

Statement Definitions

A declarative sentence which is either true or false is called a **statement**.

A statement is said to be an **Atomic** Statement if it cannot be divided into smaller statements, otherwise it is called a **Molecular** Statement.

These statements are examples of atomic statements:

- Mobile numbers in Nigeria have 11 digits.
- 5 is larger than 7.
- 12 is a perfect square

Molecular statements

- Would you like some ice cream?
- The product of two numbers.
- $1 + 3 + 5 + 7 + \dots + 2n + 1$.
- Go to the lecture room!
- $4 + x = 12$

We can also build a **complicated (molecular) sentence** by combining more than one or more simple atomic or molecular sentences by using Logical Connectives.

The five connectives we can consider are “**and**”, “**or**”, “**if... then**”, “**if and only if**”, and “**not**”.

propositional variables (also called **sentential** variables) P, Q, R, S, ... are used to represent each atomic statements in the molecular statement.

Logical Connectives

- $P \wedge Q$ is read as “P and Q,” and it is called a **conjunction**. True when both are true
- $P \vee Q$ is read as “P or Q,” and it is called a **disjunction**. True when P or Q or both are true
- $P \rightarrow Q$ is read as “if P then Q,” and it is called an **implication** or **conditional**. True when P is false or Q is true or both
- $P \leftrightarrow Q$ is read as “P if and only if Q,” and it is called a **bi-conditional**. True when P and Q are both true or both false.
- $\neg P$ is read as “not P,” and it is called a **negation**. True when P is false.

Definition of **Set**: an unordered collections of objects. Objects of the set are called elements. The number of objects in a set can be finite or infinite.

Notations

A single set, **A** can be expressed with the following notations:

$A = \{1, 2\}$; $A = \{2, 1\}$; $A = \{1, 2, 1, 2\}$; $A = \{x \mid x \text{ is an integer, } 1 \leq x \leq 2\}$

The symbol “ $\mid$ ” (or “ $:$ ” or “ such that ”), implies “such that”

The symbol “ $\in$ ” implies “is in” or “is an element of.”

$A \subseteq B$ asserts that A is a subset of B | every element of A is also an element of B.

$A \subset B$ asserts that A is a proper subset of B | every element of A is also an element of B, but every element of B is not an element of A.

U: A fixed set which contains all other sets under investigation is called universal set.

Operations on Sets

- i. $\cup$: $A \cup B$ is the union of A and B: is the set containing all elements which are elements of A or B or both.
- ii. $\cap$: $A \cap B$ is the intersection of A and B: the set containing all elements which are elements of both A and B.
- iii. $\setminus$: $A \setminus B$ is A minus B: the set containing all elements of A which are not elements of B.
- iv. A^c or $A^{\bar{}}$: The complement of A is the set of everything in U which is not an element of A.
- v. $|A|$: The cardinality (or size) of A is the number of elements in A.
- vi. $\times$: $A \times B$ is the Cartesian product of two non-empty sets A and B: the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.

Rules of Set Theory

Let P, Q and R be sets.

- i. **Commutative Law:** $(P \cup Q) = (Q \cup P)$ and $(P \cap Q) = (Q \cap P)$.
- ii. **Associative Law:** $(P \cup (Q \cup R)) = ((P \cup Q) \cup R)$ and $(P \cap (Q \cap R)) = ((P \cap Q) \cap R)$.
- iii. **Distributive Law:** $(P \cup (Q \cap R)) = (P \cup Q) \cap (P \cup R)$ and $(P \cap (Q \cup R)) = (P \cap Q) \cup (P \cap R)$.
- iv. **De Morgan's Law:** $(P \cup Q)^c = (P^c \cap Q^c)$ and $(P \cap Q)^c = (P^c \cup Q^c)$

Some special sets we will consider in this unit:

- $\emptyset$ The empty set that contains no element (also denoted as $\{ \}$).
- U The universe set is the set of all elements
- $\mathbb{N}$ $\{0, 1, 2, 3, \dots\}$, the non-negative integers
- $\mathbb{N}^+$ $\{1, 2, 3, \dots\}$, the positive integers
- $\mathbb{Z}$ $\{\dots -2, -1, 0, 1, 2 \dots\}$, the integers
- $\mathbb{Q}$ $\{q \mid q = a/b, a, b \in \mathbb{Z}, b \neq 0\}$, the rational numbers
- $\mathbb{Q}^+$ $\{q \mid q \in \mathbb{Q}, q > 0\}$, the positive rational
- $\mathbb{R}$ The real numbers
- $\mathbb{R}^+$ The positive reals

- $P(A)$ The power set of any set A is the set of all subsets of A .

Disjoint Set

Sets X and Y are said to be disjoint sets, if they have no element in common, that is, no element of X is in Y and no element of Y is in X

Power Set

We call the set of all subsets of A , the power set of A , and write it as $P(A)$

Example: Let $A = \{1, 2, 3\}$. Find $P(A)$.

Solution $P(A)$ is a set of sets, all of which are subsets of A .

So, $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Note: The power set of a set A is normally, 2^n ,

Venn Diagrams

A Venn Diagram is a tool used to visualize and represent operations on sets. It is used to display sets as intersecting circles. The region under consideration is where we carry out an operation. The cardinality of a set can be represented by putting numbers in the corresponding area.

Relations

A relation on a single set S is a subset of $S \times S$. A relation on sets S and T is a subset of $S \times T$.

Definition: A relation R on a set S is:

- Reflexive** if for all $x \in S$, $(x, x) \in R$.
- Symmetric** if for all $x, y \in S$, whenever $(x, y) \in R$, $(y, x) \in R$.
- Transitive** if for all $x, y, z \in S$, whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

A relation that is reflexive, symmetric and transitive is called an Equivalence relation and is denoted by the symbol " $\equiv$ ".

An equivalence class is a maximal subset E of the set S such that any two elements in the set E is related.

PROOFS AND INDUCTION

Mathematical Induction a technique that is used to prove certain types of mathematical statements and propositions which assert that for all positive integers or for all positive integers from some point on something is true.

Basic Proof Techniques

Proof techniques can either be direct (Proof by Construction), indirect or by induction (proof by contra-positive and proof by contradiction).

Direct Proof (Proof by Construction)

In order to prove a mathematical statement, we have to show that for a given premise X, the conclusion Y given can be derived. through a series of logical steps

Example: Let n be an integer. If n is even, then n^2 is even. If n is odd, then n^2 is odd.

Soln:

If n is even, then $n = 2k$, and

$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even.

If n is odd, then $n = 2k + 1$, and

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd.

The initial step

Firstly, prove that the proposition is true for $n = 1$. If the claim is that the proposition is true for $n \geq a$, first prove it for $n = a$.

LOGIC

Logic is a formal study of mathematics; it is the study of mathematical reasoning and proofs itself.

Propositional Logic

Logic is the study of consequences. Given a few mathematical statements or facts, we would like to be able to draw some conclusions. The truth or falsehood of a proposition is called its **truth value**.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Logical Equivalence

Two molecular statements P and Q are logically equivalent provided P is true precisely when Q is true. $P \equiv Q$.

Deduction Rule

An argument is valid provided the conclusion must be true given that the premises are true.

First Order Logic

First order logic operates over a set of objects (e.g., real numbers, people, etc.). It allows us to express properties of individual objects, to define relationships between objects, and, most important of all, to quantify over the entire set of objects.

Example:

All men are mortal.

Adam is a man.

Therefore, Adam is a mortal.

BOOLEAN ALGEBRA AND LATTICES

A partially ordered set $(L, \preceq)$ is called a lattice if every pair of elements a and b in L has both a Least Upper Bound (LUB) or Supremum and a Greatest Lower Bound (GLB) or Infimum.

Let Y be a subset of a poset X .

An element u in X is an **upper bound** of Y if $a \preceq u$ for every element $a \in Y$.

