

Chapter 1: Limit and continuity

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Chapter Outline

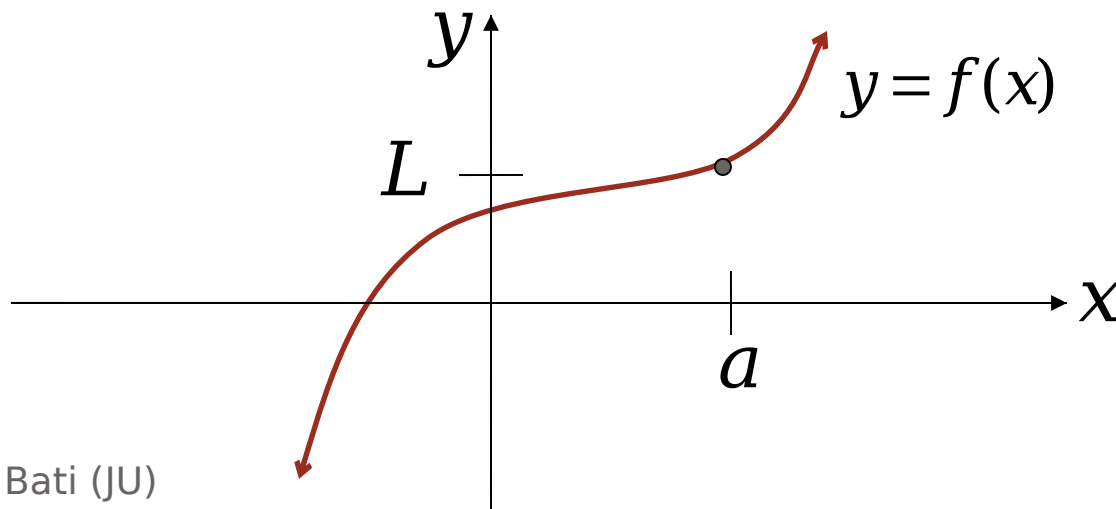
- 1.1 Definitions and examples
- 1.2 Different types of limits
- 1.3 Limit theorems
- 1.4 Techniques for Limit Evaluation
- 1.5 Continuity, Discontinuity
- 1.6 properties of continuity
- 1.5 Intermediate value theorem

Informal Definition of Limit

The limit of $f(x)$, as x approaches a , equals L

written $\lim_{x \rightarrow a} f(x) = L$

if we can make the value $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a .



Example 1

Set $f(x) = 4x + 5$ and take $c = 2$. As x approaches 2, $4x$ approaches 8 and $4x + 5$ approaches $8 + 5 = 13$. We conclude that

$$\lim_{x \rightarrow 2} f(x) = 13$$

Example 2

Set

$$f(x) = \sqrt{1-x} \text{ and take } a = -8.$$

As x approaches -8 , $1 - x$ approaches $\sqrt{9}$ and approaches 3. We conclude that

$$\lim_{x \rightarrow -8} f(x) = 3$$

If for that same function we try to calculate

$$\lim_{x \rightarrow 2} f(x)$$

we run into a problem. The function $f(x) = \sqrt{1-x}$ is defined only for $x \leq 1$. It is therefore not defined for x near 2, and the idea of taking the limit as x approaches 2 makes no sense at all:

$$\lim_{x \rightarrow 2} f(x) \text{ does not exist.}$$

Example 3

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} = 12.$$

The function $f(x) = \frac{x^3 - 8}{x - 2}$ is undefined at $x = 2$. But, as we said before, that doesn't matter. For all $x \neq 2$,

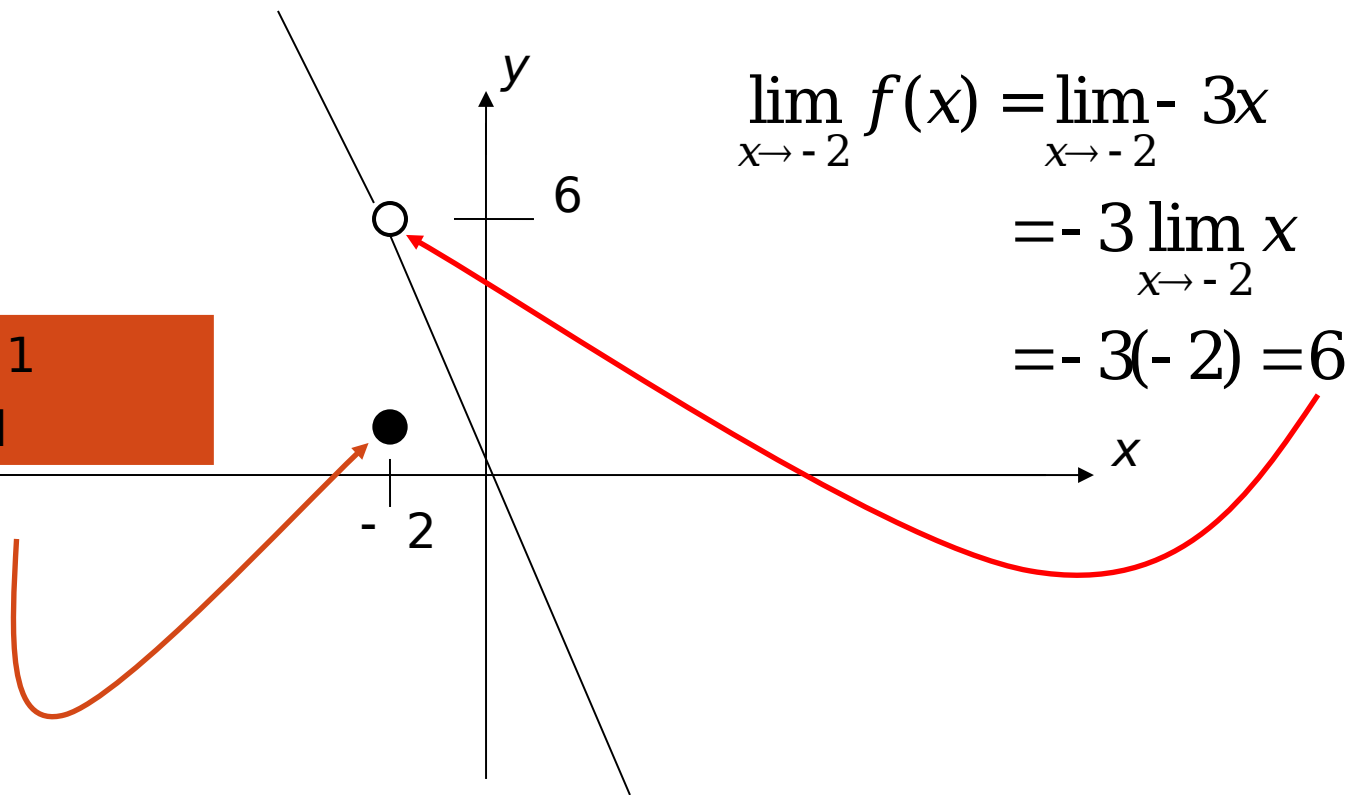
$$\frac{x^3 - 8}{x - 2} = \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} = x^2 + 2x + 4.$$

Therefore,

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} = \lim_{x \rightarrow 2} (x^2 + 2x + 4) = 12.$$

Example 4. $\lim_{x \rightarrow -2} f(x)$ where $f(x) = \begin{cases} -3x & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$

Note: $f(-2) = 1$
is not involved



$$\begin{aligned} \lim_{x \rightarrow -2} f(x) &= \lim_{x \rightarrow -2} -3x \\ &= -3 \lim_{x \rightarrow -2} x \\ &= -3(-2) = 6 \end{aligned}$$

One-Sided Limits

Numbers x near a fall into two natural categories: those that lie to the left of a and those that lie to the right of a .

We write $\lim_{x \rightarrow a^-} f(x) = L$

to indicate that *as x approaches a from the left, $f(x)$ approaches L .*

We write $\lim_{x \rightarrow a^+} f(x) = L$

to indicate that *as x approaches a from the right, $f(x)$ approaches L .*

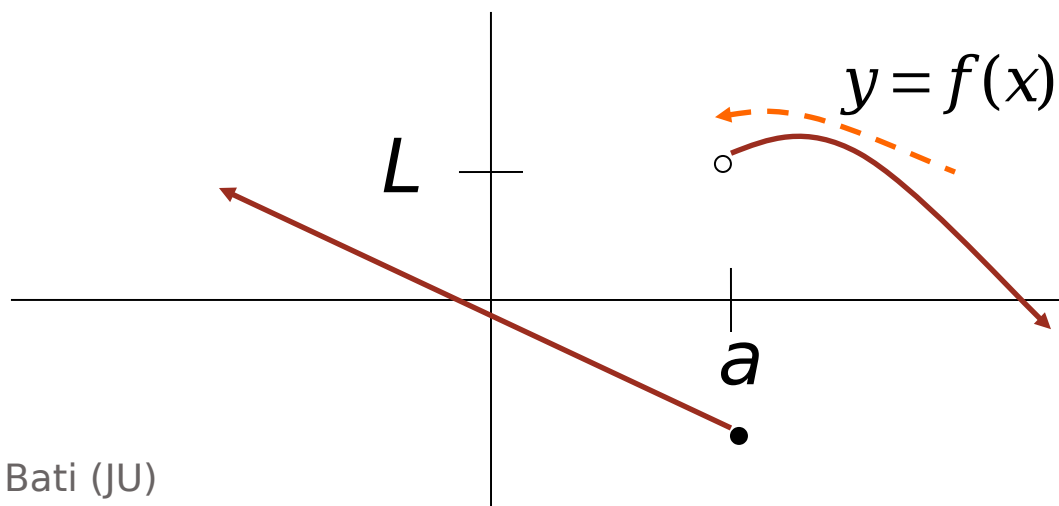
The **limit from the right** (or right-hand limit) means that x approaches c from values greater than c .

The **limit from the left** (or left-hand limit) means that x approaches c from values less than c .

