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Subject: Probability and Random Process
(ECE 3103)

Topic: Random Variables

BY

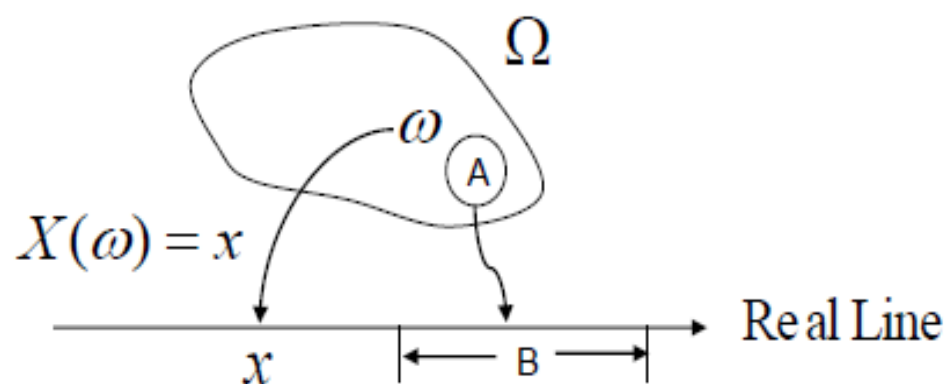
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Outline

- Introduction
- The Cumulative Distribution Function
- Probability Density and Mass Functions
- Expected Value, Variance and Moments
- Some Special Distributions

Introduction

- A random variable X is a function that assigns a real number $X(\omega)$ to each outcome ω in the sample space Ω of a random experiment.
- The sample space Ω is the domain of the random variable and the set R_X of all values taken on by X is the range of the random variable.
- Thus, R_X is the subset of all real numbers.



Introduction Cont'd.....

- If X is a random variable, then $\{\omega: X(\omega) \leq x\} = \{X \leq x\}$ is an event for every X in R_X .

Example: Consider a random experiment of tossing a fair coin three times. The sequence of heads and tails is noted and the sample space Ω is given by:

$$\Omega = \{HHH, HHT, HTH, THH, THT, HTT, TTH, TTT\}$$

Let X be the number of heads in three coin tosses. X assigns each possible outcome ω in the sample space Ω a number from the set $R_X = \{0, 1, 2, 3\}$.

$\omega:$	<i>HHH</i>	<i>HHT</i>	<i>HTH</i>	<i>THH</i>	<i>THT</i>	<i>HTT</i>	<i>TTH</i>	<i>TTT</i>
$X(\omega):$	3	2	2	2	1	1	1	0

The Cumulative Distribution Function

For discrete random variable

- The cumulative distribution function (cdf) of a random variable X is defined as the probability of the event $\{X \leq x\}$.

$$F_X(x) = P(X \leq x) = \sum_{X \leq x_i} p(x_i)$$

Properties of the cdf, $F_X(x)$:

- The cdf has the following properties.

i. $F_X(x)$ is a non - negative function, i.e.,

$$0 \leq F_X(x) \leq 1$$

ii. $\lim_{x \rightarrow \infty} F_X(x) = 1$

iii. $\lim_{x \rightarrow -\infty} F_X(x) = 0$

The Cumulative Distribution Function Cont'd.....

iv. $F_X(x)$ is a non - decreasing function of X , i.e.,

If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$

v. $P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1)$

vi. $P(X > x) = 1 - F_X(x)$

1. Let the random variable X denote the number of heads in three independent tosses of a fair coin. Find the PMF of X . Also find the CDF of X .

PMF of X :

x	p(x)
0	$p(0) = P(X = 0) = P(TTT) = 1/8$
1	$p(1) = P(X = 1) = P(HTT) + P(THT) + P(TTH) = 3/8$
2	$p(2) = P(X = 2) = P(HHT) + P(HTH) + P(THH) = 3/8$
3	$p(3) = P(X = 3) = P(HHH) = 1/8$

CDF of X:

$$F(0) = P(X \leq 0) = P(0 \text{ or fewer heads}) = 1/8$$

$$F(1) = P(X \leq 1) = P(1 \text{ or fewer heads}) = (1/8) + (3/8) = 4/8$$

$$F(2) = P(X \leq 2) = P(2 \text{ or fewer heads}) = (1/8) + (3/8) + (3/8) = 7/8$$

$$F(3) = P(X \leq 3) = P(3 \text{ or fewer heads}) = (1/8) + (3/8) + (3/8) + (1/8) = 1$$

Hence, the CDF of X is:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/8 & 0 \leq x < 1 \\ 4/8 & 1 \leq x < 2 \\ 7/8 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

The Probability Mass Function

- The probability mass function (pmf) of a discrete random variable X is defined as:

$$P_X(X = x_i) = P_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

Properties of the pmf, $P_X(x_i)$:

i. $0 \leq P_X(x_i) \leq 1, \quad k = 1, 2, \dots$

ii. $P_X(x) = 0$, if $x \neq x_k, k = 1, 2, \dots$

iii. $\sum_k P_X(x_k) = 1$

vi. $P(X > x) = 1 - F_X(x)$

2. Let X denote the number of busy servers at the checkout counters in a store at 5pm. Suppose that the CDF of X is:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.20 & 0 \leq x < 1 \\ 0.50 & 1 \leq x < 2 \\ 0.80 & 2 \leq x < 3 \\ 0.90 & 3 \leq x < 4 \\ 1 & 4 \leq x \end{cases}$$

(a) Find the PMF of X.

