

CHAPTER ONE

REAL SEQUENCES

1.1 Definition and examples of sequences

In informally speaking, the term “sequences” in mathematics is used to describe an unending succession of numbers whose order is determined by a rule or a function.

For instance, 1, 2, 3, 4, ...

2, 4, 6, 8, ...

$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$ are sequences.

The numbers in the sequence are called terms of the sequence and described as 1st term, 2nd term, 2nd term etc. according to the position they occupy.

Consider $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ i.e the set A consists of all numbers of the form $\frac{1}{n}$, where n is a positive integer. The set A is also described by defining a function f by the rule $f(n) = \frac{1}{n}$ where the domain f is a positive integer. Then the set A is precisely the set of values taken by the function f . In general we have the following definition.

Definition: - A sequence of real numbers is a function whose domain is the set of non-negative

integers greater than or equal to a given integer m (usually 0 or 1)

i. e. $a: \{0, 1, 2, \dots\} \rightarrow R$

Notation:- Terms of the sequence are denoted as

$a_1, a_2, a_3, \dots, a_n, \dots$, named as 1st, 2nd, 3rd, ..., n^{th} terms & so on

Or equivalently we can write as $\{a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots\}, \{a_n\}, \{a_n\}_{n=1}^{\infty}$

Usually (but not always) sequences have recognizable patterns and can be described by a formula.

Example-1:- a) If $a_n = \frac{1}{n}$, then $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, \dots$ for all $n \in N$

b) If $a_n = \frac{1}{2^n}$, then $a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}, \dots$ for all $n \in N$

c) If $a_n = \frac{1}{2n+1}$, then $a_1 = \frac{1}{3}, a_2 = \frac{1}{5}, a_3 = \frac{1}{7}, \dots$ for all $n \in N$

Example-2:- Find a formula a_n for each of the following

a) $\{1, -1, 1, -1, \dots\}$

b) $\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\}$

c) $\{2, 2, 2, 0, 0, \dots\}$

Solution: - a) $a_n = (-1)^n$ for all positive integers n

b) $a_n = \frac{2^n - 1}{2^n}$ for all positive integer n

c) $a_n = 2$ if $n \leq 3, a_n = 0$ if $n \geq 4$

Example-3:- Write down the first few terms of each of the following sequences

a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$ b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

Solution:- a) If $n = 1$, then $a_1 = 2$, If $n = 2$, then $a_2 = \frac{3}{4}$, If $n = 3$, then $a_3 = \frac{4}{9}$, etc.

$$\therefore \left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, \dots\right\}$$

b) If $n = 0$, then $a_0 = -1$, If $n = 1$, then $a_1 = \frac{1}{2}$, If $n = 2$, then $a_2 = \frac{-1}{4}$, etc.

$$\therefore \left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \dots\right\}$$

Note:- Since the terms of the sequence in (b) alternate in signs it is sometimes called an alternating sequence.

1.2 Convergence and Divergence of real Sequences

Definition:- A Sequence $\{a_n\}$ is said to have a limit L , written as $\lim_{n \rightarrow \infty} a_n = L$, if for each $\epsilon > 0$,

there exists a natural number N such that for every $n > N$, we have $|a_n - L| < \epsilon$. If such a

number L exists we say that the sequence $\{a_n\}$ converges (or converges to L).

If such a number L does not exist, we say that the sequence $\{a_n\}$ diverges or

that $\lim_{n \rightarrow \infty} a_n$ does not exist.

Remark:- As with limits of functions, the limit of the sequence is unique if it exists.

Example-1:- Show that the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges and that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Solution:- Let $\epsilon > 0$; $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon}$

Choose $N = \left[\frac{1}{\epsilon}\right]$

$$\forall \epsilon > 0 \exists N = \frac{1}{\epsilon} \text{ such that } n \geq N \Rightarrow \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

Then by definition of sequence, the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Note:- The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is known as the *Harmonic Sequence*.

Example-2:- Show that the sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$ converges and that $\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$

Solution:- Let $\epsilon > 0$ be given arbitrary small.

We need to find $N > 0$ such that $\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \epsilon$ whenever $n > N$

$$\text{But } \left| \frac{n}{2n+1} - \frac{1}{2} \right| < \varepsilon \Rightarrow \left| \frac{2n-2n-1}{2(2n+1)} \right| = \left| \frac{-1}{4n+2} \right| = \frac{1}{4n+2} < \varepsilon$$

$$\Rightarrow \frac{1}{\varepsilon} < 4n + 2$$

$$\Rightarrow \frac{1}{\varepsilon} - 2 < 4n$$

$$\Rightarrow n > \frac{1}{4\varepsilon} - \frac{1}{2}$$

$$\text{Choose } N = \left[\frac{1}{4\varepsilon} - \frac{1}{2} \right]$$

$$\text{Then } n > N \Rightarrow n > \left[\frac{1}{4\varepsilon} - \frac{1}{2} \right]$$

$$\Rightarrow n > \frac{1}{4\varepsilon} - \frac{1}{2}$$

$$\Rightarrow \frac{1}{4n+2} < \varepsilon$$

$$\Rightarrow \left| \frac{-1}{4n+2} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{n}{2n+1} - \frac{1}{2} \right| < \varepsilon$$

$$\text{Hence } \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$$

Example-3:- show that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges

Solution:-
$$(-1)^n = \begin{cases} 1; & n \text{ is even} \\ -1; & n \text{ is odd} \end{cases}$$

Assume that $\{(-1)^n\}_{n=1}^{\infty}$ is convergent Sequence

$$\text{Choose } 0 < \varepsilon \leq 1$$

Case 1) $L \geq 0$, then for odd values of n , we have

$$|(-1)^n - L| = |-1 - L| = |-(1 + L)| = 1 + L \geq 1$$

Case 2) $L < 0$, then even values of n , we have

$$|(-1)^n - L| = |1 - L| = 1 - L \geq 1 \text{ This is contradiction our assumption.}$$

i.e. $\forall \varepsilon > 0$ and $0 < \varepsilon \leq 1$, $\nexists$ a number L satisfying the condition of convergent sequence.

$$\therefore \{(-1)^n\}_{n=1}^{\infty} \text{ diverges.}$$

Note:- The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is an example of **oscillating sequence**.

It oscillate between 1 and -1.

Example-4 Show that the sequence $\{n^2\}_{n=0}^{\infty}$ also diverges

Solution:- As n is arbitrarily large n^2 also arbitrarily large i.e. $n \rightarrow \infty \Rightarrow n^2 \rightarrow \infty$

$$\therefore \lim_{n \rightarrow \infty} n^2 = \infty$$

Hence $\{n^2\}_{n=0}^{\infty}$ diverges.

Definition:- Let $\{a_n\}$ be a sequence. $\lim_{n \rightarrow \infty} a_n = \infty$ If for each positive number M , there is a natural number N such that $a_n > M$ for all $n > N$. In this case we say that $\{a_n\}$ diverges to ∞ . Similarly, If for each number M , there is an integer N such that $a_n < M$ for all $n > N$.

