

CHAPTER ONE

INTRODUCTION

- 1.1. *Deterministic and non-deterministic models*
 - 1.2. *Review of set theory: sets, union, intersection, complementation, De-Morgan's rules*
 - 1.3. *Random experiments, sample space and events*
 - 1.4. *Finite sample spaces and equally likely outcomes*
 - 1.5. *Counting techniques*
 - 1.6. *Definitions of probability*
 - 1.7. *Derived theorems of probability*
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1.1. Deterministic and non-deterministic models

Models are integral parts of both social and natural sciences. In both cases we construct or fit models to represent the interrelationship between two or more variables. Particularly, in the fields like Statistics and Economics models are fitted for the sake of forecasting. It is possible to classify models in to different groups based on varied attributes or criteria. Based on the type of experiment for which we fit the model, we classify model as Deterministic and Non-deterministic models.

Definition: Experiment is any activity (process or action) that we intended to do under certain condition to obtain a well defined results, usually called the outcome of an experiment. The possible results of an experiment may be one or more. Based on the number of possible results, we may classify an experiment as Deterministic and Non-deterministic experiment.

A- Deterministic experiment: this is an experiment for which the outcomes can be predicted in advance and is known prior to its conduct. For this type

of experiment we have only one possible result (certain and unique). The result of an experiment is assumed to be dependent on the condition under which an experiment is performed. A mathematical form of equations to be defined on this experiment is called a deterministic model.

Definition: Deterministic model is a model that stipulates the condition under which an experiment is performed determine the outcome of that experiment.

Examples: An experiment conducted to verify the Newton Laws of Motion: $F = ma$; an experiment conducted to determine the economic law of demand: $Q_t = a + bP_t$ where Q is a quantity demand, P is the price and t is a time; etc...

Note: Most of the experiments that we conduct to verify science law is an example of deterministic experiment.

B- Non-deterministic experiment: this is an experiment for which the outcome of a given trial cannot be predicted in advance prior to its conduct. We also call this experiment as unpredictable or probabilistic or stochastic or random experiment. Usually the result of this experiment is subjected to chance and is possibly more than one. In this experiment, whatever the condition under which an experiment is performed, one cannot tell with certainty which outcome occurs at any particular execution of an experiment though it is possible to list those outcomes.

This types of experiments are characterized by the following three properties:

- I. The experiment is repeatable under identical conditions.
- II. The outcome in any particular trial is variable, i.e., it depends on some chance or random mechanism.
- III. If the experiment is repeated a large number of times, then some regularity becomes apparent in the outcomes obtained. This regularity enables us to set some mathematical form of equations called **non-deterministic model**.

Definition: Non-deterministic model is a mathematical description of an uncertain situation.

Example: The experiment of tossing a coin: there are two possible outcomes i.e. getting a Head or Tail. Though we know the possible outcomes, we cannot for sure predict that we can get for example a Tail if we flip the coin in a particular manner and so on.

Note: the quantitative measure of uncertainty regarding one or more outcomes of a random experiment is called a probability.

Exercise: Identify the following models as deterministic or Non-deterministic

- a) Equation of a straight line.
- b) Variable cost=(Cost per unit)(Quantity produced)

c) $Y_i = AX_i + e_i$ where e_i is the error term (or random component) associated with the i^{th} observation.

1.2. Review of set theory: sets, union, intersection, complementation, De-Morgan's rules

Definition and Types of Sets

Set: A set is a well-defined collection or list of objects.

An object that belongs to a particular set is called an element. Sets are usually denoted by capital letters (A, B, C etc). On the other hand, elements of sets are usually denoted by small letters (a, b, c etc).

Examples: The set of students in a class; the set of even numbers; the set of possible outcomes of an experiment; etc.

Note: If X belongs to set A we write $X \in A$, and if X does not belong to set A we write $X \notin A$

Two Special Sets

Universal set (U): Universal set is the collection of all objects under consideration. Universal set for a given discussion is fixed and pre-determined.

Example: * The set of real numbers can be seen to be universal set of numbers.

