CHAPTER 5

PRESSURE VESSELS

1. Introduction

The cylindrical pressure vessels are divided into two groups - **thin** and **thick** cylinders. A cylinder is considered thin when the ratio of its inner diameter to the wall thickness is more than 15. Boiler shells, pipes, tubes, and storage tanks are treated as thin cylinders.
When the ratio of the inner diameter of the cylinder to the wall thickness is less than 15, the cylinder is called a thick-walled cylinder or simply thick cylinder. Hydraulic cylinders, high pressure pipes and gun barrels are examples of thick cylinders.
The difference between the analysis of stresses in thin and thick cylinders is as follows:

(i) In thin cylinders, it is assumed that the tangential stress (σ_t) is uniformly distributed over the cylinder wall thickness. In thick cylinders, the tangential stress (σ_t) has highest magnitude at the inner surface of the cylinder and gradually decreases towards the outer surface.
(ii) The radial stress (σ_r) is neglected in thin cylinders, while it is of significant magnitude in case of thick cylinders.

2. Thin cylindrical vessel subjected to internal pressure

Fig. 1 shows a thin cylindrical vessel in which a fluid under pressure is stored.

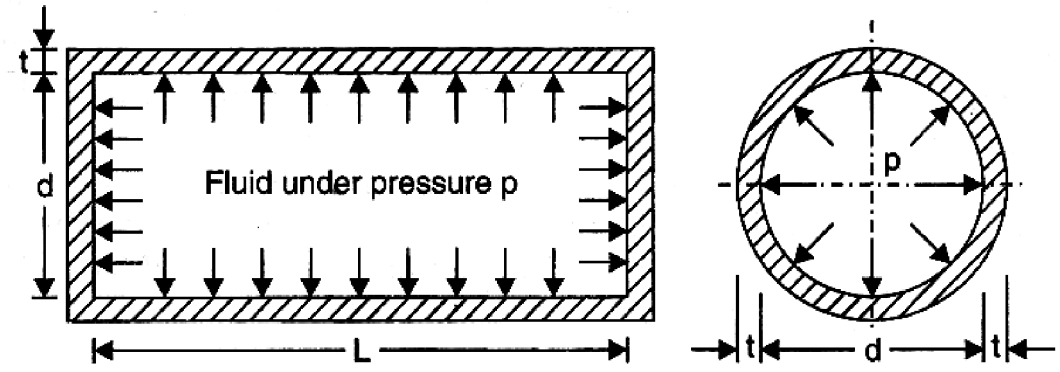


Fig. 1

On account of the internal pressure p, the cylindrical vessel may fail by splitting up in any one of the two ways as shown in Fig. 2 (a) and 2 (b).
The forces, due to pressure of the fluid acting vertically upwards and downwards on the thin cylinder, tend to burst the cylinder as shown in Fig. 2 (a).
The forces, due to pressure of the fluid, acting at the ends of the thin cylinder, tend to burst the thin cylinder as shown in Fig. 2 (b).
Let d = Internal diameter of the thin cylinder
t = Thickness of the wall of the cylinder
p = Internal pressure of the fluid
L = Length of the cylinder.

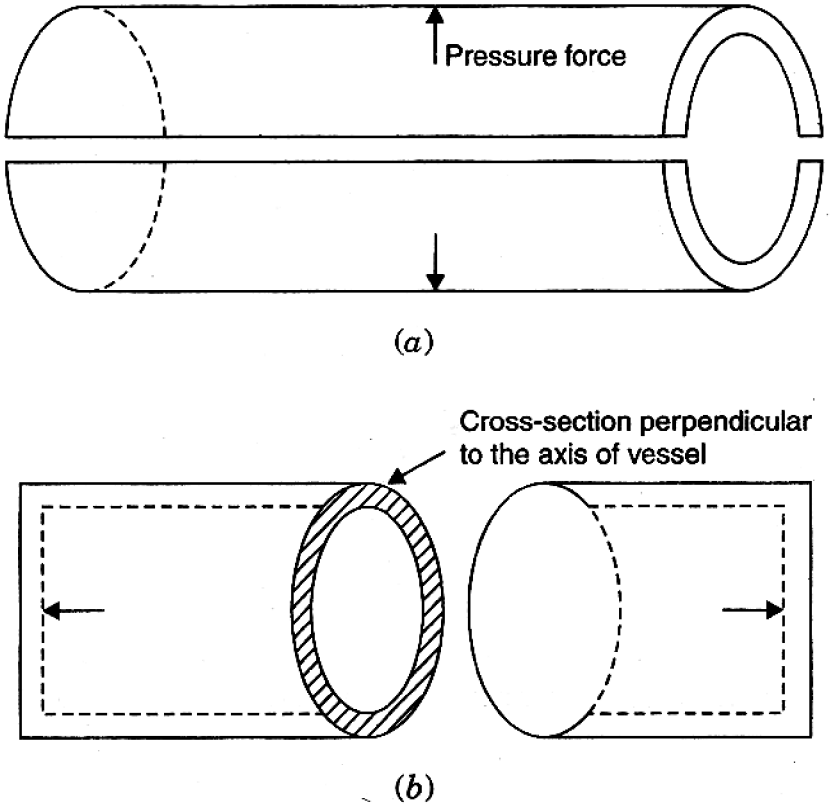


Fig. 2

3. Stresses in a thin cylindrical vessel subjected to internal pressure

When a thin cylindrical vessel is subjected to internal fluid pressure, the stresses in the wall of the cylinder on the cross-section along the axis and on the cross-section perpendicular to the axis are set up. These stresses are tensile and are known as
1. Circumferential stress (or hoop stress) and
2. Longitudinal stress.
The name of the stress, is given according to the direction in which the stress is acting. The stress acting along the circumference of the cylinder is called circumferential stress whereas the stress acting along the length of the cylinder (i.e., in the longitudinal

direction) is known as longitudinal stress. The circumferential stress is also known as hoop stress. The stress set up in Fig. 2 (a) is circumferential stress whereas the stress set up in Fig. 2 (b) is longitudinal stress.

