

## CHAPTER FOUR: POWER SERIES

## 4.1. Definition of Power Series

**Definition 4.1**(Power series in  $x$  or centered at 0)

A series of the form  $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots$  where  $x$  is a variable and the constants  $a_k$ 's are the coefficients, is called a **power series** in  $x$ .

**Definition 4.2** (Power series in  $x - c$  or centered at  $c$ )

A series of the form  $a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_k(x - c)^k + \cdots = \sum_{k=0}^{\infty} a_k(x - c)^k$  is called a **power series in  $(x - c)$**  or a power series **centered at  $c$**  or about  $c$ , where  $c$  is any constant.

**Note:** Any finite partial sum of the above power series is simply a polynomial in  $x$ .

**Examples:**

1.  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is a power series in  $x$
2.  $\sum_{k=0}^{\infty} \frac{(x+1)^k}{2^k}$  is a power series about  $-1$ , or centered at  $-1$ .
3.  $\sum_{k=0}^{\infty} \frac{(x-1/2)^k}{(k+1)!}$  is a power series in  $x - 1/2$  or centered at  $1/2$ .

## 4.2. Convergence, Divergence, Radius and Interval of a power series

The major characteristic of a power series is that **it involves a variable**, and its convergence depends on the value that  $x$  takes. Consequently, it is important to know the set of numbers that make the power series converge.

**Example:** For what value(s) of  $x$  does the power series  $\sum_{n=0}^{\infty} ax^n$  converge, for  $a \neq 0$ ?

**Solution:**  $\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \cdots$  is a geometric series with first term  $a_1 = a$  and common

ratio  $r = x$  which converges to  $\frac{a_1}{1-r}$  for  $|r| < 1$ . i.e., for  $|x| < 1$ .

Thus,  $\sum_{n=0}^{\infty} ax^n$  converges to  $\frac{a}{1-x}$  for  $|x| < 1$ .

**Theorem 4.1:** For any given power series of the form  $\sum_{k=0}^{\infty} a_k (x-c)^k$ , there are only the following three possibilities:

- (a) the series converges only for  $x = c$  or
- (b) the series converges for all  $x \in \mathbb{R}$ , or
- (c) There is a positive number  $R$  such that the series converges if  $|x - c| < R$  and diverges if  $|x - c| > R$ .

**Definition 4.3:** The set  $\{a \in \mathbb{R} : \text{the power series converges when } x \text{ is replaced by } a\}$  is called the **interval of convergence** of the power series and the number  $R$  in the previous theorem is called the **radius of convergence** for the given power series.

**Remarks:**

1. In case (a) of *theorem 2.1*, the radius of convergence is 0 and in case (b), the radius is  $\infty$ .
2. If  $R = \infty$ , then its interval of convergence is  $(-\infty, \infty)$ .
3. There are four possibilities for the interval of convergence of the power series with respect to its radius of convergence  $R$ ; these are:  
 $(c - R, c + R)$ ,  $[c - R, c + R]$ ,  $[c - R, c + R)$  or  $(c - R, c + R]$ .
4. To determine the radius and interval of convergence of a power series, we can apply the **absolute ratio test** or the **generalized convergence test**.

Furthermore, the radius of convergence of the power series can be computed using the

formula 
$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

**Example:** Determine the radius and interval of convergence of each of the following power series.

a)  $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k}$

b)  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

c)  $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

d)  $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$

e)  $\sum_{n=0}^{\infty} n! x^n$

**Solutions:**

$$\text{a) Let } a_n = \frac{(x-3)^n}{n} \Rightarrow a_{n+1} = \frac{(x-3)^{n+1}}{n+1}$$

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ &= |x-3| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x-3| \cdot 1 \\ &= |x-3| = r \end{aligned}$$

Then by absolute ratio test, the given series converges for  $r < 1$ ,

$$\text{i.e. } r = |x-3| < 1 \Leftrightarrow 2 < x < 4,$$

But the *absolute ratio test* tells us nothing if  $r = |x-3| = 1$ . In this case, we use direct substitution to determine whether the power series converges or diverges at the end points of the interval, with  $x = 2$  and  $x = 4$ .

Thus, if  $x = 2$ , the series becomes  $\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  which is a convergent alternating series

And if  $x = 4$ , the series becomes  $\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ , which is the divergent harmonic series.

Therefore, the power series converges for all  $x \in [2, 4)$ .

