



Wisdom at the source of the Blue Nile

BAHIR DAR UNIVERSITY

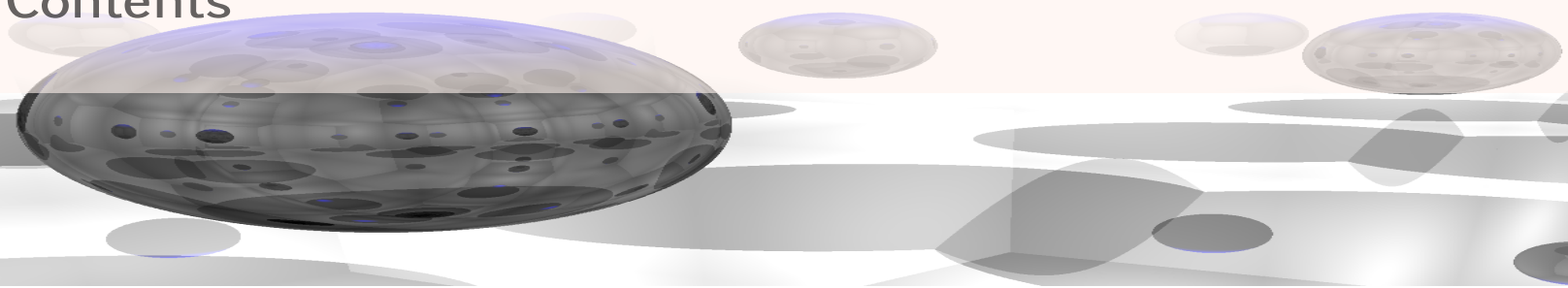
BAHIR DAR INSTITUTE OF TECHNOLOGY (BIT)

APPLIED MATHEMATICS-I PART ONE

EB ENEYEW, ME BEKELE, DW

BAHIR DAR UNIVERSITY

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1. Vectors and Their Applications

1.1 Scalars and Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition 1.1 Quantities which can be described by magnitude only are called scalars for example mass, length, speed, $\dots$

Definition 1.2 Quantities that have both magnitude and direction are described mathematically by vectors. Hence, a vector is a physical quantity which has both magnitude and direction. Vectors are usually denoted by lowercase boldface letters, such as $\mathbf{a}$ or $\vec{a}$, especially in handwriting. For instance velocity ($\vec{v}$), acceleration ($\vec{a}$), force ($\vec{F}$), e.t.c

Alternative definition of a Vector: is a directed line segment that has an initial and terminal point. If the vector represents a directed distance or displacement from a point P to a point Q it can be denoted as $\overrightarrow{PQ}$ or PQ .

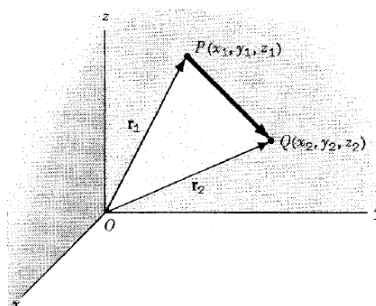
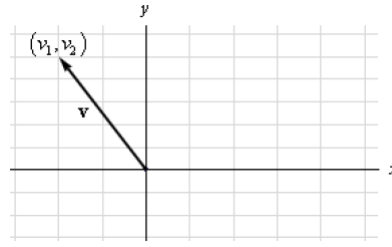


Figure 1.1: Vector in three dimensional space.

Vectors are usually shown in graphs or other diagrams as arrows (directed line segments), as illustrated in the figure. Here the point P is called the origin, tail, base, or initial point; point Q is called the head, tip, endpoint, terminal point or final point. The length of the arrow is proportional to the vector's magnitude, while the direction in which the arrow points indicates the vector's direction.

Definition 1.3 If $\vec{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2) , then the component form of vector $\vec{v}$ is given by $\vec{v} = \langle v_1, v_2 \rangle$. Similarly if $\vec{u}$ is a vector in the space whose initial point is the origin and whose terminal point is (u_1, u_2, u_3) , then the component form of vector $\vec{u}$ is given by $\vec{u} = \langle u_1, u_2, u_3 \rangle$.

As an example in two dimensions, see Figure 1.2, the vector from the origin $O = (0,0)$ to the point $v = (v_1, v_1)$ is simply written as $\vec{v} = \langle v_1, v_1 \rangle$.

Figure 1.2: Vector in $\mathbb{R}^2$

In three dimensional euclidean space, $\mathbb{R}^3$, vectors are identified with triples of scalar components as $\vec{a} = \langle a_1, a_2, a_3 \rangle$ or $\vec{a} = \langle a_x, a_y, a_z \rangle$. This can be generalized to n-dimensional euclidean space, $\mathbb{R}^n$, as $\vec{a} = \langle a_1, a_2, a_3, \dots, a_{n-1}, a_n \rangle$.

1.2 Operations on Vectors

1.2.1 Vector Addition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the sum of vector $\vec{a}$ and $\vec{b}$ is given by $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. The addition may be represented graphically by placing the tail of the arrow $\vec{b}$ at the head of the arrow $\vec{a}$, and then drawing an arrow from the tail of $\vec{a}$ to the head of $\vec{b}$. The new arrow drawn represents the vector $\vec{a} + \vec{b}$, as illustrated in Figure 1.3 below: This addition method is sometimes called

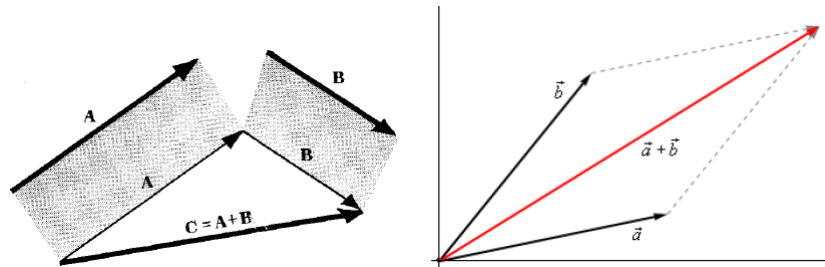


Figure 1.3: Addition of two vectors

the parallelogram rule because $\vec{a}$ and $\vec{b}$ form the sides of a parallelogram and $\vec{a} + \vec{b}$ is one of the diagonals. Having the same base point, this point will also be the base point of $\vec{a} + \vec{b}$. One can check geometrically that $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ and $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

1.2.2 Scalar multiplication

The operation of multiplying a vector by any scalar c is called scalar multiplication. For instance, if $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is a vector in $\mathbb{R}^3$, then the resulting scalar multiple is $c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$. Another way to represent a vector in n -dimensions is to introduce the standard basis vectors. In three dimensional space, there are three standard vectors $\vec{e}_1 = \mathbf{i} = \langle 1, 0, 0 \rangle$, $\vec{e}_2 = \mathbf{j} = \langle 0, 1, 0 \rangle$, $\vec{e}_3 = \mathbf{k} = \langle 0, 0, 1 \rangle$. These have the intuitive interpretation as vectors of unit length pointing up the x , y , and z axis of a Cartesian coordinate system, respectively. Any vector $\vec{a}$ in $\mathbb{R}^3$ can be expressed in terms of standard unit vectors as:

$$\begin{aligned}\vec{a} &= \langle a_1, a_2, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle, \\ &= a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3,\end{aligned}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are called the components of vector $\vec{a}$ (or vector projections) on the basis vectors or, equivalently, on the corresponding Cartesian axes x , y , and z , see Figure 1.4, while a_1 , a_2 , a_3 are the respective scalar components or scalar projections.

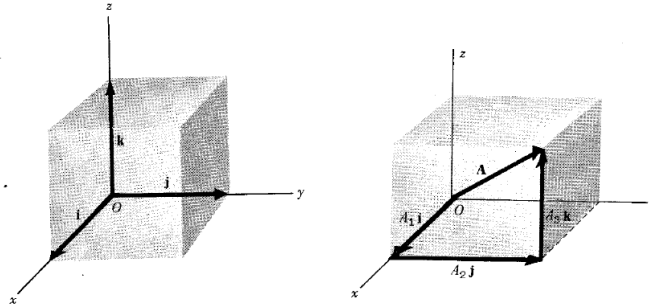


Figure 1.4: The vector notation in the form of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ axis

In introductory physics textbooks, the standard base vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are often used. Thus, $\mathbf{A} = \langle \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \rangle = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}$.

Note:-

1. Two vectors are said to be equal iff they have equal magnitude and the same direction. Equivalently they will be equal if their coordinates are equal. Thus, two vectors $\vec{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3$ and $\vec{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3$ are equal iff $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$.
2. Two vectors are opposite iff they have opposite direction but not necessarily the same magnitude. The $\vec{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3$ and $\vec{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3$ are opposite iff $a_1 = -b_1$, $a_2 = -b_2$, $a_3 = -b_3$. (does not this mean they have the same magnitude? may be we can say $\text{sign}(a_1) = -\text{sign}(b_1)$)
3. Two vectors are parallel if one is the scalar multiple of the other. That is, if $\vec{a}$ and $\vec{b}$ are parallel vectors, then there must be a number c so that $\vec{a} = c\vec{b}$.



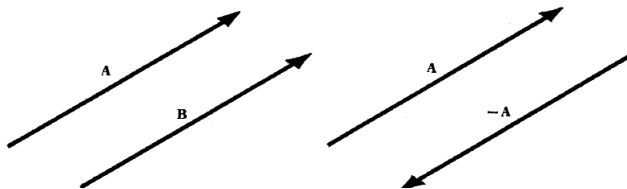


Figure 1.5: Parallel and opposite vectors

Example 1.1 Determine if the set of vectors are parallel or not.

a) $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$

b) $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$

Solution:

a) These two vectors are parallel since $\vec{b} = -3\vec{a}$.

b) These two vectors aren't parallel. This can be seen by noticing that

$$4\left(\frac{1}{2}\right) = 2 \text{ and yet } 10\left(\frac{1}{2}\right) = 5 \neq -9$$

In other words we can't make $\vec{a}$ be a scalar multiple of $\vec{b}$.

1.2.3 Vector Subtraction

The difference of $\vec{a}$ and $\vec{b}$ is $\vec{a} - \vec{b} = (a_1 - b_1)\mathbf{e}_1 + (a_2 - b_2)\mathbf{e}_2 + (a_3 - b_3)\mathbf{e}_3$. Subtraction of two vectors may also be performed by adding the opposite of the second vector to the first vector, that is, $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$. Geometrically, the difference $\vec{a} - \vec{b}$ can be obtained by placing the tail of $\vec{a}$ and $\vec{b}$ at the same point and draw an arrow from the head of $\vec{b}$ to the head of $\vec{a}$. This new arrow or vector represents $\vec{a} - \vec{b}$ as illustrated in Figure 1.5.

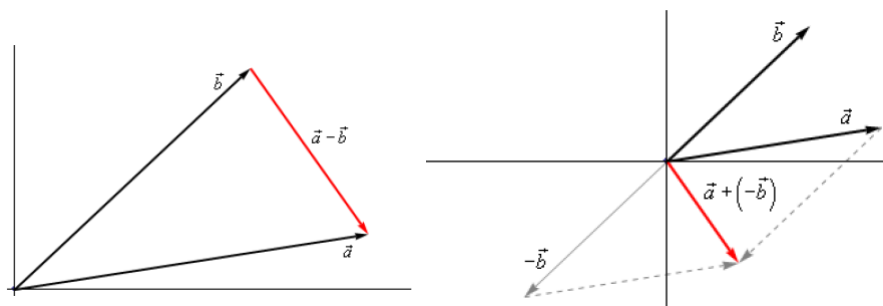


Figure 1.6: Subtraction of vectors

Example 1.2 1. Add the following two vectors

a. $\vec{u} = \langle -1, 2, 1 \rangle$ and $\vec{v} = \langle 2, -5, 0 \rangle$

b. $\vec{u} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\vec{v} = 2\mathbf{i} - 5\mathbf{j}$

2. Let $\vec{u} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, then find

a. $\frac{2}{3}\vec{u}$

b. $-1\vec{u}$

Solution:-

1. a.

$$\begin{aligned}\vec{u} + \vec{v} &= \langle -1, 2, 1 \rangle + \langle 2, -5, 0 \rangle \\ &= \langle -1 + 2, 2 - 5, 1 + 0 \rangle \\ &= \langle 1, -3, 1 \rangle\end{aligned}$$

b.

$$\begin{aligned}\vec{u} + \vec{v} &= (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) + (2\mathbf{i} - 5\mathbf{j}) \\ &= (1 + 2)\mathbf{i} + (4 - 5)\mathbf{j} + (-1 + 0)\mathbf{k} \\ &= 3\mathbf{i} - \mathbf{j} - \mathbf{k}\end{aligned}$$

2. a.

$$\begin{aligned}\frac{2}{3}\vec{u} &= \frac{2}{3}(3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \\ &= 2\mathbf{i} + \frac{8}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\end{aligned}$$

b.

$$\begin{aligned}-1\vec{u} &= -(3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \\ &= -3\mathbf{i} - 4\mathbf{j} + \mathbf{k}\end{aligned}$$

Theorem 1.1 Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors and α and β are any scalars then the following holds true:

1. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

2. $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.

3. $\vec{v} + \vec{0} = \vec{v}$.

4. $\vec{v} + -\vec{v} = \vec{0}$.