Use formula $p(x) = F_X(x_i) - F_X(x_{i-1})$

x	p(x)
0	$p(0) = P(X = 0) = F(0) - F(-1) = 0.20 - 0 = 0.20$
1	$p(1) = P(X = 1) = F(1) - F(0) = 0.50 - 0.20 = 0.30$
2	$p(2) = P(X = 2) = F(2) - F(1) = 0.80 - 0.50 = 0.30$
3	$p(3) = P(X = 3) = F(3) - F(2) = 0.90 - 0.80 = 0.10$
4	$p(4) = P(X = 4) = F(4) - F(3) = 1 - 0.90 = 0.10$

(b) Find $P(X > 3)$.

$$P(X > 3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - 0.90 = 0.10$$

Continuous Random Variable Distributions

For a continuous random variable X we have a function f , called the probability density function (pdf). Every probability density function must satisfy

$$(i) \ f(x) \geq 0, \text{ and}$$

$$(ii) \ \int_{-\infty}^{\infty} f(x)dx = 1.$$

□ Cumulative distribution function (CDF) of continuous random variable is defined as

$$F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(x)dx.$$

Notes:

$$(i) \ P(X = x) = 0 \text{ for a continuous random variable } X.$$

$$(ii) \ \frac{d}{dx} F(x) = f(x).$$

$$(iii) \ P(a \leq X \leq b) = F(b) - F(a) = \int_{-\infty}^b f(x)dx - \int_{-\infty}^a f(x)dx = \int_a^b f(x)dx$$

Types of Random Variables

- There are two basic types of random variables.

i. Continuous Random Variable

- ✓ A continuous random variable is defined as a random variable whose cdf, $F_X(x)$, is continuous every where and can be written as an integral of some non-negative function $f(x)$, i.e.,

$$F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(x)dx.$$

ii. Discrete Random Variable

- ✓ A discrete random variable is defined as a random variable whose cdf, $F_X(x)$, is a right continuous, staircase function of X with jumps at a countable set of points $x_0, x_1, x_2, \dots$

Expected Value, Variance and Moments

i. Expected Value (Mean)

- The expected value (mean) of a continuous random variable X , denoted by μ_X or $E(X)$, is defined as:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} xf_X(x)dx$$

- Similarly, the expected value of a discrete random variable X is given by:

$$\mu_X = E(X) = \sum_k x_k P_X(x_k)$$

- Mean represents the average value of the random variable in a very large number of trials.

Expected Value, Variance and Moments Cont'd.....

ii. Variance

- The variance of a continuous random variable X , denoted by σ^2_X or $VAR(X)$, is defined as:

$$\sigma^2_X = Var(X) = E[(X - \mu_X)^2]$$

$$\Rightarrow \sigma^2_X = Var(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

- Expanding $(x - \mu_X)^2$ in the above equation and simplifying the resulting equation, we will get:

$$\sigma^2_X = Var(X) = E(X^2) - [E(X)]^2$$

Expected Value, Variance and Moments Cont'd.....

- The variance of a discrete random variable X is given by:

$$\sigma^2_X = Var(X) = \sum_k (x_k - \mu_X)^2 P_X(x_k)$$

- The standard deviation of a random variable X , denoted by σ_X , is simply the square root of the variance, i.e.,

$$\sigma_X = \sqrt{E(X - \mu_X)^2} = \sqrt{Var(X)}$$

iii. Moments

- The n^{th} moment of a continuous random variable X is defined as:

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x) dx, \quad n \geq 1$$

Discrete Random variable Example

Example: An urn contains 7 balls, numbered from 1 to 7. Two balls are selected randomly from the urn. Let Random variable X be the difference of the numbers (that is, the largest number minus the smallest number) of the selected balls. Find the probability distribution or PMF of X , CDF and its mean, variance and standard deviation.

If (a) the balls are selected from the urn with replacement.

(a) First, determine the sample space:

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), \\ (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), \\ (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (3, 7), \\ (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (4, 7), \\ (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (5, 7), \\ (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6), (6, 7), \\ (7, 1), (7, 2), (7, 3), (7, 4), (7, 5), (7, 6), (7, 7)\}$$

Step II: find random variable X (difference of highest to lowest value)

Difference is 0 $\{X = 0\} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)\}$

Difference is 1 $\{X = 1\} = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (2, 1), (3, 2), (4, 3), (5, 4), (6, 5), (7, 6)\}$

----- is 2 $\{X = 2\} = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), (3, 1), (4, 2), (5, 3), (6, 4), (7, 5)\}$

$\{X = 3\} = \{(1, 4), (2, 5), (3, 6), (4, 7), (4, 1), (5, 2), (6, 3), (7, 4)\}$

$\{X = 4\} = \{(1, 5), (2, 6), (3, 7), (5, 1), (6, 2), (7, 3)\}$

$\{X = 5\} = \{(1, 6), (2, 7), (6, 1), (7, 2)\}$

$\{X = 6\} = \{(1, 7), (7, 1)\}$

Step III. Find PMF (or probability distribution for discrete random variable)

