

CHAPTER THREE

LAPLACE TRANSFORM

3.1 Definition of Laplace Transform

Definition 3.1.1 let f be a function defined for all $t \geq 0$. The Laplace Transform of $f(t)$, denoted by $L[f(t)] = F(s)$, is defined by

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (3.1)$$

Provided the integral converges.

Definition 3.1.2 (Sufficient Conditions for Existence of Laplace Transform)

A function f is piecewise continuous on $[0, \infty)$ if, in any interval $0 \leq a \leq t \leq b$, there are at most a finite number of points $t_k, k = 1, 2, 3, \dots, n$ ($t_{k-1} < t_k$) at which f has finite discontinuities and is continuous on each open interval (t_{k-1}, t_k) .

Definition 3.1.3 (Exponential Order)

A function f is said to be of exponential order if there exist $c \geq 0$ and constants for all $T, M > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$ (3.2)

Example 3.1 Find the Laplace transform of the following

- | | |
|-----------------------|---|
| a) $f(t) = 1$ | e) $f(t) = \sinh(at)$ |
| b) $f(t) = t$ | f) $f(t) = \cos(at)$ |
| c) $f(t) = e^{at}$ | g) $f(t) = \sin(at)$ |
| d) $f(t) = \cosh(at)$ | h) $f(t) = t^n$ for $n = 0, 1, 2, 3, \dots$ |

Solution:

- a) By definition of Laplace transform, we have

$$\begin{aligned} L[f(t)] = F(s) &= \int_0^{\infty} e^{-st} dt = \lim_{k \rightarrow \infty} \int_0^k e^{-st} dt = \lim_{k \rightarrow \infty} \left(\frac{-1}{s} e^{-st} \right) \Big|_0^k \\ &= \lim_{k \rightarrow \infty} \left(\frac{-1}{s} (e^{-sk} - 1) \right) \\ &= \frac{1}{s}, s > 0 \end{aligned}$$

- b) From (3.1) we obtain

$$\begin{aligned} L[f(t)] = F(s) &= \int_0^{\infty} e^{-st} t dt \\ &= \lim_{k \rightarrow \infty} \int_0^k e^{-st} t dt \\ &= \lim_{k \rightarrow \infty} \left(\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right) \Big|_0^k \end{aligned}$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} \left(e^{-sk} \left(\frac{-k}{s} - \frac{1}{s} \right) + \frac{1}{s^2} \right) \\
&= \frac{1}{s^2}, \quad s > 0
\end{aligned}$$

c) By definition we get

$$\begin{aligned}
L[f(t)] &= L[e^{at}] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt \\
&= \lim_{k \rightarrow \infty} \int_0^k e^{-(s-a)t} dt \\
&= \lim_{k \rightarrow \infty} \left(\frac{-1}{(s-a)} e^{-(s-a)t} \right) \Big|_0^k \\
&= \frac{1}{s-a} \text{ for } s - a > 0
\end{aligned}$$

d) Recall that $\cosh(at) = \frac{1}{2}(e^{at} + e^{-at})$

$$\begin{aligned}
L[\cosh(at)] &= L\left[\frac{e^{at} + e^{-at}}{2}\right] = \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2}\right) dt \\
&= \lim_{k \rightarrow \infty} \int_0^k e^{-st} \left(\frac{e^{at} + e^{-at}}{2}\right) dt \\
&= \frac{1}{2} \lim_{k \rightarrow \infty} \left(\int_0^k e^{-(s-a)t} dt + \int_0^k e^{-(s+a)t} dt \right) \\
&= \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right) \\
&= \frac{s}{s^2 - a^2}, \text{ for } s > a
\end{aligned}$$

e) Left as an exercise

f) Since $L[e^{at}] = \frac{1}{s-a}$, put $a = ia$ with $i = \sqrt{-1}$

$$\text{Then } L[e^{iat}] = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$$

From Euler's formula we have

$$\begin{aligned}
L[e^{iat}] &= L[\cos(at) + i\sin(at)] \\
\frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2} &= L[\cos(at)] + iL[\sin(at)]
\end{aligned}$$

Equate like terms. We obtain

$$L[\cos(at)] = \frac{s}{s^2+a^2} \text{ and } L[\sin(at)] = \frac{a}{s^2+a^2}$$

Theorem 3.1.1 Laplace Transform of some functions for any constant a

- a. $L[1] = \frac{1}{s}, s > 0$
- b. $L[t] = \frac{1}{s^2}, s > 0$
- c. $L[e^{at}] = \frac{1}{s-a}, s > a$
- d. $L[\cosh(at)] = \frac{s}{s^2 - a^2}, s > a$
- e. $L[\sinh(at)] = \frac{a}{s^2 - a^2}, s > a$
- f. $L[\cos(at)] = \frac{s}{s^2 + a^2}$
- g. $L[\sin(at)] = \frac{a}{s^2 + a^2}$
- h. $L[t^n] = \frac{n!}{s^{n+1}}, \text{ for } n = 0, 1, 2, 3, \dots$