5. $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$

6. $(\alpha + \beta) \cdot \vec{v} = \alpha \cdot \vec{v} + \beta \cdot \vec{v}$.

$$7. \alpha \cdot (\vec{v} + \vec{w}) = \alpha \cdot \vec{v} + \alpha \cdot \vec{w}.$$

$$8. 1 \cdot \vec{v} = \vec{v}.$$

Proof Exercise

1.3 Scalar or Dot product of vectors

Definition 1.4 The dot product of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ (sometimes called the inner product, or, since its result is a scalar, the scalar product) is denoted by $\vec{a} \cdot \vec{b}$ and is defined as:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$



Note: The scalar product is also called a dot product or inner product.

Example 1.3 Compute the dot product for each of the following

a) $\vec{v} = 5\mathbf{i} - 8\mathbf{j}$, $\vec{w} = \mathbf{i} + \mathbf{j}$

a) $\vec{a} = \langle 0, 3, -7 \rangle$, $\vec{b} = \langle 2, 3, 1 \rangle$

Solution: Not much to do other than using the formula

a) $\vec{v} \cdot \vec{w} = 5 \times 1 + (-8 \times 2) = 5 - 16 = -11$

b) $\vec{a} \cdot \vec{b} = 0 \times 2 + 3 \times 3 + (-7 \times 1) = 0 + 9 - 7 = 2$

1.4 Norm of a vector

Definition 1.5 The magnitude (or norm) of a vector $\vec{u}$ is simply the length of the vector $\vec{u}$, which is the distance between its initial and terminal points, denoted by $\|\vec{u}\|$ and is defined as

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}. \quad (1.1)$$

Equation (1.1) is a consequence of the Pythagorean Theorem since the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are

orthogonal unit vectors. This is also equal to the square root of the dot product of the vector with itself:

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}.$$

Definition 1.6 Zero vector:- The zero vector is the vector with length zero.

Written out in coordinates, the zero vector is $\langle 0, 0, 0 \rangle$, and it is commonly denoted by $\mathbf{0}$. It has an arbitrary or indeterminate direction, and cannot be normalized

Definition 1.7 Unit vector:- A unit vector is any vector with a length or magnitude of one; normally unit vectors are used simply to indicate direction.

To create a unit vector parallel to any non zero vector of arbitrary length can be divided by its length. This is known as normalizing a vector. A unit vector is often indicated with a hat as in $\hat{u}$. To normalize a non- zero vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$, we multiply the vector by the reciprocal of its length $\|\vec{u}\|$. That is:

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{u_1}{\|\vec{u}\|} \mathbf{e}_1 + \frac{u_2}{\|\vec{u}\|} \mathbf{e}_2 + \frac{u_3}{\|\vec{u}\|} \mathbf{e}_3.$$

Note that there is no unit vector that is parallel to zero vector .

Theorem 1.2 Properties of dot product and magnitude. If $\vec{v}$, $\vec{u}$ and $\vec{w}$ are vectors in a plane or space and α is scalar, then the following properties hold true.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ commutative property
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$ distributive property of dot product over vector addition.
3. $\alpha(\vec{u} \cdot \vec{v}) = (\alpha\vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha\vec{v})$ associative property of scalar multiplication and dot product.
4. $\mathbf{0} \cdot \vec{u} = \mathbf{0}$ zero vector dot with other vector.
5. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$ the relation between the dot product and magnitude.
6. $\vec{u} = \mathbf{0}$ if and only if $\|\vec{u}\| = 0$ property of zero vector.
7. $\|\alpha\vec{u}\| = |\alpha| \|\vec{u}\|$

Proof exercise

Example 1.4 a). Find the component form and the length of vector with initial point and terminal point $P = (3, -7, 1)$ and $Q = (-2, 5, 3)$

b). Find the unit vector in the direction of $\vec{v} = \langle -2, 5, 3 \rangle$

c). Let $\vec{u} = \langle 2, 5, -1 \rangle$ and $\vec{v} = \langle -1, 2, 3 \rangle$, find the scalar product of $\vec{u}$ and $\vec{v}$

Solution:

a). The component form is given by $\overrightarrow{PQ} = Q - P = (-2, 5, 3) - (3, -7, 1) = \langle -2 - 3, 5 - (-7), 3 - 1 \rangle = \langle -5, 12, 2 \rangle$ and the length or magnitude of the vector is given by $\|\overrightarrow{PQ}\| = \sqrt{(-5)^2 + (12)^2 + 2^2} = \sqrt{173}$

b). To find the unit vector in the direction of $\vec{v}$ we first compute its magnitude as $\|\vec{v}\| = \sqrt{-2^2 + 5^2 + 3^2} = \sqrt{38}$ and then divide the vector by its magnitude, i.e.,

$$\begin{aligned}\hat{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{38}} \langle -2, 5, 3 \rangle \\ &= \left\langle \frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}} \right\rangle\end{aligned}$$

c). The dot or scalar product of the two vectors is given by:

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \langle 2, 5, -1 \rangle \cdot \langle -1, 2, 3 \rangle \\ &= 2 \times (-1) + 5 \times 2 + (-1) \times 3 \\ &= -2 + 10 - 3 \\ &= 5\end{aligned}$$

1.5 Angle between two vectors

Theorem 1.3 If θ is the angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

Proof exercise



Note:-

1. The cosine of the angle between two non-zero vectors $\vec{u}$ and $\vec{v}$ is $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \Rightarrow \theta = \cos^{-1} \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right)$
2. If an angle between two vectors $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{2}$, then $\vec{u} \cdot \vec{v} = 0$ and the vectors $\vec{u}$ and $\vec{v}$ are known as orthogonal vectors. In general the dot product of two non-zero vectors is zero only if they are orthogonal.

Example 1.5 1. Find the cosine of the angle and the angle between the following vectors

a) $\vec{u} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $\vec{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

b) $\vec{v} = \langle -\sqrt{3}, 1 \rangle$ and $\vec{w} = \langle 0, 5 \rangle$

c) $\vec{v} = \langle 0, -1, 0 \rangle$ and $\vec{w} = \langle 0, 0, 3 \rangle$

2. Find the value of k such that the vectors $\langle 2, 3, -2 \rangle$ and $\langle 5, k, 2 \rangle$ are perpendicular.

3. Find the value of β such that the vectors $\langle \beta, -3, 1 \rangle$ and $\langle \beta, \beta, 2 \rangle$ are perpendicular.

Solution:- exercise

Remark:- Suppose $\vec{u}$ and $\vec{v}$ are two non-zero vectors and θ is the angle between $\vec{u}$ and $\vec{v}$:



i) if θ is acute, then $\vec{u} \cdot \vec{v} > 0$.

ii) if θ is obtuse, then $\vec{u} \cdot \vec{v} < 0$.

iii) if θ is right angle, then $\vec{u} \cdot \vec{v} = 0$.

1.6 Projection of Vectors

Suppose that two non-zero vectors $\mathbf{A}$ and $\mathbf{u}$ are positioned as in the Figure 1.7 below and that the light casts a shadow on the line containing the vector $\mathbf{u}$. Informally we think of the shadow as determining a vector parallel to vector $\mathbf{u}$, we call this vector the projection of vector $\mathbf{A}$ on to vector $\mathbf{u}$ which is denoted by $proj_{\mathbf{u}}\mathbf{A}$ and defined as $proj_{\mathbf{u}}\mathbf{A} = \left(\frac{\mathbf{u} \cdot \mathbf{A}}{\|\mathbf{u}\|^2} \right) \mathbf{u}$.



Figure 1.7: Projections with acute and obtuse angles



Note:- The scalar projection of vector A along the vector u is $\|proj_u A\| = \left| \frac{\vec{u} \cdot A}{\|\vec{u}\|} \right|$

Example 1.6 :- Let vector $\vec{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\vec{v} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, then find:

- a) $proj_{\vec{v}} \vec{u}$
- b) $proj_{\vec{u}} \vec{v}$
- c) $proj_{\frac{1}{2}\vec{v}} 3\vec{u}$
- d) The scalar projection of $proj_{\vec{v}} \vec{u}$ and $proj_{\vec{u}} \vec{v}$

Solution:-

a)

$$\begin{aligned}
 proj_{\vec{v}} \vec{u} &= \left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2} \right) \vec{v} \\
 &= \left(\frac{(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})}{\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\|^2} \right) (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\
 &= \left(\frac{2 - 2 + 3}{1^2 + 2^2 + 1^2} \right) (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\
 &= \frac{3}{6} (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\
 &= \frac{1}{2} \mathbf{i} + \mathbf{j} + \frac{1}{2} \mathbf{k}
 \end{aligned}$$

b)

$$\begin{aligned}
 \text{proj}_{\vec{u}} \vec{v} &= \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \right) \vec{u} \\
 &= \left(\frac{(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\|2\mathbf{i} - \mathbf{j} + 3\mathbf{k}\|^2} \right) (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \\
 &= \left(\frac{2 - 2 + 3}{1^2 + (-2)^2 + 3^2} \right) (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \\
 &= \frac{3}{14} (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \\
 &= \frac{6}{14} \mathbf{i} - \frac{3}{14} \mathbf{j} + \frac{9}{14} \mathbf{k}.
 \end{aligned}$$

c)

$$\begin{aligned}
 \text{proj}_{\frac{1}{2}\vec{v}} 3\vec{u} &= \left(\frac{\frac{1}{2}\vec{v} \cdot 3\vec{u}}{\|\frac{1}{2}\vec{v}\|^2} \right) \frac{1}{2}\vec{v} \\
 &= \left(\frac{\frac{1}{2}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot 3(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})}{\frac{1}{4}\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\|^2} \right) \frac{1}{2}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\
 &= \left(\frac{\frac{3}{2}(2 - 2 + 3)}{\frac{1}{4}(1^2 + 2^2 + 1^2)} \right) \frac{1}{2}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\
 &= 3(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}
 \end{aligned}$$

d)

$$\begin{aligned}
 \|\text{proj}_{\vec{v}} \vec{u}\| &= \left| \left(\frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|} \right) \right| \\
 &= \left| \left(\frac{(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})}{\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\|} \right) \right| \\
 &= \left| \left(\frac{2 - 2 + 3}{\sqrt{1^2 + 2^2 + 1^2}} \right) \right| \\
 &= \frac{3}{\sqrt{6}}
 \end{aligned}$$

and

$$\begin{aligned}
 \|\text{proj}_{\vec{u}} \vec{v}\| &= \left| \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \right) \right| \\
 &= \left| \left(\frac{(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\|2\mathbf{i} - \mathbf{j} + 3\mathbf{k}\|} \right) \right| \\
 &= \left| \frac{2 - 2 + 3}{\sqrt{1^2 + (-2)^2 + 3^2}} \right| \\
 &= \frac{3}{\sqrt{14}}
 \end{aligned}$$

1.6.1 Direction angles and direction cosine

Definition 1.8 Let $\mathbf{p}$ be a vector and α , β and γ are angles that a vector $\mathbf{p}$ makes with the positive x , y and z axis respectively. These angles are called directional angles of the vector $\mathbf{p}$ and $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are directional cosines of vector $\mathbf{p}$.

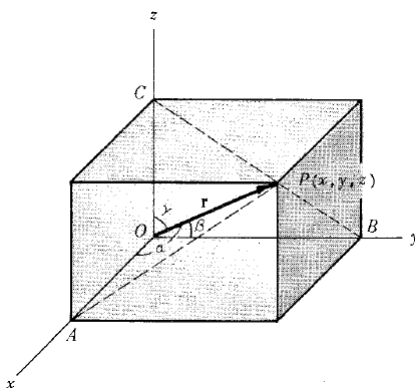


Figure 1.8: direction angeles



Note:-

1. $\cos \alpha = \frac{x}{r}$, $\cos \beta = \frac{y}{r}$ and $\cos \gamma = \frac{z}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$.
2. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

Example 1.7 1. consider the following vectors

a. $\vec{u} = \langle 2, 3 \rangle$

b. $\vec{v} = \langle 1, 1, 1 \rangle$

c. $\vec{w} = \langle 3, 5, 2 \rangle$

Then, find the direction angles and direction cosines.

2. If the direction angle of a vector $\vec{u}$ makes an angle of $\frac{\pi}{3}$ with the positive x -axis and an angle of $\frac{\pi}{4}$ with the positive y -axis, then what angle will it make with the positive z -axis?

Solution:- Exercise

1.7 Cross Product of Vectors

Definition 1.9 :- Let $\mathbf{a}$ and $\mathbf{b}$ be two arbitrary vectors, with $\hat{n}$ a unit vector normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ chosen so that $\mathbf{a}$, $\mathbf{b}$, and $\hat{n}$, in this order, obey the right-hand rule. Then, the cross product of vectors $\mathbf{a}$ and $\mathbf{b}$, written $\mathbf{a} \times \mathbf{b}$ is defined as the vector

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta) \hat{n}. \quad (1.2)$$

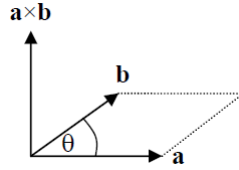


Figure 1.9: cross product

Properties of the cross product:- The following results are consequences of the definition of the cross product. Cross product is not commutative

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}. \quad (1.3)$$

The cross product is distributive over addition

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}). \quad (1.4)$$

Parallel vectors ($\theta = 0$):- If vectors $\vec{a}$ and $\vec{b}$ are parallel, then

$$\vec{a} \times \vec{b} = 0. \quad (1.5)$$

Orthogonal vectors ($\theta = \pi/2$):- If vectors $\vec{a}$ and $\vec{b}$ are orthogonal, then

$$\vec{a} \times \vec{b} = \|\vec{a}\| \|\vec{b}\| \hat{n}. \quad (1.6)$$

Product of unit vectors:- If $\vec{a}$ and $\vec{b}$ are unit vectors, then

$$\vec{a} \times \vec{b} = \sin \theta \hat{n}. \quad (1.7)$$

An immediate consequence of properties (1.2), (1.3), and (1.5) is that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \quad (1.8)$$

and

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \quad (1.9)$$

To obtain a more convenient expression for the cross product that can be used when $\vec{a}$ and $\vec{b}$ are known in terms of their components, we proceed as follows.