4. Expression for circumferential stress (or hoop stress)

Consider a thin cylindrical vessel subjected to an internal fluid pressure. The circumferential stress will be set up in the material of the cylinder, if the bursting of the cylinder takes place as shown in Fig. 3 (a).

The expression for hoop stress or circumferential stress is obtained as given below.

Let p = Internal pressure of fluid
d = Internal diameter of the cylinder
t = Thickness of the wall of the cylinder
σ_t = Circumferential or hoop stress in the material.

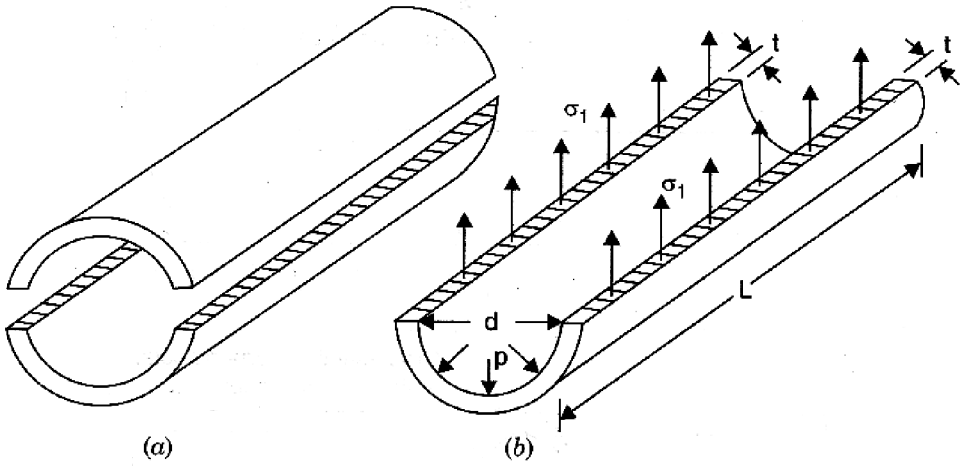


Fig. 3

The bursting will take place if the force due to fluid pressure is more than the resisting force due to circumferential stress set up in the material. In the limiting case, the two forces should be equal.

Force due to fluid pressure = p x Area on which p is acting
= p x (d x L) (i)
(p is acting on projected area d x L)

Force due to circumferential stress
= σ_t x Area on which σ_t is acting
= σ_t x (L x t + L x t)
= σ_t x 2Lt = 2σ_t x L x t (ii)

Equating (i) and (ii), we get
p x d x L = 2σ_t x L x t
σ_t = pd/2t (cancelling L) ...(1)

This stress is tensile as shown in Fig. 3 (b).

5. Expression for longitudinal stress

Consider a thin cylindrical vessel subjected to internal fluid pressure. The longitudinal stress will be set up in the material of the cylinder, if the bursting of the cylinder takes place along the section AB of Fig. 4 (a).

Let p = Internal pressure of fluid stored in thin cylinder
d = Internal diameter of cylinder
t = Thickness of the cylinder
σ_L= Longitudinal stress in the material.

The bursting will take place if the force due to fluid pressure acting on the ends of the cylinder is more than the resisting force due to longitudinal stress σ_L developed in the material as shown in Fig. 4 (b). In the limiting case, both the forces should be equal.
Force due to fluid pressure = p x Area on which p is acting

