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3$$

Solution

$$\begin{aligned} f(Sx_1, Sx_2) &= (Sx_1)^3 + 3(Sx_1)^2(Sx_2) + 3(Sx_1)(Sx_2)^2 + (Sx_2)^3 \\ &= S^3x_1^3 + S^33x_1^2x_2 + S^33x_1x_2^2 + S^3x_2^3 \\ &= S^3(x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3) \\ &= S^3 f(x_1, x_2) \end{aligned}$$

Thus, from the above definition f is homogeneous with degree 3

$$b) y = f(x_1, x_2) = \sqrt{2x_1^2 + 7x_1x_2 + 6x_2^2}$$

solution

$$\begin{aligned} f(Sx_1, Sx_2) &= \sqrt{2(Sx_1)^2 + 7(Sx_1)(Sx_2) + 6(Sx_2)^2} \\ &= \sqrt{S^2(2x_1^2 + 7x_1x_2 + 6x_2^2)} \\ &= \sqrt{S^2} \sqrt{2x_1^2 + 7x_1x_2 + 6x_2^2} \\ &= |S| f(x_1, x_2) \end{aligned}$$

Thus, f is homogeneous with degree 1

A function f which is homogeneous with degree 1 is said to be **linearly homogeneous function**

$$c) y = f(x_1, x_2) = 3x_1^2 + 6x_2$$

solution

$$\begin{aligned} f(Sx_1, Sx_2) &= 3(Sx_1)^2 + 6(Sx_2) \\ &= 3S^2x_1^2 + 6Sx_2 = S(3Sx_1^2 + 6x_2) \end{aligned}$$

Thus, since “ S ” cannot completely factor out, f is not homogeneous.

Remark

One can usually tell whether a specific function is homogeneous just by looking the degree of each term in its formula. A function f is homogeneous with degree K if the degrees of each term are equal to K . For instance, the function in part (a) of example above is homogeneous with degree 3 since the degree of each term is 3, and the function in part (c) of this example is not homogeneous since the degree of the first term is 2 and that of the second term is one.

Homogeneity is useful in many applications in economics; one of the most useful applications is in production theory because of the relationship between homogeneity and return to scale. In particular, if a production function $y = f(x_1, x_2, \dots, x_n)$ is homogeneous of degree k , then the production function exhibits

- i) Increasing return to scale (or IRS) if $K > 1$
- ii) Constant return to scale (CRS) if $K = 1$
- iii) Decreasing return to scale (DRS) if $K < 1$.

Example Show that the Cobb Douglas production function is homogeneous

$$Q = f(L, K) = AL^\alpha K^\beta, A > 0, 0 < \alpha, \beta < 1 \text{ and } \alpha + \beta = 1 \text{ exhibits CRS.}$$

Solution

$$\begin{aligned} f(SL, SK) &= A(SL)^\alpha (SK)^\beta \\ &= AS^\alpha L^\alpha S^\beta K^\beta \\ &= S^{\alpha+\beta} AL^\alpha K^\beta \\ &= SAL^\alpha K^\beta \text{ since } \alpha + \beta = 1 \\ &= SQ \end{aligned}$$

Thus, Cobb Douglas function is homogeneous with degree one & it display constant return to scale.

Exercise: Show the following functions are homogeneous and comment on its returns to scale

a. $Q = 7KL^2$

b. $Q = 50K^{1/4}L^{3/4}$

1.2.6 Applications of Functions in Economics

A. Linear Function

The graph of each linear function is a *straight line*, and conversely every straight line is the graph of a linear function. Knowing that the graph of a linear function is a line makes it quite easy to sketch the graph of the linear function. It is sufficient to find two points at which the graph crosses the axes. These two points are called *intercepts*.

- The point where the graph crosses the x-axis is called the *x-intercept* and it occurs when $y=0$, and
- The point where the graph crosses the y-axis is called the *y-intercept* and it occurs when $x = 0$

The slope of a line through two points (x_1, y_1) and (x_2, y_2) on the line is given by a formula

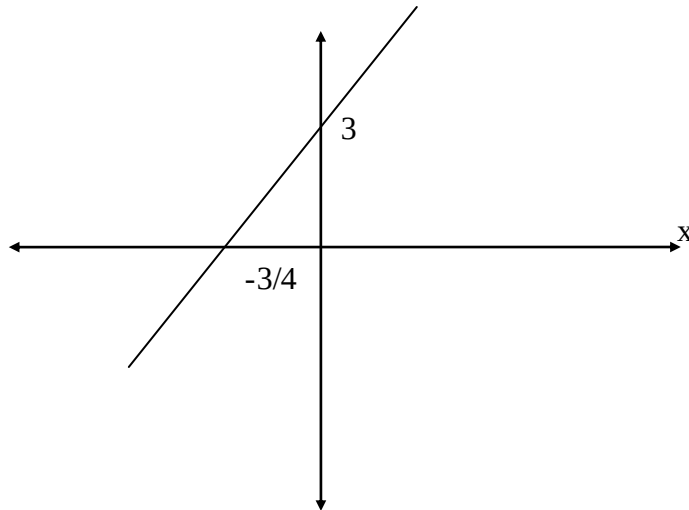
$$\text{Slope (m)} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example Plot the graph of $Y = 3 + 4x$

The y-intercept at $x=0$ is 3 or $(0, 3)$

The x- intercept at $y=0$ is $-3/4$ or $(-3/4, 0)$

We locate the two points on the plane and then draw the line connecting the two points



Notice that the slope (m) is 4 which is positive

➤ To define the equation of line whose slope (m) and one point (x, y) are given use the formula

$$Y - Y_1 = m (x - x_1)$$

Example : Write equation of the line whose slope $\frac{1}{2}$ and passes through $(-4, 3)$

$$y - y_1 = m (x - x_1) = \frac{1}{2} (x - (-4)) = \frac{1}{2} (x + 4) = \frac{1}{2} x + 2 \rightarrow y = \frac{1}{2} x + 5$$

➤ To derive the equation of a line when two points, (x_1, y_1) and (x_2, y_2) , are given use the formula:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

Example Derive equation of the line that passes through $(1, 7)$ and $(6, -3)$

$$Y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) = \frac{-3 - 7}{6 - 1} (x - 1) = -2(x - 1) = -2x + 2 \rightarrow y = -2x + 9$$

Exercise

1. Plot the graph of each of the following

a) $Y = 2x - 3$

b) $y + 3x = 5$

2. Find an equation of the line ($y = a + bx$), given the information in each problem.

a) Line passes through $(6, -2)$ with slope $\frac{2}{3}$

b) Line passes through $(2, -3)$ and $(4, 3)$

Some Common Linear Models (Functions) In Economics

A. Demand Function

Demand function shows the relationship between quantity demanded of a commodity (x) and those factors affecting its demand like its own price (P_x), consumer's income (I), price of other related commodities (P_o), tastes and preferences (T), number of consumers (N), consumers expectation about future price and income (E) etc.

Mathematically, demand function is expressed as

$$Q_{Dx} = f(p_x, I, P_o, T, N, E)$$

The demand function and its curve can only be specified if all the above factors except its own price (P_x) are kept as constant. This means that demand function shows the relationship between quantity demanded of a commodity x and its own price (P_x), other factors remain constant. Mathematically, demand function for a commodity X can be written as

$$Q_{dx} = f(P_x), \text{ Ceteris Paribus}$$

The normal or usual demand function is monotonic decreasing i.e as P_x increases, Q_x decreases and vice – versa. Then the linear demand function or curve is given by

$$Q_{dx} = a - b p_x \quad (a, b > 0), \text{ where } Q_{dx} = \text{quantity demand for x and } P_x = \text{price of x}$$

Example

For a given commodity x suppose that when its price is 20 Birr per unit, 10 Units of x are demanded by the consumers and when its price becomes 10 per unit, consumer demands 40 units of x. Derive a linear demand function for commodity x.

Since the graph of linear demand function is a line and the points $(P_1, Q_1) = (20, 10)$ and $(P_2, Q_2) = (10, 40)$ lie on this line, the equation of the line can be obtained using the formula.

$$Q - Q_1 = \frac{Q_2 - Q_1}{P_2 - P_1} (P - P_1)$$

$$Q - 10 = \frac{40 - 10}{10 - 20} (P - 20)$$

$$Q - 10 = \frac{30}{-10} (P - 20)$$

$$Q - 10 = -3(p - 20)$$

$$Q - 10 = -3p + 60$$

$$Q = 70 - 3p$$

The demand function for commodity X is $Q_{dx} = 70 - 3p_x$

B. Supply Function

Supply function describes that quantity supplies of a commodity x depends on factors like price of the same commodity (p_x), price of inputs or cost of inputs (c), technology (T) number of producers (N), government policies (G), producers expectation about future price (E), whether (W), price of other goods (P_Q) and so on. Mathematically, this supply function has a form of

$$Q_{sx} = f(P_x, C, T, N, G, E, W, P_o)$$

The usual or normal supply function describes Q_s as a function of P_x , holding other factors constant. This function in the normal condition is monotonic increasing, i.e, as P_x increases Q_x increases and vice-versa. Mathematically, it has a form of

$$Q_{sx} = f(P_x), \text{ ceteris paribus}$$

A linear supply function or model is given by

$$Q_{sx} = -C + dp_x \quad (C, d > 0)$$

For example $Q_s = -20 + 3p$ is a linear supply function.

C. Cost Function

Cost function describes the relationship between total cost (C) and output (Q) other factors remain constant. Mathematically it can be expressed as.

$$C = f(Q), \text{ ceteris paribus}$$

In short run, costs are divided into fixed and variable costs.

Fixed costs are those costs involved in the absence of any production i.e. those costs which are independent of the quantity of output produced.

Variable costs are those costs involved in the actual manufacturing process, i.e., those costs which are dependent on the quantity of output produced.

A linear cost function (model) is given by

$$C = F + VQ$$

Where F is fixed cost

V is variable cost per unit of output Q

VQ is total variable cost, C is total cost

Q is quantity produced

For example, in a linear cost function $C = 500 + 12Q$, 500 Birr is fixed cost, per unit variable cost is 12 Birr and $12Q$ Birr is total variable cost.

Example Establish a linear relationship for the total cost function if the variable cost is 6 Birr per unit and the fixed cost is 150 Birr.

$$C = F + VQ$$

$$C = 150 + 6Q$$

D. Revenue Function

$$R = PQ_s$$

Where R is total revenue, p is price and Q_s is the amount of quantity sold. For example, $R = 6Q$

E. Budget function (line)

$M = P_x$ is price of commodity x

Where M is the amount of money allotted to purchase commodity x and y

P_x is price of commodity x and P_y is price of commodity y

For example, $100 = 5x + 10Y$

F. Consumption function

$$C = a + by, \quad a > 0, \quad 0 < b < 1$$

Where C is total consumption

a is autonomous consumption which is independent of y (GNP)

b is marginal propensity to consume from the GNP,

Y is Gross National product (GNP).

Exercise

1. When the output is 100 units a company would have to charge a price 100 Birr per unit; for an output of 300 units the corresponding price would be 70 Birr. Derive a linear supply function for a company.
2. A firm makes a cabinet. The selling price is 160 Birr per unit. The variable cost cabinet is 60 birr and the fixed cost for the period is 12,000 Birr. Find
 - a. a linear cost function of a firm, and
 - b. a linear net revenue or profit function of a firm.

Basic Economic Applications of Linear Functions**Market-equilibrium analysis**

Market-equilibrium analysis involves in price determination of a particular commodity in an isolated market. It helps us in identifying the level of the price of the product at which the demand and the supply for the product are equal or the market is cleared. This level of price is called *equilibrium price*, or *market clearing price*. The equilibrium price together with its correspondence equilibrium quantity is said to be *(market) equilibrium - point*. Above this point there is shortage and below it there is *surplus*.

$$Q_d = a - bp \quad (a, b > 0) \quad \text{----- (1)}$$

$$Q_s = -C + dP \quad (c, d > 0) \quad \text{----- (2)}$$

At the equilibrium point:

$$Q_d = Q_s \text{ ----- (3)}$$

$a - bp = -c + dp$ by substituting (1) and (2) in (3), we get

$$-bp - dp = -c - a$$

$$(b+d)p = a+c$$

$$P = \frac{a+c}{b+d} \text{ (4)}$$

By inserting (4) in to (1) we get the equilibrium quantity as follows:

$$Q_d = a - bp$$

$$\bar{Q} = a - b\bar{P}$$

$$\bar{Q} = a - b\left(\frac{a+c}{b+d}\right)$$

$$\bar{Q} = \frac{a(b+d) - b(a+c)}{b+d} \text{ (5)}$$

Notice that we get the same result for $\bar{Q}$ by substituting (4) in (2)

Example Suppose that the market demand for a particular product is given by $Q_d = 500 - 0.5P$ and its market supply is given by the equation $Q_s = -100 + 2P$

- What will be the equilibrium price and equilibrium quantity?
- Determine whether there is shortage or surplus when the price is 200 birr.

Solution

$$a) Q_d = 500 - 0.5P \quad Q_s = -100 + 2P \quad a = 500, b = 0.5 \quad c = 100, d = 2$$

$$\bar{P} = \frac{a+c}{b+d} \quad \text{and} \quad \bar{Q} = \frac{a(b+d) - b(a+c)}{b+d}$$

$$\bar{P} = \frac{500+(100)}{0.5+2} \quad \text{and} \quad \bar{Q} = \frac{500(2)-0.5(+100)}{0.5+2}$$

$$\bar{P} = \frac{600}{2.5} = 240 \text{ birr and } Q = \frac{950}{2.5} = 380 \text{ unit}$$

b) At $P = 200$ birr

$$\begin{aligned} Q_d &= 500 - 0.5P & Q_s &= -100 + 2P \\ &= 500 - 0.5(200) & &= -100 + 2(200) \\ &= 400 \text{ units} & &= 300 \text{ units} \end{aligned}$$

Since $Q_d > Q_s$, there is a surplus in the market

Break - even analysis

The basic problem of break-even analysis is to determine how many units of a product that the company or firm will have to produce in order to break-even, i.e, in order to get neither a profit nor a loss. This level of output or product is called break - even output and at this level of output the firm earns zero profit. Break-even analysis is a useful planning tool to commence or expand a business or introduce a new product.

$$R = PQ \text{ ----- (1) and } C = F + VQ \text{ ----- (2)}$$

At break - even point the total revenue from sales will be equal to the total costs of producing and selling Q units. Thus, at break - even point:

$$R = C \Rightarrow PQ = F + VQ \Rightarrow PQ - VQ = F \Rightarrow Q(P - V) = F \Rightarrow Q = \frac{F}{P - V}$$

Then using a unique symbol Q_e for break - even quantity we get

$$Q_e = \frac{F}{P - V}$$

$P - V$ (Price - unit variable cost) is termed as *the contribution per unit of product* as it is the net revenue received by producing and selling each extra unit of Q . Thus, equivalently, we have

$$Q_e = \frac{\text{Fixed cost}}{\text{Contribution per unit of product}}$$

Example

Suppose that the total production cost of a firm is given by $C = 33 + 2Q$

- a) At what level of output Q a firm will break - even if the price is fixed at birr 13 per unit?
- b) What would be the net revenue at an output of 15 units?

Solution

a) $C = 33 + 2Q$

b) $R = PQ = 13Q$

$F = 33, V = 2, P = 13$

Net revenue = $R - C$

$$Q_e = \frac{F}{P - V}$$

$$= 13Q - (33 + 2Q)$$

$$= \frac{33}{13 - 2}$$

$$= 11Q - 33$$

$$= \frac{33}{11}$$

$$= 11(15) - 33$$

$$= 3 \text{ units}$$

$$= 165 - 33 = 132 \text{ unit}$$

Exercise

- Demand and supply functions for sugar are estimated as $Q_d = 1 - P$ and $Q_s = P$ respectively, find the equilibrium price and equilibrium quantity.
- The production cost, in Birr, of manufacturing Q units of an item is given by $C = 4Q + 200$ and the selling price is Birr 9 per unit find
 - The net contribution per unit of the item?
 - The break-even quantity
 - The break - even sales

B. Quadratic Functions

Quadratic function is a function that can be reduced to the form $f(x) = ax^2 + bx + c$, where a, b and c are fixed numbers and $a \neq 0$. A solution or root of a quadratic equation is a value of the unknown variable x which satisfies the equation, namely, a value of x which makes the equation $ax^2 + bx + c = 0$ a true statement. There are many ways of solving or finding the solution of a quadratic equation. But the most general method is using *the general quadratic formula*

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

And if the discriminant $b^2 - 4ac$ is

- greater than zero, then equation has two distinct real roots
- equal to zero, then the equation has one real root.
- less than zero, then the equation has no real root.

Example 13 Find the solution of

a) $3x^2 + 10x + 3 = 0$

b) $4x^2 + 12x + 9 = 0$

Solutions

- a. $3x^2 + 10x + 3$ since $b^2 - 4ac = 10^2 - 4 \times 3 \times 3 = 64 > 0$, the equation has two solutions and then by substituting $a = 3$, $b = 10$ and $c = 3$ in the quadratic formula, we get

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{64}}{6} = \frac{-10 \pm 8}{6} \Rightarrow X = -\frac{1}{3} \text{ or } X = -3$$

b. $4x^2 + 12x + 9 = 0$

Since $b^2 - 4ac = 12^2 - 4(4)9 = 0$, the equation has one solution and by substituting $a = 4$, $b = 12$ and

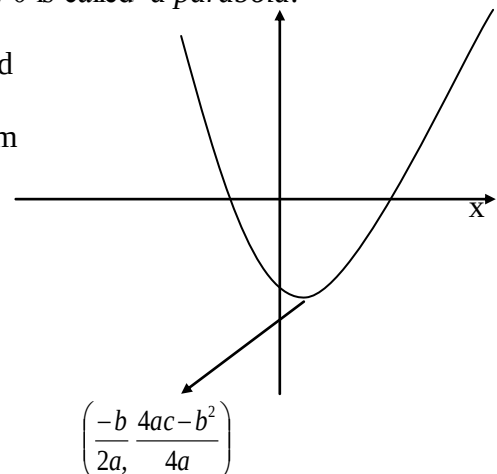
$c = 9$ in the quadratic formula we get $X = \frac{-12 \pm \sqrt{(12)^2 - 4(4)9}}{2(4)} = \frac{-12 \pm \sqrt{0}}{8} = \frac{-12}{8} = \frac{-3}{2}$

The graph of the Quadratic function $f(x) = ax^2 + bx + c$ with $a \neq 0$ is called a *parabola*.

I) If $a > 0$ (positive), then the graph is a parabola opens upward

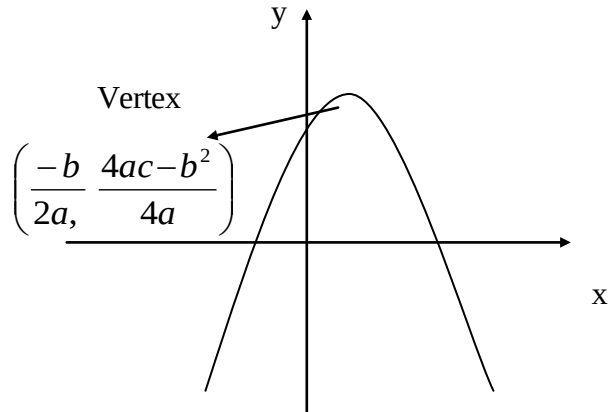
The vertex or the turning point of the parabola is the minimum

point of the graph of f and it is given by $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right)$



II) If $a < 0$ (negative) then

. The graph opens downward



Example Find the maximum or minimum value of the following quadratic functions and indicate the nature of their graph

a) $f(x) = -2x^2 - 2x + 4$ or

b) $f(x) = x^2 - 3x + 2$

Solution

a. $f(x) = -2x^2 - 2x + 4$ $a = -2$ $b = -2$ and $c = 4$

Since $a = -2 < 0$, it is a parabola opens downward and the function has a maximum value at $x = \frac{-b}{2a}$

Again since $x = \frac{-b}{2a} = \frac{-(-2)}{2(-2)} = \frac{2}{-4} = -\frac{1}{2}$, the maximum value of f is $f\left(-\frac{1}{2}\right) = -2\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) + 4 = \frac{9}{2}$

b) $f(x) = x^2 - 3x + 2$ $a = 1$, $b = -3$ and $c = 2$

Since $a = 1 > 0$, it is upward parabola and the function has a minimum value at $X = \frac{-b}{2a} = \frac{-(-3)}{2(1)} = \frac{3}{2}$

Hence, the minimum value of f is $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 = -\frac{1}{4}$

Exercise

1. Find the solutions (s), if exists, for the following

a) $x^2 + x - 2 = 0$

b) $x^2 - 2x + 1 = 0$

c) $x^2 - 4x + 5 = 0$

2. Sketch the graph of each of the following functions & determine its optimum value.

a) $Y = x^2 - 10x + 25$

b) $Y = -x^2 + 4x - 3$

Economic Applications of Quadratic Functions

Our knowledge on the quadratic functions can be used in finding the maximum or minimum value of an economic function which usually takes a quadratic form. We can also extend market equilibrium and break - even analysis for the case when there is one or more quadratic functions. In this sub-section we will illustrate these applications with the help of examples

Example

Suppose that the firm's inverse demand function for a product is given by $P = 100 - 0.1Q$

- How many units of a product the firm should produce to get maximum total revenue?
- Find the maximum possible total revenue of the firm?

Solution

- The firm's total revenue R is given by $R = PQ$ then by substituting the inverse demand equation $P = 100 - 0.1Q$ in this equation, we get $R = (100 - 0.1Q)Q = 100Q - 0.1Q^2$.

Thus the revenue R is a quadratic function in Q with $a = -0.1$, $b = 100$ and $c = 0$.

Since $a = -0.1 < 0$, R will attain its maximum at $Q = \frac{-b}{2a} = \frac{-100}{2(-0.1)} = \frac{1000}{2} = 500 \text{ units}$

Hence the firm should produce 500 units of a product to maximize its total revenue.

- The maximum R can be obtained by substituting $Q = 500$ units in the R function.

$$R(500) = 100(500) - 0.1(500)^2 = 50000 - 25,000 = 25,000$$

Example

The supply and the demand functions for a product are given by $Q_s = 2P + P^2$ and $Q_d = 40 - P^2$, where P is the price of the product in Birr. Find the equilibrium price and equilibrium quantity.

Solution

At market - equilibrium point, we have

$$Q_d = Q_s \quad 40 - P^2 = 2P + P^2 \Rightarrow 2P + P^2 = 40 - P^2 \Rightarrow 2P^2 + 2P - 40 = 0 \Rightarrow P^2 + P - 20 = 0$$

But $P^2 + P - 20 = 0$ is a quadratic equation with $a = 1$, $b = 1$ and $c = -20$, so its solutions can be found as

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(-20)}}{2(1)} = \frac{-1 \pm \sqrt{81}}{2} \quad P = \frac{-1+9}{2} \quad \text{and} \quad P = \frac{-1-9}{2}$$

$$P = \frac{8}{2} \quad \text{and} \quad P = \frac{-10}{2} \Rightarrow P = 4 \quad \text{and} \quad P = -5$$

Since there is no negative price of a product, the equilibrium price is equal to $P = 4$ birr.

To get the equilibrium quantity, we substitute $\bar{P} = 4$ in the demand equation and we get

$$\bar{Q} = 40 - \bar{P}^2 = 40 - 4^2 = 40 - 16 = 24 \text{ unit}$$

C. Exponential and Logarithmic Functions

Example: an economy is forecasted to grow continuously, so that the GNP in billions of dollars, after t years is given by $GNP = 80e^{0.02t}$. After how many years is the GNP is forecast to be Birr 88 billion?

Solution: for simplicity we cancelled billion on both sides

$$88 = 80e^{0.02t} \quad \text{----- divided both by 80}$$

$$1.1 = e^{0.02t} \quad \text{----- take log of both side}$$

$$\log 1.1 = \log e^{0.02t}$$

$$\log 1.1 = 0.02t \log e \quad \text{--- applying log rule}$$

$$\log 1.1 = 0.02t \quad \text{--- since } \log e = 1$$

$$\frac{\log 1.1}{0.02} = t \quad \text{--- dividing both by 0.02}$$

$$\frac{0.09531}{0.02} = t = 4.77 \quad \text{--- since } \log 1.1 = 0.09531$$

Therefore the economy will be 88 billion Birr after 4.77 years

Exercise

1. Supposed that the cost of producing a product is given by the cost function $c = 20 + 5Q$ and the total revenue obtained by selling the product is given by the revenue function

$R = 100Q - 0.5Q^2$ where Q is the level of a product

- Find the net revenue or profit function
- Find the level of output Q which gives maximum profit
- Find the break-even level of output

UNIT TWO

LIMITS AND CONTINUITY

2.1 The concept of limit

A. Definition of Limit

Many problems in algebra involve the behavior of functions at particular number (i.e, finding the value of the function f at $x = c$). But in various situations, the main interest is usually on the behavior of the function f near $x = c$ rather than at $x=c$. The behavior of $f(x)$ when x is close to but not equal to a particular value c will be investigated with the help of the concept of limit. Before defining the concept of limit, it may be worth full to explain it intuitively either numerically or graphically through the examples illustrated below.

Example (numerical method)

Consider the function $f(x) = \frac{x^2-4}{x-2}$. What happens to the values $f(x)$ as x gets closer and closer to 2?

Solution

One way of investigating this question is through finding the values of $f(x)$ for values of x which are very near to 2 from either side of 2 (i.e, from both left and right side of 2).

	X approaches 2 from the $\rightarrow$					2		X approaches 2 from the right				
X	1.8	1.9	1.999	1.999	$\rightarrow$	?	$\leftarrow$	2.001	2.01	2.1	2.2	
$f(x)$	3.8	3.9	3.99	3.999	$\leftarrow$		$\rightarrow$	4.001	4.01	4.1	4.2	

From this table one can easily observe the followings:

- The values of $f(x)$ approaches 4 as x approaches 2 ($x \rightarrow 2$) from either side of 2, and
- As x becomes closer enough to 2, the corresponding values of $f(x)$ will be closer enough to 4.

Statements i and ii are usually expressed mathematically by saying that "The limit of $f(x)$ as x approaches 2 is the number 4" and can be written symbolically as

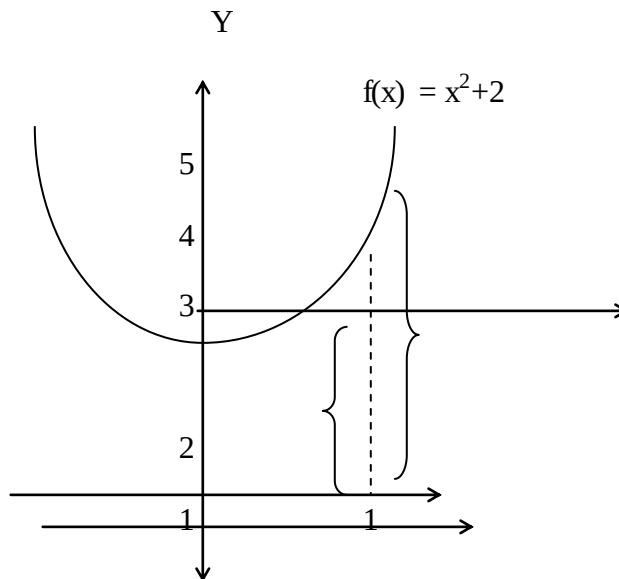
$$\lim_{x \rightarrow 2} f(x) = 4 \quad \text{or} \quad \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$$

Notice that $f(x)$ is not defined at $x = 2$ since the denominator is 0 at $x = 2$ and division by 0 is not permitted or meaning less. But $f(x)$ has a limit (=4) as x approaches 2.

