Electronics & Communication Eng. Department, School of Electrical Engineering & Computing, Adama Science & Technology University, Adama, Ethiopia

CHAPTER: 1 & 2

Subject: Probability and Random Process

(ECE 3103)

Topic: Review of Basic Probability Theory

BY

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Basic Concepts of Probability Theory

Outline:

- **Basic Set Operation**
- **Introduction Sample Space and Events**
- **Axioms and Properties of Probability**
- **Conditional Probability**
- **Independence of Events**

Basic Set Operation

The concept of sets play an important role in probability:

Definition of Set:

A set is a collection of objects called elements. The elements of a set can also be sets. Sets are usually represented by uppercase letters A, and elements are usually represented by lowercase letters a.

$$A = \{a_1, a_2, \ldots, a_n\}$$

will mean that the set A contains the elements $a_1, a_2, \ldots, a_n$. Conversely, we can write that a_k is an element of A as

$$a_k \in A$$

and a_k is not an element of A as

$$a_k \notin A$$

A finite set contains a finite number of elements, for example, $S = \{2,4,6\}$. Infinite sets will have either countably infinite elements such as $A = \{x : x \text{ is all positive integers}\}$ or uncountably infinite elements such as $B = \{x : x \text{ is real number } \leq 20\}$.

Example 1.1.1 The set A of all positive integers less than 7 is written as

$$A = \{x : x \text{ is a positive integer } < 7\}: \text{ finite set.}$$

Example 1.1.2 The set N of all positive integers is written as

$$N = \{x : x \text{ is all positive integers}\}: \text{ countably infinite set.}$$

Example 1.1.3 The set R of all real numbers is written as

$$R = \{x : x \text{ is real}\}: \text{ uncountably infinite set.}$$

Example 1.1.4 The set R^2 of real numbers x, y is written as

$$R^2 = \{(x,y) : x \text{ is real, } y \text{ is real}\}$$

Example 1.1.5 The set C of all real numbers x,y such that $x+y \leq 10$ is written as

$$C = \{(x,y) : x+y \leq 10\}: \text{ uncountably infinite set.}$$

Empty Set

An empty is a set that contains no element. It plays an important role in set theory and is denoted by $\varnothing$. The set $A = \{0\}$ is not an empty set since it contains the element 0.

Cardinality

The number of elements in the set A is called the *cardinality* of set A, and is denoted by $|A|$. If it is an infinite set, then the cardinality is ∞.

Example 1.1.6 The cardinality of the set $A = \{2,4,6\}$ is 3, or $|A| = 3$. The cardinality of set $R = \{x : x \text{ is real}\}$ is ∞.

Example 1.1.7 The cardinality of the set $A = \{x : x \text{ is positive integer } <7\}$ is $|A| = 6$.

Example 1.1.8 The cardinality of the set $B = \{x : x \text{ is a real number } <10\}$ is infinity since there are infinite real numbers <10.

Subset and Proper subset

A set B is a subset of A if every element in B is an element of A and is written as $B \subseteq A$. B is a proper subset of A if *every* element of A is not in B and is written as $B \subset A$.

Equality of Sets:

Two sets A and B are equal if $B \subseteq A$ and $A \subseteq B$, that is, if every element of A is contained in B and every element of B is contained in A. In other words, sets A and B contain exactly the same elements.

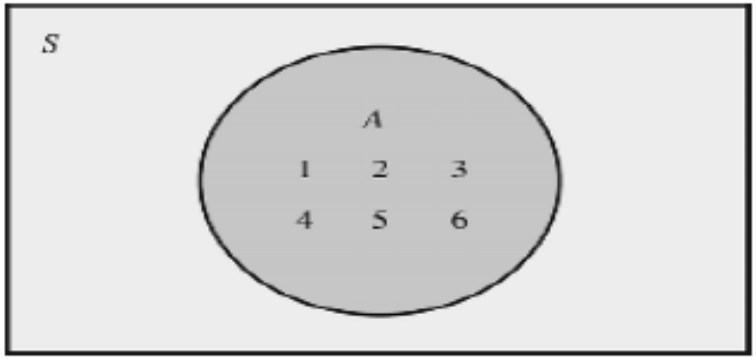
Power Set

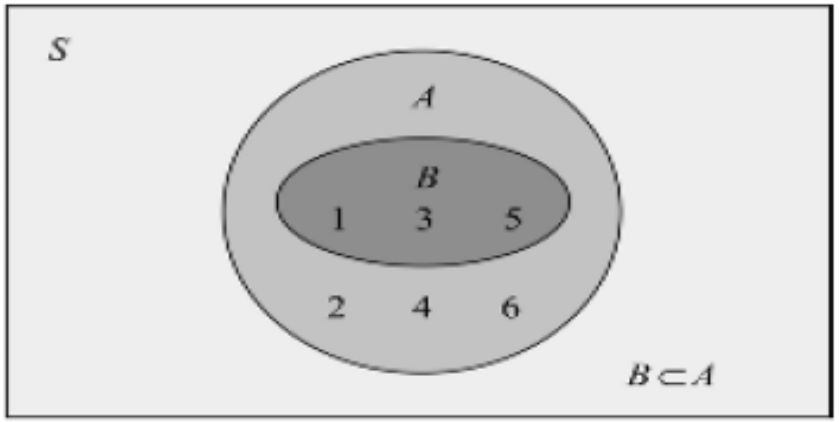
The power set of any set A is the set of all possible subsets of A and is denoted by $PS(A)$. Every power set of any set A must contain the set A itself and the empty set $\varnothing$. If n is the cardinality of the set A, then the cardinality of the power set $|PS(A)| = 2^n$.

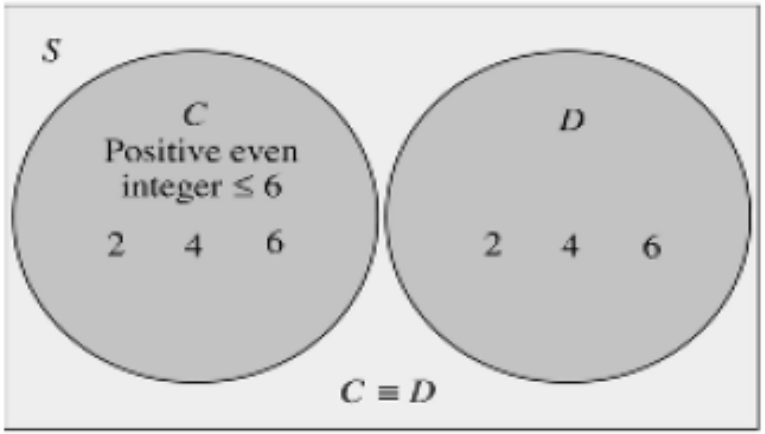
Example 1.1.10 If the set $A = \{1,2,3\}$ then $PS(A) = \{\varnothing, (1,2,3), (1,2), (2,3), (3,1), (1), (2), (3)\}$. The cardinality $|PS(A)| = 8 = 2^3$.

Venn Diagram

Sets can be represented graphically by means of a Venn diagram. In this case we assume tacitly that S is a universal set under consideration.