$$= p \times \frac{\pi}{4} d^2$$

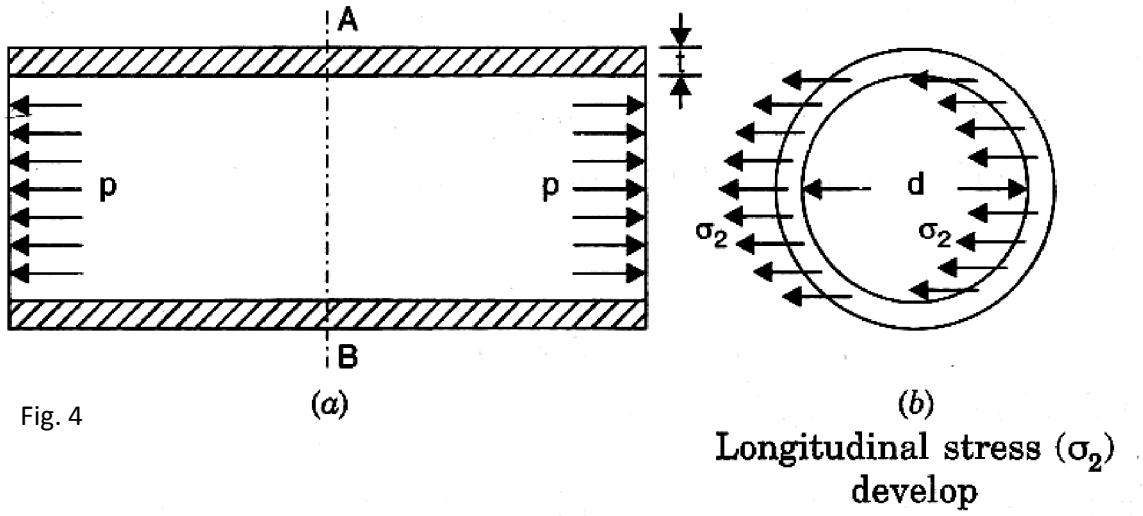


Fig. 4

Resisting force = σ_LX Area on which σ_L is acting

$$= \sigma_L \text{ X } \pi d \text{ x } t$$

:. Hence in the limiting case
Force due to fluid pressure = Resisting force

$$p \times \frac{\pi}{4} d^2 = \sigma_L \times \pi d \times t$$

$$\therefore \quad \sigma_L = \frac{p \times \frac{\pi}{4} d^2}{\pi d \times t} = \frac{pd}{4t} \qquad (2)$$

The stress σ_L is also tensile. Equation (2) can be written as

$$\sigma_L = \frac{pd}{2 \times 2t} = \frac{1}{2} \times \sigma_t \qquad \left(\because \quad \sigma_t = \frac{pd}{2t} \right)$$

or Longitudinal stress = Half of circumferential stress.

This also means that circumferential stress is two times the longitudinal stress (σ_L). Hence in the material of the cylinder the permissible stress should be less than the circumferential stress. Or in other words, the circumferential stress should not be greater than the permissible stress.

Maximum shear stress: At any point in the material of the cylindrical shell, there are two principal stresses, namely a circumferential stress of magnitude $\sigma_t = \frac{pd}{2t}$ acting circumferentially and a longitudinal stress of magnitude $\sigma_L = \frac{pd}{4t}$ acting parallel to the axis of the shell. These two stresses are tensile and perpendicular to each other.

$\therefore$ Maximum shear stress $$\tau_{\max} = \frac{\sigma_t - \sigma_L}{2} = \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2} = \frac{pd}{8t} \qquad \text{(2A)}$$

Note. (i) If the thickness of the thin cylinder is to be determined then equation (1) should be used.
(ii) If maximum permissible stress in the material is given. This stress should be taken circumnateferential stress (σ_t).
(iii) While using equations (1) and (2), the units of p, σ_1 and σ_2 should be same. They should be expressed either in N/mm^2 or N/m^2. Also the units of d and t should be same. They may be in metre (m) or millimetre (mm).

Examples

1. A cylindrical pipe of diameter 1.5 m and thickness 1.5 cm is subjected to an internal fluid pressure of 1.2 N/mm^2. Determine:
(i) Longitudinal stress developed in the pipe, and
(ii) Circumferential stress developed in the pipe.

Solution: Given: Diameter of pipe, d = 1.5 m, Thickness, t = 1.5 cm = 1.5 x 10^{-2} m
Internal fluid pressure, p = 1.2 N/mm^2
As the ratio

$\frac{t}{d} = \frac{1.5 \times 10^{-2}}{1.5} = \frac{1}{100}$, which is less than 1/15, hence this is a case of thin cylinder.

Here unit pressure (p) is in N/ mm^2. Hence the unit of σ_t and σ_L will also be in N/mm^2.

(1) The longitudinal stress (σ_L) is given by equation (2) as,

$$\sigma_2 = \frac{p \times d}{4t}$$

$$= \frac{1.2 \times 1.5}{4 \times 1.5 \times 10^{-2}} = \mathbf{30\ N/mm^2.}\quad \mathbf{Ans.}$$

(ii) The circumferential stress(σ_t) is given by equation (1) as

$$\sigma_1 = \frac{pd}{2t}$$

$$= \frac{1.2 \times 1.5}{2 \times 1.5 \times 10^{-2}} = \mathbf{60\ N/mm^2.}\quad \mathbf{Ans.}$$

2. A cylinder of internal diameter 2.5 m and of thickness 5 cm contains a gas. If the tensile stress in the material is not to exceed 80 N/mm^2, determine the internal pressure of the gas.

Solution: Given: Internal dia. of cylinder, d = 2.5 m, Thickness of cylinder, t = 5 cm = 5 x 10^{-2}m, Maximum permissible stress = 80 N/mm^2,
As maximum permissible stress is given. Hence this should be equal to circumferential stress(σ_t).
We know that the circumferential stress should not be greater than the maximum permissible stress. Hence take circumferential stress equal to maximum permissible stress,

σ_t = 80 N/mm^2

Let p = Internal pressure of the gas

Using equation (1),

$$\sigma_1 = \frac{pd}{2t}$$

or

$$p = \frac{2t \times \sigma_1}{d} = \frac{2 \times 5 \times 10^{-2} \times 80}{2.5}$$ (Here unit of σ_1 is in N/mm^2, hence unit of p will also be in N/mm^2)

$$= \mathbf{3.2\ N/mm^2.}\quad \mathbf{Ans.}$$

3. A cylinder of internal diameter 0.50 m contains air at a pressure of 7N/mm^2 (gauge). If the maximum permissible stress induced in the material in 80 N/mm^2, find the thickness of the cylinder.