Example (Graphical method)

Given $f(x) = x^2 + 2$, find $\lim_{x \rightarrow 2} f(x)$ with the help of its graph.

As you remember from section 1.7, the graph of f is a parabola opening up with vertex (0,2) as shown in the figure below.



For each number x , the functional value $f(x)$ represents the vertical distance from the x -axis to the point $(x, f(x))$ on the graph of f . If x becomes closer and closer to 1, the vertical distance will be closer and closer to 3 (the vertical distance from the x -axis to the point (1,3)). Thus, it is possible to conclude that $f(x)$ approaches 3 as x approaches 1 from both side of 1. In other words the number 3 is the limit of $f(x)$ as x approaches 1 and written as

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{or} \quad \lim_{x \rightarrow 1} f(x) x^2 + 2 = 3$$

Definition: The real number L is said to be "the limit of $f(x)$ as x approaches C " and written as $\lim_{x \rightarrow c}$

$f(x) = L$, if the functional values of a function f become closer and closer to the single real number L whenever x gets closer and closer but not equal to a real number c from both side of c .

Exercise

- What do we mean by $\lim_{x \rightarrow 2} f(x) = 5$?
- Find a reasonable estimate for each of the following limits numerically.

$$\text{a. } \lim_{x \rightarrow 1} \frac{x-2}{x^2-4} \quad \text{b. } \lim_{x \rightarrow 1} \frac{x^2-2x+1}{x-1} \quad \text{c. } \lim_{x \rightarrow -1} \frac{x^3+1}{x+1}$$

- Find each of the following limits with the help of the graph.

$$\text{a. } \lim_{x \rightarrow 2} 2x+1 \quad \text{b. } \lim_{x \rightarrow 5} (2x^2-3) \quad \text{c. } \lim_{x \rightarrow -1} x^2-8x+2$$

Properties (Rules) of Limits**Rule 1 (constant Rule)**

If $f(x) = K$ where K is a constant, then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} K = K$$

Remember that $f(x) = K$ is said to be a constant function.

Example

- a) Given $f(x) = 5$, find $\lim_{x \rightarrow 2} f(x)$ b) Evaluate $\lim_{x \rightarrow -4} 9$ c) Find $\lim_{x \rightarrow 3} b$, b is a constant

Solutions

- a) $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} 5 = 5$ b) $\lim_{x \rightarrow -4} 9 = 9$ c) $\lim_{x \rightarrow 3} b = b$ since b is a constant.

Rule -2 (Identity Rule)

If $f(x) = x$ (an identity function), then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c$

Example

- a) Given $f(x) = x$, find $\lim_{x \rightarrow 4} f(x)$? b) Evaluate i) $\lim_{x \rightarrow 2} x$ ii) $\lim_{x \rightarrow b} x$

Solutions

- a) $\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} x = 4$

b) i) $\lim_{x \rightarrow 4} x = 2$

ii) $\lim_{x \rightarrow b} x = b$

Rule -3 (Constant multiplication Rule)

If $f(x)$ is a function with $\lim_{x \rightarrow c} f(x) = L$ and k is a constant, then

$$\lim_{x \rightarrow c} Kf(x) = K \lim_{x \rightarrow c} f(x) = KL$$

Example

a) Given $f(x) = 5x$. find $\lim_{x \rightarrow 3} f(x)$?

b) Evaluate $\lim_{x \rightarrow -2} 6x$

Solutions

a) $\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} 5x$

$$= 5 \lim_{x \rightarrow 3} x \quad (\text{Rule 3})$$

$$= 5(3) \quad (\text{Rule 2})$$

$$= 15$$

b) $\lim_{x \rightarrow -2} 6x = 6 \lim_{x \rightarrow -2} x \quad (\text{Rule -3})$

$$= 6(-2) = 12 \quad (\text{Rule-2})$$

Rule-4 (Power Rule)

If $\lim_{x \rightarrow c} f(x) = L$ and n is any real number, then

$$\lim_{x \rightarrow c} [f(x)]^n = \left[\lim_{x \rightarrow c} f(x) \right]^n = L^n$$

Notice that x should be restricted to avoid even roots of negative numbers.

Example

Evaluate each of the followings

$$\text{a) } \lim_{x \rightarrow 2} x^3 \quad \text{b) } \lim_{x \rightarrow 3} \frac{1}{x^2} \quad \text{c) } \lim_{x \rightarrow -3} \sqrt[3]{x^2} \quad \text{d) } \lim_{x \rightarrow -1} f(x) \text{ given } f(x) = 6x^4$$

Solutions

$$\text{a) } \lim_{x \rightarrow 2} x^3 = \left[\lim_{x \rightarrow 2} x \right]^3 \quad (\text{Rule 4})$$

$$= (2)^3 = 8 \quad (\text{Rule 2})$$

$$\text{b) } \lim_{x \rightarrow 3} \frac{1}{x^2} = \lim_{x \rightarrow 3} x^{-2} \quad (\text{Exponent rule})$$

$$= \left[\lim_{x \rightarrow 3} x \right]^{-2} \quad (\text{Rule 4})$$

$$= [3]^{-2} \quad (\text{Rule 2})$$

$$= \frac{1}{3^2} = \frac{1}{9}$$

$$\text{c) } \lim_{x \rightarrow -3} \sqrt[3]{x^2} = \lim_{x \rightarrow -3} x^{2/3} \quad (\text{Exponent Rule})$$

$$= \left[\lim_{x \rightarrow -3} x \right]^{2/3} \quad (\text{Rule 4})$$

$$= (-3)^{2/3} \quad (\text{Rule 2})$$

$$= \sqrt[3]{(-3)^2} = \sqrt[3]{9}$$

$$\text{d) } \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} 6x^4 = 6 \lim_{x \rightarrow 1} x^4 \quad (\text{Rule 3})$$

$$= 6 \left(\lim_{x \rightarrow 1} x \right)^4 \quad (\text{Rule 4})$$

$$= 6(1)^4 \quad (\text{Rule 2})$$

$$= 6(1) = 6$$

Rule- 5 (Sum and Difference Rule)

$$\text{Given } \lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = M$$

Rule 5.1 (Sum Rule)

$$\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = L + M$$

Rule 5.2 (Difference Rule)

$$\lim_{x \rightarrow c} [f(x) - g(x)] = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x) = L - M$$

Notice that the sum and difference rules can be extended for three or more functions. For examples,

$$\lim_{x \rightarrow c} [f(x) + g(x) + h(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) + \lim_{x \rightarrow c} h(x)$$

Example

Evaluate each of the followings

$$\text{a) } \lim_{x \rightarrow 2} (x^3 - 2x + 3) \qquad \text{b) } \lim_{x \rightarrow 9} (x^2 + 2\sqrt{x}) \qquad \text{c) } \lim_{x \rightarrow 4} \sqrt{x^2} - 4$$

Solutions

$$\text{a) } \lim_{x \rightarrow 2} (x^3 - 2x + 3)$$

$$= \lim_{x \rightarrow 2} x^3 - \lim_{x \rightarrow 2} 2x + \lim_{x \rightarrow 2} 3 \quad (\text{Rule 5})$$

$$= \left(\lim_{x \rightarrow 2} x \right)^3 - 2 \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 3 \quad (\text{Rule 3 and 4})$$

$$= 2^3 - 2(2) + 3 = 7$$

$$\text{b) } \lim_{x \rightarrow 9} (x^2 + 2\sqrt{x}) = \lim_{x \rightarrow 9} x^2 + \lim_{x \rightarrow 9} 2\sqrt{x} \quad (\text{Rule 5})$$

$$= \left[\lim_{x \rightarrow 9} x \right]^2 + 2\sqrt{\lim_{x \rightarrow 9} x} \quad (\text{Rule 3 and 4})$$

$$= 9^2 + 2\sqrt{9} \quad (\text{Rule 2})$$

$$= 87$$

$$\begin{aligned}
 \text{c) } \lim_{x \rightarrow 4} \sqrt{x^2 - 4} &= \sqrt{\lim_{x \rightarrow 4} (x^2 - 4)} \quad (\text{Rule 4}) \\
 &= \sqrt{\lim_{x \rightarrow 4} x^2 - \lim_{x \rightarrow 4} 4} \quad (\text{Rule 5}) \\
 &= \sqrt{4^2 - 4} \quad (\text{Rule 1, 2 and 4}) \\
 &= \sqrt{12} = 2\sqrt{3}
 \end{aligned}$$

Rule- 6 (Product Rule)

If $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, then

$$\lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = LM$$

Notice that this product rule can be extended to three or more functions.

Example

a) Evaluate $\lim_{x \rightarrow 2} (x+1)^3 (2x-1)^2$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 2} (x+1)^3 (2x-1)^2 &= \lim_{x \rightarrow 2} (x+1)^3 \cdot \lim_{x \rightarrow 2} (2x-1)^2 \quad (\text{Rule 6}) \\
 &= \left[\lim_{x \rightarrow 2} (x+1) \right]^3 \cdot \left[\lim_{x \rightarrow 2} (2x-1) \right]^2 \quad (\text{Rule 4}) \\
 &= (2+1)^3 (2(2)-1)^2 \quad (\text{Rule 1, 2, 3 and 5}) \\
 &= 3^3 \cdot (3^2) = 243
 \end{aligned}$$

Rule- 7 (Quotient Rule)

If $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$ where $M \neq 0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{L}{M}$$

Notice that this rule works whenever the limit of the function in the denominator is different from zero (i.e $\lim_{x \rightarrow c} g(x) \neq 0$)

Example

Evaluate $\lim_{x \rightarrow 4} \frac{2x+5}{3x-7}$

Solution

$$\lim_{x \rightarrow 4} 3x-7 = 3\lim_{x \rightarrow 4} x - \lim_{x \rightarrow 4} 7 = 3(4) - 7 = 5 \neq 0.$$

So, the quotient rule can be applied since the limit of the denominator is different from zero. Thus

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{2x+5}{3x-7} &= \frac{\lim_{x \rightarrow 4} (2x+5)}{\lim_{x \rightarrow 4} (3x-7)} && \text{(Rule 7)} \\ &= \frac{\lim_{x \rightarrow 4} 2x + \lim_{x \rightarrow 4} 5}{\lim_{x \rightarrow 4} 3x - \lim_{x \rightarrow 4} 7} && \text{(Rule 5)} \\ &= \frac{2(4)+5}{2(4)-7} && \text{(Rule 1, 2, and 3)} \\ &= \frac{13}{5} \end{aligned}$$

Exercise

1. Evaluate each of the following limit

$$\text{a) } \lim_{x \rightarrow 5} (2x^2 - 3) \qquad \text{b) } \lim_{x \rightarrow 1} (x^3 - 8x + 2) \qquad \text{c) } \lim_{x \rightarrow 1} \sqrt{x^3 + 6x^2 + 7x}$$

$$\text{d) } \lim_{x \rightarrow 2} \frac{5x}{2+x^2} \qquad \text{e) } \lim_{x \rightarrow 1} (x+2)^2 (2x-4)^3$$

3. Given $\lim_{x \rightarrow 3} f(x) = 4$ and $\lim_{x \rightarrow 3} g(x) = 2$, find the following limits.

$$\begin{array}{ll}
 \lim_{x \rightarrow 3} [f(x) + g(x)] & \text{b) } \lim_{x \rightarrow 3} f(x) [g(x)]^3 \\
 \text{a) } \lim_{x \rightarrow 3} \frac{f(x)}{g(x)} & \text{d) } \lim_{x \rightarrow 3} \frac{5g(x)}{\sqrt{f(x) + g(x)}}
 \end{array}$$

B. Basic Techniques for Evaluating Limits

I. Limits of Polynomial Functions

Recall from unit one that a polynomial function is a function which can be expressed as

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

Where the coefficients $a_0, a_1, a_2, \dots, a_n$ are real numbers and n is non-negative integer.

If $f(x)$ is a polynomial function and c is any real number, then

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Example

Evaluate $\lim_{x \rightarrow -2} (5x^4 + 2x^3 + x^3 + 3x^2 - x + 1)$

Solution

Since $f(x) = 5x^4 + 2x^3 + 3x^2 - x + 5$ is a polynomial function, its limit can be obtained as

$$\lim_{x \rightarrow -2} f(x) = f(-2) = 5(-2)^4 + 2(-2)^3 + 3(-2)^2 - (-2) + 1 = 89$$

II. Limits of Rational Functions

From unit one you recall that a rational function is a quotient of two polynomial functions. In other words, $R(x)$ is a rational function if it can be expressed as

$$R(x) = \frac{P(x)}{Q(x)} \text{ where } P(x) \text{ and } Q(x) \text{ are polynomials functions and } Q(x) \neq 0.$$

Remember that the Quotient rule can be applied only if the limit of the denominator is different from zero (i.e, $\lim_{x \rightarrow c} Q(x) \neq 0$). In other words, the limit of $R(x)$ cannot be calculated by Quotient

Rule if $\lim_{x \rightarrow c} Q(x) = 0$. So, the basic question is that how can one find $\lim_{x \rightarrow c} R(x)$ if $\lim_{x \rightarrow c} Q(x) = 0$?

At this point it may be better to re-emphasize the theorem described below since it is a base to answer the above question.

2.2 The Limit Theorems

Limit theorem 1; If f and g are functions that have limits as x approaches c and $f(x) = g(x)$ for all $x \neq c$, then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x)$

You can use the above limit theorem and some algebraic methods of simplification (like factoring and canceling, rationalization and so on) to evaluate the limit of a rational function whose limit of the denominator is zero as it is illustrated in the following examples.

Example

Evaluate the following limits

$$\text{a) } \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} \qquad \text{b) } \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$$

Solutions

$$\text{a) } \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+2)}{(x-3)} = \lim_{x \rightarrow 3} (x+2) = 5$$

$$\text{b) } \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)}{(x-9)} \cdot \frac{(\sqrt{x} + 3)}{(\sqrt{x} + 3)}$$

$$= \lim_{x \rightarrow 9} \frac{(x-9)}{(x-9)(\sqrt{x} + 3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} = \frac{\lim_{x \rightarrow 9} 1}{\lim_{x \rightarrow 9} \sqrt{x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

Exercise Evaluate each of the following limits

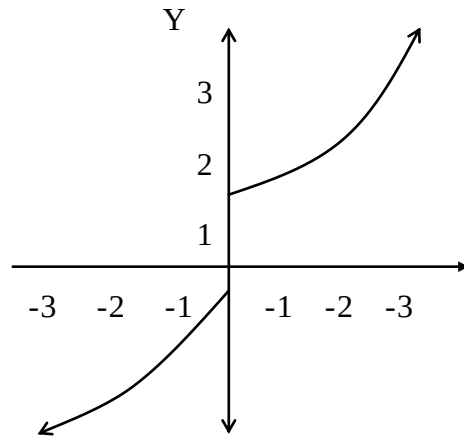
$$\text{a) } \lim_{x \rightarrow 0} \frac{x^2 - 3x}{x} \qquad \text{b) } \lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} \qquad \text{c) } \lim_{x \rightarrow -2} \frac{x^2 - x - 6}{x + 2}$$

$$\text{e) } \lim_{x \rightarrow 25} \frac{5 - \sqrt{x}}{x + 25} \qquad \text{d) } \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$$

Non Existence of Limits: Limit of a function f as x approaches c do not exist means that there is no any real number L that makes the statement " $\lim_{x \rightarrow c} f(x) = L$ " true.

One - Sided Limits

Consider the function whose graph is shown below. It is defined for all values of x except $X = 0$



Observe from the above graph of $f(x)$ that $f(x) < -1$ whenever x is negative and $f(x) > 1$ when ever x is positive or zero. So, as x takes values near 0 on both sides if 0, the corresponding values of $f(x)$ cannot get closer and closer to the same real number.

However, as you see the right half of the above graph, as x takes values closer and closer to 0 from the right side, the corresponding values of $f(x)$ get closer and closer to 1. In this case it is possible to say that "the number 1 is the limit of $f(x)$ as x approaches 0 from the right" and written as

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad (\text{Right - hand limit})$$

Similarly, from the left half of the above graph, it is clear that as x takes values closer and closer to 0 from the left side of it, the corresponding values of $f(x)$ get closer and closer to -1. In this case, it is possible to say that "the limit of $f(x)$ as approaches to 0 from the left is -1" and written as

$$\lim_{x \rightarrow 0^-} f(x) = -1 \quad (\text{Left-hand limit})$$

Definitions

Right-Hand Limit: The number L is said to be “the limit of $f(x)$ as x approaches c from the right” and written as $\lim_{x \rightarrow c^+} f(x) = L$ if the values of $f(x)$ get closer and closer to the number L whenever x takes values closer and closer (but not equal) to c on the right side of c .

Left-Hand Limit; The number L is said to be “the limit of $f(x)$ as x approaches c from the left” and written as $\lim_{x \rightarrow c^-} f(x) = L$ if the values of $f(x)$ get closer and closer to the number L whenever x takes values closer and closer (but not equal) to c on the left side of c .

Remarks

1. One side limits of a function means either its left-hand limit or right hand limit or both.
2. All rules and theorem of limit remain valid and “ $\lim$ ” is replaced either “ $\lim_{x \rightarrow c^+}$ ” or “ $\lim_{x \rightarrow c^-}$ ”.
3. One sided limits are usually important in investigating the behavior of a piece-wise defined function f near at end points for different ranges of x in its domain.

Example

Given $f(x) = \begin{cases} 1+x, & \text{if } x < 1 \\ x-1, & \text{if } x \geq 1 \end{cases}$ Find a) $\lim_{x \rightarrow 1^-} f(x)$ b) $\lim_{x \rightarrow 1^+} f(x)$

Solutions

- a. Finding $\lim_{x \rightarrow 1^-} f(x)$ means finding the limit of $f(x)$ as x approaches 1 from the left (that is, assuming $x < 1$). To do this use $1+x$ for $f(x)$. Thus, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 1+x = 1+1 = 2$
- b. To get $\lim_{x \rightarrow 1^+} f(x)$, take the expression of $f(x)$ for $x \geq 1$ from the given function ($f(x)$). In other words, use $f(x) = x-1$. Thus, $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x-1 = 1-1 = 0$

Limit Theorems 2 (Two Side)

Let f be a function and let c and L be real numbers, then $\lim_{x \rightarrow c} f(x) = L$ iff $\lim_{x \rightarrow c^-} f(x) = L = \lim_{x \rightarrow c^+} f(x)$.

Notice that from the above theorem, one can easily observe the following facts.

- i) If $\lim_{x \rightarrow c} f(x) = L$, then $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$. That is, if a function f has two-sided limit L at $x = c$, then it necessarily has L as both its left-hand and right-hand limit at $x = c$.
- ii) If $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$, then $\lim_{x \rightarrow c} f(x) = L$. That is, if a function f has the same number L as left-hand and right-hand limit at $x = c$, then the number L must be a two sided limit.
- iii) Two sided limit of a function f does not exist whenever its left-hand and right-hand limits are different. That is, $\lim_{x \rightarrow c} f(x)$ does not exist if $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$.

Infinite Limits

Infinite limits refers to the situation where the values of $f(x)$ become larger and larger without bound (∞) or become smaller and smaller without bound ($-\infty$) as x approaches c from both side of c .

Example

Find $\lim_{x \rightarrow 0} f(x)$ for $f(x) = \frac{1}{x^2}$.

Solution

By evaluating the values of $f(x)$ for some values of x near to 0 from both sides, you can construct the following table of values.

	X approaches 0 from left $\rightarrow$				$\leftarrow$ x approaches 0 from right		
X	-0.04	-0.005	-0.0001	0	0.0001	0.001	0.01
f(x)	625	40,00	100,000,000	?	100,000,000	1,000,000	10,000

As you see from the above table, as x takes values closer and closer to 0 from both sides, the corresponding values of $f(x)$ become larger and larger without bound.

Algebraically, you can guess that as x gets closer and closer to 0, x^2 also gets closer and closer to 0.

But the smaller x^2 means the larger $\frac{1}{x^2}$. Thus, as x approaches 0 from both sides, $\frac{1}{x^2}$ becomes larger and larger without bound.

Therefore, from the table and algebraic analysis, it is possible to say that “the limit of $f(x)$ as x approaches 0 is infinity” and can be written as $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$

Definition

1. If the values of $f(x)$ get larger and larger without bound as x takes values closer and closer to c from both side of c , then $\lim_{x \rightarrow c} f(x) = \infty$ and read as “the limit of $f(x)$, as x approaches C , is infinity”
2. If the values of $f(x)$ get smaller and smaller without bound as x takes values closer and closer to c from both side of c , then $\lim_{x \rightarrow c} f(x) = -\infty$ and read as “limit of $f(x)$, as x approaches C , is negative infinity,”

The limit of a function f as x approaches c (i.e, $\lim_{x \rightarrow c} f(x)$) may fail to exist if:

1. Both left-hand and right-side limits of $f(x)$ exist and have different values. or,
2. Either left-hand or right-hand limit of $f(x)$ is ∞ or $-\infty$.,
3. Both left-hand and right hand limit of $f(x)$ is ∞ or $-\infty$ so that two sided limit of $f(x)$ is ∞ or $-\infty$.,

Example

Determine whether the limit exists or not for each of the following.

$$\text{a) } \lim_{x \rightarrow 0} f(x) \text{ for } f(x) = \frac{|x|}{x} \quad \text{b) } \lim_{x \rightarrow -1} f(x) \text{ for } f(x) = \frac{1}{x+1} \quad \text{c) } \lim_{x \rightarrow 1} f(x) \text{ for } f(x) = \frac{x^2-1}{x-1}$$

Solutions

- a. Since the numerator of the function $f(x)$ is the absolute value expression $(|x|)$ which can be expressed as

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases} \quad f(x) \text{ can be further expressed as}$$

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{x} = 1, & x \geq 0 \\ \frac{x}{|x|} = \frac{x}{-x} = -1, & x < 0 \end{cases}$$

Since $f(x)$ is piecewise function, calculate the one – sided limits as

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} 1 = 1, \text{ and}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} -1 = -1$$

Thus, $\lim_{x \rightarrow 0} f(x)$ does not exist since $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$

b. As x becomes closer and closer to -1 from the left, $x + 1$ gets closer and closer to 0 from the left

$(x + 1 \rightarrow 0^-)$ and then $\frac{1}{x+1}$ gets smaller and smaller without bound. That is,

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{1}{x+1} = -\infty. \text{ Similarly, as } x \text{ becomes closer and closer } -1 \text{ from the right, } x+1$$

gets closer and closer to 0 from the right ($x+1 \rightarrow 0^+$) and then $\frac{1}{x+1}$ gets larger and larger without

bound. That is, $\lim_{x \rightarrow -1^+} f(x) = \infty$. Thus, $\lim_{x \rightarrow -1} f(x)$ does not exist.

$$\text{c. } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^-} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \rightarrow 1^-} x + 1 = 2 \text{ and}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \rightarrow 1^+} x + 1 = 2$$

Thus, $\lim_{x \rightarrow 1} f(x)$ exists and equal to 2 .

Exercise

1. Give an intuitive meaning for each of the followings.

$$\text{a) } \lim_{x \rightarrow c^-} f(x) = L \quad \text{b) } \lim_{x \rightarrow c^+} f(x) = L \quad \text{c) } \lim_{x \rightarrow c} f(x) = -\infty$$

2. Write two situations where $\lim_{x \rightarrow c} f(x)$ may fail to exist.

3. For each of the following functions find each limit if it exists.

$$\text{i. } f(x) = \begin{cases} -x + 1 & \text{if } x \leq 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases}$$

a) $\lim_{x \rightarrow 0} f(x)$

b) $\lim_{x \rightarrow 1} f(x)$

$$\text{ii) } \begin{cases} 2-x & \text{if } x \leq -1 \\ x^2 - 1 & \text{if } -1 < x < 1 \\ 3 & \text{if } x \geq 1 \end{cases}$$

a) $\lim_{x \rightarrow -1} f(x)$

b) $\lim_{x \rightarrow 0} f(x)$

c) $\lim_{x \rightarrow 1} f(x)$

4. For each of the following find the limit if it exists.

a) $\lim_{x \rightarrow 0} \frac{1}{x}$

b) $\lim_{x \rightarrow 1^+} (2x-1)$

c) $\lim_{x \rightarrow 2^-} \frac{x+2}{x-2}$

d) $\lim_{x \rightarrow 2} \frac{x+2}{x-2}$

e) $\lim_{x \rightarrow 5/2} \frac{4x^2 - 25}{2x - 5}$

f) $\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$

C. Limits at Infinity

In some types of economic functions, the main interest may be to determine the behavior the function (x) as x assumes large positive and large negative values.

Example 5-15

Consider the function $f(x) = \frac{2x^2}{1+x^2}$. What happens to the functional value of f(x) as x assumes larger and larger positive values?

Solution

To make the idea simple and clear investigate the values of f(x) for increasingly large positive values of x by constructing table of values as given below.

X assumes larger and larger +ve values							
X	0	5	10	50	100	500	1000
f(x)	0	1.92	1.98	1.9992	1.9998	1.999992	1.999998

As it can be seen from the above table, as the values of x continue to increase, the value of $f(x)$ approaches the number 2.

Definition

Furthermore, it is possible to make the value of $f(x)$ very close to 2 by taking a sufficiently large value of x . Then by using the symbol " $x \rightarrow \infty$ " to indicate that x assumes larger and larger positive values, it is possible to write the behavior of $f(x)$ as

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{2x^2}{1+x^2} = 2$$

The real number L is said to be "the limit of $f(x)$ as x approaches to ∞ " and written as $\lim_{x \rightarrow \infty} f(x) = L$ if the functional value $f(x)$ is close to the single number L whenever x is a very large positive number.

Similarly, $\lim_{x \rightarrow -\infty} f(x) = L$ if the functional value $f(x)$ is close to the single number L whenever x is a very small negative number (i.e, x is a negative number with very large absolute value).

Remarks

All properties of limits discussed above are valid (true) if the " $\lim_{x \rightarrow c}$ " replaced with either " $\lim_{x \rightarrow \infty}$ " or " $\lim_{x \rightarrow -\infty}$ ". These limits properties together with the theorem given below can be used to evaluate limits at infinity for many functions without resorting the use of tables and graphs.