SET OPERATIONS

Union

Let A and B be sets belonging to the universal set S. The *union* of sets A and B is another set C whose elements are those that are in either A or B, and is denoted by $A \cup B$. Where there is no confusion, it will also be represented as $A + B$:

$$A \cup B = A + B = \{x : x \in A \text{ or } x \in B\}$$

$$A \bigcup B = \{\omega : \omega \in A \text{ or } \omega \in B\} \Leftrightarrow A \bigcup B = \{\omega : \omega \in A \vee \omega \in B\}$$

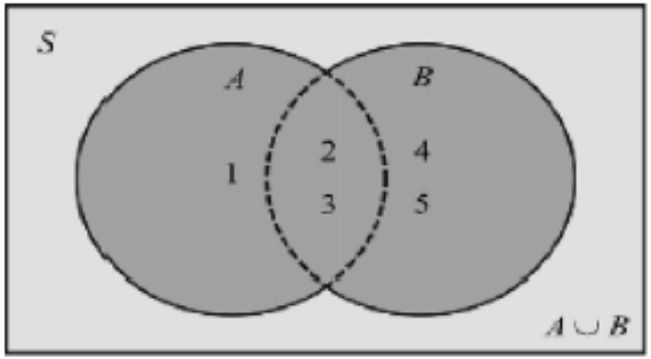
INTERSECTION

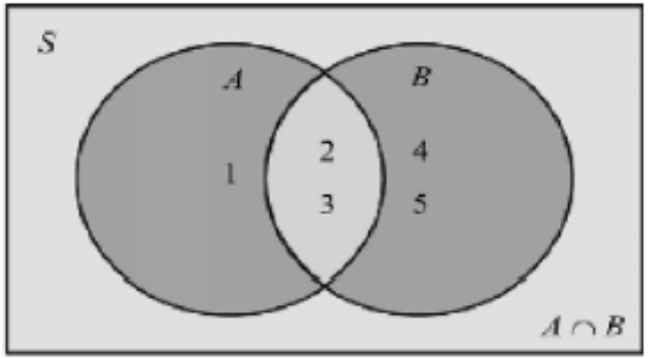
The *intersection* of the sets A and B is another set C whose elements are the same as those in both A and B and is denoted by $A \cap B$. Where there is no confusion, it will also be represented by AB.

$$A \cap B = AB = \{x : x \in A \text{ and } x \in B\}$$

$$A \bigcap B = \{\omega : \omega \in A \text{ and } \omega \in B\} \Leftrightarrow A \bigcap B = \{\omega : \omega \in A \wedge \omega \in B\}$$

Example 1.2.1 The union of sets $A = \{1,2,3\}$ and $B = \{2,3,4,5\}$ is the set $C = A \cup B = \{1,2,3,4,5\}$. $C = A \cap B = \{2,3\}$

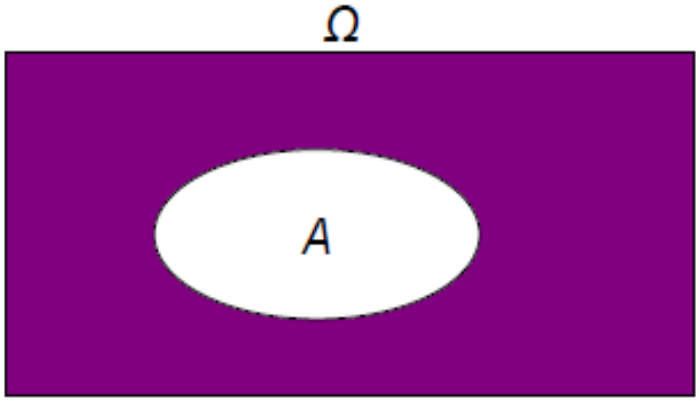




Complement:

The complement of an event A is defined as the set of all outcomes that are not in A and is denoted by $\overline{A}$.

$$\overline{A} = \{\omega : \omega \in \Omega \text{ and } \omega \notin A\} \Leftrightarrow \overline{A} = \{\omega : \omega \in \Omega \wedge \omega \notin A\}$$

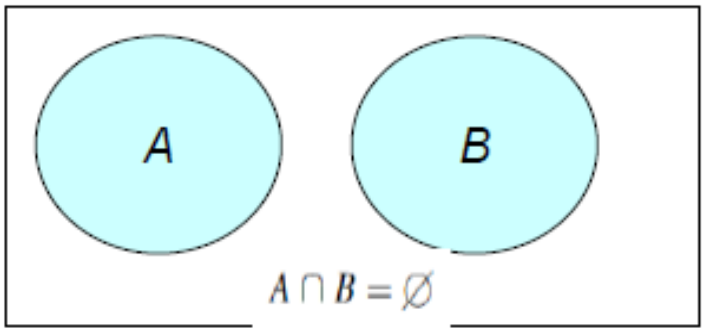


Mutually Exclusive (disjoint) Sets

Two sets A and B are called *mutually exclusive* if their intersection is empty. Mutually exclusive sets are also called *disjoint*.

$$A \cap B = \varnothing$$

One way to determine whether two sets A and B are mutually exclusive is to check whether set B can occur when set A has already occurred and vice versa. If it cannot, then A and B are mutually exclusive. For example, if a single coin is tossed, the two sets, {heads} and {tails}, are mutually exclusive since {tails} cannot occur when {heads} has already occurred and vice versa.



Independence:

Two sets A and B can be called functionally independent if the occurrence of B does not in any way influence the occurrence of A and vice versa.

As an example, the tossing of a coin is functionally independent of the tossing of a die because they do not depend on each other. However, the tossing of a coin and a die are not mutually exclusive since any one can be tossed irrespective of the other.

Table of set Properties OR Properties of Probability

Property	Equation
Identity	$A \cup \varnothing = A$ $A \cap S = A$
Domination	$A \cap \varnothing = \varnothing$ $A \cup S = S$
Idempotent	$A \cap A = A$ $A \cup A = A$
Complementation	$\overline{\overline{A}} = A$
Commutative	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's law	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Factorial, Permutation and Combination

Factorial: The notation n! or represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as n!. We read this symbol as 'n factorial'.

Thus, $n! = n(n-1)(n-2)\ldots\ldots\ldots 3 \times 2 \times 1$

$1 = 1!$

$1 \times 2 = 2!$

$1 \times 2 \times 3 = 3!$

$1 \times 2 \times 3 \times 4 = 4!$ and so on.

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$

$= 5 \times 4 \times 3 \times 2 \times 1!$

Clearly, for a natural number n

$n! = n(n-1)!$

$= n(n-1)(n-2)!$ [provided $(n \geq 2)$]

$= n(n-1)(n-2)(n-3)!$ [provided $(n \geq 3)$]

Permutation OR Arrangements

Distinctly ordered sets are called **arrangements** or **permutations**.

The number of permutations of n objects taken r at a time is given by

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} $$

Example

a) In how many ways can 6 people be arranged in a row?

Solution : $6.5.4.3.2.1 = 6! = {}^{6}P_{6}$

b) How many arrangements are possible if only 3 of them are chosen?

Solution: $6.5.4 = 120 = {}^{6}P_{3}$

Eg 1. A maths debating team consists of 4 speakers.

a) In how many ways can all 4 speakers be arranged in a row for a photo?

Solution : $4.3.2.1 = 4!$ or ${}^4P_4$



b) How many ways can the captain and vice-captain be chosen?

Solution : $4.3 = 12$ or ${}^4P_2$

Eg 2. A flutter on the horses
There are 7 horses in a race.

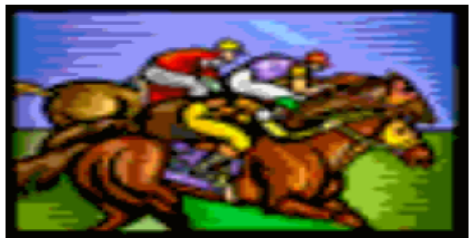


a) In how many different orders can the horses finish?

Solution : $7.6.5.4.3.2.1 = 7!$ or ${}^7P_7$

b) How many trifectas (1^{st} , 2^{nd} and 3^{rd}) are possible?