In this case we say that $\{a_n\}$ diverges to $-\infty$ and we write as $\lim_{n \rightarrow \infty} a_n = -\infty$

Example-4:- prove that $\lim_{n \rightarrow \infty} e^n = \infty$.

Solution:- Let $M > 0$ be given.

We need to find $N > 0$ such that $e^n > M$ whenever $n > N$

$$\text{But } e^n > M \Rightarrow \ln e^n > \ln M$$

$$\Rightarrow n > \ln M$$

$$\text{Choose } N = [\ln M]$$

$$\text{Then } n > N \Rightarrow n > [\ln M] \Rightarrow n > \ln M$$

$$\Rightarrow e^n > M \text{ whenever } n > N$$

$$\therefore \lim_{n \rightarrow \infty} e^n = \infty$$

Example-5:- Show that $\lim_{n \rightarrow \infty} (-2n^3) = -\infty$

Solution:- Let $M < 0$ be given.

We need to find $N > 0$ such that $-2n^3 < M$.

$$\text{But } -2n^3 < M \Rightarrow n^3 > \frac{M}{-2} \Rightarrow n > \sqrt[3]{\frac{M}{-2}}$$

$$\text{Choose } N = \left[\sqrt[3]{\frac{M}{-2}} \right]$$

$$\text{Then } n > N \Rightarrow n > \left[\sqrt[3]{\frac{M}{-2}} \right] \Rightarrow n > \sqrt[3]{\frac{M}{-2}}$$

$$\Rightarrow n^3 > \frac{M}{-2} \Rightarrow -2n^3 < M \text{ whenever } n > N$$

Theorem:- Let $\{a_n\}$ be a sequence, L a number and f a function defined for every positive

integer such that $f(n) = a_n$

a) If $\lim_{x \rightarrow \infty} f(x) = L$, then $\{a_n\}$ converges and $\lim_{x \rightarrow \infty} a_n = L$

b) If $\lim_{x \rightarrow \infty} f(x) = \infty$ (or $\lim_{x \rightarrow \infty} f(x) = -\infty$), then $\{a_n\}$ diverges and

$$\lim_{x \rightarrow \infty} a_n = \infty \text{ (or } \lim_{x \rightarrow \infty} a_n = -\infty \text{)}.$$

Proof:- a) assume that if $\lim_{x \rightarrow \infty} f(x) = L$, and let $\varepsilon > 0$.

Then there is some integer N such that if $x > N$, then $|f(x) - L| < \varepsilon$

$$\Rightarrow \text{If } n > N, \text{ then } |a_n - L| = |f(n) - L| < \varepsilon. \text{ So } \lim_{x \rightarrow \infty} a_n = L$$

b) If $\lim_{x \rightarrow \infty} f(x) = \infty$, then for any M there is an integer N such that

if $x > N$, then $f(x) > M$. So that, if $n > N$, then $f(n) = a_n > M$.

$$\Rightarrow \lim_{x \rightarrow \infty} a_n = \infty \text{ (by definition)}$$

$$\text{Similarly, } \lim_{x \rightarrow \infty} f(x) = -\infty, \Rightarrow \lim_{x \rightarrow \infty} a_n = -\infty$$

Example-6:- Show that $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right) = 1$.

Solution:- Let $f(x) = \frac{x}{x+1}$; $x \geq 1$

$$\text{Then } f(n) = \frac{n}{n+1} = a_n \text{ for } n \geq 1$$

$$\text{Since } \lim_{n \rightarrow \infty} \frac{x}{x+1} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{x}} = 1.$$

$$\text{Hence by the above theorem } \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right) = 1.$$

Example-7:- Show that $\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0$, for $x > 0$

Solution:- Let $f(x) = \frac{1}{x^r}$; $r > 0$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{1}{x^r} = \lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)^r = 0^r = 0$$

$$\text{Then by the above theorem } \lim_{n \rightarrow \infty} \frac{1}{n^r} = 0.$$

$$\text{In particular, } \lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} = 0; \quad \lim_{n \rightarrow \infty} \frac{1}{n^3} = 0; \quad \lim_{n \rightarrow \infty} \frac{1}{n^c} = 0 \text{ etc.}$$

i.e. the sequences $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$, $\left\{\frac{1}{n^3}\right\}_{n=1}^{\infty}$, $\left\{\frac{1}{n^e}\right\}_{n=1}^{\infty}$, etc converges to 0.

Example-8:- Show that $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$.

Solution :- Let $f(x) = (1 + \frac{1}{x})^x$; $x \geq 1$

$$\text{Then } f(n) = (1 + \frac{1}{n})^n; n \geq 1$$

$$\text{We know that } \lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e$$

$$\text{Then by the above theorem } \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e.$$

Example-9:- Let r be any number. Show that $\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{for } |r| < 1 \\ 1 & \text{for } r = 1 \end{cases}$ and that $\{r^n\}_{n=1}^{\infty}$

diverges for all other values of r .

Solution:- Case 1) if $r \geq 0$

$$\text{Let } f(x) = r^x; x \geq 1; \text{ so that } f(n) = r^n; n \geq 1.$$

$$\lim_{x \rightarrow \infty} r^x = \begin{cases} 0 & \text{for } 0 \leq r < 1 \\ 1 & \text{for } r = 1 \\ \infty & \text{for } r > 1 \end{cases} \text{ Sequence}$$

$$\text{Then by the above theorem } \lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{for } 0 \leq r < 1 \\ 1 & \text{for } r = 1 \\ \infty & \text{for } r > 1 \end{cases} \text{ ----- (1)}$$

Case 2) if $r < 0$

$$r = -1 \Rightarrow \{r^n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty} \text{ that is oscillating sequence which diverges.}$$

$$r \neq -1 \Rightarrow |r^n| = |r|^n$$

$$\text{From (1) } \lim_{n \rightarrow \infty} |r^n| = \lim_{n \rightarrow \infty} |r|^n = \begin{cases} 0 & -1 < r < 0 \\ \infty & r < -1 \end{cases} \text{ diverges.}$$

$$\therefore \{r^n\}_{n=1}^{\infty} \text{ converge for } |r| \leq 1 \text{ and diverges for } |r| > 1$$

$$\text{In particular, } \{(\frac{1}{2})^n\}_{n=1}^{\infty} \text{ converges to } 0$$

$$\{(1)^n\}_{n=1}^{\infty} \text{ converges to } 1$$

$$\{(1.1)^n\}_{n=1}^{\infty} \text{ diverges.}$$

Note:-For any nonzero constant c , the sequence $\{cr^n\}_{n=m}^{\infty}$ is called a **geometric sequence**.

The first term of the sequence, cr^m , is the initial term. r is called the common ratio of the sequence.

$$(i. e. \frac{cr^{n+1}}{cr^n} = r \text{ for } n \geq m)$$

Example:10 Show that $\lim_{n \rightarrow \infty} \sqrt[n]{c} = 1$.