This means, its radius of convergence is  $R = 1$  ( $R = \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k \rightarrow \infty} \frac{1/k}{1/k+1} = 1$ ) and its interval of convergence is  $[2, 4)$ .

$$\text{b) Let } r = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

$\Rightarrow r = 0 < 1$ , Thus, the series converges for all  $x \in \mathbb{R}$ ,

Thus, its radius of convergence is  $R = \infty$  and its interval of convergence is  $(-\infty, \infty)$ .

$$\text{c) } r = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1} x^{n+1}}{(\sqrt{n+2}) (-3x)^n} \cdot \sqrt{n+1} \right| = 3|x|$$

Then by the absolute ratio test, the given series converges for  $r < 1$ ,

$$\Rightarrow 3|x| < 1 \Rightarrow |x| < \frac{1}{3}. \quad \Rightarrow R = \frac{1}{3}.$$

If  $x = -\frac{1}{3}$ , the series becomes  $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$  which can be shown to be a divergent series.

But if  $x = \frac{1}{3}$ , the series becomes  $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$  which is a convergent alternating series.

Hence, the interval of convergence of the power series  $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$  is  $\underline{\underline{\left(-\frac{1}{3}, \frac{1}{3}\right]}}$ .

$$\text{d) Let } a_k = \frac{(x-5)^k}{k^2} \Rightarrow a_{k+1} = \frac{(x-5)^{k+1}}{(k+1)^2}$$

$$\text{Then } r = \lim_{k \rightarrow \infty} \left| \frac{(x-5)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x-5)^k} \right| = |x-5| \lim_{k \rightarrow \infty} \left( \frac{k}{1+k} \right)^2 = |x-5|$$

Then, by absolute ratio test, the series converges absolutely if  $|x-5| < 1$ ,

i.e, for  $4 < x < 6$ , and diverges for  $x < 4$  and  $x > 6$ .

To determine the convergence behavior of the power series at the end points,  $x = 4$  and  $x = 6$ , we substitute the numbers directly instead of  $x$  as follows:

$$\text{- If } x = 4, \sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}, \text{ which converges.}$$

$$\text{- If } x = 6, \text{ then } \sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(6-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}, \text{ which also converges.}$$

Hence, for the given power series, its radius of convergence is  $\underline{\underline{R=1}}$ , and its interval of convergence is  $\underline{\underline{[4,6]}}$ .

$$\text{e) } r = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right|$$

$$= \lim_{n \rightarrow \infty} |x|(n+1) = \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\Rightarrow$  The series converges only for  $\underline{\underline{x=0}}$  which means  $\underline{\underline{R=0}}$ .

## 4.3. Algebraic Operations On convergent power Series

*(Reading Assignment for Learners)*

## Sum and Difference of power series

Suppose that we have two functions defined by power series

$$f(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots$$

with radius of convergence  $R_1$  and

$$g(x) = b_0 + b_1(x - c) + b_2(x - c)^2 + b_3(x - c)^3 + \cdots$$

with radius of convergence  $R_2$ ,

then

$$f(x) \pm g(x) = \sum_{n=1}^{\infty} (a_n \pm b_n)(x - c)^n$$

and the radius of convergence is

(i)  $\min\{R_1, R_2\}$  if  $R_1 \neq R_2$

(ii)  $\geq R_1$  if  $R_1 = R_2$

## Product of power series

Suppose that we have two functions defined by power series

$$f(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots$$

with radius of convergence  $R_1$  and

$$g(x) = b_0 + b_1(x - c) + b_2(x - c)^2 + b_3(x - c)^3 + \cdots$$

with radius of convergence  $R_2$ ,

then

$$\begin{aligned} f(x)g(x) = & a_0 b_0 + [a_1 b_0 + a_0 b_1](x - c) + \\ & [a_2 b_0 + a_1 b_1 + a_0 b_2](x - c)^2 + \\ & [a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3](x - c)^3 + \cdots \end{aligned}$$

and the radius of convergence is  $\min\{R_1, R_2\}$  unless either  $f(x) = 0$  or  $g(x) = 0$ .

## Reciprocal of a Power series

Theorem 2.2:

Suppose that we have a function defined by power series

$$f(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots$$

with radius of convergence  $R$  and  $a_0 \neq 0$ , then

$$\frac{1}{f(x)} \text{ has a power series of the form } \sum_{n=0}^{\infty} q_n (x - c)^n$$

with positive radius of convergence  $\leq R$  and further more

$$q_0 = \frac{1}{a_0}$$

## Using Long Division to find the reciprocal of a power series

Example:

We know that  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$  ( see page 13 )

and  $\cos 0 \neq 0$ , therefore we have a power series of  $\sec x$  around the point  $x = 0$ .