Limit Theorem 3

For any positive real number P , $\lim_{x \rightarrow \infty} \frac{1}{x^P} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x^P} = 0$

Provided that x restricted to avoid even roots of negative numbers,

Example Evaluate a) $\lim_{x \rightarrow \infty} \left(\frac{x + 2x^2}{x^2} \right)$

Solutions

$$\lim_{x \rightarrow \infty} \left(\frac{x + 2x^2}{x^2} \right) = \lim_{x \rightarrow \infty} \left(\frac{x}{x^2} + \frac{2x^2}{x^2} \right) = 0 + 2 = 2$$

Remarks

1. For any polynomial function $f(x)$, $\lim_{x \rightarrow \infty} f(x) = \infty$ (or $-\infty$) and $\lim_{x \rightarrow -\infty} f(x) = \infty$ (or $-\infty$)
2. The limits at infinity for a rational function $R(x) = \frac{p(x)}{Q(x)}$ can be calculated by dividing both the numerator and denominator by the highest power of x that occurs either in the numerator or the denominator and applying the above theorem on the simplified function.

Example Evaluate each of the following limits

a) $\lim_{x \rightarrow \infty} \frac{5x+4}{2x+3}$

b) $\lim_{x \rightarrow -\infty} \frac{4x^2+3x+2}{2x^3+5}$

Solutions

- a) since the highest power of x is x_1 divide both the denominator and numerator by x as

$$\lim_{x \rightarrow \infty} \frac{5x+4}{2x+3} = \lim_{x \rightarrow \infty} \frac{\frac{5x+4}{x}}{\frac{2x+3}{x}} = \lim_{x \rightarrow \infty} \frac{5+\frac{4}{x}}{2+\frac{3}{x}} = \frac{\lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{4}{x}}{\lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} \frac{3}{x}} = \frac{5+0}{2+0} = \frac{5}{2}$$

- b) Divide both the numerator and denominator by x^3 since it is the highest power as

$$\lim_{x \rightarrow \infty} \frac{4x^2+3x+2}{2x^3+5} = \lim_{x \rightarrow \infty} \frac{\frac{4x^2+3x+2}{x^3}}{\frac{2x^3+5}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{4}{x} + \frac{3}{x^2} + \frac{2}{x^3}}{2 + \frac{5}{x^3}} = \frac{0+0+0}{2+0} = 0$$

Given a rational function $R(x)$

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \dots + b_0}, \text{ then}$$

$$\lim_{x \rightarrow \infty (-\infty)} = \begin{cases} 0 & \text{if } n < m \\ \frac{a_n}{b_n} & \text{if } n = m \\ \text{does not exist} & \text{if } n > m \end{cases}$$

Exercise

1. Write an intuitive definition for

a) $\lim_{x \rightarrow \infty} f(x) = L$

b) $\lim_{x \rightarrow \infty} f(x) = L$

2. Evaluate each of the following limits if it exists.

a) $\lim_{x \rightarrow \infty} \left(\frac{5x^2 + 3}{3x^2 + 4} \right)$

b) $\lim_{x \rightarrow \infty} \left(\frac{2x^5 + 3x}{6x^3 + 4} \right)$

c) $\lim_{x \rightarrow \infty} \left(\frac{5x^4 + 9}{8x^6 + 3} \right)$

2.3 Continuity

In the previous sections you learned the concepts of limit and different ways of evaluating limits of functions. In this section you will use the limit concept to learn an important property of functions called *continuity*. The concept of continuity and the properties of continuous functions are essentials for understanding many key concepts in calculus.

Definition of continuity (continuous function)

Intuitively speaking, the function f is said to be *continuous at* $x = c$ if you can draw the graph of f at and near the point $(c, f(c))$ without lifting your pencil from the paper. On the other hand, the function f is said to be *discontinuous (not continuous) at* $x = c$ if its graph is disconnected (i.e, if it has a break ,gap, hole or jump at $x=c$).

Identifying the points of discontinuity will be easy if the graph of a function is known in advance. But for some complicated functions it may be difficult or impossible to know their graph in advance. Thus, it is necessary to have method of identifying the points of discontinuity without looking the graph of a function.

Definition: A function f is *continuous* at the point $x = c$ if

1. $f(c)$ exists. That is, if f is defined at $x=c$

2. $\lim_{x \rightarrow \infty} f(x)$ exists. That is, $\lim_{x \rightarrow \infty} f(x)$ and

3. $\lim_{x \rightarrow \infty} F(X) = f(c) -$

Notice that if one or more of the three conditions in the definition fails, then a function is discontinuous at $x = c$.

Example

Determine the continuity of each of the following functions at the indicated value of c .

a) $f(x) = \frac{x^2 - 4}{x - 2}$ at $c = 1$ and $c = 2$ b) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ at $c = 1$

Solutions

a) Since $f(x)$ is defined for all points except $x = 2$, $f(1)$ is defined and $f(1) = \frac{1^2 - 4}{1 - 2} = 3$

Moreover, $\lim_{x \rightarrow 1} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 1} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 1} (x + 2) = 3$ and $\lim_{x \rightarrow 1} f(x) = f(1) = 3$

Thus, f is continuous at $x = 1$ since all the three conditions in the definition are satisfied.

But $f(x)$ is not continuous at $x = 2$ since $f(2)$ is not defined

b) $f(1) = 1$ and thus $f(1)$ is defined.

$\lim_{x \rightarrow 1} f(x)$ exists and equal to 1, because $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1$.

Furthermore $\lim_{x \rightarrow 1} f(x) = 1 = f(1) = 1$, the f is continuous at $x = 1$.

Remarks

1. A function f is *continuous from the right* at $x = a$ if $\lim_{x \rightarrow a^+} f(x) = f(a)$ and

A function f is *continuous from the left* at $x = b$ if $\lim_{x \rightarrow b^-} f(x) = f(b)$

2. A function f is said to be *continuous on an open interval* (a, b) if f is continuous at every number in the interval.

3. A function f is *continuous on a closed interval* $[a, b]$ if f is continuous from the right at $x = a$, continuous from the left at $x = b$, and continuous at every number in the open interval (a, b) .

For example, $f(x) = \sqrt{x-2}$ is defined for $x \geq 2$ or on the interval $[2, \infty)$. Moreover $f(x)$ is continuous from the right $x = 2$ since $f(2) = \sqrt{2-2} = 0$ and $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \sqrt{x-2} = 0 = f(2)$, and also continuous at every number on open interval $(2, \infty)$. Thus, $f(x)$ is continuous on the interval $[2, \infty)$.

Properties of continuous functions

Using the formal definition of continuity may not always be the most convenient way to determine the continuity of a function, particularly on a given interval. Thus, it is better to use the properties of continuous functions which described below in determining interval of continuity of a function. These properties are extremely useful in investigating the continuity nature for large classes of functions without looking at their graphs or the three conditions in the definition.

1. A constant function $f(x) = k$, is continuous for all value of x .
2. Polynomial functions are continuous for all values of x .
3. Rational functions are continuous for all values of x except those that make a denominator 0.
4. If $f(x)$ and $g(x)$ are both continuous functions, then
 - a. $f(x) + g(x)$ is continuous
 - b. $f(x) - g(x)$ is continuous
 - c. $f(x).g(x)$ is continuous
 - d. $\frac{f(x)}{g(x)}$, when $g(x) \neq 0$ is continuous
 - e. The absolute value of $f(x)$, (i.e, $|f(x)|$) is continuous.
5. If $f(x)$ is continuous, then $\sqrt[n]{f(x)}$ is continuous for all values of x when n is an odd positive integer greater than 1.
6. If $f(x)$ is continuous and non-negative, then $\sqrt[n]{f(x)}$ is continuous when n is an even positive integer.

Example

Determine whether each of the following function is continuous or not

$$\text{a) } f(x) = 2x^3 - x^2 + 3x + 1 \qquad \text{b) } f(x) = \frac{x}{(x+2)(x-3)}$$

$$\text{c) } f(x) = \sqrt[3]{x^2 - 4} \qquad \text{d) } f(x) = \sqrt{3 - x}$$

Solutions

a) Since f is a polynomial function, it is continuous for all values of x .

b) Since f is a rational function, it is continuous for all x except -2 and 3 (values of x that make the denominator 0). Using interval notation, f is continuous on $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$.

c) The polynomial function $x^2 - 4$ is continuous for all x . Since $n=3$ is odd, f is continuous for all x .

d) The polynomial function $3 - x$ is continuous for all x and non-negative for $x \leq 3$. Since $n=2$ is even, f is continuous for $x \leq 3$ or on the interval $(-\infty, 3]$.

Exercise: Determine whether each of the following function is continuous or not at indicated values

$$\text{a) } f(x) = \frac{x^2 - 1}{x - 1} \text{ at } c = -1 \text{ and } c = 1 \qquad \text{b) } f(x) = \begin{cases} -x & \text{if } x < 0 \\ 2 - x & \text{if } x \geq 0 \end{cases} \text{ at } c = 0$$

$$\text{c) } f(x) = \frac{x^2}{|x|} \text{ at } c = 0 \qquad \text{d) } f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x & \text{if } x > 2 \end{cases} \text{ at } x = 2$$

2.4 Economic Application of Limit and Continuous Functions

Example 1: As advancement in technology results in increasing in powerful and compact calculators, the price of calculator currently in the market decrease. Suppose that x months from now, the price of a certain model will be $P(x) = 40 + \frac{30}{x+1}$ Birr. Based on the information given calculate:

- what will be the price after 5 months
- what happen to the price in the long-run (as X become very large)

Solution:

$$a. \quad P(x) = 40 + \frac{30}{x+1} = 40 + \frac{30}{5+1} = 40 + 5 = 45$$

$$b. \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(40 + \frac{30}{x+1} \right) = \lim_{x \rightarrow \infty} 40 + \lim_{x \rightarrow \infty} \frac{30}{x+1} = 40 + 0 = 40 \text{ Birr}$$

Example 2: a manufacturer can sale a certain product for Birr 80 per unit and it incurs an overhead cost Birr 4,500 and per unit production cost of Birr 50, then

- How many units of the manufacturer product to break-even
- what is the manufacturer's profit or loss if 200 units are sold.

Solution: a. at break-even $P=AC$ i.e. $80=450/Q+50$

$$\Rightarrow 80 - 50 = \frac{450}{Q} \Rightarrow 30 = \frac{450}{Q} \Rightarrow 30Q = 450 \Rightarrow Q = 15$$

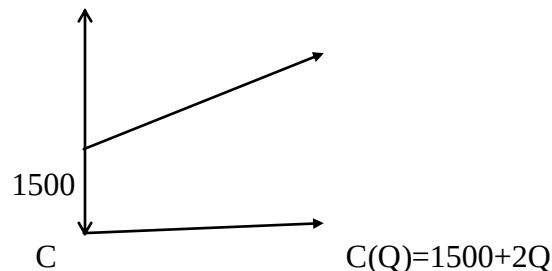
$$\begin{aligned} \lim_{Q \rightarrow 200} \pi &= \lim_{Q \rightarrow 200} (TR - TC) = \lim_{Q \rightarrow 200} (80Q - [50Q + 4500]) = \lim_{Q \rightarrow 200} 30Q - \lim_{Q \rightarrow 200} 4500 \\ &= 6000 - 4500 = 1500 \text{ Birr} \end{aligned}$$

Example 3: A manufacturer is capable of producing 500 units per day. To do so he incurred Birr 1500 per day and a variable cost of Birr 2 per unit produced then, sketch the graph of the cost function. Is the cost function continuous? If no, where does it discontinuous?

Solution:

First let us find the function of cost: $C(Q) = TFC + TVC = 1500 + 2Q$

The X-intercept at $C=0$, $Q=750$ and Y-intercept is at $C=1500$. Then the graph will be given as



As we can observe from the cost function, it is a linear function which is continuous. Therefore $C(Q)$ is continuous for all real numbers for which $0 \leq Q \leq 500$

UNIT THREE

THE DERIVATIVES

In this section you will learn how the ideas and techniques of limits developed in the previous unit can be used to define one of the fundamental concepts of calculus, the derivative, and two related concepts; rates of change and the slope of tangent line.

3.1 Rates of Change and Slope of Tangent Line

I. Rates of Change

Let a function be defined as $y = f(x)$, if the independent variable x changes from x_0 to x_1 , then the dependent variable y will change from $y_0 = f(x_0)$ to $y_1 = f(x_1)$. In this case,

- The change in x is said to be *an increment in x* and denoted as Δx (read as “delta x ”). That is,, an increment in x from x_0 to x_1 can be represented as $\Delta x = x_1 - x_0$ or $x_1 = x_0 + \Delta x$.
- The corresponding change in y is said to be *an increment in y* and denoted by Δy . That is, an increment in y from y_0 to y_1 can be represented as $\Delta y = y_1 - y_0 = f(x_1) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$

Definitions

Given $y = f(x)$

1. The average rate of change (ARC) of y with respect to x is the change in y resulted from some change in x . the average rate of change of y between $x=x_0$ and $x=x_1$ is given by

$$\text{ARC} = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

It is also termed as *the difference quotient* of the function

2. *The instantaneous rate of change* of y with respect to x is a change in y as a result of an instant change in x . It represents a change in y at a given point of x and given by .

$$\text{Instantaneous rate} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

It is also termed as *marginal rate of change* of y at $x = x_0$

Example

Given the function $y = f(x) = 3x^2$. Find

- Average rate of change of y for x changes from 1 to 4
- Instantaneous rate of change of y at $x = 1$

Solutions

- $x_0 = 1$ and $x_1 = 4$, and thus $\Delta x = x_1 - x_0 = 4 - 1 = 3$

Moreover, $\Delta y = y_1 - y_0 = f(x_1) - f(x_0) = f(4) - f(1) = 3(4^2) - 3(1^2) = 48 - 3 = 45$

Thus, average rate of change $= \frac{\Delta y}{\Delta x} = \frac{45}{3} = 15$

- by making $x_0 = 1$ calculate the instantaneous rate of change as

$$\begin{aligned}
 & \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3(1 + \Delta x)^2 - 3(1)^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3[1 + 2\Delta x + (\Delta x)^2] - 3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{6\Delta x + 3(\Delta x)^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} (6 + \Delta x) \\
 &= \lim_{\Delta x \rightarrow 0} (6 + \Delta x) = \lim_{\Delta x \rightarrow 0} 6 + \lim_{\Delta x \rightarrow 0} \Delta x = 6 + 3(0) = 6
 \end{aligned}$$

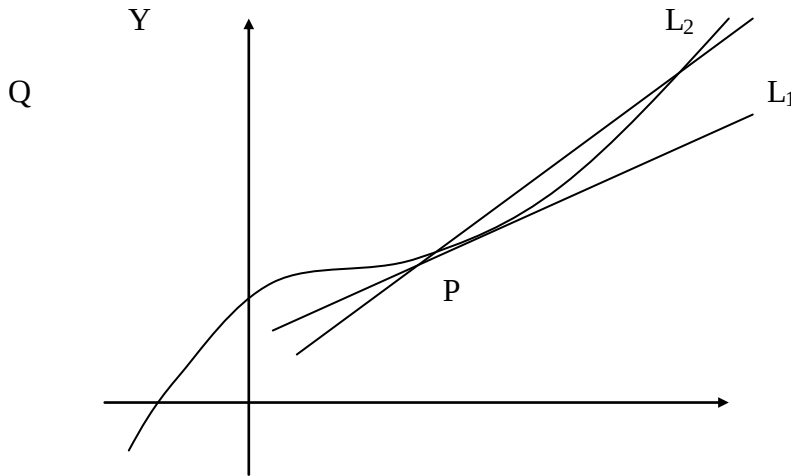
II. Slope of a Tangent line

Suppose the figure given below be the graph of $y = f(x)$

A secant line is a straight line which passes through two points on the graph of $y = f(x)$. For instance, line L_2 which passes through P and Q in the above graph is a secant line. The slope of the secant line which passes through the point $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

A **tangent line** is a straight line which touches (but not crosses) the graph of $y=f(x)$ at a given point. For instances, line L_1 in the above graph is a tangent line.



As it can be seen from the above graph, as Δx becomes closer and closer to 0 (as $\Delta x \rightarrow 0$), the point Q approaches to point P. In other words, as $\Delta x \rightarrow 0$ the secant line L_2 will approach to line L_1 and its slope will approach to the slope of L_1 . This leads to the following formal definition of a tangent line.

Definition

Given the graph of $Y = f(x)$, the tangent line at $(x_0, f(x_0))$ is a line that passes through this point whose slope is given by tangent line slope = $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ if the limit exists.

Remarks

1. The slope of the tangent line at $(x_0, f(x_0))$ is always equal to the slope of the graph or the function $y = f(x)$ at $(x_0, f(x_0))$.
2. If $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 0$, then the graph of f has a horizontal tangent line at $(x_0, f(x_0))$.
3. If $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \infty$ (or $-\infty$), then graph of f has a vertical tangent line at $(x_0, f(x_0))$.

Exercise

1. Given $y=f(x)$ write the formula to calculate

- a) average rate of change of y with respect to x ,
- b) instantaneous rate of change of y with respect to x

2. Find each of the following for $y = f(x) = x^2 + x$

- a) the average rate of change of y for x changes from 1 to 3
- b) the simplified form of the average rate of change of y for changes from x to $x + \Delta x$.
- c) the slope of the secant line joining $(1, f(1))$ and $(3, f(3))$

3.2 Definition of the Derivative

In the previous subsection, you learned the special limit $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ if it exists, gives the slope of the tangent line to the graph of $y = f(x)$ at $(x_0, f(x_0))$. It also gives the instantaneous rate of change of y at $x = x_0$

Definition Given a function $Y = f(x)$, the derivative of the function f at x , denoted as $f'(x)$ or $\frac{dy}{dx}$, is

defined as $\frac{dy}{dx}$ or $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

Remarks

1. $f'(x)$ is read as “ f prime of x ” and $\frac{dy}{dx}$ is read as “the derivative of y with respect to x ”

Both $f'(x)$ and $\frac{dy}{dx}$ will be used interchangeably to denote the derivative of function f at x .

2. The derivative of a function f at x , $f'(x)$, is a function known as a *derived function*.

The original function $f(x)$ is known as *primitive function*.

3. The process of finding the derivative of a function f at x is called *differentiation*.

That is, $f'(x)$, is obtained by differentiating the primitive function $f(x)$.

Example

Find, $f'(x)$, for $f(x) = 4x - x^2$

Solution

Using the above definition, is obtained as

$$\frac{dy}{dx} \text{ or } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

To make the computation easier, you can use the following two-step process.

Step1 find $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$ and simplify.

$$\begin{aligned} & \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \frac{[4(x + \Delta x) - (x + \Delta x)^2] - (4x - x^2)}{\Delta x} \\ &= \frac{4x + 4\Delta x - x^2 - 2x\Delta x - (\Delta x)^2 - 4x + x^2}{\Delta x} \\ &= \frac{4\Delta x - 2x\Delta x - (\Delta x)^2}{\Delta x} \\ &= \frac{\Delta x}{\Delta x} (4 - 2x - \Delta x) = 4 - 2x - \Delta x \end{aligned}$$

Step 2 find the limit of the simplified result in steps to get $f'(x)$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (4 - 2x - \Delta x) \\ &= 4 - 2x \end{aligned}$$

If our interest is to find the derivative of $y=f(x)$ at a particular point (say at $x = x_0$), denoted as $f'(x_0)$ you can do either of the following

- i) First find $f'(x_0)$, the derivative of f at x , using the above definition and then substitute x_0 in place of x in the resulted expression of $f'(x)$, or

ii) Find $f'(x_0)$, directly as $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ if the limit exists.

Notice that the limit expression for $f'(x_0)$ is the same as the one that is used to get the slope of the tangent line to the graph of f at $(x_0, f(x_0))$ and the instantaneous rate of change of y at $x = x_0$. Thus, the relationship between the three concepts can be summarized as follows.

1) $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ if the limit exists, measures

- The slope of the tangent line to the graph of $f(x)$ at $(x_0, f(x_0))$, or
- The instantaneous rate of change of y at $x = x_0$
- The instantaneous rate of change of y at $x = x_0$

2) $\frac{dy}{dx}$ or $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ if the limit exists, measure

- The slope of the tangent line to the graph of f at any point on the graph or
- The slope of the graph of f at any point of $(x, f(x))$, or
- The instantaneous rate of change of y at any point x in its domain.

Example

Given $f(x) = x^2 + 3$, find

- a) $f'(2)$ b) the slope of the tangent line at $x = 2$

Solutions

a) $f'(2)$ can be obtained as

$$\begin{aligned}
 f'(2) &= \lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(2 + \Delta x)^2 + 3] - (2^2 + 3)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[4 + 4\Delta x + (\Delta x)^2 + 3] - 7}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{4\Delta x + (\Delta x)^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} (4 + \Delta x) \\
 &= \lim_{\Delta x \rightarrow 0} 4 + \Delta x = \lim_{\Delta x \rightarrow 0} 4 + \lim_{\Delta x \rightarrow 0} \Delta x = 4
 \end{aligned}$$

b) The slope of the tangent line at $x=2$ can be measured by $f'(2)$ and $f'(2) = 4$ (see part a)

Thus, the slope of the tangent line at $x=2$ is 4.

Exercise

1- Write the formula to find the derivative of $y = f(x)$

a) at $x = x_0$

b) at any point x in its domain.

2- Find $f'(2)$ for each of the following functions by using the definition of the derivative

a) $f(x) = 2x - 3$

b) $f(x) = 6x - x^2$

c) $f(x) = \sqrt{x} - 3$

d) $f(x) = \frac{1}{x}$

3- Given $y = f(x) = 8x - 2x^2$, find

a) the slopes of the graph of f at $x = 1$ and $x = 2$

b) the equations of the tangent lines at $x = 1$

3.3 Continuity and Differentiability of a Function

In the previews section, you learned how the derivative of a function $y = f(x)$ can be calculated at a single point of x , specifically, you learned that the derivative of f at $x=x_0$ can be obtained as

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \text{ if the limit exists.}$$

However, there are many situations when this limit can fail to exist and thus the function f do not have a derivative at $x=x_0$. In this section you will see some common situations where the above limit for $f'(x_0)$ can fail to exist.

Given $y=f(x)$ if $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists, then the function f is said to be *differentiable at $x = x_0$* or $f'(x_0)$ exists. If the limit does not exist, then f is said to be *non-differentiable at $x = x_0$* or $f'(x_0)$ does not exist.

Conditions for Non Differentiability

I. a function f is not differentiable at $x=x_0$ if f is not continuous at $x=x_0$. In other words, to be differentiable at $x= x_0$, a function must be continuous at $x=x_0$.

Example

Given $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ 4-x & \text{if } x \geq 2 \end{cases}$, determine whether f is differentiable or not at $x=2$.

Solution

First check whether or not the three conditions in the continuity definition are satisfied.

$f(2)=4-2=2$, thus $f(2)$ is defined.

$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 2x = 4$ and $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 4-x = 2$. this means that f is not continuous at $x=2$,

and thus f is not differentiable at $x=2$.

Notice that if the function f is differentiable at $x = x_0$, then f must be continuous at $x=x_0$. But the converse is not always true. That is, there are functions which are continuous at $x = x_0$ but are not differentiable at $x=x_0$.

II. The function f is not differentiable at $x=x_0$ if

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \neq \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Example

Determine whether the function $f(x) = |x|$ is differentiable or not at $x=0$.

Solution

From the definition of absolute value function $|x|$, $f(x)$ can be rewritten as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Then evaluate the one-sided limits as follows

$$\begin{aligned} \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &= \lim_{\Delta x \rightarrow 0^-} \frac{-(x + \Delta x) - (-x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{-x - \Delta x + x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} -1 = -1 \text{ and} \end{aligned}$$

$$\begin{aligned} \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &= \lim_{\Delta x \rightarrow 0^+} \frac{(x + \Delta x) - x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^+} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} 1 = 1 \end{aligned}$$

Thus, f is not differentiable at $x=0$ since the left-hand and right-hand limits are not the same.

But you can check that $f(x) = |x|$ is continuous at $x=0$ and it can be considered as an example to show that continuity does not guarantee differentiability.

Notice also that the graph of $f(x) = |x|$ has a *sharp corner* at $x=x_0$. It is generally true that if the graph of f has a *sharp corner* at $x = x_0$, then f is not differentiable at $x=x_0$.

III. The function f is not differentiable at $x=x_0$ if

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x)}{\Delta x} = \infty \text{ or } \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x)}{\Delta x} = -\infty$$

Geometrically, $f(x)$ is not differentiable at $x=x_0$ if graph of $f(x)$ has vertical tangent line at $x=x_0$.

Remarks

1. A function f is said to be *differentiable on the interval (a, b)* if it is differentiable at each point between a and b .
2. If a function f is differentiable at each number in its domain, then f is said to be a *differentiable function*. Thus *the graph of a differentiable function is a continuous curve with no sharp corners and no vertical tangent lines*.

Exercise

1. Write the three possible situations where a function f is non differentiable at $x=x_0$.
2. Explain why continuity of a function at $x=x_0$ is a necessary but not a sufficient condition for differentiability of f at $x=x_0$.
3. For each of the following function determine whether f is differentiable or not at indicated value of x

$$a) f(x) = \frac{x+1}{x-2}, \text{ at } x=2$$

$$b) f(x) = \begin{cases} 2x & \text{if } x < 2 \\ 6x & \text{if } x \geq 2 \end{cases}, \text{ at } x=2$$

$$c) f(x) = x + |x| \text{ at } x=0$$

$$d) f(x) = x^{-1/3} \text{ at } x=0$$

3.4 Rules of Differentiation

Given $y=f(x)$, then $f'(x)$, $\frac{dy}{dx}$, or y' all represents the derivative of f at x .

Rule 1 Constant Function Rule

If $y=f(x) = K$, where k is any constant, then $f'(x) = 0$ or $\frac{dy}{dx} = 0$

That is, the derivative of a constant or a constant function is zero.

Rule 2 The Power Rule

If $y = f(x) = x^n$, where n is any real number, then $f'(x)$ or $\frac{dy}{dx} = nx^{n-1}$

Example

Find the derivative of the following functions

a) $f(x) = x^3$

b) $f(x) = \frac{1}{x^2}$

c) $f(x) = \sqrt{x}$

Solutions

a) for $f(x) = x^3$, $f'(x) = 3x^{3-1} = 3x^2$

b) $f(x) = \frac{1}{x^2} = x^{-2}$ (by exponent rule) then, $f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$

c) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ (by exponent rule), then $f'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1}$
 $= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

Rule - 3 Constant Times a function Rule

If $g(x) = K f(x)$ where k is any constant, then $g'(x) = kf'(x)$

That is, the derivative of constant times a function is a constant times a derivative of a function.

Example

Find the derivative of $f(x) = 3x^2$.