The right-hand limit of $f(x)$, as x approaches a , equals L

written: $\lim_{x \rightarrow a^+} f(x) = M$

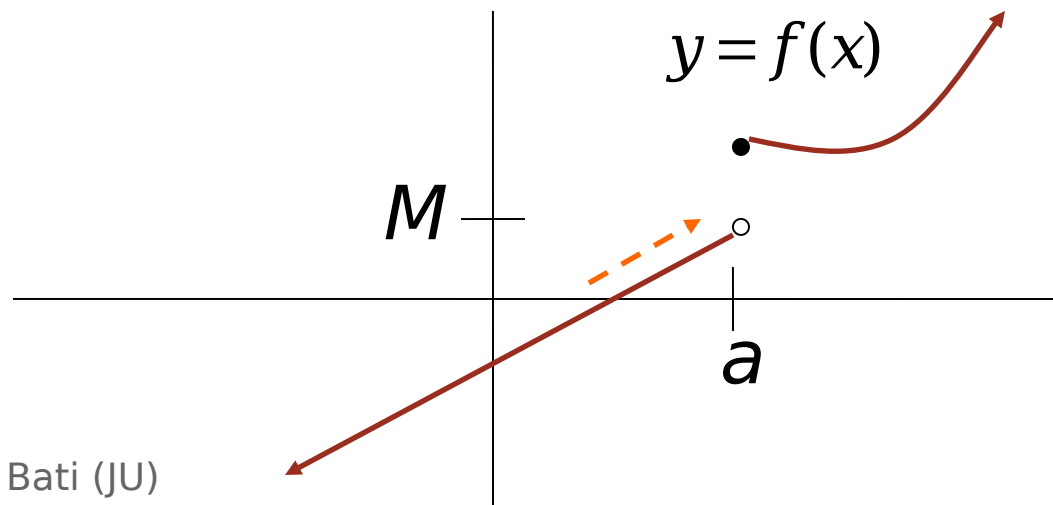
if we can make the value $f(x)$ arbitrarily close to L by taking x to be sufficiently close to the right of a .



The left-hand limit of $f(x)$, as x approaches a , equals M

written: $\lim_{x \rightarrow a^-} f(x) = M$

if we can make the value $f(x)$ arbitrarily close to M by taking x to be sufficiently close to the left of a .



Example

S

1. Given $f(x) = \begin{cases} x^2 & \text{if } x \leq 3 \\ 2x & \text{if } x > 3 \end{cases}$

Find $\lim_{x \rightarrow 3^+} f(x)$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} 2x = 6$$

Find $\lim_{x \rightarrow 3} f(x)$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} x^2 = 9$$

2. Let $f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x \leq 0. \end{cases}$

Find the limits:

a) $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x+1) = 0+1=1$

b) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x-1) = 0-1=-1$

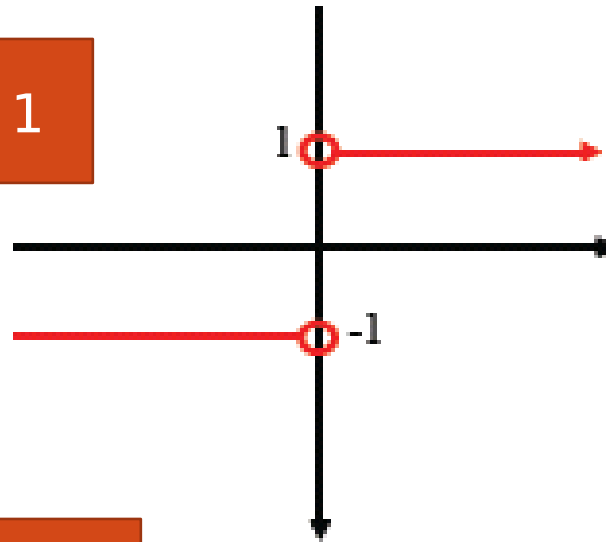
c) $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x+1) = 1+1=2$

d) $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x+1) = 1+1=2$

Example 3

Find

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$$



$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$$

Therefore, the limit as x approaches 0
DNE!!

A Theorem

$\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^+} f(x) = L$ and $\lim_{x \rightarrow a^-} f(x) = L$.

This theorem is used to show a limit does not exist.

For the
function

$$f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x \leq 0. \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$ does not exist because $\lim_{x \rightarrow 0^+} f(x) = 1$ and $\lim_{x \rightarrow 0^-} f(x) = -1$.

But

$\lim_{x \rightarrow 1} f(x) = 2$ because $\lim_{x \rightarrow 1^+} f(x) = 2$ and $\lim_{x \rightarrow 1^-} f(x) = 2$.

Example 2 For the function f indicated in figure 2.1.8,

$$\lim_{x \rightarrow (-2)^-} f(x) = 5 \quad \text{and} \quad \lim_{x \rightarrow (-2)^+} f(x) = 5$$

In this case $\lim_{x \rightarrow -2} f(x) = 5$

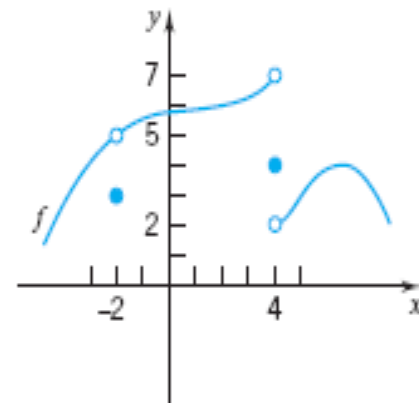


Figure 2.1.8

It does not matter that $f(-2) = 3$.

Examining the graph of f near $x = 4$, we find that

$$\lim_{x \rightarrow 4^-} f(x) = 7 \quad \text{whereas} \quad \lim_{x \rightarrow 4^+} f(x) = 2$$

Since these one-sided limits are different, $\lim_{x \rightarrow 4} f(x)$

does not exist.

Exercises. Find the following limits if it exists.

a. $\lim_{x \rightarrow 1} f(x); f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases}$

b. Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

Limits at Infinity

Definition: We say that a real number L is the limit of a function f as x approaches to infinity iff $f(x)$ gets closer to L as x increases without bound and is denoted by

$$\lim_{x \rightarrow \infty} f(x) = L$$

An infinite limit will exist as x approaches a finite value when direct substitution produces

$$\frac{\text{not zero}}{0}$$

For all $n > 0$

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0$$

provided that $\frac{1}{x^n}$ is defined.

Example 1.

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 5x + 1}{2 - 4x^2} = \lim_{x \rightarrow \infty} \frac{3 + \frac{5}{x} + \frac{1}{x^2}}{\frac{2}{x^2} - 4}$$

Divide by x^2

$$= \frac{\lim_{x \rightarrow \infty} 3 + \lim_{x \rightarrow \infty} \left(\frac{5}{x} \right) + \lim_{x \rightarrow \infty} \left(\frac{1}{x^2} \right)}{\lim_{x \rightarrow \infty} \left(\frac{2}{x^2} \right) - \lim_{x \rightarrow \infty} 4} = \frac{3 + 0 + 0}{0 - 4} = -\frac{3}{4}$$

$$2. \quad \lim_{x \rightarrow \infty} \left(\frac{4x^2 - 5x + 21}{7x^3 + 5x^2 - 10x + 1} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{\frac{4x^2}{x^3} - \frac{5x}{x^3} + \frac{21}{x^3}}{\frac{7x^3}{x^3} + \frac{5x^2}{x^3} - \frac{10x}{x^3} + \frac{1}{x^3}} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{\frac{4}{x} - \frac{5}{x^2} + \frac{21}{x^3}}{7 + \frac{5}{x} - \frac{10}{x^2} + \frac{1}{x^3}} \right)$$

$$= \frac{0}{7}$$

$$= 0$$

$$3. \quad \lim_{x \rightarrow \infty} \left(\frac{x^2 + 2x - 4}{12x + 31} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{\frac{x^2}{x} + \frac{2x}{x} - \frac{4}{x}}{\frac{12x}{x} + \frac{31}{x}} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{x + 2 - \frac{4}{x}}{12 + \frac{31}{x}} \right)$$

$$= \frac{\infty + 2}{12}$$

$$= \infty$$

$$4. \quad \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{(\sqrt{x^2 + 1} - x) \sqrt{x^2 + 1} + x}{1 \cdot \sqrt{x^2 + 1} + x} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{x^2 + 1 - x^2}{\sqrt{x^2 + 1} + x} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{1}{\sqrt{x^2 + 1} + x} \right)$$

$$= \frac{1}{\infty + \infty} = \frac{1}{\infty} = 0$$

Example - 5

Evaluate: $\lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7}$

Solution :

$$\text{We have } \lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7}$$

$$= \lim_{x \rightarrow \infty} \frac{3 - \frac{4}{x} + \frac{6}{x^2} - \frac{1}{x^3}}{2 + \frac{1}{x} - \frac{5}{x^2} + \frac{7}{x^3}} \quad \left[\text{Dividing numerator and denominator by } x^3 \right]$$

$$= \frac{3 - 0 + 0 - 0}{2 + 0 - 0 + 0} = \frac{3}{2}$$

Example - 6

Evaluate: $\lim_{x \rightarrow \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}}$

Solution :

We have $\lim_{x \rightarrow \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}}$

$$= \lim_{x \rightarrow \infty} \frac{x \left(5 - \frac{6}{x} \right)}{x \left(\sqrt{4 + \frac{9}{x^2}} \right)} = \lim_{x \rightarrow \infty} \frac{5 - \frac{6}{x}}{\sqrt{4 + \frac{9}{x^2}}}$$

$$= \frac{5 - 0}{\sqrt{4 + 0}} = \frac{5}{2}$$

Example - 7

Evaluate: $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1} \right)$

Solution :

We have $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1} \right)$

$$= \lim_{x \rightarrow \infty} \frac{\left(\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1} \right)}{\left(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1} \right)} \times \left(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 + x + 1 - x^2 - 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}}$$

Solution Cont.