Solution : $7.6.5 = 210$ or ${}^7P_3$



Permutations with Restrictions

Eg. In how many ways can 5 boys and 4 girls be arranged on a bench if



a) there are no restrictions?

Solution : $9!$ or $^9P_9$

c) boys and girls alternate?

Solution : A boy will be on each end

BGBGBGBGB = $5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$

$= 5! \times 4!$ or $^5P_5 \times {}^4P_4$

Eg. In how many ways can 5 boys and 4 girls be arranged on a bench if

c) boys and girls are in separate groups?

Solution : Boys & Girls or Girls & Boys

$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$

or $^5P_5 \times {}^4P_4 \times 2$

d) Anne and Jim wish to stay together?



Solution : (AJ) _ _ _ _ _ _ _

$= 2 \times 8!$ or $2 \times {}^8P_8$

Arrangements with Repetitions

If we have **n** elements of which x are alike of one kind, y are alike of another kind, z are alike of another kind,
..... then the number of ordered selections or permutations is given by:

$$\frac{n!}{x!\ y!\ z!}$$

Eg.1 How many different arrangements of the word **PARRAMATTA** are possible?

Solution : **10 letters but note repetition (4 A's, 2 R's, 2 T's)**

P

A A A A

R R

M

T T

No. of arrangements $= \frac{10!}{4!\ 2!\ 2!}$

$= 37\,800$



Arrangements with Restrictions

Eg 1. How many arrangements of the letters of the word REMAND are possible if:

a) there are no restrictions?

Solution : $^6P_6 = 720$ or $6!$

b) they begin with RE?

Solution : R E _ _ _ _ $= {}^4P_4 = 24$ or $4!$

c) they do not begin with RE?

Solution : Total – (b) = $6! - 4! = 696$

Eg 1. How many arrangements of the letters of the word REMAND are possible if:

d) they have RE together in order?

Solution : (RE) _ _ _ _ $= {}^5P_5 = 120$ or $5!$

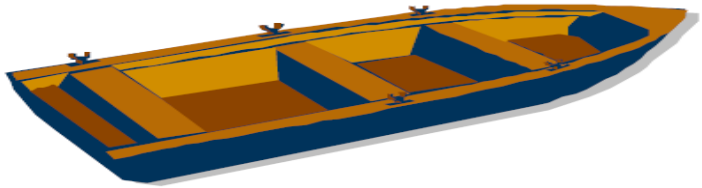
e) they have REM together in any order?

Solution : (REM) _ _ _ $= {}^3P_3 \times {}^4P_4 = 144$

f) R, E and M are not to be together?

Solution : Total – (e) = $6! - 144 = 576$

Eg 2. There are 6 boys who enter a boat with 8 seats, 4 on each side. In how many ways can



a) they sit anywhere?

Solution : $^8P_6$

b) two boys A and B sit on the port side and another boy W sit on the starboard side?

Solution : A & B = $^4P_2$

W = $^4P_1$

Others = $^5P_3$

Total = $^4P_2 \times {}^4P_1 \times {}^5P_3$



Eg 3. From the digits 2, 3, 4, 5, 6

a) how many numbers greater than 4 000 can be formed?

Solution : 5 digits (any) = $^5P_5$

4 digits (must start with digit $\geq$ 4) = $^3P_1 \times {}^4P_3$

Total = $^5P_5 + {}^3P_1 \times {}^4P_3$

b) how many 4 digit numbers would be even?

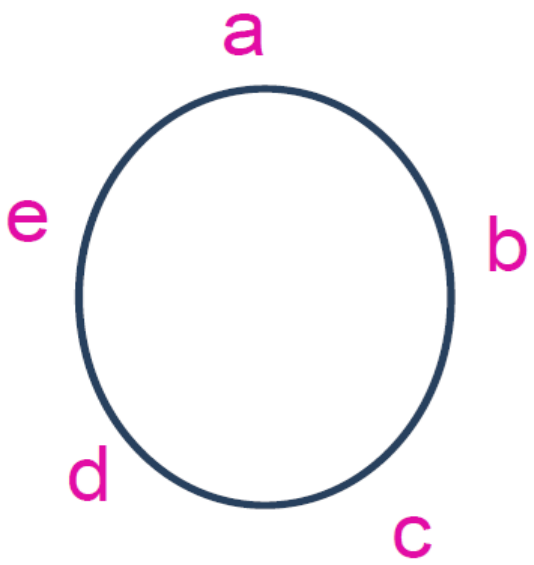
Even (ends with 2, 4 or 6) = _ _ _ $^3P_1$

= $^4P_3 \times {}^3P_1$

Circular Arrangements

Circular arrangements are permutations in which objects are arranged in a circle.

Consider arranging 5 objects (a, b, c, d, e) around a circular table. The arrangements



abcde

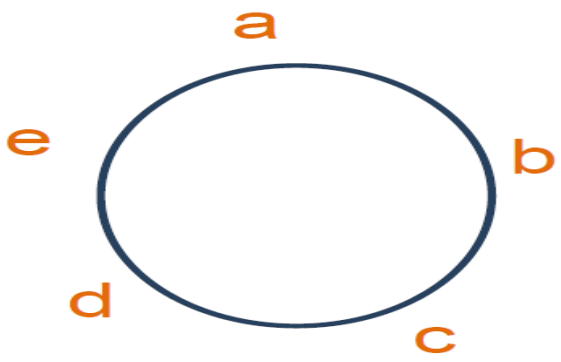
bcdea

cdeab

deabc

eabcd

are different in a line, but are **identical** around a circle.



To calculate the number of ways in which n objects can be arranged in a circle, we arbitrarily fix the position of one object, so the remaining (n-1) objects can be arranged as if they were on a straight line in (n-1)! ways.

i.e. the number of arrangements in a circle = $(n-1)!$

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:



a) there are no restrictions

Solution :

$(12-1)! = 11!$

b) men and women alternate

Solution : $(6-1)! \times 6! = 5! \times 6!$

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

c) Ted and Carol must sit together

Solution : (TC) & other 10 = $2! \times 10!$

d) Bob, Ted and Carol must sit together

Solution : (BTC) & other 9 = $3! \times 9!$

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

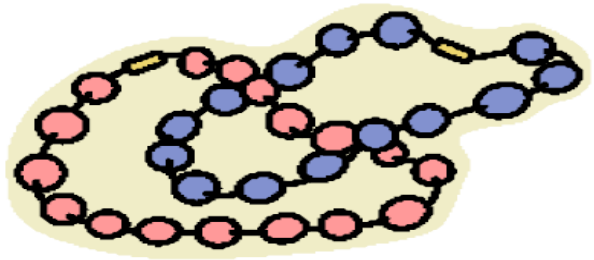
d) Neither Bob nor Carol can sit next to Ted.

Solution : Seat 2 of the other 9 people next to Ted in (9×8) ways or ${}^9P_2$

Then sit the remaining 9 people (including Bob and Carol) in $9!$ ways

Ways = $(9 \times 8) \times 9!$ or ${}^9P_2 \times 9!$

Eg 2. In how many ways can 8 differently coloured beads be threaded on a string?



Solution :

As necklace can be turned over, clockwise and anti-clockwise arrangements are the same

$= (8-1)! \div 2 \quad = 7! \div 2$

Combinations

The number of different **combinations** (i.e. unordered sets) of **r** objects from **n** distinct objects is represented by :

$$\text{No. of Combinations} = \frac{\text{number of permutations}}{\text{arrangements of r objects}}$$

and is denoted by

$$^{n}C_{r} = \frac{^{n}P_{r}}{r!} = \frac{n!}{r!\,(n-r)!}$$

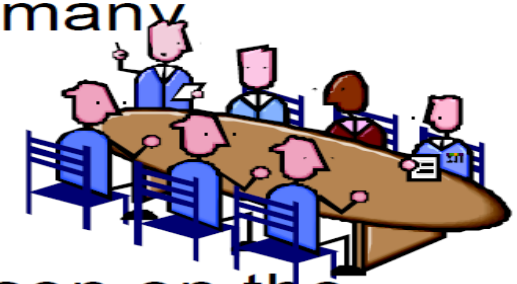
Eg 1. How many ways can a basketball team of 5 players be chosen from 8 players?