Solution:- Note that $\sqrt[n]{c} = c^{\frac{1}{n}} = e^{\left(\frac{1}{n}\right) \ln c}$

Let $f(x) = e^{\left(\frac{1}{x}\right) \ln c}$; $x \geq 1$ so that f is continuous and

$$f(n) = e^{\left(\frac{1}{n}\right) \ln c}; n \geq 1.$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\left(\frac{1}{x}\right) \ln c} = e^{\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) \ln c} = e^0 = 1$$

Then by the above theorem $\lim_{n \rightarrow \infty} \sqrt[n]{c} = 1$

Example:11 Show that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

Solution:- Note that $\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\left(\frac{1}{n}\right) \ln n}$

Let $f(x) = e^{\left(\frac{1}{x}\right) \ln x}$; $x \geq 1$ so that f is continuous and $f(n) = e^{\left(\frac{1}{n}\right) \ln n}$; $n \geq 1$

$\lim_{x \rightarrow \infty} x = \infty$ and $\lim_{x \rightarrow \infty} \ln x = \infty \Rightarrow$ by *L'Hopital Rule*

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0$$

Then $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\left(\frac{1}{x}\right) \ln x} = e^{\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) \ln x} = e^0 = 1$

$\therefore$ using the above theorem $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

Activities

Find the limit of each of the following

a) $\lim_{n \rightarrow \infty} \left(1 + \frac{0.05}{n}\right)^n$

b) $\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n$

c) $\lim_{n \rightarrow \infty} \frac{n^2 + 2n - 1}{2n^2 - 5n + 3}$

1.3 Properties of convergent real Sequences

Since Sequences are functions, we may add, subtract, multiply, and divide as we do for functions. Rules for computing the limits of combinations of Sequences are analogous to the rules for limits of combinations of functions

Theorem:-suppose that the sequence a_n and b_n converges to limits L and M , and c is a constant, then

- a) $\lim_{n \rightarrow \infty} c = c$
- b) $\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n = cL$
- c) $\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n = L + M$
- d) $\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n = L \cdot M$
- e) $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{L}{M}$ if $M \neq 0$

Proof:-Left as an exercise.

Example-1:-Find a) $\lim_{n \rightarrow \infty} \frac{2n-1}{3n+5}$ b) $\lim_{n \rightarrow \infty} \frac{3n^2}{2n-1} \sin\left(\frac{\pi}{n}\right)$

Solution a) The highest power of 'n' in both in the numerator and the denominator is 1.

Then divide the numerator and the denominator by 'n'.

$$\text{i.e. } \frac{(2n-1) \times \frac{1}{n}}{(3n+5) \times \frac{1}{n}} = \frac{2 - \frac{1}{n}}{3 + \frac{5}{n}}$$

Then by applying convergence properties, we have

$$\lim_{n \rightarrow \infty} \frac{2n-1}{3n+5} = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{3 + \frac{5}{n}} = \frac{\lim_{n \rightarrow \infty} 2 - \lim_{n \rightarrow \infty} \frac{1}{n}}{\lim_{n \rightarrow \infty} 3 + \lim_{n \rightarrow \infty} \frac{5}{n}} = \frac{2-0}{3+0} = \frac{2}{3}$$

$$\begin{aligned} \text{b) } \lim_{n \rightarrow \infty} \frac{3n^2}{2n-1} \sin\left(\frac{\pi}{n}\right) &= \lim_{n \rightarrow \infty} \frac{3n}{2n-1} \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{3n}{2n-1} \cdot \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} = \frac{3}{2} \pi \end{aligned}$$

Theorem:- (squeezing theorem for sequences)

Given three sequences a_n , b_n and c_n such that $a_n \leq b_n \leq c_n$ for every n and

$\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$, then $\lim_{n \rightarrow \infty} b_n = L$

Proof:-For every $\varepsilon > 0$, there is N_1 and N_2 such that $|c_n - L| < \varepsilon$ for $n > N_1$

and $|a_n - L| < \varepsilon$ for $n > N_2$. Let $N = \{N_1, N_2\}$.

For $n > N$, we have $|a_n - L| < \varepsilon$ which implies $-\varepsilon < a_n - L < \varepsilon$

$\Rightarrow L - \varepsilon < a_n < L + \varepsilon$ and hence $L - \varepsilon < a_n \dots\dots\dots (*)$

From $|c_n - L| < \varepsilon$, we have $-\varepsilon < c_n - L < \varepsilon$
 $\Rightarrow L - \varepsilon < c_n < L + \varepsilon \Rightarrow c_n < L + \varepsilon \dots\dots\dots(**)$
 Since $a_n \leq b_n \leq c_n$ by hypothesis, from (*) and (**) it follows that
 $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \Rightarrow |b_n - L| < \varepsilon$, for every $n > N$.
 Therefore $\lim_{n \rightarrow \infty} b_n = L$ by the definition of limits.

Example-2:- show that $\lim_{n \rightarrow \infty} \frac{\sin^2 n}{n} = 0$

Solution:- We know that $0 \leq \frac{\sin^2 n}{n} \leq \frac{1}{n}$, since $\lim_{n \rightarrow \infty} 0 = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Then by show that squeezing theorem we have $\lim_{n \rightarrow \infty} \frac{\sin^2 n}{n} = 0$

1.4 Bounded and Monotonic Sequences

Definition:- Let $\{a_n\}_{n=m}^{\infty}$ be a sequence, then

- The sequence $\{a_n\}_{n=m}^{\infty}$ is bounded above if there is a number M_1 such that $a_n \leq M_1$ for every positive integer $n \geq m$.
- The sequence $\{a_n\}_{n=m}^{\infty}$ is bounded below if there is a number M_2 such that $a_n \geq M_2$ for every positive integer $n \geq m$.
- The sequence $\{a_n\}_{n=m}^{\infty}$ is bounded if there is a number M such that $|a_n| \leq M$ for every positive integer $n \geq m$. Otherwise the sequence is unbounded.

Example:-1) the sequence $\{(-1)^n\}_{n=1}^{\infty}$ has an upper bound 1 and a lower bound -1.

So it is bounded.

Example:-2) the sequences $\{\frac{1}{n}\}_{n=2}^{\infty}$ is bounded, since $|\frac{1}{n}| \leq \frac{1}{2}; n \geq 2$.

Example:-3) the sequences $\{n^2\}_{n=1}^{\infty}$ and $\{2^n\}_{n=1}^{\infty}$ are unbounded because no matter what number M we choose, there is value n such that $n^2 > M$ and $2^n > M$.

Theorem:- a) $\{a_n\}_{n=m}^{\infty}$ converges $\Rightarrow \{a_n\}_{n=m}^{\infty}$ is bounded.

b) $\{a_n\}_{n=m}^{\infty}$ is unbounded $\Rightarrow \{a_n\}_{n=m}^{\infty}$ diverges.

Proof a) Suppose $\lim_{n \rightarrow \infty} a_n = L$.

$$\exists N \exists n \geq N \Rightarrow |a_n - L| < 1$$

$$\therefore n \geq N \Rightarrow |a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|$$

$$\text{Let } M = \max\{|a_m|, |a_{m+1}|, \dots, |a_{N-1}|, 1 + |L|\}$$

Then $|a_n| \leq M$ for $n \geq m$, and thus the sequence is bounded.

$$\text{b) } (P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

i.e. part (b) is logically equivalent to part(a)

Example:-4) the sequences $\{\ln n\}_{n=1}^{\infty}$ and $\{2^n\}_{n=0}^{\infty}$ are unbounded and hence by the above theorem the sequences diverge.