$$= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x^2}}} \quad \left[\text{Dividing numerator and denominator by } x \right]$$

$$= \frac{1}{\sqrt{1 + 0 + 0} + \sqrt{1 + 0}}$$

$$= \frac{1}{1 + 1} = \frac{1}{2}$$

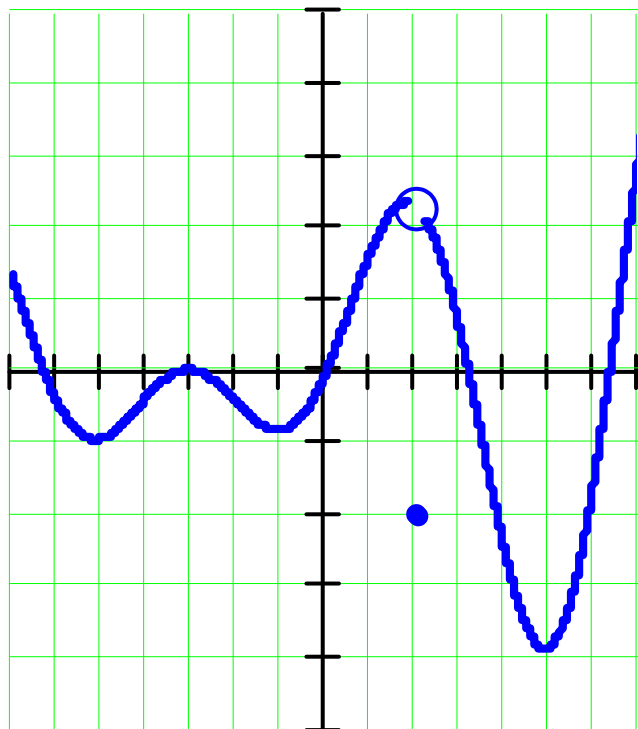
Exercise

Evaluate : $\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4}$

Solution :

$$\begin{aligned} \text{We have } \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} &= \lim_{n \rightarrow \infty} \frac{n^2 (n+1)^2}{4n^4} \quad \left[\because \sum n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2 \right] \\ &= \lim_{n \rightarrow \infty} \frac{n^2 (n^2 + 1 + 2n)}{4n^4} = \lim_{n \rightarrow \infty} \frac{n^4 \left(1 + \frac{1}{n^2} + \frac{2}{n} \right)}{4n^4} \\ &= \frac{1}{4} \times \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2} + \frac{2}{n} \right) \\ &= \frac{1}{4} \times (1 + 0 + 0) = \frac{1}{4} \end{aligned}$$

Graph of $f(x)$



True or false

(a) $x = 2$ is in the domain of f

(b) $\lim_{x \rightarrow 2} f(x)$ exists

(c) $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$

Limit Theorems

Suppose $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a} g(x) = M$ and c is any real number.

Then,

$$1. \lim_{x \rightarrow a} [f(x)]^r = L^r \quad r, a \text{ real number}$$

$$2. \lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x) = cL \quad c, a \text{ real number}$$

$$3. \lim_{x \rightarrow a} [f(x) \pm g(x)] = L \pm M$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = LM$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$$

Provided that $M \neq 0$

$$6. \lim_{x \rightarrow a} c = c$$

$$7. \lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{L} \quad (L > 0)$$

$$8. \lim_{x \rightarrow a} x = a$$

Example 1

Let $\lim_{x \rightarrow 5} f(x) = 4$ and $\lim_{x \rightarrow 5} g(x) = -2$.
Find the following limits.

1. $\lim_{x \rightarrow 5} [f(x) + 5g(x)]$

$$\lim_{x \rightarrow 5} f(x) + \lim_{x \rightarrow 5} [5g(x)]$$

$$\lim_{x \rightarrow 5} f(x) + 5 \lim_{x \rightarrow 5} g(x)$$

$$4 + 5(-2)$$

$$\boxed{-6}$$

2. $\lim_{x \rightarrow 5} [f(x) \cdot g(x)]$

$$\lim_{x \rightarrow 5} f(x) \cdot \lim_{x \rightarrow 5} g(x)$$

$$4(-2)$$

$$\boxed{-8}$$

3. $\lim_{x \rightarrow 5} \frac{f(x)}{g(x)}$

$$\frac{\lim_{x \rightarrow 5} f(x)}{\lim_{x \rightarrow 5} g(x)}$$

$$\frac{4}{-2}$$

$$\boxed{-2}$$

Example 2

Evaluate $\lim_{x \rightarrow 2} (2x^5 - 9x^3 + 3x^2 - 11)$

$$= \lim_{x \rightarrow 2} (2x^5) - \lim_{x \rightarrow 2} (9x^3) + \lim_{x \rightarrow 2} (3x^2) - \lim_{x \rightarrow 2} (11)$$

Sum/Difference Property

$$= 2 \left[\lim_{x \rightarrow 2} x^5 \right] - 9 \left[\lim_{x \rightarrow 2} x^3 \right] + 3 \left[\lim_{x \rightarrow 2} x^2 \right] - 11$$

Multiple and Constant Properties

$$= 2 \left[\lim_{x \rightarrow 2} x \right]^5 - 9 \left[\lim_{x \rightarrow 2} x \right]^3 + 3 \left[\lim_{x \rightarrow 2} x \right]^2 - 11$$

Power Property

$$= 2(2)^5 - 9(2)^3 + 3(2)^2 - 11 = -7$$

Limit of x Property

Direct Substitution

Example 3

$$\begin{aligned}\lim_{x \rightarrow 3} (x^2 + 1) &= \lim_{x \rightarrow 3} x^2 + \lim_{x \rightarrow 3} 1 \\ &= \left(\lim_{x \rightarrow 3} x \right)^2 + \lim_{x \rightarrow 3} 1 \\ &= 3^2 + 1 = 10\end{aligned}$$

Example 4.

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\sqrt{2x-1}}{3x+5} &= \frac{\sqrt{\lim_{x \rightarrow 1} (2x-1)}}{\lim_{x \rightarrow 1} (3x+5)} = \frac{\sqrt{2 \lim_{x \rightarrow 1} x - \lim_{x \rightarrow 1} 1}}{3 \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 5} \\ &= \frac{\sqrt{2-1}}{3+5} = \frac{1}{8}\end{aligned}$$

Example

5

1. Suppose $\lim_{x \rightarrow 3} f(x) = 4$ and $\lim_{x \rightarrow 3} g(x) = -2$. Find

$$\begin{aligned} \text{a) } \lim_{x \rightarrow 3} (f(x) + g(x)) &= \lim_{x \rightarrow 3} f(x) + \lim_{x \rightarrow 3} g(x) \\ &= 4 + (-2) = 2 \end{aligned}$$

$$\begin{aligned} \text{b) } \lim_{x \rightarrow 3} (f(x) - g(x)) &= \lim_{x \rightarrow 3} f(x) - \lim_{x \rightarrow 3} g(x) \\ &= 4 - (-2) = 6 \end{aligned}$$

$$\begin{aligned} \text{c) } \lim_{x \rightarrow 3} \left(\frac{2f(x) - g(x)}{f(x)g(x)} \right) &= \frac{\lim_{x \rightarrow 3} 2f(x) - \lim_{x \rightarrow 3} g(x)}{\lim_{x \rightarrow 3} f(x) \cdot \lim_{x \rightarrow 3} g(x)} = \frac{2 \cdot 4 - (-2)}{4 \cdot (-2)} = \frac{-5}{4} \end{aligned}$$

Exercises:

1. *Iff* $\lim_{x \rightarrow -2} f(x) = 2$ *and* $\lim_{x \rightarrow -2} g(x) = 1$ *Find*

a) $\lim_{x \rightarrow -2} [f(x) + 5g(x)]$

b) $\lim_{x \rightarrow -2} [fg(x)]$

c) $\lim_{x \rightarrow -2} \left[\frac{f}{g}(x) \right]$

2. Find $\lim_{x \rightarrow 0^-} f(x)$ *iff* $f(x) = \begin{cases} \frac{x - |x|}{x} & \text{for } x < 0 \\ 2 & \text{for } x \geq 0 \end{cases}$

Techniques for Limit Evaluation

1. Direct substitution

First substitute the value of x being approached into the function $f(x)$. If this is a real number then the limit is that number.

$$1. \lim_{x \rightarrow 4} \sqrt{x} = \sqrt{4} = 2$$

**Substitute 4
for x .**

$$2. \lim_{x \rightarrow 6} \frac{x^2}{x+3} = \frac{6^2}{6+3} = \frac{36}{9} = 4$$

**Substitute 6
for x .**

2. By factorization followed by cancellation

Example 1

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \frac{0}{0} \quad \text{Which is undefined!}$$

But the limit exist!!!!

Graph it.

What happens at $x = 2$?

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \rightarrow 2} (x+3) = 5$$

NOTE: $f(x) = \frac{x^2 + x - 6}{x - 2}$ graphs as a straight line.

Example 2

Find the limit:

Solution: begin by factoring the numerator and dividing out any common factors.

$$\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \rightarrow -3} \frac{(x - 2)(x + 3)}{x + 3} \quad \text{Factor numerator.}$$

$$= \lim_{x \rightarrow -3} (x - 2) \quad \text{Divide out common factor and simplify.}$$

$$= -3 - 2 = -5 \quad \text{Direct substitution}$$

Example 3 . Find $\lim_{x \rightarrow -5} \frac{x+5}{x^2-25}$

$$\lim_{x \rightarrow -5} \frac{x+5}{x^2-25}$$

Notice form $\frac{0}{0}$

$$= \lim_{x \rightarrow -5} \frac{\cancel{x+5}}{(x-5)\cancel{(x+5)}}$$

Factor and cancel
common factors

$$= \lim_{x \rightarrow -5} \frac{1}{(x-5)} = \frac{1}{-10}$$

Exercise: Evaluate

a) $\lim_{x \rightarrow 0} \frac{1}{x-1} \left(\frac{1}{x+3} - \frac{2x}{3x+5} \right)$

b) $\lim_{x \rightarrow 4} \left(\frac{2x - 8\sqrt{x} + 8}{2 - \sqrt{x}} \right)$

c) $\lim_{x \rightarrow 3} \left(\frac{x^2 - 3x}{\sqrt{x} - \sqrt{3}} \right)$

Example 4

Evaluate the limit analytically:

$$\lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3} \cdot \frac{3x}{3x} = \lim_{x \rightarrow 3} \frac{3 - x}{(x - 3)3x}$$

Cancel the
denominator
s of the
fractions in
the
numerator

$$= \lim_{x \rightarrow 3} \frac{-(x - 3)}{(x - 3)3x}$$

← If the subtraction is
backwards, Factoring a
negative 1 to flip the signs

$$= \lim_{x \rightarrow 3} \frac{\cancel{-(x - 3)}}{\cancel{(x - 3)}3x}$$

← Cancel common factors

$$= \lim_{x \rightarrow 3} \frac{-1}{3x}$$

$$= \frac{-1}{3 \cdot 3} = -\frac{1}{9}$$

← Direct substitution

Example 5

Evaluate the limit analytically:

$$\lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x} = \lim_{x \rightarrow \pi/4} \frac{1 - \frac{\sin x}{\cos x}}{\sin x - \cos x} \cdot \frac{\cos x}{\cos x}$$

