

Fredholm Integral equation of the second kind
with separable (or Degenerate) kernels

Ex 1 The solution of the integral equation

$$g(s) = s + \int_0^1 s u^2 g(u) du \text{ is given by}$$

(A) $g(t) = 3t/4$ (B) $g(t) = 4t/3$ (C) $g(t) = 3t/2$

Ans (B) given $g(s) = s + s \int_0^1 u^2 g(u) du$ — (1)

Let $C = \int_0^1 u^2 g(u) du$ — (2)

using (2), (1) yields $g(s) = s + Cs = s(1+C)$

from (2), $g(u) = u(1+C)$ — (3)

using (3) (2) yields $C = \int_0^1 u^3 (1+C) du$
 $= (1+C) \times \left[\frac{u^4}{4} \right]_0^1$

or $C = (1+C) \times (1/4) \Rightarrow C = 1/4 + C/4$

so that $C = 1/3$

Hence (3), $g(u) = u(1+C)$

$$g(u) = s(1 + 1/3)$$

$$g(u) = s(3 + 1/3)$$

$$= s \cdot 4/3 = \frac{4t}{3}$$

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Ex 4. Solve: $y(x) = f(x) + \lambda \int_0^1 x t y(t) dt$

Ans given $y(x) = f(x) + \lambda \int_0^1 x t y(t) dt$

$$y(x) = f(x) + \lambda x \int_0^1 t y(t) dt \quad \text{--- (1)}$$

$$C = \int_0^1 t y(t) dt \quad \text{--- (2)}$$

Then (1) reduces to

$$y(x) = f(x) + \lambda C x \quad \text{--- (3)}$$

$$\text{From (3), } y(t) = f(t) + \lambda C t \quad \text{--- (4)}$$

using (4), (2) reduces to

$$C = \int_0^1 t [f(t) + \lambda C t] dt$$

$$C = \int_0^1 t f(t) dt + \lambda C \left[\frac{t^3}{3} \right]_0^1$$

$$C = \int_0^1 t f(t) dt + \lambda \frac{C}{3}$$

$$\text{or } C \left(1 - \frac{\lambda}{3} \right) = \int_0^1 t f(t) dt$$

$$\text{or } C = \frac{3}{3-\lambda} \int_0^1 t f(t) dt, \text{ where } \lambda \neq 3$$

putting this value of C in (3), the required solution is

$$y(x) = f(x) + \frac{3\lambda x}{3-\lambda} \int_0^1 t f(t) dt, \text{ where } \lambda \neq 3$$

S13 Resolvent kernel or resolvent kernel :-

i) Suppose solution of Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt$$

It can takes the form

$$y(x) = f(x) + \lambda \int_a^b R(x,t; \lambda) f(t) dt$$

$$\text{or } y(x) = f(x) + \lambda \int_a^b r(x,t; \lambda) f(t) dt$$

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ii) The n th iterated kernel $k_n(x,t)$ satisfies the relation

$$k_n(x,t) = \int_a^b k_1(x,y) k_{n-1}(y,t) dy$$

where n is any positive integer less than m .

Let $R(x,t; \lambda)$ be the resolvent (or resolvent) kernel of a Fredholm integral equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt$$

then the resolvent kernel satisfies the integral equation

$$R(x,t; \lambda) = K(x,t) + \lambda \int_a^b K(x,z) R(z,t; \lambda) dz$$

the series for the resolvent kernel $R(x,t; \lambda)$

$$R(x,t; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$$

is absolutely and uniformly convergent for all values of x and t in the circle $|\lambda| < B^{-1}$.

Q1. find the iterated kernel (or function) for the following kernels.

i) $k(x,t) = \sin(x-2t)$, $0 \leq x \leq 2\pi$, $0 \leq t \leq 2\pi$

So iterated kernel $k_n(x,t)$ are given by

$$k_1(x,t) = k(x,t) \quad \text{--- (1)}$$

$$\text{and } k_n(x,t) = \int_0^{2\pi} k(x,z) k_{n-1}(z,t) dz, \quad (n=2,3,\dots) \quad \text{--- (2)}$$

$$[\because 0 \leq x \leq 2\pi, 0 \leq t \leq 2\pi \\ a=0, b=2\pi]$$

$$\text{From (1), } k_1(x,t) = k(x,t) = \sin(x-2t) \quad \text{--- (3)}$$

putting $n=2$ in (2), we have

$$k_2(x,t) = \int_0^{2\pi} k(x,z) k_1(z,t) dz$$

$$= \int_0^{2\pi} \sin(x-2z) \sin(z-2t) dz \quad \text{--- (2) using (3)}$$

$$= \frac{1}{2} \int_0^{2\pi} [\cos(x-2t-3z) - \cos(x-2t-z)] dz$$

$$= \frac{1}{2} \int_0^{2\pi} \cos$$

$$= \frac{1}{2} \left[-\frac{1}{3} \sin(x-2t-3z) + \sin(x-2t-z) \right]_0^{2\pi}$$

$$= 0$$

$$\therefore k_2(x,t) = 0 \quad \text{--- (4)}$$

putting $n=3$ in (2), we have

$$k_3(x,t) = \int_0^{2\pi} k(x,z) k_2(z,t) dz = 0$$

$$\therefore k_0(x, t) = 0$$

$$\text{Thus } k_1(x, t) = \sin(x-2t)$$

$$\text{and } k_n(x, t) = 0 \text{ for } n = 2, 3, 4, \dots$$

Note:- 1. $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

2. $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

3. $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

4. $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

1. $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$

2. $\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$

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Solution of Fredholm Integral equation with the help of Resolvent kernel:

$\Rightarrow y(x) = f(x) + \lambda \int_a^b k(x, t) y(t) dt$ be given Fredholm Integral equation. ^① let $k_m(x, t)$ be the m th iterated kernel and let $R(x, t; \lambda)$ be the resolvent kernel of ①.

Then we have

$$R(x, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x, t) \quad \text{--- ②}$$

Suppose the sum of infinite series ② exist and so $R(x, t; \lambda)$ can be obtained in the closed form. Then the required solution of ① is given by

$$y(x) = f(x) + \lambda \int_a^b R(x, t; \lambda) f(t) dt$$

Q94 The integral equation $y(x) = 1 + \lambda \int_0^{x/2} \cos(x-t)y(t)dt$ has

- a) a unique solution for $\lambda \neq 4/(x+2)$
- b) a unique solution for $\lambda \neq \frac{4}{x-2}$
- c) no solution for $\lambda = \frac{4}{x+2}$, but the corresponding homogeneous equation has non-trivial solutions.