* Consider an experiment of rolling a balanced die. Then, the set of all possible outcomes of this experiment can be considered as the universal set. $U = \{1, 2, 3, 4, 5, 6\}$

Empty set (ϕ or $\{\}$): A set with no element is called empty set.

Example: In the experiment of tossing two dice at the same time, define A to be a set whose elements are pairs of outcomes from the two dice with sum greater than 20. Obviously, the set A is empty set because the maximum possible sum is 12, which is attained when both the outcomes are 6.

Subset: $A \subseteq B$ iff $X \in A \Rightarrow X \in B$ for all X element in the universal set.

1.2.1 Set operations

One can identify three basic set operations. These are:

a. Complement: For any set A, the complement of A denoted by A' , or A^c or $\bar{A}$ is given by

$$\{X \in U / X \notin A\}$$

Note: $U' = \phi$; $\phi' = U$; $(A')' = A$

b. Union: Given two sets A and B, the union of A and B denoted by $A \cup B$ is the set of all elements, which belong to set A or B, or both.

$A \cup B = \{X: X \in A \vee X \in B\}$; here the logical connective $\vee$ (OR) is used in its inclusive sense.

In general, if we have a sequence of sets $A_1, A_2, \dots, A_n$ then

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i = \{X \in U : X \in A_i \text{ for at least one } i \text{ between } 1 \text{ and } n\}$$

Recall that $A \cup \phi = A$ and $A \cup U = U$

c. Intersection: For any two sets A and B the intersection of A and B is defined to be the set of all elements that occur in both set A and also set B. Symbolically, we write

$$A \cap B = \{X \in U / X \in A \wedge X \in B\}$$

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i = \{X \in U : X \in A_i \text{ for all } i \text{ between } 1 \text{ and } n\}$$

Sets with no intersection are called disjoint sets. A and B are disjoint if $A \cap B = \phi$.

1.2.2 Properties of set operations

I. Commutative law

i) $A \cup B = B \cup A$, and

ii) $A \cap B = B \cap A$

II. Associative law

i) $A \cup (B \cup C) = (A \cup B) \cup C$, and

ii) $A \cap (B \cap C) = (A \cap B) \cap C$

III. Distributive law

$$\text{i) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$\text{ii) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

More generally,

$$A \cap \left(\bigcup_{i=1}^n B_i \right) = \bigcup_{i=1}^n (A \cap B_i), \text{ and}$$

$$A \cup \left(\bigcap_{i=1}^n B_i \right) = \bigcap_{i=1}^n (A \cup B_i)$$

IV. Demorgan's law

$$(A \cup B)' = A' \cap B' \text{ and } (A \cap B)' = A' \cup B'$$

More generally,

$$\left(\bigcup_{i=1}^n A_i \right)' = \bigcap_{i=1}^n A_i', \text{ and } \left(\bigcap_{i=1}^n A_i \right)' = \bigcup_{i=1}^n A_i'$$

1.3. Random experiments, sample space and events

Introduction

Perhaps the concept of sample space and events may be considered as the corner stones for the whole discussion in probability. The concept of probability can never be addressed without furnishing the idea of sample space and events. The good news is that one can easily grasp the concept of sample space and events provided there is a clear understanding of universal set and subsets, which are elementary concepts in set theory. We have a perfect analogy. The concepts of universal set and subsets in the theory of sets directly correspond, respectively, to the concepts sample space and events when it comes to the theory of probability.

1. Random Experiment: It is an experiment that can be repeated any number of times under similar conditions and it is possible to enumerate the total number of outcomes without predicting an individual outcome. Random experiments are associated with probability models. As the experiment is repeated a large number of times a certain patterns appear.

Some features of a random experiment

- ✓ Each experiment is capable of being repeated indefinitely under essentially unchanged conditions,
- ✓ Although we are, in general, unable to determine which possible outcome will result in a single trial of the experiment, we are able to describe the set of all possible outcomes.
- ✓ As the experiment is repeated a large number of times a definite pattern or regularity is expected to appear. From this regularity we can construct a precise mathematical model.