Eliminate the embedded fraction

Rewrite the tangent function using cosine and sine

$$= \lim_{x \rightarrow \pi/4} \frac{\cos x - \sin x}{(\sin x - \cos x) \cos x}$$

$$= \lim_{x \rightarrow \pi/4} \frac{- (\cancel{\sin x - \cos x})}{(\cancel{\sin x - \cos x}) \cos x}$$

If the subtraction is backwards, Factoring a negative 1 to flip the signs

$$= \lim_{x \rightarrow \pi/4} \frac{-1}{\cos x}$$

$$= \frac{-1}{\cos(\frac{\pi}{4})}$$

Direct substitution

$$= \frac{-1}{\sqrt{2}/2} = -\sqrt{2}$$

3 – Rationalizing Technique

Find the $\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$

Solution: By direct substitution, you obtain the $\frac{0}{0}$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{0+1} - 1}{0} = \frac{0}{0} \quad \text{Indeterminate form}$$

In this case, rewrite the fraction by rationalizing the numerator.

$$\begin{aligned} \frac{\sqrt{x+1} - 1}{x} &= \left(\frac{\sqrt{x+1} - 1}{x} \right) \left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \right) \\ &= \frac{(x+1) - 1}{x(\sqrt{x+1} + 1)} \quad \text{Multiply.} \end{aligned}$$

$$= \frac{x}{x(\sqrt{x+1} + 1)}$$

Simplify.

$$= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0$$

Divide out common factor.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1} + 1}$$

$$= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0$$

Simplify.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1} + 1}$$

$$= \frac{1}{\sqrt{0+1} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

Example 2

Evaluate the limit analytically:

$$\lim_{y \rightarrow 2} \frac{\sqrt{y+2} - 2}{y - 2} \cdot \frac{\sqrt{y+2} + 2}{\sqrt{y+2} + 2} \quad \leftarrow \text{Rationalize the numerator}$$

$$= \lim_{y \rightarrow 2} \frac{(y+2) - 4}{(y-2)(\sqrt{y+2} + 2)}$$

$$= \lim_{y \rightarrow 2} \frac{\cancel{y-2}}{(\cancel{y-2})(\sqrt{y+2} + 2)} \quad \leftarrow \text{Cancel common factors}$$

$$= \lim_{y \rightarrow 2} \frac{1}{\sqrt{y+2} + 2} = \frac{1}{\sqrt{2+2} + 2} = \frac{1}{4}$$

Direct substitution

Example 5

$$\begin{aligned} \text{a) } \lim_{x \rightarrow 9} \left(\frac{\sqrt{x} - 3}{x - 9} \right) &= \lim_{x \rightarrow 9} \left(\frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} \right) \\ &= \lim_{x \rightarrow 9} \left(\frac{x - 9}{(x - 9)(\sqrt{x} + 3)} \right) = \lim_{x \rightarrow 9} \left(\frac{1}{\sqrt{x} + 3} \right) = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \text{b) } \lim_{x \rightarrow -2} \left(\frac{4 - x^2}{2x^2 + x^3} \right) &= \lim_{x \rightarrow -2} \left(\frac{(2 - x)(2 + x)}{x^2(2 + x)} \right) \\ &= \lim_{x \rightarrow -2} \left(\frac{2 - x}{x^2} \right) \\ &= \frac{2 - (-2)}{(-2)^2} = \frac{4}{4} = 1 \end{aligned}$$

Exercises: Evaluate

$$a) \lim_{x \rightarrow 3} \left(\frac{\sqrt{x} - \sqrt{3}}{x - 3} \right)$$

$$b) \lim_{x \rightarrow 2} \left(\frac{\sqrt{x+7} - 3}{x - 2} \right)$$

$$c) \lim_{x \rightarrow 2} \left(\frac{\sqrt{x-3} \sqrt{x+4} - 2}{\sqrt{x} - 3} \right)$$

$$d) \lim_{x \rightarrow \infty} (\sqrt{4x^2 + x} - \sqrt{4x^2 - x})$$

By using important Limits

$$1. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Proof (Assignment)

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \ln a$$

$$2. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e$$

$$\lim_{x \rightarrow 0} \ln \left(\frac{1+x}{x} \right) = 1$$

$$\lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x - a} \right) = n(a)^{n-1}$$

Example 1

Evaluate the limit analytically:

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{5x} \cdot \frac{3}{3} = \lim_{x \rightarrow 0} \frac{3 \sin 3x}{5 \cdot 3x}$$

If $3x$ is the
input of the
sine function
then $3x$
needs to be
in the
denominator

$$= \lim_{x \rightarrow 0} \frac{3}{5} \cdot \frac{\sin 3x}{3x}$$

← Isolate the “freebie”

$$= \frac{3}{5} \cdot \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}$$

← Scalar Multiple Property

$$= \frac{3}{5} \cdot 1$$

← Assumed Trig Limit

$$= \frac{3}{5}$$

Example 2

Evaluate the limit analytically:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 + \cos x - \cos x - \cos^2 x}{x \sin x (1 + \cos x)}$$

Try
multiplying
by the
reciprocal

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x \sin x (1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \sin x (1 + \cos x)}$$

Use the
Trigonometry
Laws

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x (1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{1 + \cos x}$$

Split up the limits

$$= 1 \cdot \frac{1}{1 + \cos 0}$$

A freebie limit and Direct
substitution

$$= \frac{1}{2}$$

Exercises: Evaluate

$$a) \lim_{x \rightarrow 0} \left(\frac{6x - \sin 2x}{2x + 3\sin 4x} \right)$$

$$b) \lim_{x \rightarrow 0} \left(\frac{\cos(ax) - \cos(bx)}{x^2} \right)$$

$$c) \lim_{x \rightarrow 0} \left(\frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} \right)$$

$$d) \lim_{x \rightarrow 0} \left(\frac{7x^2 + \tan^2(3x)}{x^2} \right)$$

$$e) \lim_{x \rightarrow \infty} \left(\frac{3x+6}{3x-9} \right)^{2x-5}$$

$$f) \lim_{x \rightarrow \infty} \frac{(x+1)^x}{x^{x+1}}$$

$$g) \text{ If } \lim_{x \rightarrow \infty} \left(\frac{x+a}{x-4} \right)^{3x} = e^{36}, \text{ then find the value of } a.$$

Exercises: Evaluate

$$a) \lim_{x \rightarrow \infty} \left(\frac{3x+6}{3x-9} \right)^{2x}$$

$$b) \lim_{x \rightarrow \infty} \left(\frac{x+3}{x} \right)^x$$

$$c) \text{ If } \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = 1, \text{ prove that}$$

$$i) \lim_{x \rightarrow 0} \left(\frac{e^{-ax} - e^{-bx}}{x} \right) = b - a$$

$$ii) \lim_{x \rightarrow 0} \left(\frac{a^x - b^x}{x} \right) = \ln\left(\frac{a}{b}\right)$$

The Squeezing Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a , and if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L, \text{ then } \lim_{x \rightarrow a} g(x) = L$$

Example: Show that $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$.

Note that we cannot use product rule because $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{x}\right)$ DNE!

But - $1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$ and so - $-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2$.

Since $\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} (-x^2) = 0$, we use the Squeezing Theorem to conclude

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0.$$

Exercise

1. $\lim_{x \rightarrow \infty} \left(\frac{6x^2 + x \sin \sqrt{x}}{x^2 + 12} \right)^{2x}$

2. Suppose $\sqrt{\frac{1 - \cos 4x}{2}} \leq \sqrt{f(x) - 4} \leq \sin 2x, \quad \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$

Then evaluate $\lim_{x \rightarrow \frac{\pi}{4}} f(x)$

3. Evaluate

a) $\lim_{x \rightarrow 0} \left(x^2 \sin\left(\frac{1}{x}\right) \right)$

b) $\lim_{x \rightarrow 0} \left(\frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} \right)$

The Formal () Definition of Limit

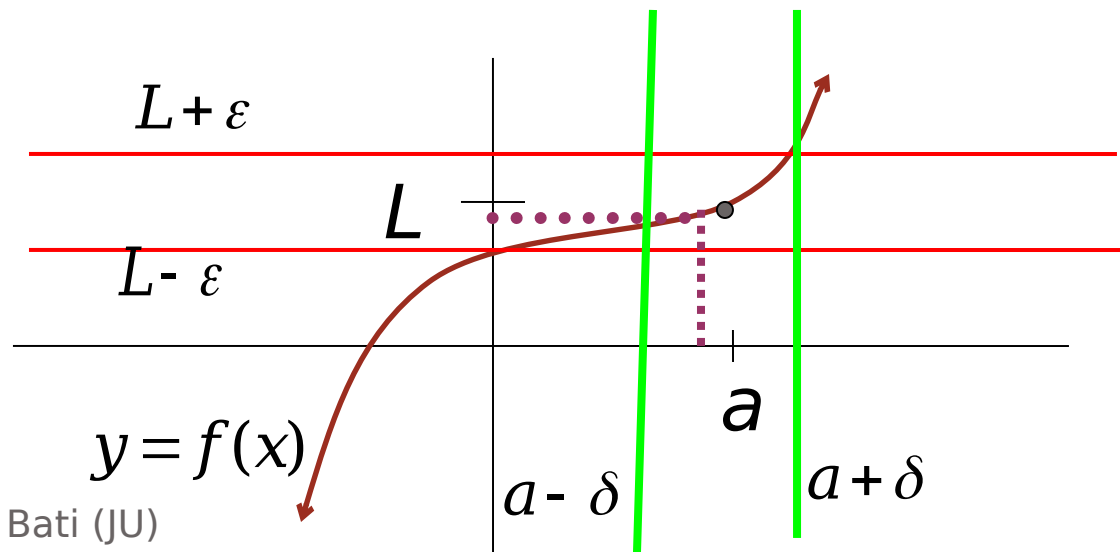
$\varepsilon - \delta$

Let f be a function defined on an interval containing " a " except possibly at a itself.