If u is an upper bound of Y such that $u \preceq v$ for every other upper bound v of Y , then u is called an **LUB** of Y .

An element l in X is said to be a lower bound of Y if $l \preceq a$ for all $a \in Y$.

If l is a lower bound of Y such that $k \preceq l$ for every other lower bound k of Y , then l is called a GLB of Y .

The least upper bound is also called the **join** of a and b , denoted by $a \vee b$. The greatest lower bound is called the **meet** of a and b , and is denoted by $a \wedge b$.

If $(L, \preceq)$ is a lattice and $a, b, c, d \in L$, then the meet and join have the following order properties:

- i. $a \wedge b \preceq \{a, b\} \preceq a \vee b$,
- ii. $a \preceq b$ if and only if $a \wedge b = a$,
- iii. $a \preceq b$ if and only if $a \vee b = b$,
- iv. if $a \preceq b$, then $a \wedge c \preceq b \wedge c$ and $a \vee c \preceq b \vee c$
- v. if $a \preceq b$ and $c \preceq d$, then $a \wedge c \preceq b \wedge d$ and $a \vee c \preceq b \vee d$

Let Y be a nonempty subset of a poset X . If Y has a least upper bound, then Y has a unique least upper bound. If Y has a greatest lower bound, then Y has a unique greatest lower bound.

Principle of Duality: Any statement that is true for all lattices remains true when $\leq$ is replaced by $\geq$ and $\vee$ and $\wedge$ are interchanged throughout the statement.

If L is a lattice, then the binary operations $\vee$ and $\wedge$ satisfy the following properties for $a, b, c \in L$.

- i. Commutative laws: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
- ii. Associative laws: $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
- iii. Idempotent laws: $a \vee a = a$ and $a \wedge a = a$.
- iv. Absorption laws: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Boolean Algebras

A lattice L is distributive if and only if

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for all $a, b, c \in L$.

A **Boolean algebra** is a lattice B with a greatest element I and a smallest element O such that B is both distributive and complemented.

A set B is a Boolean algebra if and only if there exist binary operations $\vee$ and $\wedge$ on B satisfying the following axioms.

- i. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for $a, b \in B$.
- ii. $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ for $a, b, c \in B$.
- iii. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for $a, b, c \in B$.
- iv. There exist elements I and O such that $a \vee O = a$ and $a \wedge I = a$ for all $a \in B$.
- v. For every $a \in B$ there exists an $a' \in B$ such that $a \vee a' = I$ and $a \wedge a' = O$.

Let B be a Boolean algebra. Then,

- i. $a \vee I = I$ and $a \wedge O = O$ for all $a \in B$.
- ii. If $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$ for $a, b, c \in B$ then, $b = c$.
- iii. If $a \vee b = I$ and $a \wedge b = O$, then $b = a'$.
- iv. $(a')' = a$ for all $a \in B$.
- v. $I' = O$ and $O' = I$.
- vi. $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ (De Morgan's Laws).

Finite Boolean Algebras

A Boolean algebra is a finite Boolean algebra if it contains a finite number of elements as a set.

Let B and C , be Boolean algebras. A bijective map $\phi: B \rightarrow C$ is an isomorphism of Boolean algebras if

$$\phi(a \vee b) = \phi(a) \vee \phi(b)$$

$$\phi(a \wedge b) = \phi(a) \wedge \phi(b)$$

for all a and b in B .

Let B be a finite Boolean algebra. If b is an element of B with $b \neq 0$, then there is an atom a in B such that $a \leq b$.

Let a and b be atoms in a finite Boolean algebra B such that $a \neq b$. Then $a \wedge b = 0$.

Let B be a Boolean algebra and $a, b \in B$. The following statements are equivalent.

- i. $a \leq b$,
- ii. $a \wedge b' = 0$,
- iii. $a' \vee b = 1$.

Let B be a Boolean algebra and b and c be elements in B such that $b \not\leq c$. Then there exists an atom $a \in B$ such that $a \leq b$ and $a \not\leq c$.