Solution: Given: Internal dia. of cylinder, d = 0.50 m, Internal pressure of air, p = 7 N/mm^2.

Maximum permissible stress in the material means the circumferential stress (σ_t).
As stated earlier that the circumferential stress should not be greater than the maximum permissible stress. Hence take circumferential stress equal to maximum permissible stress.
Circumferential stress, $\sigma_t = 80$ N/mm^2
Let t = Thickness of the cylinder
Using equation (1),

$$\sigma_1 = \frac{pd}{2t}$$

$$\therefore \quad t = \frac{pd}{2 \times \sigma_1} = \frac{7 \times 0.50}{2 \times 80} = 0.021875 \text{ m}$$

(Here p and σ_1 are in N/mm^2, d is in m hence t will be in m)

$$= 2.188 \text{ cm.} \quad \text{Ans.}$$

If the value of t is taken 2.1875 cm, the stress induced will be 80 N/mm^2. If the value of is less than 2.1875 cm, the stress induced will be more than 80 N/mm^2. But the stress induced should not be more than 80 N/mm^2. If the value of t is taken more than 2.1875 cm (say t = 2.188 cm), the stress induced will be less than 80 N/mm^2.
Hence take t = 2.188 cm or say 2.2 cm. Ans.

4. A thin cylinder of internal diameter 1.25 m contains a fluid at an internal pressure of 2 N/mm^2. Determine the maximum thickness of the cylinder if:
(i) The longitudinal stress is not to exceed 30 N/mm^2.
(ii) The circumferential stress is not to exceed 45 N/mm^2.
Solution: Given: Internal dia. of cylinder, d = 1.25 m, Internal pressure of fluid, p = 2 N/mm^2, Longitudinal stress, $\sigma_L = 30$ N/mm^2, Circumferential stress, $\sigma_t = 45$ N/mm^2
Using equation (1),

$$\sigma_1 = \frac{pd}{2t}$$

$$\therefore \quad t = \frac{p \times d}{2 \times \sigma_1} = \frac{2 \times 1.25}{2 \times 45} = 0.0277 \text{ m}$$

$$= 2.77 \text{ cm.} \qquad \ldots(i)$$

Using equation (2),

$$\sigma_2 = \frac{pd}{4t}$$

$$t = \frac{pd}{4 \times \sigma_2} = \frac{2 \times 1.25}{4 \times 30} = 0.0208 \text{ m}$$

$$= 2.08 \text{ cm.} \qquad \ldots(ii)$$

The longitudinal or circumferential stresses induced in the material are inversely proportional to the thickness (t) of the cylinder. Hence the stress induced will be less if the value of 't' is more. Hence take the maximum value of 't' calculated in equations (i) and (ii).
From the equations (i) and (ii) it is clear that t should not be less than 2.77 cm.
Take **t = 2.80 cm. Ans.**

5. A water main 80cm diameter contains water at a pressure head of l00m. If the weight density of water is 9810 N/m^3, find the thickness of the metal required for the water main. Given the permissible stress as 20 N/mm^2.

Solution: Given: Dia. of main, d = 80 cm, Pressure head of water, h = 100 m
Weight density of water, w = ρ x g = 1,000 x 9.81 = 9810 N/m^3, Permissible stress = 20 N/mm^2, Permissible stress is equal to circumferential stress (σ_t), or σ_t = 20 N/mm^2
Pressure of water inside the water main,

P = ρ x g x h = wh = 9810 x 100N/m^2

Here σ_t is in N/mm^2, hence pressure (p) should also be N/mm^2. The value of p in N/mm^2 is given as

$$p = \frac{9810 \times 100}{1000^2 \text{ mm}^2} \text{ N/mm}^2 \qquad (\because \quad 1 \text{ m} = 1000 \text{ mm})$$
$$= 0.981 \text{ N/mm}^2$$

Let t = Thickness of the metal required.
Using equation (1),

$$\sigma_1 = \frac{p \times d}{2 \times t} \qquad \text{(Here 'd' is in cm hence 't' will also be in cm)}$$

$$\therefore \quad t = \frac{p \times d}{2 \times \sigma_1} = \frac{0.981 \times 80}{2 \times 20} = 2 \text{ cm.} \quad \textbf{Ans.}$$

6. Efficiency of a joint

The cylindrical shells such as boilers are having two types of joints namely longitudinal joint and circumferential joint. In case of a joint, holes are made in the material of the shell for the rivets. Due to the holes, the area offering resistance decreases. Due to the decrease in area, the stress (which is equal to the force divided by the area) developed in the material of the shell will be more.
Hence in case of riveted shell the circumferential and longitudinal stresses are greater than what are given by equations (1) and (2). If the efficiency of a longitudinal joint and circumferential joint are given then the circumferential and longitudinal stresses are obtained as:
Let η_t = Efficiency of a longitudinal joint, and
η_c = Efficiency of the circumferential joint.

