

CHAPTER FIVE

5. TWO DIMENSIONAL RANDOM VARIABLES

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5.1 Definition of Two Dimensional Random Variables

Up to now we have observed only one-dimensional characteristics of an object as a random variable, for example a person's height. In this chapter, however, we will discuss more than one characteristic of an object for example, a person's weight in addition to the person's height.

Perhaps, it is advantageous to borrow a concept from mathematics to explain the idea of two-dimensional random variables. In mathematics, whenever we talk of the real number system we are referring to the real number line, which is one-dimensional involving only the X-axis. On the other hand, whenever we talk of a set of ordered pairs, relations, we are referring to the coordinate plane, which is two-dimensional involving both the X and Y-axes.

In the case of random variables as well, one can furnish the same analogy as in the one stated above. Whenever we talk of a random variable whose possible values are subsets of the set of real numbers, the random variable under consideration is one-dimensional. Likewise, when a given random variable assumes values that are subsets of the cartesian product of real numbers, set of ordered pairs (points) in the X-Y plane, that random variable is referred to as two dimensional random variable. By extending the analogy, one can also have other higher order dimensional random variables.

Definition (Two-Dimensional Random Variables): Let S be a sample space associated with a random experiment E . Let $X(s) = x$ and $Y(s) = y$ be two functions each assigning a real number to each outcome $s \in S$. Then we call (X, Y) a two dimensional random variable. Equivalently, we call X and Y bivariate random variables.

Examples: It may be of one's interest to measure

- ✓ The amount of precipitate P and volume V of a gas from controlled experiment (P, V)

- ✓ The total rain fall R and the average temperature T of a locality during a specified period (R, T)
- ✓ The distribution of monthly temperature readings in a given region over a given year.
- ✓ The hardness H and tensile strength T of copper, resulting in the outcomes (h, t) .

Extensions to cases involving more than two random variables are straightforward.

5.2 Joint Distributions for Discrete and Continuous Random Variables

5.2.1 Joint Distributions for Discrete Random Variable

Definition (Two-Dimensional Discrete Random Variables): (X, Y) is a two-dimensional discrete random variable if the possible values of (X, Y) are finite or countably infinite, i.e., if the possible values of (X, Y) can be written as $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$

If X and Y are two discrete random variables, the probability distribution for their simultaneous occurrence can be represented by a function with values $f(x, y)$ for any pair of values (x, y) within the range of the random variables X and Y . It is customary to refer to this function as the **joint probability distribution** of X and Y .

Definition: The function $f(x, y)$ is a joint probability distribution or probability mass function of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y) ,
2. $\sum \sum f(x, y) = 1$

The set of triplets $(x, y, f(x, y))$ is called the joint probability distribution of (X, Y) .

For any region A in the xy plane, $P[(X, Y) \in A] = \sum_A f(x, y)$

Example

1. Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote, respectively, the number of red and white balls chosen, then find the joint probability mass function of X and Y .
2. Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find
 - a) the joint probability function $f(x, y)$,
 - b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.
3. Two production lines manufacture a certain type of item. Suppose that the capacity per day is 5 items for line I, and 3 items for line II. Assume that the number of items actually produced by either production line is a random variable. Let (X, Y) represents the two - dimensional random variable yielding the number of items produced by line I and II, respectively.

	X						
Y	0	1	2	3	4	5	Total
0	0	0.01	0.03	0.05	0.07	0.09	0.25
1	0.01	0.02	0.04	0.05	0.06	0.08	0.26
2	0.01	0.03	0.05	0.05	0.05	0.06	0.25
3	0.01	0.02	0.04	0.06	0.06	0.05	0.24
Total	0.03	0.08	0.16	0.21	0.24	0.28	1

- Show that the above probability distribution is a legitimate probability distribution
- What is the probability that both lines produce the same number of items?
- What is the probability that more items are produced by line II than line I?