We say $\lim_{x \rightarrow a} f(x) = L$ if and only if

given a positive number ε , there exists a positive δ such that

if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.



This means that if we are given a small interval $(L - \varepsilon, L + \varepsilon)$ centered at L , then we can find a (small) interval $(a - \delta, a + \delta)$ such that for all $x \neq a$ in $(a - \delta, a + \delta)$, $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$.

Examples

1. Show that $\lim_{x \rightarrow 2} (3x + 4) = 10$.

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that if $|x - 2| < \delta$, then $|(3x + 4) - 10| < \varepsilon$.

$$\text{But } |(3x + 4) - 10| = |3x - 6| = 3|x - 2| < \varepsilon$$

$$\text{if } |x - 2| < \frac{\varepsilon}{3} \quad \text{So we choose } \delta = \frac{\varepsilon}{3}.$$

2. Show that $\lim_{x \rightarrow 1} \frac{1}{x} = 1$.

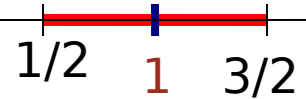
Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\text{if } |x - 1| < \delta, \text{ then } \left| \frac{1}{x} - 1 \right| < \varepsilon.$$

$$\text{But } \left| \frac{1}{x} - 1 \right| = \left| \frac{x - 1}{x} \right| = \frac{1}{x} |x - 1|.$$

What do we do with the x ?

If we decide $|x - 1| < \frac{1}{2}$, then $\frac{1}{2} < x < \frac{3}{2}$.



And so $\frac{1}{x} < 2$.

Thus $|\frac{1}{x} - 1| = \frac{1}{x} |x - 1| < 2 |x - 1|$.

Now we choose $\delta = \min \left\{ \frac{\varepsilon}{2}, \frac{1}{2} \right\}$

Exercises: Prove that

a) $\lim_{x \rightarrow 3} (4x - 5) = 7$

b) $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$

c) $\lim_{x \rightarrow 3} x^2 = 9$

d) $\lim_{x \rightarrow 1} \left(\frac{2x^4 - 6x^3 + x^2 + 3}{x - 1} \right) = -8$

Definition of Limit

DEFINITION 2.2.7 LEFT-HAND LIMIT

Let f be a function defined at least on an open interval of the form $(c - p, c)$. We say that

$$\lim_{x \rightarrow c^-} f(x) = L$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if } c - \delta < x < c, \quad \text{then } |f(x) - L| < \epsilon.$$

DEFINITION 2.2.8 RIGHT-HAND LIMIT

Let f be a function defined at least on an open interval of the form $(c, c + p)$. We say that

$$\lim_{x \rightarrow c^+} f(x) = L$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if } c < x < c + \delta \quad \text{then } |f(x) - L| < \epsilon.$$

Exercise. Prove that

$$a) \quad \lim_{x \rightarrow 0^+} \sqrt{x} = 0$$

$$b) \quad \lim_{x \rightarrow 4^-} \sqrt{4 - x} = 0$$

Asymptotes

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L.$$

The line $x = c$ is called a **vertical asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow c^-} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow c^+} f(x) = \pm\infty.$$

Examples

Find the asymptotes of the graphs of the functions

$$1. f(x) = \frac{x^2 + 1}{x^2 - 1}$$

$$(i) \lim_{x \rightarrow 1^-} f(x) = -\infty$$

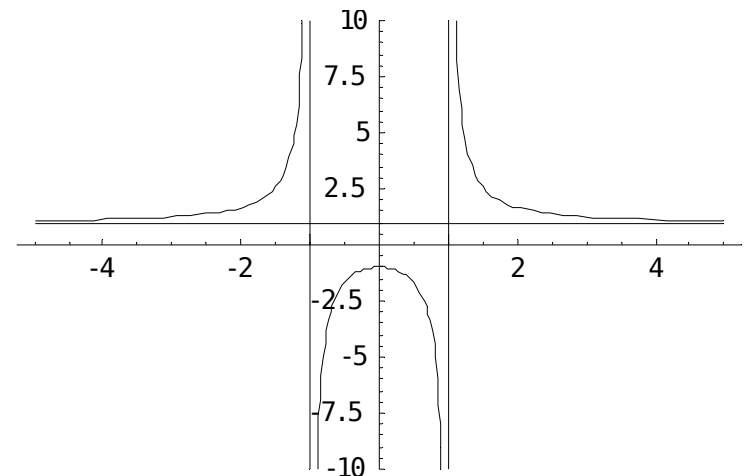
Therefore the line $x = 1$
is a vertical asymptote.

$$(ii) \lim_{x \rightarrow -1^-} f(x) = +\infty.$$

Therefore the line $x = -1$
is a vertical asymptote.

$$(iii) \lim_{x \rightarrow \infty} f(x) = 1.$$

Therefore the line $y = 1$
is a horizontal asymptote.



$$2. \quad f(x) = \frac{x-1}{x^2-1}$$

$$\begin{aligned} \text{(i)} \quad \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \left(\frac{x-1}{x^2-1} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{x-1}{(x-1)(x+1)} \right) = \lim_{x \rightarrow 1} \left(\frac{1}{x+1} \right) = \frac{1}{2}. \end{aligned}$$

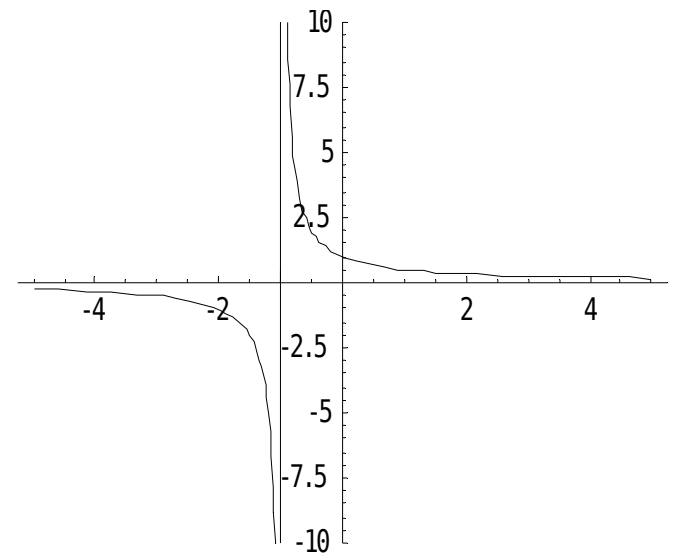
Therefore the line $x = 1$
is **NOT** a vertical asymptote.

$$\text{(ii)} \quad \lim_{x \rightarrow -1^+} f(x) = +\infty.$$

Therefore the line $x = -1$
is a vertical asymptote.

$$\text{(iii)} \quad \lim_{x \rightarrow \infty} f(x) = 0.$$

Therefore the line $y = 0$
is a horizontal asymptote.

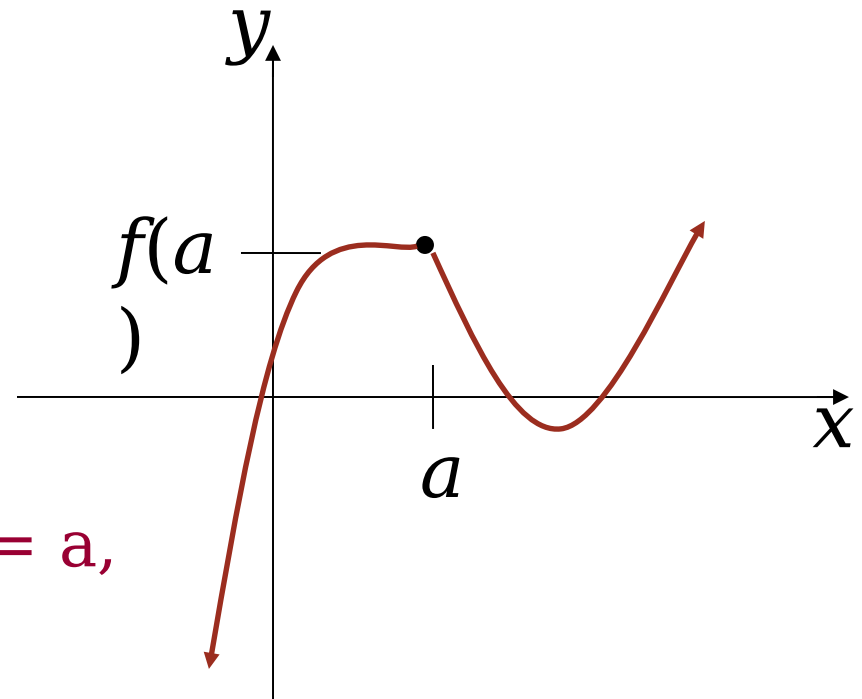


Continuity

1. Continuity of a Function at a point

A function f is **continuous** at the point $x = a$ if the following three conditions are satisfied:

- i) $f(a)$ is defined
- ii) $\lim_{x \rightarrow a} f(x)$ exists
- iii) $\lim_{x \rightarrow a} f(x) = f(a)$



If $f(x)$ is not continuous at $x = a$,
then it is said to be
discontinuous at $x = a$.

N.B To say a function is continuous at $x = c$ means that there is NO interruption in the graph of f at c . The graph has no holes, gaps, or jumps.

Examp

le $f(x) = x - 1$ at $x = 2$.

a. $f(2) = 1$ $f(2)$ is defined

b. $\lim_{x \rightarrow 2} (x - 1) = 1$ The limit exist!

c. $f(2) = 1 = \lim_{x \rightarrow 2} (x - 1)$

Therefore the function is continuous at $x = 2$.

Example

$$f(x) = (x^2 - 9)/(x + 3) \text{ at } x = -3$$

a. $f(-3) =$ Is undefined!

b. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$ The limit exist!

c. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} \neq f(-3)$

Therefore the function is not continuous at $x = -3$.

Example -3

Determine the continuity of the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{at } x = 0.$$

Solution :

$$(\text{LHL at } x = 0) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$$

$$= 0 \times (\text{a finite oscillating number between } -1 \text{ and } 1) = 0$$

$$(\text{RHL at } x = 0) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$$

$$= 0 \times (\text{a finite oscillating number between } -1 \text{ and } 1) = 0 \quad \text{and } f(0) = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

So, $f(x)$ is continuous at $x = 0$.