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^2) = 3 \frac{d}{dx}(x^2) \quad (\text{Rule 3}) \\ &= 3(2x^{2-1}) \quad (\text{Rule 1}) \\ &= 2(2x) \\ &= 6x \end{aligned}$$

Rule - 4 the Sum and Difference Rule

Given two functions $f(x)$ and $g(x)$ with both $f'(x)$ and $g'(x)$ exist,

If $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$, and

If $h(x) = f(x) - g(x)$ then $h'(x) = f'(x) - g'(x)$

In other words, the derivative of a sum (or difference) of two differentiable functions is the sum (or difference) of the individual function derivatives. That is,

$$\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) = f'(x) \pm g'(x)$$

Example

Find the derivatives of the following functions

$$\text{a) } f(x) = 3x^2 + 2x \quad \text{b) } y = \sqrt[3]{x} - 3x^2$$

Solutions

$$\begin{aligned} \text{a) } f'(x) &= \frac{d}{dx}(3x^2 + x) \\ &= \frac{d}{dx}(3x^2) + \frac{d}{dx}(2x) \quad (\text{Rule-4}) \\ &= 3 \frac{d}{dx}(x^2) + 2 \frac{d}{dx}(x) \quad (\text{Rule-3}) \\ &= 3(2x^{2-1}) + 2(1x^{1-1}) \quad (\text{Rule-2}) \\ &= 3(x) + 2(x^0) \\ &= 6x + 2 \quad (\text{since } x^0 = 1) \end{aligned}$$

$$\begin{aligned} \text{b) } f'(x) &= \frac{d}{dx}(\sqrt[3]{x} - 3x^2) \\ &= \frac{d}{dx}\sqrt[3]{x} - \frac{d}{dx}(3x^2) \quad (\text{Rule-4}) \\ &= \frac{d}{dx}(x^{\frac{1}{3}}) - 3 \frac{d}{dx}(x^2) \quad (\text{Rule-3}) \\ &= \frac{1}{3}x^{\frac{1}{3}-1} - 3(2x^{2-1}) \quad (\text{Rule-2}) \\ &= \frac{1}{3}x^{-\frac{2}{3}} - 3(2x) \\ &= \frac{1}{3\sqrt[3]{x^2}} - 6x \end{aligned}$$

Remarks; The sum and difference rule can be generalized for any three or more differentiable functions. For instances, for four differentiable functions $f(x)$, $g(x)$, $r(x)$ and $s(x)$, the sum and difference rule can be extended as

$$\frac{d}{dx}(f(x) \pm g(x) \pm r(x) \pm s(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) \pm \frac{d}{dx} r(x) \pm \frac{d}{dx} s(x)$$

Example

Find the derivatives of the following functions

$$a) f(x) = 3x^4 - 2x^3 + x^2 - 5x + 7$$

$$b) y = \frac{5}{3x^2} - \frac{2}{x^4} + \frac{x^2}{9}$$

Solutions

$$\begin{aligned} a) \frac{dy}{dx} &= \frac{d}{dx} (3x^4 - 2x^3 + x^2 - 5x + 7) \\ &= \frac{d}{dx} (3x^4) - \frac{d}{dx} (2x^3) + \frac{d}{dx} x^2 - \frac{d}{dx} (5x) + \frac{d}{dx} (7) && \text{(Rule - 4)} \\ &= 3 \frac{d}{dx} x^4 - 2 \frac{d}{dx} x^3 + \frac{d}{dx} x^2 - 5 \frac{d}{dx} (x) + \frac{d}{dx} (7) && \text{(Rule - 3)} \\ &= 3(4x^3) - 2(3x^2) + 2x - 5(1) + 0 && \text{(Rules 1 and 2)} \\ &= 12x^3 - 6x^2 + 2x - 5 \end{aligned}$$

$$\begin{aligned} b) \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{5}{3x^2} - \frac{2}{x^4} + \frac{x^2}{9} \right) \\ &= \frac{d}{dx} \left(\frac{5x^{-2}}{3} - 2x^{-4} + \frac{x^2}{9} \right) && \text{(ExponentRule)} \\ &= \frac{d}{dx} \left(\frac{5}{3} x^{-2} \right) - \frac{d}{dx} (2x^{-4}) + \frac{d}{dx} \left(\frac{x^2}{9} \right) && \text{(Rule - 4)} \\ &= \frac{5}{3} \frac{d}{dx} (x^{-2}) - 2 \frac{d}{dx} (x^{-4}) + \frac{1}{9} \frac{d}{dx} (x^2) && \text{(Rule - 3)} \\ &= \frac{5}{3} (-2x^{-3}) - 2(-4x^{-5}) + \frac{1}{9} (2x) && \text{(Rule - 2)} \\ &= \frac{10}{3x^3} + \frac{8}{x^5} + \frac{2x}{9} \end{aligned}$$

Remark; The value of the derivative of $y = f(x)$ at any specified value of x (say at $x=a$), denoted as

$f'(a)$ or $\left. \frac{dy}{dx} \right|_{x=a}$ can be obtained through the following steps.

Step. 1. Find $f'(a)$ or $\left. \frac{dy}{dx} \right|_{x=a}$ by applying the necessary rules of differential.

Step. 2 evaluate $f'(a)$ by substituting a in place of x in the result of $f'(a)$ or $\left. \frac{dy}{dx} \right|_{x=a}$ from step 1

Exercise

1. Find the derivatives of the following functions by using the above rules of differentiations.

$$\begin{array}{lll} \text{a) } f(x) = -\sqrt{3} & \text{b) } f(x) = x & \text{c) } Y = \frac{-4}{x^5} \\ \text{d) } f(x) = 3x^{1/3} - 5\sqrt{x} & \text{e) } Y = 5x^4 + 6x - 3 & \end{array}$$

2. Find $f'(a)$ for

$$\text{a) } f(x) = 3x^{-4} \text{ at } a = 8 \quad \text{b) } f(x) = 5 \text{ at } a = 1 \quad \text{c) } f(x) = 2x^3 - 3\sqrt{x} + 4 \text{ at } a = 2$$

3. Give $f(x) = x^3 - 3^2 + 2$, then find

a) the slope of $f(x)$ at $x = 2$

b) the equation of the tangent line to the graph at $x = 2$

Rule-5 The Product Rule

If both the derivatives of $f(x)$, $f'(x)$ and the derivative of $g(x)$, $g'(x)$, exist, then

$$\begin{aligned} \frac{d}{dx} [f(x) \cdot g(x)] &= f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x) \\ &= f(x) \cdot g'(x) + g(x) f'(x) \end{aligned}$$

That is, the derivative of the product of two differentiable functions, $f(x)$ and $g(x)$, is equal to the first function, $f(x)$, times the derivative of the second function, $g'(x)$, plus the second function, $g(x)$, times the derivative of the first function $f'(x)$.

Example

Find the derivative for a) $y = 2x^2 (3x^4 - 2)$ b) $f(x) = (x^2 + 4)(5x^4 - 5)$

Solution

$$\text{Let } f(x) = 2x^2 \text{ and } g(x) = 3x^4 - 2 \text{ then } f'(x) = 4x \text{ and } g'(x) = 12x^3$$

To get $\frac{dy}{dx}$ use Rule-5 and substitute the above values as

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[f(x).g(x)] = f'(x)g(x) + g'(x)f(x) = 2x^2(12x^3) + (3x^4-2)(4x) \\ &= 24x^5 + 12x^5 - 8x = 36x^5 - 8x\end{aligned}$$

Notice that you can also differentiate $y = 2x^2(3x^4-2)$ by first multiplying the two expressions and then applying the previous rules on the simplified single expression. For instances,

$$Y = 2x^2(3x^4-2) = 6x^6 - 4x^2$$

$$\begin{aligned}\text{then } \frac{dy}{dx} &= \frac{d}{dx}(6x^6 - 4x^2) \\ &= 6\frac{d}{dx}x^6 - 4\frac{d}{dx}(x^2) \quad (\text{Rule 3 and 4}) \\ &= 6(6x^5) - 4(2x) \quad (\text{Rule-2}) \\ &= 36x^5 - 8x\end{aligned}$$

b) Let $y = f(x) = (x^2+4)(5x^4-5)$, then

$$\begin{aligned}\frac{dy}{dx} &= f'(x) = \frac{d}{dx}[(x^2+4)(5x^4-5)] \\ &= (x^2+4)\frac{d}{dx}(5x^4-5) + (5x^4-5)\frac{d}{dx}(x^2+4) \quad (\text{Rule-5}) \\ &= (x^2+4)(20x^3) + (5x^4-5)(2x) \quad (\text{Rule 1, 2, 3 and 4}) \\ &= 20x^5 + 80x^3 + 10x^5 - 10x \\ &= 30x^5 + 80x^3 - 10x\end{aligned}$$

Remark

The product rule can be extended for more than two differentiable functions. For instances, if

$$y = f(x).g(x).h(x), \text{ then } \frac{dy}{dx} = f'(x)g(x)h(x) + g'(x)f(x)h(x) + h'(x)f(x).g(x)$$

Rule - 6 The Quotient Rule

If $y = r(x) = \frac{f(x)}{g(x)}$ with $g(x) \neq 0$ and both $f'(x)$ and $g'(x)$ exist, then

$$\begin{aligned}
 r'(x) &= \frac{dy}{dx} = \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] \\
 &= \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \cdot \frac{d}{dx} g(x)}{[g(x)]^2} \\
 &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}
 \end{aligned}$$

Notice that since subtraction is not commutative, interchanging the numerator in the quotient rule as $f(x) \cdot g'(x) - f'(x) \cdot g(x)$ will lead to a wrong result.

Example

Find the derivatives for the following functions

$$a) y = \frac{x^2}{2x-1} \qquad b) f(x) = \frac{x^2 - x}{x^3 + 1}$$

Solutions

a) Let $f(x) = x^2$ and $g(x) = 2x-1$, here $f'(x) = 2x$ and $g'(x) = 2$ then

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
 &= \frac{(2x)(2x-1) - x^2(2)}{(2x-1)^2} \\
 &= \frac{4x^2 - 2x - 2x^2}{(2x-1)^2} \\
 &= \frac{2x^2 - 2x}{(2x-1)^2}
 \end{aligned}$$

b) Let $y = f(x) = \frac{x^2 - x}{x^3 + 1}$, then

$$\frac{dy}{dx} = \frac{(x^3 + 1) \frac{d}{dx}(x^2 - x) - \left[(x^2 - x) \frac{d}{dx}(x^3 + 1) \right]}{(x^3 + 1)^2}$$

$$\begin{aligned}
&= \frac{(x^3+1)(2x-1) - [(x^2-x)(3x^2)]}{(x^3+1)^2} \\
&= \frac{2x^4 - x^3 + 2x - 1 - (3x^4 - 3x^3)}{(x^3+1)^2} \\
&= \frac{-x^4 + 2x^3 + 2x - 1}{(x^3+1)^2}
\end{aligned}$$

Notice that it may sometimes advantageous to check whether or not an expression can be simplified algebraically before blindly using the quotient rule.

Example

Find $f'(x)$ for $f(x) = \frac{2x^4 - x^3}{x^2} = \frac{2x^4}{x^2} - \frac{x^3}{x^2} = 2x^2 - x$

Solution

Apply the necessary rules of differentiation to get $f'(x)$ as $f'(x) = \frac{d}{dx}(2x^2 - x) = 4x - 1$

Exercise

1. Differentiate the following functions by using product rule or quotient rule

a) $f(x) = 2x^3(x^2 - 2)$ b) $f(x) = (3x+2)(4x-5)$ c) $f(x) = (x^2+1)(5x-1)(3x^2)$ d) $f(x) = \frac{x^2+1}{2x-3}$

2. Find and simplify each of the followings

a) $f'(x)$ for $f(x) = (2x+1)(x^2-3x)$

b) $f'(2)$ for $f(x) = \frac{x}{x^2+1}$

c) $\left. \frac{dy}{dx} \right|_{x=0}$ for $Y = \frac{x}{x^2+1}$

Rule 7: The Inverse Function Rule

Definition

The function $y=f(x)$ has an inverse provided that there is a function g such that the domain of g is the range of f and $y=f(x)$ if and only if $x = f(y)$. If $y=f(x)$ has an inverse function, then its inverse function is denoted as $f^{-1}(x)$ (read as “f inverse”).

The interest of this sub section is to find $\frac{dx}{dy}$ for $y=f(x)$. That is, finding the derivative of the inverse function for $y = f(x)$. One possible way of getting $\frac{dx}{dy}$ for $y = f(x)$ is that first finding a formula for the inverse function, $f^{-1}(x)$, and then computing its derivative. But finding a formula for $f^{-1}(x)$ may be tedious or impossible for many types of functions. Thus, for such and other cases you can use the following general rule to find the derivative of inverse function.

If $y = f(x)$ has the inverse function, $f^{-1}(x)$, and $f^{-1}(x) \neq 0$, then $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

The derivative of inverse function is the reciprocal of the derivative of the original function.

Example

Find $\frac{dx}{dy}$ for $y=5x^2+4x^3$

Solution

Since $\frac{dy}{dx} = 10x + 12x^2$, $\frac{dx}{dy}$ can be obtained as

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{10x + 12x^2}$$

3.5 Derivative of Logarithmic and Exponential Function Rule

3.5.1 Derivative of Logarithmic Function Rule

The function $y = f(x) = \log_a^x$ with $a > 0$ and $a \neq 1$ is said to be the *logarithmic function*. The number a is called the base of the logarithm

$$\text{If } y = f(x) = \log_a^x m, \text{ then } \frac{dy}{dx} = f'(x) = \frac{1}{x} \log_a^e$$

Where e is an irrational number approximately equals to 2.718281

$$\text{For instances, If } y = \log_2^x \text{ then } \frac{dy}{dx} = f'(x) = \frac{1}{x} \log_2^e$$

Natural Logarithmic function Rule

The logarithmic function with base e , i.e., $f(x) = \log_e^x = \ln x$ is called *natural logarithm function*.

If $y = f(x) = \ln x$, then

$$\begin{aligned} \frac{dy}{dx} &= f'(x) = \frac{d}{dx}(\ln x) \\ &= \frac{d}{dx}(\log_e^x) \\ &= \frac{1}{x} \log_e^e \\ &= \frac{1}{x} \text{ since } \log_e^e = 1 \end{aligned}$$

$$\text{Thus, } \frac{d(\ln x)}{dx} = \frac{1}{x}$$

Example

Find $f'(x)$ for $f(x) = 2x^4 + 3\ln x$

Solution

Let $Y = f(x) = 2x^4 + 3\ln x$, then

$$\begin{aligned}
 f'(x) &= \frac{dy}{dx} = \frac{d}{dx}(2x^4 + 3\ln x) \\
 &= 2 \frac{d}{dx}(x^4) + 3 \frac{d}{dx}(\ln x) \\
 &= 8x^3 + 3\left(\frac{1}{x}\right) = 8x^3 + \frac{3}{x}
 \end{aligned}$$

3.5.2 Derivative of Exponential Function Rule

A function $y=f(x) = a^x$ with $a>0$, and $a \neq 1$ is said to be *an exponential function* with base a

If $y=f(x)=a^x$, then

$$f'(x) = \frac{dy}{dx} = a^x \log_e a = a^x \ln a$$

For example,

$$\text{If } y=3^x, \text{ then } \frac{dy}{dx} = 3^x \ln 3$$

Natural Exponential Function Rule

The exponential function with base e , i.e, $f(x) = e^x$ is said to be *natural exponential function*.

If $y= f(x) = e^x$, then

$$\begin{aligned}
 \frac{dy}{dx} &= f'(x) = e^x \ln e \\
 &= e^x \log_e e \\
 &= e^x
 \end{aligned}$$

$$\text{Thus, } \frac{d}{dx}(e^x) = e^x$$

Example 6-17

Given $y = x^3 + 2e^x - 5x$, find $\left. \frac{dy}{dx} \right|_{x=a}$

solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^3 + 2e^x - 5x) \\ &= \frac{d}{dx}(x^3) + \frac{d}{dx}(2e^x) - \frac{d}{dx}(5x) \\ &= 3x^2 + 2\frac{d}{dx}(e^x) - 5 \\ &= 3x^2 + 2e^x - 5 \end{aligned}$$

$$\text{Thys, } \left. \frac{dy}{dx} \right|_{x=0} = 3(0)^2 + 2e^0 - 5 = 2$$

Exercise

1-Differentiate each of the following functions.

$$a) f(x) = \log_5 x \quad b) f(x) = 4(3^x) \quad c) y = \frac{2+e^x}{1-2x}$$

3.6 Higher order Derivatives

In the previous sections, you learned the definition and different ways of computing $f'(x)$ called the *first derivative* for $y=f(x)$. In this section, you will learn the concept of second derivative, third derivative and so on.

The derivative of a function $y = f(x)$, denoted as $f'(x)$ or $\frac{dy}{dx}$, is said to be the *first derivative* of the function f .

Since the first derivative, $f'(x)$, is a function, you can find its derivative. The derivative of the first derivative is said to be the *second derivative* of the function f and denoted by

$$f''(x) \text{ or } \frac{d^2y}{dx^2}$$

Since the second derivative is again a function, you can find its derivative. The derivative of the second derivative is said to be *the third derivative* of the function f and denoted by

$f'''(x)$ or $f^3(x)$ or $\frac{d^3y}{dx^3}$ to indicate that $f(x)$ has differentiated three times with respect to x .

This process may continue to get the fourth derivative denoted as $f^4(x)$ or $\frac{d^4y}{dx^4}$, the fifth derivative, denoted as $f^5(x)$ or $\frac{d^5y}{dx^5}$ and so on.

Generally, the n th-derivative of the function f is denoted by $f^n(x)$ or $\frac{d^n y}{dx^n}$ to indicate that this function is found by n successive operations of differentiations, starting with the function f

Example

Find the first, second and third derivative of the function $f(x) = 2x^3 - 4x^2 + 6x + 5$

Solution

The first derivative is $f'(x) = 6x^2 - 8x + 6$,

The second derivative is $f''(x) = \frac{d}{dx}(6x^2 - 8x + 6) = 12x - 8$,

The third derivative is $f'''(x) = \frac{d}{dx}(12x - 8) = 12$

Example

Find the first five derivatives of $f(x) = 2x^4 - 6$

$$f'(x) = 8x^3$$

$$f''(x) = 24x^2$$

$$f'''(x) = 48x$$

$$f^4(x) = 48$$

$$f^5(x) = 0$$

Exercise

1. Find the first and the second derivatives of the following functions.

A) $f(x) = x^3 - 2x^2 - 1$

b) $f(x) = 2x^5 - 3$

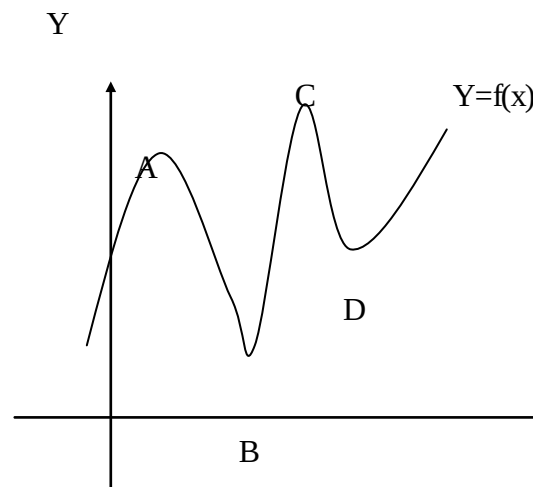
c) $f(x) = \frac{1}{x}$

d) $f(x) = e^{(2x^2-1)}$

e) $f(x) = \ln(3x^2 + 1)$

3.7 Relative Maximum and Minimum

A *relative maximum* of a function is a *peak*, a point on the graph of the function that is higher than any neighboring point on the graph. A *relative minimum* is the *bottom of a valley*, a point on the graph that is lower than any neighboring point. The relative maxima and minima of a function are collectively termed as *relative extrema*.



The above graph shows the relative maxima and minima of a function. The function has a relative maximum point A and C and a relative minimum at points B and D. Each of these points represents an extreme in the immediate neighborhood of the point only.

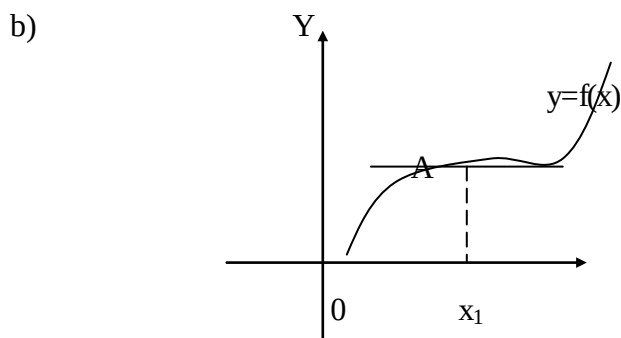
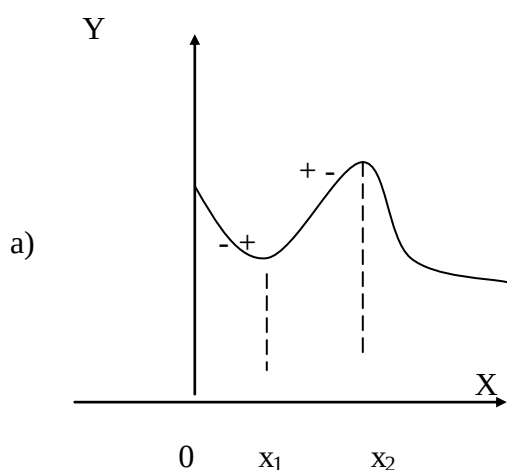
3.7.1 First Order Derivative Test for Relative Extreme

Given $y=f(x)$ which is continuous and possess a continuous derivative, then the first derivative, $f'(x)$ plays a major role in our search of its extreme values.

Definition; The values of x where the first derivative $f'(x)=0$ are called *critical points* and the corresponding values of $f(x)$ are called *critical values*.

The value of the function at the critical value, $f(x_0)$, will be

- a relative maximum if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.
- a relative minimum if $f'(x)$ changes its sign from negative to positive from the immediate left of x_0 to its immediate right.
- neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of point x_0 .



In (a) the value of the function at x_1 , $f(x_1)$, is a relative minimum because the sign of the first derivative changes $f'(x)$ its sign from negative to positive. The value of the function at x_2 , $f(x_2)$, is a relative maximum because the sign of the first derivative $f'(x)$ changes its sign from positive to negative. In (b) the value of the first derivative at x_1 , $f'(x_1)$ is zero, the derivative does not change its sign from one side of $x=x_1$, to the other side. Point A is called an *inflection point*.

The point $(x_1, f(x_1))$ and $(x_2, f(x_2))$ in both (a) and (b) are called *stationary points*. The value of the function $f(x)$ at these stationary points (i.e, $f(x_1)$ and $f(x_2)$) are called *stationary values* of y .

Note that a relative extremum must be a stationary value, but a stationary value may be associated with either a relative extremum or an inflection point.

Example Find the relative extrema of the function $f(x) = x^2 - 4x + 5$

Solution

First find the first derivative of the function; $f'(x) = 2x - 4$, then equate $f'(x)$ to zero

$$f'(x) = 2x - 4 = 0 \Rightarrow 2x = 4 \Rightarrow x = 2 \text{ is the only critical value.}$$

Now check the sign of $f'(x)$ by taking numbers from the immediate left and right of $x = 2$

Let's take 1.9 from the left and 2.1 from the right.

$$\text{At } x = 1.9 \Rightarrow f'(x) = 2(1.9) - 4 = 3.8 - 4 = -0.2 < 0$$

$$\text{At } x = 2.1 \Rightarrow f'(x) = 2(2.1) - 4 = 4.2 - 4 = 0.2 > 0$$

Therefore, the sign of $f'(x)$ changes from negative to positive, hence $f(2) = 1$ is a relative minimum.

Example: Find the relative extrema for $f(x) = x^3 - 12x^2 + 36x + 8$.

Solution

The first derivative of $f(x)$ is $f'(x) = 3x^2 - 24x + 36$ equating this to zero,

$$3x^2 - 24x + 36 = 0$$

$$x_1 = 2 \text{ and } x_2 = 6 \text{ are the critical values.}$$

Next check the signs of $f'(x)$ by taking numbers from the left and right of $x = 2$ and $x = 6$

For $x = 2$ take $x = 1.9$ and $x = 2.1$

$$\text{At } x = 1.9, f'(x) = 3(1.9)^2 - 24(1.9) + 36 = 1.23$$

$$\text{At } x = 2.1, f'(x) = 3(2.1)^2 - 24(2.1) + 36 = -1.17$$

Since the sign of $f'(x)$ changes from positive to negative, $f(20) = 40$ is a relative maximum.

For $x = 6$ take $x = 5.9$ and $x = 6.1$

$$\text{At } x = 5.9, f'(x) = 3(5.9)^2 - 24(5.9) + 36 = -1.17$$

$$\text{At } x = 6.1, f'(x) = 3(6.1)^2 - 24(6.1) + 36 = 1.23$$

Since $f'(x)$ changes its sign from negative to positive, $f(6) = 8$ is a relative minimum.

Example

Find the relative extrema for $f(x) = (x^3 - 1)^4$

Solution

The critical values of the function obtained as

$$f'(x) = 4(x^3 - 1)^3 (3x^2) = 0$$

$$(x^3 - 1)^3 = 0 \text{ or } 3x^2 = 0$$

$$x^3 = 1 \text{ or } x = 0 \Rightarrow x = 1 \text{ or } x = 0 \text{ are the critical values.}$$

Take numbers to the immediate right and left of $x = 1$, for example, $x = 0.8$ and $x = 1.1$

$$f'(0.8) = 4(0.8^3 - 1)^3 (3(0.8)^2) = -0.8925 < 0$$

$$f'(1.1) = 4(1.1^3 - 1)^3 (3(1.1)^2) = 0.52656 > 0$$

Therefore, since the sign of $f'(x)$ changes from negative to positive, $f(1) = 0$ is a relative minimum.

For $x = 0$, take numbers from the immediate right and left of $x = 0$, say $x = -0.1$ and $x = 0.1$

$$f'(-0.1) = 4(-0.1^3 - 1)^3 (3(-0.1)^2) = -0.12$$

$$f'(0.1) = 4((0.1)^3 - 1)^3 (3(0.1)^2) = -0.119$$

Since $f'(x)$ does not change its sign, $f(0) = 1$ is neither a relative maximum nor a relative minimum.

It is an inflection point.

Exercise

1. Define a) an inflection point b) Stationary point
 2. For the following functions Find
 - i) the critical values of f
 - ii) the relative extremum by using first order test
- a) $f(x) = x^3 - 6x^2 + 9x + 1$ b) $f(x) = 2x^2 - x^4$ c) $f(x) = 2x^3 + 4$

3.7.2 Second Order Derivative Test for Relative Extremum

The second derivative can be used to find the relative maxima or relative minima of a function. This test is more convenient than the first derivative test, because it does not require you to check the derivative sign to both the left and the right of x_0 .

If first derivative of a function f at $x = x_0$ is $f'(x) = 0$, then the value of function at x_0 , $f(x_0)$, will be

- i) a relative maximum if the second derivative value at x_0 is *negative*. That is, if $f''(x) < 0$.
- ii) a relative minimum if the second derivative value at x_0 is *positive*. That is, if $f''(x) > 0$.

But if $f''(x_0) = 0$, the stationary value $f(x_0)$ can be either a relative maximum, or a relative minimum, or even an inflection value. In this case we must revert to the first derivative test, or to another test.:

Example

Find the relative extremum of the function $f(x) = 4x^2 - x$ by using the second derivative test.