3.1.2 Properties of Laplace Transform

Let $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$

Then

- 1) Linearity of the Laplace Transform

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)], \text{ for any constant } a \text{ and } b.$$

- 2) First shifting property

$$L[e^{at}f(t)] = F(s - a)$$

- 3) Second shifting property

$$L[g(t)] = e^{-as}L[f(t)] = e^{-as}F(s), \text{ where } g(t) = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

- 4) Change of scale property

$$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right) \text{ for } a \neq 0$$

- 5) Multiplication of $f(t)$ by t^n

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

- 6) Division of $f(t)$ by t

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$$

Provided the integral exists.

- 7) Laplace Transform of integrals

$$L\left[\int_0^t f(u)du\right] = \frac{F(s)}{s}$$

- 8) Let $f(t)$ be periodic function with period $p > 0$ so that $f(t+p) = f(t)$, then

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)dt$$

Example 3.2 Find the Laplace transform of the following given functions:

a) $f(t) = 4e^{5t} + 6t^3 - 3\sin(4t) + 2\cos(t)$

e) $f(t) = \sin(3t)$

b) $g(t) = e^{-2t}(\cos(6t) - 5\sin(6t))$

f) $h(t) = t^2 \sin(t)$

c) $g(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & , \quad t > \frac{2\pi}{3} \\ 0 & , \quad t < \frac{2\pi}{3} \end{cases}$

g) $f(t) = \begin{cases} \sin(\omega t), & \text{for } 0 \leq t < \frac{\pi}{\omega} \\ 0, & \text{for } \frac{\pi}{\omega} < t \leq \frac{2\pi}{\omega} \end{cases}$

d) $f(t) = \frac{1-e^t}{t}$

h) $f(t) = e^{7+t}$

Solution:

a) Let $f(t) = 4e^{5t} + 6t^3 - 3\sin(4t) + 2\cos(t)$, then the Laplace Transform gives as

$$\begin{aligned} F(s) &= L[4e^{5t} + 6t^3 - 3\sin(4t) + 2\cos(t)] \\ &= L[4e^{5t}] + L[6t^3] - L[3\sin(4t)] + L[2\cos(t)] \\ &= \frac{4}{s-5} + 6 \frac{3!}{s^4} - 3 \frac{4}{s^2+16} + 2 \frac{s}{s^2+1} \\ &= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+1} \end{aligned}$$

b) Let $g(t) = e^{-2t}(\cos(6t) - 5\sin(6t))$

Since $F(s) = L[\cos(6t) - 5\sin(6t)] = \frac{6s}{s^2+36} - \frac{30}{s^2+36}$ we have

$$\begin{aligned} L[g(t)] &= F(s+2) = \frac{6(s+2)}{(s+2)^2+36} - \frac{30}{(s+2)^2+36} \\ G(s) &= \frac{6(s-3)}{(s+2)^2+36} \end{aligned}$$

c) Since $a = \frac{2\pi}{3}$ and $f(t) = \cos t$, then $F(s) = \frac{s}{s^2+1}$

By second shifting property, we obtain

$$\begin{aligned} L[g(t)] &= e^{-\frac{2\pi s}{3}} L[f(t)] = e^{-\frac{2\pi s}{3}} F(s) \\ &= \frac{s}{s^2+1} e^{-\frac{2\pi s}{3}} \end{aligned}$$

d) Since $F(s) = L[1 - e^t] = L[1] - L[e^t] = \frac{1}{s} - \frac{1}{s-1}$ we have

$$\begin{aligned} L\left[\frac{1-e^t}{t}\right] &= \int_s^\infty F(s) ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds \\ &= (\ln(s) - \ln(s-1)) \Big|_s^\infty \\ &= \ln\left(\frac{s-1}{s}\right) \end{aligned}$$

e) We know that $L[\sin t] = \frac{1}{s^2+1} = F(s)$ and $a = 3$

By change of scale property, we get

$$L[\sin(3t)] = \frac{1}{3} F\left(\frac{s}{3}\right) = \frac{1}{3} \left(\frac{1}{\left(\frac{s}{3}\right)^2 + 1} \right) = \frac{3}{s^2+9}$$