Exercise

- Let (X, Y) be a two-dimensional random variable with probability distribution $p(x, y) = 2^{-(x+y)}$ $x, y = 1, 2, 3, \dots$
 - Show that $p(x, y)$ is a probability distribution
 - Find $P(X > Y)$
- Two tablets are selected at random from a bottle containing three aspirin, two sedative and four laxatives. If X and Y are, respectively the number of aspirin tablets and the number sedative tablets included among the tablets drawn from the bottle. Find joint probability distribution of (X, Y) .

5.2.2. Joint Distributions for Continuous Random Variable

Definition: Two-Dimensional Continuous Random Variables : (X, Y) is a two-dimensional continuous random variable if it assumes all values in some region R of the Euclidian plane.

Definition: Probability Density Function: Let (X, Y) be a two-dimensional continuous random variable assuming all values in the some region R^2 (2-dimentional set). Then the joint probability density function (p.d.f) of (X, Y) is a function that satisfies the following two conditions.

$$i) \quad f(x, y) \geq 0 \text{ for all } (x, y) \in R$$

$$ii) \quad \iint_R f(x, y) dx dy = 1$$

Remark:

$$\triangleright \quad \iint_R f(x, y) dx dy = 1 \text{ indicates that the volume under the surface is one.}$$

$\triangleright$ If B is the range space of (X, Y) we have

$$P(B) = \iint_B f(x, y) dx dy$$

Example 1: Suppose (X, Y) has a joint P.d.f given by

$$f(x, y) = \begin{cases} c, & 0 < x < 2, 0 < y < 4 \\ 0, & \text{elsewhere} \end{cases}$$

- Determine the constant c
- Find $P(X < 1, Y < 3)$
- Find $P(X > Y)$

Example 2: Suppose that a two-dimensional random variable (X, Y) has joint p.d.f given by

$$f(x, y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{Otherwise} \end{cases}$$

- Show that $f(x, y)$ is a p.d.f
- Find $P(X < Y)$

Exercise

A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y , respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} 2/5(2x + 3y) & 0 \leq x \leq 1; \\ 0 & \text{otherwise} \end{cases}$$

- Verify that $f(x, y)$ is a p.d.f
- Find $P(0 < X < 1/2, 1/4 < Y < 1/2)$

5.3 Cumulative Distribution Function (CDF) of Two Dimensional Random Variable

Definition: CDF: Let (X, Y) be a two-dimensional random variable. The cumulative distribution function F of (X, Y) is defined as $F(x, y) = P(X \leq x, Y \leq y)$

Case 1: If (X, Y) is discrete, then

$$F(x, y) = P(X \leq x, Y \leq y) = \sum \sum P(X = x_i, Y = y_j) \forall x_i \leq x, y_j \leq y$$

Case 2: If (X, Y) is continuous, then

$$F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f(s, t) ds dt$$

Properties of a CDF

- i) $0 \leq F(x, y) \leq 1$
- ii) $P(a < X < b, Y < y) = F(b, y) - F(a, y)$
- iii) $P(X \leq x, c < Y < d) = F(x, d) - F(x, c)$
- iv) $P(a < X < b, c < Y < d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$
- v) $F(x, y)$ is non-decreasing function
- vi) $F(-\infty, y) = 0; F(x, -\infty) = 0$
- vii) $F(-\infty, +\infty) = 1$
- viii) $\frac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y)$

Example1: Consider the example on the two production lines, given previously, and find $F(1,2)$

Example 2: Let the p.d.f of X and Y be given as follows

$$f(x, y) = \begin{cases} x + y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the corresponding joint cdf of (X, Y)

Exercise 1: The joint probability function (X, Y) is given by

$$P(x, y) = k(2k + 3y); x = 0, 1, 2; y = 1, 2, 3$$

- i. Compute the value of k
- ii. Find the joint cdf of (X, Y)

Exercise 2: Suppose that the joint cdf of a two dimensional continuous random variable is given by

$$F(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)} & \text{if } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then compute a) $P(X \leq 1; Y \leq 1)$

b) $P(X + Y \leq 1)$

Exercise 3

$$\text{Let } f(x, y) = \begin{cases} 1 & 0 < x < 2, 0 < y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find $F(1, 3)$

5.4. Marginal Probability Distributions and Conditional Distributions

5.4.1. Marginal Probability Distributions

How do you find the p.d.f of a random variable X or Y from the given joint p.d.f of (X,Y) ?