Let $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and consider the cross product $\vec{a} \times \vec{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$. Expanding this expression term by term is justified in (1.8) and (1.9) the result is

$$\begin{aligned}\vec{a} \times \vec{b} = & a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) + \\ & a_2b_1(\mathbf{j} \times \mathbf{i}) + a_2b_2(\mathbf{j} \times \mathbf{j}) + a_2b_3(\mathbf{j} \times \mathbf{k}) + \\ & a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k}).\end{aligned}$$

Result (1.8) makes three terms in the above equation on the right-hand side to vanish, and result (1.9) allows the remaining six terms to be collected into three groups as follows to give

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

This alternative expression for the cross product in terms of the Cartesian components of vectors $\vec{a}$ and $\vec{b}$ can be further simplified by making formal use of the third-order determinant,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}\tag{1.10}$$

Example 1.8 Given that $\vec{a} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\vec{b} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, find $\vec{a} \times \vec{b}$ and a unit vector $\hat{n}$ normal to the plane containing $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$, and $\hat{n}$, in this order, obey the right-hand rule.

Solution:- Using Equation (1.10) we get

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 1 & 4 & 2 \end{vmatrix} \\ &= ((-2) \cdot 2 - 4 \cdot (-1))\mathbf{i} - (3 \cdot 2 - 1 \cdot (-1))\mathbf{j} + (3 \cdot 4 - 1 \cdot (-2))\mathbf{k} \\ &= -7\mathbf{j} + 14\mathbf{k}.\end{aligned}$$

The required unit vector $\hat{n}$ is simply the unit vector in the direction of $\mathbf{a} \times \mathbf{b}$, hence we have

$$\begin{aligned}\hat{n} &= \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \\ &= \frac{(-7\mathbf{j} + 14\mathbf{k})}{7\sqrt{5}} \\ &= \left(\frac{-1}{\sqrt{5}}\right)\mathbf{j} + \left(\frac{2}{\sqrt{5}}\right)\mathbf{k}.\end{aligned}$$

Exercise 1.1 1. If A and B are parallel, what is $A \times B$?

2. If A and B are orthogonal, What is $\|A \times B\|$?

3. Find a unit vector perpendicular to both $A = (2, -3, 1)$ and $B = (1, 2, -4)$.

1.8 Application of cross and dot product

1.8.1 Area

Area of a parallelogram:- The area of a parallelogram defined by the side vectors $\vec{a}$ and $\vec{b}$ is determined by the formula

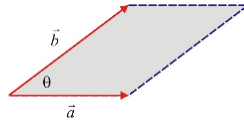


Figure 1.10: A parallelogram with sides $\vec{a}$ and $\vec{b}$

$$\begin{aligned} A &= \|\vec{a}\| \|\vec{b}\| \sin \theta \\ &= \|\vec{a} \times \vec{b}\|, \end{aligned} \tag{1.11}$$

where θ is an angle between $\vec{a}$ and $\vec{b}$.

Example 1.9 : Find the area of the parallelogram with sides the vectors $\vec{a} = \langle 1, -1, 0 \rangle$ and $\vec{b} = \langle 0, 1, 2 \rangle$.

Solution: The area of the parallelogram is given by the norm of the cross product of the two given vectors. Hence we have,

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{vmatrix} \\ &= ((-1)(2) - (1)(0))\mathbf{i} + ((0)(0) - 1(2))\mathbf{j} + ((1)(1) - (-1)(0))\mathbf{k} \\ &= -2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \end{aligned}$$

Thus, the area of the parallelogram is given by

$$\begin{aligned} A &= \|\vec{a} \times \vec{b}\| \\ &= \sqrt{(-2)^2 + (-2)^2 + 1^2} \\ &= \sqrt{4 + 4 + 1} = \sqrt{9} = 3 \end{aligned}$$

The area of a triangle:- The area of a triangle with the two side vectors $\vec{a}$ and $\vec{b}$ is given by:

$$\begin{aligned} A &= \frac{1}{2} \|\vec{a}\| \|\vec{b}\| \sin \theta \\ &= \frac{1}{2} \|\vec{a} \times \vec{b}\| \end{aligned}$$

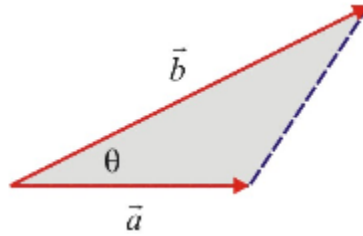


Figure 1.11: the area of triangle with sides $\vec{a}$ and $\vec{b}$

Example 1.10 Find the area of the triangle $\triangle ABC$ where $A(0, 1, 2)$, $B(-1, 0, 2)$, and $C(1, -2, 0)$.

Solution:- First, let's find the two side vectors that determine the triangle from the three given points as:

$$\vec{AB} = (-1, 0, 2) - (0, 1, 2) = \langle -1, -1, 0 \rangle$$

$$\vec{AC} = (1, -2, 0) - (0, 1, 2) = \langle 1, -3, -2 \rangle$$

Next, we compute the cross product of these vectors as:

$$\begin{aligned} \vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 0 \\ 1 & -3 & -2 \end{vmatrix} \\ &= (2 - 0)\mathbf{i} + (0 - 2)\mathbf{j} + (3 + 1)\mathbf{k} \\ &= 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Finally, the area of the triangle is obtained by computing half the norm of the above cross product as

$$\begin{aligned} A &= \frac{1}{2} \|\vec{AB} \times \vec{AC}\| \\ &= \frac{1}{2} \|2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}\| = \frac{1}{2} \sqrt{2^2 + (-2)^2 + 4^2} \\ &= \frac{1}{2} \sqrt{4 + 4 + 16} \\ &= \frac{1}{2} \sqrt{24} \\ &= \sqrt{6}. \end{aligned}$$

1.8.2 The scalar triple product

The scalar triple product (also called the box product or mixed triple product) is not really a new operator, but a way of applying the other two multiplication operators to three vectors. The scalar triple product is sometimes denoted by $(\vec{a} \vec{b} \vec{c})$ and defined as:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \quad (1.12)$$

1.8.3 Volume

The volume of a parallelepiped defined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by:

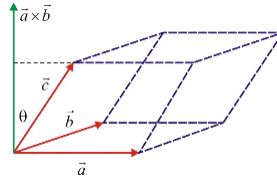


Figure 1.12: the volume of parallelepiped

$$\begin{aligned} V &= |\vec{c} \cdot (\vec{a} \times \vec{b})| \\ &= |\vec{a} \cdot (\vec{b} \times \vec{c})| \\ &= |\vec{b} \cdot (\vec{c} \times \vec{a})| \end{aligned}$$

The above result can be obtained as follows:

$$\begin{aligned} V &= A_{base} \times h \\ &= \|(\vec{a} \times \vec{b})\| |Sproj_{\vec{a} \times \vec{b}} \vec{c}|, \quad \text{where } Sproj \text{ is the scalar projection} \\ &= \|(\vec{a} \times \vec{b})\| \left| \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{\|(\vec{a} \times \vec{b})\|} \right| \\ &= |(\vec{a} \times \vec{b}) \cdot \vec{c}| \end{aligned}$$

Example 1.11 Find the volume of the parallelepiped defined by the vectors $\vec{a} = \langle 0, 1, -3 \rangle$, $\vec{b} = \langle 1, 2, 3 \rangle$ and $\vec{c} = \langle -1, 0, 1 \rangle$.

Solution:-First, let's compute the cross product as:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -3 \\ 1 & 2 & 3 \end{vmatrix} \\ &= (3+6)\mathbf{i} + (3-0)\mathbf{j} + (0-1)\mathbf{k} \\ &= 9\mathbf{i} + 3\mathbf{j} - \mathbf{k} \end{aligned}$$

Then, the volume of the parallelepiped is given by the triple product as:

$$\begin{aligned} V &= |(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}| \\ &= |(9\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{k})| \\ &= |9(-1) + 3(0) + (-1)(1)| \\ &= |-9 - 1| \\ &= |-10| \\ &= 10 \end{aligned}$$

Exercise 1.2 1. Find the volume and surface area of the parallelepiped formed by the vectors $\vec{\mathbf{a}} = 2\mathbf{i}$, $\vec{\mathbf{b}} = 3\mathbf{j}$, $\vec{\mathbf{c}} = \mathbf{j} + \mathbf{k}$

2. Find the volume of the parallelepiped spanned by the vectors $\mathbf{u} = \langle 1, 0, 2 \rangle$, $\mathbf{v} = \langle 0, 2, 3 \rangle$, $\mathbf{w} = \langle 0, 1, 3 \rangle$.

1.9 Equation of Lines and planes

Consider a line through the point $P_0(x_0, y_0, z_0)$ in the direction defined by the vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$, as shown in Figure 1.13.

Let $P(x, y, z)$ be any other point on the line, then we have a vector $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$.

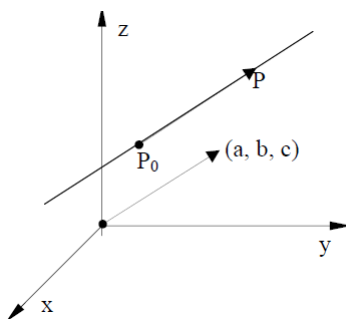


Figure 1.13: A parallel vector of a line

The vectors $\overrightarrow{P_0P}$ and $\vec{v}$ are parallel, hence there exists a scalar t such that $\overrightarrow{P_0P} = t\vec{v}$ which gives us

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle. \quad (1.13)$$

This is called *vector equation of the line*. Comparing the components of the vectors on the left and right hand side of Equation (1.13) results in $x - x_0 = ta$, $y - y_0 = tb$, $z - z_0 = tc$. Rearranging these equations as follows gives us the *parametric equations of a line* in R^3 :

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc, \quad -\infty < t < \infty. \quad (1.14)$$

Example 1.12 Find the vector and parametric equation of the line that pass through the point $(1, 2, 5)$ in the direction of $\langle 4, 3, 2 \rangle$, and determine any two points on the line.

Solution: Let $\vec{b} = \langle a, b, c \rangle = \langle 4, 3, 1 \rangle$ and $P_0 = (x_0, y_0, z_0) = (1, 2, 5)$ then from Equation (1.13) we can write the vector equation of the line as

$$\langle x - 1, y - 2, z - 5 \rangle = t\langle 4, 3, 1 \rangle$$

And from Equation (1.14) we get the parametric equation of the line as

$$x = 1 + 4t, \quad y = 2 + 3t, \quad z = 5 + t, \quad -\infty < t < \infty.$$

To find two points on the line we give t two arbitrary values, for instance $t = 1$ leads to the point $(5, 5, 6)$, and $t = -1$ leads to the point $(-3, -1, 4)$.

Example 1.13 Find the parametric equation of the line pass through the points $(-1, 2, 6)$ and $(1, 5, 4)$.

Solution: Let $P_0 = (x_0, y_0, z_0) = (-1, 2, 6)$. The direction of the line is given by the vector $\vec{v} = \langle a, b, c \rangle = (1, 5, 4) - (-1, 2, 6) = \langle 2, 3, -2 \rangle$. Consequently the parametric equations of the line are

given by

$$x = -1 + 2t, \quad y = 2 + 3t, \quad z = 6 - 2t, \quad -\infty < t < \infty.$$

Symmetric Equations of a Line: From equation (1.14) we can clear the parameter t by writing it as

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

provided that the three numbers a , b , and c are non-zero. The resulting equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

are said to be *symmetric equations* for the line pass through the points P_0 and P .

Example 1.14 Find the symmetric equations for the line that pass through the points $(4, 10, -6)$ and $(7, 9, 2)$.

Solution: First let us find the reference vector as below

$$\begin{aligned} \langle a, b, c \rangle &= (4, 10, -6) - (7, 9, 2) \\ &= \langle -3, 1, -8 \rangle. \end{aligned}$$

Then if we let $(x_0, y_0, z_0) = (7, 9, 2)$ the symmetric equation of the line is given by.

$$\frac{x - 7}{-3} = \frac{y - 9}{1} = \frac{z - 2}{-8}$$



If one of the numbers a, b , or c is zero in Equation (1.14), we use the remaining two equations to eliminate the parameter t . For example if $a = 0$, $b \neq 0$, and $c \neq 0$, then (1.14) yields the symmetric equations for the line to be

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Theorem 1.4 Let L be a line parallel to a vector $\vec{v}$ and p_1 be a point not on a line L then the distance “ D ” between p_1 and L is given by

$$D = \frac{\| \vec{v} \times \overrightarrow{p_0 p_1} \|}{\| \vec{v} \|},$$

where p_0 is any point on L and $(\| \vec{v} \| \neq 0)$.

Proof exercise

Example 1.15 Find the distance from the point $p = (1, 1, 5)$ to the line $L : x = 1 + t, y = 3 - t, z = 2t$

Solution: We see from the equation for L that passes through $p_0 = (1, 3, 0)$ parallel to $\vec{v} = i - j + 2k$ with $\overrightarrow{p_0 p} = -2j + 5k$ and $\overrightarrow{p_0 p} \times \vec{v} = i + 5j + 2k$. Then from the above theorem,

$$\begin{aligned} D &= \frac{\|\overrightarrow{p_0 p} \times \vec{v}\|}{\|\vec{v}\|} \\ &= \frac{\sqrt{30}}{\sqrt{6}} \\ &= \sqrt{5} \end{aligned}$$

Example 1.16 Find the distance between the following two parallel lines with the given parametric equations $L_1 : x = 2 - t, y = 2t, z = 1 + t$ and $L_2 : x = 1 + 2t, y = 3 - 4t, z = 5 - 2t$.