This power series can be found by the long division

$$\left[ 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \overline{)1}$$

$$\begin{array}{r}
 \left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right] \overline{) \begin{array}{l} 1 + \frac{x^2}{2} + \frac{5x^4}{24} \\ 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \\ \hline \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \dots \\ \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48} - \frac{x^8}{1440} + \dots \\ \hline \frac{5x^4}{24} - \frac{14x^6}{720} + \dots \end{array} }
 \end{array}$$

$$\therefore \sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots$$

#### 4.4. Differentiation and Integration of power series

If the power series  $\sum_{n=0}^{\infty} a_n (x-c)^n$  has radius of convergence  $R > 0$ , then the function  $f$  defined by

$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$  is differentiable on the interval  $(c-R, c+R)$  and its derivative and integral

are as given below:

(i)  $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$  (Using term by term differentiation of the series

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_k(x-c)^k + \dots = \sum_{k=0}^{\infty} a_k (x-c)^k)$$

(ii)  $\int f(x) dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + a_2 \frac{(x-c)^3}{3} + \dots$

i.e.,  $\int \left[ \sum a_n (x-c)^n \right] dx = \sum_{n=0}^{\infty} \int a_n (x-c)^n dx = C + \sum_{n=0}^{\infty} \frac{1}{n+1} a_n (x-c)^{n+1}$  where  $C$  is the constant of integration.

**Example 1:** we know that the series  $\sum_{n=0}^{\infty} x^n$  converges to  $\frac{1}{1-x}$  for  $|x| < 1$ .

$\Rightarrow R=1$  and it is differentiable on the interval  $(-1,1)$ .

$$\text{i.e., } f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1 \dots \dots \dots (1)$$

Differentiating both sides of (1) with respect to the variable  $x$  as  $\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} (1 + x + x^2 + \dots)$

gives:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=1}^{\infty} n x^{n-1}$$

Integrating both sides of (1) yields:

$$-\ln|1-x| = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$\Rightarrow -\ln|1-x| = C + \sum_{n=1}^{\infty} \frac{x^n}{n}, \text{ for } |x| < 1.$$

To get the value of the constant  $C$ , we can put in any number in the interval  $(-1,1)$ .

For instance, if we insert  $x=0$ , we get:  $-\ln|1-0| = c \Rightarrow c=0$ .

$$\therefore \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \text{ for } |x| < 1 \dots \dots \dots (2)$$

This is the power series representation of the function  $f(x) = \ln(1-x)$  for  $|x| < 1$ .

Similarly, if we insert  $x = \frac{1}{2}$  in equation (2), we obtain the power series representation for  $\ln 2$

$$\text{to be } \ln 2 = \sum_{n=1}^{\infty} \frac{1}{n 2^n}$$

Again if we put  $-x$  in terms of  $x$  in (2), we get:

$$\ln(1+x) = -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \dots \dots \text{power series representation for } f(x) = \ln(1+x).$$

**Exercise:** Find the power series representation of  $\frac{1}{1+x^2}$  for  $|x| < 1$ .

**Example 2:** Find the power series representation of  $f(x) = \tan^{-1} x$ .

**Solution:** In  $\tan^{-1} x = \int_0^x \frac{1}{1+t^2} dt$  and  $\frac{1}{1-x} = 1 + x + x^2 + \dots$  for  $|x| < 1$ , replacing  $x$  by  $-t^2$ , we get:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \dots = \sum_{n=0}^{\infty} (-1)^n t^{2n} \quad \text{for } |t^2| < 1 \dots \dots \dots (3)$$

Integrating both sides of (3) yields  $\int_0^x \frac{1}{t^2+1} dt = \int_0^x \left( \sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt$

$$\Leftrightarrow \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\therefore \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1} x, \quad |x| < 1.$$

This implies that the power series  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$  converges to  $\tan^{-1} x$  for  $|x| < 1$  or the power series

expansion of  $f(x) = \tan^{-1} x$  is  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1} x, \quad |x| < 1.$

**Exercise:** Verify that the power series  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$  also converge for  $|x| = 1$

and as a result  $\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$

#### 4.5. Taylor and Maclaurin Series

Given a function  $f$  and a point  $c$  on the  $x$  – axis.