Solution :

$^{8}C_{5}$



Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if



a) there are no restrictions?

Solution : $^{10}C_{5}$

b) one particular person must be chosen on the committee?

Solution : $1 \times {}^{9}C_{4}$

c) one particular woman must be excluded from the committee?

Solution : $^{9}C_{5}$

Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if:

d) there are to be 3 men and 2 women?

Solution : **Men & Women = $^{6}C_{3} \times {}^{4}C_{2}$**

e) there are to be men only?

Solution : $^{6}C_{5}$

f) there is to be a majority of women?

Solution :

3 Women & 2 men Or 4 Women & 1 man

$= {}^{4}C_{3} \times {}^{6}C_{2} + {}^{4}C_{4} \times {}^{6}C_{1}$

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf:



a) If there are no restrictions?

Solution : $^{6}C_{4} \times {}^{5}C_{3} \times 7!$

c) If the 4 Maths books remain together?

Solution : = (MMMM) _ _ _

= $^{6}P_{4} \times {}^{5}C_{3} \times 4!$ or $({}^{6}C_{4} \times 4!) \times {}^{5}C_{3} \times 4!$

CHAPTER:2

Introduction: Sample Space and Events

- Events can in general be classified as:
  - **Random**-the event may or may not occur
  - **Certain**- the occurrence of the event is inevitable
  - **Impossible**-the event will never occur
- Probability is the study of **randomness** and **uncertainty**.
- Probability theory provides powerful tools to explain, model and design real world physical systems with some degree of uncertainty.
- Some application areas of probability theory include:
  - Data communication systems
  - Wireless communication systems
  - Control systems, etc…

Sample Space and Events

i. Random Experiment

- A random experiment is an experiment in which the outcome varies in an unpredictable manner when the experiment is repeated under the same conditions.

Examples:

- Tossing a coin
- Rolling a die
- Picking a card from a deck

ii. Sample Space

- The sample space is the set of all possible outcomes of a random experiment.

Sample Space and Events Cont'd….

- The sample space is denoted by Ω and the possible outcomes are represented by ω_i

$$\Omega = \{\omega_1, \omega_2, \ldots\ldots \omega_n\}$$

iii. Event

- An event is any subset of the sample of the sample space, Ω
- Events can be represented by A, B, C, ……

Examples of Discrete Sample Space

Example-1:

Consider a random experiment of rolling a die once.

i. Sample Space $\Omega = \{1, 2, 3, 4, 5, 6\}$

Sample Space and Events Cont'd….

ii. Some possible events

- An event of obtaining even numbers

$$A = \{2, 4, 6\}$$

- An event of obtaining numbers less than 4

$$B = \{1, 2, 3\}$$

Example-2:

Consider a random experiment of flipping a fair coin twice.

i. Sample space

$$\Omega = \{HH, \text{HT}, \text{TH}, \text{TT}\}$$

Sample Space and Events Cont'd….

ii. Some possible events

- An event of getting exactly one head

$$A = \{HT, \text{TH}\}$$

- An event of getting at least one tail

$$B = \{HT, \text{TH}, \text{TT}\}$$

- An event of getting at most one tail

$$C = \{HH, HT, \text{TH}\}$$

Examples of Continuous Sample Space

Example 2.1.4 The set of all real numbers between 0 and 2 is the sample space $S = \{x : 0 \leq x \leq 2\}$.

Example 2.1.5 The set of all real numbers x and y such that x is between 0 and 1 and y is between 1 and 2 is given by $S = \{x, y : 0 \leq x \leq 1, 1 \leq y \leq 2\}$.



Fig: For example 2.1.4

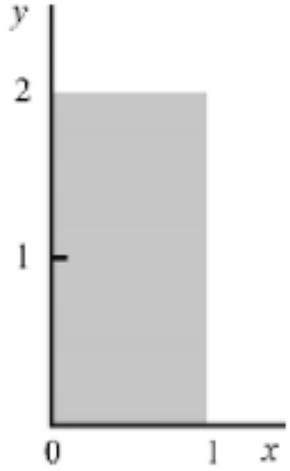


Fig: For example 2.1.5

Axioms and Properties of Probability

- Probability is a rule that assigns a number to each event ***A*** in the sample space, ***Ω***.
- In short , the probability of any event ***A*** is given by

$$P(A) = \frac{n(A)}{n(\Omega)}$$

where

$n(A)$ - is the number of elements in the event A

$n(\Omega)$ - is the number of elements in the sample space Ω

Axioms and Properties of Probability Cont'd…..

- The probability of an event ***A*** is a real number which satisfies the following axioms.

1. Probability is a non-negative number, i.e.,

$$P(A) \geq 0$$

2. Probability of the whole set is unity, i.e.,

$$P(\Omega) = 1$$

From axioms (1) and (2), we obtain

$$0 \leq P(A) \leq 1$$

Axioms and Properties of Probability Cont'd…..

3. Probability of the union of two mutually exclusive (disjoint) events is the sum of the probability of the events, i.e.,

   If $A \cap B = \Phi$, then $P(A \cup B) = P(A) + P(B)$

- We can generalize axiom (3) for ***n*** pairwise mutually exclusive (disjoint) events.
- If $\boldsymbol{A_1}, \boldsymbol{A_2}, \boldsymbol{A_3}, \ldots, \boldsymbol{A_n}$ is a sequence of ***n*** pairwise mutually exclusive (disjoint) events in the sample space $\boldsymbol{\Omega}$ such that $A_i \cap A_j = \Phi$, for $i \neq j$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Axioms and Properties of Probability Cont'd…..

- By using the above probability axioms, other useful properties of probability can be obtained.

1. $P(\overline{A}) = 1 - P(A)$

Proof:

$$A \cap \overline{A} = \Phi \Rightarrow P(A \cup \overline{A}) = P(A) + P(\overline{A}), \text{ but } A \cup \overline{A} = \Omega$$

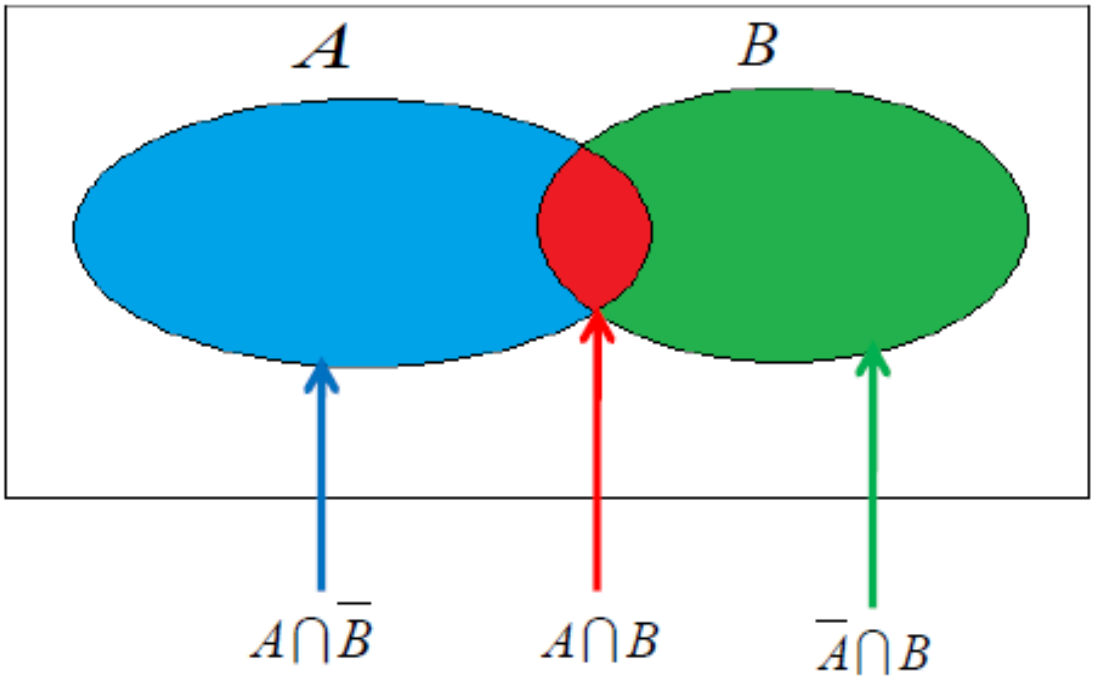
$$\Rightarrow P(\Omega) = P(A) + P(\overline{A}),\ P(\Omega) = P(A \cup \overline{A})$$