Soln: given integral equation is

$$y(x) = 1 + \lambda \int_0^{x/2} \cos(x-t)y(t)dt$$

$$\Rightarrow y(x) = 1 + \lambda \int_0^{x/2} (\cos x \cos t + \sin x \sin t)y(t)dt$$

$$\Rightarrow y(x) = 1 + \lambda \int_0^{x/2} \cos x \cos t y(t)dt + \int_0^{x/2} \sin x \sin t y(t)dt$$

$$\text{where } C_1 = \int_0^{x/2} \cos t y(t)dt$$

$$y(x) = 1 + \lambda [\cos x C_1 + \sin x C_2]$$

$$\text{where } C_1 = \int_0^{x/2} \cos t y(t)dt \quad \text{--- (i)}$$

$$C_2 = \int_0^{x/2} \sin t y(t)dt \quad \text{--- (ii)}$$

from eq (i) we have

$$C_1 = \int_0^{x/2} \cos t [1 + \lambda (\cos t C_1 + \sin t C_2)]dt$$

$$= \int_0^{x/2} \cos t dt + \lambda C_1 \int_0^{x/2} \cos^2 t dt + \lambda C_2 \int_0^{x/2} \cos t \sin t dt$$

$$C_1 = [\sin t]_0^{\pi/2} + \lambda C_1 \int_0^{\pi/2} \left(1 + \frac{\cos 2t}{2}\right) dt + \lambda \frac{C_2}{2} \int_0^{\pi/2} \sin 2t dt$$

$$C_1 = 1 + \frac{\lambda C_1}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{\pi/2} + \lambda \frac{C_2}{2} \left[-\frac{\cos 2t}{2} \right]_0^{\pi/2}$$

$$C_1 = 1 + \frac{\lambda C_1}{4} \left[\frac{\pi}{2} \right] - \frac{\lambda C_2}{4} [-1, 1]$$

$$C_1 = 1 + \frac{\lambda \pi}{4} C_1 + \frac{\lambda C_2}{2}$$

det A = 0

(not-invariant)

$$\Rightarrow C_1 \left(1 - \frac{\lambda \pi}{4}\right) - \frac{\lambda}{2} C_2 = 1 \quad \text{--- (v)}$$

Now,

$$C_2 = \int_0^{\pi/2} \sin t dt + \lambda C_1 \int_0^{\pi/2} \cos t \sin t dt + \lambda C_2 \int_0^{\pi/2} \sin^2 t dt$$

$$C_2 = [-\cos t]_0^{\pi/2} + \frac{\lambda C_1}{2} \left[-\frac{\cos 2t}{2} \right]_0^{\pi/2} + \frac{\lambda C_2}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{\pi/2}$$

$$\Rightarrow C_2 = 1 + \frac{\lambda C_1}{2} + \frac{\lambda \pi}{4} C_2$$

$$\Rightarrow \left(C_2 - \frac{\lambda \pi}{4} C_2 - \frac{\lambda C_1}{2} \right) = 1$$

$$\Rightarrow \left(1 - \frac{\lambda \pi}{4}\right) C_2 - \frac{\lambda C_1}{2} = 0$$

$$\Rightarrow \left(-\frac{\lambda}{2}\right) C_1 + \left(1 - \frac{\lambda \pi}{4}\right) C_2 = 1 \quad \text{--- (vi)}$$

Thus from equation (v) and (vi) we get a system of equation in two variables C_1 and C_2 as

$$\left. \begin{aligned} \left(1 - \frac{\lambda \pi}{4}\right) C_2 - \frac{\lambda}{2} C_1 &= 1 \\ \left(-\frac{\lambda}{2}\right) C_1 + \left(1 - \frac{\lambda \pi}{4}\right) C_2 &= 1 \end{aligned} \right\} \quad \text{--- (vii)}$$

This system of equations has unique solution of

$$\begin{vmatrix} \frac{4-\lambda x}{4} & -\frac{\lambda}{2} \\ -\lambda/2 & \frac{4-\lambda x}{4} \end{vmatrix} = 0$$

$$\text{implies } \left(\frac{4-\lambda x}{4} \right)^2 - \frac{\lambda^2}{4} = 0$$

$$\text{implies } \frac{16 + \lambda^2 x^2 - 8\lambda x}{16} - \frac{\lambda^2}{4} = 0$$

$$\Rightarrow 16 + \lambda^2 x^2 - 8\lambda x - 4\lambda^2 = 0$$

$$\text{implies } \lambda = \frac{4}{x-2}, \frac{4}{x+2}$$

Hence option (A) and (B) are incorrect

Now, putting $\lambda = \frac{4}{x+2}$ in system of eq (vii), we get

$$\begin{aligned} & \left(\frac{4 - \frac{4}{x+2} \times x}{4} \right) \\ &= \frac{4(x+2) - 4x}{4(x+2)} \\ &= \left(\frac{2}{x+2} \right) e_1 - \left(\frac{2}{x+2} \right) e_2 = 1 \end{aligned}$$

$$\left(-\frac{4^2}{\pi+2} \times \frac{1}{2}\right) c_1 + \left(\frac{2}{\pi+2}\right) c_2 = 1.$$

$$\left. \begin{aligned} c_1 - c_2 &= \frac{\pi+2}{2} \\ -c_1 + c_2 &= \frac{\pi+2}{2} \end{aligned} \right\} \text{--- (ix)}$$

which has no solution

Now, the corresponding homogeneous equation of integral equation (i) is $y(x) = \lambda \int_0^{x/2} \cos(x-t) y(t) dt$

which has homogeneous system of equation as given by

$$\left. \begin{aligned} \left(\frac{4-\lambda\pi}{4}\right) c_1 - \frac{\lambda}{2} c_2 &= 0 \\ \left(-\frac{\lambda}{2}\right) c_1 + \left(\frac{4-\lambda\pi}{4}\right) c_2 &= 0 \end{aligned} \right\} \text{--- (x)}$$

So putting $\lambda = \frac{4}{\pi+2}$, we get

$$\left(\frac{2}{\pi+2}\right) c_1 - \left(\frac{2}{\pi+2}\right) c_2 = 0$$

$$\left(-\frac{2}{\pi+2}\right) c_1 + \left(\frac{2}{\pi+2}\right) c_2 = 0$$

$$c_1 - c_2 = 0$$

$$-c_1 + c_2 = 0$$

implies $\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1-1=0$ --- (xi)

Hence, system of eq (vi) has non-trivial solution.

Hence, given integral equation has no solution for

$$\lambda = \frac{4}{\pi+2}$$

But the corresponding homogeneous equation has non-trivial solution

Hence opt (c) is correct.