Solution

First find the first derivative of the function and then the critical value

$$f'(x) = 8x - 1 \text{ and thus } x = 1/8 \text{ is the critical value.}$$

Then find the second derivative $f''(x) = 8$ and check its sign at the critical value $\Rightarrow f''(1/8) = 8 > 0$

Therefore, $f(1/8) = -1/16$ is the relative minimum

Example

Find the relative extrema for $y=g(x)=x^3-3x^2+2$ using the second derivative test.

Solution

First derivative of the function is $g'(x)=3x^2-6x$, equate this to zero to get

$$3x^2-6x = 0 \Rightarrow x(3x-6) = 0 \Rightarrow x=0 \text{ or } x=2 \text{ are critical values}$$

Stationary values are $g(0)=2$ and $g(2)=-2$

To check whether these are the relative maximum or minimum values, find the second derivative

$g''(x)=6x-6$ and check its sign at $x=0$ and $x=2$ as

$$g''(0)=6(0)-6=-6<0 \text{ and } g''(2)=6(2)-6=6>0$$

Hence $g(0)=2$ is a relative maximum and $g(2)=-2$ is a relative minimum.

Steps for finding a minimum or a maximum value by using second derivative test;

Step 1 Find the first derivative, $f'(x)$ for the given function $y=f(x)$

Step 2 Find the critical values of the function (values of x that make $f'(x)=0$)

Step 3 Find the second derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$ of the function $y=f(x)$

Step 4 Put $x=x_0$ in step 3 (i.e. evaluate $f''(x)$ at x_0).

From step 4, if $f''(x_0) > 0$, $f(x_0)$ is a relative minimum.

if $f''(x_0) < 0$, $f(x_0)$ is a relative maximum.

Example

Find the relative extrema of the function $f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$

Solution

Follow the above steps

Step 1: $f'(x) = 4x^3 + 6x^2 - 6x - 4$

Step 2: solve $f'(x) = 4x^3 + 6x^2 - 6x - 4 = 0$ to get

$$x_0 = 1, x_0 = -1/2 \text{ and } x_0 = -2 \text{ as the critical values.}$$

Step 3: $f''(x) = 12x^2 + 12x - 6$

Step 4: at $x_0 = 1$, $f''(1) = 12 + 12 - 6 = 18 > 0$

$$\text{at } x_0 = -1/2, f''(-1/2) = 12(1/4) + 12(-1/2) - 6 = -9 < 0$$

$$\text{at } x_0 = -2, f''(-2) = 12(-2)^2 + 12(-2) - 6 = 18 > 0$$

Therefore, $f(x)$ has a relative minimum at $x = 1$, $f(1) = 0$ and $x = -2$, $f(-2) = 0$. It has a relative maximum at $x = -1/2$, $f(-1/2) = 81/16$

Exercise; Use the second derivative test to find the relative extrema of the following functions.

a) $f(x) = 5x^2 - 2x + 1$ b) $f(x) = -x^3 + 3x^2 - 3x + 3$ c) $f(x) = x + \frac{16}{x}$ d) $f(x) = \frac{x^2}{4} + \frac{4}{x}$

3.8 Economic Applications of Derivatives

In this section you see the importance of derivative in solving some standard economic problems like deriving marginal function from the total function and calculating point-price elasticity of demand or supply for a good.

3.8.1 Marginal Analysis

You learned, in the previous sections, that the derivative gives the instantaneous or marginal rate of change of the dependent variable, y , at a given point of x . Much of economics in fact is concerned with just this sort of analysis. For example, the major concern of theory of firms is to study how a change in firm's output level affects its costs and revenues. Thus, using derivatives one can perform what economists refers as *marginal analysis*, which is of central importance in economics.

Given a total function $y = f(x)$ (which may be total cost function, total revenue function, total product function etc),

i) Its average value function can be obtained as $A(x) = \frac{f(x)}{x}$

ii) Its marginal value function can be obtained as $mf(x) = f'(x) = \frac{dy}{dx}$

I. Relationship between Total, Average, and Marginal Cost Functions

Given $C = f(x)$ as total cost function, $AC = \frac{C}{Q}$ and $MC = \frac{dC}{dQ}$

Example

Suppose that the short-run total cost function of a given firm is given as

$$C = Q^3 - 3Q^2 + 15Q + 7, \text{ Then find}$$

- Average cost function
- Marginal cost function
- The rate of change of average variable cost (AVC) with respect to change in output level

Solutions

$$a) AC = \frac{C}{Q} = Q^2 - 3Q + 15 + \frac{27}{Q}$$

$$b) MC = \frac{dC}{dQ} = 3Q^2 - 6Q + 15$$

$$c) \text{ First calculate average variable cost using the formula } AVC = \frac{TVC}{Q}$$

$$\text{That is, } AVC = \frac{TVC}{Q} = \frac{Q^3 - 3Q^2 + 15Q}{Q} = Q^2 - 3Q + 15$$

$$\text{Then rate of change of AVC with respect to } Q \text{ as } \frac{d(AVC)}{dQ} = \frac{d}{dQ}(Q^2 - 3Q + 15) = 2Q - 3$$

Example

Given the total cost function as $C = f(Q)$, describe the relationship between MC and AC by finding the rate of change of AC with respect to Q.

Solution

Since $AC = \frac{C}{Q} = \frac{f(Q)}{Q}$, its rate of change can be calculated through quotient rule as

$$\begin{aligned}\frac{d(AC)}{dQ} &= \frac{d}{dQ} \left(\frac{f(Q)}{Q} \right) = \frac{Qf'(Q) - \frac{d}{dQ}(Q)f(Q)}{Q^2} \\ &= \frac{Qf'(Q) - f(Q)}{Q^2} \quad \text{since } \frac{d}{dQ}(Q) = 1 \\ &= \frac{f'(Q)}{Q} - \frac{f(Q)}{Q^2} \\ &= \frac{1}{Q} \left[f'(Q) - \frac{f(Q)}{Q} \right] = \frac{1}{Q} [MC - AC]\end{aligned}$$

Thus, $\frac{d(AC)}{dQ} = \frac{1}{Q} [MC - AC]$ and this result leads to the following important relationships between MC and AC.

- i) When AC is declining, $MC < AC$. That is, when AC falls, MC is below AC.
- ii) When AC is constant, $MC = AC$.
- iii) When AC is rising, $MC > AC$. That is, when AC rises, MC is above AC.

II. Relationship between Total, Average, and Marginal Revenue Functions

The total revenue (TR) of either a competitive firm or a simple monopolist is the unit price of its output (P) times the quantity it produce or sell (Q). That is, total revenue for any type of firm can be obtained as $TR = P \cdot Q$

Moreover, average revenue is given by $AR = \frac{TR}{Q}$ and marginal revenue (MR), which represents the

change in total revenue for each additional unit of sales (Q) can be obtained as $MR = \frac{dTR}{dQ}$

A competitive firm treats the price as constant value, equal to the market price ($\bar{P}$). Thus, in a competitive firm $TR = \bar{P} \cdot Q$ and $AR = MR = \bar{P}$.

But a monopolist is the only firm in the industry and so recognizes that the amount it can sell is determined by the price it sets according to the market demand function, $Q = d(P)$. Thus, to get the total revenue of a monopolist firm, first write its inverse demand function as $P = D^{-1}(Q) = f(Q)$ and then multiply by the level of output Q . That is, the monopolist total revenue function can be obtained as

$$TR = PQ = f(Q).Q$$

Since, $AR = \frac{TR}{Q} = \frac{PQ}{Q} = P = f(Q)$, the average revenue function (AR) of a monopolist firm always equal to its inverse demand function (P).

Again since TR is the product two functions, $f(Q)$ and Q , the marginal revenue function can be further simplified through the product rule as follows.

$$\begin{aligned} MR &= \frac{dTR}{dQ} = \frac{d}{dQ}(f(Q).Q) \\ &= Q \cdot \frac{d}{dQ}(f(Q)) + f(Q) \frac{d}{dQ}(Q) \\ &= Q \cdot f'(Q) + f(Q) \\ &= Q \cdot f'(Q) + AR \quad \text{since } f(Q) = P = AR \end{aligned}$$

Thus, in a monopolist firm the relationship between MR and AR can be expressed as $MR = Q \cdot f'(Q) + AR$. moreover, under the usual assumption of negatively sloped demand curve (i.e, $f'(Q) < 0$), the term $Q \cdot f'(Q)$ will be negative and thus $MR < AR$. In other words, when the monopolist's demand curve is downward sloping, MR is always lies below AR.

Example

Suppose that the demand function of a monopolist firm is given by $Q = 20 - 0.5P$, then find MR.

Solution

First find the inverse demand function in terms of Q as $Q = 20 - 0.5P \Rightarrow 0.5P = 20 - Q \Rightarrow P = 40 - 2Q$

TR as $TR = PQ = (40 - 2Q).Q = 40Q - 2Q^2$, and then calculate MR as

$$MR = \frac{dTR}{dQ} = \frac{d}{dQ}(40Q - 2Q^2) = 40 - 4Q.$$

3.8.2 Point-Price Elasticity of Demand

In your microeconomics course you learned the concept of price elasticity of demand. Price elasticity of demand measures the responsiveness of buyer(s) to changes in price of a good. In this sub section you will see how derivative helps in computing price elasticity of demand for any type of demand function.

Given a demand function as $Q = f(P)$, the price elasticity of demand between two points on the demand curve can be obtained as

$$\epsilon_d = \frac{\Delta Q}{\Delta P} \frac{P}{Q} = (\text{slope}) \frac{P}{Q} \text{ and it is termed as average or arc Price elasticity of demand.}$$

The *point-price elasticity of demand* measures price elasticity of demand at particular level of P or at a point on the demand curve. Since the derivative measures the slope a function at a given point, the derivative $\frac{dQ}{dP}$ can be used instead of $\frac{\Delta Q}{\Delta P}$ to calculate point-price elasticity of demand.

Thus, the point-price elasticity of demand can be obtained as $\epsilon_d = \frac{dQ}{dP} \frac{P}{Q}$

Example

Find the price elasticity of demand at $P = 4$ for demand function of $Q = 25 - 4P + P^2$.

Solution

First find $\frac{dQ}{dP}$ and evaluate its value at $P = 4$ as

$$\frac{dQ}{dP} = \frac{d}{dP}(25 - 4P + P^2) = -4 + 2P \text{ and } \left. \frac{dQ}{dP} \right|_{P=4} = -4 + 2(4) = 4.$$

Then find the value of Q at $P = 4$ as $Q(4) = 25 - 4(4) + 4^2$. Finally, compute ϵ_d as

$$\epsilon_d = \left. \frac{dQ}{dP} \right|_{P=4} \cdot \frac{P}{Q(4)} = 4 \left(\frac{4}{25} \right) = 0.64.$$

Exercise

1. Suppose that the total cost function of a given firm is given by

$$C = 2Q^3 - Q^2 + 4Q, \text{ Then find}$$

a) the AC function b) the marginal cost function c) the slope AC

2. Suppose that the demand function of a monopolist firm is given by $Q = 125 - 0.5P$, then find

a) the average revenue function

b) the marginal revenue function

c) the values of AC and MR at $Q = 25$ and compare the results.

3. Find the price elasticity of demand and state the nature of demand for the following demand function at the indicated values of P .

a) $Q = 10 - P$ at $P = 4$

b) $Q = \frac{20}{P+1}$ at $P = 3$

3.8 Optimization: Unconstrained and constrained**Example (unconstrained)**

Find the relative extreme of the average cost function by using the first derivative test

$AC = f(Q) = Q^2 - 5Q + 8$ where Q is the level of output

Solution

The first derivative of the function is $f'(Q) = 2Q - 5$ and equating this to zero gives the critical number $2Q - 5 = 0$, $Q = 5/2 = 2.5$

Taking numbers from the immediate left and right of $Q = 2.5$, say $Q = 2.4$ and $Q = 2.6$.

$$f'(2.4) = 2(2.4) - 5 = 4.8 - 5 = -0.2 < 0$$

$$f'(2.6) = 2(2.6) - 5 = 5.2 - 5 = 0.2 > 0$$

Therefore, since $f'(Q)$ changes its sign from negative to positive AC at $Q = 2.5$ (i.e, $f(2.5)$) is a relative minimum

Example

Let the $R(Q)$ and $C(Q)$ be the revenue and cost functions respectively and

$$R(Q) = 1200Q - 2Q^2 \text{ and } C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$

Then find

- a) the profit maximizing level of output
- b) the maximum profit

Solutions

There are two ways to solve this problem

- i) by using the second derivative test or
- ii) by using the marginal revenue = -marginal cost approach.

You can solve the problem by using (i) as follows.

$$a) \text{ Profit} = \text{Total Revenue} - \text{Total Cost} = Q^3 + 59.25Q^2 - 328.5Q - 2000$$

Next find the critical values from the profit function by finding the first derivative of the profit function and equating to zero.

$$\pi'(Q) = -3Q^2 + 118.5Q - 328.5 = 0 \quad Q = 3 \text{ and } Q = 36.5 \text{ are the critical values}$$

Then check the sign of the second derivative π'' at $Q=3$ and $Q=36.5$

$$\pi''(Q) = \frac{d^2\pi}{dQ^2} = -6Q + 118.5$$

$$\text{At } Q = 3, \frac{d^2\pi}{dQ^2} = -6(3) + 118.5 = 100.5 > 0$$

$$\text{At } Q = 36.5, \frac{d^2\pi}{dQ^2} = -6(36.5) + 118.5 = -100.5 < 0$$

Therefore $Q = 36.5$ is the profit maximizing level of output.

b) The maximum profit can be found by directly substituting $Q = 36.5$ in to the profit function as

$$\pi = -(36.5)^3 + 59.25(36.5)^2 - 328.5(36.5) - 2000 = 16318.44$$

Remark The above problem can also be solved by MR- MC Approach (ii). This requires equating MR and MC and solving for Q (the profit maximizing level of output) the result will be the same.

Example (constrained for all values of $q > 0$)

Suppose the total cost of producing a certain commodity is given by $C(q) = 3q^2 + q + 48$

- Find the level of output at the smallest value of the average cost.
- Find the level of output, which equates the average cost and marginal cost.

Solutions

$$a) \quad AC = A(q) = \frac{C(q)}{q} = \frac{3q^2 + q + 48}{q} = 3q + 1 + \frac{48}{q}$$

$$A'(q) = 3 - \frac{48}{q^2} = \frac{3q^2 - 48}{q^2} = \frac{3(q^2 - 16)}{q^2} = \frac{3(q-4)(q+4)}{q^2}, q > 0$$

By equating the first derivative of the average cost function to zero, you get

$$\frac{3(q-4)(q+4)}{q^2} = 0 \Rightarrow q = 4 \text{ for } q > 0$$

Then find the second derivative of the average cost function

$$A''(q) = \frac{96}{q^3} \text{ and it is positive at } q = 4. \text{ Thus, AC will be minimal at } Q = 4 \text{ units}$$

- First find the average cost and marginal cost function of the total cost function and equate them.

$$C(q) = A(q)$$

$$6q + 1 = 3q + 1 + \frac{48}{q} \Rightarrow 3q = \frac{48}{q}$$

$$\Rightarrow q^2 = 16$$

$$\Rightarrow q = 4 \text{ since } q > 0 \text{ which is the same as in (a)}$$

Exercise

1. Suppose that the total cost of a firm is given by

$$C = 100 + 2Q + \frac{1}{10}Q^2, \text{ Then find}$$

a) the level of output Q at which the average cost (AC) is minimal

b) the minimum average cost

c) the level of output at which marginal cost is equal average cost

2. Suppose that the total cost of a firm is $C = 100 + 2Q + \frac{1}{10}Q^2$, and its demand function is

$$P = 20 - \frac{1}{5}Q, \text{ Then find}$$

a) the level of output (Q) and the price (P) required to maximize the total revenue (TR)

b) the maximum value of total revenue

c) the level of output (Q) and the price (P) required to maximize the profit

d) the maximum profit

e) the level of output which equates marginal revenue and marginal cost.

UNIT FOUR

Differential Calculus of Multivariable Functions

In the previous section you have discussed optimization with single independent variable give mainly as $Y = f(x)$. Many economic activities, however involve functions of more than one independent variables given in the form of $Z = f(x, y, \dots)$. Thus, in this section you will discuss a way of finding the extreme values of a function that involves two or more choice variables. Particularly, the discussion will concentrate on function with of two choice variables, $z = f(x, y)$.

4.1 Partial Derivatives

Partial derivative is the derivative of functions of several variables. Let $Z = f(x, y)$ be a function of two independent variables. If the variable y is held fixed at a value of $y = y_0$ the relation $Z = f(x, y_0)$ expresses Z as a function of one variable x .

So long as the curve of the function is smooth, there will be a tangent line in each point of the curve and the slope of this tangent line can be calculated by differentiating z with respect to x from the relation $Z = f(x, y_0)$. This derivative is found in usual way as a limit according to the following expression:

$$\frac{df(x, y_0)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y_0) - f(x, y_0)}{\Delta x} \text{ if the limit exists}$$

This limit is called the *partial derivative of Z with respect to x* and usually denoted by $\frac{\partial Z}{\partial x}$.

4.1.1 First Order Partial Derivatives

Definition: let $Z = f(x, y)$ be a function of Z with x and y . Then

The *partial derivative of Z with respect to x* , denoted as $\frac{\partial Z}{\partial x}$, is defined as $\frac{\partial Z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y_0) - f(x, y_0)}{\Delta x}$

The *partial derivative of Z with respect to y* , denoted as $\frac{\partial Z}{\partial y}$, is defined as $\frac{\partial Z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y + \Delta y) - f(x_0, y)}{\Delta y}$.

Notice that calculating the partial derivative requires all other independent variables to be constant except the variable in which the derivative is being carried out. For example, if $Z = f(x, y, w, m)$ and

if one wants to calculate $\frac{\partial Z}{\partial y}$, all the independent variables should be kept constant except variable y and by doing this the response of the dependent variable Z only due to the change in variable y can be measured.

Remarks

Partial derivatives can be evaluated using essentially the same techniques as those used for evaluating ordinary derivative (i.e, derivative of $y = f(x)$). To compute the partial derivatives apply any of the necessary techniques simply by treating all the independent variables except the one with respect to which you are differentiating as constant. Therefore, you can use the power rule, product and quotient rules, and chain rules in the usual way whenever necessary.

Example: Calculate $\frac{\partial Z}{\partial y}$ when $Z = x^3 + 5xy^2 + 2y^3$.

Solution

To finding $\frac{\partial Z}{\partial y}$, you should treat x as a constant and differentiate Z with respect to y .

$$\frac{\partial Z}{\partial y} = 0 + 5x(2y) + 2(3y^2) = 10xy + 6y^2$$

Example: Calculate $\frac{\partial Z}{\partial x}$ for $Z = \sqrt{x^2 + y^2}$.

Solution

Treat y as constant and use the chain rule to get $\frac{\partial Z}{\partial x}$ as

$$\frac{\partial Z}{\partial x} = \frac{\partial (x^2 + y^2)^{\frac{1}{2}}}{\partial x} = \frac{1}{2}(x^2 + y^2)^{\frac{1}{2}-1}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

Exercise

Calculate $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$ for the following functions

a) $Z = 3e^{2x} - 5\ln^x + 7$ b) $Z = xy^2 + x^2y$ c) $Z = x\ln^y + y^2\ln^x$ d) $Z = e^{2x+3y}$.

4.1.2 Higher-order partial Derivatives

Note that the partial derivative obtained from the original function $y = f(x_1, x_2, \dots, x_n)$ are said to be first-order partial derivatives just to distinguish from higher – order partial derivatives. Moreover, from the higher – order partial derivatives the ones which have significance importance to describe various characteristics of economic functions are second-order partial derivatives. So, due to this reason, we will confine our discussions on such type of partial derivatives.

Now, let us start our discussion for $y = f(x_1, x_2)$ and we will generalize it for n-variable case latter.

As we discussed in the previous subsection, the function $y = f(x_1, x_2)$ has two first-order partial derivatives which denoted as $\frac{\partial y}{\partial x_1}$ or f_1 and $\frac{\partial y}{\partial x_2}$ or f_2 . Moreover, each of these partial derivatives are a gain a function of x_1 and x_2 . So, we can continue differentiating each of them partially with respect to x_1 and x_2 . We can get the following four possible second – order partial derivatives.

1. When $\frac{\partial y}{\partial x_1}$ is differentiated with respect to x_1 while holding x_2 as a constant, we have

$$\frac{\partial}{\partial x_1} \left(\frac{\partial y}{\partial x_1} \right) \text{ and denoted as } \frac{\partial^2 y}{\partial x_1^2} \text{ or } f_{11}$$

2. When $\frac{\partial y}{\partial x_1}$ is differentiated with respect to x_2 keeping x_1 as a constant, we have

$$\frac{\partial}{\partial x_2} \left(\frac{\partial y}{\partial x_1} \right) \text{ and denoted as } \frac{\partial^2 y}{\partial x_2 \partial x_1} \text{ or } f_{12};$$

3. When $\frac{\partial y}{\partial x_2}$ is differentiated partially with respect to x_1 we have,

$$\frac{\partial}{\partial x_1} \left(\frac{\partial y}{\partial x_2} \right) \text{ and denoted as } \frac{\partial^2 y}{\partial x_1 \partial x_2} \text{ or } f_{21} \text{ and}$$

4. When $\frac{\partial y}{\partial x_2}$ is differentiated with respect to x_2 , holding x_1 as a constant, we have

$$\frac{\partial}{\partial x_2} \left(\frac{\partial y}{\partial x_2} \right) \text{ and denoted as } \frac{\partial^2 y}{\partial x_2^2} \text{ or } f_{22}$$

- i. To make the notations simple and clear, we will often use f_1 and f_2 to denote the first-order partial derivatives with respect to x_1 and x_2 respectively and f_{11}, f_{12}, f_{22} and f_{21} to denote the second-order partial derivatives.
- ii. f_{11} and f_{22} are called pure or own partial derivatives. f_{11} indicates that the given function $y = f(x_1, x_2)$ is first partially differentiated w.r.t. x_1 . f_{22} indicates that $y = f(x_1, x_2)$ is first differentiated partially with respect to x_2 .
- iii. f_{12} and f_{21} are called cross or mixed partial derivatives. f_{12} indicates that $y = f(x_1, x_2)$ is first partially differentiated w.r.t. x_2 and then with respect to x_1 .

Now, before defining the second-order partial derivatives for n-variable case, let us see how we compute them for $y = f(x_1, x_2)$ with the help of the following example.

Example

Find the second – order partial derivatives for $3x_1 + 2x_2^3 - x_1^3 x_2^2$

a) First, compute f_1 as

$$f_1 = \frac{\partial y}{\partial x_1} = \frac{\partial}{\partial x_1} (3x_1 + 2x_2^3 - x_1^3 x_2^2) = 3 + 0 - 3x_1^2 x_2^2 = 3 - 3x_1^2 x_2^2$$

The differentiate f_1 w.r.t. x_1 to get f_{11} as

$$f_{11} = \frac{\partial}{\partial x_1} (f_1) = \frac{\partial}{\partial x_1} (3 - 3x_1^2 x_2^2) = \frac{\partial}{\partial x_1} (3) - \frac{\partial}{\partial x_1} (3x_1^2 x_2^2) = 0 - 6x_1 x_2^2 = -6x_1 x_2^2$$

and also differentiate f_1 w.r.t. x_2 to get f_{12} as

$$f_{12} = \frac{\partial y}{\partial x_2} (f_1) = \frac{\partial}{\partial x_2} (3 - 3x_1^2 x_2^2) = \frac{\partial}{\partial x_2} (3) - \frac{\partial}{\partial x_2} (3x_1^2 x_2^2) = 0 - 6x_1^2 x_2 = -6x_1^2 x_2.$$

Second, compute f_2 as $f_2 = \frac{\partial y}{\partial x_2} = \frac{\partial}{\partial x_2} (3x_1 + 2x_2^3 - x_1^3 x_2^2) = 6x_2^2 - 2x_1^3 x_2$

Then differentiate f_2 w.r.t x_1 to get f_{21} as

$$f_{21} = \frac{\partial}{\partial x_1}(f_2) = \frac{\partial}{\partial x_1}(6x_1^2 - 2x_1^3x_2) = \frac{\partial}{\partial x_1}(6x_1^2) - \frac{\partial}{\partial x_1}(2x_1^3x_2) = 0 - 6x_1^2x_2 = -6x_1^2x_2 \text{ and}$$

also differentiate f_2 w.r.t x_2 to get f_{22} as

$$f_{22} = \frac{\partial}{\partial x_2}(f_2) = \frac{\partial}{\partial x_2}(6x_1^2 - 2x_1^3x_2) = \frac{\partial}{\partial x_2}(6x_1^2) - \frac{\partial}{\partial x_2}(2x_1^3x_2) = 12x_1^2 - 2x_1^3$$

Therefore, the second-order partial derivatives for this function are

$$\begin{aligned} f_{11} &= -6x_1x_2^2 & f_{21} &= -6x_1^2x_2 \\ f_{12} &= -6x_1^2x_2 & f_{22} &= 12x_1^2 - 2x_1^3 \end{aligned}$$

Notice that the cross partial derivatives are equal. That is,

$$f_{12} = f_{21} = -6x_1^2x_2.$$

b) First compute the first-order partial derivatives as

$$\begin{aligned} f_1 &= \frac{\partial y}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^3e^{2x_2}) = \frac{\partial}{\partial x_1}(x_1^3)e^{2x_2} = 3x_1^2e^{2x_2}, \text{ and} \\ f_2 &= \frac{\partial y}{\partial x_2} = \frac{\partial}{\partial x_2}(x_1^3e^{2x_2}) = x_1^3 \frac{\partial}{\partial x_2}(e^{2x_2}) = x_1^3 2e^{2x_2} = 2x_1^3e^{2x_2} \end{aligned}$$

Then obtain the second –order partials as follows: $3x_1^2e^{2x_2}$

$$\begin{aligned} f_{11} &= \frac{\partial}{\partial x_1}(f_1) = \frac{\partial}{\partial x_1}(3x_1^2e^{2x_2}) = \frac{\partial}{\partial x_1}(3x_1^2)e^{2x_2} = 6x_1e^{2x_2}; \\ f_{12} &= \frac{\partial}{\partial x_2}(f_1) = \frac{\partial}{\partial x_2}(3x_1^2e^{2x_2}) = \frac{\partial}{\partial x_2}(e^{2x_2})3x_1^2 = 2e^{2x_2}(3x_1^2) = 6x_1^2e^{2x_2}; \\ f_{21} &= \frac{\partial}{\partial x_1}(f_2) = \frac{\partial}{\partial x_1}(2x_1^3e^{2x_2}) = \frac{\partial}{\partial x_1}(2x_1^3)e^{2x_2} = 6x_1^2(e^{2x_2}) = 6x_1^2e^{2x_2}; \text{ and} \\ f_{22} &= \frac{\partial}{\partial x_2}(f_2) = \frac{\partial}{\partial x_2}(2x_1^3e^{2x_2}) = \frac{\partial}{\partial x_2}(e^{2x_2})2x_1^3 = 2e^{2x_2}(2x_1^3) = 4x_1^3e^{2x_2} \end{aligned}$$

Therefore, the second – order partial for the function are

$$\begin{aligned} f_{11} &= 6x_1e^{2x_2} & f_{21} &= 6x_1^2e^{2x_2} \\ f_{12} &= 6x_1^2e^{2x_2} & f_{22} &= 4x_1^3e^{2x_2} \end{aligned}$$

Again notice that the cross-partial derivatives are equal. That is,

$$f_{12} = f_{21} = 6x_1^2 e^{2x_2}$$

Just as we have seen for $Y = f(x_1, x_2)$, the second-order partial derivatives for n-variable function can be defined analogously as follows:

Definition

The function $y = f(x_1, x_2, \dots, x_n)$, has n first-order partial derivatives and n^2 second-order partial derivatives each denoted as

$$\frac{\partial^2 y}{\partial x_j \partial x_i} \text{ or } f_{ij} = \frac{\partial}{\partial x_j} (f_i)$$

Where f_{ij} is the second-order partial obtained first differentiating the function with respect to x_i and then differentiating the result f_i with respect to x_j for $i, j = 1, 2, \dots, n$.