Examples

At which value(s) of x is the given function discontinuous?

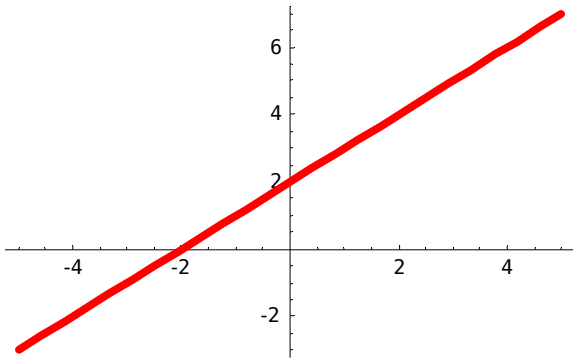
1. $f(x) = x + 2$

Continuous

everywhere

$$\lim_{x \rightarrow a} (x + 2) = a + 2$$

and so $\lim_{x \rightarrow a} f(x) = f(a)$

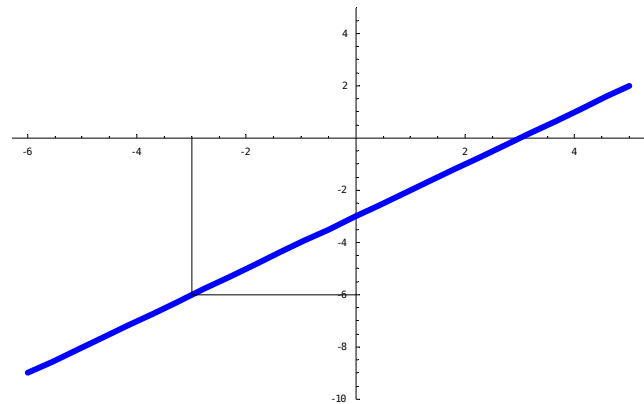


2. $g(x) = \frac{x^2 - 9}{x + 3}$

Continuous everywhere except at

$$x = -3$$

$g(-3)$ is undefined

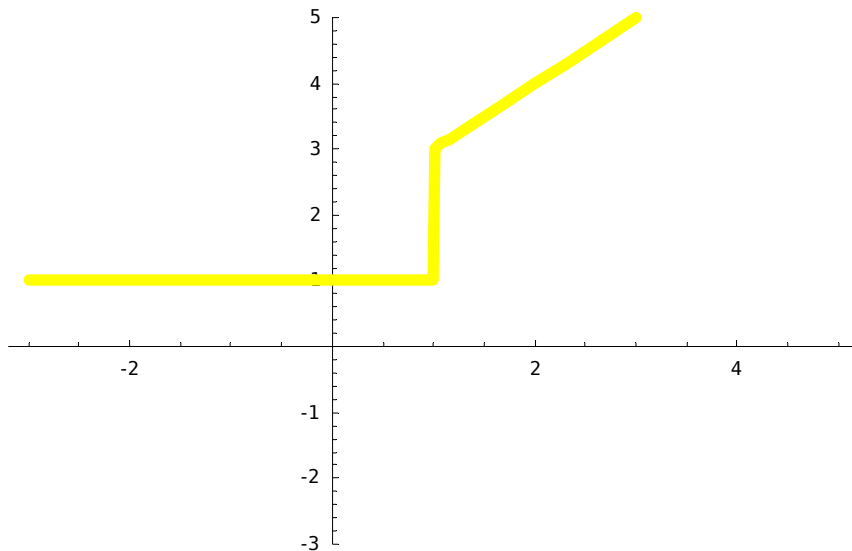


$$3. \quad h(x) = \begin{cases} x+2, & \text{if } x > 1 \\ 1, & \text{if } x \leq 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} h(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 1^+} h(x) = 3$$

Thus h is not cont. at $x=1$.

h is continuous everywhere else

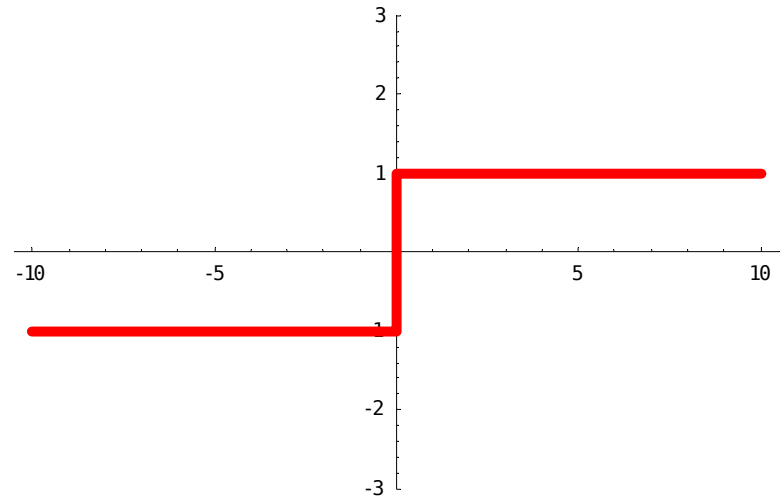


$$4. \quad F(x) = \begin{cases} -1, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} F(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} F(x) = -1$$

Thus F is not cont. at $x=0$.

F is continuous everywhere else



Exercises

1. Determine the value of the constant k so that the function

$$f(x) = \begin{cases} \frac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases} \quad \text{is continuous at } x = 0.$$

2. Determine the continuity of the function

$$f(x) = \begin{cases} \frac{e^x - 1}{\ln(1 + 2x)}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0.$$

3. Find the value of k iff $f(x)$ is continuous at $x = 2$, where

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$$

Solution for exercise number 2

Determine the continuity of the function

$$f(x) = \begin{cases} \frac{e^x - 1}{\ln(1 + 2x)}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0.$$

Solution :

$$(\text{LHL at } x = 0) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} \frac{e^x - 1}{\ln(1 + 2x)} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \times \frac{x}{\ln(1 + 2x)}$$

$$= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \times \frac{1}{\lim_{x \rightarrow 0} \frac{\ln(1 + 2x)}{2x} \times 2} = 1 \times \frac{1}{1 \times 2} = \frac{1}{2}$$

Solution (Cont.)

$$(\text{RHL at } x = 0) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} \frac{e^x - 1}{\ln(1 + 2x)} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \times \frac{x}{\ln(1 + 2x)}$$

$$= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \times \frac{1}{\lim_{x \rightarrow 0} \frac{\ln(1 + 2x)}{2x} \times 2} = 1 \times \frac{1}{1 \times 2} = \frac{1}{2}$$

and $f(0) = 7$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \neq f(0)$$

So, $f(x)$ is discontinuous at $x = 0$.

Discontinuity

- A function is discontinuous at c if f is defined on an interval (a, b) containing c (except maybe at c) and f is not continuous at c .

Types of Discontinuity

Discontinuities fall into two categories: **removable** and **non-removable**.

1. **Removable** : A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining $f(c)$).
You can factor/cancel out, therefore making it continuous by redefining $f(c)$.
2. **Non-Removable**: You can't remove it/cancel it

Two Types of Discontinuities

1. Removable

Point Discontinuity

2. Non-removable

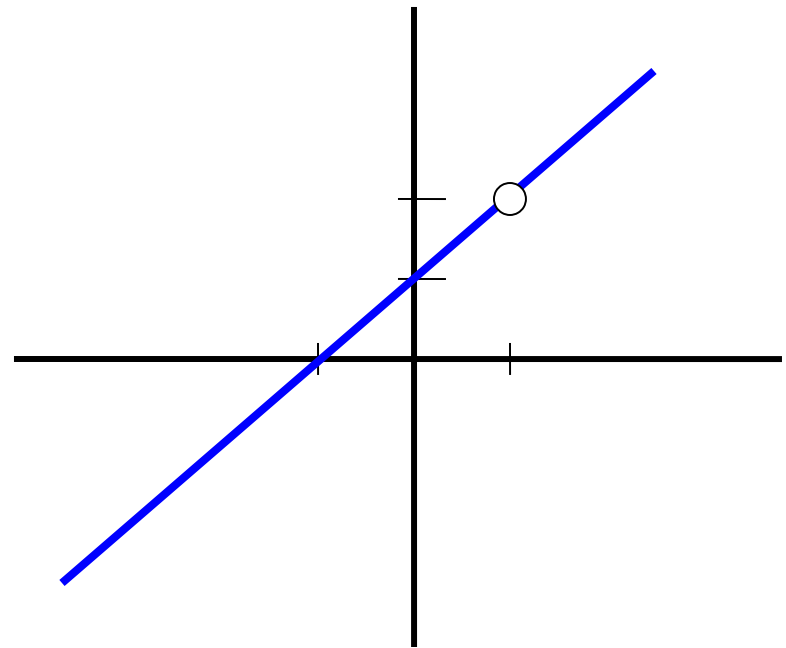
Jump and Infinite

Removable example

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$$f(x) = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