Solution: Here the two lines are parallel, (i.e they do not intersect and they are on the same plane) (show!). Now to find the distance, take $\vec{v} = \langle -1, 2, 1 \rangle$ the vector parallel to a line L_1 , the point $p_0(2, 0, 1)$ on the line L_1 and $p_1(1, 3, 5)$ on the line L_2 such that $\overrightarrow{p_0 p_1} = \langle -1, 3, 4 \rangle$. But

$$\begin{aligned} \vec{v} \times \overrightarrow{p_0 p_1} &= \begin{vmatrix} i & j & k \\ 1 & 3 & 5 \\ -1 & 3 & 4 \end{vmatrix} \\ &= (8 - 9)i - (-4 + 3)j + (-3 + 2)k \\ &= -i + j - k, \end{aligned}$$

and hence

$$\|\vec{v} \times \overrightarrow{p_0 p_1}\| = \sqrt{3}.$$

Also we have

$$\|\vec{v}\| = \sqrt{6}.$$

Thus, the distance between the two parallel lines is given by:

$$\begin{aligned} D &= \frac{\|\vec{v} \times \overrightarrow{p_0 p_1}\|}{\|\vec{v}\|} \\ &= \frac{\sqrt{3}}{\sqrt{6}} \\ &= \frac{1}{\sqrt{2}} \end{aligned} \tag{1.15}$$



Remark: Two lines in space are

1. Coincident if they are the same lines
2. parallel if they have no point in common but belong to the same plane.
3. Intersecting if they have only one point in common.
4. Skew lines if they have no points in common and belong to different planes.
5. Two lines are said to be in the same plane if $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, where $\mathbf{u}$ is parallel to the first line, $\mathbf{v}$ is parallel to the second line and $\mathbf{w}$ is a vector containing a point of each line.

Exercise 1.3 1. Find the vector, parametric, and symmetric equations for the line through the point $(1, 0, -3)$ and parallel to the vector $2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$.

2. Find the parametric and symmetric equations of the line passing through the point $(-3, 5, 4)$ and parallel to the line $x = 1 + 3t, y = -1 - 2t, z = 3 + t$.

3. Find the shortest distance and the equation of line that contains the shortest path between the lines $r = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and $r = -\mathbf{i} - \mathbf{j}\mathbf{k} + \gamma(7\mathbf{i} - 6\mathbf{j} + \mathbf{k})$.

4. Find the shortest distance between the following lines:

$$r = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \text{ and } r = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}).$$

5. Find λ such that $\vec{a}$ and $\vec{b}$ are perpendicular, where $\vec{a} = \lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\vec{b} = 5\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$.

6. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \mathbf{0}$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

7. Find the vector in the direction of the vector $5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ which has the magnitude 8 units.

8. Show that the points $A(3, -5, 1), B(-1, 0, 8)$ and $C(7, -10, -6)$ are collinear.

9. a) Prove that the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ are coplanar. b) Show that the points $(1, 0, 1), (1, 1, 0), (0, 1, 1)$ and $(0, 0, 2)$ are coplanar

10. If $\vec{a}$ and $\vec{b}$ are unit vectors and θ be the angle between them then show that

i. $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}\|\vec{a} - \vec{b}\|$

ii. $\cos\theta = \frac{1}{2}\|\vec{a} + \vec{b}\|$

iii. $\tan\theta = \frac{\|\vec{a} - \vec{b}\|}{\|\vec{a} + \vec{b}\|}$

11. Find the area of parallelogram whose diagonals are along the vectors $\mathbf{i} + 2\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{k}$.

12. Show that the points A, B, C with position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ respectively are the vertices of right triangle. Also find remaining angles of the triangle.
13. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of a triangle $\triangle ABC$ respectively. Find an expression for the area of $\triangle ABC$ and deduce the condition for the points to be collinear.
14. Vectors $\vec{a}$ and $\vec{b}$ are such that $\|\vec{a}\| = \sqrt{3}, \|\vec{b}\| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.
15. Find λ if $(2\mathbf{i} + 6\mathbf{j} + 14\mathbf{k}) \times (\mathbf{i} - \lambda\mathbf{j} + 7\mathbf{k}) = \mathbf{0}$.
16. If $\vec{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \vec{b} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\vec{c} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.
17. Let $\vec{a} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}, \vec{b} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$ and $\vec{c} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 18$.
18. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} \cdot \vec{b}| = \|\vec{a} \times \vec{b}\|$, then what is the angle between $\vec{a}$ and $\vec{b}$.
19. Find the vector whose length is 3 and which is perpendicular to the vectors $\vec{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}, \vec{b} = 6\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$.
20. If $A = (1, 1, 1), C(0, 1, -1)$ are two given vectors, then find a vector B satisfying the equation $A \times B = C$ and $A \cdot B = 3$.
21. If A, B, C and D are any four points in spaces prove that

$$\|AB \times CD + BC \times AD + CA \times BD\| = 4(\text{area of } \triangle ABC)$$

1.9.1 Equations of Planes in $\mathbb{R}^3$

Let $p_0(x_0, y_0, z_0)$ be a point in a plane. Let $\langle a, b, c \rangle$ be a vector perpendicular to the plane, called **normal** to the plane. These two quantities, namely a point in a plane and a normal vector to the plane characterize the plane. There is only one plane through a given point and having a given normal. We will now derive the equation of a plane passing through the point $p_0(x_0, y_0, z_0)$ and having normal $\langle a, b, c \rangle$. Let $P(x, y, z)$ be any arbitrary point in the plane. Then, we have

$$\begin{aligned} \overrightarrow{p_0P} &= (x, y, z) - (x_0, y_0, z_0) \\ &= \langle x - x_0, y - y_0, z - z_0 \rangle \end{aligned}$$

which lies on the plane. Thus the vectors $\langle a, b, c \rangle$ and $\overrightarrow{p_0 p}$ are orthogonal, as a result, their dot product is zero. This observation leads to a vector equation of the plane given by

$$\begin{aligned}\langle a, b, c \rangle \cdot \overrightarrow{p_0 p} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0\end{aligned}$$

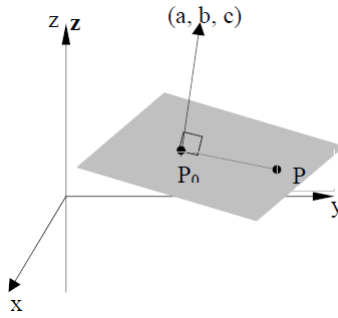


Figure 1.14: Normal vector and plane

Specifically the above equation yields the *point-normal form* of the equation of the plane as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \quad (1.16)$$

In addition, expanding Equation (1.16) and letting $d = ax_0 + by_0 + cz_0$ we obtain *the general form* of the equation of the plane as

$$ax + by + cz = d. \quad (1.17)$$

Observe that the components of $\langle a, b, c \rangle$ appear as coefficients in (1.16) and (1.17), and the coordinates of the points $p_0(x_0, y_0, z_0)$ in the plane appear inside the parenthesis in Equation (1.16).

Example 1.17 : Find the point-normal and general forms of the equation of the plane passing through the point $(1, 2, 3)$ and having normal $\langle -1, 4, 6 \rangle$.

Solution: Let $(x_0, y_0, z_0) = (1, 2, 3)$ and $\langle a, b, c \rangle = \langle -1, 4, 6 \rangle$. Then the point normal form equation of the plane is given by

$$-(x - 1) + 4(y - 2) + 6(z - 3) = 0.$$

Multiplying and simplifying the above equation results in the general form of equation of the line as

$$x + 4y + 6z = 25.$$

Example 1.18 Determine the equation of the plane through the three points $P(2, -1, 1)$, $Q(-1, 1, 3)$ and $R(2, 0, -3)$.

Solution: The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the plane and $\overrightarrow{PQ} \times \overrightarrow{PR}$ will be normal to the plane. Hence, since $\overrightarrow{PQ} = (-1, 1, 3) - (2, -1, 1) = \langle -3, 2, 2 \rangle$ and $\overrightarrow{PR} = (2, 0, -3) - (2, -1, 1) = \langle 0, 1, -4 \rangle$ we have $\overrightarrow{PQ} \times \overrightarrow{PR} = -10i - 12j - 3k$. Finally letting $(x_0, y_0, z_0) = (2, -1, 1)$ and $\langle a, b, c \rangle = \langle -10, -12, -3 \rangle$ we get the point-normal equation as

$$-10(x - 2) - 12(y + 1) - 3(z - 1) = 0$$

or the general equation as

$$-10x - 12y - 3z = -11.$$

1.9.2 The angle between two planes

In three dimensions, two planes are either parallel or they intersect in a straight line. (Think about this some.) Suppose that two planes having normal vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, intersect. Then the angle between the planes is the same as the angle between $\mathbf{a}$ and $\mathbf{b}$.

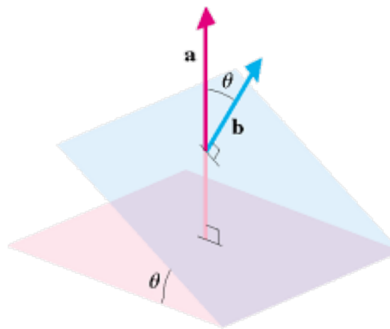


Figure 1.15: Angle between planes

Note: Two planes are parallel whenever their normal vectors are parallel and the planes are orthogonal whenever their normal vectors are orthogonal.



Example 1.19 Determine whether the planes $x - 3y + 6z = 4$ and $5x + y - z = 4$ are orthogonal, parallel, or neither. Find the angle of intersection and the set of parametric equations for the line of intersection of the plane.

Solution: For the plane $x - 3y + 6z = 4$ the normal vector is $\vec{n}_1 = \langle 1, -3, 6 \rangle$ and for the plane $5x + y - z = 4$ the normal vector is $\vec{n}_2 = \langle 5, 1, -1 \rangle$. The two planes will be orthogonal only if their corresponding normal vectors are orthogonal, that is, if $\vec{n}_1 \cdot \vec{n}_2 = 0$. However, we see that

$$\vec{n}_1 \cdot \vec{n}_2 = \langle 1, -3, 6 \rangle \cdot \langle 5, 1, -1 \rangle = (1)(5) + (-3)(1) + (6)(-1) = 5 - 3 - 6 = -4 \neq 0$$

Hence, the planes are not orthogonal.

If the planes are parallel, then their corresponding normal vectors must be parallel. For that to occur, there must exist a scalar k where

$$\vec{n}_2 = k\vec{n}_1.$$

Rearranging this equation as $k\vec{n}_1 = \vec{n}_2$ and substituting for $\vec{n}_1$ and $\vec{n}_2$ gives us $k\langle 1, -3, 6 \rangle = \langle 5, 1, -1 \rangle$ or $\langle k, -3k, 6k \rangle = \langle 5, 1, -1 \rangle$. Equating components results in the equations $k = 5$, $-3k = 1$, $6k = -1$. This gives $k = 5$, $k = -1/3$, $k = -1/6$. Since the values of k are not the same for each component to make the vector $\vec{n}_2$ a scalar multiple of the vector $\vec{n}_1$, and hence the planes are not parallel.

Thus, the planes must intersect in a straight line at a given angle. To find this angle, we use the equation

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|}$$

since $\vec{n}_1 \cdot \vec{n}_2 = \langle 1, -3, 6 \rangle \cdot \langle 5, 1, -1 \rangle = -4$, $\|\vec{n}_1\| = \sqrt{1^2 + (-3)^2 + 6^2} = \sqrt{46}$ and $\|\vec{n}_2\| = \sqrt{5^2 + 1^2 + (-1)^2} = \sqrt{27}$ we have

$$\begin{aligned} \cos \theta &= \frac{-4}{\sqrt{46}\sqrt{27}} \\ \Rightarrow \theta &= \cos^{-1}\left(\frac{-4}{\sqrt{46}\sqrt{27}}\right) = 1.68 \text{ radians} \\ &= 96.5^\circ. \end{aligned} \tag{1.18}$$

To find the equation of the line of intersection between the two planes, we need a point on the line and a parallel vector. To find a point on the line, we can consider the case where the line touches the $x - y$ plane, that is, at $z = 0$. In the two equations of the planes

$$x - 3y + 6z = 4$$

$$5x + y - z = 4$$

if we substitute $z = 0$, we obtain the system of equations

$$x - 3y = 4 \tag{1}$$

$$5x + y = 4 \tag{2}$$

Multiplying the first equation by -5 and adding it to the second equation results in $16y = -16$ or $y = \frac{-16}{16} = -1$. Substituting $y = -1$ back into equation (1) gives us $x = 1$. Thus, the point $(1, -1, 0)$ is on the plane. Since the line is on both planes, it is orthogonal to both normal vectors $\vec{n}_1$ and $\vec{n}_2$. Hence, the cross product $\vec{n}_1 \times \vec{n}_2$ will be a parallel vector for the line. Thus, we have

$$\begin{aligned}\vec{v} = \vec{n}_1 \times \vec{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 6 \\ 5 & 1 & -1 \end{vmatrix} \\ &= (3 - 6)\mathbf{i} - (-1 - 30)\mathbf{j} + (1 + 15)\mathbf{k} \\ &= -3\mathbf{i} + 31\mathbf{j} + 16\mathbf{k}.\end{aligned}$$

Hence, using the point $(1, -1, 0)$ and the parallel vector $\vec{v} = -3\mathbf{i} + 31\mathbf{j} + 16\mathbf{k}$, we find the parametric equations of the line are

$$x = 1 - 3t, \quad y = -1 + 31t, \quad z = 16t.$$

Exercise 1.4 1. Determine whether the planes $3x + y - 4z = 3$ and $-9x - 3y + 12z = 4$ are orthogonal, parallel, or neither. Find the angle of intersection and the set of parametric equations for the line of intersection of the plane.