Example: Consider the random experiments given above

- i) For the experiment of flipping a coin the possible outcomes are either Head (H) or Tail (T). Hence, the sample space will be $S = \{H, T\}$
- ii) For the experiment of rolling a die the possible outcomes are the six sides of the die that are numbered from 1 to 6. Thus, the sample space will be $S = \{1, 2, 3, 4, 5, 6\}$
- iii) Toss a coin twice and observe the face up. The sample space of this experiment is $S = \{HH, HT, TH, TT\}$. What would be the sample space if you were to toss three times?
- iv) For the experiment of drawing a card from a deck of cards the following sets can be possible sample spaces.
 $S = \{\text{Heart, Spider, Diamond, Flower}\}$
 $S = \{1s, 2s, 3s, \dots, Qs, Ks, \text{jokers}\}$
 $S = \{\text{Black card, Red card}\}$

As the last case of the above example indicates it is possible to have more than one sample space for a given random experiment. Hence, we can state the following remark.

There are different types of sample space:-

- ✓ **Finite sample space:** if they have a finite number of elements. Example:
 - Tossing a coin. $S = \{\text{Heads, Tails}\}$
 - Throwing a die. $S = \{1, 2, 3, 4, 5, 6\}$
 - Throwing a coin and a die. $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.
- ✓ **Countably infinite sample space:** if there is a one-to-one mapping between the elements of S and the natural numbers;
 - $S = \{0, 1, 2, 3, \dots\}$
 - $S = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$

- $S = \{p: p \text{ is prime}\} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$
- $S = \{1/r: r \text{ is a positive natural number}\} = \{1/2, 1/3, 1/4, 1/5, \dots\}$

✓ **Uncountable sample space:** if there are an infinite number of elements in S and there is not a one-to-one mapping between the elements and the natural numbers.

- $S = (0, \infty)$
- $S = (-\infty, \infty)$
- $S = \{(x, y): 0 \leq x \leq 1; y > 0\}$

2. Outcome: The result of a single trial of experiment

3. Event: It is a subset of sample space. It is a statement about one or more outcomes of a random experiment. It is denoted by capital letter.

Examples:

- In the experiment of flipping a die we can define an event $E_1 = \text{Heads shows up} = \{H\}$
- In the experiment of rolling a die we can define an event $C_2 = \text{The number is odd} = \{1, 3, 5\}$
- In tossing a coin twice we can define an event $D_1 = \text{At least one head occurs} = \{HH, TT, TH\}$

Note:

- ✓ If S has n members, then there are exactly 2^n subsets or events.
- ✓ Recall that any set is the subset of itself. Accordingly, the sample space S by itself is an event called a sure event or certain event.
- ✓ An event occurs if any one of its elements turns out to be the outcome of the experiment.
- ✓ Empty set (ϕ) is called an impossible event.
- ✓ Any event which consists of a single outcome in the sample space is called **an elementary or simple event** whereas events which consist of more than one outcome are called **compound events**.

4. Mutually Exclusive events: Two events A and B are said to be mutually exclusive if they cannot occur simultaneously (together) or $A \cap B = \phi$

Example: Experiment -Toss a coin twice

$$S = \{HH, HT, TH, TT\}$$

$$\text{Let } A = \text{Two heads occur } \{HH\}$$

B= Two tails occur {TT}

C= At least one head occur {HH, HT, TH}

$A \cap B = \phi \Rightarrow A$ and B are mutually exclusive events

$B \cap C = \phi \Rightarrow B$ and C are mutually exclusive events

$A \cap C = \{HH\} \Rightarrow A$ and C are not mutually exclusive

5. **Independent Events:** Two events are said to be independent if the occurrence of one does not affect probability of the other occurring.

6. **Dependent Events:** Two events are dependent if the first event affects the outcome or occurrence of the second event in a way the probability is changed.

7. Combination of events

a) **Union:** The event $A \cup B$ occurs if either A or B or both occur.

If $A_i, i=1, 2, 3, \dots, n$ is any finite collection of events, then $\bigcup_{i=1}^n A_i$ occurs if at least one of the events (A_i) occurs.

b) **Intersection:** The event $A \cap B$ occurs if both A and B occur.