The answer is simple! You simply find the Marginal Probability Distributions.

With two-dimensional random variable (X, Y) , we associate two one-dimensional random variables X and Y individually. That is, we may be interested in finding the probability distribution of X and Y from their joint distribution. We call these distributions the marginal distributions of X and Y .

Definition: Marginal Distribution (Discrete case): Let (X,Y) be a two-dimensional discrete random variable with joint p.m.f $P(X = x_i, Y = y_j)$. Then the marginal probability functions of X and Y are:

$$P(x_i) = P(X = x_i) = \sum_{j=1}^{\infty} P(X = x_i, Y = y_j)$$
$$\text{and } q(y_j) = P(Y = y_j) = \sum_{i=1}^{\infty} P(X = x_i, Y = y_j) \text{ respectively}$$

Example1. Let X and Y have a joint p.m.f given by:

$\begin{matrix} \text{X} \\ \text{Y} \end{matrix}$	0	1	2
0	0.25	0.15	0.10
1	0.10	0.08	0.10
2	0.05	0.07	0.10

Find the marginal distribution of X and Y

Example 2: Let (X,Y) be a two-dimensional discrete random variable with p.m.f

$$f(x,y) = \begin{cases} \frac{x+y}{21}; & \text{if } x = 1,2,3 \text{ and } y = 1,2 \\ 0, & \text{else where} \end{cases}$$

Find the respective p.m.f s of X and Y .

Definition: Marginal Distribution (Continuous Case): Let (X,Y) be a two-dimensional continuous random variable with joint p.d.f $f_{XY}^{(x,y)}$. Then the marginal distribution of X and Y , respectively, are:

$$f_X^{(x)} = \int_{-\infty}^{\infty} f_{XY}^{(x,y)} dy \text{ and } f_Y^{(y)} = \int_{-\infty}^{\infty} f_{XY}^{(x,y)} dx$$

Note: -To find the marginal p.d.f of X, integrate the joint p.d.f $f_{XY}^{(x,y)}$ with respect to y and evaluate it for all values of y in R.

- To find the marginal p.d.f of Y, integrate the joint p.d.f $f_{XY}^{(x,y)}$ with respect to x and evaluate it for all values of x in R.

Example 1: Two characteristics of a rocket engine's performance are thrust X and mixture ratio Y. Suppose that (X,Y) is a two-dimensional continuous random variable with joint p.d.f given by

$$f_{XY}^{(x,y)} = \begin{cases} 2(x+y-2xy) & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal p.d.f of X and Y

Example 2: Consider the following joint p.d.f of X and Y.

$$f_{XY}^{(x,y)} = \begin{cases} 12x; 0 < y < x < 1, & 0 < x^2 < y < 1 \\ 0, & \text{else where} \end{cases} \quad \text{Find the marginal p.d.f for X and Y.}$$

Exercise : Let (X,Y) be a two-dimensional continuous random variable with P.d.f

$$f_{XY}^{(x,y)} = \begin{cases} 2; & \text{if } 0 < x < y < 1 \\ 0, & \text{else where} \end{cases} \quad \text{Find the respective p.d.f s of X and Y.}$$

5.4.2. Conditional Probability Distributions

Definition: Conditional Probability (Discrete Case): Let (X,Y) be a two-dimensional discrete random variable with joint p.m.f $P(X = x_i, Y = y_j)$. Let $P(x_i)$ and $q(y_j)$ be marginal probability distributions of X and Y, respectively.