Remark:- boundedness does not necessarily imply convergence i.e. all bounded sequences may not converge.

Example:-5) the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but it diverges.

Activity

1. Show that whether the following sequences are convergent or divergent

$$\text{a) } \left\{ \frac{2n^2+n}{3n+1} \right\}_{n=1}^{\infty}$$

$$\text{b) } \left\{ \frac{n^2-n}{n^2+1} \right\}_{n=1}^{\infty}$$

$$\text{c) } \left\{ (-1)^n \frac{n}{n^2+1} \right\}_{n=1}^{\infty}$$

Definition:- A sequence $\{a_n\}_{n=m}^{\infty}$ is said to be

- Monotone increasing if $a_n \leq a_{n+1} \forall n \geq m$.
- Monotone decreasing if $a_n \geq a_{n+1} \forall n \geq m$.
- Monotonic if it is either monotone increasing or monotone decreasing.
- Strictly increasing if $a_n < a_{n+1} \forall n > m$.
- Strictly decreasing if $a_n > a_{n+1} \forall n > m$.

A sequence is also said to be strictly monotonic if it is either strictly increasing or decreasing.

Example:-6 a) $\{1 - \frac{1}{n}\}_{n=1}^{\infty}$ and $\{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\}_{n=1}^{\infty}$ are increasing sequences.

$$\text{b) } \left\{ \sin \frac{1}{n} \right\}_{n=1}^{\infty} \text{ is decreasing sequence.}$$

$$\text{c) } \{(-1)^n\}_{n=m}^{\infty} \text{ is neither increasing nor decreasing because it oscillates between -1 and 1}$$

Theorem:-Every bounded monotonic sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

Proof:-Assume that $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence.

Since $\{a_n\}_{n=1}^{\infty}$ is bound, the set $S = \{a_1, a_2, \dots, a_n, \dots\}$ has an upper bound.

By completeness axiom it has a least upper bound L .

Let $\varepsilon > 0$, then $L - \varepsilon$ is not an upper bound of S since L is the least upper bound.

Thus $\exists$ a number a_N in S such that $L - \varepsilon < a_N \leq L$ for some integer N .

Since the given sequence is increasing, we have $a_N \leq a_n$ for $n \geq N$.

Hence $L - \varepsilon < a_N \leq a_n \leq L; \quad \forall n \geq N$.

Thus $-\varepsilon < a_n - L \leq a_n - L \leq L - L; \quad \forall n \geq N$

$$\Rightarrow -\varepsilon < a_n - L \leq 0; \quad \forall n \geq N$$

$$\Rightarrow -\varepsilon < a_n - L < \varepsilon; \quad \forall n \geq N$$

$$\Rightarrow |a_n - L| < \varepsilon; \quad \forall n \geq N$$

$$\therefore \lim_{n \rightarrow \infty} a_n = L.$$

Hence $\{a_n\}_{n=1}^{\infty}$ converges to L .

Example:-7) Let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by $a_0 = 0$ and $a_{n+1} = a_n^2 + \frac{1}{4}, \quad \forall n \geq 0$.

Show that $\{a_n\}_{n=0}^{\infty}$ converges.

Solution:-Using the above theorem it is enough to prove that the sequence is bounded and increasing.

(i) To prove boundedness

Claim that $|a_n| < \frac{1}{2}; \quad n \geq 0$

$$a_0 = 0 \Rightarrow |a_0| = 0 < \frac{1}{2}$$

Assume that $|a_n| < \frac{1}{2}; \quad \forall n \geq 0$.

$$\text{Then } |a_{n+1}| = a_n^2 + \frac{1}{4} < \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2} \quad [\because a_n^2 = |a_n|^2 < \left(\frac{1}{2}\right)^2]$$

Then $\{a_n\}_{n=0}^{\infty}$ is bounded

(ii) To prove the given sequence is increasing we must show that $\forall n \geq 0, a_n < a_{n+1}$

$$a_n < a_{n+1} \Leftrightarrow a_n < a_n^2 + \frac{1}{4}$$

$$\Leftrightarrow a_n^2 - a_n + \frac{1}{4} > 0$$

$$\Leftrightarrow (a_n - \frac{1}{2})^2 > 0 \text{ is always true since } a_n \neq \frac{1}{2}.$$

Then the given sequence is increasing.

Using (i) and (ii) and applying the above theorem the given sequence converges.

To find $\lim_{n \rightarrow \infty} a_n$

Let $L = \lim_{n \rightarrow \infty} a_n$ then $L = \lim_{n \rightarrow \infty} a_{n+1}$

$$\therefore L = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} (a_n^2 + \frac{1}{4}) = \lim_{n \rightarrow \infty} a_n^2 + \frac{1}{4} = (\lim_{n \rightarrow \infty} a_n)^2 + \frac{1}{4}$$

$$= L^2 + \frac{1}{4}$$

$$\Rightarrow L^2 - L + \frac{1}{4} = 0$$

$$\Rightarrow (L - \frac{1}{2})^2 = 0 \Rightarrow L = \frac{1}{2}$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \frac{1}{2}$$

Note: If we specify a_0 and express a_{n+1} in terms of $\{a_0, a_1, a_2, \dots, a_n, \dots\}$ we say that $\{a_n\}_{n=0}^{\infty}$

is defined *recursively*.

Activity

Show that whether the following sequences are convergent or divergent

a) $\left\{2 + \frac{1}{n+1}\right\}_{n=1}^{\infty}$

b) $\left\{n - \frac{1}{n^2+1}\right\}_{n=1}^{\infty}$

CHAPTER 2

2.1 Infinite Series

We now wish to extend addition to infinitely many numbers and to define what we mean by such a sum.

Definition:-consider a sequence with terms $a_1, a_2, a_3, \dots, a_n, \dots$ we may form in succession

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$\vdots$$

$$s_n = a_1 + a_2 + a_3 + \dots + a_n$$

The sum $s_1, s_2, s_3, \dots, s_n$ are called **partial sums**. The sequence of **partial sums** $\{s_n\}_{n=1}^{\infty}$ is called an infinite series or simply series. It is denoted by $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \dots + a_n + \dots$

Therefore, an infinite series or just series is equal to the limit of the partial sums, that is,

$$\sum_{i=1}^{\infty} a_i = \lim_{n \rightarrow \infty} s_n$$

Example:-1) If $a_n = n$, Then $s_1 = 1, s_2 = 1 + 2 = 3, s_3 = 1 + 2 + 3 = 6$

$$\therefore s_n = 1 + 2 + 3 + \dots + n - 1 + n = \frac{n(n+1)}{2}$$

$\therefore$ the sequence of a partial Sum is $\{\frac{n(n+1)}{2}\}$

Example:-2) If $a_n = (-1)^n$, Then $s_1 = -1, s_2 = -1 + 1 = 0, s_3 = -1 + 1 + (-1) = -1$

$$\therefore s_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \text{ is the sequence of a partial Sums.}$$

2.2 Convergence and Divergence real series

Consider the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ of the series $\sum_{n=1}^{\infty} a_n$. If this sequence converge to a number s , then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to s .