$X = x_k$	0	1	2	3	4	5	6
$P_X(x_k)$	$\frac{7}{49}$	$\frac{12}{49}$	$\frac{10}{49}$	$\frac{8}{49}$	$\frac{6}{49}$	$\frac{4}{49}$	$\frac{2}{49}$
	$P(0)$	$p(1)$	$p(2)$	$p(3)$	$p(4)$	$p(5)$	$p(6)$

❖ Find CDF: Do your self

❖ Mean value or Expected of discrete random variable X

$$\mu_X = E(X) = \sum_k x_k P_X(x_k)$$

$$\mu = 0 \cdot \frac{7}{49} + 1 \cdot \frac{12}{49} + 2 \cdot \frac{10}{49} + 3 \cdot \frac{8}{49} + 4 \cdot \frac{6}{49} + 5 \cdot \frac{4}{49} + 6 \cdot \frac{2}{49} = \frac{112}{49} \approx 2.29,$$

❖ Variance of discrete random variable X

$$\sigma^2_X = Var(X) = \sum_k (x_k - \mu_X)^2 P_X(x_k)$$

$$\sigma^2 = (0-2.29)^2 \cdot \frac{7}{49} + (1-2.29)^2 \cdot \frac{12}{49} + (2-2.29)^2 \cdot \frac{10}{49} + (3-2.29)^2 \cdot \frac{8}{49} + (4-2.29)^2 \cdot \frac{6}{49} + (5-2.29)^2 \cdot \frac{4}{49} + (6-2.29)^2 \cdot \frac{2}{49} \approx 2.78.$$

❖ Standard deviation of discrete random variable X.

$$\sigma_X = \sqrt{E(X - \mu_X)^2} = \sqrt{Var(X)}$$

$$\sigma = \sqrt{2.78} \approx 1.67.$$

Example-2:

Consider a discrete random variable X whose pmf is given by:

$$P_X(x_k) = \begin{cases} 1/3, & x_k = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance of X .

Solution:

i. Mean

$$\mu_X = E(X) = \sum_{k=-1}^1 x_k P_X(x_k) = 1/3(-1 + 0 + 1) = 0$$

ii. Variance

$$\sigma_X^2 = \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{k=-1}^1 x_k^2 P_X(x_k) = 1/3[(-1)^2 + (0)^2 + (1)^2] = 2/3$$

$$\Rightarrow \sigma_X^2 = \text{Var}(x) = 2/3 - (0)^2 = 2/3$$

Random Variable Examples

Continuous

Example-1:

The pdf of a continuous random variable is given by:

$$f_X(x) = \begin{cases} kx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

- a. Determine the value of k .
- b. Find the corresponding cdf of X .
- c. Find $P(1/4 \leq X \leq 1)$
- d. Evaluate the mean and variance of X .

Random Variable Examples Cont'd.....

Solution:

$$a. \quad \int_{-\infty}^{\infty} f_X(x) dx = 1 \Rightarrow \int_0^1 kx dx = 1$$

$$\Rightarrow k \left(\frac{x^2}{2} \right) \Big|_0^1 = 1$$

$$\Rightarrow \frac{k}{2} = 1$$

$$\therefore k = 2$$

$$\Rightarrow f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution:

b. The cdf of X is given by :

$$F_X(x) = \int_{-\infty}^x f_X(u) du$$

Case 1: for $x < 0$

$$F_X(x) = 0, \text{ since } f_X(x) = 0, \text{ for } x < 0$$

Case 2: for $0 \leq x < 1$

$$F_X(x) = \int_0^x f_X(u) du = \int_0^x 2u du = u^2 \Big|_0^x = x^2$$

Random Variable Examples Cont'd.....

Solution:

Case 3 : for $x \geq 1$

$$F_X(x) = \int_0^1 f_X(u) du = \int_0^1 2u du = u^2 \Big|_0^1 = 1$$

$\therefore$ The cdf is given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Random Variable Examples Cont'd.....

Solution:

c. $P(1/4 \leq X \leq 1)$

i. Using the pdf

$$P(1/4 \leq X \leq 1) = \int_{1/4}^1 f_X(x) dx = \int_{1/4}^1 2x dx$$

$$\Rightarrow P(1/4 \leq X \leq 1) = x^2 \Big|_{1/4}^1 = 15/16$$

$$\therefore P(1/4 \leq X \leq 1) = 15/16$$

ii. Using the cdf

$$P(1/4 \leq X \leq 1) = F_X(1) - F_X(1/4)$$

$$\Rightarrow P(1/4 \leq X \leq 1) = 1 - (1/4)^2 = 15/16$$

$$\therefore P(1/4 \leq X \leq 1) = 15/16$$

Solution:

d. Mean and Variance

i. Mean

$$\mu_X = E(X) = \int_0^1 x f_X(x) dx = \int_0^1 2x^2 dx$$
$$\Rightarrow \mu_X = \left. \frac{2x^3}{3} \right|_0^1 = 2/3$$

ii. Variance

$$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2$$
$$E(X^2) = \int_0^1 x^2 f_X(x) dx = \int_0^1 2x^3 dx = 1/2$$
$$\Rightarrow \sigma_X^2 = Var(x) = 1/2 - (2/3)^2 = 1/18$$