Let B be a finite Boolean algebra. Then there exists a set X such that B is isomorphic to $P(X)$.

GRAPH THEORY

Graphs

Graphs are made up of a collection of dots that are called **vertices** and lines connecting those dots that are called **edges**. When two vertices are connected by an edge, we say that they are **adjacent**.

A graph is an ordered pair $G = (V, E)$ consisting of a nonempty set V (**vertices**) and a set E (**edges**) of two-element subsets of V .

A **directed graph** G is a pair (V, E) where V is a set of vertices (or nodes), and $E \subseteq V \times V$ is a set of edges. The order of the two connected vertices is important.

An **undirected graph** additionally has the property that $(u, v) \in E$ if and only if $(v, u) \in E$.

Isomorphic Graphs

An isomorphism between two graphs G_1 and G_2 is a bijection, $f: V_1 \rightarrow V_2$ between the vertices of the graphs such that $\{a, b\}$ is an edge in G_1 if and only if $\{f(a), f(b)\}$ is an edge in G_2 . Two graphs are isomorphic if there is an isomorphism between them. In this case we write $G_1 \cong G_2$.

Subgraphs

We say that $G' = (V', E')$ is a subgraph of $G = (V, E)$, and write $G' \subseteq G$, provided $V' \subseteq V$ and $E' \subseteq E$.

We say that $G' = (V', E')$ is an induced subgraph of $G = (V, E)$ provided $V' \subseteq V$ and every edge in E whose vertices are still in V' is also an edge in E' .

Bipartite Graphs

A graph is bipartite if the vertices can be divided into two sets, A and B , with no two vertices in adjacent in A and B .

If each vertex in A is adjacent to all the vertices in B , then the graph is a complete bipartite graph,

Union and Intersection of a Graph:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs

- i. The union of G_1 and G_2 , denoted by $G_1 \cup G_2$, is the graph G_3 defined as $G_3 = (V_1 \cup V_2, E_1 \cup E_2)$.
- ii. The intersection of G_1 and G_2 , denoted by $G_1 \cap G_2$, is the graph G_4 defined as $G_4 = (V_1 \cap V_2, E_1 \cap E_2)$.

Complement of a Graph: This operation that is used with a single graph.

Let $G = (V, E)$ be a subgraph of $K|V|$. The complement of G in $K|V|$, denoted as $G^c = (V, E^c)$, is the subgraph of $K|V|$ with $V^c = V$ and $E^c = K|V| (E) - E$

The Handshaking Problem

Let G be a graph with at least two vertices. At least two vertices of G have the same degree.

Euler Paths and Circuits

An Euler path, in a graph or multigraph can be defined as a walk through the graph which uses every edge exactly once. While an Euler circuit is an Euler path which starts and stops at the same vertex.

we can conclude the followings:

- i. A graph has an Euler circuit if and only if the degree of every vertex is even.
- ii. A graph has an Euler path if and only if there are at most two vertices with odd degree.

MATRICES AND DETERMINANTS

MATRIX

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is said to have dimension $m \times n$.

A set of mn numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [] is called $m \times n$ matrix (read “ m by n matrix”) .

Order of a Matrix: The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix.

Matrices A and B are equal, $A = B$, if A and B have the same dimensions and each entry of A is equal to the corresponding entry of B .

Types of Matrices

1. **Row Matrix and Column Matrix:** A matrix consisting of a single row is called a row matrix or a row vector, whereas a matrix having single column is called a column matrix or a column vector.
2. **Null or Zero Matrix:** A matrix in which each element is “0” is called a Null or Zero matrix.
3. **Square matrix:** A matrix A having same numbers of rows and columns is called a square matrix
4. **Main or Principal Diagonal:** The principal (leading) diagonal of a square matrix is the ordered set of elements a_{ij} , where $i = j$, extending from the upper left-hand corner to the lower right-hand corner of the matrix.