**Claim:** To represent  $f$  as a power series in  $x$  or  $(x - c)$  (if possible)

That is, to find the constants  $a_0, a_1, \dots, a_n, \dots$  such that

$$f(x) = \sum_{k=0}^n a_k (x-c)^k \quad \text{or} \quad f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k.$$

To do this, we assume  $f$  has derivatives of  $n$  – orders at  $c$ , and suppose  $f$  can be represented by the polynomial,

$$f(x) = P_n(x) = \sum_{k=0}^n a_k (x-c)^k \dots \dots \dots (4)$$

To determine the coefficients in  $a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$ , we proceed as follows:

Let  $x = c$  in (4)

Then  $f(c) = a_0$ .

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots, |x-c| < R$$

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-c) + \cdots$$

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x-c) + \cdots$$

$$\vdots$$

$$f^{(n)}(c) = n! a_n$$

$$\Rightarrow f(c) = a_0, f'(c) = a_1, f''(c) = 2a_2$$

$$f'''(c) = 3 \cdot 2a_3, \cdots, f^{(n)}(c) = n! a_n$$

This implies that the coefficients  $a_k$ 's are:

$$a_0 = f(c)$$

$$a_1 = f'(c)$$

$$a_2 = \frac{f''(c)}{2!}$$

$$a_3 = \frac{f'''(c)}{3!}$$

$$\vdots$$

$$a_k = \frac{f^{(k)}(c)}{k!} \Rightarrow a_n = \frac{f^{(n)}(c)}{n!}$$

**Theorem 4.3:** If a function  $f$  has a polynomial approximation at  $x = c$ ,

$$f(x) = \sum_{k=0}^n a_k (x-c)^k, \text{ for } |x-c| < R, \text{ then the coefficients } a_k \text{'s are given by the formula } a_k = \frac{f^{(k)}(c)}{k!}$$

Then putting this formula of  $a_k$  in to (4), we obtain the following definition for

$n^{\text{th}}$  – Taylor polynomial.

**Definition 4.4:** If  $f$  can be differentiable  $n$  – times at  $x = c$ , then we define the  $n^{\text{th}}$  – Taylor polynomial for the function  $f$  about  $x = c$  to be:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(c) (x-c)^k}{k!}.$$

**Definition 4.5:** If  $f$  has derivatives of all orders (i.e., infinitely differentiable) at  $x=c$ , then the **Taylor series** for  $f$  about  $x=c$ , is defined as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c) (x-c)^n}{n!} \dots\dots\dots (5)$$

**Definition 4.6:** If we put  $c=0$  in particular in (5) above it is reduced to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \text{ which is called the } \mathbf{Maclaurin Series} \text{ representation for } f.$$

Furthermore, if  $f$  has derivatives of  $n$  – orders, the  **$n^{\text{th}}$  – Maclaurin polynomial** for  $f(x)$  is given

$$\text{by: } f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k$$

**Examples:**

1. Find the third Taylor polynomial approximation for  $f(x) = \ln x$  about  $x=1$ .
2. Find the Maclaurin series of (i)  $f(x) = \sin x$  (ii)  $f(x) = e^x$
3. Find the Taylor series representation for the function  $g(x) = \cos x$  about  $x = \pi$ .

**Solutions:**

1. From  $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x-c)^k$ , In this case, we have  $f(x) = \ln x$ ,  $c=1$ , and  $n=3$ .

Here,  $f(x) = \ln x$  is differentiable around  $x=1$ ,

$$\text{and } f'(x) = \frac{1}{x} \Rightarrow f'(c) = f'(1) = 1.$$

$$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1.$$

$$f'''(x) = \frac{+2}{x^3} \Rightarrow f'''(1) = 2, \text{ \& } f(1) = \ln(1) = 0$$

Then the third Taylor polynomial for  $f(x) = \ln x$  is:

$$\begin{aligned}
P_3(x) &= \sum_{k=0}^3 \frac{f^{(k)}(1)(x-1)^k}{k!} \\
&= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 \\
&= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 \\
&= \ln(1) + 1(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 \\
\Rightarrow \underline{\underline{P_3^{(3)}(x)} &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}
\end{aligned}$$

$$\begin{aligned}
2. \text{ (i) } f(x) &= \sin x \Rightarrow f(0) = \sin(0) = 0 \\
f'(x) &= \cos x \Rightarrow f'(0) = \cos 0 = 1 \\
f''(x) &= -\sin x \Rightarrow f''(0) = -\sin 0 = 0 \\
f'''(x) &= -\cos x \Rightarrow f'''(0) = -\cos(0) = -1. \\
f^{(iv)}(x) &= \sin x \Rightarrow f^{(iv)}(0) = 0 \\
f^{(v)}(x) &= \cos x \Rightarrow f^{(v)}(0) = 1 \\
&\vdots \qquad \qquad \qquad \vdots
\end{aligned}$$