The conditional probability of X for given a value $Y = y_j$ is defined by

$$P(x_i|y_j) = P(X = x_i|Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{P(X = x_i, Y = y_j)}{q(y_j)}, \quad q(y_j) > 0$$

And the conditional probability of Y given $X=x_i$ is:

$$P(y_j|x_i) = P(Y = y_j|X = x_i) = \frac{P(X = x_i, Y = y_j)}{p(X = x_i)} = \frac{P(X = x_i, Y = y_j)}{p(x_i)}, \quad p(x_i) > 0$$

Note: For given j, $P(x_i|y_j)$ satisfies all the conditions for a probability distribution. We have

$$P(x_i|y_j) \geq 0 \text{ and}$$

$$\sum_{i=1}^{\infty} P(x_i | y_j) = \sum_{i=1}^{\infty} \frac{P(x_i, y_j)}{q(y_j)} = \frac{q(y_j)}{q(y_j)} = 1$$

Example: Suppose (X, Y) is a discrete random variable and $P(X = 1, Y = 2) = 0.03$ and $P(Y=2) = 0.25$. Find $P(X = 1 | Y = 2)$

Definition: Conditional Probability (Continuous Case): Let (X, Y) be a two-dimensional continuous random variable with joint p.d.f $f_{XY}^{(x,y)}$. Let $f_X^{(x)}$ and $f_Y^{(y)}$ be the marginal p.d.f's of X and Y , respectively.

The conditional p.d.f of X given $Y=y$ is given by:

$$f_X^{(x/y)} = \frac{f_{XY}^{(x,y)}}{f_Y^{(y)}}; f_Y^{(y)} > 0$$

Similarly, the conditional p.d.f of Y given $X=x$ is given by:

$$f_Y^{(y/x)} = \frac{f_{XY}^{(x,y)}}{f_X^{(x)}}; f_X^{(x)} > 0$$

Remarks:

$$\int_{-\infty}^{\infty} f_X^{(x/y)} dx = 1$$

$$\int_{-\infty}^{\infty} f_Y^{(y/x)} dy = 1$$

Example: Let (X, Y) be jointly distributed with p.d.f

$$f_{XY}^{(x,y)} = \begin{cases} 2 & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the conditional distributions of X given y and Y given x

Example: Suppose that (X, Y) is a two-dimensional random variable with p.d.f

$$f_{XY}^{(x,y)} = \begin{cases} 2, & x > 0, y > 0, x + y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find a. $P(X < 1/2 | Y = 1/4)$

b. $P(Y > 1/3 | X = 1/2)$

5.5. Independent Random Variables

Definition: Independent Discrete Random Variables : Let (X, Y) be a two-dimensional discrete random variable, we say that X and Y are independent random variables if and only if

$$P(X = x_i, Y = y_j) = P(X = x_i) \cdot P(Y = y_j) \text{ for all } i \text{ and } j.$$

Theorem: Let (X, Y) be a two-dimensional discrete random variable. Then X and Y are independent random variables if and only if $P(x_i | y_j) = P(x_i)$ or equivalently iff $P(y_j | x_i) = q(y_j)$ for all i and j .

Definition: Independent Continuous Random Variables: Let (X, Y) be a two-dimensional continuous random variable. We say X and Y are independent random variables iff

$$f_{XY}^{(x,y)} = f_X^{(x)} f_Y^{(y)}$$

Theorem: Let (X, Y) be a two-dimensional continuous random variable. We say X and Y are independent random variables iff $f_X^{(x|y)} = f_X^{(x)}$ or equivalently iff $f_Y^{(y|x)} = f_Y^{(y)}$

Example: Suppose

$$P(X = i, Y = j) = \begin{cases} 2^{-(i+j)}, & i, j = 1, 2, \dots \\ 0, & \text{elsewhere} \end{cases} \quad . \text{ Are } X \text{ and } Y \text{ independent?}$$

Example: Suppose the joint distribution of X and Y is given as follows. Are X and Y independent?

$$f_{XY}^{(x,y)} = \begin{cases} 4xy, & 0 < x < 1 \quad 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

5.6 Distributions of Functions of Two Dimensional Random Variables

Let us now consider $Z = H(x, y)$, a function of two random variables X and Y . It should be clear that Z is again a random variable. Consider the following sequence of steps:

- Perform the experiment E and obtain the outcomes.
- Evaluate random numbers $X(s)$ and $Y(s)$
- Evaluate the numbers $Z = H(X(s), Y(s))$

The value of Z clearly depends on s , the original outcome of the experiment. Given the distribution of (X, Y) , what is the probability distribution of $Z = H(x, y)$?