If $A_i, i=1, 2, 3, \dots, n$ is any finite collection of events, then $\bigcap_{i=1}^n A_i$ occurs if all A_i occur.

c) **Complement of an Event:** the complement of an event A means nonoccurrence of A and is denoted by A' or A^c or $\bar{A}$ contains those points of the sample space which don't belong to A .



$\Rightarrow A \cup A' = S$

$\Rightarrow A \cap A' = \phi$

Exercise 1: A person is selected at random from a population of a given town

A: be the event that the person is male

B: be the event that the person is under 30 years

C: be the event that the person speaks foreign language

Describe the following events symbolically

a) A male under 30 years who does not speak foreign language. $(A \cap B) \cap C'$

- b) A female who is either under 30 or speaks foreign language. $A' \cap (B \cup C)$
 c) A person who is either under 30 or female but not both. $(B \cup A') \cap (B \cap A')^c$
 d) Male who is either under 30 or speaks foreign language but not both.
 $A \cap [(B \cup C) \cap (B \cap C)^c]$

Exercise 2: Let A, B and C be three events associated with an experiment. Describe the following in symbols.

- a) at least one of the events occurs
 b) only B occurs
 c) exactly one of the events occur
 d) exactly two of the events occur
 e) at least two of the events occur
 f) at most two of the events occur
 g) none of the events occur

Answer:

- a) $A \cup B \cup C$ b) $B \cap A' \cap C'$ c) $(A \cap B' \cap C') \cup (B \cap A' \cap C') \cup (C \cap B' \cap A')$
 d) $(A \cap B \cap C') \cup (B \cap A' \cap C) \cup (C \cap B' \cap A)$
 e) $((A \cap B) \cup C) \cup (A \cap C) \cup B) \cup (C \cap B) \cup A) \cup (A \cap B \cap C)$
 f) $(A \cap B \cap C)' = A' \cup B' \cup C'$ g) $(A' \cap B' \cap C')$

1.4. Counting Techniques

In order to calculate probabilities, we have to know

- The number of elements of an event
- The number of elements of the sample space.

That is, in order to judge what is **probable** we have to know what is **possible**.

- In order to determine the number of outcomes, one can use several rules of counting.
 - ☞ The addition rule
 - ☞ The multiplication rule
 - ☞ Permutation rule
 - ☞ Combination rule

I. Addition Rule

Let E1 and E2 be any two **mutually exclusive** events (i.e. there are no common outcomes).

Let an event E describes the situation where either event E1 **or** event E2 will occur. Then the number of times event E will occur can be given by the expression:

$$n(E) = n(E1) + n(E2)$$

where

$n(E)$ = Number of outcomes of event E

$n(E1)$ = Number of outcomes of event E1

$n(E2)$ = Number of outcomes of event E2

In general, if there are k procedures and the i^{th} procedure can be performed in n_i ways, $i = 1, 2, \dots, k$, then the number of ways in which we may perform procedure 1 or 2 or..... or procedure k is given by $n_1 + n_2 + \dots + n_k = \sum_{i=1}^k n_i$, assuming no procedures may be performed together.

Example 1: Suppose that we are planning a trip and deciding between bus and train transportation. If there are three bus routes and two train routes. How many different routes are available for the trip?

Solution: Let A= {an event for bus transportation}, B= {an event for train transportation}.

Then, we have $n_1 = 3$ and $n_2 = 2$; assuming that we are going to use only one route, by the addition rule we have $n_1 + n_2 = 3 + 2 = 5$ different possible routes are available for the trip.

Example 2: Consider a set of numbers $S = \{-4, -2, 1, 3, 5, 6, 7, 8, 9, 10\}$

Let the events E1, E2 and E3 be defined as:

E = choosing a negative or an odd number from S;

E1= choosing a negative number from S;

E2 = choosing an odd number from S.

Find $n(E)$.

Solution:

E1 and E2 are mutually exclusive events (i.e. no common outcomes).

$$n(E) = n(E1) + n(E2)$$

$$= 2 + 5$$

$$= 7$$

Example 3

In how many ways can a number be chosen from 1 to 22 such that it is a multiple of 3 or 8?