In this case we write $\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = s$

On the other hand if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ diverges, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges. In this case $\lim_{n \rightarrow \infty} s_n = \pm\infty$ or $\lim_{n \rightarrow \infty} s_n$ is not unique, then $\sum_{i=1}^{\infty} a_i$ is *diverges*.

Note:-If $\lim_{n \rightarrow \infty} s_n$ is not unique then $\sum a_n$ **oscillates**.

Example:-1) Determine whether the series $\sum_{n=1}^{\infty} n$ is convergent or divergent.

If it converges determine its value.

Solution:-The sequence of partial sum is $s_n = \sum_{i=1}^n i$

This is known series and its value can be shown to be

$$s_n = \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{ \frac{n(n+1)}{2} \right\}_{n=1}^{\infty} \text{ is convergent or divergent.}$$

Now,
$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2} = \infty.$$

Therefore, the sequence of partial sums diverges to ∞ . So the series *diverges*.

Example:-2) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots$ converges and find the sum.

Solution:-By partial fraction $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ for $n \geq 1$.

$$\begin{aligned} \text{Hence } s_n &= \sum_{i=1}^n \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \dots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1$$

$$\text{Hence } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges and } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Note:- The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is called a telescoping series because when we write the partial sums all but the first and the last terms cancel.

Example:-3) Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent or divergent. If it converges determine its value .

Solution: - The sequence of partial sum

$$s_n = \sum_{i=1}^n \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{ 1 - \frac{1}{2^n} \right\}_{n=1}^{\infty} \text{ is convergent or divergent.}$$

In this case the limit of the sequence of partial sums is,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} 1 - \frac{1}{2^n} = 1$$

Therefore, the sequence of partial sums is convergent and the series will also be convergent. The value of the series is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Example:-4) consider the series $1 + 2 - 3 + 1 + 2 - 3 + \cdots$

Here $\lim_{n \rightarrow \infty} s_n = 1 \text{ or } 3 \text{ or } 0.$

Hence $\lim_{n \rightarrow \infty} s_n$ is not unique. The series oscillates.

Activities

Determine whether the following series is convergent or divergent. If it converges determine its value .

a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

c) $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

b) $\sum_{n=1}^{\infty} (-1)^n$

d) $\sum \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}}$

Properties of convergent series

Theorem: - If $\sum a_n$ and $\sum b_n$ are convergent series, then

- i. $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$ (where c is a constant)
- ii. $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- iii. $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example:- 5) Find the sum of the series $\sum_{n=1}^{\infty} [\frac{3}{n(n+1)} + \frac{1}{2^n}]$

Solution:- The series $\sum 1/2^n$ is a geometric series with $a = 1/2$ and $r = 1/2$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \quad \text{----- (1)}$$

We have $s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n [\frac{1}{i} - \frac{1}{i+1}]$

$$= \left[1 - \frac{1}{2} \right] + \left[\frac{1}{2} - \frac{1}{3} \right] + \left[\frac{1}{3} - \frac{1}{4} \right] + \dots + \left[\frac{1}{n} - \frac{1}{n+1} \right] = 1 - \frac{1}{n+1}$$

and so $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} [1 - \frac{1}{n+1}] = 1 - 0 = 1$

Therefore, the given series is convergent and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ ----- (2)

So, by properties of convergent series

$$\sum_{n=1}^{\infty} [\frac{3}{n(n+1)} + \frac{1}{2^n}] = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3 \cdot 1 + 1 = 4$$

Theorem: -If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} (a_n + b_n)$ is divergent

Proof:- Let $b_n = (a_n + b_n) - a_n$, then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n) - \sum_{n=1}^{\infty} a_n$

Since $\sum_{n=1}^{\infty} a_n$ is convergent by hypothesis, $\sum_{n=1}^{\infty} (a_n + b_n)$ cannot be convergent for otherwise $\sum_{n=1}^{\infty} b_n$ would be convergent by the above theorem which contradicts the hypothesis. Hence $\sum_{n=1}^{\infty} b_n$ is divergent

Note: - If both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent, then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ may or may not be convergent

Example:- 6) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n = \sum_{n=1}^{\infty} b_n$, then $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} (n + n)$
 $= \sum_{n=1}^{\infty} 2n = 2 \sum_{n=1}^{\infty} n$ is divergent since $\sum_{n=1}^{\infty} n$ is divergent.

But if we let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-n)$,

We have $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} (n + (-n)) = 0$ which is convergent.

Theorem:- Removal of a finite number of terms from a series doesn't affect its convergence or divergence.

Proof:- Consider $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \dots + a_{k-1} + a_k + \dots + a_n + \dots$

Let $\sum_{k=1}^{\infty} a_i = a_k + a_{k+1} + \dots + a_n + \dots$

And $\sum_{i=1}^{k-1} a_i = M$ (the partial sum of the 1st (k-1) terms.

Let $\{s_n\}$ be any sequence of partial sums of $\geq n$

Then the partial sum $\sum_{k=1}^{\infty} a_i = s_n - M$

Then $\lim_{n \rightarrow \infty} (s_n - M)$ exists iff $\lim_{n \rightarrow \infty} s_n$ exists

Hence $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=k}^{\infty} a_i$ converges or diverges together.

Note:- If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series differing only in their first m terms (i.e. $\geq m$), then either both diverges or converges.

Theorem:- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$ (the n^{th} term of a convergent series tends to zero)

Proof:- Let $s_n = \sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1} + a_n$

$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1}$

Then $a_n = s_n - s_{n-1}$

Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent.

Let $\lim_{n \rightarrow \infty} s_n = s$.

Since $n - 1 \rightarrow \infty$ as $n \rightarrow \infty$, we also have $\lim_{n \rightarrow \infty} s_{n-1} = s$.

Therefore, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0$

Therefore, $\lim_{n \rightarrow \infty} a_n = 0$ proved

Note:- The converse of the above theorem is not true in general.

i.e If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum a_n$ may or may not converge.

Example: -7) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge but $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Theorem :-(The Test for Divergence)

If $\lim_{n \rightarrow \infty} a_n \neq 0$ (does not exist) then the series $\sum_{n=1}^{\infty} a_n$ is diverges.

Proof:- Suppose $\sum_{n=1}^{\infty} a_n$ is diverges, then $\lim_{n \rightarrow \infty} a_n = 0$ by the above theorem

Hence by contra positive $\sum_{n=1}^{\infty} a_n$ is diverges

Example:- 8) Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution:- $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{5n^2+4} = \lim_{n \rightarrow \infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$

So the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges by the Divergence Test theorem.

Activities

Show that whether the following series are convergent or divergent.

$$\text{a) } \sum_{n=1}^{\infty} \frac{n^2-2}{n^2+5n} \qquad \text{b) } \sum_{n=1}^{\infty} n \sin \left(\frac{1}{n} \right)$$

Some types of infinite series

1. Harmonic series

Definition:- the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots = 1$ is called a harmonic series.