The partial derivatives in form of f_{ii} or f_{jj} where $i = j$ are called pure or own partial derivatives and those in a form of f_{ij} where $i \neq j$ are called cross or mixed partial derivatives.

Example

Find the second-order partial derivatives for $y = 2x_1^2 + 3x_1x_2 + x_2^2 - 5x_1x_3 + 4x_3^2$

Solution for this function we have a group of $3^2 = 9$ second-order partial derivatives which are

First order partials

$$f_1$$

$$f_2$$

$$f_3$$

Second-order partials

$$f_{11} \quad f_{12} \quad f_{13}$$

$$f_{21} \quad f_{22} \quad f_{23}$$

$$f_{31} \quad f_{32} \quad f_{33}$$

Out of the nine second-order partials, f_{12} , f_{21} , f_{13} , f_{31} , f_{23} and f_{32} are cross partial derivatives and

$$f_{12} = f_{21}, f_{13} = f_{31} \text{ and } f_{23} = f_{32}$$

$$f_{11} = \frac{\partial}{\partial x_1}(f_1) = \frac{\partial}{\partial x_1}(4x_1 + 3x_2 - 5x_3) = 4$$

$$f_{12} = \frac{\partial}{\partial x_2}(f_1) = \frac{\partial}{\partial x_2}(4x_1 + 3x_2 - 5x_3) = 3 = f_{21}$$

$$f_{13} = \frac{\partial}{\partial x_3}(f_1) = \frac{\partial}{\partial x_3}(4x_1 + 3x_2 - 5x_3) = -5 = f_{31}$$

$$f_{22} = \frac{\partial}{\partial x_2}(f_2) = \frac{\partial}{\partial x_2}(3x_1 + 2x_2) = 2$$

$$f_{23} = \frac{\partial}{\partial x_3}(f_2) = \frac{\partial}{\partial x_3}(3x_1 + 2x_2) = 0 = f_{32} \text{ and}$$

$$f_{33} = \frac{\partial}{\partial x_3}(f_3) = \frac{\partial}{\partial x_3}(-5x_1 + 8x_3) = 8$$

Exercises

1. List the first-order and second-order partial derivatives for a function having four independent variables with their notations.

3. Find the second-order partial derivatives of each of the following functions

a) $y = x_1 x_2$

b) $y = x_1 x_2^2 - 3x_1 - 5x_2$

c) $y = \ln(3x_1 + x_2)$

d) $y = Ax_1^\alpha x_2^\beta$, Where A, α and β are constants.

e) $y = x_1^2 x_2^4 x_3^5$

4.1.3 Composite Function (The Chain Rule)

Example: Find $\frac{dy}{dx}$ if $y=2u^2+1$ and $u= 3x-1$.

Solution

First find $\frac{dy}{du}$ and $\frac{du}{dx}$ as

$$\frac{dy}{du} = \frac{d}{du}(2u^2+1)=4u \text{ and } \frac{du}{dx} = \frac{d}{dx}(3x-1)=3$$

Then apply the chain rule to get $\frac{dy}{dx}$ as $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u(3) = 12u$

Since Y is regarded as a function of x, $\frac{dy}{dx}$ should be expressed in terms of x.

To do this, substitute the value of u in $\frac{dy}{dx} = 12u$ to get $\frac{dy}{dx} = 12u = 12((3x-1)) = 36x-12$

The chain rule is particularly very useful to differentiate the following kinds of functions.

a) When $y=[u(x)]^n$ and multiplying $u(x)$ n times becomes a tedious process. In this case, you can apply the power and chain rules to get $\frac{dy}{dx}$ as $\frac{dy}{dx} = n[u(x)]^{n-1}u'(x)$ for any integer n.

Example

Find $\frac{dy}{dx}$ for $y=(x^3+4)^7$

Solution

Let $u(x)=x^3+4$, then $u'(x)=3x^2$

$$\begin{aligned}\frac{dy}{dx} &= n[u(x)]^{n-1}u'(x) \\ &= 7(x^3+4)^6 \cdot (3x^2) \\ &= 21x^2(x^3+4)^6\end{aligned}$$

Or equivalently you can find $\frac{dy}{dx}$ as

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3+4)^7 \\ &= 7(x^3+4)^{7-1} \cdot \frac{d}{dx}(x^3+4) \\ &= 7(x^3+4)^6 \cdot (3x^2) \\ &= 21x^2(x^3+4)^6\end{aligned}$$

b) when $y=\sqrt[n]{u(x)}$

In this case by rewriting $f(x)$ as $f(x) = \sqrt[n]{u(x)} = [u(x)]^{\frac{1}{n}}$ and applying the power and chain rules, you can get $\frac{dy}{dx}$ as

$$\frac{dy}{dx} = \frac{1}{n} [u(x)]^{\frac{1}{n}-1} \cdot u'(x)$$

Example Find $f'(x)$ for $f(x) = \sqrt[3]{4x^2-1}$

Solution Let $u(x) = 4x^2 - 1$ so that $u'(x) = 8x$

then since $f(x) = \sqrt[3]{4x^2-1} = (4x^2-1)^{\frac{1}{3}} = [u(x)]^{\frac{1}{3}}$, you can get $f'(x)$ as

$$\begin{aligned} f'(x) \frac{dy}{dx} &= \frac{1}{n} (u(x)) \\ &= \frac{1}{3} [4x^2-1]^{\frac{1}{3}-1} \cdot 8x \\ &= \frac{1}{3} (4x^2-1)^{-\frac{2}{3}} \cdot 8x \\ &= \frac{1}{3(4x^2-1)^{\frac{2}{3}}} \cdot 8x = \frac{8x}{3(4x^2-1)^{\frac{2}{3}}} \end{aligned}$$

Or equivalently, you can get $f'(x)$ as

$$\begin{aligned} f'(x) &= \frac{dy}{dx} = \frac{d}{dx} \sqrt[3]{4x^2-1} \\ &= \frac{d}{dx} (4x^2-1)^{\frac{1}{3}} \\ &= \frac{1}{3} (4x^2-1)^{\frac{1}{3}-1} \cdot \frac{d}{dx} (4x^2-1) \\ &= \frac{1}{3} (4x^2-1)^{-\frac{2}{3}} \cdot (8x) \\ &= \frac{8x}{3(4x^2-1)^{\frac{2}{3}}} \end{aligned}$$

c) When $y = \ln(u(x))$

In this case, apply the natural logarithmic function and chain rules to get $\frac{dy}{dx}$ as $\frac{dy}{dx} = \frac{1}{u(x)} \cdot u'(x)$

Example

Find $\frac{dy}{dx}$ for $y = \ln(7x^2 - 5x)$

Solution Let $u(x) = 7x^2 - 5x$ so that $u'(x) = 14x - 5$

Since $Y = \ln(7x^2 - 5x) = \ln(u(x))$, you can get $\frac{dy}{dx}$ as

$$\frac{dy}{dx} = \frac{1}{u(x)} u'(x)$$

$$= \frac{1}{(7x^2 - 5x)} (14x - 5) = \frac{14x - 5}{7x^2 - 5x}$$

or equivalently you can find $\frac{dy}{dx}$ as

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \ln(7x^2 - 5x) \\ &= \frac{1}{(7x^2 - 5x)} \cdot \frac{d}{dx} (7x^2 - 5x) \\ &= \frac{1}{(7x^2 - 5x)} (14x - 5) = \frac{14x - 5}{7x^2 - 5x} \end{aligned}$$

d) When $Y = e^{u(x)}$

In this case, you can apply natural exponential function and

chain rules to get $\frac{dy}{dx}$ as

$$\frac{dy}{dx} = e^{u(x)} \cdot u'(x)$$

Example

Find $f'(x)$ for $f(x) = e^{(3x^4 + x^3 - 2x)}$

Solution

Let $u(x) = 3x^4 + x^3 - 2x$, then $u'(x) = 12x^3 + 3x^2 - 2$. More over, $f(x) = e^{(3x^4 + x^3 - 2x)} = e^{u(x)}$, thus you can get $f'(x)$ as

$$\begin{aligned} f'(x) &= e^{u(x)} \cdot u'(x) \\ &= e^{(3x^4 + x^3 - 2x)} \cdot (12x^3 + 3x^2 - 2) \\ &= (12x^3 + 3x^2 - 2)e^{(3x^4 + x^3 - 2x)} \end{aligned}$$

Remark

The chain rule can be extended for composition of three or more functions. For instance for case of

three functions with $y = f(u)$, $u = g(r)$ and $r = h(x)$, $\frac{dy}{dx}$ can be obtained by $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dr} \cdot \frac{dr}{dx}$

Example

Find $\frac{dy}{dx}$ for $y = \ln \sqrt{1-x^2}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (\ln \sqrt{1-x^2}) \\ &= \frac{1}{\sqrt{1-x^2}} \cdot \frac{d}{dx} \sqrt{1-x^2} \cdot \frac{d}{dx} (1-x^2) \\ &= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2} (1-x^2)^{\frac{1}{2}-1} \cdot (-2x) \\ &= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \cdot (-2x) \\ &= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) \\ &= \frac{-2x}{2\sqrt{1-x^2} \cdot \sqrt{1-x^2}} = \frac{-2x}{2(1-x^2)} = \frac{-x}{(1-x^2)}\end{aligned}$$

Exercise

1. Differentiate each of the following by using the chain rule

a) $f(x) = (x^4 - 5)^5$ b) $Y = \frac{1}{(x^2 + 4)^2}$

c) $f(x) = \frac{4}{\sqrt{x^2 - 3x^1}}$ d) $Y = e^{(2x^2 + 5)}$

e) $f(x) = \ln(x^2 - 4x + 2)$ f) $Y = e^{1 + \ln x^2}$

2) Find the derivative of each of the following and simplify the result whenever it is possible.

a) $f(x) = 4x^2 \sqrt{x^2 - 1}$ b) $f(x) = (x^2 + 1) \ln(x^2 + 1)$ c) $y = \ln \left(\frac{1+x^2}{1-x^2} \right)$

4.2 Total Differencial and Derivatives

4.2.1 Total Differential

In the first part of this chapter we learned about partial derivative. Partial derivative indicates or measures the change in the dependent variable, y , when there is an infinitesimal (or a very small) change in one independent variable while others are kept as constant. The change in the dependent variable may also come from a simultaneous change in all independent variables. Now the question is how we find or measure the change in the dependent variable when all independent variables are changes simultaneously?

In this section, therefore, we will address this problem with the concept known as **total differential**.

Now just as we did in the previous sections, let us first elaborate the concept for $y = f(x_1, x_2)$ and we will extend it for n -variable function.

Given $y = f(x_1, x_2)$, we learned that $\frac{\partial y}{\partial x_1}$ or f_1 gives the change in y when there is a small change in x_1 ($\Delta x_1 \rightarrow 0$) holding x_2 as constant. Now if we suppose that x_1 changes by some amount, Δx_1 , while x_2 holding as constant, the change in y will be $\frac{\partial y}{\partial x_1} \cdot \Delta x_1$.

Similarly, since $\frac{\partial y}{\partial x_2}$ or f_2 represents the change in y for a small change in x_2 ($\Delta x_2 \rightarrow 0$), the change in y as a result of change in x_2 by some amount, Δx_2 will be $\frac{\partial y}{\partial x_2} \cdot \Delta x_2$.

Therefore, the total change in y (say, Δy) due to small change in x_1 and x_2 will be:

$$\Delta y = \frac{\partial y}{\partial x_1} \Delta x_1 + \frac{\partial y}{\partial x_2} \Delta x_2$$

Notice that, when the change in x_1 and x_2 becomes infinitesimal (That is, as $\Delta x_1 \rightarrow 0$ and $\Delta x_2 \rightarrow 0$), Δx_1 becomes dx_1 and Δx_2 becomes dx_2 . Moreover, as $\Delta x_1 \rightarrow 0$ and $\Delta x_2 \rightarrow 0$, the corresponding change in y becomes infinitesimal (i.e., $\Delta y \rightarrow 0$) and thus Δy becomes dy .

So, using these notational relationships and from the above formula for Δy , the total change in $y(dy)$ as a result of an infinitesimal change in both x_1 and x_2 (dx_1 and dx_2) will be equal to:

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \text{ or } dy = f_1 dx_1 + f_2 dx_2.$$

In this formula,

$\frac{\partial y}{\partial x_1} dx_1$ is called the **partial differential of $y = f(x_1, x_2)$ w.r.t x_1**

$\frac{\partial y}{\partial x_2} dx_2$ is called the **partial differential of $y = f(x_1, x_2)$ w.r.t x_2** , and

dy or $\frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$ is called the **total differential of $y = f(x_1, x_2)$** .

The definition and interpretation of differentials can be extended in an analogous manner to function of several variables as follows:

For a general function $y = f(x_1, x_2, \dots, x_n)$, the total differential in y is given by

$$dy = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$$

And it measures the total change in y when all variables $x_1, x_2, \dots, x_n$ are changes simultaneously.

Remarks

1. In the process of finding total differentials, we assumed that all independent variables are mutually exclusive (That is, each independent variable can change with out affecting or changing the others).

2. There are two techniques or methods of finding the total differential of a function. These are:

i. Using partial derivative method

In this method, first we compute partial derivatives of the function and then we use the formula given above for dy to get the total differential.

ii. Using rules of (total) differential

In this method, we consider each independent variable as a function and we apply the appropriate rules of differentiation for univariate function like power rule, sum or difference rule, product rule, quotient rule, chain rule and so on based on the way the variable or the combination of variables are presented in a given function. In other words, in this method, we use the following rules of total differential.

Rules of Total differential

Let k be a constant and u and v be each considered as a variable or a function. Then

Rule 1: Constant Rule: $dk = 0$

According to this rule, the total differential of a constant is zero. For instance, $d(3) = 0$, $d(-500) = 0$, and $dA = 0$ if A is considered as a constant.

Rule 2: Power Rule: $d(cu^n) = cnu^{n-1}du$, where c is a constant.

For instance, $d(5u^3) = 5(3u^2)du = 15u^2du$

Notice that we can apply this rule for x_1 and x_2 if they are presented in power form. For example,

$$d(x_1^3) = 3x_1^2dx_1, \quad d(4x_2^5) = 4(5x_2^4)dx_2 = 20x_2^4dx_2, \quad d(x_1) = 1dx_1 = dx_1 \quad \text{and} \quad d(x_2) = dx_2.$$

Rule 3: Sum or difference Rule: $d(u \pm v) = du \pm dv$

For instance, $d(u^2 + 5v^3) = d(u^2) + d(5v^3) = 2udu + 15v^2dv$

In terms of x_1 and x_2 , this rule for example, can be applied as

$$d(3x_1^2 + 4x_2^3) = d(3x_1^2) + d(4x_2^3) = 6x_1dx_1 + 12x_2^2dx_2 \quad \text{and} \quad d(5x_1^3 - 6x_2) = 15x_1^2dx_1 - 6dx_2.$$

Rule 4: Product Rule: $d(uv) = vdu + u dv$

For instance, $d(u^3v^2) = d(u^3)v + d(v^2)u = 3u^2vdu + 2uvdv$.

Note that this rule will help us to find the total differential of a term which contains x_1 and x_2 in their multiplicative form.

For example, $d(3x_1^2x_2) = d(x_1^2)3x_2 + d(x_2)3x_1^2 = (2x_1)(3x_2)dx_1 + 3x_1^2dx_2 = 6x_1x_2dx_1 + 3x_1^2dx_2$

Rule 5: Quotient Rule: $d\left(\frac{u}{v}\right) = \frac{vdu - u dv}{v^2}$, where $v \neq 0$

For instance,
$$d\left(\frac{u^3}{v^2}\right) = \frac{d(u^3)v^2 - d(v^2)u^3}{(v^2)^2} = \frac{3u^2v^2du - 2u^3v dv}{v^4}$$

Note also that we will use this rule when the function $y = f(x_1, x_2)$ is given in quotient form or to get the total differential of a term which contains x_1 and x_2 in quotient form. For example,

$$d\left(\frac{3x_1^2}{x_2}\right) = \frac{d(3x_1^2)x_2 - d(x_2)3x_1^2}{x_2^2} = \frac{6x_1x_2dx_1 - 3x_1^2dx_2}{x_2^2}$$

Again notice that the exponential, the logarithmic and the chain rules will also applied to get the total differential of a function $y = f(x_1, x_2)$ or any of its terms if its form entails the necessity of these rules.

For example, $d(e^{x_1}) = e^{x_1}dx_1$, $d(\ln x_2) = \frac{1}{x_2}dx_2$ and $d(\ln x_1^3) = \frac{1}{x_1^3}d(x_1^3) = \frac{1}{x_1^3}(3x_1^2dx_1) = \frac{3}{x_1}dx_1$.

Remark

All the above rules of total differential can also applied in analogous manner to find total differential of a function with more than two independent variables.

Now, let use see how we use the above two methods to find the total differential of multivariate function with the help of the following example:

Example

Find the total differential of each of the following functions using the partial derivative method and rules of total differential.

a) $y = 3x_1^2 + x_1^2x_2 - 2x_2^3$

b) $y = \ln(x_1^2 + 3x_2)$

c) $y = x_1^2x_2x_3^2$.

Solutions

a) Let us first use the partial derivative method, first we compute the partial derivatives as

$$f_1 = 6x_1 + 2x_1x_2 \text{ and } f_2 = x_1^2 - 6x_2^2$$

Then using the formula for dy as given above, we have

$$dy = f_1 dx_1 + f_2 dx_2 = (6x_1 + 2x_1x_2)dx_1 + (x_1^2 - 6x_2^2)dx_2.$$

The same result can be obtained using rules of total differential as follows:

$$\begin{aligned} dy &= d(3x_1^2 + x_1^2x_2 - 2x_2^3) = d(3x_1^2) + d(x_1^2x_2) - d(2x_2^3) \\ &= 6x_1dx_1 + d(x_1^2)x_2 + d(x_2)x_1^2 - 6x_2^2dx_2 = 6x_1dx_1 + 2x_1x_2dx_1 + x_1^2dx_2 - 6x_2^2dx_2 \\ &= (6x_1 + 2x_1x_2)dx_1 + (x_1^2 - 6x_2^2)dx_2. \end{aligned}$$

b) To apply the partial derivative method, first compute f_1 and f_2 as

$$\begin{aligned} f_1 &= \frac{\partial y}{\partial x_1} \ln(x_1^2 + 3x_2) = \frac{1}{x_1^2 + 3x_2} \frac{\partial}{\partial x_1} (x_1^2 + 3x_2) = \frac{1}{x_1^2 + 3x_2} (2x_1) = \frac{2x_1}{x_1^2 + 3x_2}, \text{ and} \\ f_2 &= \frac{\partial y}{\partial x_2} \ln(x_1 + 3x_2) = \frac{1}{x_1^2 + 3x_2} \frac{\partial}{\partial x_2} (x_1^2 + 3x_2) = \frac{1}{x_1^2 + 3x_2} (3) = \frac{3}{x_1^2 + 3x_2}. \end{aligned}$$

Then use the formula for dy as

$$dy = f_1 dx_1 + f_2 dx_2 = \frac{2x_1}{x_1^2 + 3x_2} dx_1 + \frac{3}{x_1^2 + 3x_2} dx_2.$$

The same result can be obtained using rules of total differential as

$$\begin{aligned} dy &= d[\ln(x_1^2 + 3x_2)] = \frac{1}{x_1^2 + 3x_2} d(x_1^2 + 3x_2) = \frac{1}{x_1^2 + 3x_2} d(x_1^2) + d(3x_2) \\ &= \frac{1}{x_1^2 + 3x_2} (2x_1dx_1 + 3dx_2) = \frac{2x_1}{x_1^2 + 3x_2} dx_1 + \frac{3}{x_1^2 + 3x_2} dx_2. \end{aligned}$$

c) Using partial derivative method, first we have

$$f_1 = \frac{\partial}{\partial x_1} (x_1^2 x_2 x_3^2) = 2x_1 x_2 x_3^2, \quad f_2 = \frac{\partial}{\partial x_2} (x_1^2 x_2 x_3^2) = x_1^2 x_3^2, \quad \text{and} \quad f_3 = \frac{\partial}{\partial x_3} (x_1^2 x_2 x_3^2) = 2x_1^2 x_2 x_3$$

Then using the formula for dy , we have

$$dy = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = (2x_1 x_2 x_3^2) dx_1 + (x_1^2 x_3^2) dx_2 + (2x_1^2 x_2 x_3) dx_3.$$

We can also get the same result using rules of total differential as

$$\begin{aligned} dy &= d(x_1^2 x_2 x_3^2) = d(x_1^2) x_2 x_3^2 + d(x_2) x_1^2 x_3^2 + d(x_3^2) x_1^2 x_2 \\ &= 2x_1 x_2 x_3^2 dx_1 + x_1^2 x_3^2 dx_2 + 2x_1^2 x_2 x_3 dx_3 \end{aligned}$$

Remark

As we have seen in the above example, both methods give the same result for all functions. Moreover, partial derivative method may be easier than using the rules of total differential in some function and the latter may be easier in other types of functions. But since economic text books may use either or both of the two methods to explain various concepts and to solve different problems, it will be worth full to know the two methods.

Exercises

1. Write the formula to get the total differential of $y = f(x_1, x_2)$ and give its interpretation.
2. Explain the two methods of finding the total differential and identify the one which is easier for you.
3. Find the total differential of each of the following functions using both method.

a) $y = x_1^2 + 3x_1^2 x_2 + 6x_1 x_2^2 + 2x_1^3$

b) $y = x_1^5 + x_2^4 - x_1^4 x_2^5$

c) $y = \frac{x_1}{x_1 + x_2}$

d) $y = \ln \sqrt{x_1^2 + x_2}$

e) $y = 2x_1^2 x_2 - x_1 x_3 + 3x_2^2 x_3^2$

4.2.2 Total Derivative

Under partial derivative and total differential, we assumed that all independent variables are mutually exclusive. In other words, we use partial derivative or total differential to measure the change in the dependent variable as far as the independent variables are not functionally related with each other or with other external variable (or parameter).

But these may not always be happened in many economic functions or models. There are many economic functions whose independent variables are functionally related either with each other or with another external variable and thus we cannot use our previous techniques to measure the

change in dependent variable for such types of functions. So, our question is that is there a technique which measures the change in the dependent variable for functions whose independent variables are functionally related? The answer is “yes” as we will see in **total derivative**.

As we did in the previous sections, let us first introduce the technique for $y = f(x_1, x_2)$ and we will extend it in analogous manner for n-variable function.

In $y = f(x_1, x_2)$, the independence between x_1 and x_2 may fail in the following three cases.

Case I: When $y = f(x_1, x_2)$ and $x_1 = g(x_2)$

In this case, x_1 is functionally related with x_2 which means that the dependent variable x_1 in a function g cannot change by itself. Moreover, x_1 can change and brings a corresponding change in y only x_2 changes. So, y changes only x_2 changes and for this reason we call x_2 as an **ultimate source** of change in y .

The change in y as a result of an infinitesimal change in x_2 is said to be the **total derivative of y with respect to x_2** and denoted as $\frac{dy}{dx_2}$.

To get $\frac{dy}{dx_2}$, we can use the following two step process.

First, we find the total differential of y as

$$dy = f_1 dx_1 + f_2 dx_2$$

Then, we divide both side of the above equation by dx_2 to get $\frac{dy}{dx_2}$ as

$$\frac{dy}{dx_2} = f_1 \frac{dx_1}{dx_2} + f_2 \frac{dx_2}{dx_2} = f_1 \frac{dx_1}{dx_2} + f_2$$

So, the total derivative, $\frac{dy}{dx_2}$, in this case, can be obtained as $\frac{dy}{dx_2} = f_1 \frac{dx_1}{dx_2} + f_2$

In this formula,

- $f_1 \frac{dx_1}{dx_2}$ is called an **indirect effect** of x_2 on y or f ,
- f_2 is called a **direct effect** of x_2 on y or f , and
- $\frac{dy}{dx_2}$ is called a **total effect or change** of x_2 on y .

Example; Find the total derivative given the function as $y = 5x_1^2 - 3x_2 + 1$, where $x_1 = 2x_2$.

Solution

The total derivative for this function is $\frac{dy}{dx_2}$ since x_2 is the independent variable in the second

function, $x_1 = 2x_2$, and it is given by $\frac{dy}{dx_2} = f_1 \frac{dx_1}{dx_2} + f_2$

But $\frac{dx_1}{dx_2}$ is the derivative of x_1 with respect to x_2 in $x_1 = 2x_2$ and it is equal to 2. Moreover, from the

first function, $y = 5x_1^2 - 3x_2 + 1$, we have $f_1 = \frac{\partial y}{\partial x_1} = 10x_1$ and $f_2 = \frac{\partial y}{\partial x_2} = -3$.

Thus, we have $\frac{dy}{dx_2}$ as $\frac{dy}{dx_2} = f_1 \frac{dx_1}{dx_2} + f_2 = 10x_1(2) - 3 = 20x_1 - 3$.

Case II : when $y = f(x_1, x_2)$, $x_1 = g(t)$ and $x_2 = h(t)$

In this case, x_1 and x_2 are not functionally related but both are related with another external variable t . So, both x_1 and x_2 cannot change y by themselves. The only source of change for y is the change in t and thus t is the ultimate source of change in y .

In this case also, we can use the two step process to get the total differential or change of y with respect to t as

$$\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt}$$

Notice that, $f_1 \frac{dx_1}{dt}$ and $f_2 \frac{dx_2}{dt}$ measure the indirect effect of t on y through x_1 and x_2 respectively.

There is no direct effect of t on y since t is not given as independent variable in the function f .