2. Find the point where the line $x = 1 + t, y = 2t$, and $z = -3t$ intersects the plane $-4x + 2y - 4z = -2$.

1.9.3 Distance between Points and a Plane

Suppose that we wanted to find the distance from the plane $ax + by + cz + d = 0$ to a point $P_0(x_0, y_0, z_0)$ not on the plane. Notice that the distance is measured along a line segment connecting the point to the plane that is orthogonal to the plane as show in Figure 1.16. To compute this distance, pick any point $P_1(x_1, y_1, z_1)$ that lies on the plane and let $\vec{a} = \langle a, b, c \rangle$ denote a vector normal to the plane. Next, notice from the figure that the distance from P_0 to the plane is simply the norm of $proj_{\vec{a}} \overrightarrow{P_0P_1}$ and also $\overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

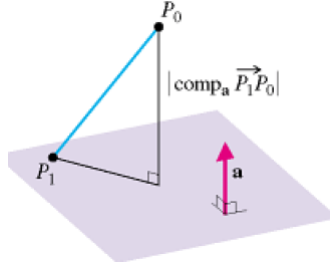


Figure 1.16: The distance of the plane and the point

Thus, the distance between the plane and the given point is obtained by

$$\begin{aligned}
 D &= \| \text{proj}_{\vec{a}} \overrightarrow{P_0P_1} \| \\
 &= | \overrightarrow{P_0P_1} \cdot \frac{\vec{a}}{\|\vec{a}\|} | \\
 &= \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}
 \end{aligned}$$



Note: To find the distance between two parallel planes in space, take any point of one of the planes and find the distance between the point and the other plane.

Example 1.20 Find the distance between the point $(1, 2, 3)$ and plane $2x - y + z = 4$.

Solution: Since we are given the point $Q = (1, 2, 3)$, we need to find a point on the plane $2x - y + z = 4$ in order to find the vector $\overrightarrow{PQ}$. We can simply do this by setting $y = 0$ and $z = 0$ in the plane equation $2x - y + z = 4$ and solving for x . Thus we have $2x - 0 + 0 = 4 \Rightarrow 2x = 4 \Rightarrow x = 4/2 \Rightarrow x = 2$. Thus get the point $P(2, 0, 0)$. The vector $\overrightarrow{PQ}$ is then $\langle 1 - 2, 2 - 0, 3 - 0 \rangle = \langle -1, 2, 3 \rangle$. The normal vector for the plane is $\vec{n} = \langle 2, -1, 1 \rangle$, hence the distance **D** between the point Q and the plane is

$$\begin{aligned}
 D &= | \overrightarrow{PQ} \cdot \frac{\vec{n}}{\|\vec{n}\|} | \\
 &= \frac{| \langle -1, 2, 3 \rangle \cdot \langle 2, -1, 1 \rangle |}{\sqrt{2^2 + (-1)^2 + 1^2}} \\
 &= \frac{| -2 - 2 + 3 |}{\sqrt{4 + 1 + 1}} \\
 &= 1/\sqrt{6}
 \end{aligned}$$

Example 1.21 Find the distance between the following two parallel planes given by $3x - y + 2z = 6$ and $6x - 2y + 4z = 4$.

Solution: To find the distance between the two planes take a point, say $(x_0, y_0, z_0) = (2, 0, 0)$ from the first plane and the normal vector of the second plane, i.e., $a = 6$, $b = -2$ and $c = 4$. Hence the distance D is

$$\begin{aligned} D &= \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(6)(2) + (-2)(0) + (4)(0) - (-4)|}{\sqrt{6^2 + (-2)^2 + 4^2}} \\ &= \frac{12 + 0 + 0 + 4}{\sqrt{36 + 4 + 16}} \\ &= \frac{16}{\sqrt{56}} \end{aligned}$$

Exercise 1.5 1. In the triangle that determined by the points P, Q and R , find a unit vector perpendicular to plane PQR , where

- a. $P(1, 1, 1), Q(2, 1, 3)$ and $R(3, -1, 1)$ b. $P(-2, 2, 0), Q(0, 1, -1)$ and $R(-1, 2, -2)$

2. Find equations for the lines:

- The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
- The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$.
- The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$.
- The line through the point $(3, -2, 1)$ parallel to the line

$$x = 1 + 2t, y = 2 - t, z = 3t$$

- The line through $(1, 1, 1)$ parallel to the z -axis.
- The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$.
- The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$.
- The line through $(2, 3, 0)$ perpendicular to the vectors $\vec{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

3. Find equations of the planes:

- The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.
- The plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$.
- The plane through $(1, 1, -1), (2, 0, 2)$ and $(0, -2, 1)$

d. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, y = 1 + 3t, z = 4t$$

e. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A .

4. Find the equation of plane determined by the intersecting lines:

$$L1 : x = -1 + t, y = 2 + t, z = 1 - t, \quad -\infty < t < \infty$$

$$L2 : x = 1 - 4s, y = 1 + 2s, z = 2 - 2s, \quad -\infty < s < \infty$$

5. Find the equation of plane through $P_0(2, 1, -1)$ perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.

6. Find the equation of plane through the points $P_1(1, 2, 3), P_2(3, 2, 1)$ perpendicular to the Plane $4x - y + 2z = 7$.

7. Find the angle between the planes:

a. $x + y = 1, \quad 2x + y - 2z = 2$

b. $5x + y - z = 10, \quad x - 2y + 3z = -1$

8. Find the point in which the line meets the plane.

a. $x = 1 - t, y = 3t, z = 1 + t, \quad 2x - y + 3z = 6$

b. $x = 2, y = 3 + 2t, z = -2 - 2t, \quad 6x + 3y - 4z = -12$

9. Find the vector equation of the line parallel to the line $\frac{(x-1)}{5} = \frac{(3-y)}{2} = \frac{(z+1)}{4}$ and passing through $(3, 0, -4)$. Also find the distance between these two lines.

10. Find the shortest distance between the lines, whose equations are $\frac{(x-8)}{3} = \frac{(y+9)}{-16} = \frac{(10-z)}{-7}$ and $\frac{(x-15)}{-16} = \frac{(58-2y)}{-16} = \frac{(z-5)}{-5}$.

11. Find the equation of the plane which passes through the points $(0, 0, 0)$ and $(3, -1, 2)$ and is parallel to the line $\frac{(x-4)}{1} = \frac{(y+3)}{-4} = \frac{(z+1)}{7}$.

12. Find the foot of the perpendicular from $(1, 2, 3)$ on the line $\frac{(x-6)}{3} = \frac{(y-7)}{2} = \frac{(z-7)}{-15}$. Also obtain the equation of the plane containing the line and the point $(1, 2, 3)$.

13. Prove that the image of the point $(3, -2, 1)$ in the plane $3x - y + z = 2$ lies on the plane, $x + y + z + 4 = 0$.

14. Find the distance of the point $(3, 4, 5)$ from the plane $x + y + z = 2$ measured parallel to the line $2x = y = z$.

15. Prove that if a plane has intercepts a, b and c and is at a distance of p units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.
16. Show that $\frac{(x-1)}{2} = \frac{(y-3)}{4} = -z$ and $\frac{(x-4)}{3} = \frac{(1-y)}{2} = \frac{(z-1)}{1}$ are coplanar. Also, find the equation of the plane containing the lines.
17. Find the image of point $(1, 6, 3)$ in the line $\frac{x}{1} = \frac{(y-1)}{2} = \frac{(z-2)}{3}$.
18. Show that line L whose equation is $r = 2i - 2j + 3k + \lambda(i - j + 4k)$ is parallel to plane Π whose vector equation is $r \cdot (i + 5j + k) = 5$ and find distance between them.
19. Find the equation of the plane containing the lines $r = i + j + \lambda(i + 2j - k)$ and $r = i + j + \mu(-i + j - 2k)$. Find the distance this plane from origin and also the point $(1, 1, 1)$.
20. Find the equation of the plane passing through the intersection of the planes $2x + 3y - z + 1; x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z - 4 = 0$. Also find the inclination of this plane with the xy plane.
21. Find the length and the equations of the line of the shortest distance between the lines $\frac{(x-3)}{3} = \frac{(y-8)}{-1} = \frac{(z-3)}{1}$ and $\frac{(x+3)}{-3} = \frac{(y+7)}{2} = \frac{(z-6)}{4}$. Also, find the points where the line of shortest distance meets the given lines.
22. Find the equation of the plane which contains the two parallel lines $\frac{(x-3)}{3} = \frac{(y+4)}{2} = \frac{(z-1)}{1}$ and $\frac{(x+1)}{3} = \frac{(y-2)}{2} = \frac{z}{1}$.
23. Find the equation of the plane through the intersection of the planes $3x - 4y + 5z = 10; 2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.
24. Find the equation of the plane containing the line $\frac{(x+1)}{-3} = \frac{(y-3)}{2} = \frac{(z+2)}{1}$ and the point $(0, 7, -7)$ and show that the line $\frac{x}{1} = \frac{(y-7)}{-3} = \frac{(z+7)}{2}$ also lie in the same plane.
25. Find the image of the point having position vector $i + 3j + 4k$ in the plane $r \cdot (2i - j + k) + 3 = 0$.
26. Find the vector equation of the plane through the intersection of the planes $r \cdot (2i - j + k) = 3$ and passing through the point $(2, 1, -2)$.
27. Find the distance of the point $(1, -2, 3)$ from a plane $x - y + z = 5$ measured parallel to the line $\frac{(x+1)}{2} = \frac{(y+3)}{3} = \frac{(z+1)}{-6}$.
28. Find the foot of the perpendicular from the point $2i - j + 5k$ to the line $r = 11i - 2j - 8k + \lambda(10i - 4j - 11k)$. Also find the length of the perpendicular.
29. Show that the lines $\frac{x}{1} = \frac{(y-7)}{-3} = \frac{(z+7)}{2}$ and $\frac{(x+1)}{-3} = \frac{(y-3)}{2} = \frac{(z+2)}{1}$ are coplanar. Also find the equation of the plane containing them.

30. Find the distance from the plane $2x - y + 2z + 1 = 0$ to the origin.
31. Find the equation of the plane passing through the point $P(1, 1, 1)$ and containing the line $L_1 = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} - 5\mathbf{k})$. Also, show that the plane contains the line $L_2 = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 5\mathbf{k})$.
32. Find the point on the line $\frac{(x+2)}{3} = \frac{(y+1)}{2} = \frac{(z-3)}{2}$ at a distance of 5 units from the point $P(1, 3, 3)$.
33. Find the distance of the point $P(6, 5, 9)$ from the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$.
34. Find the equation of the plane through three points $A(0, -1, -1)$, $B(4, 5, 1)$ and $C(3, 9, 4)$.
35. Find the coordinate of the foot of the perpendicular drawn from the point $A(1, 8, 4)$ to the line joining $B(0, -1, 3)$ and $C(2, -3, -1)$.
36. Find the distance of the point $(3, 4, 5)$ from the plane $x + y + z = 2$ measured parallel to the line $2x = y = z$.
37. Find the equation of the plane passing through the intersection of the planes $2x + 3y - z + 1$; $x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z - 4 = 0$. Also find the inclination of this plane with the xy plane.
38. The set of all points equidistant from $(0, 1, 5)$ and $(5, -1, 3)$ is a plane. Find the equation.
39. A kite is pulled with a force $\vec{F} = \langle 2, 1, 4 \rangle$. It has velocity $\vec{v} = \langle 1, -1, 1 \rangle$. The dot product of $\vec{F}$ with $\vec{v}$ is called power.
- Find the angle between the force and the velocity.
 - Find the vector projection of the force onto the velocity vector.
40. Light shines long the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and reflects at the three coordinate planes where the angle of incidence equals the angle of reflection. Verify that the reflected ray is $\vec{a}$.
- Hint.** Reflect first at the xy -plane and find the components of the ray after reflection. You can assume that in that case, the reflected vector is in the plane spanned by $\vec{k} = \langle 0, 0, 1 \rangle$ and $\vec{a}$.
41. Given the vectors $\vec{a} = \langle 1, 2, 1 \rangle$, $\vec{b} = \langle 1, -1, 1 \rangle$, $\vec{c} = \langle 0, 1, 1 \rangle$, $\vec{d} = \langle 2, 3, 4 \rangle$, compute all possible dot products and determine which pairs are perpendicular.
42. a. Find the angle between a diagonal of a cube and the diagonal in one of its faces.