the open circle
can be filled in
to make it
continuous

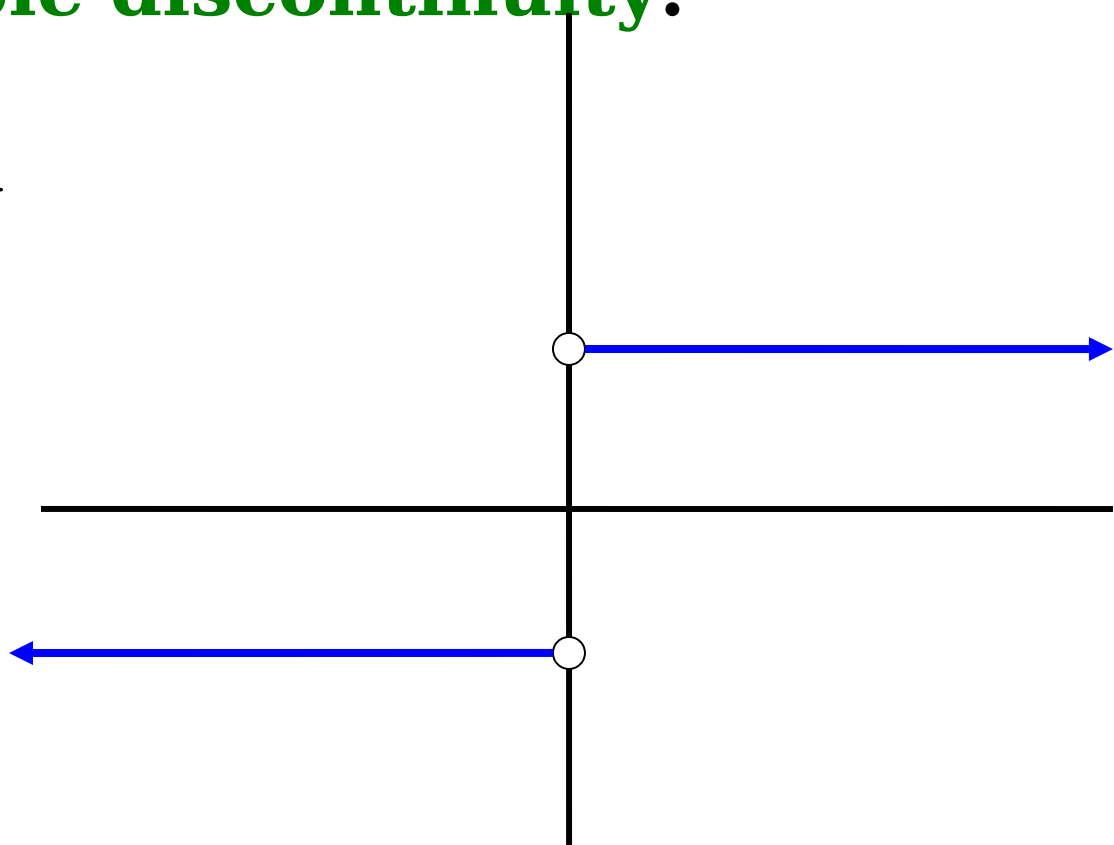


Non-removable discontinuity.

Ex. $\lim_{x \rightarrow 0} \frac{|x|}{x}$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$$

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$$



Examples:

- 1. Removable:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

We “removed” the $(x - 2)$.

Therefore, we have a removable discontinuity when

- $x = 2$. Non-Removable:

We can't remove/cancel out this discontinuity, so we have a NON-Removable discontinuity when $x = 1$.

If a function can be made to be continuous by defining or redefining a single point, the function has a

removable discontinuity. Example $f(x) =$

we can make the function continuous by defining $g(4) =$

Exercises

Discuss the continuity of each function.

a. $f(x) = \frac{1}{x}$

b. $g(x) = \frac{x^2 - 1}{x - 1}$

c. $h(x) = \begin{cases} x + 1, & x \leq 0 \\ x^2 + 1, & x > 0 \end{cases}$

d. $y = \sin x$

Ans. a) has non-removable discontinuity at $x=0$

b) Has removable discontinuity at $x=1$

c) And d) are continuous at everywhere.

One Sided Continuity

i. $f(x)$ is said to be continuous from the left at $x = a$ if

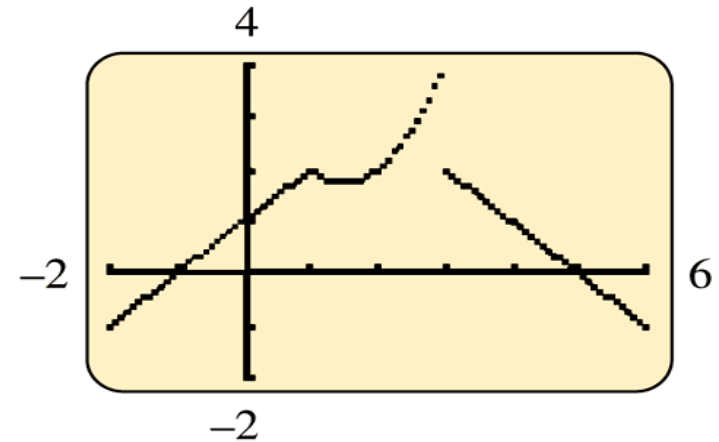
$$\lim_{x \rightarrow a^-} f(x) \text{ exists and } \lim_{x \rightarrow a^-} f(x) = f(a)$$

ii. $f(x)$ is said to be right continuous at $x = a$ if

$$\lim_{x \rightarrow a^+} f(x) \text{ exists and } \lim_{x \rightarrow a^+} f(x) = f(a)$$

1. Determine the continuity of the ff function at $x=1$ and at $x = 3$

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \leq x \leq 3 \\ 5 - x & \text{if } x > 3 \end{cases}$$



$$2. g(x) = \begin{cases} 5 - x, & -1 \leq x \leq 2 \\ x^2 - 1, & 2 < x < 3 \end{cases}$$

Is $g(x)$ continuous at $x = 2$?

2. Continuity Over an Open/Closed Interval

a) A function $f(x)$ is said to be **continuous** on an open interval (a, b) if $f(x)$ is continuous at every x -value in the interval.

b) A function $f(x)$ is said to be **continuous** on a closed interval $[a, b]$ if

(1) $f(x)$ is continuous on (a, b) .

$$(2) \lim_{x \rightarrow a^+} f(x) = f(a)$$

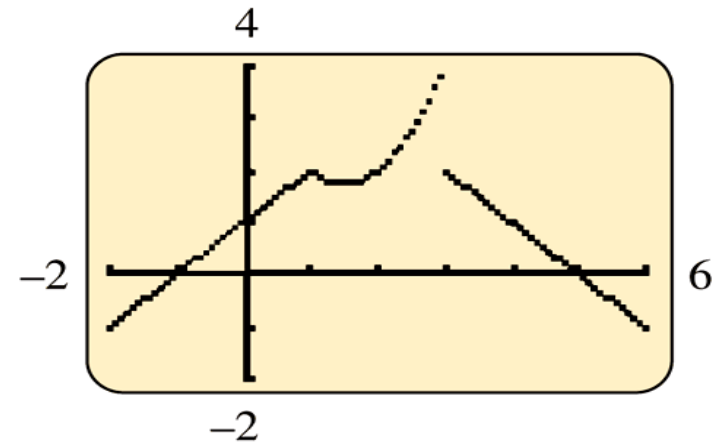
$$(3) \lim_{x \rightarrow b^-} f(x) = f(b)$$

CONTINUITY ON A CLOSED INTERVAL

A function is **continuous on a closed interval** $[a, b]$ if

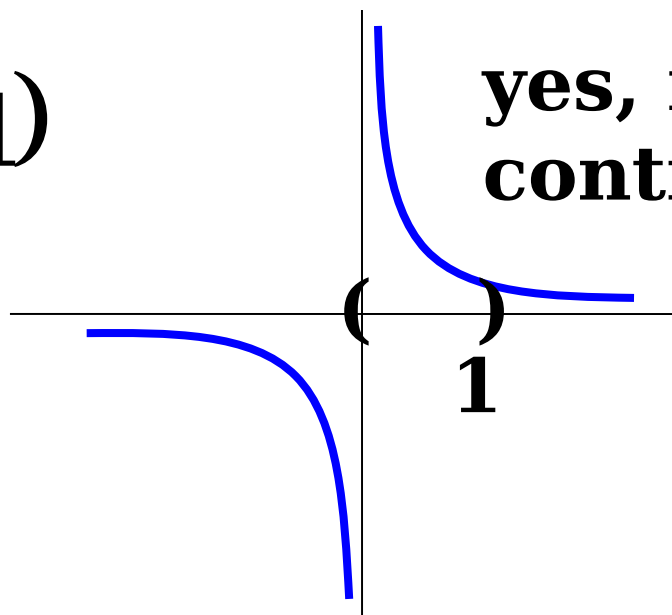
1. it is continuous on the open interval (a, b) ,
2. it is continuous from the right at $x = a$, and
3. it is continuous from the left at $x = b$.

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \leq x \leq 3 \\ 5 - x & \text{if } x > 3 \end{cases}$$



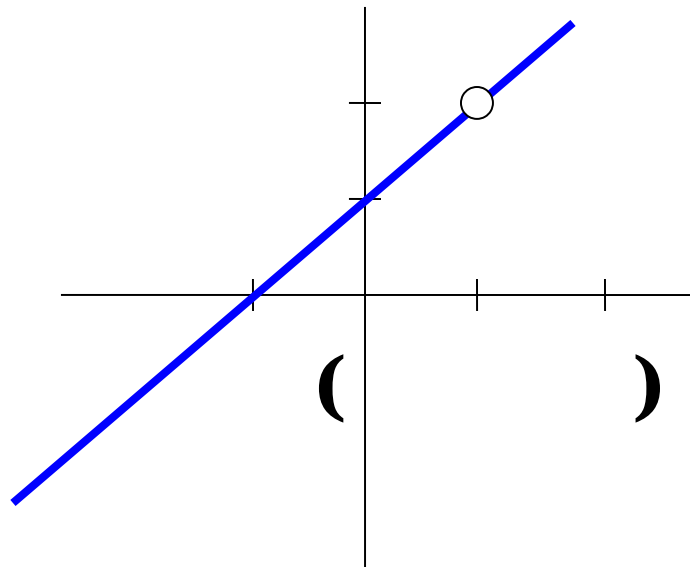
determine whether the following functions are continuous on the given interval.

$$f(x) = \frac{1}{x}, \quad (0,1)$$



**yes, it is
continuous**

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad (0, 2)$$



discontinuous at $x = 1$

removable discontinuity since filling in $(1, 2)$
could make it continuous.

Example – *Continuity on a Closed Interval*

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

Solution:

The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$ the function is continuous.

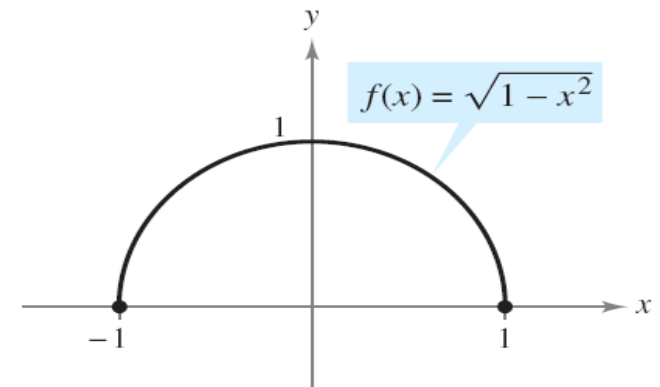
Moreover, $\lim_{x \rightarrow -1^+} \sqrt{1 - x^2} = 0 = f(-1)$

Continuous from the right

$$\lim_{x \rightarrow 1^-} \sqrt{1 - x^2} = 0 = f(1)$$

Continuous from the left

You can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure



f is continuous on $[-1, 1]$.