- A Gaussian random variable is one whose probability density function can be written in the general form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).$$

The $Q(\cdot)$ Function

- The $Q(\cdot)$ Function Tells Us The Probability A Zero-Mean Unit-Variance Gaussian RV Is Greater Than Or Equal To Some Number: $P(x \geq A) = Q(A)$

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-x^2/2} dx = 1 - F_X(y)$$

- In General

$$P(X > x) = Q\left(\frac{x - m}{\sigma}\right)$$

$$P(X \leq x) = 1 - Q\left(\frac{x - m}{\sigma}\right)$$

Special Distribution:

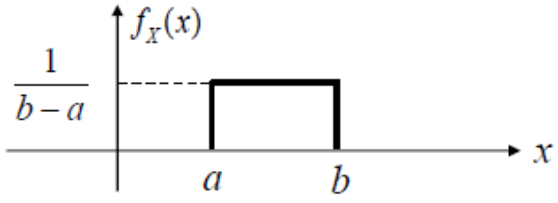
Continuous Probability Distributions

❖ UNIFORM DISTRIBUTION

The pdf $f_X(x)$ of a random variable uniformly distributed in the interval $(a, b]$, (notation $X \sim U(a, b]$), is given by

PDF

$$f_X(x) = \begin{cases} 0 & x \leq a \\ \frac{1}{b-a} & a < x \leq b \\ 0 & x > b \end{cases}$$



CDF

$$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$

Fig. Uniform Distribution

Application: 1. The distribution of a telephone call occurring at random in a given interval.
2. Such distributions occur in transmission of sinusoids where the phase angle will be uniformly distributed in $(0, 2\pi)$.

Example 6.2.1 A sinusoidal signal with random phase angle Θ , uniformly distributed between $(0, 2\pi]$ is sent along a communications channel. We want to find the probability that the phase angle is between $(\pi/4, 3\pi/4]$.

$$P\left\{\frac{\pi}{4} < \theta \leq \frac{3\pi}{4}\right\} = \frac{1}{2\pi} \frac{3\pi - \pi}{4} = \frac{1}{4}$$

❖ EXPONENTIAL DISTRIBUTION

PDF

$$f_X(x) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

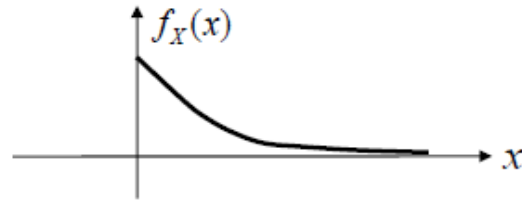


Fig. Exponential Distribution

CDF

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$

where the parameter λ is the average rate at which events occur.

Application: Exponential distributions find wide applicability in queuing theory, reliability theory, and communication theory

❖ Normal (Gaussian) Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

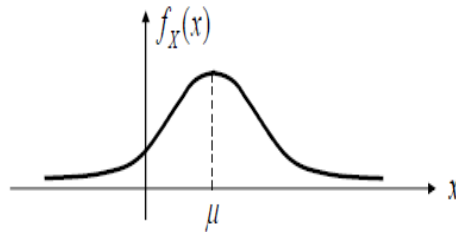


Fig. Normal or Gaussian Distribution

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} dy$$
$$= G\left(\frac{x-\mu}{\sigma}\right)$$

where $G(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$

Application: A random variable X consisting of a number of components, each with a general distribution tends to a normal distribution as the number becomes very large.

LAPLACE DISTRIBUTION

The Laplace density, also called a double-exponential density, is defined by

$$f_X(x) = \begin{cases} \frac{\lambda}{2} e^{-\lambda|x|} & \lambda > 0 \\ -\infty < x < +\infty. \end{cases} \quad F_X(x) = \begin{cases} \frac{1}{2} e^{-\lambda x} & x \leq 0 \\ 1 - \frac{1}{2} e^{-\lambda x} & x > 0 \end{cases}$$

Applications: To model navigation and pilot errors, speech, and image processing.

Cauchy Distribution

$$f_X(x) = \frac{\alpha / \pi}{\alpha^2 + (x - \mu)^2}, \quad -\infty < x < +\infty.$$

Application: A Cauchy distribution is a good indicator of how sensitive the tests are to heavy-tail departures from normality.

Assignment-II

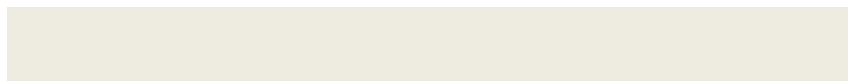
1. The continuous random variable X has the pdf is given by:

$$f_X(x) = \begin{cases} k(2x - x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

Find :

- a. the value of k .
- b. the cdf of X .
- c. $P(X > 1)$



2. The cdf of continuous random variable X is given by:

$$f_X(x) = \begin{cases} 0, & x < 0 \\ k \sin^{-1}(x), & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

where k is a constant.

Determine :

- a. the value of k .
- b. the pdf of X .