f) Let $L[\sin t] = \frac{1}{s^2+1} = F(s)$ and $n = 2$

By property (5) we have

$$L[t^2 \sin t] = (-1)^2 \frac{d^2}{ds^2} F(s) = \frac{d^2}{ds^2} \left(\frac{1}{s^2+1} \right) = \frac{d}{ds} \left(\frac{-2s}{(s^2+1)^2} \right) = \frac{6s^2-2}{(s^2+1)^3}$$

g) Since f is periodic function with period $p = \frac{2\pi}{\omega}$ we have

$$\begin{aligned} L[f(t)] &= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t) dt = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left(\int_0^{\frac{\pi}{\omega}} e^{-st} f(t) dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st} f(t) dt \right) \\ &= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} \sin(\omega t) dt \\ &= \frac{\omega \left(1 + e^{-\frac{\pi s}{\omega}} \right)}{1 - e^{-\frac{2\pi s}{\omega}}} \end{aligned}$$

Exercise 3.1 Find the Laplace Transform of:

$$1) f(t) = \begin{cases} 0 & \text{for } 0 \leq t < 3 \\ 2 & \text{for } t \geq 3 \end{cases}$$

$$5) f(t) = t^2 e^{-3t} \cos(4t)$$

$$2) f(t) = \sin(2t) \cos(3t)$$

$$6) f(t) = \frac{\sin t}{t}$$

$$3) f(t) = \sin(4t + 5)$$

$$7) f(t) = \frac{e^{at} - e^{-bt}}{t}$$

$$4) f(t) = \begin{cases} \sin t & \text{for } 0 \leq t < \pi \\ 2 & \text{for } \pi < t \leq 2\pi \end{cases}$$

$$8) f(t) = \frac{\cos(at) - \cos(bt)}{t}$$

3.2 Inverse Laplace Transform

Definition 3.2.1 Let $F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st} dt$ be the Laplace Transform of $f(t)$. We say that $f(t)$ is the inverse Laplace Transform of $F(s)$ and write

$$f(t) = L^{-1}(F(s)) \quad (3.3)$$

Theorem 3.3.1 some inverse Laplace Transforms

$$a. 1 = L^{-1}\left(\frac{1}{s}\right)$$

$$e) \sinh(at) = L^{-1}\left(\frac{a}{s^2-a^2}\right)$$

$$b. t = L^{-1}\left(\frac{1}{s^2}\right)$$

$$f) \cos(at) = L^{-1}\left(\frac{s}{s^2+a^2}\right)$$

$$c. e^{at} = L^{-1}\left(\frac{1}{s-a}\right)$$

$$g) \sin(at) = L^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$d. \cosh(at) = L^{-1}\left(\frac{s}{s^2-a^2}\right)$$

$$h) t^n = L^{-1}\left(\frac{n!}{s^{n+1}}\right)$$

3.2.2 Properties of inverse Laplace Transform

Let $F(s)$ and $G(s)$ are Laplace Transform of $f(t)$ and $g(t)$, respectively

i.e. $F(s) = L[f(t)] \Rightarrow f(t) = L^{-1}(F(s))$ and $G(s) = L[g(t)] \Rightarrow g(t) = L^{-1}(G(s))$ then

- 1) The inverse Laplace Transform is linear

$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F(s)) + bL^{-1}(G(s))$$

- 2) Shifting property

$$L^{-1}(F(s-a)) = e^{at} f(t)$$

- 3) Inverse Laplace Transform of derivatives

$$L^{-1}(s^n F(s)) = \frac{d^n}{dt^n} f(t)$$

Where $f(0) = f'(0) = f''(0) = \dots = f^{n-1}(0) = 0$, which is a consequence of the transform of derivatives.

- 4) $L^{-1}\left((-1)^n \frac{d^n}{ds^n} F(s)\right) = t^n f(t)$ for $n = 1, 2, 3, \dots$

- 5) Inverse Laplace Transform of integration

$$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(u) du$$

- 6) Inverse Laplace Transform of Division of $f(t)$ by t

$$L^{-1}\left(\int_s^\infty F(s) ds\right) = \frac{f(t)}{t}$$

Example 3.3 Evaluate the following

$$1) L^{-1}\left(\frac{1}{s^5}\right)$$

$$4) L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$2) L^{-1}\left(\frac{1}{s^2+64}\right)$$

$$5) L^{-1}\left(\frac{(s+1)^3}{s^4}\right)$$

$$3) L^{-1}\left(\frac{3s+5}{s^2+7}\right)$$

$$6) L^{-1}\left(\frac{3s-137}{s^2+2s+401}\right)$$

Solution:

- 1) Since $t^n = L^{-1}\left(\frac{n!}{s^{n+1}}\right)$

We identify $n+1 = 5$ and $n = 4$, then multiply and divided by $4!$

$$L^{-1}\left(\frac{1}{s^5}\right) = \frac{1}{4!} L^{-1}\left(\frac{4!}{s^5}\right) = \frac{1}{24} t^4$$

- 2) Since $\sin(at) = L^{-1}\left(\frac{a}{s^2+a^2}\right)$

Here $a = 8$, then we obtain

$$L^{-1}\left(\frac{1}{s^2+64}\right) = \frac{1}{8} L^{-1}\left(\frac{8}{s^2+64}\right) = \frac{1}{8} \sin(8t)$$

$$\begin{aligned}
3) \quad L^{-1}\left(\frac{3s+5}{s^2+7}\right) &= L^{-1}\left(\frac{3s}{s^2+7}\right) + L^{-1}\left(\frac{5}{s^2+7}\right) \\
&= 3L^{-1}\left(\frac{s}{s^2+7}\right) + 5L^{-1}\left(\frac{1}{s^2+7}\right) \\
&= 3L^{-1}\left(\frac{s}{s^2+(\sqrt{7})^2}\right) + \frac{5}{\sqrt{7}}L^{-1}\left(\frac{\sqrt{7}}{s^2+(\sqrt{7})^2}\right) \\
&= 3\cos(\sqrt{7}t) + \frac{5}{\sqrt{7}}\sin(\sqrt{7}t)
\end{aligned}$$

$$4) \text{ Let } f(t) = \sin(at) = L^{-1}\left(\frac{a}{s^2+a^2}\right), \text{ so that } L[f(t)] = L[\sin(at)] = \frac{a}{s^2+a^2} = F(s),$$

By property, we have

$$\begin{aligned}
tf(t) &= L^{-1}\left(-\frac{d}{ds}F(s)\right) \\
t\sin(at) &= L^{-1}\left(-\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right) \\
&= L^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right) \\
\frac{1}{2a}t\sin(at) &= L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)
\end{aligned}$$

$$\begin{aligned}
5) \quad L^{-1}\left(\frac{(s+1)^3}{s^4}\right) &= L^{-1}\left(\frac{s^3+3s^2+3s+1}{s^4}\right) \\
&= L^{-1}\left(\frac{s^3}{s^4} + \frac{3s^2}{s^4} + \frac{3s}{s^4} + \frac{1}{s^4}\right) \\
&= L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{3}{s^2}\right) + L^{-1}\left(\frac{3}{s^3}\right) + L^{-1}\left(\frac{1}{s^4}\right) \\
f(t) &= 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3
\end{aligned}$$

$$\begin{aligned}
6) \quad L^{-1}\left(\frac{3s-137}{s^2+2s+401}\right) &= L^{-1}\left(\frac{3(s+1)-140}{(s+1)^2+20^2}\right) \\
&= L^{-1}\left(\frac{3(s+1)}{(s+1)^2+20^2}\right) - L^{-1}\left(\frac{140}{(s+1)^2+20^2}\right) \\
&= (3\cos(20t) - 7\sin(20t))e^{-t}
\end{aligned}$$

Exercise 3.2 Evaluate

$$a) \quad L^{-1}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+2}\right)$$

$$b) \quad L^{-1}\left(\frac{s}{s^2+2s-3}\right)$$

$$c) \quad L^{-1}\left(\frac{s}{s^2+4s+13}\right)$$

$$d) \quad L^{-1}\left(\frac{2s+4}{s^2+4s+19}\right)$$

$$e) \quad L^{-1}\left(\frac{1}{s-4} - \frac{8}{(s-4)^2}\right)$$

$$f) \quad L^{-1}\left(\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right)$$

Exercise 3.3 Find the inverse Laplace transform of $\ln\left(1 + \frac{1}{s}\right)$.

3.3 Laplace Transform of Derivatives

Theorem 3.3.1 If $f(t)$, $f'(t)$, $f''(t)$, ..., $f^{n-1}(t)$ are continuous on $[0, \infty]$ and are of exponential and if $f^n(t)$ is piecewise continuous on $[0, \infty]$, then

$$L[f^n(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0) \quad (3.4)$$

Where $L[f(t)] = F(s)$

Remark: The Laplace Transform of the first and second derivatives of $f(t)$ satisfies:

- a) $L[f'(t)] = sF(s) - f(0)$
- b) $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

Example 3.3 Find the Laplace transform of

- a) $f(t) = \sin^2(t)$
- b) $f(t) = t\sin(at)$
- c) $f(t) = \cos^2(t)$

Solution:

- a) We have $f(0) = 0$ and $f'(t) = 2 \sin(t) \cos(t) = \sin(2t)$

By the above theorem, we obtain

$$L[f'(t)] = sF(s) - f(0)$$

$$L[\sin(2t)] = sF(s) - 0$$

$$\frac{2}{s^2+4} = sF(s)$$

$$F(s) = \frac{2}{s(s^2+4)}$$

$$\text{Therefore } F(s) = L[\sin^2(t)] = \frac{2}{s(s^2+4)}$$

- b) We obtain $f(0) = 0$, $f'(t) = \sin(t) + at\cos(at)$ and $f'(0) = 0$
 $f''(t) = 2 \cos(at) - a^2 t \sin(at)$

From (3.4) we get

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$$L[2 \cos(at) - a^2 t \sin(at)] = s^2(t \sin(at)) - s \cdot 0 - 0$$

$$L[2 \cos(at)] - a^2 L[t \sin(at)] = s^2(t \sin(at))$$

$$\frac{2as}{s^2+a^2} = (s^2 + a^2)L[t \sin(at)]$$

$$L[t \sin(at)] = \frac{2as}{(s^2+a^2)^2}$$

Example 3.4 Solve the IVP by using Laplace transform

- a) $y'' + 4y = 0, y(0) = 1, y'(0) = 2$
 b) $y'' - 3y' + 2y = e^{3t}, y(0) = 1, y'(0) = 0$

Solution:

- a) We apply both sides Laplace Transforms

$$L[y'' + 4y] = L[0]$$

$$L[y''] + L[4y] = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 0$$

$$s^2Y(s) - s - 2 + 4Y(s) = 0$$

$$(s^2 + 4)Y(s) = s + 2$$

$$Y(s) = \frac{s+2}{s^2+4}$$

$$Y(t) = L^{-1}\left(\frac{s+2}{s^2+4}\right)$$

$$Y(t) = L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$Y(t) = \cos(2t) + \sin(2t)$$

- b) $y'' - 3y' + 2y = e^{3t}, y(0) = 1, y'(0) = 0$ Taking Laplace transform both sides

$$L[y'' - 3y' + 2y] = L[e^{3t}]$$

$$L[y''] - 3L[y'] + L[2y] = L[e^{3t}]$$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s-3}$$

$$s^2Y(s) - s - 0 - 3(sY(s) - 1) + 2Y(s) = \frac{1}{s-3}$$

$$(s^2 - 3s + 2)Y(s) - s + 3 = \frac{1}{s-3}$$

$$(s^2 - 3s + 2)Y(s) = (s - 3) + \frac{1}{s-3}$$

$$Y(s) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

By partial fraction

$$\text{Let } \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\text{Therefore } A = \frac{5}{2}, B = -2, C = \frac{1}{2}$$

$$Y(s) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{\frac{5}{2}}{s-1} + \frac{-2}{s-2} + \frac{\frac{1}{2}}{s-3}$$

$$\text{Finally, } Y(t) = \frac{5}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}$$

Example 3.5 solve

a) $ty'' - ty' - y = 0, y(0) = 0, y'(0) = 3$

Solution:

a) $ty'' - ty' - y = 0, y(0) = 0, y'(0) = 3$, taking both sides Laplace transform

$$L[ty'' - ty' - y] = L[0]$$

$$L[ty''] - L[ty'] - L[y] = 0$$

$$-\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) - \left(-\frac{d}{ds}(sY(s) - y(0))\right) - Y(s) = 0$$

$$-\frac{d}{ds}(s^2Y(s) - s \cdot 0 - 3) - \left(-\frac{d}{ds}(sY(s) - 0)\right) - Y(s) = 0$$