Properties of Continuous Functions

If f and g are continuous at $x = a$,
then

$f \pm g$, fg , and $\frac{f}{g}$ ($g(a) \neq 0$) are continuous at $x = a$.

Continuity of Special Functions

(1) A polynomial function is continuous at everywhere.

(2) A rational function is continuous at all x values in its domain.

(3) Trigonometric functions are continuous in their respective domains.

(4) The exponential function a^x , $a > 0$ is continuous everywhere.

(5) The logarithmic function is continuous in its domain.

(6) Inverse trigonometric functions are continuous in their domains.

(7) The composition of two continuous functions is a continuous function.

Exercise

Describe the interval(s) on which each function is continuous

$$\begin{array}{lll} \text{a. } f(x) = \tan x & \text{b. } g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} & \text{c. } h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \end{array}$$

$$\begin{array}{lll} \text{a. } f(x) = \tan x & \text{b. } g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} & \text{c. } h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \end{array}$$

Ans. A) The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$

At all other points it is continuous.

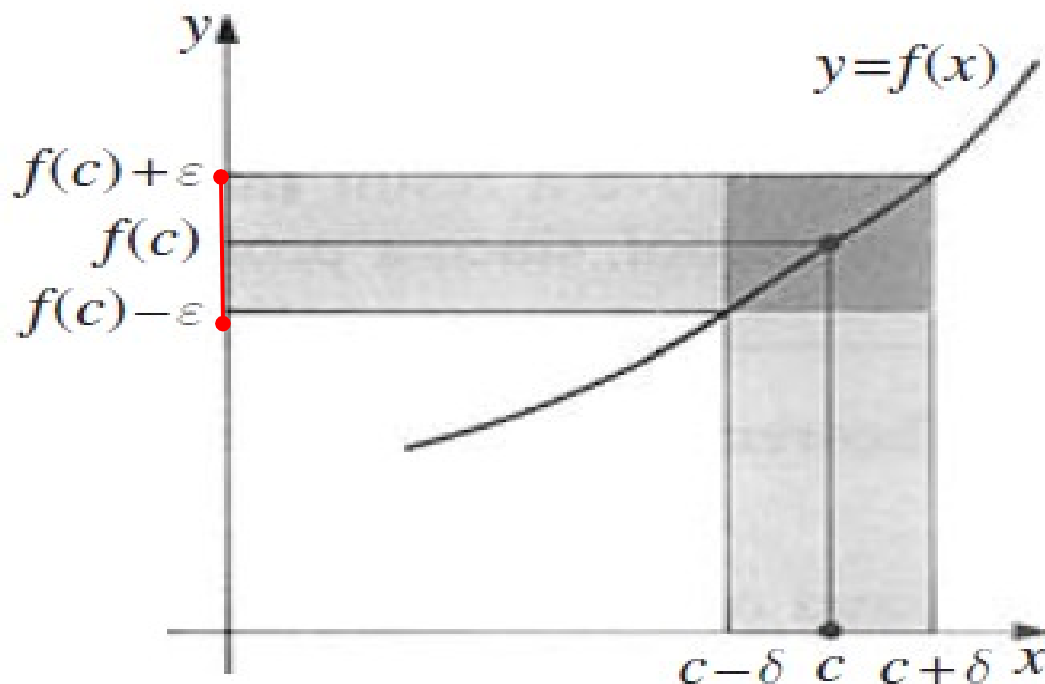
b) $y = \sin (1/x)$ is continuous at all real values except $x = 0$.

g is continuous on the interval $(-\infty, 0)$ and $(0, \infty)$,

c) h is continuous on the entire real line.

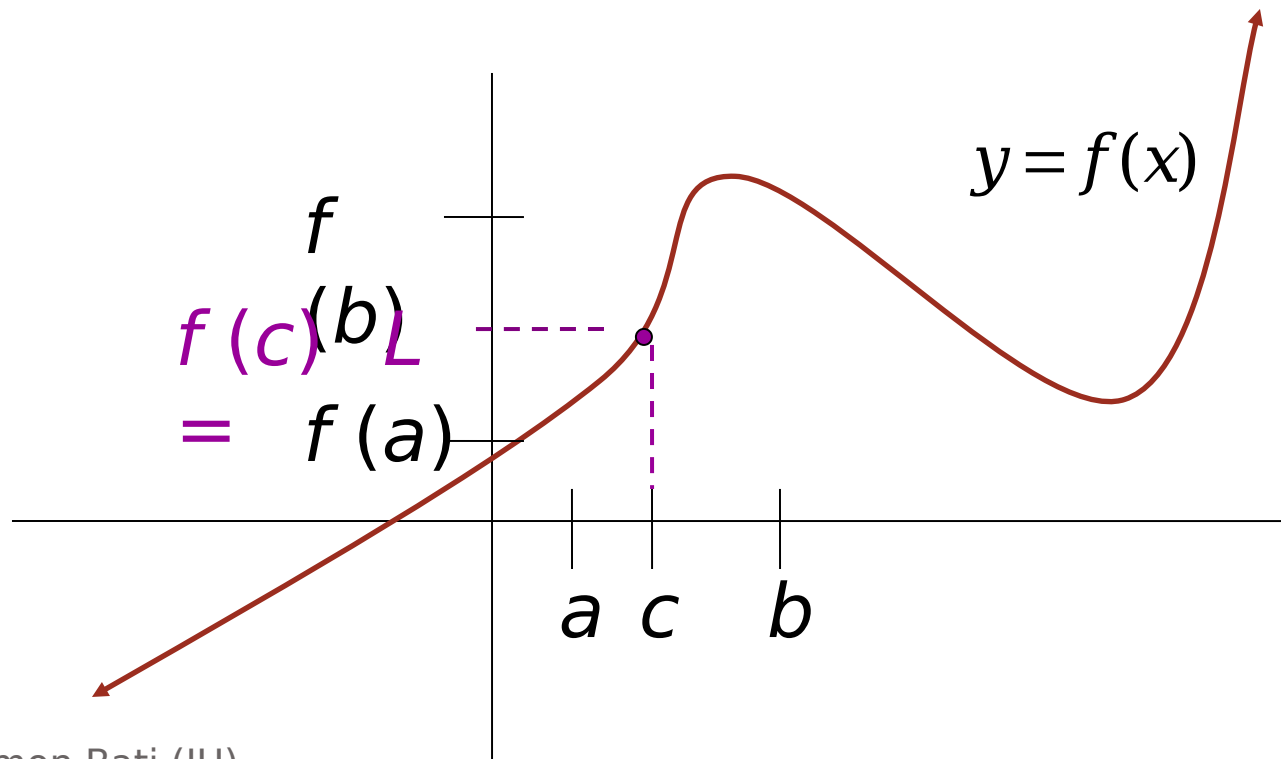
The Definition of continuity

A function f defined on an open interval contains c is **continuous at c** if $\forall \varepsilon > 0, \exists \delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.



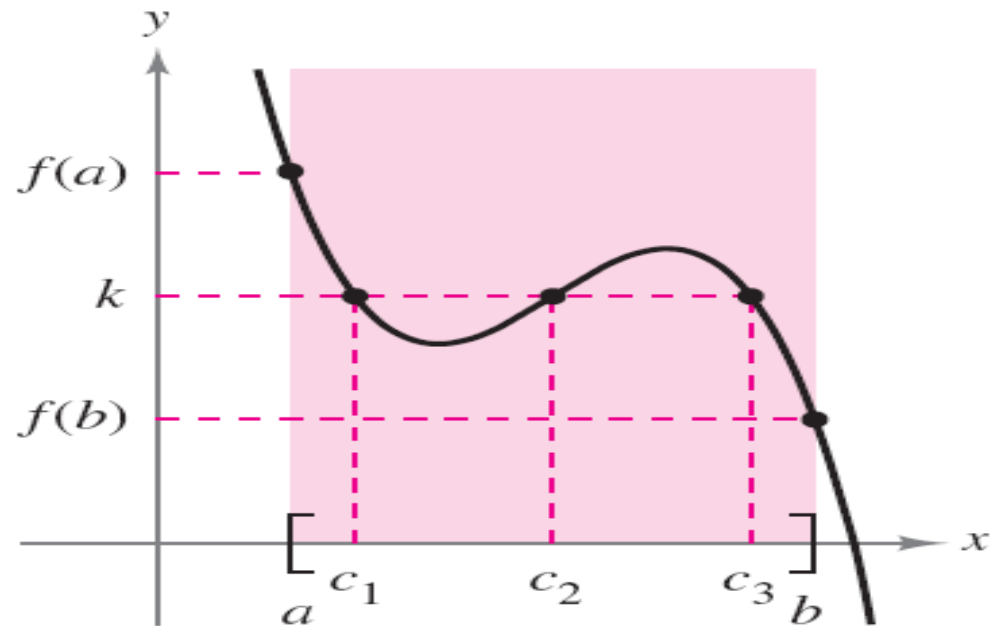
Intermediate Value Theorem

If f is a continuous function on a closed interval $[a, b]$, and L is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = L$.



The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$.

There may, of course, be more than one number c such that $f(c) = k$ as shown in Figure



f is continuous on $[a, b]$.

[There exist three c 's such that $f(c) = k$.]

The Intermediate Value Theorem states that for a continuous function f , if x takes on all values between a and b , $f(x)$ must take on all values between $f(a)$ and $f(b)$.

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.

Intermediate Value Theorem for Continuous Functions

If f is continuous, $f(a) < 0$ and $f(b) > 0$, then there is a point c between a and b so that $f(c) = 0$.

Example 1

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

Solution:

Note that f is continuous on the closed interval $[0, 1]$.

Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$.

You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

Example Given $f(x) = 3x^2 - 2x - 5$,

Show that $f(x) = 0$ has a solution on $[1, 2]$.

$$f(1) = -4 < 0$$

$$f(2) = 3 > 0$$

$f(x)$ is continuous (polynomial) and since $f(1) < 0$ and $f(2) > 0$, by the Intermediate Value Theorem there exists a c on $[1, 2]$ such that $f(c) = 0$.

Exercise. Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.