Then the circumferential stress ($б_t$) is given as

$$\sigma_t = \frac{p \times d}{2t \times \eta_l} \qquad (3)$$

and the longitudinal stress ($б_L$) is given as

$$\sigma_L = \frac{p \times d}{4t \times \eta_c} \qquad (4)$$

Note. (i) In longitudinal joint, the circumferential stress is developed whereas in circumferential joint, the longitudinal stress is developed.
(ii) Efficiency of a joint means the efficiency of a longitudinal joint.
(iii) If efficiencies of a joint are given, the thickness of the thin shell is determined from equation (3).

Examples

6. A boiler is subjected to an internal steam pressure of 2 N/mm^2. The thickness of boiler plate is 2.6 cm and permissible tensile stress is 120 N/mm^2. Find out the maximum diameter, when efficiency of longitudinal joint is 90% and that of circumferential joint is 40%.

Solution. Given: Internal steam pressure, p = 2 N/mm^2, Thickness of boiler plates, t = 2.0 cm, Permissible tensile stress = 120 N/mm^2;
In case of joints, the permissible stress may be circumferential stress or longitudinal stress.
Efficiency of longitudinal joint, η_t = 90% = 0.90
Efficiency of circumferential joint, η_c = 40% = 0.40.
Max. diameter for circumferential stress is given by equation (3).
Using equation (3

$$\sigma_t = \frac{p \times d}{2 \times \eta_l \times t}$$

where σ_t = Given permissible stress = 120 N/mm^2

$$\therefore \quad 120 = \frac{2 \times d}{2 \times 0.90 \times 2.0}$$ (Here p and σ_1 are in same units. Thickness is in cm hence 'd' will be in cm)

$$\therefore \quad d = \frac{120 \times 2 \times 0.9 \times 2.0}{2} = 216.0 \text{ cm.} \qquad ...(i)$$

Max. diameter for longitudinal stress is given by equation (4).
Using equation (4),

$$\sigma_L = \frac{p \times d}{4 \times \eta_c \times t}$$

where σ_L = Given permissible stress
= 120 N/mm^2.

$$\therefore \quad 120 = \frac{2 \times d}{4 \times 0.4 \times 2.0}$$

$$\therefore \quad d = \frac{120 \times 4 \times 0.4 \times 2.0}{2} = 192 \text{ cm.} \qquad ...(ii)$$

The longitudinal or circumferential stresses induced in the material are directly proportional to diameter (d). Hence the stress induced will be less if the value of 'd' is less. Hence take the minimum value of 'd' calculated from equations (i) and (ii).

Maximum diameter of the boiler is equal to the minimum value of diameter given by equation (i) and (ii).
Hence maximum diameter, **d = 192 cm. Ans.**

(Please note that if d is taken as equal to 216.0 cm, the longitudinal stress (σ_L) will be more than the given permissible value as shown below:

$$\sigma_L = \frac{p \times d}{4 \times \eta_c \times t} = \frac{2 \times 216}{4 \times 0.4 \times 2.0} = 135 \ N/m^2)$$

7. A boiler shell is to be made of 15mm thick plate having a limiting tensile stress of 120 N/mm2. If the efficiencies of the longitudinal and circumferential joints are 70% and 30% respectively determine

(i) The maximum permissible diameter of the shell for an internal pressure of 2 N/mm^2, and
(ii) Permissible intensity of internal pressure when the shell diameter is 1.5 m.

Solution: Given:

Thickness of boiler shell, t = 15 mm

Limiting tensile stress = 120 N/mm^2

Limiting tensile stress may be circumferential stress or longitudinal stress.

Efficiency of longitudinal joint, η_L = 70% = 0.70

Efficiency of circumferential joint, η_t = 30% = 0.30.

(i) Maximum Permissible diameter for an internal pressure,

p = 2 N/mm^2.

The boiler shell should be designed for the limiting tensile stress of 120 N/mm^2. First consider the limiting tensile stress as circumferential stress and then as longitudinal stress. The minimum diameter of the two case will satisfy the condition.

(a) Taking limiting tensile stress = Circumferential stress = $б_t$ =120 N/mm^2.

But $б_t$ is also given by equation (3) as

$$\sigma_t = \frac{p \times d}{2 \times \eta_t \times t} \Rightarrow 120 = \frac{2 \times d}{2 \times 0.7 \times 15} \qquad (Here\ t\ is\ in\ mm\ hence\ 'd'\ will\ be\ in\ mm)$$

$$\therefore d = \frac{120 \times 2 \times 0.7 \times 15}{2} = 1260\,mm \qquad (i)$$

(b) Taking limiting tensile stress = Longitudinal stress

$б_L$= 120 N/mm^2.

Using equation (4),

$$\sigma_L = \frac{p \times d}{4 \times \eta_c \times t} \Rightarrow 120 = \frac{2 \times d}{4 \times 0.30 \times 15}$$

$$\therefore \qquad d = \frac{120 \times 4 \times 0.3 \times 15}{2} = 1080\ mm \qquad (ii)$$

Thus the maximum diameter of the shell, in order both the conditions may be satisfied

= 1080 mm. **Ans.**

(ii) Permissible intensity of internal pressure when the shell diameter is 1.5 m.

or d =1.5 m = 1500 mm.