Now, let us see how we compute the total derivative for such type of case with the help of the following example;

Example

Find the total derivative $\frac{dy}{dt}$ for $y = 3x_1 + 2x_2$, where $x_1 = 2t^2$ and $x_2 = 5t + 1$

Solution

Here we use the formula $\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt}$. To do this, first compute f_1 and f_2 from

$$y = 3x_1 + 2x_2 \text{ as } f_1 = \frac{\partial y}{\partial x_1} = 3 \text{ and } f_2 = \frac{\partial y}{\partial x_2} = 2.$$

Then compute $\frac{dx_1}{dt}$ from $x_1 = 2t^2$ as $\frac{dx_1}{dt} = 4t$ and $\frac{dx_2}{dt}$ from $x_2 = 5t + 1$ as $\frac{dx_2}{dt} = 5$.

Finally, we substitute the results of $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in the formula to get $\frac{dy}{dt}$ as

$$\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt} = 3(4t) + 2(5) = 12t + 10.$$

Case III: When $y = f(x_1, x_2, t)$, $x_1 = g(t)$ and $x_2 = h(t)$

In this case, t is given as an independent variable in the function f and thus it has a direct effect on

y . So, using the two step process, the total derivative $\frac{dy}{dt}$ can be obtained using the formula

$$\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt} + f_t$$

Here the total change of derivative of y with respect to t is the sum of two indirect effect of t on y through x_1 and x_2 , and the direct effect of t on y .

Now, let us see how we apply the above formula using the following example:

Example

Find the total derivative for $y = 4x_1 + x_2t$, where $x_1 = t^2$ and $x_2 = 2t^3 + 1$.

Solution
$$\begin{aligned}\frac{dy}{dt} &= f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt} + f_t \\ &= 4(2t) + t(6t^2) + x_2\end{aligned}$$

But the answer for $\frac{dy}{dt}$ should be in terms of t and thus x_2 should be replaced by $2t^3 + 1$ to get the

final answer of $\frac{dy}{dt}$ as
$$\frac{dy}{dt} = 4(2t) + t(6t^2) + (2t^3 + 1) = 8t + 6t^3 + 2t^3 + 1 = 8t^3 + 8t + 1$$

Exercise

1. What is the major assumption on the nature of independent variables to apply total derivative in multivariate functions.
2. For each of the following functions, identify the ultimate source variable and find the total derivative with respect to this variable.

a) $y = 2x_1^2 - 3x_1x_2$, and $x_1 = 3x_2^2 + 5$

b) $y = x_2^2 + 5x_1x_3$, where $x_1 = 2t$, $x_2 = 3t^2$, $x_3 = t - 1$

c) $y = 4x_1t + x_2t^2$, where $x_1 = t$ and $x_2 = 3t^3 - 1$.

4.3 Derivatives of Implicit and Inverse Functions**4.3.1 Implicit Functions**

Generally an implicit function with n independent variables will be denoted as

$$F(x_1, x_2, \dots, x_n, y) = 0$$

Notice that we use the capital letter F to denote implicit function and a small letter f to denote explicit function. For instance, we use $y = f(x_1, x_2)$ and $F(x_1, x_2, y) = 0$ to denote an explicit and implicit function with two independent variable x_1 and x_2 respectively.

There are two methods of finding the (first) derivatives of implicit functions. These are:

- i) Partial derivative method and

ii) Implicit differentiation method

In the first method, we use a formula in terms of partial derivative which is stated as a rule for univariate and multivariate cases as follows:

Rule 1: For uni-variate implicit function

If $F(x, y) = 0$ and F_x and F_y both exist and $F_y \neq 0$, then the explicit function $y = f(x)$ exists and the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Where, $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$.

Rule 2: For Multivariate implicit function

If $F(x_1, x_2, \dots, x_n, y) = 0$ has continuous first order partial derivatives F_i and $F_y \neq 0$, then the explicit function $y = f(x_1, x_2, \dots, x_n)$ exists and the partial derivative $\frac{\partial y}{\partial x_i}$ is given by

$$\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}$$

Where $F_i = \frac{\partial F}{\partial x_i}$ and $F_y = \frac{\partial F}{\partial y}$

In the implicit differentiation method, we apply the technique of total differential as follows:

i. In univariate implicit function, $F(x, y) = 0$

➤ Take the total differential of both sides of the equation and then rearrange the terms to get $\frac{dy}{dx}$.

ii) In multivariate implicit function, $F(x_1, x_2, \dots, x_n, y) = 0$

➤ Compute the total differential of both sides of the equation and then rearrange the terms to get

$\frac{\partial y}{\partial x_i}$ considering all other variable $x_j \neq x_i$ as constant (or making all $dx_j = 0$ for $x_j \neq x_i$).

Example

Find $\frac{dy}{dx}$ for $2y + xy - 4x = 0$ using both methods

Solution

To apply the partial derivative method, let $F(x, y) = 2y + xy - 4x$, and compute F_x and F_y as

$$F_x = \frac{\partial F}{\partial x} = y - 4 \text{ and } F_y = \frac{\partial F}{\partial y} = 2 + x$$

Then use the formula given in rule 1 to get $\frac{dy}{dx}$ as $\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(y-4)}{2+x} = \frac{4-y}{2+x}$

To apply the implicit differentiation method, take the total differential of both sides of the equation and rearrange the terms to get $\frac{dy}{dx}$ as follows

$$\begin{aligned} d(2y + xy - 4x) &= d(0) \Rightarrow d(2y) + d(xy) - d(4x) = 0 \Rightarrow 2dy + xdy + ydx - 4dx = 0 \\ \Rightarrow (2+x)dy + (y-4)dx &= 0 \Rightarrow (2+x)dy = -(y-4)dx \Rightarrow \frac{dy}{dx} = \frac{-(y-4)}{2+x} = \frac{4-y}{2+x} \end{aligned}$$

Example

Find the first order partial derivatives for $3x_1x_2 + x_2y^2 - 10 = 0$

Solution

Since the function is an implicit function with two independent variables x_1 and x_2 , the required derivatives are the partial derivatives $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$.

Now, to apply the partial derivative method, we first let $F(x_1, x_2, y) = 3x_1x_2 + x_2y^2 - 10$, and compute F_1, F_2 and F_y as $F_1 = \frac{\partial F}{\partial x_1} = 3x_2$, $F_2 = \frac{\partial F}{\partial x_2} = 3x_1 + y^2$ and $F_y = \frac{\partial F}{\partial y} = 2x_2y$.

And then, we use the formula give in rule 2 to get $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$ as

$$\frac{\partial y}{\partial x_1} = \frac{-F_1}{F_y} = \frac{-3x_2}{2x_2y} = \frac{-3}{2y} \text{ and } \frac{\partial y}{\partial x_2} = \frac{-F_2}{F_y} = \frac{-(3x_1 + y^2)}{2x_2y}.$$

The same results for both $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$ is obtained using the implicit differentiation method as

First take the total differential of both sides as

$$\begin{aligned} d(3x_1x_2 + x_2y^2 - 10) &= d(0) \Rightarrow d(3x_1x_2) + d(x_2y^2) - d(10) = 0 \\ &\Rightarrow 3x_2dx_1 + 3x_1dx_2 + y^2dx_2 + 2x_2ydy = 0 \end{aligned}$$

Then to get $\frac{\partial y}{\partial x_1}$, in the last equation make $dx_2 = 0$ since x_2 is considered as a constant and change

d by ∂ . This will make the equation as

$$3x_2\partial x_1 + 2x_2y\partial y = 0 \Rightarrow 2x_2y\partial y = -3x_2\partial y \Rightarrow \frac{\partial y}{\partial x_1} = \frac{-3x_2}{2x_2y} = \frac{-3}{2y}$$

Similarly, to get $\frac{\partial y}{\partial x_2}$, set $dx_1 = 0$ and change d by ∂ in the last equation obtained from taking

total differential. This will make the equation as

$$\begin{aligned} 3x_1\partial x_2 + y^2\partial x_2 + 2x_2y\partial y &= 0 \Rightarrow (3x_1 + y^2)\partial x_2 + 2x_2y\partial y = 0 \Rightarrow 2x_2y\partial y = -(3x_1 + y^2)\partial x_2 \\ &\Rightarrow \frac{\partial y}{\partial x_2} = \frac{-(3x_1 + y^2)}{2x_2y} \end{aligned}$$

Exercise

1. Differentiate each of the following functions implicitly (find $\frac{dy}{dx}$).

a) $x^2 - 2x^2y + 3xy^2 - 9 = 0$ b) $x^2 \ln y - e^{xy} - 1 = 0$

2. Find the first order partial derivatives for each of the following functions.

a) $x_1^2 - x_2^2 + x_1x_2 + 12y = 0$ b) $3x_1^2 - x_2y + x_1x_2y^2 = 0$ c) $x_1^2 \ln y + ye^{x_1x_2} - y - 1 = 0$

5. Find $\frac{dx_1}{dx_2}$ for $x_1^2 + 4x_1x_2 + x_2^2 = 0$.

4.3.2 Derivatives of Inverse Function

Definition

The function $y=f(x)$ has an inverse provided that there is a function g such that the domain of g is the range of f and $y=f(x)$ if and only if $x = f(y)$. If $y=f(x)$ has an inverse function, then its inverse function is denoted as $f^{-1}(x)$ (read as “f inverse”)

The interest of this sub section is to find $\frac{dx}{dy}$ for $y=f(x)$. That is, finding the derivative of the

inverse function for $y = f(x)$. One possible way of getting $\frac{dx}{dy}$ for $y = f(x)$ is that first finding a formula for the inverse function, $f^{-1}(x)$, and then computing its derivative. But finding a formula for $f^{-1}(x)$ may be tedious or impossible for many types of functions. Thus, for such and other cases you can use the following general rule to find the derivative of inverse function.

If $y = f(x)$ has the inverse function, $f^{-1}(x)$, and $f^{-1}(x) \neq 0$, then $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$. That is the derivative of

the inverse function is the reciprocal of the derivative of the original function.

Example

Find $\frac{dx}{dy}$ for $y=5x^2+4x^3$

Solution

Since $\frac{dy}{dx} = 10x + 12x^2$, $\frac{dx}{dy}$ can be obtained as

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{10x + 12x^2}$$

4.4 Optimization of Multivariable Functions: Lagrange – multiplier method

The *Lagrange method* combines the objective function and the constraints into a single function called the **Lagrange function**, or simply the **Lagrangian**. The Lagrangian is usually denoted with the symbol $\mathcal{L}$ and is formed by rewriting the constraint so that it is nonnegative, multiplying it by a new variable λ called the **Lagrange** (or **Lagrangian**) **multiplier**, and adding the product to the objective function.

Consider the above constrained maximization problem of $U = x_1x_2 + x_1$, subject to the constraint $4x_1 + 2x_2 = 60$. Let us write what is referred to as the *Lagrangian function*, which is a modified version of the objective function that incorporates the constraint as follows:

$$L = x_1x_2 + 2x_1 + \lambda(60 - 4x_1 + 2x_2)$$

The symbol λ (the Greek letter lambda), representing some as yet undetermined number, is called a *Lagrange (undetermined) multiplier*. If somehow be assured that $4x_1 + 2x_2 = 60$, so that the constraint will be satisfied, then the last term in Lagrange equation will vanish regardless of the value of λ . This is because of $60 - 4x_1 + 2x_2 = 0$, and multiplying λ by zero will give a value equal to zero. In that event, L will be identical with U (initial objective function). Moreover, with the constraint out of the way, we only have to seek the free maximum of L , in lieu of the *constrained* maximum of U , with respect to the two variables x_1 and x_2 . The question is: how can we make the parenthetical expression in the above Lagrange equation vanish?

The tactic that will accomplish this is simply to treat λ as an additional variable in the Lagrange equation, i.e., to consider $L = L(\lambda, x_1, x_2)$. For then the first-order condition for free extreme will consist of the set of simultaneous equations

$$L_{\lambda} \left(\equiv \frac{dz}{d\lambda} \right) = 60 - 4x_1 - 2x_2 = 0$$

$$L_{x_1} \left(\equiv \frac{dz}{dx_1} \right) = x_2 + 2 - 4\lambda = 0$$

$$L_{x_2} \left(\equiv \frac{dz}{dx_2} \right) = x_1 - 2\lambda = 0$$

and the first equation will automatically guarantee the satisfaction of the constraint. Thus, by incorporating the constraint into the Lagrangian function L and by treating the Lagrange multiplier as an extra variable, we can obtain the constrained extremum $\bar{U}$ (two choice variables) simply by screening the stationary values of L , taken as a *free* function of three choice variables: x_1 , x_2 , and λ .

Solving the above first order equation for the critical values of the variables, starting from the last equation we have, $\lambda = 1/2x_1$

Substituting the value of λ into the second equation, we have

$$x_2 + 2 - 4\lambda = 0 \Rightarrow x_2 + 2 - 4\left(\frac{1}{2}x_1\right) = 0$$

$$x_2 = 2x_1 - 2$$

Substituting again this equation for x_2 into the first equation, we get

$$60 - 4x_1 - 2x_2 = 0 \Rightarrow 60 - 4x_1 - 2(2x_1 - 2) = 0$$

$$60 - 8x_1 + 4 = 0$$

$$\bar{x}_1 = 8$$

$$\bar{x}_2 = 2x_1 - 2 \Rightarrow x_2 = 2(8) - 2 = 14$$

Similarly, we have $\lambda = \frac{1}{2x_1} \Rightarrow \bar{\lambda} = \frac{1}{2}(8) = 4$

The solution will then be $\bar{x}_1 = 8$, $\bar{x}_2 = 14$ (and $\bar{\lambda} = 4$).

Example: Find the extremum of

$$Z = xy \quad \text{subject to} \quad x + y = 6$$

Solution

The Lagrangian function is given by $L = xy - \lambda(6 - x - y)$

For a stationary value of L , it is necessary that

$$L_\lambda = 6 - x - y = 0 \quad x + y = 6$$

$$L_x = y - \lambda = 0 \quad -\lambda + y = 0 \rightarrow \lambda = y$$

$$L_y = x - \lambda = 0 \quad -\lambda + x = 0 \rightarrow \lambda = x = y$$

Substituting the value $x=y$ on constraint function we get the value of x and y .

$$\text{i.e. } 6 - x - x = 0 \rightarrow x = 3 = y = \lambda$$

The stationary value is $\bar{L} = \bar{Z} = 9$, which needs to be tested against a second-order condition before we can tell whether it is a maximum or minimum (or neither). That will be taken up later.

Excercise

$$\text{Optimize } Z = x_1^2 + x_2^2 \quad \text{subject to} \quad x_1 + 4x_2 = 2$$

Minimization problem

The principles of minimizations are the same with that of maximization. The basic difference lies on the formulation of the problem. We can use the Lagrangian function to solve such problems. The Lagrangian for the problem of minimizing $f(x, y)$ subject to the constraint that $g(x, y) = g_0$ is

$$\mathcal{L}_{(x,y,\lambda,g_0)} = f(x, y) + \lambda(g_0 - g(x, y))$$

The only difference between this Lagrangian and the one for a maximization problem is that we write the constraint so it is nonpositive. (As in the case with maximization problems, the only reason this matters is for the interpretation of the Lagrange multiplier)

We derive the first-order conditions of the minimization of the above equation exactly as in the maximization case, by setting equal to zero the derivatives of the Lagrangian with respect to the Lagrangian multiplier and each of the choice variables.

Example

$$\text{Minimize: } \min Z = 8x + 6y \text{ subject to } 108 = x^{0.4}y^{0.6}.$$

Solution

The Lagrangian function is then

$$\min \mathcal{L} = (8x + 6y) + \lambda(108 - x^{0.4}y^{0.6})$$

The first-order condition for this minimization problem can be obtained

$$\frac{\partial \mathcal{L}}{\partial x} = 8 - 0.4\lambda x^{0.4-1}y^{0.6} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 6 - 0.6\lambda x^{0.4}y^{0.6-1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 108 - x^{0.4}y^{0.6} = 0$$

To eliminate the Lagrangian multiplier λ from the first two of the first-order functions,

$$\lambda = \frac{8}{0.4x^{-0.6}y^{0.6}} = \frac{20x^{0.6}}{y^{0.6}}$$

$$\lambda = \frac{6}{0.6x^{0.4}y^{-0.4}} = \frac{10y^{0.4}}{x^{0.4}}$$

Eliminating λ and equating the two functions

$$\frac{20x^{0.6}}{y^{0.6}} = \frac{10y^{0.4}}{x^{0.4}}$$

$$20x^{0.6}x^{0.4} = 10y^{0.4}y^{0.6} \rightarrow y = 2x$$

Substituting the value of x in in the third first-order function,

$$108 - (x)^{0.4}(2x)^{0.6} = 0 \Rightarrow (2)^{0.4}y = 108$$

$$x^* = 81.85 \text{ and } y^* = 2x = 2 \times 81.85 = 163.7$$

The stationary values of the minimization problem is thus, $x^* = 81.85$, $y^* = 163.7$ and the minimum objective value will then be $Z^* = 8x + 6y = 8 * 81.85 + 6 * 163.7 = 1800.7$

Importance of Lagrange Multiplier Technique

Lagrange-multiplier technique gives you the value of λ as a by-product. The value of λ conveys an important information. It measures the sensitivity of the $\bar{L}$ (or $\bar{Z}$) to a shift of the constraint. Therefore, the Lagrange-multiplier method offers the advantage of containing certain built-in comparative statistic information in the solution. If, for example, the objective function is to maximize utility subject to budget constraint, the value of λ in the solution will tell you by how

much the optimal solution – in this particular case the maximum utility - will increase when the total budget allocated will increase by one unit. In other words, it shows the extent to which the optimal solution will change when *the constraint is relaxed*. Hence, it shows the marginal effect of a relaxation of its associated constraint on the value of objective function.

Meaning of the value of Lagrange - λ

The value of the Lagrange multiplier on the constraint at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the constraint is relaxed. If the constraint is more than one and if the j^{th} constraint arises because of a limit on the amount of some resource, then we refer to $\lambda_j(c)$ as the shadow price of the j^{th} resource.

Thus, the value of λ shows the amount of change in utility due to a shift in the budget line. As you remember from microeconomics course, when the budget line shifts to the right due to an increase in income of consumer or reduction in tax, the consumer's optimal allocation will change and the consumer will achieve a higher level of satisfaction. The value of λ , therefore, tells you the amount of increase in the level of optimum utility due to a one Birr increase in the budget of the consumer. For instance, in the example above, the value of λ which is equal to 4 tells us that if the constraint increases by one unit (from 60 to 61), the optimum level of utility will increase by 4 units (from the initial optimum point of $\bar{L} = 128$ to $\bar{L} = 132$).

4.5 Economic Applications of Partial Derivatives

Constrained optimization is very important for the following two applications i.e. utility and Profit analysis

I. Utility Analysis

The main objective of rational consumer is maximization his utility given his limited income and market prices of goods and services. Mathematically, the aim of the consumer can be written as

$$\text{Max } U = f(x, y, z)$$

S.t

$$I = P_x \cdot x + p_y \cdot y + p_z \cdot z$$

where U is a total utility

I is consumer limited income

P_x, p_y , and p_z are per unit price of good x, y, and z respectively

The main interest here is to find the consumer equilibrium (i.e, the consumption level that gives maximum utility to the consumer). This in fact means that maximizing the above constrained utility function through the Lagrange Multiplier method.

To do this the first step is to find the critical points for the corresponding Lagrange Multiplier function through first order derivative test as follows.

Let $V(x, y, z, \lambda) = U(x, y, z) - \lambda (p_x \cdot x + p_y \cdot y + p_z \cdot z - \bar{I})$ be the Lagrange Multiplier function,

then find the first order partial derivatives and equating them to zero to get the critical points.

$$\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - \lambda \cdot p_x = 0$$

$$\frac{\partial V}{\partial y} = \frac{\partial U}{\partial y} - \lambda \cdot p_y = 0$$

$$\frac{\partial V}{\partial z} = \frac{\partial U}{\partial z} - \lambda \cdot p_z = 0$$

$$\frac{\partial u}{\partial \lambda} = - \left(p_x \cdot x + p_y \cdot y + p_z \cdot z - \bar{I} \right) = 0$$

But $\frac{\partial U}{\partial x} = MU_x$, $\frac{\partial U}{\partial y} = MU_y$, and $\frac{\partial U}{\partial z} = MU_z$

So the above equations can be re-arranged as: $MU_x = \lambda \cdot P_x$, $MU_y = \lambda \cdot P_y$ and $MU_z = \lambda \cdot P_z$

And from these equations you can derive the *first condition for maximization* as

$$\lambda = \frac{MU_x}{P_x} = \frac{MU_y}{P_y} = \frac{MU_z}{P_z}$$

The second condition for maximization from the second order derivative test will be

$$\frac{\partial^2 V}{\partial x^2} < 0, \frac{\partial^2 V}{\partial y^2} < 0 \text{ and } \frac{\partial^2 V}{\partial z^2} < 0$$

Moreover, $\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 U}{\partial x^2} < 0$ and $\frac{\partial^2 U}{\partial x^2}$ is slope of MU_x

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 U}{\partial y^2} < 0 \text{ and } \frac{\partial^2 U}{\partial y^2} \text{ is slope of } MU_y$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 U}{\partial z^2} < 0 \text{ and } \frac{\partial^2 U}{\partial z^2} \text{ is slope of } MU_z$$

Therefore, at equilibrium *the marginal utilities of each commodity must be positive and decline.*

Example

Find the levels of x and y that can maximize the utility given by

$$\text{Max } U = x^{0.6} y^{0.25}$$

S. t

$$8x + 5y = 680$$

Solution

$$V(x, y, \lambda) = x^{0.6} y^{0.25} - \lambda(8x + 5y - 680)$$

$$\frac{\partial V}{\partial x} = 0.6 x^{0.6-1} y^{0.25} - 8\lambda = 0$$

$$\frac{\partial V}{\partial y} = 0.25 x^{0.6} y^{0.25-1} - 5\lambda = 0$$

$$\frac{\partial V}{\partial \lambda} = -(8x + 5y - 680) = 0$$

$$0.6 x^{-0.4} y^{0.25} = 8\lambda \dots\dots\dots(1)$$

$$0.25 x^{0.6} y^{-0.75} = 5\lambda \dots\dots\dots(2)$$

Then divide equation (1) by (2), t

$$\frac{0.6 x^{-0.4} y^{0.25}}{0.25 x^{0.6} y^{-0.75}} = \frac{8}{5} \rightarrow \frac{60}{25} \frac{y}{x} = \frac{8}{5} \rightarrow \frac{y}{x} = \frac{8}{5} \times \frac{25}{60} = \frac{40}{60} \text{ and } \frac{x}{y} = \frac{60}{40} = 1.5 \rightarrow x = 1.5y.$$

Substitute $x = 1.5y$. on budget line equation as

$$\begin{array}{ll}
 8(1.5y) + 5y = 680 & \text{and } x = 1.5y \\
 12y + 5y = 680 & + 1.5(40) \\
 17y = 680 & x = 60 \text{ units} \\
 y = 40 \text{ units} &
 \end{array}$$

Moreover, $\frac{\partial^2 V}{\partial x^2} = -0.4(0.6)x^{-0.4-1}y^{0.25} = -0.24x^{-1.4}y^{0.25} < 0$ at $x = 60$ and $y = 40$

$$\frac{\partial^2 V}{\partial y^2} = -0.75(0.25)x^{0.6}y^{-0.75-1} = -0.1875x^{0.6}y^{-1.75} < 0 \text{ at } x = 60 \text{ and } y = 40$$

Therefore, $x = 60$ units and $y = 40$ units are utility maximization combinations.

II. Profit Analysis

Profit maximization of the firm is achieved in two ways:

- Maximization of output subject to a cost constraint.
- Minimization of cost for a given level of output.

A) Maximization of output subject to a cost constraint

In this situation cost is given or remaining constant. Mathematically, it can be represented as

$$\text{Max } Q = f(L, K)$$

$$\text{S.t}$$

$$\bar{C} = wL + rK$$

Here also, you can use the method of Lagrange Multiplier to find the level of input combination that can maximize the profit of the firm.

Let $F(L, K, \lambda) = f(L, k) - \lambda(wL + rK - \bar{C})$ be the Lagrange Multiplier function, then find the

first order partial derivatives and equating them to zero to get the critical points.

$$\frac{\partial F}{\partial L} = \frac{\partial f}{\partial L} - \lambda w = 0$$

$$\frac{\partial F}{\partial K} = \frac{\partial f}{\partial K} - \lambda r = 0$$

$$\frac{\partial F}{\partial \lambda} = -\left(wL + rK - \bar{c}\right) = 0$$

But the expressions $\frac{\partial f}{\partial L}$ and $\frac{\partial f}{\partial K}$ are the marginal products of labor and capital respectively and thus the first two of the above equations can be re-arranged as $MP_L = \lambda w$ and $MP_K = \lambda r$

If you solve these equations for λ , we will get the expression first condition for maximization as

$$\lambda = \frac{MP_L}{w} = \frac{MP_K}{r}$$

The second condition for maximization will be

$$\frac{\partial^2 F}{\partial L^2} = \frac{\partial^2 f}{\partial L^2} < 0, \text{ and } \frac{\partial^2 f}{\partial L^2} = \text{slope of } MP_L$$

$$\frac{\partial^2 F}{\partial K^2} = \frac{\partial^2 f}{\partial K^2} < 0, \text{ and } \frac{\partial^2 f}{\partial K^2} = \text{slope of } MP_K$$

Therefore, at equilibrium,

- i) The ratio of marginal product to the price of the input must be equal for all inputs and
- ii) Each marginal product must be positive but declining (slope of MP must be negative)

B) Minimization of cost for a given level of output

In this case output is given or remaining constant and the problem can be written mathematically as

$$\text{Min } C = wL + rK$$

$$\text{s.t.}$$

$$\bar{Q} = f(L, K)$$

Let $V(L, K, \lambda) = wL + rK - \lambda(f(L, K) - \bar{Q})$ be the Lagrange function.

First apply the first order test on the Lagrange function, V , to get the first order condition for maximization.

$$\frac{\partial V}{\partial L} = w - \lambda \cdot \frac{\partial f}{\partial L} = 0$$

$$\frac{\partial V}{\partial K} = r - \lambda \cdot \frac{\partial f}{\partial K} = 0$$

$$\frac{\partial V}{\partial \lambda} = - \left(f(L, K) - \bar{Q} \right) = 0$$

Using the marginal product notations, the first two of the above equations can be written as

$$\begin{aligned} w &= \lambda \cdot MP_L \\ r &= \lambda \cdot MP_K \end{aligned}$$

If you solve these equations for λ , you will get the expression

$$\lambda = \frac{w}{MP_L} = \frac{r}{MP_K} \text{ or } \frac{1}{\lambda} = \frac{MP_L}{w} = \frac{MP_K}{r} \text{ as the first condition for maximization.}$$

The second order condition for V to be maximized will be

$$\frac{\partial^2 V}{\partial L^2} = -\lambda \cdot \frac{\partial^2 f}{\partial L^2} > 0 \text{ which implies that } \frac{\partial^2 f}{\partial L^2} < 0 \text{ or the slope of } MP_L \text{ must be negative}$$

$$\frac{\partial^2 V}{\partial K^2} = -\lambda \cdot \frac{\partial^2 f}{\partial K^2} > 0 \text{ which implies that } \frac{\partial^2 f}{\partial K^2} < 0 \text{ or the slope of } MP_K \text{ must be negative}$$

Example

The cost of manufacturing x units at the first plant and y units at the second plant is given by $C(x, y) = x^2 + 2y^2 + 5xy + 700$. If the firm has a supply order of 500 units, how many units should be produced at each plant to minimize the total cost?