Two Random Variables

- In many applications, it is very important to study two or more random variable defined on the same sample space.
- In this lecture, we will consider only two random variables and this concept can be extended to three or more random variables.
- Let Ω be the sample space of a random experiment and let X and Y be two random variables.
- Then, the pair (X, Y) is called a two dimensional random variable if each of X and Y associates a real number with every element of Ω .
- Thus, a two dimensional random variable (X, Y) is a function that assigns a point (x, y) in the xy -plane to each possible outcome ω in the sample space.

Example: 1

Consider the experiment of tossing a coin twice. The sample space is

$$S = \{HH, HT, TH, TT\}.$$

Let X denotes the number of heads obtained in the first toss and Y denote the number of heads in the second toss. Then

s	HH	HT	TH	TT
$X(s)$	1	1	0	0
$Y(s)$	1	0	1	0

(X, Y) is a two-dimensional random variable or Bi-Variate random variable.

The range space X is $R_X = (0, 1)$ and Y is $R_Y = (0, 1)$

The range space of R_{XY} is $\{(1,1), (1,0), (0,1), (0,0)\}$ which is finite and so (X, Y) is a two-dimensional discrete random variables.

Example 2

Consider an experiment of tossing a fair coin twice. Let (X, Y) be a bivariate r.v., where X is the number of heads that occurs in the two tosses and Y is the number of tails that occurs in the two tosses.

- (a) What is the range R_X of X , R_Y of Y ?
- (b) Find and sketch the range R_{XY} of (X, Y) .
- (c) Find $P(X = 2, Y = 0)$, $P(X = 0, Y = 2)$, and $P(X = 1, Y = 1)$.

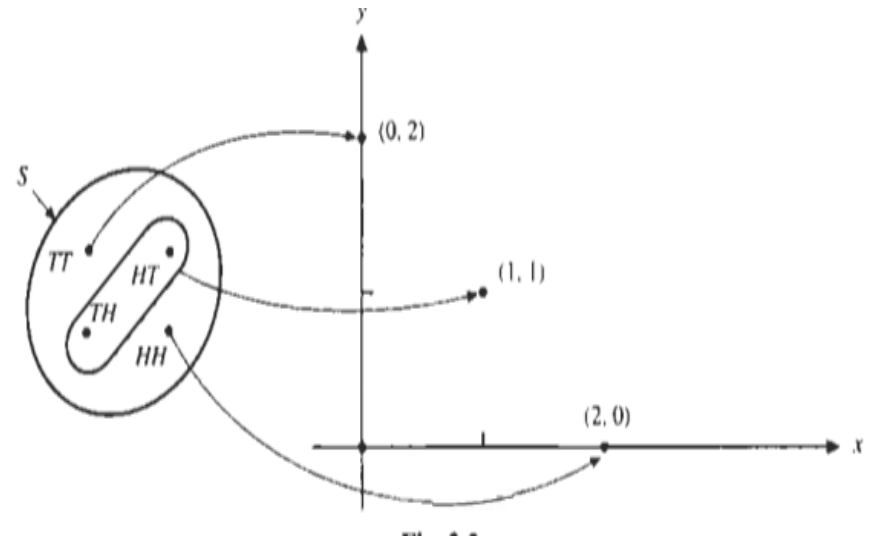
The sample space S of the experiment is $S = \{HH, HT, TH, TT\}$

- (a) $R_X = (0, 1, 2)$, $R_Y = (0, 1, 2)$
- (b) $R_{XY} = ((2, 0), (1, 1), (0, 2))$ which is sketched in Fig. 3-2.
- (c) Since the coin is fair, we have

$$P(X = 2, Y = 0) = P\{HH\} = 1/4$$

$$P(X = 0, Y = 2) = P\{TT\} = 1/4$$

$$P(X = 1, Y = 1) = P\{HT, TH\} = 1/2$$



The Joint Cumulative Distribution Function

- The joint CDF of two random variables X and Y denoted by $F_{XY}(x, y)$ is a function defined by:

$$F_{XY}(x, y) = P[X(\omega) \leq x \text{ and } Y(\omega) \leq y]$$

$$\Leftrightarrow F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

where x and y are arbitrary real numbers.

Properties of the Joint CDF, $F_{XY}(x, y)$:

- i. $0 \leq F_{XY}(x, y) \leq 1$
- ii. $\lim_{\substack{x \rightarrow \infty \\ y \rightarrow \infty}} F_{XY}(x, y) = F_{XY}(\infty, \infty) = 1$
- iii. $\lim_{\substack{x \rightarrow -\infty \\ y \rightarrow -\infty}} F_{XY}(x, y) = F_{XY}(-\infty, -\infty) = 0$

The Joint Cumulative Distribution Function Cont'd...

$$iv. \quad \lim_{x \rightarrow -\infty} F_{XY}(x, y) = F_{XY}(-\infty, y) = 0$$

$$v. \quad \lim_{y \rightarrow -\infty} F_{XY}(x, y) = F_{XY}(x, -\infty) = 0$$

$$vi. \quad P(x_1 \leq X \leq x_2, Y \leq y) = F_{XY}(x_2, y) - F_{XY}(x_1, y)$$

$$vii. \quad P(X \leq x, y_1 \leq Y \leq y_2) = F_{XY}(x, y_2) - F_{XY}(x, y_1)$$

$$viii. \quad P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = F_{XY}(x_2, y_2) \\ - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)$$

The Joint Probability Density Function

- The joint probability function (PDF) of two continuous random variables X and Y is defined as:

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

- Thus, the joint cumulative distribution function (CDF) is given by:

$$F_{XY}(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{XY}(x, y) dx dy$$

The Joint Probability Density Function Cont'd.....