Then the Maclaurin series for  $f(x) = \sin x$  is

$$\begin{aligned}
f(x) = \sin x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!} \\
&= \frac{0.x^0}{0!} + \frac{1.x^1}{1!} + \frac{0.x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{0.x^4}{4!} + \frac{1.x^5}{5!} + \dots \\
\Rightarrow \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
\end{aligned}$$

Using sigma notation, the Maclaurin series for  $f(x) = \sin x$  is:

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}(-1)^k}{(2k+1)!}$$

$$\text{(ii). } f(x) = e^x \Rightarrow f(0) = e^0 = 1.$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$$

$$\vdots$$

$$f^{(k)}(0) = 1 \quad \forall k.$$

$$\text{So, } e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\Rightarrow e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ i.e., the Maclaurin series for } e^x \text{ is } \underline{\underline{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}}$$

$$3. \quad g(x) = \cos x \Rightarrow g^{(0)}_{(\pi)} = g(\pi) = \cos \pi = -1$$

$$g'(x) = -\sin x \Rightarrow g'(\pi) = 0$$

$$g''(x) = -\cos x \Rightarrow g''(\pi) = 1$$

$$g'''(x) = \sin x \Rightarrow g'''(\pi) = 0$$

$$g^{(iv)}(x) = \cos x \Rightarrow g^{(iv)}(\pi) = -1$$

$$\vdots$$

Then its Taylor series representation at  $x = \pi$  is:

$$\begin{aligned} g(x) = \cos x &= \sum_{k=0}^{\infty} \frac{g^{(k)}(\pi)(x-\pi)^k}{k!} \\ &= g^{(0)}(\pi) + g'(\pi)(x-\pi) + \frac{g''(\pi)(x-\pi)^2}{2!} + \frac{g'''(\pi)(x-\pi)^3}{3!} + \frac{g^{(iv)}(\pi)(x-\pi)^4}{4!} + \dots \\ &= -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 + \dots \end{aligned}$$

Using sigma notation, it can be written as  $\underline{\underline{\cos x = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(x-\pi)^{2k}}{(2k)!}}}$  (Taylor series representation

for  $\cos x$  about  $x = \pi$ ).

### The Binomial Series (*Reading assignment*)

**Definition:** If  $m$  is a real number, then the Maclaurin series for  $(1+x)^m$  is called the **binomial series**, which is given by:

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \frac{m(m-1)(m-2)(m-3)}{4!}x^4 + \dots = \sum_{n=0}^{\infty} \binom{m}{n} x^n = (1+x)^m \text{ where}$$

$$\binom{m}{n} = \frac{m(m-1)(m-2)\dots(m-n+1)}{n!}, \text{ which is the coefficient of } x^n \text{ in the expansion.}$$

- If  $m$  is a non – negative integer, then ,  $f(x) = (1+x)^m$  is clearly a polynomial of degree  $m$ , and hence,
- $f^{(m+1)}(0) = f^{(m+2)}(0) = \dots = f^{(k)}(0) = 0$  for all  $k > m$ , and the binomial series reduces to

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + x^m, \text{ which is valid for } -\infty < x < \infty.$$

### Examples:

1. Find the binomial series expansion for  $(1+x)^5$
2. Find the coefficient of  $x^4$  in the binomial expansion of  $(1+x)^6$ .

### Solutions

$$1. (1+x)^5 = 1 + 5x + \frac{20}{2!}x^2 + \frac{60}{3!}x^3 + \frac{120}{4!}x^4 + x^5 \text{ (since } m=5)$$

$$\text{Thus, } (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

2. Coefficient of  $x^4$  is: ( $m=6$ )

$$\begin{aligned} & \frac{m(m-1)(m-2)(m-3)}{4!} \\ &= \frac{6(6-1)(6-2)(6-3)}{4!} = \frac{6!}{4!2!} = \frac{6 \cdot 5 \cdot 4!}{4!2!} = \underline{\underline{15}} \end{aligned}$$