$$\Rightarrow 1 = P(A) + P(\overline{A}),\ P(\Omega) = 1$$

$$\therefore P(\overline{A}) = 1 - P(A)$$

Axioms and Properties of Probability Cont'd…..

- We can decompose the events A, B and AUB as unions of mutually exclusive (disjoint) events as follows.



We can see that $A \cap B$, $A \cap \overline{B}$ and $\overline{A} \cap B$ are disjoint events.

Axioms and Properties of Probability Cont'd…..

- From the above Venn diagram, we can write the following relations.

i. $A = (A \cap B) \cup (A \cap \overline{B})$

$\Rightarrow P(A) = P(A \cap B) + P(A \cap \overline{B})$

$\Rightarrow P(A \cap \overline{B}) = P(A) - P(A \cap B)$

ii. $B = (A \cap B) \cup (\overline{A} \cap B)$

$\Rightarrow P(B) = P(A \cap B) + P(\overline{A} \cap B)$

$\Rightarrow P(\overline{A} \cap B) = P(B) - P(A \cap B)$

iii. $A \cup B = A \cup (\overline{A} \cap B)$

$\Rightarrow P(A \cup B) = P(A) + P(\overline{A} \cap B)$

iv. $A \cup B = B \cup (A \cap \overline{B})$

$\Rightarrow P(A \cup B) = P(B) + P(A \cap \overline{B})$

v. $A \cup B = (A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B)$

$\Rightarrow P(A \cup B) = P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B)$

Axioms and Properties of Probability Cont'd…..

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

$P(A \cup B) = P(A) + P(\overline{A} \cap B)$

But, $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- We can generalize the above property for three events ***A***, ***B*** and ***C*** as follows.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Axioms and Properties of Probability Cont'd…..

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

$P(A \cup B) = P(A) + P(\overline{A} \cap B)$ BY (iii)

But, $P(\overline{A} \cap B) = P(B) - P(A \cap B)$ BY (ii)

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- We can generalize the above property for three events ***A***, ***B*** and ***C*** as follows.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Axioms and Properties of Probability Cont'd…..

- For **n** events $A_1, A_2, A_3, \ldots, A_n$ the above property can be generalized as:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} P(A_j) - \sum_{j<k}^{n} P(A_j \cap A_k) + \ldots + (-1)^n P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

3. $P(A \cup B) \leq P(A) + P(B)$

Proof:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

But, $P(A \cap B) \geq 0$

$\therefore P(A \cup B) \leq P(A) + P(B)$

Axioms and Properties of Probability Cont'd…..

Example-1:

A box contains 10 identical balls numbered 0, 1, 2,…,9. A single ball is selected from the box at random. Consider the following events.

A: number of ball selected is odd

B: number of ball selected is multiple of 3

C: number of ball selected is less than 5

Find the following probabilities.

a. $P(A)$

b. $P(B)$

c. $P(C)$

d. $P(A \cap B)$

e. $P(A \cup B \cup C)$

Axioms and Properties of Probability Cont'd…..

Solution:

- The sample space and the events are given by:

$\Omega = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $\quad C = \{0, 1, 2, 3, 4\}$

$A = \{1, 3, 5, 7, 9\}$ $\quad A \cap B = \{3, 9\}$

$B = \{3, 6, 9\}$ $\quad A \cup B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$

- The number of elements in the sample space and events are:

$n(\Omega) = 10$ $\quad n(C) = 5$

$n(A) = 5$ $\quad n(A \cap B) = 2$

$n(B) = 3$ $\quad n(A \cup B \cup C) = 9$

Axioms and Properties of Probability Cont'd…..

- Thus, probabilities of the given events are given by:

$$a.\ P(A)=\frac{n(A)}{n(\Omega)}=\frac{5}{10}=\frac{1}{2} \qquad d.\ P(A\cap B)=\frac{n(A\cap B)}{n(\Omega)}=\frac{2}{10}=\frac{1}{5}$$

$$b.\ P(B)=\frac{n(B)}{n(\Omega)}=\frac{3}{10} \qquad e.\ P(A\cup B\cup C)=\frac{n(A\cup B\cup C)}{n(\Omega)}=\frac{9}{10}$$

$$c.\ P(C)=\frac{n(C)}{n(\Omega)}=\frac{5}{10}=\frac{1}{2}$$

Example-2:

Given $P(A)=0.9, P(B)=0.8$ and $P(A\cap B)=0.75$, find :

$a.\ P(A\cup B)$ $\quad c.\ P(\overline{A}\cap\overline{B})$ $\quad e.\ P(\overline{A}\cup B)$

$b.\ P(A\cap\overline{B})$ $\quad d.\ P(\overline{A}\cup\overline{B})$ $\quad f.\ P(\overline{B})$

Axioms and Properties of Probability Cont'd…..

Solution:

a. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) = 0.9 + 0.8 - 0.75$

$\therefore P(A \cup B) = 0.95$

b. $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

$\Rightarrow P(A \cup B) = 0.9 - 0.75$

$\therefore P(A \cup B) = 0.15$

c. $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$\Rightarrow P(\overline{A} \cap \overline{B}) = 1 - 0.95$

$\therefore P(\overline{A} \cup \overline{B}) = 0.05$

d. $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$

$\Rightarrow P(\overline{A} \cup \overline{B}) = 1 - 0.75$

$\therefore P(\overline{A} \cup \overline{B}) = 0.25$

e. $P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B)$

$= 1 - P(A) + P(B) - P(B) + P(A \cap B)$

$= 1 - P(A) + P(A \cap B)$

$\Rightarrow P(\overline{A} \cup B) = 1 - 0.9 + 0.75$

$\therefore P(\overline{A} \cup B) = 0.85$

f. $P(\overline{B}) = 1 - P(B)$

$\Rightarrow P(\overline{B}) = 1 - 0.8$

$\therefore P(\overline{B}) = 0.2$

Axioms and Properties of Probability Cont'd…..

Exercise:

1. If $A \cap B = \Phi$, then show that $P(A) \leq P(B)$.
2. If $P(A) = P(B) = P(A \cap B)$, then show that
   $P[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = 0.$
3. If $P(A) = P(B) = 1$, then show that $P(A \cap B) = 1.$
4. If $P(A) = 0.9$ and $P(B) = 0.8$, then show that $P(A \cap B) \geq 0.7$

Conditional Probability

- The conditional probability of an event ***A*** given ***B***, denoted by ***P(A/B)***, is defined as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0 \qquad (1)$$

- Similarly, the conditional probability of an event ***B*** given ***A***, denoted by ***P(B/A)***, is given by

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0 \qquad (2)$$

- From equations (1) and (2), we will get

$$P(A \cap B) = P(A/B)P(B) = P(B/A)P(A) \qquad (3)$$

Conditional Probability Cont'd…….