Now, putting $\lambda = 4/\pi+2$ in eq (vii), we get

$$-\left(\frac{2}{2-\pi}\right) c_1 + \left(\frac{2}{\pi+2}\right) c_2 = 1$$

$$-\left(\frac{2}{\pi+2}\right) c_1 + \left(\frac{2}{2-\pi}\right) c_2 = 1$$

$$\Rightarrow \left. \begin{aligned} c_1 + c_2 &= \left(\frac{2-\pi}{2}\right) \\ c_1 + c_2 &= \left(\frac{\pi-2}{2}\right) \end{aligned} \right\}$$

which has no solution

But, the corresponding homogeneous equation is

$$c_1 + c_2 = 0$$

$$c_1 + c_2 = 0$$

$$\text{Hence, } \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$

Hence integral eq (i) has no solution

$$\text{for } \lambda = \frac{4}{\pi+2}$$

but the corresponding homogeneous equation has non-trivial solutions.

195. For the integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t) dt$$

to have a continuous solution in the interval $a \leq x \leq b$, which of the following assumptions are necessary?

- (a) $K(x,t) \neq 0$, is real and continuous in the region $a \leq x \leq b$, $a \leq t \leq b$ with $|K(x,t)| \leq M$
- (b) $f(x) \neq 0$, is real and continuous in the interval $a \leq x \leq b$
- (c) λ is constant
- (d) $|\lambda| < \frac{1}{M(b-a)}$

196. $K(x,t)$ Given integral equation is

$$u(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt$$

and solution interval is $a \leq x \leq b$

$K(x,t) \neq 0$ is real and continuous in the region $a \leq x \leq b$, $a \leq t \leq b$

For if $K(x,t) \equiv 0$, the integral eq (i) doesn't exist or there is no integral sign.

Again, $|K(x,t)| \leq M$ i.e. $K(x,t)$ is bounded real value continuous function

for $\lambda |K(x,t)| \leq At$,

for, if $K(x,t)$ is unbounded, then $\int_a^b K(x,t) y(t) dt$ becomes an improper integral which shows that the integral eq (1) doesn't exist.

Hence option (a) is correct.

(ii) $f(x) \neq 0$ is real and continuous in the interval $a \leq x \leq b$ is also true. If $f(x) \equiv 0$, the integral eq (1) becomes homogeneous. Hence opt (b) is correct.

(iii) By definition of integral equation λ is always constant as λ is the eigenvalue of the integral equation.

Hence, opt (c) is correct.

(iv) Consider the integral equation

$$y(x) = 1 + \lambda \int_0^{\pi/2} \sin x y(t) dt$$

$$\text{with } y\left(\frac{\pi}{2}\right) = -1$$

$$\text{Then } y(x) = 1 + \lambda c \sin x$$

$$\text{where } c = \int_0^{\pi/2} y(t) dt$$

$$= \int_0^{\pi/2} [1 + \lambda c \sin t] dt$$

$$c = \left[t \right]_0^{\pi/2} + \lambda c [-\cos t]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \lambda c$$

$$\Rightarrow (c - c\lambda) = \pi/2$$

$$\Rightarrow c(1-\lambda) = \pi/2$$

$$\Rightarrow c = \frac{\pi}{2(1-\lambda)}$$

$$\text{Hence } y(x) = 1 + \lambda \sin x \cdot \frac{\pi}{2(1-\lambda)}$$

$$\text{i.e. } y(x) = 1 + \frac{\lambda \pi}{2(1-\lambda)} \sin x$$

but given that

$$y(\pi/2) = -1$$

$$-1 = 1 + \frac{\lambda \pi}{2(1-\lambda)} \Rightarrow 2 = \frac{\lambda \pi}{2(1-\lambda)}$$

$$\lambda = \frac{4}{4-\pi} \quad (\text{since } |\sin x| \leq 1)$$

$$\Rightarrow \frac{1}{M(b-a)} = \frac{1}{1(\pi/2 - 0)}$$

$$= \frac{2}{\pi}$$

$$|\lambda| = \left| \frac{4}{4-\pi} \right| > \frac{2}{\pi} = \frac{1}{M(b-a)}$$

$$|\lambda| > \frac{1}{M(b-a)}$$

Hence opt (d) ~~is~~ incorrect

Q40 For the Volterra type linear integral equation

$$\phi(x) = x + 2 \int_0^x e^{x-\xi} \phi(\xi) d\xi$$

the resolvent kernel $R(x, \xi, \lambda)$ of the kernel $e^{x-\xi}$ is

a) $(x-\xi)^2 e^{2(x-\xi)}$ b) $(x-\xi) e^{x-\xi}$

c) $e^{2(x-\xi)}$ d) $e^{x-\xi}$

Soln: The Volterra type linear integral equation is

$$\phi(x) = x + 2 \int_0^x e^{x-\xi} \phi(\xi) d\xi$$

Then $K(x, \xi) = e^{x-\xi}$

$$K_1(x, \xi) = K(x, \xi) = e^{x-\xi}$$

$$\begin{aligned} K_2(x, \xi) &= \int_{\xi}^x K(x, z) K_1(z, \xi) dz \\ &= \int_{\xi}^x e^{x-z} e^{z-\xi} dz \\ &= e^{x-\xi} (x-\xi) \\ &= \int_{\xi}^x e^{x-\xi} dz = e^{x-\xi} (x-\xi) \end{aligned}$$

$$\begin{aligned} K_3(x, \xi) &= \int_{\xi}^x K(x, z) K_2(z, \xi) dz \\ &= \int_{\xi}^x e^{x-z} \cdot e^{z-\xi} \cdot (z-\xi) dz \end{aligned}$$

$$\begin{aligned}
&= \int_{\xi}^x e^{x-\xi} (x-\xi) d\xi \\
&= e^{x-\xi} \int_{\xi}^x (x-\xi) d\xi \\
&= e^{x-\xi} \left[\int_{\xi}^x x d\xi - \int_{\xi}^x \xi d\xi \right] \\
&= e^{x-\xi} \left\{ x(x-\xi) - \frac{\xi^2}{2} \right\}_{\xi}^x \\
&= e^{x-\xi} \left(x(x-\xi) - \left(\frac{x^2-\xi^2}{2} \right) \right) \\
&= e^{x-\xi} \left[\left(\frac{x^2-\xi^2}{2} - \xi(x-\xi) \right) \right] \\
&= e^{x-\xi} \left[\frac{x^2-\xi^2 - 2\xi x + 2\xi^2}{2} \right] \\
&= e^{x-\xi} \frac{(x-\xi)^2}{2}
\end{aligned}$$

and so on.

$$\Rightarrow R(x, \xi; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x, \xi) = k_1(x, \xi) + \lambda k_2(x, \xi) + \lambda^2 k_3(x, \xi) + \dots$$

Hence, Resolvent Kernel

$$\lambda = 2$$

$$\begin{aligned}
R(x, \xi, \lambda) &= R(x, \xi, 2) \\
&= k_1(x, \xi) + 2k_2(x, \xi) + 2^2 k_3(x, \xi) + \dots \\
&= e^{x-\xi} + 2e^{x-\xi}(x-\xi) + 2^2 e^{x-\xi} \frac{(x-\xi)^2}{2} + \dots \\
&= e^{x-\xi} \left[1 + 2(x-\xi) + 2^2 \frac{(x-\xi)^2}{2} + \dots \right] \\
&= e^{x-\xi} \cdot e^{2(x-\xi)} = e^{3(x-\xi)}
\end{aligned}$$