Properties of the Joint pdf, $f_{XY}(x, y)$:

1. $f_{XY}(x, y) \geq 0$

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$

3. $P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x, y) dx dy$

The Joint Probability Mass Function

- The joint probability mass function (pmf) of two discrete random variables X and Y is defined as:

$$P_{XY}(x_i, y_j) = P(X = x_i, Y = y_j)$$

- The joint cdf can be written as:

$$F_{XY}(x, y) = \sum_{x_i \leq x} \sum_{y_j \leq y} P_{XY}(x_i, y_j)$$

Properties of the Joint pmf, $P_{XY}(x_i, y_j)$:

1. $0 \leq P_{XY}(x_i, y_j) \leq 1$
2. $\sum_{x_i} \sum_{y_j} P_{XY}(x_i, y_j) = 1$

Condition Distributions

i. Conditional Probability Density Functions

- If $\mathbf{X}$ and $\mathbf{Y}$ are two continuous random variables with joint pdf $f_{XY}(\mathbf{x}, \mathbf{y})$, then the conditional pdf of $\mathbf{Y}$ given that $\mathbf{X}=\mathbf{x}$ is defined by:

$$f_{Y/X}(y / x) = \frac{f_{XY}(x, y)}{f_X(x)}, \quad f_X(x) > 0$$

- Similarly, the conditional pdf of $\mathbf{X}$ given that $\mathbf{Y}=\mathbf{y}$ is defined by:

$$f_{X/Y}(x / y) = \frac{f_{XY}(x, y)}{f_Y(y)}, \quad f_Y(y) > 0$$

Condition Distributions Cont'd.....

ii. Conditional Probability Mass Functions

- If X and Y are two discrete random variables with joint pmf $P_{XY}(x_i, y_j)$, then the conditional pmf of X given that $Y=y_j$ is defined by:

$$P_{X/Y}(x_i / y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}, \quad P_Y(y_j) > 0$$

- Similarly, the conditional pmf of Y given that $X=x_i$ is defined by:

$$P_{Y/X}(y_j / x_i) = \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}, \quad P_X(x_i) > 0$$

Marginal Statistics of Two Random Variables

- In the case of two or more random variables, the statistics of each individual variable are called marginal statistics.

i. Marginal cdf of X and Y

$$F_X(x) = \lim_{y \rightarrow \infty} F_{XY}(x, y) = F_{XY}(x, \infty)$$

$$F_Y(y) = \lim_{x \rightarrow \infty} F_{XY}(x, y) = F_{XY}(\infty, y)$$

ii. Marginal pdf of X and Y

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$$

Marginal Statistics of Two Random Variables Cont'd.....

iii. Marginal PMF of X and Y

$$P(X = x_i) = P_X(x_i) = \sum_{y_j} P_{XY}(x_i, y_j)$$

$$P(Y = y_j) = P_Y(y_j) = \sum_{x_i} P_{XY}(x_i, y_j)$$

Independence of Two Random Variables

- If two random variables X and Y are independent, then

i. from the joint cdf

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

ii. from the joint pdf

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

iii. from the joint pmf

$$P_{XY}(x_i, y_j) = P_X(x_i)P_Y(y_j)$$

Examples on Two Random Variables

Example-1:

The joint pdf of two continuous random variables X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} kxy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

- a. Find the value of k .
- b. Find the marginal pdf of X and Y .
- c. Are X and Y independent?
- d. Find $P(X + Y < 1)$
- e. Find the conditional pdf of X and Y .

Examples on Two Random Variables Cont'd.....

Solution:

$$a. \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1 \Rightarrow \int_0^1 \int_0^1 kxy dx dy = 1$$

$$\Rightarrow k \int_0^1 y \left(\frac{x^2}{2} \right) \Big|_0^1 = 1$$

$$\Rightarrow \frac{k}{2} \int_0^1 y dy = k \left(\frac{y^2}{4} \right) \Big|_0^1 = \frac{k}{4} = 1$$

$$\therefore k = 4$$

Examples on Two Random Variables Cont'd.....

Solution:

b. Marginal pdf of X and Y

i. Marginal pdf of X

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_0^1 4xy dy$$

$$\Rightarrow f_X(x) = 4x \left(\frac{y^2}{2} \right) \Big|_0^1 = 2x$$

$$\therefore f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Solution:

b. Marginal pdf of X and Y

ii. Marginal pdf of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \int_0^1 4xy dx$$

$$\Rightarrow f_Y(y) = 4y \left(\frac{x^2}{2} \right) \Big|_0^1 = 2y$$

$$\therefore f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Solution:

$$c. \quad f_{XY}(x, y) = f_X(x) f_Y(y)$$

$\therefore X$ and Y are independent

$$\begin{aligned} d. \quad P(X + Y < 1) &= \int_0^1 \int_0^{1-y} 4xy dx dy = \int_0^1 4y \left(\frac{x^2}{2} \right) \bigg|_0^{1-y} dy \\ &= \int_0^1 4y [1/2(1-y)^2] dy = \int_0^1 2(y - 2y^2 + y^3) dy \\ &= 2(y^2/2 - 2y^3/3 + y^4/4) = 1/6 \end{aligned}$$

$$\therefore P(X + Y < 1) = 1/6$$

Examples on Two Random Variables Cont'd.....