Solution

$$\text{Min } C = x^2 + 2y^2 + 5xy + 700$$

$$\text{Subject to } x + y = 500$$

Let $V(x, y, \lambda) = x^2 + 2y^2 + 5xy + 700 - \lambda(x + y - 500)$ be the Lagrange function, then the first conditions will be

$$\frac{\partial V}{\partial x} = 2x + 5y - \lambda = 0$$

$$\frac{\partial V}{\partial y} = 5x + 4y - \lambda = 0$$

$$\frac{\partial V}{\partial \lambda} = -(x + y - 500) = 0$$

Then rearrange the first equations as

$$\begin{aligned} 2x + 5y &= \lambda \\ 4y + 5x &= \lambda \end{aligned}$$

Again solve these equations simultaneously to get $y = 3x$ and substitute it in the supply order constraint function as follows to get the critical point.

$$x + 3x = 500 \Rightarrow 4x = 500 \Rightarrow x = 125 \text{ and } y = 3(125) = 375$$

Moreover, $\frac{\partial^2 V}{\partial x^2} = 2 > 0$ and $\frac{\partial^2 V}{\partial y^2} = 4 > 0$

Therefore, $x = 125$ units and $y = 375$ units are cost minimizing levels of output

Exercise: find the values of the function that can optimize the following functions

a) $Z = 4x^2 - 2xy + 6y^2$ subject to $x + y = 72$

b) $Z = 4x^2 - 3x + 5xy - 8y + 2y^2$ subject to $x = 2y$

UNIT FIVE

INTEGRATION

5.1 The Concept and Properties of Integration

Integral calculus and differential calculus have inverse meanings and processes. Differentiation is the process of converting the total function in to marginal functions but integration is the process of converting marginal function in to aggregate functions. Integration is mainly divided in to two parts; indefinite and definite integrals. In this unit you will discuss the meaning of integration, rules of integration and application of integration on economic variables.

5.1.1 The Concept of Integration

Definition: The function F is an anti-derivative of the continuous function f if and only if for all X in the domain of f such that $F'(X) = f(X)$.

Example: The function $f(X) = \frac{1}{3}x^3$ is the ant derivative of $f(X) = X^2$ because $\frac{d}{dx}\left(\frac{1}{3}X^3\right) = X^2$.

Observe that the function $f(x) = x^2$ has other anti-derivatives. Some of these are:

$$F_1(x) = \frac{1}{3}x^3 + 7 \quad F_2(x) = \frac{1}{3}x^3 - 132 \quad F_3(x) = \frac{1}{3}x^3 + \frac{52}{3} \text{ and so on.}$$

The function, $F(x) = \frac{1}{3}x^3 + C$ where C is any arbitrary constant, is the anti derivative of $f(X) = X^2$.

Example: For the function $f(x) = \frac{-2}{X^3}$, if we find the anti derivation of this function, we have to

find a function whose derivative can generate the value $\frac{-2}{X^3}$. Some of these are

$$F_1(x) = \frac{1}{x^2} + 2 \quad F_2(x) = \frac{1}{x^2} - 30 \quad F_3(x) = \frac{1}{x^2} + \frac{93}{4} \text{ and so on}$$

The function $F(x) = \frac{1}{x^2} + C$, where C is any constant, is the anti derivative of the function $f(x) = \frac{-2}{x^3}$

Theorem 1 If $F(x)$ is anti-derivative function of $f(x)$, then $F(x) + C$ where C is an arbitrary constant, is also the anti-derivative of $f(x)$

Theorem 2 If $F(x)$ and $G(x)$ are anti-derivatives of the function $f(x)$ in the interval $[a, b]$, then $F = G(x) + C$, for all $x \in [a, b]$

Exercise

1) Find the anti-derivatives of the following functions;

$$\begin{array}{ll} \text{a). } f(x) = x^5 + 6 & \text{b). } f(x) = \frac{1}{x} \\ \text{c). } f(x) = 8(2x + 3)^3 & \text{d). } f(x) = \frac{1}{2}x^2 + 3x + 5 \end{array}$$

2) Find the anti-derivatives of the following function

$$\text{a). } f(x) = e^{2x} \quad \text{b). } f(x) = 2xe^{x^2} \quad \text{c). } f(x) = \log a^x$$

Definition: The set of all anti derivatives of a continuous function is called the indefinite integral of f and is denoted symbolically by $\int f(x) dx$.

The function f is called the integrand and x is called the variable of integration.

$$\int f(x) dx = F(x) + C$$

Example: Find the integral of the function $f(x) = 3x^2$

Solution: - we have to find the anti-derivation of this function

$$\int f(x) dx = F(x) + C \text{ from the definition}$$

Therefore, $\int 3x^2 dx = x^3 + C$, more over, Form the definition of integration, it is clear that:

$$\frac{d}{dx} \left[\int f(x) dx \right] = f(x)$$

5.1.2 Basic Rules (Properties) of Integration

Just as there are rules of derivation, there are also certain rules of integration. The following are important rules of integration.

1) **Rule of power function.** If $f(x)$ is given in certain power function, its integration can be easily

calculated with this rule. If $f(x) = x^n$ then $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

This is because $\frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = x^n (n+1)$

Example

Find the integration of the following functions

$$\text{a). } f(x) = x^3 \quad \text{b). } f(x) = \frac{1}{x^2} \quad \text{c). } f(x) = dx \quad \text{d). } f(x) = \frac{1}{x}$$

Solution

Since most of them are power functions, we can use power rule for finding their integration.

$$\text{a). } \int x^3 dx = \frac{x^{3+1}}{3+1} + C = \frac{1}{4} x^4 + C$$

$$\text{b). } \int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} = \frac{-1}{x} + C$$

$$\text{c). } \int dx = \int 1 dx = \int x^{0dx} = \int x^0 dx = \frac{x^{0+1}}{0+1} + C = x + C$$

$$\text{d). } \int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} \quad \text{which is undefined because any number divided by zero}$$

is not defined. So, this function can't be closed under the rule of this power. That is why we

said that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for all values of n except $n = -1$. Therefore, we have to

look for other way to find the integration. $\int \frac{1}{x} dx = \ln |x| + c$

2). **Coefficient rule** ÷ the integral of the product of a constant and a function of x is equal to the constant times the integral of a function.

$$\int K \cdot f(x) dx = K \int f(x) dx$$

Example

$$\text{a) } \int 3x^2 dx = 3 \int x^2 dx = 3 \left(\frac{1}{3} x^3 \right) + C = x^3 + C$$

$$\text{b) } \int 2e^x dx = 2 \int e^x dx = 2e^x + C$$

3) **Summation Rule**: - The integral of the sum of two functions is the sum of their integrals.

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$$\int 2x dx = \int x dx + \int x dx = \frac{1}{2} x^2 + \frac{1}{2} x^2 + C = x^2 + C$$

4). **Difference Rule**: - Integral of the difference of two functions is the difference of their integrals

$$\int (f(x) - g(x)) dx = \int f(x) dx - \int g(x) dx$$

$$\text{Example: } \int (x - 6) dx = \int x dx - \int 6 dx = \frac{1}{2} x^2 - 6x + C$$

5) **The exponential rule**: It states that $\int e^x dx = e^x + C$

5.2 Techniques of Integration

5.2.1 Integration by substitution

This rule helps to find the integration of functions whose derivative is found by using chain rule.

$$\int f(g(x)) \cdot g'(x) = F(g(x)) + C \text{ for any differentiable function } g \text{ that is not a constant function.}$$

$$\text{If } u = g(x), \int f(u) du = F(u) + C = F(g(x)) + C$$

To find integral of a function by substitution we can follow the following steps

1. Introduce the letter u stands for some expression in x that is chosen with the simplification of the integral as the goal
2. Rewrite the integral in-terms of u , to rewrite dx compute du/dx and algebraically as if the symbol du/dx were a quotient
3. Evaluate the resulting integral and then replace u by its expression in terms of x in the answer

Example Find the integration of $2xe^{x^2} dx$

Solution

Step 1. Let $u = x^2$ $du = 2x dx$ and solve for $dx = \frac{du}{2x}$

$$\text{Step 2. } \int 2xe^{x^2} dx = \int 2xe^{x^2} dx = \int 2xe^u \frac{du}{2x} = \int e^u du = e^u + C$$

$$\text{Step 3. Substitute } u \text{ by } x^2 \text{ then } \int 2xe^{x^2} dx = e^{x^2} + C$$

Example Find the integration of $2x(x^2 + 1)^4 dx$

Solution

Step 1. let $u = x^2 + 1$ and $du = 2x dx$ and solve for $dx = \frac{du}{2x}$

$$\text{Step 2. } \int 2x(x^2 + 1)^4 dx = \int 2xu^4 \frac{du}{2x} = \int u^4 du = \frac{1}{5}u^5 + C,$$

$$\text{Step 3. Substitute } u \text{ by } x^2 + 1 \text{ and the final solution is given as } \int 2x(x^2 + 1)^4 dx = \frac{1}{5}(x^2 + 1)^5 + C$$

Exercise: find the integral of

$$\text{a. } (x^2 + 3x + 5)^9 (2x + 3) dx = (x^2 + 3x + 5)^9 + c$$

$$\text{b. } x^3 e^{x^4 + 2} dx = \frac{1}{4} x^3 e^{x^4 + 2} + c$$

5.2.2 Integration by Parts

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

Here, we have to be systematic enough when we are assigning which one is $f(x)$ and which one is $g'(x)$. Assign a function whose anti derivative is found easily as $g'(x)$ and a function whose derivative is found easily as $f(x)$.

Example Find the integration of xe^{2x}

Solution: let $f(x) = x$ and $g'(x) = e^{2x} \Rightarrow g(x) = \frac{1}{2} e^{2x}$

$$\begin{aligned} \int xe^{2x} dx &= x \frac{1}{2} e^{2x} - \int 1 \cdot \frac{1}{2} e^{2x} dx = \frac{1}{2} xe^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{1}{2} xe^{2x} - \frac{1}{2} \left(\frac{1}{2} e^{2x} \right) + C \\ &= \frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} + C = \frac{1}{4} e^{2x} (2x - 1) + C \end{aligned}$$

Example: Find the integration of $x^2 \ln x dx$

Solution

Let $f(x) = \ln x$ and $g'(x) = x^2$, implying that $g(x) = \frac{1}{3} x^3$

$$\int x^2 \ln x dx = \ln x \cdot \frac{1}{3} x^3 - \int \frac{1}{x} \cdot \frac{1}{3} x^3 dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \left(\frac{1}{3} x^3 \right) + C = \frac{1}{9} x^3 (3 \ln x - 1) + C$$

Steps to find the integral of a function by integration part

1. Select one of the factors of the product as the one to be integrated and the other to be differentiated based on easiness to find them
2. Integrate the designed factor and multiply it by the other factor
3. Differentiate the designed factor and multiply it by the integrated factor from step 2 and subtract the integral of this product from the result of step 2

4. Complete the producer by finding the new integral that was formed in step 3 and add the constant term c only at the very end.

➤ Here we used $f(x)$ to denote the factor to be differentiated and $g(x)$ to factor to integrate.

Theorem3: If $\int f(x)dx = F(x) + C$, then $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$ and

$$\int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$$

The integration of $\frac{1}{ax+b} dx$ is $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln^{ax+b} + C$

Example

Compute the integration of $(3x-7)^5 dx$

Solution

$\int (3x-7)^5 dx$ can be found in two ways; one is by substitution rule and the other is by theorem3.

$$\int (3x-7)^5 dx = \frac{1}{3} \frac{(3x-7)^6}{6} + C = \frac{1}{18} (3x-7)^6 + C$$

Exercise; Compute the following by

1. Applying substitution rule

a) $\int x\sqrt{x+5} dx$ b) $\int x^2 e^x dx$ c) $\int \ln x dx$

2. Applying substitution rule or theorem 3

a) $\int \frac{3}{x} dx$ b) $\int (2ax+b)^2 dx$ c) $\int \frac{1}{2x+3} dx$ d) $\int (2ax+b)(ax^2+bx)^7 dx$

5.2.3 Integration by Partial Fraction

It is a technique of integration applicable when the function is given in fractional form of two polynomial functions (rational functions).

Steps to followed in this technique of integration

1. find the common denominator for the fraction
2. add factors and equate to the numerators
3. find the value of A and B
4. Use the general equation
5. Integrate the general equation you get in step 4

Example: Compute

a. $\int \frac{1}{x^2 - 4} dx$ b. $\int \frac{2x+3}{x^2 - 9} dx$ c. $\int \frac{2-x}{x^2 + 5x} dx$

Solution

a. **Step1.** Find the common denominator $\int \frac{1}{x^2 - 4} dx = \int \frac{1}{(x+2)(x-2)} dx = \left(\frac{A}{x+2} + \frac{B}{x-2} \right) dx$

Step2. Add factors and equate to nominator $A(x-2) + B(x+2)=1$

Step3. Find the value of A and B

$$\text{Let } x=-2 \Rightarrow A(-2-2) + B(-2+2)=1 \Rightarrow -4A(x-2)=1 \Rightarrow A=-1/4$$

$$\text{Let } x=2 \Rightarrow A(2-2) + B(2+2)=1 \Rightarrow 4B=1 \Rightarrow B=1/4$$

Step 4 the general equation is $\int \frac{1}{x^2 - 4} dx = \int \left(\frac{-1/4}{x+2} + \frac{1/4}{x-2} \right) dx =$

Step5. Integrate the general equation $\int \left(\frac{-1/4}{x+2} + \frac{1/4}{x-2} \right) dx = \int \left[-\frac{1}{4} \left(\frac{1}{x+2} \right) + \frac{1}{4} \left(\frac{1}{x-2} \right) \right] dx$

$$= -1/4 \int \left(\frac{1}{x+2} \right) dx + 1/4 \int \left(\frac{1}{x-2} \right) dx = \frac{-1}{4} \ln|x+2| + \frac{1}{4} \ln|x-2|$$

$$= \frac{1}{4} (\ln|x-2| - \ln|x+2|) + c = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + c$$

b. **Step1.** Find the common denominator $\int \frac{2x+3}{x^2-9} dx = \int \frac{2x+3}{(x+3)(x-2)} dx = (\frac{A}{x+3} + \frac{B}{x-3}) dx$

Step2. Add factors and equate to nominator $A(x-3) + B(x+3) = 2x+3$

Step3. Find the value of A and B

$$\text{Let } x=-3 \Rightarrow A(-3-3) + B(-3+3) = 2(-3)+3 \Rightarrow -6A = -3 \Rightarrow A = 1/2$$

$$\text{Let } x=3 \Rightarrow A(3-3) + B(3+3) = 2(3)+3 \Rightarrow 4B = 1 \Rightarrow B = 3/2$$

Step 4 the general equation is $(\frac{A}{x+3} + \frac{B}{x-3}) dx = (\frac{1/2}{x+3} + \frac{3/2}{x-3}) dx$

Step5. Integrate the general equation $(\frac{1/2}{x+3} + \frac{3/2}{x-3}) dx = \int [\frac{1}{2}(\frac{1}{x+3}) + \frac{3}{2}(\frac{1}{x-3})] dx$

$$= \frac{1}{2} \int (\frac{1}{x+3}) dx + \frac{3}{2} \int (\frac{1}{x-3}) dx = \frac{1}{2} \ln|x+3| + \frac{3}{2} \ln|x-3| + c$$

c. **Step1.** Find the common denominator $\int \frac{2-X}{x^2+5X} dx = \int \frac{2-x}{x(x+5)} dx = \int \frac{A}{x} + \frac{B}{x+5} dx$

Step2. Add factors and equate to nominator $A(x+5) + B(x) = 2-x$

Step3. Find the value of A and B

$$\text{Let } x=0 \Rightarrow A(0+5) + B(0) = 2-0 \Rightarrow 5A = 2 \Rightarrow A = 2/5$$

$$\text{Let } x=-5 \Rightarrow A(-5+5) + B(-5) = 2-(-5) \Rightarrow -5B = 7 \Rightarrow B = -7/5$$

Step 4 the general equation is $\int \frac{2-x}{x(x+5)} dx = \int \frac{A}{x} + \frac{B}{x+5} dx$

Step5. Integrate the general equation $\int \frac{2-x}{x(x+5)} dx = \int \frac{2/5}{x} + \frac{-7/5}{x+5} dx$

$$= \frac{2}{5} \int \frac{1}{x} dx - \frac{7}{5} \int \frac{1}{x+5} dx = \frac{2}{5} \ln|x| - \frac{7}{5} \ln|x+5| + c$$

Exercise: find the integral of

a. $\frac{x^2}{x^2-16}$

b. $\frac{x^2+x-1}{x(x^2-1)}$

5.3 Definite Integrals

Meaning of Definite Integrals

All the integrals cited in the preceding section are of the indefinite variety: each is a function of a variable and hence possesses no definite numerical value. After here, for a given indefinite integral of a continuous function $f(x)$, $\int f(x)dx = F(x) + C$. If we choose two values of x in the domain say a and b , the indefinite integral can be changed in to definite integral.

Definition: - Let $f(x)$ be a function with an anti-derivative that we denote it by $F(x)$. Let a and b be any two real numbers such that $f(x)$ and $F(x)$ exist for all values of x in the closed interval with end points a and b . Then definite integral of $f(x)$ from $x=a$ to $x=b$

Is denoted by: $\int_a^b f(x)dx$ and is defined by $\int_a^b f(x)dx = F(b) - F(a)$. The numbers a and b are called limits of integration, a the lower limit and b the upper limit. Usually $a < b$ but not necessarily.

Example

$$a) \int_a^b x^4 dx = \left[\frac{1}{5} x^5 \right]_a^b = \frac{1}{5} (b^5 - a^5)$$

$$b) \int_1^3 \frac{1}{t} dt = [\ln t]_1^3 = \ln^3 - \ln^1 = \ln^3$$

$$c) \int_0^2 e^{3x} dx = \left(\frac{1}{3} e^{3x} \right)_0^2 = \frac{1}{3} e^6 - \frac{1}{3} e^0 = \frac{1}{3} (e^6 - 1)$$

Note: In a definite integral since constants are canceling of each other, there is no need to include a constant term C .

Exercise

Find the integral of following functions

$$a) \int_1^e \ln^x dx$$

$$b) \int_2^4 3x dx$$

$$c) \int_0^6 e^{3x} dx$$

5.3.1 Properties of Definite Integral

Property I: - The interchange of the limits of integration changes the sign of the definite integral.

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

Example

$$\begin{aligned} \int_2^4 3x dx &= - \int_4^2 3x dx \\ &= \left(\frac{3}{2} x^2 \right)_2^4 = - \left(\frac{3}{2} x^2 \right)_4^2 \\ &= \frac{3}{2}(4) - \frac{3}{2}(2)^2 = - \left(\frac{3}{2}(2)^2 - \frac{3}{2}(4)^2 \right) \\ &= \frac{3}{2}(16) - \frac{3}{2}(4) = - \left(\frac{3}{2}(4) - \frac{3}{2}(16) \right) \\ &= 24 - 6 = -(6 - 24) \\ &= 18 = -(-18) \\ &18 = 18 \end{aligned}$$

Property II- A definite integral has a value of zero when the two limits of integration are identical.

$$\int_a^a f(x)dx = F(a) - F(a) = 0$$

Property III: - A definite integral can be expressed as a sum of finite number of definite sub integrals.

$$\int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx$$

Property IV:-
$$\int_a^b -f(x)dx = - \int_a^b f(x)dx$$

Property V:- $\int_a^b Kf(x)dx = K \int_a^b f(x)dx$

Property VI:- $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Exercise

Find the integral of the following

a). $\int_3^2 3x^2 dx$

b). $\int_2^2 Ln^x dx$

c). $\int_1^3 -(6x^2 + 5x + 4)dx$

5.3.2 Integral and area under the curve

One of the application of definite integration is finding the area of a region bounded by $f(x)$, the X-axis and the vertical lines $X = a$ and $x=b$.

Note: The area of the region R, denoted by $A(R)$, bounded by f , the X-axis, the vertical lines $X=a$

and $X=b$ is given by; $A(R) = \int_a^b f(x)dx$ provided that two conditions hold. These are:

1. f must be continuous in $[a, b]$

2. $f(x) \geq 0$ for all $x \in [a, b]$

If f is continuous and non-positive in the interval $[a, b]$, the area of the region bounded by f , x-axis

and the lines $X = a$ and $X = b$ is given by: $A(R) = -\int_a^b f(x)dx$

Fundamental Theorem of calculus

Let $f(x)$ be a continuous non-negative function in $a \leq X \leq b$ and let $F(x)$ be an anti derivative of $f(x)$.

Then A, the area between $Y = f(x)$, the X-axis and vertical lines $X = a$ and $X = b$ is given by the

definite integral: $A = \int_a^b f(x)dx = F(b) - F(a)$

Example

Evaluate the area of the region bounded by the function

$$y = 3x^2 + 2x + 5, \text{ the } x\text{-axis and the lines } x = 1 \text{ and } x = 3$$

Solution

$$A = \int_1^3 (3x^2 + 2x + 5) dx = \left[x^3 + x^2 + 5x \right]_1^3$$

$$A = (3)^3 + (3)^2 + 5(3) - (1^3 + 1^2 + 5(1)) = 27 + 9 + 15 - 7 = 51 - 7 = 44 \text{ square units}$$

5.4 Economic Applications of Integrals

Integrals have several uses in economics variable analysis. The following are some of the areas:

5.4.1 Developing a total function from marginal function

The process of integration can yield a total function if the marginal function is given.

Example

If the marginal cost (Mc) of a firm is given by an equation $C'(Q) = -4Q^3 + 6Q - 80$ and the fixed cost of production is 90 Birr, we can get the total cost of production that can be computed from MC function by integrating it

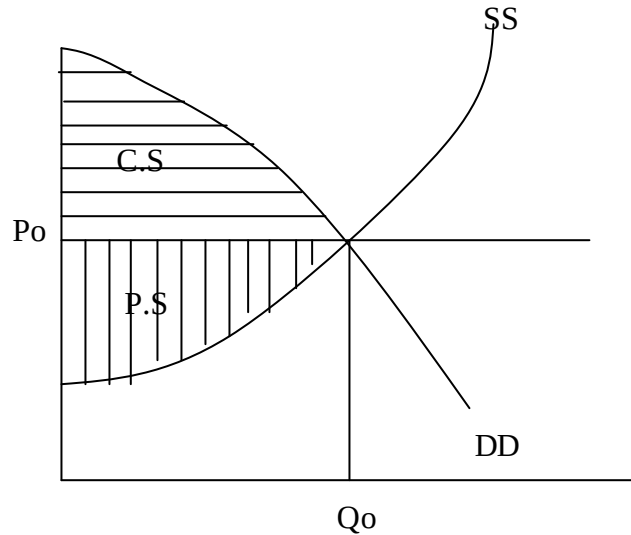
$$\int MC dQ = TC = \int (-4Q^3 + 6Q - 80) dQ = \int -4Q^3 dQ + \int 6Q dQ + \int -80 dQ$$

$$TC = -Q^4 + 3Q^2 - 80Q + C \text{ Since } C \text{ is the fixed cost, it can be replaced by 90.}$$

$$\text{Therefore, the total cost equation is given as: } TC = -Q^4 + 3Q^2 - 80Q + 90$$

5.4.2 Consumer's and Producer's Surplus

The other application of definite integral is to find the consumer's and producer's surplus which is explained as follows: Let the demand curve be $P = f(Q)$ for a certain commodity and the supply curve for the same commodity be given by $P = h(Q)$. Q denotes the quantity of a commodity that can be sold or supplied at price P per unit. In general, the demand function is a decreasing function and the supply function is an increasing function of price.



The market is at its equilibrium (Q_o, P_o) where the demand and supply curves intersect each other. From the graph of the demand curve, it is clear that as the price increases, demand decreases: This implies that there are some consumers who will be willing to buy the commodity at a higher price than the market equilibrium price, P_o that they actually have to pay. Consider the quantity ΔQ of units that lie between Q_1 and $Q_1 + \Delta Q$. The area $P_1 \Delta Q$ of the rectangle ACDB in the above figure can be interpreted as the total amount of money that the consumer would pay for these ΔQ units if the price were $P_1 = f(Q_1)$ per unit. At the market equilibrium price P_o , the actual amount spent by the consumers on these ΔQ units. In other words the consumer saves equal to $P_1 \Delta Q - P_o \Delta Q = [f(Q_1) - P_o] \Delta Q$. This saving is equal to the area of shaded rectangle ABFE. If we divide the range from 0 to $Q = Q_o$ in to a large number intervals of length ΔQ , we obtain a similar result for each interval. This is known as consumer's surplus. The consumer's surplus is given as:

$$C.S = \int_0^{Q_o} (f(Q) - P_o) dQ = \int_0^{Q_o} f(Q) dQ - P_o Q_o$$

Similarly, in the free market there are also producers who would be willing to sell the commodity at price lower than the market price P_o that the consumers actually pay. In such a situation, the

producers also are benefited; this benefit to the producers is called the producer's surplus. The producer's surplus is given as:

$$P.S = \int_0^{Q_0} (P_0 - h(Q)) dQ$$

$$P.S = P_0 Q_0 - \int_0^{Q_0} h(Q) dQ$$

Example

The supply and demand functions for a certain product are given by:

$$P = 52 + 2Q$$

$$P = 100 - Q^2$$

Assuming that market equilibrium has been established, then determine the consumer's and producer's surplus?

Solution

First we have to find the equilibrium price and quantity since these are the base for computing both consumers and producer's surplus.

At equilibrium, demand = Supply

$$52 + 2Q + 100 - Q^2 \Rightarrow Q^2 + 2Q - 48 = 0 \Rightarrow (Q+8)(Q-6) = 0 \Rightarrow Q_0 = 6 \text{ units, eq. qty}$$

$$P_0 = 52 + 2Q_0 = 52 + 2(6) = 64 \text{ birr, equilibrium price}$$

$$C.S = \int_0^6 f(Q) dx - P_0 Q_0 = \int_0^6 (100 - Q^2) dx - 6 \times 64 = \left[100Q - \frac{1}{3} Q^3 \right]_0^6 - 384$$

$$= 600 - \frac{1}{3} (6)^3 - 384 = 600 - \frac{1}{3} (216) - 384 = 600 - 72 - 384 = 600 - 456 = 144 \text{ Birr}$$

$$P.S = P_0 Q_0 - \int_0^6 (52 + 2Q) dQ = 64 \times 6 - (52Q + Q^2)_0^6$$

$$= 384 - (52(6) + (6)^2) = 384 - 312 - 36 = 384 - 348 = 36 \text{ birr}$$

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