Q100 The integral equation involving a parameter λ ,

$$\phi(x) = \cos 2x + \lambda \int_0^\pi \cos(x+\xi) d\xi \text{ has}$$

- a) a unique solution, if $\lambda = 1$ and an infinite number of solutions, if $\lambda = \frac{2}{\pi}$
 b) a unique solution, if $\lambda = -1$ and an infinite number of solutions, if $\lambda = -\frac{2}{\pi}$
 c) unique solution if $\lambda \neq \frac{2}{\pi}$
 d) no solution, if $\lambda = \pm \frac{2}{\pi}$

Soln $\phi(x) = \cos 2x + \lambda \int_0^\pi \cos(x+\xi) d\xi$

$$\phi(x) = \cos 2x + \lambda \int_0^\pi (\cos x \cos \xi - \sin x \sin \xi) d\xi$$

$$\phi(x) = \cos 2x + \lambda [\cos x \int_0^\pi \cos \xi d\xi - \sin x \int_0^\pi \sin \xi d\xi]$$

$$\phi(x) = \cos 2x + \lambda [\cos x (\sin \xi) - \sin x (-\cos \xi)]_0^\pi$$

$$\Rightarrow \phi(x) = \cos 2x - 2\lambda \sin x$$

Question 98
has correct

(i) at $\lambda = 1$

$$\phi(x) = \cos 2x - 2\sin x$$

(ii) at $\lambda = \frac{2}{\pi}$,

$$\phi(x) = \cos 2x - \frac{2 \times 2}{\pi} \sin x$$

$$\phi(x) = \cos 2x + \lambda \int_0^{\pi} \cos(x+\xi) \phi(\xi) d\xi$$

$$\phi(x) = \cos 2x + \lambda [C_1 \cos x - C_2 \sin x] \quad \text{--- (1)}$$

$$\therefore \phi(x) = \cos 2x + \lambda \int_0^{\pi} (\cos x \cos \xi - \sin x \sin \xi) \phi(\xi) d\xi$$

$$= \cos 2x + \lambda \left(\int_0^{\pi} \cos x \cos \xi d\xi - \int_0^{\pi} \sin x \sin \xi d\xi \right)$$

$$\text{where } C_1 = \int_0^{\pi} \cos \xi \phi(\xi) d\xi = \int_0^{\pi} \cos \xi \phi(\xi) d\xi$$

$$C_2 = \int_0^{\pi} \sin \xi \phi(\xi) d\xi$$

$$C_1 = \int_0^{\pi} \cos \xi [\cos 2\xi + \lambda \{C_1 \cos \xi - C_2 \sin \xi\}] d\xi$$

$$C_2 = \int_0^{\pi} \sin \xi [\cos 2\xi + \lambda \{C_1 \cos \xi - C_2 \sin \xi\}] d\xi$$

$$C_1 =$$

$$C_1 + A_2 C_1$$

$$C_2$$

$$\lambda = \pm \frac{2}{\pi}$$

$$\lambda^2 = \frac{4}{\pi^2}$$

$$\lambda = \pm \frac{2}{\pi}$$

Q3 The eigenvalues of λ of the following Fredholm integral equation

$$y(x) = \lambda \int_0^1 x^2 t y(t) dt, \quad \text{is}$$

- a) -2 b) $\frac{5}{2}$ ☒ c) 4 d) -4

$$y(x) = \lambda \int_0^1 x^2 t y(t) dt$$

$$y(x) = \lambda x^2 \int_0^1 t y(t) dt \quad \text{--- (i)}$$

$$C = \int_0^1 t y(t) dt \quad \text{--- (ii)}$$

Then eq (i) reduces to

$$y(x) = \lambda C x^2 \quad \text{--- (iii)}$$

From eq (iii), $y(t) = \lambda C t^2 \quad \text{--- (iv)}$

$$C = \int_0^1 t y(t) dt$$

$$= \int_0^1 t (\lambda C t^2) dt$$

$$C = \lambda C \int_0^1 t^3 dt$$

$$C = \frac{\lambda C}{4}$$

$$\left[1 - \frac{\lambda}{4} \right] C = 0 \quad \text{--- (v)}$$

If $C = 0$, then from eq (iv) $y(x) = 0$

We assume for non-zero solution of eq (i), $C \neq 0$. Then eq (v) gives

$$1 - \frac{\lambda}{4} = 0 \Rightarrow \lambda = 4$$

which is an eigenvalue of eq (i).

Q5 The resolvent kernel for the integral equation

$$u(x) = f(x) + \int_{\log 2}^x e^{(t-x)} u(t) dt$$

is

- a) $\cos(x-t)$ b) 1 c) e^{t-x} d) $e^{2(t-x)}$

Soln, on given

$$u(x) = f(x) + \int_{\log 2}^x e^{t-x} u(t) dt \quad \text{--- (i)}$$

on comparing eq (i), with

$$u(x) = f(x) + \lambda \int_{\log 2}^x k(x,t) u(t) dt$$

$$\lambda = 1$$

$$\text{and } k(x,t) = e^{t-x} \quad \text{--- (ii)}$$

Let $k_m(x,t)$ be the m^{th} iterated kernel, then

$$k_1(x,t) = k(x,t) \quad \text{--- (iii)}$$

$$\text{and } k_m(x,t) = \int_t^x k(x,z) k_{m-1}(z,t) dz$$

$$m = 2, 3, \dots \quad \text{--- (iv)}$$

we have now,

$$k_2(x,t) = k(x,t) = e^{t-x}$$

putting $m=2$ in eq (iv) and using eq (iii), we get

$$k_2(x,t) = \int_t^x k(x,z) k_1(z,t) dz$$

$$= \int_t^x e^{z-x} \cdot e^{t-z} dz$$

$$= e^{t-x} \int_t^x dz$$

$$= (x-t) e^{t-x} \text{ --- (vi)}^{\circ}$$

Next putting $m=3$, in eq (v), we get

$$\begin{aligned} k_3(x,t) &= \int_t^x k(x,z) k_2(z,t) dz \\ &= \int_t^x e^{z-x} (z-t) e^{t-z} dz \\ &= e^{t-x} \int_t^x (z-t) dz \\ &= e^{t-x} \left[\frac{(z-t)^2}{2} \right]_t^x \\ &= e^{t-x} \cdot \frac{(x-t)^2}{2!} \text{ --- (vi)}^{\circ} \end{aligned}$$

and soon. observing v eq (v), (vi)^o, and (vi)^o etc.
by mathematical induction, we have

$$k_m(x,t) = e^{t-x} \cdot \frac{(x-t)^{m-1}}{(m-1)!}$$

$$m = 1, 2, 3, \dots$$

Now, by the definition of resolvent kernel,

$$\begin{aligned} R(x,t;\lambda) &= \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) = \sum_{m=1}^{\infty} k_m(x,t) \\ &= \sum_{m=1}^{\infty} e^{t-x} \cdot \frac{(x-t)^{m-1}}{(m-1)!} \\ &= e^{t-x} \sum_{m=1}^{\infty} \frac{(x-t)^{m-1}}{(m-1)!} \\ &= e^{t-x} \cdot e^{x-t} \\ &= e^{t-x+x-t} = e^0 = 1 \end{aligned}$$