Particular cases of a square matrix

1. **Diagonal matrix:** A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix
2. **Scalar Matrix:** A diagonal matrix in which all the diagonal elements are same, and the remaining elements are zero is called a scalar matrix.
3. **Identity Matrix or Unit Matrix:** A scalar matrix in which each diagonal element is 1 (unity) is called a unit matrix. An identity matrix of order n is denoted by I_n .
4. **Equal Matrices:** Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B . this implies that for each i, j , $a_{ij} = b_{ij}$.
5. **The Negative of a Matrix:** The negative of the matrix $A_{m \times n}$, denoted by $-A_{m \times n}$, is the matrix formed by replacing each element in the matrix $A_{m \times n}$ with its additive inverse.

Operations on Matrices

1. **Multiplication of a Matrix by a Scalar:** If A is a matrix and k is a scalar

(constant), then kA is a matrix whose elements are the elements of A , each multiplied by k .

2. **Addition and subtraction of Matrices:** If A and B are two matrices of same order $m \times n$ then their sum $A + B$ is defined as C , $m \times n$ matrix such that each element of C is the sum of the corresponding elements of A and B

3. **Product of Matrices:** Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B . Then the product matrix AB has the same number of rows as A and the same number of columns as B . Thus the product of the matrices $A_{m \times p}$ and $B_{p \times n}$ is the matrix $(AB)_{m \times n}$.

- i. Multiplication of matrices is not commutative i.e., $AB \neq BA$ in general.
- ii. For matrices A and B if $AB = BA$ then A and B commute to each other.
- iii. A matrix A can be multiplied by itself if and only if it is a square matrix. The product $A \times A$, in such cases is written as A^2 .
- iv. In the product AB , A is said to be pre multiple of B and B is said to be post multiple of A .

Determinant

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified, operations, which is characteristic of the matrix. Determinant is denoted by $\det(A)$ or $|A|$ for a square matrix A .

Minor and Cofactor of Element

The minor M_{ij} of the element a_{ij} in a given determinant is the determinant of order $(n - 1 \times n - 1)$ obtained by deleting the i th row and j th column of $A_{n \times n}$.

Special Matrices

1. **Transpose of a Matrix:** If $A = [a_{ij}]$ is $m \times n$ matrix, then the matrix of order $n \times m$ obtained by interchanging the rows and columns of A is called the transpose of A .
2. **Symmetric Matrix:** A square matrix A is called symmetric if $A = A^t$.
3. **Skew Symmetric:** A square matrix A is called skew symmetric if $A = -A^t$
4. **Singular and Non-singular Matrices:** A square matrix A is called singular if $|A| = 0$ and is non-singular if $|A| \neq 0$.
5. **Adjoint of a Matrix:** Let $A = (a_{ij})$ be a square matrix of order $n \times n$ and (c_{ij}) is a matrix obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij} then $(c_{ij})^t$ is called the adjoint of A . It is written as $\text{Adj}(A)$.
6. **Adjoint of a 2x2 Matrix:** Interchange the elements of the principal diagonal. Just change (but do NOT interchange) the signs of the elements of the other diagonal.
7. **Inverse of a Matrix:** If A is a non-singular square matrix then, $A^{-1} = \text{adj}(a)/|A|$

APPLICATIONS TO COUNTING

The Product and Sum Rules

Suppose there are $n(A)$ ways to perform task A , and regardless of how task A is performed, there are $n(B)$ ways to perform task B .

Then, there are $n(A) \times n(B)$ ways to perform both task A and task B ; this is the

product rule.

The sum rule

Suppose there are $n(A)$ ways to perform task A, and distinct from these, there are $n(B)$ ways to perform task B. Then, there are $n(A) + n(B)$ ways to perform task A or task B.

Permutations and Combinations

Given n distinct objects, how many ways are there to “choose” r of them?

Permutations

A permutation of a set A is an ordered arrangement of the elements in A . An ordered arrangement of just r elements from A is called an r -permutation of A . For non-negative integers $r \leq n$, $P(n, r)$ denotes the number of r -permutations of a set with n elements.