Solution:

e. Conditional pdf of X and Y

ii. Conditional pdf of Y

$$f_{Y/X}(y/x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{4xy}{2x} = 2y$$

$$\therefore f_{Y/X}(y/x) = \begin{cases} 2y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Example-2:

The joint pdf of two continuous random variables X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} k, & 0 < y \leq x < 1 \\ 0, & \text{otherwise} \end{cases} \quad \begin{matrix} 0 < y < 1, \\ y < x < 1 \end{matrix}$$

where k is a constant.

- a. Determine the value of k .
- b. Find the marginal pdf of X and Y .
- c. Are X and Y independent?
- d. Find $P(0 < X < 1/2)$
- e. Find the conditional pdf of X and Y .

Examples on Two Random Variables Cont'd.....

Solution:

$$a. \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1 \Rightarrow \int_0^1 \int_y^1 k dx dy = 1$$

$$\Rightarrow k \int_0^1 (x) \Big|_y^1 dy = 1$$

$$\Rightarrow k \int_0^1 (1 - y) dy = k \left(y - \frac{y^2}{2} \right) \Big|_0^1 = \frac{k}{2} = 1$$

$$\therefore k = 2$$

Examples on Two Random Variables Cont'd.....

Solution:

b. Marginal pdf of X and Y

i. Marginal pdf of X

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_0^x 2 dy$$

$$\Rightarrow f_X(x) = (2y) \Big|_0^x = 2x$$

$$\therefore f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Solution:

b. Marginal pdf of X and Y

ii. Marginal pdf of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \int_y^1 2 dx$$

$$\Rightarrow f_Y(y) = (2x) \Big|_y^1 = 2(1-y)$$

$$\therefore f_Y(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Solution:

$$c. \quad f_{XY}(x, y) \neq f_X(x) f_Y(y)$$

$\therefore X$ and Y are not independent

$$\begin{aligned} d. \quad P(0 < X < 1/2) &= \int_0^{1/2} \int_0^x f_{XY}(x, y) dy dx \\ &= \int_0^{1/2} \int_0^x 2 dy dx = \int_0^{1/2} (2y) \Big|_0^x dx \\ &= \int_0^{1/2} 2x dx = x^2 \Big|_0^{1/2} = 1/4 \end{aligned}$$

$$\therefore P(0 < X < 1/2) = 1/4$$

Examples on Two Random Variables Cont'd.....

Solution:

e. Conditional pdf of X and Y

i. Conditional pdf of X

$$f_{X/Y}(x/y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)}$$

$$\therefore f_{X/Y}(x/y) = \begin{cases} \frac{1}{1-y}, & 0 < y \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Solution:

e. Conditional pdf of X and Y

ii. Conditional pdf of Y

$$f_{Y/X}(y/x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}$$

$$\therefore f_{Y/X}(y/x) = \begin{cases} \frac{1}{x}, & 0 < y \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Example-3:

The joint pmf of two discrete random variables X and Y is given by:

$$P_{XY}(x_i, y_j) = \begin{cases} k(2x_i + y_j), & x_i = 1, 2; y = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

- a. Find the value of k .
- b. Find the marginal pmf of X and Y .
- c. Are X and Y independent?

Examples on Two Random Variables Cont'd.....

Solution:

$$a. \sum_{x_i} \sum_{y_j} P_{XY}(x_i, y_j) = 1$$

$$\Rightarrow \sum_{x_i=1}^2 \sum_{y_j=1}^2 k(2x_i + y_j) = 1$$

$$\Rightarrow k[(2+1) + (2+2) + (4+1) + (4+2)] = 1$$

$$\Rightarrow 18k = 1$$

$$\therefore k = 1/18$$

Examples on Two Random Variables Cont'd.....

Solution:

b. Marginal pmf of X and Y

i. Marginal pmf of X

$$P_X(x_i) = \sum_{y_j} P_{XY}(x_i, y_j) = \sum_{y_j=1}^2 \frac{1}{18} (2x_i + y_j)$$

$$\Rightarrow P_X(x_i) = \frac{1}{18} (2x_i + 1) + \frac{1}{18} (2x_i + 2)$$

$$\therefore P_X(x_i) = \begin{cases} \frac{1}{18} (4x_i + 3), & x_i = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Solution:

b. Marginal pmf of X and Y

ii. Marginal pmf of Y

$$P_Y(y_j) = \sum_{x_i} P_{XY}(x_i, y_j) = \sum_{x_i=1}^2 \frac{1}{18} (2x_i + y_j)$$

$$\Rightarrow P_Y(y_j) = \frac{1}{18} (2 + y_j) + \frac{1}{18} (4 + y_j)$$

$$\therefore P_Y(y_j) = \begin{cases} \frac{1}{18} (2y_j + 6), & y_j = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Two Random Variables Cont'd.....