Functions of Two Dimensional Discrete Random Variable

If (X, Y) is a two-dimensional discrete random variable, the following one dimensional discrete random variables might of interest

$$U = \min(X, Y); V = \max(X, Y); W = X + Y$$

Example 1: The joint probability function (X, Y) is given by

$$P(x, y) = K(2x + 3y); x = 0, 1, 2; y = 1, 2, 3$$

a) Compute the value of K

b) Find the probability distribution of

i. $W = (X + Y)$

ii. $U = \min(X, Y)$

iii. $V = \max(X, Y)$

Solution:

a) To determine the constant K we make use one of the properties of a p.m.f namely

$$\sum_{x=0}^2 \sum_{y=1}^3 P(x, y) = 1$$

$$\Rightarrow 72K = 1$$

$$\Rightarrow K = \frac{1}{72}$$

b) i. $R_W = \{1, 2, 3, 4, 5\}$

w	1	2	3	4	5
$P(W = w)$	$\frac{3}{72}$	$\frac{11}{72}$	$\frac{24}{72}$	$\frac{21}{72}$	$\frac{13}{72}$

$$P(W = 1) = P(0, 1) = \frac{3}{72}$$

$$P(W = 2) = P(0, 2) + P(1, 1) = \frac{6}{72} + \frac{5}{72} = \frac{11}{72}$$

$$P(W = 3) = P(0,3) + P(1,2) + P(2,1) = \frac{7}{72} + \frac{8}{72} + \frac{9}{72} = \frac{24}{72}$$

$$P(W = 4) = P(1,3) + P(2,2) = \frac{11}{72} + \frac{10}{72} = \frac{21}{72}$$

$$P(W = 5) = P(2,3) = \frac{13}{72}$$

ii. $R_U = \{0,1,2,3\}$

u	0	1	2
$P(U = u)$	$\frac{18}{72}$	$\frac{25}{72}$	$\frac{23}{72}$

$$P(U = 0) = P(0,1) + P(0,2) + P(0,3) = \frac{3}{72} + \frac{6}{72} + \frac{9}{72} = \frac{18}{72}$$

$$P(U = 1) = P(1,1) + P(1,2) + P(2,1) + P(1,3) = \frac{3}{72} + \frac{6}{72} + \frac{7}{72} + \frac{9}{72} = \frac{25}{72}$$

$$P(U = 2) = P(2,2) + P(2,3) = \frac{10}{72} + \frac{13}{72} = \frac{23}{72}$$

Functions of Two Dimensional Continuous Random Variable

In finding the pdf of $Z = H_1(x, y)$ it is often simplest to consider a second random variable, say $W = H_2(x, y)$, and first obtain the joint pdf Z and W , say $f_{ZW}(z, w)$. From a knowledge of $f_{ZW}(z, w)$ we can then obtain the desired pdf of Z , say $f_Z(z)$, by simply integrating $f_{ZW}(z, w)$ with respect to w . That is $f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w) dw$

Theorem: Suppose that (X, Y) is a two-dimensional continuous random variable with joint pdf $f_{XY}(x, y)$. Let $Z = H_1(x, y)$ and $W = H_2(x, y)$, and assume that H_1 and H_2 satisfy the following conditions:

- The equation $Z = H_1(x, y)$ and $W = H_2(x, y)$ can be uniquely solved for x and y in terms of z and w , say $x = G_1(z, w)$ and $y = G_2(z, w)$.
- The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}$ and $\frac{\partial y}{\partial w}$ exist.

Then the joint pdf of (Z, W) , say $f_{ZW}(z, w)$, is given by the following expression:

$$f_{ZW}(z, w) = f_{XY}(G_1(z, w), G_2(z, w)) |J(Z, W)|$$

where $J(Z, W)$ is the following 2x2 determinant:

$$J(Z, W) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

This determinant is called the Jacobian of the transformation $(x, y) \rightarrow (z, w)$

Example

$$\text{Let } f(x, y) = \begin{cases} 1/2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint pdf of $U = X+Y$ and $V = X-Y$