(a) Taking limiting tensile stress = Circumferential stress ($б_t$)

= 120 N/mm^2

Using equation (3),

$$\sigma_t = \frac{p \times d}{2 \times \eta_t \times t}$$

or $$120 = \frac{p \times 1500}{2 \times 0.7 \times 15}$$ (Here σ_1 is in N/mm^2 hence 'p' will be in N/mm^2)

$$\therefore \quad p = \frac{120 \times 2 \times 0.7 \times 15}{1500} = 1.68 \text{ N/mm}^2 \qquad ...(i)$$

(b) Taking limiting tensile stress = Longitudinal stress ($б_l$) = 120 N/mm^2
Using equation (4),

$$\sigma_L = \frac{p \times d}{4 \times \eta_c \times t}$$

$$120 = \frac{p \times 1500}{4 \times 0.30 \times 15}$$

$$\therefore \quad p = \frac{120 \times 4 \times 0.30 \times 15}{1500} = 1.44 \text{ N/mm}^2 \qquad ...(ii)$$

Hence in order both the conditions may be satisfied the maximum permissible internal pressure is equal to the minimum value of pressure given by (i) and (ii).

Maximum permissible internal pressure = 1.44 N/mm^2
If p is taken equal to 1.68 N/mm^2, then longitudinal stress $б_L$ will be,

$$\sigma_L = \frac{p \times d}{4 \times \eta_c \times t} = \frac{1.68 \times 1500}{4 \times 0.30 \times 15} = 140 \ N/mm^2$$

This value is more than the given limiting tensile stress.

***Lame's equation*:** Assuming that the longitudinal fibres of the cylindrical shell are equally strained, Lame has shown that the tangential stress at any radius *x* is,

$$\sigma_t = \frac{p_i (r_i)^2 - p_o (r_o)^2}{(r_o)^2 - (r_i)^2} + \frac{(r_i)^2 (r_o)^2}{x^2} \left[\frac{p_i - p_o}{(r_o)^2 - (r_i)^2} \right]$$

and radial stress at any radius *x*,

$$\sigma_r = \frac{p_i (r_i)^2 - p_o (r_o)^2}{(r_o)^2 - (r_i)^2} - \frac{(r_i)^2 (r_o)^2}{x^2} \left[\frac{p_i - p_o}{(r_o)^2 - (r_i)^2} \right]$$

Since we are concerned with the internal pressure ($p_i = p$) only, therefore substituting the value of external pressure, $p_o = 0$.
Tangential stress at any radius *x*,

$$\sigma_t = \frac{p\,(r_i)^2}{(r_o)^2 - (r_i)^2}\left[1 + \frac{(r_o)^2}{x^2}\right] \qquad \ldots(i)$$

and radial stress at any radius *x*,

$$\sigma_r = \frac{p\,(r_i)^2}{(r_o)^2 - (r_i)^2}\left[1 - \frac{(r_o)^2}{x^2}\right] \qquad \ldots(ii)$$

In designing a thick cylindrical shell of brittle material (*e.g.* cast iron, hard steel and cast aluminium) with closed or open ends and in accordance with the maximum normal stress theory failure, the tangential stress induced in the cylinder wall,

$$\sigma_t = \sigma_{t(max)} = \frac{p\,[(r_o)^2 + (r_i)^2]}{(r_o)^2 - (r_i)^2}$$

Since $r_o = r_i + t$, therefore substituting this value of r_o in the above expression, we get

$$\sigma_t = \frac{p\,[(r_i + t)^2 + (r_i)^2]}{(r_i + t)^2 - (r_i)^2}$$

$$\sigma_t\,(r_i + t)^2 - \sigma_t\,(r_i)^2 = p\,(r_i + t)^2 + p\,(r_i)^2$$

$$(r_i + t)^2\,(\sigma_t - p) = (r_i)^2\,(\sigma_t + p)$$

$$\frac{(r_i + t)^2}{(r_i)^2} = \frac{\sigma_t + p}{\sigma_t - p}$$

$$\frac{r_i + t}{r_i} = \sqrt{\frac{\sigma_t + p}{\sigma_t - p}} \quad \text{or} \quad 1 + \frac{t}{r_i} = \sqrt{\frac{\sigma_t + p}{\sigma_t - p}}$$

$$\therefore \quad \frac{t}{r_i} = \sqrt{\frac{\sigma_t + p}{\sigma_t - p}} - 1 \quad \text{or} \quad t = r_i \left[\sqrt{\frac{\sigma_t + p}{\sigma_t - p}} - 1 \right] \qquad ...(iii)$$

$$\tau = \tau_{max} = \frac{\sigma_{t(max)} - \sigma_{t(min)}}{2} = \frac{\dfrac{p\,[(r_o)^2 + (r_i)^2]}{(r_o)^2 - (r_i)^2} - (-p)}{2}$$

$$= \frac{p\,[(r_o)^2 + (r_i)^2] + p\,[(r_o)^2 - (r_i)^2]}{2[(r_o)^2 - (r_i)^2]} = \frac{2p\,(r_o)^2}{2[(r_o)^2 - (r_i)^2]}$$

$$= \frac{p\,(r_i + t)^2}{(r_i + t)^2 - (r_i)^2} \qquad ... (\because r_o = r_i + t)$$

or
$$\tau(r_i + t)^2 - \tau(r_i)^2 = p(r_i + t)^2$$

$$(r_i + t)^2 (\tau - p) = \tau(r_i)^2$$

$$\frac{(r_i + t)^2}{(r_i)^2} = \frac{\tau}{\tau - p}$$

$$\frac{r_i + t}{r_i} = \sqrt{\frac{\tau}{\tau - p}} \quad \text{or} \quad 1 + \frac{t}{r_i} = \sqrt{\frac{\tau}{\tau - p}}$$

$$\therefore \quad \frac{t}{r_i} = \sqrt{\frac{\tau}{\tau - p}} - 1 \quad \text{or} \quad t = r_i \left[\sqrt{\frac{\tau}{\tau - p}} - 1 \right] \qquad ...(iv)$$