Solution:

Here, E_1 = multiples of 3:

$$E_1 = \{3, 6, 9, 12, 15, 18, 21\}$$

$$n(E_1) = 7$$

E_2 = multiples of 8:

$$E_2 = \{8, 16\}$$

$$n(E_2) = 2$$

Events E_1 and E_2 are mutually exclusive.

$$n(E) = n(E_1) + n(E_2) = 7 + 2 = 9$$

II. The Multiplication Rule

If there are k procedures and the i^{th} procedure can be performed in n_i ways, $i = 1, 2, \dots, k$, then the procedure consisting of procedure 1 followed by procedure 2,..., followed by procedure k may be performed in $n_1 * n_2 * \dots * n_k = \prod_{i=1}^k n_i$ ways.

Example 1: Suppose that there are three different types of meal for lunch and four different types of soft drinks. How many choices of meal and soft drinks can be made?

Solution: $n_1 = 3$ and $n_2 = 4$ hence $n_1 * n_2 = 3 * 4 = 12$ different choices of meal and soft drinks can be made.

Example 2: How many two digit numerals can be written by choosing the ten's digit from $A = \{1, 3, 5, 7, 9\}$ and the units digit from $B = \{2, 4, 6, 8\}$

Solution: The selection consists of two steps where the 1st can be made in 5 different ways for the ten's digits and for each of these the 2nd can be made in 4 different ways for the units digit, Hence the whole selection one after the other can be made in 5×4 different ways. i.e. there are 20 two digit numerals

i.e. $n_1 = 5$ and $n_2 = 4$

Example 3: The digits 0, 1, 2, 3, and 4 are to be used in 4 digit identification card. How many different cards are possible if

- a) Repetitions are permitted.
- b) Repetitions are not permitted.

Solution:

a) There are four steps

1 st digit	2 nd digit	3 rd digit	4 th digit
5	5	5	5

1. Selecting the 1st digit, this can be made in 5 ways.
2. Selecting the 2nd digit, this can be made in 5 ways.
3. Selecting the 3rd digit, this can be made in 5 ways.
4. Selecting the 4th digit, this can be made in 5 ways.

Hence, $n_1 * n_2 * n_3 * n_4 = 5 * 5 * 5 * 5 = 625$ different cards are possible.

b) There are four steps

1 st digit	2 nd digit	3 rd digit	4 th digit
5	4	3	2

1. Selecting the 1st digit, this can be made in 5 ways.
2. Selecting the 2nd digit, this can be made in 4 ways.
3. Selecting the 3rd digit, this can be made in 3 ways.
4. Selecting the 4th digit, this can be made in 2 ways.

Hence, $n_1 * n_2 * n_3 * n_4 = 5 * 4 * 3 * 2 = 120$ different cards are possible.

Example 4: There are 6 roads between A and B and 4 roads between B and C.

- a. In how many ways can one drive from A to C by way of B?
- b. In how many ways can one drive from A to C and back to A, passing through B on both trip?
- c. In how many ways can one drive the circular trip described in (b) without using the same road more than once?

Ans. 24, 576 and 360 resp

III. Permutation

An arrangement of n objects in a specified order is called permutation of the objects.

Permutation Rules:

1. The number of permutations of n distinct objects taken all together is $n!$

Where $n! = n * (n - 1) * (n - 2) * ... * 2 * 1$. **Note that:** $0! = 1$

2. The arrangement of n objects in a specified order using r objects at a time is called the permutation of n objects taken r objects at a time. It is written as ${}_n P_r$ and the formula is

$${}_n P_r = \frac{n!}{(n-r)!}$$

3. The number of distinct permutation of n objects in which n_1 are alike, n_2 are alike ---- etc is

$$\frac{n!}{n_1! * n_2! * ... * n_k!}, \text{ where } n = n_1 + n_2 + ... + n_k$$

4. An arrangement of n distinct objects around a circle is $(n-1)!$ ways.

Example 1: Suppose we have the letters A, B, C, and D

a) How many permutations are there by taking all the four?

b) How many permutations can be formed by taking two letters at a time?