- Then using equation (3), we will get

- We know that
$$P(A/B) = \frac{P(B/A)P(A)}{P(B)} \quad \text{OR} \quad P(B/A) = \frac{P(A/B)P(B)}{P(A)} \qquad (4)$$

- Substituting equation (5) into equation (4), we will get

$$P(B) = P(A \cap B) + P(\overline{A} \cap \mathrm{B})$$

$$\Rightarrow P(B) = P(B/A)P(A) + P(B/\overline{A})P(\overline{A}) \qquad (5)$$

$$P(A/B) = \frac{P(B/A)P(A)}{P(B/A)P(A) + P(B/\overline{A})P(\overline{A})} \qquad (6)$$

Conditional Probability Cont'd…….

- Similarly,
- Equations (6) and (7) are known as ***Baye's Rule***.
- ***Baye's Rule*** can be extended for n events as follows.
- Let events $A_1, A_2, A_3, \ldots, A_n$ be pairwise mutually exclusive (disjoint) events and their union be the sample space Ω, i.e.

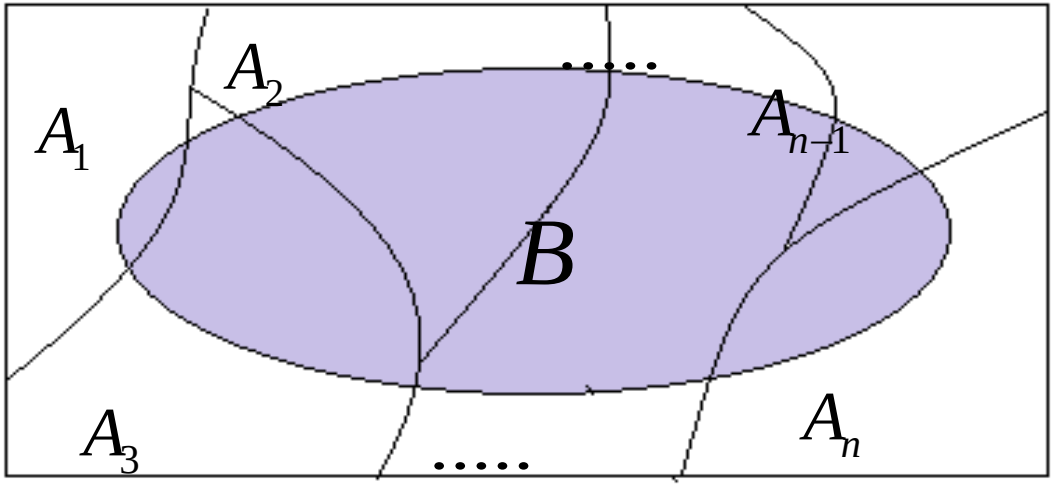
$$P(B/A) = \frac{P(A/B)P(B)}{P(A/B)P(B) + P(A/\overline{B})P(\overline{B})} \qquad (7)$$

$$A_i \cap A_j = \Phi \text{ and } \bigcup_{i=1}^{n} A_i = \Omega$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Conditional Probability Cont'd…….

- Let ***B*** be any event in ***Ω*** as shown below.



$$B = B \cap (A_1 \cup A_2 \cup .... \cup A_n)$$

$$\Rightarrow B = (B \cap A_1) \cup (B \cap A_2) \cup ... \cup (B \cap A_n)$$

$$\text{But, } A_i \cap A_j = \Phi \Rightarrow (B \cap A_i) \cap (B \cap A_j) = \Phi$$

Conditional Probability Cont'd…….

- The events are mutually exclusive events.

$$B \cap A_i \text{ and } B \cap A_j$$

- In short,

$$\Rightarrow P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)$$

- Then using equation (4), we will obtain

Total Probability

$$\Rightarrow P(B) = P(B/A_1)P(A_1) + P(B/A_2)P(A_2) + \ldots + P(B/A_n)P(A_n) \qquad (8)$$

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B/A_i)P(A_i) \qquad (9)$$

Baye's Rule can be extended for n events as follows.

$$P(A_i/B) = \frac{P(B/A_i)P(A_i)}{\sum_{i=1}^{n} P(B/A_i)P(A_i)} \qquad (10)$$

Conditional Probability Cont'd…..

Example-1:

Show that $P(\overline{A}/B) = 1 - P(A/B)$

Solution:

$$P(B) = P(A \cap B) + P(\overline{A} \cap B)$$

$$\Rightarrow P(B) = P(A/B)P(B) + P(\overline{A}/B)P(B)$$

Dividing both sides by $P(B)$, we obtain

$$1 = P(A/B) + P(\overline{A}/B)$$

$$\therefore P(\overline{A}/B) = 1 - P(A/B)$$

Conditional Probability Cont'd…..

Example-2:

Let *A* and *B* be two events such that *P(A)=x*, *P(B)=y* and *P(B/A)=z*. Find the following probabilities in terms of *x*, *y* and *z*.

a. $P(A/B)$

b. $P(\overline{A}\cup\overline{B})$

c. $P(\overline{A}/B)$

Solution:

$$P(A\cap B)=P(B/A)P(A)=xz$$

a. $P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{xz}{y}$

b. $P(\overline{A}\cup\overline{B})=P(\overline{A\cap B})=1-P(A\cap B)=1-xz$

c. $P(\overline{A}/B)=\dfrac{P(\overline{A}\cap B)}{P(B)}=\dfrac{P(B)-P(A\cap B)}{P(B)}=1-\dfrac{xz}{y}$

Conditional Probability Cont'd…..

Example-3:

A box contains two black and three white balls. In succession, two balls are selected at random from the box without replacement. Find the probability that

a. both balls are black

b. the second ball is white

Solution:

First let us define the events as follows:

B_1 : the outcome in the first selection is a black ball

Conditional Probability Cont'd…..

B_2 : the outcome in the second selection is a black ball

W_1 : the outcome in the first selection is a white ball

W_2 : the outcome in the second selection is a white ball

$$P(B_1) = 2/5 \qquad P(B_2 / B_1) = 1/4 \qquad P(W_2 / B_1) = 3/4$$
$$P(W_1) = 3/5 \qquad P(B_2 / W_1) = 2/4 \qquad P(W_2 / W_1) = 2/4$$

a. $P(B_1 \cap B_2) = P(B_2 / B_1)P(B_1) = (1/4)(2/5)$

$\therefore P(B_1 \cap B_2) = 1/10$

b. $P(W_2) = P(W_2 \cap B_1) + P(W_2 \cap W_1)$

$= P(W_2 / B_1)P(B_1) + P(W_2 / W_1)P(W_1)$

$= (3/4)(2/5) + (2/4)(3/5)$

$\therefore P(W_2) = 3/5$

Conditional Probability Cont'd…..

Example-4:

Box ***A*** contains 100 bulbs of which 10% are defective. Box ***B*** contains 200 bulbs of which 5% are defective. A bulb is picked from a randomly selected box.

a. Find the probability that the bulb is defective
b. Assuming that the bulb is defective, find the probability that it came from box ***A***.

Conditional Probability Cont'd…..

Solution:

First let us define the events as follows.