- b. The hypercube or tesseract has vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the hyper diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the space diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$.
43. a. Verify that if $\vec{a}, \vec{b}$ are nonzero, then $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a}$ and $\vec{b}$ if $\vec{c}$ is not zero.
- b. Verify the parallelogram law

$$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2.$$

44. a. Find a nonzero vector orthogonal to the plane through the points $P = (-2, 3, 1)$, $Q = (1, 5, 2)$, $R = (4, 3, -1)$ and containing P .
- b. Find the equation of this plane.
- c. Find the area of the triangle PQR .
45. a. Parametrize a line perpendicular to the plane containing $A = (1, 1, 1)$, $B = (2, 3, 4)$ and $C = (4, 5, 6)$ and passing through A.
- b. Find the equation $ax + by + cz = d$ of the plane through A, B, C .
46. a. Use volume to determine whether $A = (1, 1, 2)$, $B = (3, 1, 6)$, $C = (5, 2, 0)$ and $D = (1, 4, 12)$ are in the same plane.
- b. Find the distance between the line L through A, B and the line M through C, D .
- c. How come that whenever A, B, C, D are not in the same plane, then the distance between L and M is positive?
47. a. Find an equation of the plane containing the line of intersection of the planes $x - z = 1$ and $y + z = 3$ which is perpendicular to the plane $x + y - 2z = 1$.
- b. Find the distance of the plane found in a) to the origin $(0, 0, 0)$.
48. a. Parametrize the line L through $P = (2, 1, 2)$ that intersects the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ perpendicularly.
- b. What is the distance from this line L to the origin $(0, 0, 0)$?

2. Matrices and Determinants

2.1 Definition and Types of matrix

Definition 2.1 A matrix is a rectangular array of numbers. The numbers in the array are called entries or elements of the matrix.

Elements in the horizontal lines form rows and elements in the vertical lines form columns. The size or order of a matrix is described in terms of the number of rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

A is a matrix of size m by n (written as $m \times n$), since A has m rows and n columns. The entry in i^{th} row and j^{th} column of the matrix is denoted by a_{ij} or $(A)_{ij}$. The i^{th} row of matrix A is $R_i = [a_{i1} \ a_{i2} \ a_{i3} \ \cdots \ a_{in}] \quad 1 \leq i \leq m$

The j^{th} column of the matrix A is $c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad 1 \leq j \leq n.$

We often write matrix A as $A = [a_{ij}]_{m \times n}$, $1 \leq i \leq m$, $1 \leq j \leq n$ where i -denotes the row and j - denotes the column.

For example, in the matrix $A = \begin{bmatrix} 5 & 9 & 6 & 8 \\ 3 & 2 & 3 & 1 \\ 1 & 0 & 4 & 7 \end{bmatrix}$, there are 3 rows and 4 columns. Therefore, matrix A can

be called a matrix of order or size 3×4 . The rows are $[5 \ 9 \ 6 \ 8]$, $[3 \ 2 \ 3 \ 1]$ and $[1 \ 0 \ 4 \ 7]$,

and the columns are $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}$.

Here $a_{11} = 5$, $a_{12} = 9$, $a_{13} = 6$, $a_{14} = 8$, $a_{21} = 3$, $a_{22} = 2$, $a_{23} = 3$, $a_{24} = 1$, etc. and A has 12 elements.

Definition 2.2 : Two matrices A and B are said to be equal, written $A = B$, if they are of the same order and if all corresponding entries are equal i.e. $a_{ij} = b_{ij}$.

For example, $\begin{bmatrix} 5 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2+3 & 1 & 0 \\ 2 & 3 & 2 \times 2 \end{bmatrix}$ but $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 2 & 9 \end{bmatrix}$. Why?

Example 2.1 Given the matrix equation

$$\begin{bmatrix} x+y & 8 \\ x-y & 6 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 6 \end{bmatrix}.$$

Find x and y .

Solution: By the definition of equality of matrices, we have $x + y = 3$ and $x - y = 1$ and solving this system of equations gives $x = 2$ and $y = 1$.

Exercise 2.1 Find the values of x , y , z and w which satisfy the matrix equation

a. $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

b. $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w+6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2w \end{bmatrix}$

2.1.1 Types of Matrices

1. A matrix having exactly one row is called a **row matrix**.

Example 2.2 ; $\begin{bmatrix} 1 & 0 & 4 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 \end{bmatrix}$, are row matrices. A row matrix is often referred to as a row vector.

2. A matrix having exactly one column is called a **column matrix**

Example 2.3 $\begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ is a column matrix.

3. A **zero matrix** or **null matrix** is a matrix in which all of its elements are zero.

Example 2.4 The matrices, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ are the zero matrices.

4. A matrix in which number of rows and number of columns are equal, that is, an $n \times n$ matrix, is called a **square matrix** of order n .

Example 2.5 $\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix}$ are square matrices.

If $A = (a_{ij})_{n \times n}$ is a square matrix the elements a_{ii} 's are called the **diagonal elements**. The main diagonal or simply diagonal of A consists of the elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$.

5. A square matrix in which all the non-diagonal elements are zero is called a **diagonal matrix**.

Example 2.6 $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ are diagonal matrices.



Note that the diagonal elements in a diagonal matrix may also be zero.

Example 2.7 $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are also diagonal matrices.

6. A diagonal matrix whose diagonal elements are equal is called a **scalar matrix**

Example 2.8 $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are scalar matrices.

7. Diagonal matrix of order n in which every diagonal element is unity (one) is called the **identity matrix** or **unit matrix** of order n . The identity matrix of order n is denoted by I_n .

Example 2.9 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order two.

8. A square matrix having only zeros below its diagonal is called **upper triangular matrix**. A square matrix having only zeros above its diagonal is called **lower triangular matrix**. A matrix that is either upper triangular or lower triangular is called **triangular matrix**.

Example 2.10 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}$, and $\begin{bmatrix} 3 & 0 \\ 6 & 4 \end{bmatrix}$ are lower triangular matrices and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and $\begin{bmatrix} 3 & 7 \\ 0 & 4 \end{bmatrix}$ are upper triangular matrices.

Definition 2.3 A matrix obtained by deleting one or more rows and/or columns of A is called a **sub matrix** of A .

Example 2.11 If $A = \begin{bmatrix} 4 & 6 & 1 \\ 3 & 8 & 2 \\ 2 & 0 & 3 \end{bmatrix}$, then $\begin{bmatrix} 8 & 2 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$ and $\begin{bmatrix} 4 & 6 \\ 3 & 8 \\ 2 & 0 \end{bmatrix}$ are some of the sub matrices of A .

2.2 Operations on Matrices and their properties

2.2.1 Addition and Subtraction of matrices

Definition 2.4 If A and B are matrices of the same order, then the sum of A and B , denoted by $A + B$, is a new matrix of the same order obtained by adding the corresponding elements of A and B . Similarly, the difference of A and B , denoted by $A - B$, is the matrix obtained by subtracting the corresponding elements of A and B .

Example 2.12 If $A = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 8 & 2 \\ 4 & 6 & 1 \end{bmatrix}$, then we have that

$$\text{i. } A + B = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 3 \\ 3 & 8 & 2 \\ 4 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+0 & 8+3 \\ 4+3 & 6+8 & 0+2 \\ 1+4 & 3+6 & 5+1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 11 \\ 7 & 14 & 2 \\ 5 & 9 & 6 \end{bmatrix}$$

$$\text{ii. } A - B = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 3 \\ 3 & 8 & 2 \\ 4 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-0 & 8-3 \\ 4-3 & 6-8 & 0-2 \\ 1-4 & 3-6 & 5-1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 1 & -2 & -2 \\ -3 & -3 & 4 \end{bmatrix}.$$

2.2.2 Multiplication of Matrices by Scalar

Let A be any matrix and α be a scalar (real number), then αA is the matrix obtained from A by multiplying each element of A by α . This operation is called scalar multiplication. In particular, $-A$ is the matrix obtained from A by multiplying each element of A by -1 and is called the **additive inverse** of A .

Example 2.13 If $A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix}$, then

$$3A = 3 \begin{bmatrix} 1 & 2 & 6 \\ 5 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(6) \\ 3(5) & 3(0) & 3(4) \\ 3(3) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 18 \\ 15 & 0 & 12 \\ 9 & 3 & 6 \end{bmatrix}$$

2.2.3 Properties of Matrix addition and Scalar Multiplication

Let A , B and C be $m \times n$ matrices and $\mathbf{0}$ be a zero matrix of size $m \times n$ and α , β be scalars. Then

1. $A + B = B + A$ (Commutative law for addition).
2. $(A + B) + C = A + (B + C)$ (Associative law for addition).
3. $A + \mathbf{0} = \mathbf{0} + A = A$ ($\mathbf{0}$ is called Additive identity).
4. For each matrix A , there exists a unique $m \times n$ matrix $-A$ such that $A + (-A) = \mathbf{0} = -A + A$.
5. $\alpha(A + B) = \alpha A + \alpha B$.
6. $(\alpha\beta)A = \alpha(\beta A) = \beta(\alpha A)$.
7. $(\alpha + \beta)A = \alpha A + \beta A$.

From the above properties, the set of all matrices having the same order forms a vector space with the operations addition and scalar multiplication.

2.3 Product of matrices

Definition 2.5 Matrix Product: Let A be an $m \times r$ matrix and B be an $r \times n$ matrix. The $(ij)^{th}$ entry of $C = AB$ is the dot product of the i^{th} row vector of A and the j^{th} column vector of B :

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ir} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{rj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} = \sum_{k=1}^r a_{ik}b_{kj}$$

The product C has order $m \times n$.

Example 2.14 i. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 5 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 1(2)+1(2)+3(4) & 1(5)+1(3)+3(1) \\ 1(2)+0(2)+5(4) & 1(5)+0(3)+5(1) \\ 3(2)+2(2)+1(4) & 3(5)+2(3)+1(1) \end{bmatrix} = \begin{bmatrix} 2+2+12 & 5+3+3 \\ 2+0+20 & 5+0+5 \\ 6+4+4 & 15+6+1 \end{bmatrix} = \begin{bmatrix} 14 & 11 \\ 22 & 10 \\ 14 & 22 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 2 & 3 \\ 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 6 & 9 \\ 0 & 21 \end{bmatrix}$ and $BA = \begin{bmatrix} 21 & 0 \\ 11 & 6 \end{bmatrix}$. So, $AB \neq BA$



Note: In general matrix multiplication is not commutative.

2.3.1 Properties of Matrix Multiplication

Let A , B and C be three matrices of the appropriate sizes. Let α be a scalar. Then

1. $A(BC) = (AB)C$.
2. $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
3. $\alpha(AB) = (\alpha A)B = A(\alpha B)$

2.4 Transpose of a Matrix

Definition 2.6 A matrix obtained from a given matrix A by interchanging the rows and columns of A is called the **transpose of A** and is denoted by A^t . That is, if $A = (a_{ij})_{m \times n}$ then $A^t = (a_{ji})_{n \times m}$

Example 2.15 :

$$1. \text{ If } A = \begin{bmatrix} 2 & 5 \\ -2 & 3 \\ 4 & 1 \end{bmatrix}, \text{ then } A^t = \begin{bmatrix} 2 & -2 & 4 \\ 5 & 3 & 1 \end{bmatrix}$$

$$2. \text{ If } B = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}, \text{ then } B^t = [4 \quad 3 \quad -6]$$

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then A is said to be

i. Symmetric matrix if $A^t = A$

ii. Skew symmetric if $A^t = -A$

Example 2.16 1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 0 & -2 \\ 4 & -2 & 1 \end{bmatrix}, A^t = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 0 & -2 \\ 4 & -2 & 1 \end{bmatrix}. \implies A^t = A.$ Hence, A is symmetric.

2. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, A^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \implies A^t = -A.$ Therefore, A is skew-symmetric.

2.4.1 Properties of Transpose of a Matrix

Let A and B be matrices such that addition and multiplication is defined. Then

1. $(A^t)^t = A$

2. $(A + B)^t = A^t + B^t$ And $(AB)^t = B^t A^t$

3. $(\alpha A)^t = \alpha A^t$, where α – is a scalar

4. Any square matrix A can be written as a sum of symmetric and skew symmetric matrices:

$$A = \underbrace{\frac{1}{2}(A + A^t)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^t)}_{\text{skewsymmetric}}$$

2.5 Trace of a Matrix

Definition 2.7 Let $A = (a_{ij})_{n \times n}$, be a square matrix of order n . Then trace of A is defined to be the sum of the diagonal elements of A , i.e. $\text{trace}(A) = \sum_{i=1}^n a_{ii}$.

Notation: The trace of a matrix A is also commonly denoted as $\text{trace}(A)$ or $\text{tr}(A)$.

2.5.1 Properties of trace of a matrix

If A and B are square matrices, then

1. $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$
2. $\text{trace}(A) = \text{trace}(A^t)$
3. $\text{trace}(cA) = c(\text{trace}(A))$
4. $\text{trace}(AB) = \text{trace}(BA)$

Example 2.17 Find the trace of $A = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & 2 \\ 3 & 2 & 6 \end{bmatrix}$

Solution: $\text{tr}(A) = \sum_{i=1}^3 a_{ii} = 15 + (-4) + 6 = 17$

2.6 Elementary Row Operations and Row Echelon form

Definition 2.8 Elementary row operation: Given any matrix A of order $m \times n$, any one of the following operations on the matrix is called **elementary row operation**:

1. Interchanging any two rows of A ; $R_i \Leftrightarrow R_j$ (Interchange the i^{th} and j^{th} row)
2. Multiplying a row of A by a nonzero constant k ; $R_i \Rightarrow kR_i$ (Multiply the i^{th} row by scalar k)

3. Adding a multiple of one row of A to another row of A . $R_j \Rightarrow R_j + kR_i$ (add k times i^{th} row to j^{th} row).

Example 2.18 Give a matrix $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$

- a). Interchange rows 1 and 3 of A

$$\Rightarrow \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- b). Multiply the third row of A by $1/3$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

- c). Multiply the second row of A by -2 , then add to the third row of A

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Definition 2.9 Two matrices A and B are called **row equivalent** or simply (**equivalent matrices**) if one matrix can be obtained from the other matrix by applying finite number of elementary operations. In this case we write $A \sim B$.