Solution: a) Here $n = 4$, there are four distinct objects. Hence, there are $4! = 24$ permutations.

b) Here $n = 4$ and $r = 2$. There are ${}_4P_2 = \frac{4!}{(4-2)!} = \frac{24}{2} = 12$ permutations.

Example 2: How many different permutations can be made from the letters in the word “CORRECTION”?

Ans: 453600

Example 3: In how many ways can a party of 7 people arrange themselves

a) on a row of 7 chairs?

b) around a circular table?

c) How many ways are there if two of the persons are not allowed to sit next to each other? (The arrangement is in a row)

Ans.: 5040, 720 and 3600 respectively.

IV. Combination

A selection of r objects from n objects without regard to order is called combination.

Example: Given the letters A, B, C, and D list the permutation and combination for selecting two letters.

Solution:

Permutation	Combination
AB, BA, CA, DA,	AB, BC,
AC, BC, CB, DB,	AC, BD,
AD, BD, CD, DC	AD, DC

Note that: in permutation AB is different from BA. But in combination AB is the same as BA.

Combination Rule

The number of combinations of r objects selected from n objects is denoted by ${}_nC_r$ or $\binom{n}{r}$ and is given by the formula:

$$\binom{n}{r} = \frac{n!}{(n-r)! * r!}$$

Example 1: In how many ways can a committee of 5 people be chosen out of 9 people?

Ans: 126 ways

Example 2: Among 15 clocks there are two defectives. In how many ways can an inspector choose three of the clocks for inspection so that:

- a) there is no restriction,
- b) none of the defective clocks is included,
- c) only one of the defective clocks is included,
- d) two of the defective clocks is included.

Ans: 455, 286, 156, and 13 ways, respectively.

Example 3: A delegation of four people is selected each year from a college to attend a meeting.

- a. In how many ways can the delegation be chosen if there are 12 eligible students?
- b. In how many ways if two of the eligible students will not attend the meeting together?
- c. In how many ways if two of the eligible students are married and will only attend the meeting together?

$${}^{12}C_4=495, {}^{10}C_4=210 + 2C_1*{}^{10}C_3=240=450, {}^{10}C_4=210+ 2C_2*{}^{10}C_0 = 211$$

Some remarks

1. When we select r objects from n distinct objects, we have $(n - r)$ objects unselected and hence there are as many ways of selecting r from n as there are not selected $n - r$ objects.
Thus

- a. $\binom{n}{r} = \binom{n}{n-r}$
- b. $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

2. Binomial Coefficients

The quantity $\binom{n}{r}$ is also known as a binomial coefficient because it is the coefficient of the term $a^{n-r}b^r$ in the expansion of the expression $(b+a)^n$.

Binomial Theorem

For any two real numbers a and b and any positive integer n we have $(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$

Special cases

$$\begin{aligned} 1. (a+1)^n &= \sum_{r=0}^n \binom{n}{r} a^r & 2. (1-a)^n &= \sum_{r=0}^n \binom{n}{r} (-a)^r \\ 3. (1+1)^n &= \sum_{r=0}^n \binom{n}{r} = 2^n & 4. (1-1)^n &= \sum_{r=0}^n \binom{n}{r} (-1)^r = 0 \end{aligned}$$

1.5. Definitions of probability

In any random experiment there is always uncertainty as to whether a particular event will or will not occur. Thus, if A is an event associated with an experiment, we cannot state with certainty that an event A will occur or will not occur. Hence, it becomes very important to try to associate a number with the event A which will measure how likely is the event A will occur in the defined sample space. This leads to the theory of probability in which we are going to consider the quantitative measure of uncertainty.

Approaches to Measuring Probability

There are four different conceptual approaches to study probability theory. These are:

- ✓ The classical approach.
- ✓ The frequentist approach.
- ✓ The subjective approach.
- ✓ The axiomatic approach.

A. The classical approach

This approach is used when:

- ☞ All outcomes are equally likely and mutually exclusive.
- ☞ Total number of outcomes is finite, say N .