A: Box A is selected

B: Box B is selected

D: Bulb is defective

$$\Rightarrow \quad P(A) = P(B) = 1/2,\ P(D/A) = 1/10,\ P(D/B) = 1/20$$

a. $P(D) = P(D/A)P(A) + P(D/B)P(B)$

$= (1/10)(1/2) + (1/20)(1/20)$

$\therefore P(D) = 3/40$

b. $P(A/D) = \dfrac{P(D/A)P(A)}{P(D)} = \dfrac{1/20}{3/40} = (1/20)(40/3)$

$\therefore P(A/D) = 2/3$

Conditional Probability Cont'd…..

Example-5:

One bag contains 4 white and 3 black balls and a second bag contains 3 white and 5 black balls. One ball is drawn from the first bag and placed in the second bag unseen and then one ball is drawn from the second bag. What is the probability that it is a black ball?

Solution:

First let us define the events as follows.

B_1 : black ball is drawn from the first bag

W_1 : white ball is drawn from the first bag

Conditional Probability Cont'd…..

B_2 : black ball is drawn from the second bag

W_2 : white ball is drawn from the second bag

Then, we will have:

$$P(B_1) = 3/7 \qquad P(B_2 / B_1) = 6/9 \qquad P(B_2 / W_1) = 5/9$$

$$P(W_1) = 4/7 \qquad P(W_2 / B_1) = 3/9 \qquad P(W_2 / W_1) = 4/9$$

$$P(B_2) = P(B_2 \cap B_1) + P(B_2 \cap W_1)$$

$$\Rightarrow P(B_2) = P(B_2 / B_1)P(B_1) + P(B_2 / W_1)P(W_1)$$

$$\Rightarrow P(B_2) = (6/9)(3/7) + (5/9)(4/7)$$

$$\therefore P(B_2) = 28/63$$

Conditional Probability Cont'd…..

Exercise:

1. For three events ***A***, ***B*** and ***C***, show that:
   a. $P[(A \cap B)/C] = P[A/(B \cap C)]P(B/C)$
   b. $P(A \cap B \cap C) = P[A/(B \cap C)]P(B/C)P(C)$
2. Box ***A*** contains 3 white and 2 red balls while another box ***B*** contains 2 red and 5 white balls. A ball drawn at random from one of the boxes turns out to be red. What is the probability that it came from box ***A***?
3. **Box I contains 2 white and 3 red balls and box II contains 4 white and 5 red balls. One ball is drawn at random from one of the boxes and is found to be red. Then, the probability that it was from box II, is?**

Conditional Probability Cont'd…..

3. In a certain assembly plant, three machines, A, B, and C, make 34%, 20%, and 46%, respectively, of the products. It is known from past experience that 3%, 3%, and 2% of the products made by each machine, respectively are defective.

a) If a product is chosen at random, what is the probability that it is defective?

b) Given a randomly selected product was found to be defective, what is the probability it was made by machine C?

Solution: a) $P(A)= 0.34, P(B)= 0.20, P(C) = 0.46$

$P(D/A)= 0.03, P(D/B)= 0.03, P(D/C)= 0.02$

$P(D)= P(D/A)\times P(A)+P(D/B) \times P(B) +P(D/C) \times P(C)$

b) $P(C/D)= \dfrac{P(D/C)\times P(C)}{P(D)}$

Example 2.4.3 A communication receiver sends numbers 1,2,3. As a result of inherent noise in the channel, the conditional probabilities of receiving 1,2,3 under the condition that 1,2,3 were sent are shown in Fig. 2.4.2.

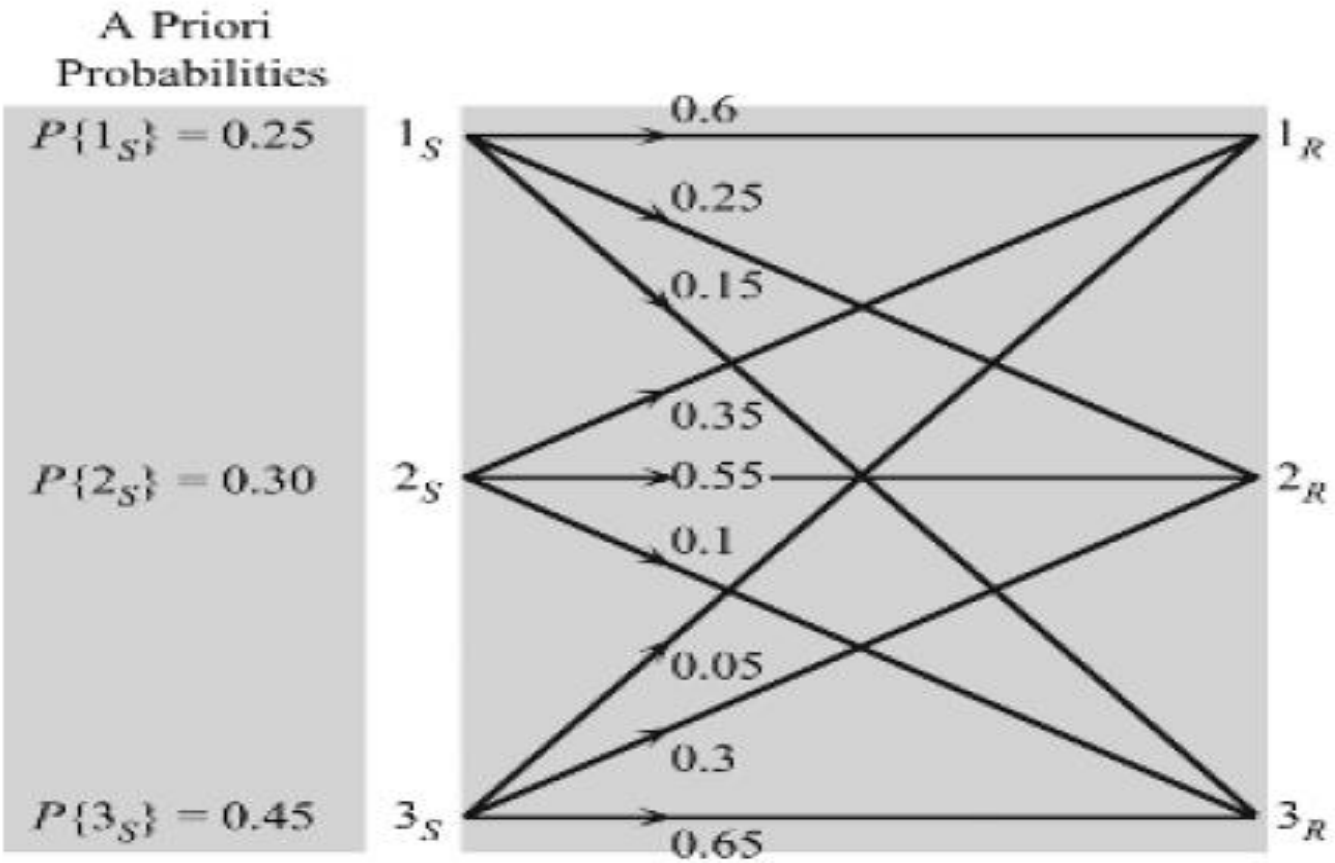


FIGURE 2.4.2

Transition Probabilities:

$P\{1_R \mid 1_S\} = 0.60$	$P\{2_R \mid 1_S\} = 0.25$	$P\{3_R \mid 1_S\} = 0.15$
$P\{1_R \mid 2_S\} = 0.35$	$P\{2_R \mid 2_S\} = 0.55$	$P\{3_R \mid 2_S\} = 0.10$
$P\{1_R \mid 3_S\} = 0.05$	$P\{2_R \mid 3_S\} = 0.30$	$P\{3_R \mid 3_S\} = 0.65$