Q10
Q13

The eigenvalues λ of the integral equation

$$y(x) = \lambda \int_0^{2\pi} \sin(x+t) y(t) dt \text{ are}$$

- (a) $\frac{1}{2\pi}, -\frac{1}{2\pi}$ (b) $\frac{1}{\pi}, -\frac{1}{\pi}$ (c) $\pi, -\pi$ (d) $2\pi, -2\pi$

Ans:

$$y(x) = \lambda \int_0^{2\pi} (\sin x \cos t + \cos x \sin t) y(t) dt$$

$$y(x) = \lambda \sin x \int_0^{2\pi} \cos t y(t) dt + \lambda \cos x \int_0^{2\pi} \sin t y(t) dt \quad \text{--- (i)}$$

$$\text{Let } C_1 = \int_0^{2\pi} \cos t y(t) dt \quad \text{--- (ii)}$$

$$C_2 = \int_0^{2\pi} \sin t y(t) dt \quad \text{--- (iii)}$$

Then eq (i) reduces to

$$y(x) = \lambda C_1 \sin x + \lambda C_2 \cos x \quad \text{--- (iv)}$$

From eq (ii), we get

$$y(t) = \lambda C_1 \sin t + \lambda C_2 \cos t \quad \text{--- (v)}$$

Using eq (v) and (ii) becomes

$$C_1 = \int_0^{2\pi} \cos t (\lambda C_1 \sin t + \lambda C_2 \cos t) dt$$

$$C_1 = \frac{\lambda C_1}{2} \int_0^{2\pi} \sin 2t dt + \frac{\lambda C_2}{2} \int_0^{2\pi} (1 + \cos 2t) dt$$

$$C_1 = \frac{\lambda C_1}{2} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi} + \frac{\lambda C_2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$C_1 = 0 + \lambda C_2 \pi$$

$$C_1 - \lambda \pi C_2 = 0 \quad \text{--- (vi)}$$

Using eq (i) and (ii) becomes

$$C_2 = \int_0^{2\pi} \sin t (\lambda C_1 \sin t + \lambda C_2 \cos t) dt$$

$$C_2 = \frac{\lambda C_1}{2} \int_0^{2\pi} (1 - \cos 2t) dt + \lambda \frac{C_2}{2} \int_0^{2\pi} \sin 2t dt$$

$$C_2 = \frac{\lambda C_1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi} + \frac{\lambda C_2}{2} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi}$$

$$C_2 = \lambda C_1 \pi \Rightarrow \pi \lambda C_1 - C_2 = 0 \quad \text{--- (vi)}$$

Thus, we have a system of homogeneous linear eq (vi) and (vii) for determining C_1 and C_2 .
For non-zero solution of this system of equation,

we must have

$$\begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0$$

$$\Rightarrow -1 + \lambda^2 \pi^2 = 0$$

$$\lambda = \pm \frac{1}{\pi}$$

Hence, the eigenvalues are given by

$$\lambda_1 = \frac{1}{\pi} \text{ and } \lambda_2 = -\frac{1}{\pi}$$

3. The initial value problem corresponding to the integral equation

$$y(x) = 1 + \int_0^x y(t) dt \text{ is}$$

(a) $y' - y = 0$, $y(0) = 1$ (b) $y' + y = 0$, $y(0) = 0$

(c) $y' - y = 0$, $y(0) = 0$ (d) $y' + y = 0$, $y(0) = 1$

Ans given $y(x) = 1 + \int_0^x y(t) dt$ — (1)

putting $x = 0$ in eq (1), we get

$$y(0) = 1 + \int_0^0 y(t) dt$$

$$y(0) = 1. \quad \text{———— (2)}$$

On differentiating both sides of eq (1), wrt x ,

we get

$$y'(x) = 0 + \int_0^x \frac{d}{dx} (1) \cdot y(t) dt + y(x) \frac{dx}{dx} \\ - y(0) \cdot \frac{d}{dx} (0)$$

$$y'(x) = 0 + \int_0^x \frac{d}{dx} (1) \cdot y(t) dt + y(x) \cdot \frac{dx}{dx} \\ - y(0) \cdot \frac{d}{dx} (0)$$

$$y'(x) = 0 + 0 \cdot y(x) \cdot 1 - 0$$

$$y'(x) = y(x)$$

$$\Rightarrow y' - y = 0$$

Hence, the required initial value problem is
 $y' - y = 0$ with initial value condition
 $y(0) = 1$.

Q99. The integral equation

$$\phi(x) = \lambda \int_0^\pi k(x, \xi) \phi(\xi) d\xi$$

where λ is the boundary value problem

$$\phi''(x) - f(\lambda)\phi(x) = 0$$

$\phi(\pi) = 0$, $\phi'(0) = 0$, where $f(\lambda)$ is known.

Then the boundary value problem has

- a) a unique solution when $f(\lambda) = 0$
- b) infinite number of solutions when $f(\lambda) > 0$
- c) no solution when $f(\lambda) < 0$
- d) unique solution when $\lambda > 1$

Soln

The integral equation

$$\text{is } \phi(x) = \lambda \int_0^\pi k(x, \xi) \phi(\xi) d\xi$$

where λ is parameter and
parameter

$$k(x, \xi) = \begin{cases} \cos x \sin \xi, & \text{for } 0 \leq x < \xi \\ \cos \xi \sin x, & \text{for } \xi \leq x \leq \pi \end{cases}$$

$$\therefore \phi(x) = \lambda \left[\int_0^\xi \cos x \sin \xi \phi(\xi) d\xi + \int_\xi^\pi \cos \xi \sin x \phi(\xi) d\xi \right]$$

Differentiating w.r.t x , we get

$$\phi'(x) = \lambda \left[\int_0^\xi -\sin x \sin \xi \phi(\xi) d\xi + \int_\xi^\pi \cos \xi \cos x \phi(\xi) d\xi \right]$$