$$-(s^2 - s)Y'(s) - 2sY(s) = 0$$

$$\frac{Y'}{Y} + \frac{2}{s-1} ds = 0 \quad (\text{Separated})$$

Integrating both sides w.r.t Y and s, respectively, we have

$$\ln(Y) + 2 \ln(s - 1) = c$$

$$(s - 1)^2 Y = c$$

$$Y = \frac{c}{(s-1)^2}$$

Taking both sides inverse Laplace Transform, we get

$$y(t) = L^{-1}\left(\frac{c}{(s-1)^2}\right)$$

$$y(t) = cte^t$$

$$y'(0) = 3 \quad \text{To find } c, y'(t) = ce^t + cte^t$$

$$y'(0) = c + 0$$

$$c = 3$$

$$\text{Therefore } y(t) = 3te^t$$

Exercise 3.4 Solve the following using Laplace transforms

1) $y'' - 6y' + 5y = 0, y(0) = 1, y'(0) = -3$

2) $y'' - 2y' + 2y = \cos(t), y(0) = 1, y'(0) = 0$

3) $2y'' + ty' - 2y = 10, y(0) = y'(0) = 0$

4) $4ty'' + 2y' + y = 0, y(0) = y'(0) = 0$

- 5) $\begin{cases} x' = 4x - 2y, & x(0) = 2 \\ y' = 5x + 2y, & y(0) = -2 \end{cases}$
- 6) $\begin{cases} x' = -3x + 4y + \cos(t), & x(0) = 0 \\ y' = -2x + 3y + t, & y(0) = 1 \end{cases}$
- 7) $\begin{cases} x' = x - 2y + t^2, & x(0) = 1 \\ y' = 4x + 5y - e^t, & y(0) = -1 \end{cases}$

3.4 Convolution

Definition 3.4.1 If functions $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$, then the convolution of $f(t)$ and $g(t)$, denoted by $f * g$, is defined by the integral

$$f * g = \int_0^t f(\tau)g(t - \tau)d\tau \quad (3.5)$$

Example 3.6 Evaluate

- a) $e^t * \sin(t)$
b) $e^{at} * 1$

Solution:

- a) By definition, we have

$$\begin{aligned} e^t * \sin(t) &= \int_0^t e^\tau \sin(t - \tau)d\tau \\ &= \frac{1}{2}(e^t - \sin t - \cos t) \end{aligned}$$

$$\begin{aligned} \text{b) } e^{at} * 1 &= \int_0^t e^{a\tau} d\tau = \frac{1}{a}(e^{a\tau})|_{\tau=0}^{\tau=t} \\ &= \frac{1}{a}(e^{at} - 1) \end{aligned}$$

Properties of convolution

- 1) $f * g = g * f$
- 2) $f * (g * h) = (f * g) * h$
- 3) $f * (g + h) = (f * g) + (f * h)$
- 4) $f * 0 = 0 * f = 0$

Theorem 3.4.1 (Convolution Theorem)

If functions $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$, and are of exponential order, Then

$$\begin{aligned} L[f * g] &= L\left[\int_0^t f(\tau)g(t - \tau)d\tau\right] \\ &= F(s)G(s) \end{aligned} \quad (3.6)$$

Example 3.7 Find the Laplace Transform of each given convolution integral

- a) $f(t) = \int_0^t \tau^3 \sin(t - \tau) d\tau$
- b) $f(t) = \int_0^t \sinh(4\tau) \cosh 5(t - \tau) d\tau$
- c) $f(t) = \int_0^t e^{17(t-\tau)} \tau^{19} d\tau$

Solution: by the above theorem, we have

- a)
$$\begin{aligned} L[f(t)] &= \left[\int_0^t \tau^3 \sin(t - \tau) d\tau \right] \\ &= L[t^3] L[\sin t] \\ &= \left(\frac{3!}{s^{3+1}} \right) \left(\frac{1}{s^2+1} \right) \\ &= \frac{6}{(s^4)(s^2+1)} \end{aligned}$$
- b)
$$\begin{aligned} L[f(t)] &= L[\sinh(4t)] L[\cosh(5t)] \\ &= \left(\frac{4}{s^2-16} \right) \left(\frac{5}{s^2-25} \right) \\ &= \frac{20}{s^2-16(s^4)(s^2-25)} \end{aligned}$$
- c) Left as an exercise

Exercise 3.5 Evaluate

- a) $L[t^2 * \cos(t)]$
- b) $L[e^{-2t} \cos(t) * e^t]$

Theorem 3.4.2 (The Inverse Convolution Theorem)

Let $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then

$$L^{-1}(F(s)G(s)) = L^{-1}(F(s)) * L^{-1}(G(s)) \quad (3.7)$$

Example 3.8 Find the inverse Laplace Transform by using convolution for

- a) $\frac{1}{(s-1)^2}$
- b) $\frac{1}{s^2(s^2+1)}$
- c) $\frac{1}{(s^2+11)^2}$

Solution:

$$\begin{aligned}
 \text{a) } L^{-1}\left(\frac{1}{(s-1)^2}\right) &= L^{-1}\left(\left(\frac{1}{s-1}\right)\left(\frac{1}{s-1}\right)\right) \\
 &= L^{-1}\left(\frac{1}{s-1}\right) * L^{-1}\left(\frac{1}{s-1}\right) \\
 &= e^t * e^t \\
 &= \int_0^t e^\tau e^{t-\tau} d\tau \\
 &= \int_0^t e^t d\tau \\
 &= (\tau e^t) \Big|_{\tau=0}^{\tau=t} \\
 &= t e^t
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } L^{-1}\left(\frac{1}{(s^2)(s^2+1)}\right) &= L^{-1}\left(\left(\frac{1}{s^2}\right)\left(\frac{1}{s^2+1}\right)\right) \\
 &= L^{-1}\left(\frac{1}{s^2}\right) * L^{-1}\left(\frac{1}{s^2+1}\right) \\
 &= t * \sin(t) \\
 &= \int_0^t \tau \sin(t-\tau) d\tau \\
 &= (\tau \cos(t-\tau) + \sin(t-\tau)) \Big|_{\tau=0}^{\tau=t} \\
 &= (t \cos(0) - \sin(0)) - (0 - \sin(t)) \\
 &= t - \sin(t)
 \end{aligned}$$