Solution:

$$c. \quad P_{XY}(x_i, y_j) \neq P_X(x_i)P_Y(y_j)$$

$\therefore X$ and Y are not independent.

- Example : Air Conditioner Maintenance
 - A company that services air conditioner units in residences and office blocks is interested in how to schedule its technicians in the most efficient manner
 - The random variable X , taking the values 1,2,3 and 4, is the service time in hours
 - The random variable Y , taking the values 1,2 and 3, is the number of air conditioner units

Y= number of units	X=service time			
	1	2	3	4
1	0.12	0.08	0.07	0.05
2	0.18	0.15	0.21	0.13
3	0.01	0.01	0.02	0.07

Joint p.m.f

$$\sum_i \sum_j p_{ij} = 0.12 + 0.18 + \dots + 0.07 = 1.00$$

Joint cumulative distribution function

$$\begin{aligned} F(2,2) &= p_{11} + p_{12} + p_{21} + p_{22} \\ &= 0.12 + 0.18 + 0.08 + 0.15 \\ &= 0.43 \end{aligned}$$

Lemma 3.9 (Variance of linear functions). *For any constants a and b*

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

Proof.

$$\begin{aligned}\text{Var}(aX + b) &= E((aX + b - E(aX + b))^2) \\ &= E((aX + b - aE(X) - b)^2) \\ &= a^2 E((X - E(X))^2) \\ &= a^2 \text{Var}(X).\end{aligned}$$

Example(3.10): The PDF of the random variable X is defined by

$$f(x) = \begin{cases} 0 & , \quad x < 0 \\ k(e^{-2x}) & , \quad x \geq 0 \end{cases}$$

a) What is the value of K ? (b) What is the CDF of X ?

(c) What is $P(2 < X < 5)$? (d) Find Mean and Variance of X .

Example(3.10) : The PDF of the random variable X is defined by

$$f(x) = \begin{cases} 0 & , \quad x < 0 \\ k(e^{-2x}) & , \quad x \geq 0 \end{cases}$$

- a) What is the value of K?
- b) What is the CDF of X?
- c) What is $P(2 < X < 5)$?
- d) Find Mean and Variance of X

$$(a) \int_0^{\infty} f(x) dx = 1$$

$$\left[\frac{k(e^{-2x})}{-2} \right]_0^{\infty} = 1$$

$$f(x) = \begin{cases} 0, & x < 0 \\ 2(e^{-2x}), & x \geq 0 \end{cases}$$

CDF of Random Variable X.

Case (1) $x < 0$

$$F_X(x) = \int_{-\infty}^0 f(x) dx$$

As $f(x) = 0$

$$F_X(x) = \int_{-\infty}^0 f(x) dx$$

So,

$$F_X(x) = 0$$

(b) CDF of Random Variable X.

Case (1) $x < 0$

$$F_X(x) = \int_{-\infty}^0 f(x) dx$$

As $f(x) = 0$

$$F_X(x) = \int_{-\infty}^0 f(x) dx$$

So,

$$F_X(x) = 0$$

Case (2) $0 \leq x < \infty$

$$F_X(x) = \int_{-\infty}^0 f(x) dx + \int_0^x f(x) dx$$

$$F_X(x) = 0 + \int_0^x 2(e^{-2x}) dx$$

$$F_X(x) = 0 + \left[\frac{2(e^{-2x})}{-2} \right]_0^x$$

$$F_X(x) = 1 - e^{-2x}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-2x} & 0 \leq x < \infty \end{cases}$$

$$(c) \quad \int_2^5 f(x) dx = \int_2^5 2(e^{-2x}) dx = \left[\frac{2(e^{-2x})}{-2} \right]_2^5 = e^{-4} - e^{-10}$$

(d) **Find Mean and Variance of X.**

$$\text{Mean} = E[X] = \int_0^{\infty} x f(x) dx$$

$$= \int_0^{\infty} (2xe^{-2x}) dx$$

$$\text{Put } y = 2x, \quad dy = 2 dx$$

$$= \frac{1}{2} \int_0^{\infty} y(e^{-y}) dy$$

$$= \frac{1}{2} \int_0^{\infty} y(e^{-y}) dy$$

$$= \frac{1}{2} [-e^{-y} - y(e^{-y})]_0^{\infty}$$

$$= \frac{1}{2}$$

$$E[X^2] = \int_0^{\infty} x^2 f(x) dx$$

$$= \int_0^{\infty} (2x^2 e^{-2x}) dx$$

$$\text{Put } y = 2x, \quad dy = 2 dx$$

$$= \frac{1}{4} \int_0^{\infty} y^2 (e^{-y}) dy$$

$$\begin{aligned}
&= \frac{1}{4} \int_0^{\infty} y^2 (e^{-y}) dy \\
&= \frac{1}{4} \left[-2e^{-y} - 2ye^{-y} - y^2(e^{-y}) \right]_0^{\infty} \\
&= \frac{2}{4} = \frac{1}{2}
\end{aligned}$$

$$\mathbf{Variance} = E[X^2] - (E[X])^2$$

$$\mathbf{Variance} = \frac{1}{2} - \frac{1}{4}$$

$$\mathbf{Variance} = \frac{1}{4}$$