Again, differentiating w.r.t x , we get

$$\phi''(x) = \lambda \left[\int_0^x \cos s \, ds \right]$$

$$\phi''(x) = \lambda \left[\int_0^x \cos s \sin s \, ds - \int_0^{\pi} \cos s \sin x \phi(s) \, ds \right]$$

$$\text{implies } \phi''(x) = -\phi(x)$$

$$\text{implies } \phi''(x) + \phi(x) = 0$$

$$\text{Also, } \phi(0) = 0, \phi(\pi) = 0$$

$$\phi'(0) = 0$$

$$\text{Hence, we get } f(\lambda) = 1$$

$$\text{Now, } f''(x) - f(\lambda)\phi(x) = 0$$

Auxiliary equation is

$$\text{implies } \phi^2 - f(\lambda) = 0$$

$$\text{implies } \phi^2 = f(\lambda)$$

$$\phi = \pm \sqrt{f(\lambda)}$$

$$\phi(x) = c_1 e^{+\sqrt{f(\lambda)}x} + c_2 e^{-\sqrt{f(\lambda)}x}$$

$$\phi'(x) = c_1 \sqrt{f(\lambda)} e^{+\sqrt{f(\lambda)}x} - c_2 \sqrt{f(\lambda)} e^{-\sqrt{f(\lambda)}x}$$

Cases: When all λ_i 's are distinct and real

$$y_c = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots + c_n e^{\lambda_n x}$$

also: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{\lambda x} + c_4 e^{\lambda_4 x} + \dots + c_n e^{\lambda_n x}$$

Now, using boundary condition, we get-

$$\phi(\pi) = 0,$$

$$\Rightarrow 0 = c_1 e^{\sqrt{f(\lambda)}\pi} + c_2 e^{-\sqrt{f(\lambda)}\pi}$$

$$\phi'(0) = 0 \Rightarrow 0 = c_1 - c_2$$

On solving, we get-

$$c_1 = c_2 = 0$$

$$\text{Hence } \phi(\pi) \equiv 0 \neq f(\lambda)$$

which is unique, Hence option (a) is correct

while option (b) and (c) are incorrect,

$$\text{Since } f(\lambda) = 1.$$

Hence option (d) is incorrect

Q. 67 For the linear integral equations

$$\phi(x) = x + \int_0^{1/2} \phi(\xi) d\xi$$

the resolvent kernel, $R(x, \xi; \lambda)$ is

- a) $1/2$ b) 2
c) $3/2$ d) 4

Soln. $\phi(x) = x + \int_0^{1/2} \phi(\xi) d\xi$

kernel $K(x, t) = 1$

$$k_1(x, t) = 1$$

$$k_2(x, t) = \int_0^{1/2} k_1(x, z) k(z, t) dz$$

$$= \int_0^{1/2} 1 \cdot dz$$

$$= [z]_0^{1/2} = \frac{1}{2} - 0 = \frac{1}{2}$$

$$k_3(x, t) = \int_0^{1/2} k_2(x, z) k(z, t) dz$$

$$= \frac{1}{2} [z]_0^{1/2} = \frac{1}{2} \cdot \frac{1}{2}$$

Similarly, $k_4(x, t) = \frac{1}{2^3}$

$k_5(x, t) = \frac{1}{2^4}$ and so on

Therefore, Resolvent kernel $R(x, \xi; \lambda)$

$$= \sum_{n=0}^{\infty} k_{n+1} \lambda^n$$

$$= k_1(x, t) + \lambda k_2(x, t) + \lambda^2 k_3(x, t) + \dots$$

Here $\lambda = 1$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

$$= \frac{1}{1 - \frac{1}{2}} = 2$$

Ques For the homogeneous Fredholm integral equation

$\phi(x) = \lambda \int_0^1 e^{x+t} \phi(t) dt$, a non-trivial solution exists, when λ has the value

(a) $\lambda = \frac{2}{e-1}$

(b) $\lambda = \frac{1}{e^2+1}$

(c) $\lambda = \frac{1}{e+1}$

(d) $\lambda = \frac{e}{e^2-1}$

Soln Given Fredholm integral equation is

$$\phi(x) = \lambda \int_0^1 e^{x+t} dt$$

$$= \lambda e^x \int_0^1 e^t dt$$

$$= \lambda e^x \cdot C$$

$$\text{where } C = \int_0^1 e^t dt = \int_0^1 e^t \phi(t) dt$$

$$= \int_0^1 e^t \lambda e^x C dt$$

$$= \lambda C \int_0^1 e^t e^t dt$$

$$= \lambda c \int_0^1 e^{2t} dt$$

$$= \lambda c \left[\frac{e^{2t}}{2} \right]_0^1$$

$$= \lambda c \left[\frac{e^2}{2} - \frac{1}{2} \right]$$

$$\Rightarrow c \left(1 - \lambda \left(\frac{e^2 - 1}{2} \right) \right) = 0$$

$$\Rightarrow 1 = \lambda \left(\frac{e^2 - 1}{2} \right)$$

$$\Rightarrow \lambda = \frac{2}{e^2 - 1}$$

32 The general solution of the differential equation $\frac{d^2 y}{dx^2} + y = f(x)$, $x \in (-\infty, \infty)$, where f is a continuous, real-valued function on $(-\infty, \infty)$ is (where A, B, c and k are arbitrary constants).

a) $y(x) = A \cos x + B \sin x + \int_0^x f(t) \sin(x-t) dt$

b) $y(x) = \cos(x+k) + c \int_0^x f(t) \sin(x-t) dt$

c) $y(x) = A \cos x + B \sin x + \int_0^x e^{-t} f(x-t) \sin t dt$

d) $y(x) = A \cos x + B \sin x + \int_0^x f(x-t) \cos t dt$

Ans 32 Correct, $x = t$
 $f(x) = f(t)$, $y'' + y = f(x)$
 $m^2 + 1 = m \pm \pm i$

C.F. = $C_1 \cos x + C_2 \sin x$

P.I. = $\frac{1}{0^2 + 1} \cdot f(x)$

$$P.I = \frac{1}{D^2+1} f(x)$$

$$= (1+D^2)^{-1} f(x)$$

Hence opt (b) is correct.

Q54 The initial value problem

$$\frac{\partial^2 y}{\partial x^2} + y = 0; x > 0$$

$$y(0) = 1$$

$$y'(0) = 0$$

is equivalent to the Volterra integral equation

(a) $y(x) = 1 + \int_0^x (t-x) y(t) dt$

(b) $y(x) = 1 + \int_0^x (t+x) y(t) dt$

(c) $y(x) = 1 + \int_0^x x + y(t) dt$

(d) $y(x) = 1 + \int_0^x (x-t) y(t) dt$

Ans. given initial value problem is

$$\frac{\partial^2 y}{\partial x^2} + y = 0; x > 0$$

$$y(0) = 1, y'(0) = 0$$

$$\frac{\partial^2 y}{\partial x^2} = -y$$

on integration, we get

$$y'(x) - y'(0) = - \int_0^x y(t) dt$$

$$y'(x) = - \int_0^x y(t) dt$$

again, on integrating, we get

$$y(x) - y(0) = - \int_0^x \int_0^x y(t) dt \quad \text{--- (1)}$$

Converting a multiple integral into a single ordinary integral

$$\boxed{\int_a^x y(t) dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} y(t) dt}$$

from (1)

$$y(x) - 1 = - \int_0^x (x-t) y(t) dt$$

$$y(x) - 1 = - \int_0^x (t-x) y(t) dt$$

$$y(x) = 1 + \int_0^x (t-x) y(t) dt$$

which is required Volterra integral equation

therefore, option (a) is correct.