Note:- the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (see example 7 above)

2. The Geometric Series

Definition:- The series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^{n-1} \dots$ is called a geometric series. r is called the ratio of the geometric series .

Theorem:- the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ where $a \neq 0$ converges if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1 \text{ and diverges if } |r| \geq 1.$$

Proof:- Consider the series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^{n-1} + \dots$

$$\text{Let } s_n = a + ar + ar^2 + \dots + ar^{n-1} \quad \text{----- (1)}$$

$$\text{and } rs_n = ar + ar^2 + \dots + ar^{n-1} + ar^n \quad \text{----- (2)}$$

Subtracting equation (2) from (1), we get

$$\begin{aligned} s_n - rs_n &= a - ar^n \\ \Rightarrow s_n(1 - r) &= a - ar^n \\ \Rightarrow s_n &= \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \sum_{n=1}^{\infty} ar^{n-1} &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r} \\ &= \lim_{n \rightarrow \infty} \left(\frac{a}{1-r} - \frac{ar^n}{1-r} \right) \\ &= \lim_{n \rightarrow \infty} \frac{a}{1-r} - \lim_{n \rightarrow \infty} \frac{ar^n}{1-r} \\ &= \frac{a}{1-r} - \frac{a}{1-r} \lim_{n \rightarrow \infty} r^n \end{aligned}$$

Case (i) if $|r| < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$

$$\therefore \lim_{n \rightarrow \infty} s_n = \frac{a}{1-r} - \frac{a}{1-r} \cdot 0 = \frac{a}{1-r}$$

$$\text{Hence } \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $\frac{a}{1-r}$

Case (ii) if $r = 1$

$$\text{Then } s_n = a + a + a + \dots + a = na$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} na = \pm \infty$$

The geometric series diverges.

Case (iii) if $r = -1$

$$\text{Then } s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$

Therefore, $\lim_{n \rightarrow \infty} s_n$ does not exist since it oscillates between 0 and a .

Hence geometric series diverges.

Case (iii) If $|r| > 1$, then $r^n \rightarrow \infty$ as $n \rightarrow \infty$,

$$\text{Then } \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(r^n - 1)}{r - 1} = \frac{a}{1 - r} \lim_{n \rightarrow \infty} r^n - \frac{a}{1 - r} = \infty$$

The series is diverges.

Therefore, for $|r| \geq 1$, the geometric series is divergent.

Hence theorem proved.

Example:- 9) Determine whether the each following series converges or diverges. In case of convergence find the sum.

a) $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}}$

b) $\sum_{n=1}^{\infty} 3 \left(\frac{-2}{3}\right)^{n-1}$

c) $\sum_{n=1}^{\infty} \frac{4 \cdot 5^{n-1}}{4^{n-1}}$

Solution:-a) $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1}$ which is a geometric series with $r = \frac{1}{4}$ and $a = 1$.

Since $|r| = \left|\frac{1}{4}\right| < 1$, the series converges

$$\therefore \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

b) $\sum_{n=1}^{\infty} 3 \left(\frac{-2}{3}\right)^{n-1}$ is a geometric series with $r = \frac{-2}{3}$ and $a = 3$.

Since $|r| = \left|\frac{-2}{3}\right| < 1$, the series converges

$$\therefore \sum_{n=1}^{\infty} 3 \left(\frac{-2}{3}\right)^{n-1} = \frac{a}{1-r} = \frac{3}{1-\left(\frac{-2}{3}\right)} = \frac{9}{5}$$

c) $\sum_{n=1}^{\infty} \frac{4 \cdot 5^{n-1}}{4^{n-1}} = \sum_{n=1}^{\infty} 4 \left(\frac{5}{4}\right)^{n-1}$ Which is a geometric series with $r = \frac{5}{4}$ and $a = 4$.

Since $|r| = \left|\frac{5}{4}\right| > 1$, the series diverges.

The above theorem can be used to express as a non terminating repeating decimals as a rational number.

Example:-10) Express decimals as a rational number

a) $0.666 \dots$

b) $0.4545 \dots$

Solution:-a) $0.666 \dots = 0.6 + 0.06 + 0.006 + \dots$

$$= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \dots$$

$$= \frac{6}{10} \left(1 + \frac{1}{10} + \frac{1}{100} + \dots \right)$$

$$= \sum_{n=1}^{\infty} \frac{6}{10} \left(\frac{1}{10} \right)^{n-1} \text{ is a geometric series with } r = \frac{1}{10} \text{ and } a = \frac{6}{10}.$$

Since $|r| = \frac{1}{10} < 1$, the series converges

$$\therefore \sum_{n=1}^{\infty} \frac{6}{10} \left(\frac{1}{10} \right)^{n-1} = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{9} = \frac{2}{3}$$

b) $0.4545 \dots = 0.45 + 0.0045 + 0.000045 + \dots$

$$= \frac{45}{100} + \frac{45}{(100)^2} + \frac{45}{(100)^3} + \dots$$

$$= \frac{45}{100} \left(1 + \frac{1}{100} + \frac{1}{(100)^2} + \dots \right)$$

$$= \sum_{n=1}^{\infty} \frac{45}{100} \left(\frac{1}{100} \right)^{n-1} \text{ is a geometric series with } r = \frac{1}{100} \text{ and } a = \frac{45}{100}.$$

Since $|r| = \frac{1}{100} < 1$, the series converges

$$\therefore \sum_{n=1}^{\infty} \frac{45}{100} \left(\frac{1}{100} \right)^{n-1} = \frac{\frac{45}{100}}{1 - \frac{1}{100}} = \frac{45}{99} = \frac{5}{11}$$

2.3 Tests for Convergent of Real Series

(Integral, Comparison, Limit comparison, Ratio and Root tests)

In general it is difficult to find the exact sum of a series. This is true when we can't find any simpler expression for the sum of the first n terms of a series as a function n . but several tests are developed which enable us to say whether the series is Convergent or divergent without explicitly finding its sum.

1. The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $f(n) = a_n$ for $n \geq 1$,

then the series $\sum_{n=1}^{\infty} a_n$ converges iff the improper integral $\int_1^{\infty} f(x) dx$ converges.

Example:1) Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution:-The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$.

So we use the Integral Test:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+1} dx = \lim_{t \rightarrow \infty} [\tan^{-1} x]_1^t \\ &= \lim_{t \rightarrow \infty} \left[\tan^{-1} t - \frac{\pi}{4} \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

Thus $\int_1^{\infty} \frac{1}{x^2+1} dx$ is a convergent improper integral and so by the Integral Test,

the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is converges.

Example:-2) Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution:-The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus $f'(x) < 0$ when $\ln x > 1$, that is, $x > e$.

It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^t = \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent by the Integral Test.

Example:-2) Show that the series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

Solution:-If $p < 1$, then $\frac{1}{n^p}$ does not tend to 0 as $n \rightarrow \infty$

So by divergent test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Assume $p > 0$

Let $f(x) = \frac{1}{x^p}$ on $[1, \infty)$.