c) Exercise

3.5 Integral Equations

An equation of the form

$$y(t) = f(t) + \lambda \int_0^t K(t, \tau) y(\tau) d\tau \quad (3.8)$$

Is called a **Volterra Integral Equation**, where λ is a parameter and $K(t, \tau)$ is called the Kernel of the integral equation. Equation of this type is often associated with the solution of initial value problems. The Laplace Transform is well suited to the solution of such integral equations when the Kernel $K(t, \tau)$ has a special form that depends on t and τ only through the difference $t - \tau$, because then $K(t, \tau) = K(t - \tau)$ and the integral in (3.8) becomes a convolution integral.

Example 3.9 Solve the Volterra integral equation

$$y(t) = 2e^{-t} + \int_0^t \sin(t - \tau) y(\tau) d\tau$$

Solution: Taking both sides Laplace Transform, we have

$$L[y(t)] = L\left[2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau\right]$$

$$Y(s) = \frac{2}{s+1} + L\left[\int_0^t \sin(t-\tau)y(\tau)d\tau\right]$$

$$Y(s) = \frac{2}{s+1} + \frac{Y(s)}{s^2+1}$$

$$\left(\frac{s^2}{s^2+1}\right)Y(s) = \frac{2}{s+1}$$

$$Y(s) = \frac{2s^2+2}{s^2(s+1)}$$

$$Y(s) = \frac{-2}{s} + \frac{2}{s^2} + \frac{4}{s-1}$$

$$y(t) = L^{-1}\left(\frac{-2}{s} + \frac{2}{s^2} + \frac{4}{s-1}\right)$$

$$y(t) = -2 + 2t + 4e^t \text{ for } t > 0$$

Example 3.10 Solve the integro-differential equation

$$y'' + y = \int_0^t \sin(\tau)y(t-\tau)d\tau \quad y(0) = 1, y'(0) = 0$$

Solution: Taking the Laplace transform in the usual way gives

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = L\int_0^t \sin(\tau)y(t-\tau)d\tau$$

$$s^2Y(s) - s - 0 + Y(s) = \frac{Y(s)}{s^2+1}$$

$$(s^2 + 1)Y(s) - s = \frac{Y(s)}{s^2+1}$$

$$\left((s^2 + 1) - \frac{1}{s^2+1}\right)Y(s) = s$$

$$\left(\frac{s^4+2s^2}{s^2+1}\right)Y(s) = s$$

$$Y(s) = \frac{s^2+1}{s(s^2+2)}$$

$$y(t) = L^{-1}\left(\frac{s^2+1}{s(s^2+2)}\right)$$

$$y(t) = L^{-1}\left(\frac{1}{s} + \frac{\frac{1}{2}s}{s^2+2}\right)$$

$$y(t) = \frac{1}{2}(1 + \cos(\sqrt{2}t))$$

Exercise 3.6 Solve

- a) $y(t) = \sin(t) + \int_0^t \sin(t - \tau) y(\tau) d\tau$
- b) $y(t) = \cos(t) + \int_0^t \sin(2(t - \tau)) y(\tau) d\tau$
- c) $y(t) = t^2 + \int_0^t \cos(t - \tau) y(\tau) d\tau$
- d) $y' + 4y = 4 \int_0^t \sin(\tau) y(t - \tau) d\tau \quad y(0) = 1$
- e) $y'' - 4y = 2 \int_0^t \sin(2\tau) y(t - \tau) d\tau \quad y(0) = 1, y'(0) = 0$

3.6 Integration of Laplace Transform

Theorem 3.6.1 The Transform of an integral

Let $f(t)$ be a piecewise continuous function such that $|f(t)| \leq Me^{kt}$ for $k > 0$ and all $t \geq 0$. Then, if $L[f(t)] = F(s)$,

$$L \int_0^t f(\tau) d\tau = \frac{F(s)}{s} \quad \text{for } s > k \quad (3.9)$$