The value of shear stress (τ) is usually taken as one-half the tensile stress (σ_t). Therefore the above expression may be written as

$$t = r_i \left[\sqrt{\frac{\sigma_t}{\sigma_t - 2p}} - 1 \right] \qquad ...(v)$$

Birnie's equation: In case of open-end cylinders (such as pump cylinders, rams, gun barrels etc.) made of ductile material (i.e. low carbon steel, brass, bronze, and aluminium alloys), the allowable stresses cannot be determined by means of maximum-stress theory of failure. In such cases, the maximum-strain theory is used. According to this theory, the failure occurs when the strain reaches a limiting value and Birnie's equation for the wall thickness of a cylinder is

$$t = r_i \left[\sqrt{\frac{\sigma_t + (1 - \mu)\, p}{\sigma_t - (1 + \mu)\, p}} - 1 \right]$$

The value of σ_t may be taken as 0.8 times the yield point stress (σ_y).

***Clavarino's equation*:** This equation is also based on the maximum-strain theory of failure, but it is applied to closed-end cylinders heads) made of ductile material. According to this equation, the thickness of a cylinder,

$$t = r_i \left[\sqrt{\frac{\sigma_t + (1 - 2\mu)\, p}{\sigma_t - (1 + \mu)\, p}} - 1 \right]$$

In this case also, the value of σ_t may be taken as 0.8 σ_y.

Barlow's equation **:** This equation is generally used for high pressure oil and gas pipes. According to this equation, the thickness of a cylinder,

$$t = p.r_o / \sigma_t$$

For ductile materials, σ_t = 0.8 σ_y and for brittle materials σ_t = 0.125 σ_u, where σ_u is the ultimate stress.

Examples

8. A cast iron cylinder of internal diameter 200 mm and thickness 50 mm is subjected to a pressure of 5 N/mm2. Calculate the tangential and radial stresses at the inner, middle (radius = 125 mm) and outer surfaces.

Solution. Given : d_i = 200 mm or r_i = 100 mm ; t = 50 mm ; p = 5 N/mm^2
We know that outer radius of the cylinder,

$$r_o = r_i + t = 100 + 50 = 150 \text{ mm}$$

Tangential stresses at the inner, middle and outer surfaces
We know that the tangential stress at any radius *x*,

$$\sigma_t = \frac{p\,(r_i)^2}{(r_o)^2 - (r_i)^2}\left[1 + \frac{(r_o)^2}{x^2}\right]$$

Tangential stress at the inner surface (*i.e.* when $x = r_i = 100$ mm),

$$\sigma_{t(inner)} = \frac{p\,[(r_o)^2 + (r_i)^2]}{(r_o)^2 - (r_i)^2} = \frac{5\,[(150)^2 + (100)^2]}{(150)^2 - (100)^2} = 13\ \text{N/mm}^2 = 13\ \text{MPa}\ \text{Ans.}$$

Tangential stress at the middle surface (*i.e.* when $x = 125$ mm),

$$\sigma_{t(middle)} = \frac{5\,(100)^2}{(150)^2 - (100)^2}\left[1 + \frac{(150)^2}{(125)^2}\right] = 9.76\ \text{N/mm}^2 = 9.76\ \text{MPa}\ \text{Ans.}$$

and tangential stress at the outer surface (*i.e.* when $x = r_o = 150$ mm),

$$\sigma_{t(outer)} = \frac{2\,p\,(r_i)^2}{(r_o)^2 - (r_i)^2} = \frac{2 \times 5\,(100)^2}{(150)^2 - (100)^2} = 8\ \text{N/mm}^2 = 8\ \text{MPa}\ \text{Ans.}$$

Radial stresses at the inner, middle and outer surfaces
We know that the radial stress at any radius *x*,

$$\sigma_r = \frac{p\,(r_i)^2}{(r_o)^2 - (r_i)^2}\left[1 - \frac{(r_o)^2}{x^2}\right]$$

Radial stress at the inner surface (*i.e.* when $x = r_i = 100$ mm),

$$\sigma_{r(inner)} = -p = -5\ \text{N/mm}^2 = 5\ \text{MPa (compressive)}\ \text{Ans.}$$

Radial stress at the middle surface (*i.e.* when $x = 125$ mm)

$$\sigma_{r(middle)} = \frac{5\,(100)^2}{(150)^2 - (100)^2}\left[1 - \frac{(150)^2}{(125)^2}\right] = -1.76\ \text{N/mm}^2 = -1.76\ \text{MPa}$$
$$= 1.76\ \text{MPa (compressive)}\ \text{Ans.}$$

and radial stress at the outer surface (*i.e.* when $x = r_o = 150$ mm),

$$\sigma_{r(outer)} = 0\ \text{Ans.}$$

9. A hydraulic press has a maximum capacity of 1000 kN. The piston diameter is 250 mm. Calculate the wall thickness if the cylinder is made of material for which the permissible strength may be taken as 80 MPa. This material may be assumed as a brittle material.