Definition: If a random experiment with N equally likely outcomes is conducted and out of these N_A outcomes are favorable to the event A , then the probability that event A occurs denoted $P(A)$ is defined as:

$$P(A) = \frac{N_A}{N} = \frac{\text{No. of outcomes favorable to the event } A}{\text{Total number outcomes}} = \frac{n(A)}{n(S)}$$

Example 1: A fair die is tossed once. What is the probability of getting

- a) Number 4?
- b) An odd number?

- c) Number greater than 4?
- d) Either 1 or 2 or or 6

Example 2: A box of 80 candles consists of 30 defective and 50 non-defective candles. If 10 of these candles are selected at random, what is the probability?

- a) All will be defective.
- b) 6 will be non defective
- c) All will be non defective

Solution: Total selection $= \binom{80}{10} = N = n(S)$

Let A be the event that all will be defective.

The total ways in which A will occur $= \binom{30}{10} \binom{50}{0} = N_A = n(A)$

Hence,

$$P(A) = \frac{N_A}{N} = \frac{n(A)}{n(S)} = \frac{\binom{30}{10} \binom{50}{0}}{\binom{80}{10}} = 0.00001825$$

Ans: 0.265, and 0.00624 for b and c, respectively.

Limitation: The classical approach cannot be employed if:

- ✓ it is not possible to enumerate all the possible outcomes for an experiment.
- ✓ the sample points (outcomes) are not mutually exclusive.
- ✓ each and every outcomes is not equally likely.

B. The Frequentist Approach

This is based on the relative frequencies of outcomes belonging to an event.

Definition: The probability of an event A is the proportion of outcomes favorable to A in the long run when the experiment is repeated under same condition.

$$P(A) = \lim_{N \rightarrow \infty} \frac{N_A}{N}$$

Example 1: If records show that 60 out of 100,000 bulbs produced are defective. What is the probability of a newly produced bulb to be defective?

$$P(A) = \lim_{N \rightarrow \infty} \frac{N_A}{N} = \frac{60}{100,000} = 0.0006$$

Limitations of frequentist approach:

- ✓ If repeated trials are not possible we cannot use this approach
- ✓ The limit as N approaches to infinity has no clear meaning (How large is large??)

- ✓ Generating infinite trials has cost imposition.

C. Subjective Approach

It is always based on some prior body of knowledge. Hence subjective measures of uncertainty are always conditional on this prior knowledge.

Definition: Subjective probability is the degree of believe assigned to the occurrence of an event by a particular individual.

Examples: - A medical Dr. may assign the probability that a patient recovers from a disease.

- Estimating the likelihood that you will earn an A grade in this course.

D. Axiomatic Approach:

Let E be a random experiment and S be a sample space associated with E . With each event A defined on S . We can associate a real number called the probability of A satisfying the following properties called axioms of probability or postulates of probability.

1. $P(A) \geq 0$ (i.e. the probability of any event is non-negative real number)
2. $P(S) = 1$ (Sure event)
3. If A and B are two mutually exclusive events, the probability that one or the other occur equals the sum of the two probabilities. i. e. $P(A \cup B) = P(A) + P(B)$.
4. If $A_1, A_2, A_3, \dots$ form a sequence of pair wise mutually exclusive events (meaning they satisfy $A_i \cap A_j = \emptyset$ for all $i \neq j$) then

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

1.6. Derived theorems of probability

1. $P(\emptyset) = 0$ for any sample space S
2. If A' is the complement of event A , then $P(A') = 1 - P(A)$
3. If A and B are any two events in S , then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. If $A \subseteq B$ then $P(A) \leq P(B)$
5. For any two events say, A and B , the probability that exactly one of the events A or B occurs but not both is :

$$P(A) + P(B) - 2P(A \cap B)$$

Examples

1) For any events A and B show that $P(A \cup B) \leq P(A) + P(B)$

2) Suppose A and B are events for which

$$P(A) = x \quad P(B) = y \quad \text{and} \quad P(A \cap B) = z$$

Find i. $P(A' \cup B)$ ii) $P(A' \cap B')$

3) If $P(A) = \frac{3}{4}$ and $P(B') = \frac{1}{4}$, can A and B be mutually exclusive?

4) If $P(A^c) = a$ $P(B') = b$ Show that $P(A \cap B) \geq 1 - a - b$