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1) = n! / (n-r)! \dots \dots \dots (1)$$

Combinations

Considering unordered selections.

An unordered arrangement of r elements from a set A is called an r -combination of A . For non-negative integers $r \leq n$, $C(n, r)$ denotes the number of r -combinations of a set with n elements. $C(n, r)$ is also called the binomial coefficients

$$C(n, r) = n! / (n-r)! r!$$

Combinatorial Identities

- i. If $0 \leq k \leq n$, then $C(n, k) = C(n, n-k)$.
- ii. If $0 < k \leq n$, then $C(n+1, k) = C(n, k-1) + C(n, k)$.
- iii. $\sum_{k=0}^n C(n, k) = 2^n$
- iv. $(x+y)^n = \sum_{k=0}^n C(n, k) x^{n-k} y^k$

Inclusion-Exclusion Principle

Suppose there are $n(A)$ ways to perform task A and $n(B)$ ways to perform task B, how many ways are there to perform task A or B, if the methods to perform these tasks are not distinct? We can cast this as a set cardinality problem. Let X be the set of ways to perform A, and Y be the set of ways to perform B. Then:

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

For three sets, we can still glean from the Venn diagram that

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|$$

More generally,

Let $A_1, \dots, A_n$ be finite sets. Then,

$$|\bigcup_{i=1}^n A_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{I, I \subseteq \{1, \dots, n\}, |I|=k} |\bigcap_{i \in I} A_i| = \sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} |\bigcap_{i \in I} A_i|$$

Pigeonhole Principle

The principle says that if we place $k + 1$ or more pigeons into k pigeon holes, then at least one pigeon hole contains 2 or more pigeons.

(Pigeonhole Principle). If we place n (or more) pigeons into k pigeon holes, then at least one box contains $\lceil n/k \rceil$ or more pigeons.

DISCRETE PROBABILITY GENERATING FUNCTION

Common Sums

Geometric Series

$$1 + z + z^2 + z^3 + z^4 + \dots = \sum_{x=0}^{\infty} z^x = \frac{1}{1-z}, \text{ when } |z| < 1$$

Binomial Theorem

Binomial theorem states that for any $p, q \in \mathbb{R}$ and integer n , then

$$(p + q)^n = \sum_{x=0}^n \binom{n}{x} p^x q^{n-x}, \text{ where } \binom{n}{x} = \frac{n!}{(n-x)!x!}$$

Exponential Series

Exponential series state that for any $\lambda \in \mathbb{R}$, then $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^\lambda$

Probability Generating Function (PGF)

Let be a random variable defined over the negative integers $\{0, 1, 2, 3, \dots\}$. The probability generating function of X is given by

$G_X(z) = p_0 + p_1 z + p_2 z^2 + \dots = \sum_{j=0}^{\infty} p_j z^j = \mathbb{E}(z^X)$ for all $z \in \mathbb{R}$ for which the sum converges. Therefore, to calculate the probability generating function, we that

$$G_X(z) = \mathbb{E}(z^X) = \sum_{x=0}^{\infty} z^x p(X=x)$$

Properties of the PGF

(1) $G_X(0) = P(X=0)$:

$$G_X(0) = 0^0 \times P(X=0) + 0^1 \times P(X=1) + 0^2 \times P(X=2) + \dots$$

$$G_X(0) = P(X=0).$$

(2) $G_X(1) = \sum_{x=0}^{\infty} 1^x p(X=x) = \sum_{x=0}^{\infty} p(X=x) = 1$

Mean (Expected value)

Let X be a discrete random variable with PGF $G_X(z)$. Then, the expectation value can be expressed by

$$E[x] = \sum_{x=1}^{\infty} xP(X = x) \\ = G'_x(1), \text{ where } G'_x(1) \text{ denotes the derivative of } G_x(z)$$

$$\text{Hence } G'_x(1) = \sum_{x=0}^{\infty} xP(X = x)z^{x-1}, = \sum_{x=1}^{\infty} xP(X = x)z^{x-1}$$