And, conversely,

$$L^{-1} \left(\frac{F(s)}{s} \right) = \int_0^t f(\tau) d\tau \quad (3.10)$$

Example 3.11 Evaluate

- a) $L^{-1} \left(\frac{1}{s(s^2+1)} \right)$
- b) $L^{-1} \left(\frac{1}{(s^2)(s^2+1)} \right)$
- c) $L^{-1} \left(\frac{1}{(s^3)(s^2+1)} \right)$

Solution:

- a) $L^{-1} \left(\frac{1}{s(s^2+1)} \right) = L^{-1} \left(\frac{\frac{1}{s^2+1}}{s} \right)$, here $F(s) = \frac{1}{s^2+1}$, $L^{-1} \left(\frac{1}{s^2+1} \right) = \sin(t)$
 $= \int_0^t \sin(\tau) d\tau$
 $= -\cos(\tau) \Big|_{\tau=0}^{\tau=t}$
 $= 1 - \cos(t)$
- b) $L^{-1} \left(\frac{1}{(s^2)(s^2+1)} \right) = L^{-1} \left(\frac{\frac{1}{s(s^2+1)}}{s} \right) = L^{-1} \left(\frac{\frac{1}{s} - \frac{s}{(s^2+1)}}{s} \right) = \int_0^t (1 - \cos(\tau)) d\tau = t - \sin(t)$

$$\begin{aligned} \text{c) } L^{-1}\left(\frac{1}{(s^3)(s^2+1)}\right) &= L^{-1}\left(\frac{\frac{1}{s^2(s^2+1)}}{s}\right) = L^{-1}\left(\frac{\frac{1}{s^2} - \frac{1}{(s^2+1)}}{s}\right) = \int_0^t (\tau - \sin(\tau)) d\tau \\ &= \frac{t^2}{2} + \cos(t) - 1 \end{aligned}$$

Exercise 3.7 Evaluate

$$\text{a) } L^{-1}\left(\frac{1}{s(s-1)}\right)$$

$$\text{b) } L^{-1}\left(\frac{1}{s^2(s-1)}\right)$$

3.7 Application to evaluate integral

We can use the Laplace Transforms of functions for evaluating complicated integrals in easy manner as illustrated by the following examples.

Example 3.12 evaluate $\int_0^\infty t^2 e^{-t} \sin(2t) dt$ by using Laplace Transform

Solution: we know that $L[\sin(2t)] = \frac{2}{s^2+4} = F(s)$

By multiplication property, we have

$$\begin{aligned} L[t^2 \sin(2t)] &= (-1)^2 \frac{d^2}{ds^2} F(s) \\ &= \frac{d^2}{ds^2} \left(\frac{2}{s^2+4} \right) \\ &= \frac{12s^2-16}{(s^2+4)^3} \end{aligned}$$

By definition of Laplace transform

$$\begin{aligned} \int_0^\infty t^2 e^{-st} \sin(2t) dt &= L[t^2 \sin(2t)] \\ &= \frac{12s^2-16}{(s^2+4)^3}, \text{ when } s = 1, \text{ we have} \\ \int_0^\infty t^2 e^{-t} \sin(2t) dt &= \frac{12-16}{(1+4)^3} \\ &= -\frac{4}{125} \end{aligned}$$

Example 3.13 Prove that $\int_0^{\infty} \frac{e^{-t}-e^{-3t}}{t} dt = \ln(3)$

Solution: apply the property of division

$$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)ds$$

Taking $f(t) = e^{-t} - e^{-3t}$ we have

$$F(s) = L[e^{-t} - e^{-3t}] = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\begin{aligned} \text{Then } L\left[\frac{e^{-t}-e^{-3t}}{t}\right] &= \int_s^{\infty} \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds \\ &= (\ln(s+1) - \ln(s+3))|_s^{\infty} \\ &= -\ln\left(\frac{s+1}{s+3}\right) \\ &= \ln\left(\frac{s+3}{s+1}\right) \end{aligned}$$

By definition we have $\int_0^{\infty} e^{-st} \left(\frac{e^{-t}-e^{-3t}}{t}\right) dt = \ln\left(\frac{s+3}{s+1}\right)$

$$\int_0^{\infty} \left(\frac{e^{-t}-e^{-3t}}{t}\right) dt = \ln(3) \text{ , when } s = 0$$

Exercise 3.8

- Prove that $\int_0^{\infty} \frac{e^{-t} \sin^2(t)}{t} dt = \frac{1}{4} \ln(5)$
- Evaluate $\int_0^{\infty} \frac{\sin(mt)}{t} dt$
- Evaluate $\int_0^{\infty} t e^{-t} \sin^4(t) dt$

3.8 Fourier Transform (Reading assignment For student's)