$f'(x) = \frac{-p1}{x^{p+1}} \leq 0 \Rightarrow f$ is decreasing on $[1, \infty)$.

So we can apply the Integral Test

$$\text{And } \int_1^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \begin{cases} \frac{x^{-p+1}}{-p+1} & \text{if } p \neq 1 \\ \ln x & \text{if } p = 1 \end{cases} \Big|_1^t$$

$$\text{Hence } \lim_{t \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t = \lim_{t \rightarrow \infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) = \begin{cases} \infty & \text{if } p < 1 \\ \frac{1}{-p+1} & \text{if } p > 1 \end{cases}$$

$$\text{And } \lim_{t \rightarrow \infty} (\ln t) \Big|_1^t = \lim_{t \rightarrow \infty} (\ln t) = \infty$$

$\therefore \sum \frac{1}{n^p}$ Converges if $p > 1$ and diverges if $p \leq 1$.

Note:-the series $\sum \frac{1}{n^p}$ is called the **p** – series

Activities

Determine whether the following series are convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+6}$

c) $\sum_{n=1}^{\infty} n e^{-n^2}$

e) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$

b) $\sum_{n=1}^{\infty} \frac{n}{n \ln n}$

d) $\sum_{n=1}^{\infty} n e^{-n}$

f) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$

2. The comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- i. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.
- ii. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

Example:-3) Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution:-For large n the dominant term in the denominator is $2n^2$ so we compare the given series with the series $\sum 5/2n^2$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

Because the left side has a bigger denominator, we know that by comparison test

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent by } p\text{-test with } p = 2 > 1$$

Therefore, $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ is convergent by comparison test.

Example:-4) Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence.

Solution:-Observe that $\ln n > 1$ for $n \geq 3$ and so $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$

We know that $\sum 1/n$ is divergent by p -test with $p = 1$.

Therefore, the given series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent by the comparison test.

Activities

Test the following series for convergence or divergence.

a) $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}+1}$

c) $\sum_{n=1}^{\infty} \frac{1}{\ln n}$

b) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

d) $\sum_{n=1}^{\infty} \frac{n}{5n^2-4}$

3. The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- i. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both diverge.
- ii. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ converges, then $\sum a_n$ also converges.
- iii. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Example:-5) Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.

Solution:-We use the limit comparison test with

$$a_n = \frac{1}{2^n-1} \quad b_n = \frac{1}{2^n}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n-1} = \lim_{n \rightarrow \infty} \frac{1}{1-\frac{1}{2^n}} = 1 > 0$$

Since this limit exist and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the limit comparison test.

Example:-6) Test the series $\sum \frac{2n^2+5}{4n^5+1}$ for convergence or divergence

Solution:- We use the limit comparison test with

$$a_n = \frac{2n^2+5}{4n^5+1} \Rightarrow a_n = \frac{(2+\frac{5}{n^2})}{n^3(4+\frac{1}{n^5})} \quad , \quad b_n = \frac{1}{n^3}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{1}{2} \text{ (finite and non zero)}$$

Then by *limit comparison test* either both series converge or both diverge.

But $\sum b_n = \sum \frac{1}{n^3}$ converges by *p – Test* since $p = 3 > 1$

Thus the given series $\sum \frac{2n^2+5}{4n^5+1}$ is convergent by limit comparison test.

Activities

Test the following series for convergence or divergence.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10n}}$

b) $\sum_{n=1}^{\infty} \frac{n+1}{n(2n-1)}$

d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

4. The Ratio Test

The Ratio Test determines if the terms of the given series are approaching 0 at a rate sufficient for convergence by considering the ratio between successive terms of the series.

Let $\sum a_n$ be a series of positive terms such that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l$ then

- i. The series converges if $l < 1$
- ii. The series diverges if $l > 1$ and
- iii. The test provides no conclusion if $l = 1$.

Example:-7) Test the convergence of $\sum \frac{x^n}{n}$

Solution:- Let $a_n = \frac{x^n}{n}$ and $a_{n+1} = \frac{x^{n+1}}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = \lim_{n \rightarrow \infty} \frac{x}{1+\frac{1}{n}} = x$$

Therefore, the series converges if $x < 1$ and diverges if $x > 1$

Put $x = 1 \Rightarrow a_n = \frac{1}{n}$

$$\sum a_n = \sum \frac{1}{n} \text{ diverges by } p - \text{test since } p = 1$$

Therefore, the given series converges if $x < 1$ and diverges if $if x \geq 1$

Example:-8) Test the series for convergence $\sum_{n=1}^{\infty} \frac{n! 2^n}{n^n}$

Solution:- Let $a_n = \frac{n! 2^n}{n^n}$ and $a_{n+1} = \frac{(n+1)! 2^{n+1}}{(n+1)^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)! 2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n! 2^n} = \frac{2}{(1+\frac{1}{n})^n}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{2}{(1+\frac{1}{n})^n} = \frac{2}{e} < 1$$

Therefore, the series is convergent.

Example:-9) Show that $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges

Solution:- Here $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left[\frac{2^{n+1}}{(n+1)^2} \div \frac{2^n}{n^2} \right] = \lim_{n \rightarrow \infty} \left[\frac{2^{n+1}}{2^n} \cdot \frac{n^2}{(n+1)^2} \right]$

$$= \lim_{n \rightarrow \infty} \left\{ 2 \cdot \left[\frac{n}{n+1} \right]^2 \right\}$$

$$= 2 \cdot \lim_{n \rightarrow \infty} \left[\frac{1}{1+\frac{1}{n}} \right]^2 = 2 > 1$$

Therefore, the given series $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges.

Example:-10) Show that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges

Solution:- Here $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left[\frac{(n+1)!}{(n+1)^{n+1}} \div \frac{n!}{n^n} \right] = \lim_{n \rightarrow \infty} \left[\frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \right]$

$$= \lim_{n \rightarrow \infty} \left\{ (n+1) \cdot \frac{1}{(n+1)} \cdot \left[\left(\frac{n}{n+1} \right)^n \right] \right\}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{1+\frac{1}{n}} \right)^n = \lim_{n \rightarrow \infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$$

Therefore, the given series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges

Activities

Investigate the following series for convergence or divergence.

a) $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$

c) $\sum_{n=1}^{\infty} \frac{5^n}{n^5}$

b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

d) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{(n-1)!}$

5. The Root Test

As Ratio Test, The Root Test can be applied without considering other series whose convergence is known. But in this test we take the limits of roots of terms instead of taking the limits of quotient of successive terms of the series.

Let $\sum a_n$ be a series of positive terms such that $\lim_{n \rightarrow \infty} a_n^{1/n} = l$ (possibly ∞) then

- i. The series converges if $l < 1$
- ii. The series diverges if $l > 1$ and
- iii. The test provides no conclusion if $l = 1$.

Example:- 11) Test the convergence of the series $\sum \frac{1}{n^n}$

Solution:- Here $a_n = \frac{1}{n^n} \Rightarrow a_n^{1/n} = \frac{1}{n}$

Thus $\lim_{n \rightarrow \infty} a_n^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1$

Therefore, the series converges.