Solution. Given : W = 1000 kN = 1000 × 10^3 N ; d = 250 mm ; σ_t = 80 MPa = 80 N/mm^2
First of all, let us find the pressure inside the cylinder (p). We know that load on the hydraulic press (W),

$$1000 \times 10^3 = \frac{\pi}{4} \times d^2 \times p = \frac{\pi}{4} (250)^2 p = 49.1 \times 10^3 p$$

$$\therefore \quad p = 1000 \times 10^3 / 49.1 \times 10^3 = 20.37 \text{ N/mm}^2$$

$$\text{Let} \quad r_i = \text{Inside radius of the cylinder} = d/2 = 125 \text{ mm}$$

We know that wall thickness of the cylinder

$$t = r_i \left[\sqrt{\frac{\sigma_t + p}{\sigma_t - p}} - 1 \right] = 125 \left[\sqrt{\frac{80 + 20.37}{80 - 20.37}} - 1 \right] \text{mm}$$

$$= 125 (1.297 - 1) = 37 \text{ mm Ans.}$$

10. A closed-ended cast iron cylinder of 200 mm inside diameter is to carry an internal pressure of 10 N/mm^2 with a permissible stress of 18 MPa. Determine the wall thickness by means of Lame's and the maximum shear stress equations. What result would you use? Give reason for your conclusion.

Solution. Given : d_i = 200 mm or r_i = 100 mm ; p = 10 N/mm^2 ; σ_t = 18 MPa = 18 N/mm^2
According to Lame's equation, wall thickness of a cylinder,

$$t = r_i \left[\sqrt{\frac{\sigma_t + p}{\sigma_t - p}} - 1 \right] = 100 \left[\sqrt{\frac{80 + 10}{80 - 10}} - 1 \right] = 87 \text{ mm}$$

According to maximum shear stress equation, wall thickness of a cylinder,

$$t = r_i \left[\sqrt{\frac{\tau}{\tau - p}} - 1 \right]$$

We know that the shear stress (τ) is usually taken one-half the tensile stress (σ_t). In the present case, $\tau = \sigma_t / 2 = 18/2 = 9$ N/mm^2. Since τ is less than the internal pressure (p = 10 N/mm^2), therefore the expression under the square root will be negative. Thus no thickness can prevent failure of the cylinder. Hence it is impossible to design a cylinder to withstand fluid pressure greater than the allowable working stress for the given material. This difficulty is overcome by using compound cylinders.
Thus, we shall use a cylinder of wall thickness, t = 87 mm **Ans.**

11. The cylinder of a portable hydraulic riveter is 220 mm in diameter. The pressure of the fluid is 14 N/mm^2 by gauge. Determine suitable thickness of the cylinder wall assuming that the maximum permissible tensile stress is not to exceed 105 MPa.

Solution. Given : d_i = 220 mm or r_i = 110 mm ; p = 14 N/mm^2 ; σ_t = 105 MPa = 105 N/mm^2
Since the pressure of the fluid is high, therefore thick cylinder equation is used.
Assuming the material of the cylinder as steel, the thickness of the cylinder wall (t) may be obtained by using Birnie's equation. We know that

$$t = r_i \left[\sqrt{\frac{\sigma_t + (1 - \mu)\, p}{\sigma_t - (1 + \mu)\, p}} - 1 \right]$$

$$= 110 \left[\sqrt{\frac{105 + (1 - 0.3)\, 14}{105 - (1 + 0.3)\, 14}} - 1 \right] = 16.5 \text{ mm} \quad \text{Ans.}$$

...(Taking Poisson's ratio for steel, μ = 0.3)

Exercise

1. A pressure vessel has an internal diameter of 1 m and is to be subjected to an internal pressure of 2.75 N/mm^2 above the atmospheric pressure. Considering it as a thin cylinder and assuming the efficiency of its riveted joint to be 79%, calculate the plate thickness if the tensile stress in the material is not to exceed 88 MPa.

2. A steam boiler, 1.2 metre in diameter, generates steam at a gauge pressure of 0.7 N/mm^2. Assuming the efficiency of the riveted joints as 75% , find the thickness of the shell. Given that ultimate tensile stress = 385 MPa and factor of safety = 5.

3. The maximum force exerted by a small hydraulic press is 500 kN. The working pressure of the fluid is 20 N/mm^2. Determine the diameter of the plunger, operating the table. Also suggest the suitable thickness for the cast steel cylinder in which the plunger operates, if the permissible stress for cast steel is 100 MPa.

4. Find the thickness of the flat end cover plates for a 1 N/mm^2 boiler that has a diameter of 600 mm. The limiting tensile stress in the boiler shell is 40 MPa.

5. The hydraulic cylinder 400 mm bore operates at a maximum pressure of 5 N/mm^2. The piston rod is connected to the load and the cylinder to the frame through hinged joints. **Design: 1.** cylinder, **2.** piston rod, **3.** hinge pin, and **4.** flat end cover.
The allowable tensile stress for cast steel cylinder and end cover is 80 MPa and for piston rod is 60 MPa.
Draw the hydraulic cylinder with piston, piston rod, end cover and O-ring.