Example 2.19 As we observe from the above example,

$$\begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2/3 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Definition 2.10 Matrix in reduced row echelon form: A matrix in reduced row echelon form has the following properties:

1. All rows consisting entirely of 0 are at the bottom of the matrix.
2. For each nonzero row, the first nonzero entry is 1. The first entry is called a **leading 1**.
3. For two successive non zero rows, the leading 1 in the higher row appears farther to the left than the leading 1 in the lower row.
4. If a column contains a leading 1, then all other entries in that column are 0.



Note: A matrix is in **row echelon form** if the matrix has the first 3 properties.

Example 2.20 $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

are matrices in reduced row echelon form. Whereas the matrix

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form since the matrix has the first 3 properties fulfilled. But it is not in reduced row echelon form since the entries above the leading 1 in the third column are nonzero. The matrix

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in row echelon form (hence not in reduced row echelon form) since the leading 1 in the second row is not in the left of the leading 1 in the third row.

If a matrix is not in row echelon form, then it is not in reduced row echelon form. A matrix in reduced row echelon form is in row echelon form.

2.7 Rank of a matrix

Definition 2.11 The rank of a matrix A , denoted by $\text{rank}(A)$, is the number of nonzero rows remaining after it has been changed to row echelon form or reduced row echelon form.

Remark: If A is a zero matrix then $\text{rank}(A)$ is 0.



Example 2.21 Determine the rank of the following matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 3 & 1 & 2 & 6 \\ -1 & 2 & 5 & -4 \\ 2 & 3 & 7 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 6 & 9 & -3 \\ 2 & 4 & 6 & -2 \end{bmatrix}$$

Solution:

$$\begin{aligned} \text{a. } A &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 3 & 1 & 2 & 6 \\ -1 & 2 & 5 & -4 \\ 2 & 3 & 7 & 2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 + R_1 \\ R_4 \rightarrow R_4 - 2R_1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & -3 \\ 0 & 3 & 7 & 0 \end{bmatrix} \begin{array}{l} R_3 \rightarrow R_3 - 2R_2 \\ R_4 \rightarrow R_4 - 3R_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 1 & -9 \end{bmatrix} \\ &\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_4 \rightarrow R_4 - R_3 \\ R_2 \rightarrow R_2 - 2R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence, $\text{rank}(A) = 3$.

b. By using elementary row operations:

$$B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 6 & 9 & -3 \\ 2 & 4 & 6 & -2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Therefore, } \text{rank}(B) = 1$$

Exercise 2.2 Transform the following matrices into reduced row echelon form and determine their rank:

$$\text{a. } A = \begin{bmatrix} 10 & -8 & 0 \\ 1 & 3 & -5 \\ 7 & 0 & 9 \end{bmatrix}$$

$$\text{c. } C = \begin{bmatrix} 2 & 6 & 7 & 9 \\ 3 & 4 & 5 & -1 \\ 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 4 \\ -1 & 2 & 2 & 10 \end{bmatrix}$$

$$\text{b. } B = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

$$\text{d. } D = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 8 & 0 & 3 \end{bmatrix}$$

2.8 Inverse of a matrix and its properties

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is invertible or non-singular and B is the inverse of A . In this situation, we write $B = A^{-1}$.

Notice that if B is the inverse of A , then we can also say that A is the inverse of B , or A and B are inverses of each other.

Example 2.22 Show that $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is an inverse for the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Solution:-By the definition there are two multiplications to confirm. (We will show later that this isn't necessary, but for now we strictly apply the definition). Hence, we have

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(2) + (-1)(1) & 2(1) + (-1)(2) \\ (-1)(1) + 1(1) & -1(1) + 1(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and similarly

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1(2) + 1(-1) & 1(-1) + 1(1) \\ 1(2) + 2(-1) & 1(-1) + 2(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

The definition for inverse of a matrix is satisfied. Hence, A and B are inverses of each other.

Example 2.23 Show that $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

Solution:- Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the inverse of matrix B . Then

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \Rightarrow 0 &= 1 \text{ which is not true.} \end{aligned}$$

Therefore, B is not invertible.

Theorem 2.1 :- Suppose that A is invertible and that both B and C are inverses of A . Then $B = C$ and we will denote the inverse as A^{-1} .

2.8.1 Computing the Inverse of a Non Singular Matrix

Suppose A is a non singular square matrix of size n . Create an $n \times 2n$ matrix M by placing the $n \times n$ identity matrix to the right of matrix A . Let N be a matrix that is row-equivalent to M and in reduced row-echelon form then the first n columns of N is I_n and the last n columns of N is A^{-1} .

Example 2.24 Compute the inverse of $B = \begin{bmatrix} -7 & 6 & 12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$

Solution: The augmented matrix is $[B | I] = \begin{bmatrix} -7 & -6 & -12 & 1 & 0 & 0 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{bmatrix}$

by applying elementary row operations, the equivalent reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -10 & -12 & 9 \\ 0 & 1 & 0 & 13/2 & 8 & 11/2 \\ 0 & 0 & 1 & 5/2 & 3 & 5/2 \end{bmatrix}$$

$$\text{So } B^{-1} = \begin{bmatrix} -10 & -12 & 9 \\ 13/2 & 8 & 11/2 \\ 5/2 & 3 & 5/2 \end{bmatrix}$$

2.8.2 Properties of inverse matrix

Let A, B, C be matrices of the appropriate sizes so that the following multiplications make sense, I a suitably sized identity matrix, and α a nonzero scalar. Then

1. (Uniqueness) The matrix A has at most one inverse, henceforth denoted as A^{-1} , provided A is invertible.
2. (Double Inverse) If A is invertible, then $(A^{-1})^{-1} = A$.
3. (2=3 Rule) If any two of the three matrices A, B and AB are invertible, then so is the third, and moreover $(AB)^{-1} = B^{-1}A^{-1}$.
4. A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$.
5. If A is invertible, then $(\alpha A)^{-1} = (\frac{1}{\alpha})A^{-1}$.
6. (Inverse/Transpose) If A is invertible, then $(A^t)^{-1} = (A^{-1})^t$.
7. (Cancellation) Suppose A is invertible. If $AB = AC$ or $BA = CA$, then $B = C$

2.9 Determinant of a matrix and its properties

Definition 2.12 Determinant of a matrix: Let A be an $n \times n$ matrix, then the determinant of A denoted by $\det(A)$ or $|A|$ is defined recursively by. If $A = [a]$ is a 1×1 matrix, then $\det(A) = a$. If A is a matrix of size n with $n \geq 2$, then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^{n+1}a_{1n}\det(A_{1n})$$

where A_{1j} a sub-matrix of A obtained by deleting the first row and the j^{th} column.

So to compute the determinant of a 5×5 matrix we must build 5 sub-matrices, each of size 4. To compute the determinants of each of the 4×4 matrices, we need to create 4 sub matrices of size 3 and so on. To compute the determinant of a 10×10 matrix would require computing the determinant of $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,628,800$ 1×1 matrices. Fortunately there are better ways. However, this suggests an excellent computer programming exercise to write a recursive procedure to compute a determinant.

Lets compute the determinant of a reasonable sized matrix by hand. Suppose that we have a 3×3

matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}$ then

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{vmatrix} \\ &= 3 \begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix} \\ &= 3(1 \cdot 2 - 6 \cdot (-1)) - 2(4 \cdot 2 - 6 \cdot (-3)) - (4 \cdot (-1) - (-3)) \\ &= 3(1(2) - 6(-1)) - 2(4(2) - 6(-3)) - (4(-1) - (-3)) \\ &= 24 - 52 + 1 \\ &= -27 \end{aligned}$$

Theorem 2.2 Exchanging Columns Changes the Sign of a Determinant: If the matrix A' is obtained from A by interchanging any two successive columns, and their determinants exist, then $|A'| = -|A|$.

2.9.1 Properties of determinants of matrix

1. If A is a triangular matrix, then the determinant of A is the product of all the diagonal elements of A .
2. If B is obtained from A by multiplying one row of A by the scalar α , then $\det(B) = \alpha(\det(A))$.
3. If B is obtained from A by adding a multiple of one row of A to another row of A , then $\det(B) = \det(A)$.
4. The matrix A is invertible if and only if $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$.
5. The determinant of a product of two matrices is the product of their determinants. That is,

$$\det(AB) = \det(A)\det(B) \implies \det(A^n) = (\det(A))^n$$

6. If B is the transpose of a matrix A , then $\det(B) = \det(A)$

2.10 Minor of a matrix

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ sub matrix formed by removing row i and column j . Then the **minor** of matrix A at location $i j$ is the determinant of the sub-matrix, $M_{ij}(A) = \det(A_{ij})$.

2.10.1 Cofactor of a matrix

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ sub matrix formed by removing row i and column j . Then the **Cofactor** of A at location $i j$ is the determinant of the sub-matrix, $C_{ij}(A) = (-1)^{i+j} \det(A_{ij})$.

Definition 2.13 Adjoint If $A = (a_{ij})$ is an $n \times n$ matrix, the adjoint of A , denoted by $\text{adj}(A)$, is the transpose of the matrix of cofactors. Hence

$$\text{Adj}(A) = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & c_{32} & \cdots & c_{n2} \\ c_{13} & c_{23} & c_{33} & \cdots & c_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix}$$

Theorem 2.3 Let A be an $n \times n$ matrix. Then

$$A[\text{Adj}(A)] = [\det(A)]I_n = \text{Adj}(A)A.$$



Note If the $\det(A) \neq 0$, then $A^{-1} = \frac{1}{\det(A)} \text{Adj}(A)$

Example 2.25 Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$ Then determine

a) The minors of all elements A .

- b) The cofactors of all elements of A .
- c) The $\text{Adj}(A)$.
- d) The inverse of A .

Solution: *Exercise*

2.11 Solving system of linear equations

Definition 2.14 A linear equation in the variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where the coefficients $a_1, a_2, \dots, a_n$ and right hand side constant term b are given constants.

Definition 2.15 A linear system of m equations in the n unknowns $x_1, x_2, \dots, x_n$ is a list of m equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\dots \\ &\dots \\ &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \tag{2.1}$$

where the values of a_{ij} , b_i and x_j are from the set of real numbers.

Example 2.26 Given the system of linear equations,

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

we have $n = 4$ variables and $m = 3$ equations. Also,

$$\begin{aligned} a_{11} &= 1 & a_{12} &= 2 & a_{13} &= 0 & a_{14} &= 1 & b_1 &= 7 \\ a_{21} &= 1 & a_{22} &= 1 & a_{23} &= 1 & a_{24} &= -1 & b_2 &= 3 \\ a_{31} &= 3 & a_{32} &= 1 & a_{33} &= 5 & a_{34} &= -7 & b_3 &= 1 \end{aligned}$$

Additionally, convince yourself that $x_1 = -2$, $x_2 = 4$, $x_3 = 2$, $x_4 = 1$ is one solution (but it is not the only one!).

Note that the above system can be written concisely as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

we may write the above simultaneous equations as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

A matrix $\mathbf{A}$ is called the coefficient matrix of the system, while the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

obtained by adjoining $\mathbf{b}$ to $\mathbf{A}$ is called the augmented matrix of the system.



Remark: If $b_i = 0, \quad \forall i = 1, 2, \dots, m$ then the linear system is called Homogenous otherwise, non Homogenous.

Theorem 2.4 If $[\mathbf{A}|\mathbf{b}]$ and $[\mathbf{C}|\mathbf{d}]$ are row equivalent, then the systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Cx} = \mathbf{d}$ have exactly the same solutions.

Remark: If $\mathbf{A}$ is an m by n matrix then the linear system $\mathbf{Ax} = \mathbf{0}$ has trivial solution $\mathbf{x} = \mathbf{0}$.

Theorem 2.5 If A is an m by n matrix then the equation $Ax = 0$ has non trivial solution only, when $\text{Rank}(A) < n$ otherwise if $\text{Rank}(A) = n$ then the trivial solution is unique.

The system of equation $Ax = b$ can be directly solved in the following cases.

i) If $A = D$ (*adiagonalmatrix*), the equation (2.1) become

$$\begin{array}{rcl} a_{11}x_1 & & = b_1 \\ & a_{22}x_2 & = b_2 \\ & \ddots & \\ & & a_{nn}x_n = b_n \end{array}$$

The solution is given by $x_i = \frac{b_i}{a_{ii}}$, $a_{ii} \neq 0$.

ii) If $A = L$ (a lower triangular matrix), then Equation (2.1) become

$$\begin{array}{rcl} a_{11}x_1 & & = b_1 \\ a_{21}x_1 + a_{22}x_2 & & = b_2 \\ & \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{array}$$

Solving the first equation and then successively solving the 2nd, 3rd and so on.

We obtain $x_1 = \frac{b_1}{a_{11}}$, $x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$, $\dots$, $x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n(n-1)}x_{n-1})}{a_{nn}}$, where $a_{ii} \neq 0$, $i = 1, 2, \dots, n$

This method of solving equation is called **forward substitution method**.

iii) If $A = U$ an upper triangular matrix, then Equation (2.1) becomes

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n & = & b_1 \\ & a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n & = b_2 \\ & \vdots & \\ & & a_{nn}x_n = b_n \end{array}$$

Solving for the unknowns in the in the order $x_n, x_{n-1}, \dots, x_1$, we get

$$x_n = \frac{b_n}{a_{nn}}, \quad x_{n-1} = \frac{b_{n-1} - a_{(n-1)nx_n}}{a_{(n-1)(n-1)}}, \quad \dots, \quad x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}}$$

This method of solving equations is called **backward substitution method**. Therefore, matrix A is solvable if it can be transformed into any one of the forms D , U or L .