Q55. The integral equation

$$\varphi(x) - \lambda \int_{-1}^1 \cos(\pi(x-t)) \varphi(t) dt = f(x)$$

- (a) a unique solution for $\lambda \neq 1$ when $f(x) = x$
 (b) no solution for $\lambda \neq 1$ when $f(x) = 1$
 (c) $y(x) = 1 + \int_0^x x t y(t) dt$
 (d) $y(x) = 1 + \int_0^x (x-t) y(t) dt$
 (e) no solution for $\lambda = 1$ when $f(x) = x$
 (f) infinite number of solutions for $\lambda = 1$ when $f(x) = 1$.

Soln. Given the integral equation

$$\varphi(x) = f(x) + \lambda \int_{-1}^1 \cos(\pi(x-t)) \varphi(t) dt$$

$$\Rightarrow \varphi(x) = f(x) + \lambda \int_{-1}^1 \cos(\pi(x-t)) \varphi(t) dt$$

The corresponding homogeneous integral equation is

$$\varphi(x) = \lambda \int_{-1}^1 [\cos \pi x \cos \pi t + \sin \pi x \sin \pi t] \varphi(t) dt$$

$$\varphi(x) = \lambda c_1 \cos \pi x + \lambda c_2 \sin \pi x$$

$$\text{where } c_1 = \int_{-1}^1 \cos \pi t \varphi(t) dt$$

$$c_2 = \int_{-1}^1 \sin \pi t \varphi(t) dt$$

$$\text{Now Now, } c_1 = \int_{-1}^1 \cos \pi t \left[f(t) + \lambda \int_{-1}^1 \cos \pi(t-s) \varphi(s) ds \right] dt$$

$$C_1 = \int_{-1}^1 \cos \pi t \phi(t) dt$$

$$= \int_{-1}^1 \cos \pi t [\lambda c_1 \cos \pi t + \lambda c_2 \sin \pi t] dt$$

$$C_1 = \lambda c_1 \int_{-1}^1 (1 + \frac{\cos 2\pi t}{2}) dt + \lambda c_2 \int_{-1}^1 \frac{\sin 2\pi t}{2} dt$$

$$C_1 = \frac{\lambda c_1}{2} \left[t + \frac{\sin 2\pi t}{2\pi} \right]_{-1}^1 + \frac{\lambda c_2}{2} \left[-\frac{\cos 2\pi t}{2\pi} \right]_{-1}^1$$

$$C_1 = \frac{\lambda c_1}{2} \left[1 + \frac{\sin 2\pi}{2\pi} - (-1 + \frac{\sin -2\pi}{2\pi}) \right] + \frac{\lambda c_2}{2} \left[-\frac{\cos 2\pi}{2\pi} + \frac{\cos -2\pi}{2\pi} \right]$$

$$C_1 = \frac{\lambda c_1}{2} \cdot 2$$

$$C_1 = \lambda c_1$$

$$C_1 (1 - \lambda) = 0$$

$$\lambda = 1$$

$$C_2 = \int_{-1}^1 \sin \pi t [\lambda c_1 \cos \pi t + \lambda c_2 \sin \pi t] dt$$

$$C_2 = \int_{-1}^1 \lambda c_1 \frac{\sin 2\pi t}{2} + \lambda c_2 \left(\frac{1 - \cos 2\pi t}{2} \right) dt$$

$$= \lambda c_1 \int_{-1}^1 \frac{\sin 2\pi t}{2} + \lambda c_2 \int_{-1}^1 \frac{(1 - \cos 2\pi t)}{2} dt$$

$$C_2 = 0 + \lambda c_2, \lambda = 1$$

Ques The integral equation

$$\phi(x) = f(x) + \int_0^1 k(x,y) \phi(y) dy$$

For $k(x,y) = xy^2$ has a solution

(a) $\phi(x) = f(x)$

(b) $\phi(x) = k(x,x)$

(c) $\phi = x^3$

(d) $\phi(x) = f(x) + \frac{4}{3} x \int_0^1 x^2 f(x) dx$

Soln $\phi(x) = f(x) + \int_0^1 xy^2 \phi(y) dy$

$$f(x) = f(x) + xc$$

$$\text{where } c = \int_0^1 y^2 \phi(y) dy$$

$$c = \int_0^1 y^2 (f(x) + xc) dy$$

$$= \int_0^1 y^2 \left(\frac{1}{3} f(y) + xy \right) dy$$

$$c = \int_0^1 [y^2 f(y) + cy^3] dy$$

$$c = \int_0^1 y^2 f(y) dy - \frac{c}{4}$$

$$c \left(1 - \frac{1}{4} \right) = \int_0^1 y^2 f(y) dy$$

$$c \left(\frac{3}{4} \right) = \int_0^1 y^2 f(y) dy$$

$$c = \frac{4}{3} \int_0^1 y^2 f(y) dy$$

from eq (i) and (ii), we get

$$\phi(x) = f(x) + \frac{4}{3} x \int_0^1 x^2 f(x) dx$$

Q.97. let λ_1, λ_2 be the characteristic numbers, and f_1, f_2 the corresponding eigenfunctions for the homogeneous integral equation

$$\phi(x) - \lambda \int_0^1 (xt + 2x^2) \phi(t) dt = 0, \text{ then}$$

(a) $\lambda_1 = -18 - 6\sqrt{10}, \lambda_2 = -18 + 6\sqrt{10}$

(b) $\lambda_1 = -36 - 12\sqrt{10}, \lambda_2 = -36 + 12\sqrt{10}$

(c) $\int_0^1 f_1(x) f_2(x) dx = 1$

(d) $\int_0^1 f_1(x) f_2(x) dx = 0$

Ans. $\phi(x) = \lambda \int_0^1 (xt + 2x^2) \phi(t) dt$

or $y(x) = \lambda \int_0^1 xt \phi(t) dt + \lambda \int_0^1 2x^2 \phi(t) dt$

$$= \frac{\lambda x \int_0^1 t \phi(t) dt}{1} + \frac{\lambda 2x^2 \int_0^1 \phi(t) dt}{\lambda}$$