The total probability that 1 was received is obtained from Eq. (2.4.3):

$$\begin{aligned} P\{1_R\} &= P\{1_R \mid 1_S\}P\{1_S\} + P\{1_R \mid 2_S\}P\{2_S\} + P\{1_R \mid 3_S\}P\{3_S\} \\ &= 0.6 \times 0.25 + 0.35 \times 0.3 + 0.05 \times 0.45 = 0.2775 \end{aligned}$$

From Bayes' theorem, Eq. (2.4.4), the a posteriori probabilities of 1, 2, and 3 being sent conditioned on 1 received are found as follows:

$$P\{1_S \mid 1_R\} = \frac{P\{1_R \mid 1_S\}P\{1_S\}}{P\{1_R\}} = \frac{0.6 \times 0.25}{0.2775} = 0.5405: \quad P\{1_S\} = 0.25$$

$$P\{2_S \mid 1_R\} = \frac{P\{1_R \mid 2_S\}P\{2_S\}}{P\{1_R\}} = \frac{0.35 \times 0.3}{0.2775} = 0.3784: \quad P\{2_S\} = 0.3$$

$$P\{3_S \mid 1_R\} = \frac{P\{1_R \mid 3_S\}P\{3_S\}}{P\{1_R\}} = \frac{0.05 \times 0.45}{0.2775} = 0.0811: \quad P\{3_S\} = 0.45$$

Independence of Events

- Two events ***A*** and ***B*** are said to be statistically independent if and only if

$$P(A \cap B) = P(A)P(B)$$

- Similarly, three events ***A***, ***B*** and ***C*** are said to be statistically independent if and only if

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

- Generally, if $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_n}$ are a sequence of independent events, then

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$$

Independence of Events Cont'd……

- If ***A*** and ***B*** are independent, then we have

i. $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$\Rightarrow P(A/B) = P(A)$$

ii. $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

$$\Rightarrow P(B/A) = P(B)$$

Example-1:

If A and B are independent, then show that A and $\overline{B}$ are also independent.

Independence of Events Cont'd……

Solution:

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

$$\Rightarrow P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$

$$\Rightarrow P(A \cap \overline{B}) = P(A)[1 - P(B)] = P(A)P(\overline{B})$$

$\therefore$ By the definition of independent events, A and $\overline{B}$ are independent.

Example-2:

The probability that a husband and a wife will be alive 20 years from now are given by 0.8 and 0.9 respectively. Find the probability that in 20 years

a. both will be alive

b. neither will be alive

c. at least one will be alive

Independence of Events Cont'd……

Solution:

- First let us define the events as follows.

- Then we will have

$$H: \text{Husband will be alive}$$
$$W: \text{Wife will be alive}$$

- The two events can be considered as independent.

$$P(H) = 0.8 \Rightarrow P(\overline{H}) = 1 - P(H) = 1 - 0.8 = 0.2$$
$$P(W) = 0.9 \Rightarrow P(\overline{W}) = 1 - P(W) = 1 - 0.9 = 0.1$$

Independence of Events Cont'd……

Solution:

$$a.\quad P(both) = P(H \cap W) = P(H)P(W)$$

$$\Rightarrow P(both) = P(H \cap W) = (0.8)(0.9)$$

$$\therefore P(both) = P(H \cap W) = 0.72$$

$$b.\quad P(neither) = P(\overline{H} \cap \overline{W}) = P(\overline{H})P(\overline{W})$$

$$\Rightarrow P(neither) = P(\overline{H} \cap \overline{W}) = (0.2)(0.1)$$

$$\therefore P(neither) = P(\overline{H} \cap \overline{B}) = 0.02$$

$$c.\quad P(\text{at least one}) = 1 - P(\text{neither})$$

$$\Rightarrow P(\text{at least one}) = 1 - 0.02$$

$$\therefore P(\text{at leat one}) = 0.98$$

Assignment-1

1. A fair die is tossed twice, and events A represents even number outcome on the first roll, and B represents an even number outcome on the second roll. Determine P(A), P(B), and P(AB).

2. We have four urns containing a number of red or green balls, according to the distribution shown below:

Urn Number	Number of green balls	Number of red balls
1	19	1
2	3	2
3	9	1
4	9	1

- The idea is to randomly select one of the urns and then randomly select one ball from the chosen urn. Determine:
- A) The probability that the selected ball is red.
- B) The probability that the selected ball is red, given that it comes from urn 2.

1. A computer manufacturer uses chips from three sources. Chips from sources A, B, and C are defective with probabilities .005, .001, and .010, respectively. If a randomly selected chip is found to be defective, assume that the proportions of chips from A, B, and C are 0.5, 0.1, and 0.4 respectively;

a. Find the probability that the manufacturer was A;
b. Find the probability that the manufacturer was C.

A= The chip is from the manufacturer A
B= The chip is from the manufacturer B
C= the chip is from the manufacturer C
D= The chip is defective"

P(A)=0.5
P(B)=0.1
P(C)=0.4
P(D|A)=0.005
P(D|B)=0.001
P(D|C)=0.010

Total Probability

$P(D)= P(D|A)P(A)+P(D|B)P(B)+P(D|C)P(C)=0.005\times0.5+0.001\times0.1+0.01\times0.4$

$P(D) =6.6x10^{-3}$

Based on Bayes rule:

a. The chip is from the manufacturer A

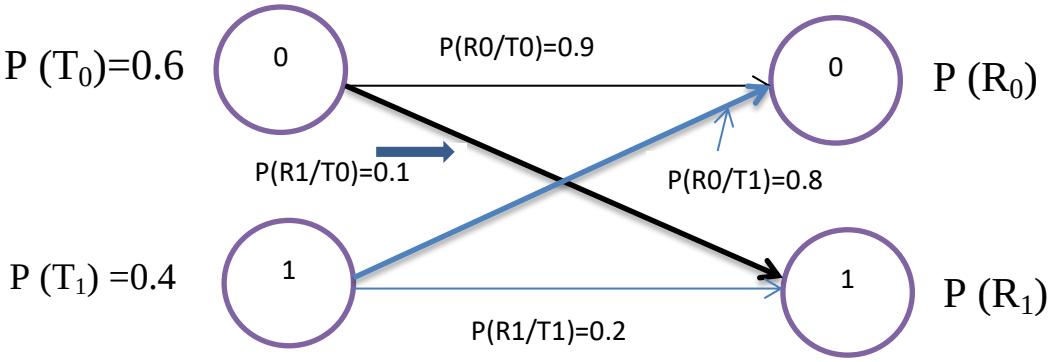
$P(A|D) =P(D|A)\,P(A) / P(D) =0.005\times0.5 / 6.6x10^{-3}=0.38$

b. Find the probability that the manufacturer was C.

$P(C|D) =P(D|C)\,P(C) / P(D) =0.01\times0.4 / 6.6x10^{-3}=0.60$

2. In a binary communication system a zero and a one is transmitted with probability 0.6 and 0.4 respectively. Due to error in the communication system a zero becomes a one with a probability 0.1 and a one becomes a zero with a probability 0.8. Determine the probability of

a) Receiving a zero
b) That a one was transmitted when the received message is one



(a) $P(R0) = P(T0)\,P(R0/T0) + P(T1)P(R0/T1)$

$= 0.6 \times 0.9 + 0.4 \times 0.8$

$= 0.86$

(b) $P(R1) = P(T0)\,P(R1/T0) + P(T1)P(R1/T1)$

$= 0.6 \times 0.1 + 0.4 \times 0.2$

$= 0.14$

$P(T1/R1) = P(R1/T1)\,P(T1) / P(R1)$

$= 0.4 \times 0.2 / 0.14$

$= 0.571$