Theorem 2.6 Consider m equations with n variables $A\mathbf{x} = \mathbf{b}$. Then,

- a) If $\text{rank}(A|\mathbf{b}) = \text{rank}(A) = n$ then the system has unique solution
- b) If $\text{rank}(A|\mathbf{b}) = \text{rank}(A) < n$ then the system has infinitely many solutions.
- c) If $\text{rank}(A|\mathbf{b}) > n$ then the system has no solution.

To illustrate this, let's consider the following system of equations:

$$\begin{array}{rcl} x_1 + 2x_2 = 1 & 3x_1 + 2x_2 = 3 & 3x_1 + 2x_2 = 3 \\ 3x_1 + x_2 = -2 & -6x_1 - 4x_2 = 0 & -6x_1 - 4x_2 = -6 \end{array}$$

The augmented matrices for these systems are, respectively,

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & -2 \end{array} \right] \quad \left[\begin{array}{cc|c} 3 & 2 & 3 \\ -6 & -4 & 0 \end{array} \right] \quad \left[\begin{array}{cc|c} 3 & 2 & 3 \\ -6 & -4 & -6 \end{array} \right]$$

Applying the row-reduction algorithm yields the row-reduced form of each of these augmented matrices. The results are, again respectively,

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array} \right] \quad \left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{array} \right] \quad \left[\begin{array}{cc|c} 1 & \frac{2}{3} & 1 \\ 0 & 0 & 0 \end{array} \right]$$

From each of these row-reduced versions of the augmented matrices, one can read off the rank of the coefficient matrix as well as the rank of the augmented matrix. Applying Theorem 2.6 to each of these tells us the number of solutions to expect for each of the corresponding systems. We summarize our findings in the table below.

System	$\text{rank}(A)$	$\text{rank}(A \mathbf{b})$	n	# of solutions
First	2	2	2	1
Second	1	2	2	0 (inconsistent)
Third	1	1	2	∞

To solve a linear system of equations, we have the following Methods:

2.11.1 Cramer's rule

If $A\mathbf{x} = \mathbf{b}$ is a linear system of n equations in n unknowns, and if $\det A \neq 0$, then the system has unique solution which can be determined by $x_i = \frac{|A_i|}{|A|}$, $i = 1, 2, \dots, n$ where A_i is the matrix obtained from A when i^{th} column of A is replaced by $\mathbf{b}$.

Example 2.27 Solve the following system by Cramer's rule:

$$2x_1 + 3x_2 + 4x_3 = 19$$

$$x_1 + 2x_2 + x_3 = 4$$

$$3x_1 - x_2 + x_3 = 9$$

Solution: The coefficient matrix is $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}$ and column matrix is $\mathbf{b} = \begin{bmatrix} 19 \\ 4 \\ 9 \end{bmatrix}$. Then

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{vmatrix} \\ &= 4 + 9 - 4 - 24 - 3 + 2 \\ &= -16 \neq 0, \end{aligned}$$

hence, the system has unique solution.

$$A_1 = \begin{bmatrix} 19 & 3 & 4 \\ 4 & 2 & 1 \\ 9 & -1 & 1 \end{bmatrix} \text{ and } \det(A_1) = \begin{vmatrix} 19 & 3 & 4 \\ 4 & 2 & 1 \\ 9 & -1 & 1 \end{vmatrix} = 38 + 27 - 16 - 72 - 12 + 19 = -16$$

$$A_2 = \begin{bmatrix} 2 & 19 & 4 \\ 1 & 4 & 1 \\ 3 & 9 & 1 \end{bmatrix} \text{ and } \det(A_2) = \begin{vmatrix} 2 & 19 & 4 \\ 1 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} = 8 + 57 + 36 - 48 - 19 - 18 = 16$$

$$A_3 = \begin{bmatrix} 2 & 3 & 19 \\ 1 & 2 & 4 \\ 3 & -1 & 9 \end{bmatrix} \text{ and } \det(A_3) = \begin{vmatrix} 2 & 3 & 19 \\ 1 & 2 & 4 \\ 3 & -1 & 9 \end{vmatrix} = 36 + 36 - 19 - 114 - 27 + 8 = -80.$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-16}{-16} = 1$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{16}{-16} = -1$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-80}{-16} = 5.$$

the solution of the system is $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

Cramer's rule as test of solutions

- i) If $\det(A) \neq 0$, the system of equation has a unique solution.
- ii) If $\det(A) = 0$ and $\det(A_i) = 0 \forall i$, the system has infinitely many solutions.
- iii) If $\det(A) = 0$ and atleast one of $\det(A_i) \neq 0$, then no solutions exist and we say the system is inconsistent.

2.11.2 Gaussian elimination method

Gauss elimination method is a method of solving linear equations by reducing the augmented matrix to row echelon form using elementary row operations. Then we solve for the variables by using backward substitution method. This method will fail if any of the pivot element a_{ii} , $i = 1, 2, \dots, n$ becomes zero. In such case we re-write the equations in such a way that pivots are non zero. This procedure is called **partial pivoting**.

Consider system $Ax = b$, to solve this system using the Gaussian elimination method we follow the following steps:

Step 1: Form the augmented matrix $[A|b]$

Step 2: Transform $[A|b]$ to row echelon form $[U|d]$ using row operations.

Step 3: Solve the system $Ux = d$ by back substitution.

Example 2.28 Solve the following system using Gauss elimination method.

$$2x_1 - 3x_2 + x_3 = 5$$

$$4x_1 + 14x_2 + 12x_3 = 10$$

$$6x_1 + x_2 + 5x_3 = 9$$

Solution: The augmented matrix of the system is
$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 4 & 14 & 12 & 10 \\ 6 & 1 & 5 & 9 \end{array} \right]$$

Applying elementary row operations on this matrix, we change into row echelon form.

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 4 & 14 & 12 & 10 \\ 6 & 1 & 5 & 9 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 0 & 20 & 10 & 0 \\ 0 & 10 & 2 & -6 \end{array} \right] R_3 \rightarrow R_3 - 1/2R_2 \left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 0 & 20 & 10 & 0 \\ 0 & 0 & -3 & -6 \end{array} \right]$$

Since $\text{rank}(A|\mathbf{b}) = \text{rank}(A) = 3 = n$ the solution exists and is unique.

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 5 \\ 20x_2 + 10x_3 &= 0 \\ -3x_3 &= -6 \end{aligned}$$

From this we get $x_3 = 2$. And using back substitution we have $x_2 = -1$ and $x_1 = 0$
Hence $(0, -1, 2)$ is the solution of the system.

2.11.3 Inverse matrix method

Let $A\mathbf{x} = \mathbf{b}$ be a system of n linear equations with n unknowns and A is invertible, then the system has unique solution given by inversion method as $\mathbf{x} = A^{-1}\mathbf{b}$.

Note:- When A is not square or is singular, the system may not have a solution or may have more than one solution.



Example 2.29 Use the inverse of the coefficient matrix to solve the following system:

$$\begin{aligned} 3x_1 + x_2 &= 6 \\ -x_1 + 2x_2 + 2x_3 &= -7 \\ 5x_1 - x_3 &= 10 \end{aligned}$$

Solution: First, let's write down the matrix form of the system.

$$\begin{bmatrix} 3 & 9 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 10 \end{bmatrix}$$

Then we find the inverse of the coefficient matrix.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix} \Rightarrow C_A = \begin{bmatrix} -2 & 9 & -10 \\ 1 & 3 & 5 \\ 2 & -6 & 7 \end{bmatrix} \Rightarrow \text{adj}(A) = \begin{bmatrix} 2 & -1 & 2 \\ 9 & -3 & -6 \\ -10 & 5 & 7 \end{bmatrix}$$

and $\det(A) = 3(-2) + 1(9) + 0(-10) = -6 + 9 = 3$, then

$$A^{-1} = 1/3 \begin{bmatrix} 2 & -1 & 2 \\ 9 & -3 & -6 \\ -10 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 3 & -1 & -2 \\ -10/3 & 5/3 & 7/3 \end{bmatrix}.$$

Therefore $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 3 & -1 & -2 \\ -10/3 & 5/3 & 7/3 \end{bmatrix} \begin{bmatrix} 6 \\ -7 \\ 10 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 5 \\ -25/3 \end{bmatrix}$ Now each of the entries of $\mathbf{x}$ are $x_1 = 1/3$, $x_2 = 5$ and $x_3 = -25/3$.

Exercise 2.3 1. Solve the following linear system of equations by using Cramer's rule, Gaussian elimination method, and inverse method.

$$\begin{array}{lll} 2x_1 + 5x_2 + 3x_3 = 9 & x + z = 1 & x + 2y + z = 3 \\ \text{a) } 3x_1 + x_2 + 2x_3 = 3 & \text{b) } 2x + y + z = 0 & \text{c) } 2x + 5y - z = -4 \\ x_1 + 2x_2 - x_3 = 6 & x + y + 2z = 1 & 3x - 2y - z = 5 \end{array}$$

2. Use rank of a matrix to determine the values of a , b and c so that the following system has:

a) no solution b) more than one solution c) a unique solution and solve it.

$$\begin{array}{lll} 1x + y - bz = 1 & x + 2y - 3z = a & x - 2y + bz = 3 \\ \text{i) } 2x + 3y + az = 3 & \text{ii) } 2x + 6y - 11z = b & \text{iii) } ax + 2z = 2 \\ x + ay + 3z = 2 & x - 2y + 7z = c & 5x + 2y = 2 \end{array}$$

2.12 Eigenvalues and Eigenvectors

Definition 2.16 Let A be a square matrix. Then if λ is a real number and $\mathbf{x}$ a non-zero column vector satisfying $A\mathbf{x} = \lambda\mathbf{x}$, we call $\mathbf{x}$ an eigenvector of A and λ is called an eigenvalue of A . We also say that $\mathbf{x}$ is an eigenvector corresponding to the eigenvalue λ .

Example 2.30 Let $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then show that $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is eigenvector of A with $\lambda = 1$.

If λ is an eigenvalue of an $n \times n$ matrix A with corresponding eigenvector $\mathbf{x}$, then $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$, with $\mathbf{x} \neq \mathbf{0}$. Hence, $\det(A - \lambda I_n) = 0$ and there are at most n distinct eigenvalues of A . Conversely, if $\det(A - \lambda I_n) = 0$, then $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\mathbf{x}$. The equation $\det(A - \lambda I_n) = 0$ is called the characteristic equation of A , while the polynomial $\det(A - \lambda I_n)$ is called the characteristic polynomial of A . The characteristic polynomial of A is often denoted by $ch_A(\lambda)$. Hence the eigenvalues of A are the roots of the characteristic polynomial of A .

Example 2.31 Find the eigenvalues and all eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution: The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = 0$, or $(\lambda - 1)(\lambda - 3) = 0$. Hence, $\lambda = 1$ or $\lambda = 3$. The eigenvector equation $(A - I_n)\mathbf{x} = \mathbf{0}$ reduces to

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$(2 - \lambda)x + y = 0$$

$$x + (2 - \lambda)y = 0$$

Taking $\lambda = 1$ gives

$$x + y = 0$$

$$x + y = 0,$$

which has solution $x = -y$, and let $y = t$ be arbitrary non zero real number. Consequently, the eigenvectors corresponding to $\lambda = 1$ are the vectors $\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $t \neq 0$ which is the scalar

multiple of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Taking $\lambda = 3$ gives

$$-x + y = 0$$

$$x - y = 0,$$

which has solution $x = y$, and let $y = t$ is arbitrary non zero real number. Consequently, the eigenvectors corresponding to $\lambda = 3$ are the vectors $\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $t \neq 0$ hence the scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore $\lambda_1 = 1$ and $\lambda_2 = 3$ are the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and the corresponding eigenvector are $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

Theorem 2.7 If A is an $n \times n$ matrix, the following statements are equivalent

- λ is an eigenvalue of A .
- The system of equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.
- There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.
- λ is a solution of the characteristics equation $\det(A - \lambda I) = 0$.

Diagonalization

Problem 2.1 Given an $n \times n$ matrix A , does there exist an invertible matrix P such that

$$P^{-1}AP$$

is diagonal?

Problem 2.2 Given an $n \times n$ matrix A , does it have n linearly independent eigenvectors?

Theorem 2.8 Let A be an $n \times n$ matrix having distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding eigenvectors $X_1, X_2, \dots, X_n$ respectively. Let P be the matrix whose columns are respectively $X_1, X_2, \dots, X_n$. Then P is non singular and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Definition 2.17 If A and B are square matrices, then we say that B is similar to A if there is an invertible matrix P such that

$$B = P^{-1}AP$$

Definition 2.18 A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix. In other words, A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to diagonalize A .

Theorem 2.9 If A is an $n \times n$ matrix, the following statements are equivalent

- a) A is diagonalizable.
- b) A has n linearly independent eigenvectors.

Procedure for Diagonalizing a Matrix

1. Confirm that the matrix is actually diagonalizable by finding n linearly independent eigenvectors. One way to do this is by finding a basis for each eigenspace and merging these basis vectors into a single set S . If this set has fewer than n vectors, then the matrix is not diagonalizable.
2. Form the matrix

$$P = [p_1 p_2 \cdots p_n]$$

that has the vectors in S as its column vectors.

3. The matrix $P^{-1}AP$ will be diagonal and have the eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

corresponding to the eigenvectors

$$p_1, p_2, \dots, p_n$$

as its successive diagonal entries.

Exercise 2.4 In each of the following, determine if the the matrix is diagonalizable

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