$$= \lambda x \int_0^1 t \phi(t) dt + \lambda 2x^2 \int_0^1 \phi(t) dt$$

$$\Rightarrow c_1 = \int_0^1 t \phi(t) dt$$

$$c_2 = \int_0^1 \phi(t) dt$$

$$y(x) = \lambda x c_1 + \lambda_2 x^2 c_2$$

$$y(t) = \lambda t c_1 + \lambda_2 t^2 c_2$$

$$c_1 = \int_0^1 t (\lambda t c_1 + \lambda_2 t^2 c_2) dt$$

$$= \int_0^1 t^2 \lambda c_1 + 2 \lambda_2 t^3 c_2 dt$$

$$= \int_0^1 t^2 \lambda c_1 dt + 2 \lambda_2 \int_0^1 t^3 c_2 dt$$

$$= \lambda c_1 \left. \frac{t^3}{3} \right|_0^1 + 2 \lambda_2 c_2 \left. \frac{t^4}{4} \right|_0^1$$

$$c_1 = \frac{\lambda c_1}{3} + \frac{2 \lambda_2 c_2}{4}$$

$$c_1 - \frac{\lambda c_1}{3} - \frac{2 \lambda_2 c_2}{4} = 0$$

$$c_1 \left(1 - \frac{\lambda}{3}\right) - \frac{2 \lambda_2 c_2}{4} = 0$$

$$c_1 \left(1 - \frac{\lambda}{3}\right) - \frac{1}{2} \lambda_2 c_2 = 0$$

Again

$$c_2 = \int_0^1 (\lambda x c_1 + \lambda_2 x^2 c_2) dx$$

$$= \int_0^1 (\lambda t c_1 + \lambda_2 t^2 c_2) dt$$

$$= \int_0^L (\lambda t c_1 + \lambda 2t^2 c_2) dt$$

$$c_2 = \lambda c_1 \frac{t^2}{2} \Big|_0^L + \lambda c_2 \cdot \frac{2t^3}{3} \Big|_0^L$$

$$c_2 = \frac{\lambda c_1}{2} + \frac{2\lambda c_2}{3}$$

$$c_2 - \frac{2\lambda c_2}{3} = \frac{\lambda c_1}{2}$$

$$c_2 \left(1 - \frac{2\lambda}{3}\right) - \frac{\lambda c_1}{2} = 0$$

$$\Rightarrow -\frac{\lambda c_1}{2} + c_2 \left(1 - \frac{2\lambda}{3}\right) = 0$$

$$\begin{pmatrix} 1 - \frac{2\lambda}{3} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & 1 - \frac{2\lambda}{3} \end{pmatrix} = 0$$

$$\Rightarrow \left(1 - \frac{2\lambda}{3}\right)^2 - \frac{\lambda^2}{4} = 0$$

$$\Rightarrow \left(1 - \frac{2\lambda}{3}\right)^2 = \frac{\lambda^2}{4}$$

$$\Rightarrow \left(1 - \frac{2\lambda}{3}\right) = \frac{\lambda}{2}$$

$$\Rightarrow 1 = \frac{\lambda}{2} + \frac{2\lambda}{3} \Rightarrow 1 = \frac{3\lambda + 4\lambda}{6} \Rightarrow \lambda = \frac{6}{7}$$

$$\begin{vmatrix} 1 - \frac{\lambda}{3} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & 1 - \frac{2}{3}\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(1 - \frac{\lambda}{3}\right) \left(1 - \frac{2}{3}\lambda\right) - \left(\frac{\lambda^2}{4}\right) = 0$$

$$\Rightarrow \frac{2}{3}\lambda^2 - \frac{\lambda^2}{4} - \lambda + 1 = 0$$

$$\Rightarrow \frac{1}{36}\lambda^2 + \lambda - 2 = 0$$

$$\Rightarrow \lambda^2 + 36\lambda - 72 = 0$$

$$\Rightarrow (\lambda + 18)^2 = 360$$

$$\lambda = -18 \pm \sqrt{360}$$

$$= -18 \pm 6\sqrt{10}$$

$$\lambda_1 = -18 - 6\sqrt{10}$$

$$\lambda_2 = -18 + 6\sqrt{10}$$

16 f_1 and f_2 are corresponding eigen functions, then they are orthogonal.

basically $f_1 = 0, f_2 = 0$ homogenous solution.

Qus The homogeneous integral equation

$$\varphi(x) - \lambda \int_0^1 (3x-2)t \varphi(t) dt = 0 \text{ has}$$

- (a) one characteristic number
- (b) three characteristic numbers
- (c) two characteristic numbers
- (d) No characteristic numbers

Soln

$$\varphi(x) = \lambda \int_0^1 (3x-2)t \varphi(t) dt$$

$$\Rightarrow \varphi(x) = \lambda c (3x-2) \quad \text{--- (1)}$$

$$\Rightarrow c = \int_0^1 t \varphi(t) dt$$

Now, from equation (1),

$$\Rightarrow c = \int_0^1 t \varphi(t) dt$$

$$\Rightarrow c = \int_0^1 \lambda c (3x-2)t dt$$

$$= \int_0^1 \lambda c (3t-2)t dt$$

$$c = \lambda c \int_0^1 (3t^2 - 2t) dt$$

$$c = \lambda c \left[\frac{3t^3}{3} - \frac{2t^2}{2} \right]_0^1$$

$$c = \lambda c [1-1] \Rightarrow c = 0 \text{ (no characteristic number).}$$

43 Let $y: [0, \infty) \rightarrow \mathbb{R}$ be twice continuously differentiable and satisfy

$$y(x) + \int_0^x (x-s)y(s)ds = \frac{x^3}{6}$$

Then

A) $y(x) = \frac{1}{6} \int_0^x s^3 \sin(x-s) ds$

B) $y(x) = \frac{1}{6} \int_0^x s^3 \cos(x-s) ds$

C) $y(x) = \int_0^x s \sin(x-s) ds$

D) $y(x) = \int_0^x s \cos(x-s) ds$

Soln Given $y: [0, \infty) \rightarrow \mathbb{R}$ be twice differentiable and satisfy

$$y(x) + \int_0^x (x-s)y(s)ds = \frac{x^3}{6}$$

$$\Rightarrow y(x) = \frac{x^3}{6} - \int_0^x (x-s)y(s)ds$$

$$= \frac{x^3}{6} + \int_0^x (s-x)y(s)ds$$

The given equation is Volterra equation of second kind on comparison with

$$y(x) = f(x) + \lambda \int_0^x k(x,s)y(s)ds$$

$$f(x) = \frac{x^3}{6}, \lambda = 1, k(x,s) = (s-x)$$

Now, we find Redvent Kernel

Let $k_m(x,s)$ be the m th iterated kernel, then
 $k_1(x,s) = k(x,s)$ and