Example:-12) Show that $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges

Solution:- Here $a_n = \frac{n}{2^n} \Rightarrow a_n^{1/n} = \frac{\sqrt[n]{n}}{2}$

Thus $\lim_{n \rightarrow \infty} a_n^{1/n} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{2} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2} < 1$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by Root Test.

Example:-13) Show that $\sum_{n=1}^{\infty} \left(\frac{3n}{n+2}\right)^n$ diverges

Solution:- Here $a_n = \left(\frac{3n}{n+2}\right)^n \Rightarrow \sqrt[n]{a_n} = \sqrt[n]{\left(\frac{3n}{n+2}\right)^n} = \frac{3n}{n+2}$

Thus $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{3n}{n+2} = 3 > 1$

Hence, the series $\sum_{n=1}^{\infty} \left(\frac{3n}{n+2}\right)^n$ diverges by Root Test.

Example:-14) Show that the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution:- $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^2}} = 1$ and $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n}} = 1$

Hence we can't conclude anything about the convergence or divergence of both series by a root test. But by other tests we can check that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series while $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series.

Activities

Investigate the following series for convergence or divergence.

$$a) \sum_{n=1}^{\infty} \left(\frac{2n}{n+2} \right)^{2n-1}$$

$$c) \sum_{n=1}^{\infty} \frac{3^{n-2}}{n^n}$$

$$b) \sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$$

$$d) \sum_{n=1}^{\infty} \frac{2^{n-1}}{(n-1)^n}$$

2.4 Alternating Series and Alternating Series Test

Alternating Series

An Alternating Series is a series whose terms are alternately positive and negative.

Here are two examples:

$$i. \quad \sum_{n=1}^{\infty} (-1)^{n+1} 3^n = 3 - 9 + 27 - 81 + \dots$$

$$ii. \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} = -\frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \dots \text{ are alternating series.}$$

The Alternating Series Test

If the Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots \quad a_n > 0 \text{ Satisfies}$$

$$i. \quad a_{n+1} \leq a_n \quad \text{for all } n \text{ (the sequence is decreasing)}$$

$$ii. \quad \lim_{n \rightarrow \infty} a_n = 0$$

Then the series is convergent.

Example:- Test the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ for convergence or divergence.

Solution:- The given series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is an alternating series. It satisfies

$$i. \quad a_{n+1} < a_n \text{ because } \frac{1}{n+1} < \frac{1}{n}$$

$$ii. \quad \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

So the series is convergent by the Alternating series test.

Example:- Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$

Solution:- $\sum a_n = \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \dots$

The given series is an alternating series since the terms are alternatively positive and negative.

Also it satisfies

i. $a_{n+1} < a_n$ because $\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$

ii. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$

So the series is convergent by the Alternating series test.

Activities

Test the following series for convergence or divergence.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n,n}}{n^2+1}$

2.5 Absolute and Conditional Convergence

Absolute Convergence

Definition:- Let $\sum a_n$ be a series of positive and negative terms. Then series $\sum a_n$ is said to be

absolutely convergent iff $\sum |a_n|$ is convergent

Example:- Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Solution:- The given series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ is absolutely convergent

because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$ is a convergent series by p -test where $p = 2 > 1$

Example:- Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^{n-1}}$ is absolutely convergent.

Solution:- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^{n-1}} = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{5^{n-1}} \right| = \sum_{n=1}^{\infty} \frac{1}{5^{n-1}}$

$= \sum_{n=1}^{\infty} \left(\frac{1}{5} \right)^{n-1}$ is a geometric series with $a = 1, |r| = \frac{1}{5} < 1$.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^{n-1}}$ is absolutely convergent.

Theorem:- Every absolutely convergent series is convergent.

Remark: - If $\sum_{n=1}^{\infty} |a_n|$ diverges, we can't say any thing about $\sum_{n=1}^{\infty} a_n$.

Theorem: - If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, then $\sum_{n=1}^{\infty} a_n \pm b_n$ and $\sum_{n=1}^{\infty} k a_n$ where k is a constant converge absolutely.

Conditionally Convergent

Definition: - Let $\sum a_n$ be a series of positive and negative terms, then $\sum a_n$ is said to be conditionally Convergent if $\sum a_n$ is Convergent and $\sum |a_n|$ is divergent.

Example:- Show that the following series conditionally convergent:

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$

Solution:- The given series is an alternating series.

$$\begin{aligned} \sum |a_n| &= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots \\ &= \sum |a_n| = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}} \end{aligned}$$

It is divergent by $p - test$ since $p = \frac{1}{2} < 1$

So $\sum a_n$ is conditionally convergent.

Example:- Show that the following series is conditionally convergent

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

Solution:- The given series is an alternating series.

$$\begin{aligned} \sum |a_n| &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \\ &= \sum |a_n| = \sum \frac{1}{n} \end{aligned}$$

It is divergent by $p - test$ since $p = 1$

So $\sum a_n$ is conditionally convergent.

2.6 Generalized Convergence Tests

Let $\sum_{n=1}^{\infty} a_n$ be a series and $\sum_{n=1}^{\infty} b_n$ a positive series then

a. Generalized comparison Test

If $|a_n| \leq b_n$ for $n \geq 1$ and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).

b. Generalized limit comparison test

If $\lim_{n \rightarrow \infty} \left| \frac{a_n}{b_n} \right| = L$, where L is a positive number, and if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).

c. Generalized Ratio Test

Suppose that $a_n \neq 0$ for $n \geq 1$ and that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r$

If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely)

If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

If $r = 1$, No conclusion about the convergence of the series.

d. Generalized Root Test

Suppose that $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r$

If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely)

If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $r = 1$, No conclusion about the convergence of the series.

Example: - Show that $\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$ Converges absolutely for $|x| < 1$,

Converges conditionally for $x = -1$ and Diverges for $x = 1$ and for $|x| > 1$

Solution: - If $x = 0 \Rightarrow \sum_{n=1}^{\infty} 0 = 0$ converges

$$\begin{aligned}
 \text{If } x \neq 0 \text{ then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \div \frac{x^n}{n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \right| \\
 &= \lim_{n \rightarrow \infty} \left| x \cdot \frac{n}{n+1} \right| \\
 &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) |x| \\
 &= 1 \cdot |x| \\
 &= |x|
 \end{aligned}$$

By Generalized Ratio Test, the given series converges for $|x| < 1$ and diverges for $|x| > 1$

For $x = -1$ the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is alternating series

Then $\sum |a_n| = \sum \frac{1}{n}$ diverges

Therefore $\sum a_n$ conditionally convergent for $x = -1$

For $x = 1$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ which is diverges by p -test.

Activities

Test the following series for convergence or divergence.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{6^n}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

Corollary:-let $\{a_n\}$ be a sequence. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = r < 1$,

then $\lim_{n \rightarrow \infty} a_n = 0$

Example: - Show that $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \forall x$

Solution: - $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)} x \right|$$

$$= \lim_{n \rightarrow \infty} \frac{1}{(n+1)} |x|$$

$$= 0 < 1, \forall x$$

Hence by the above Corollary $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \forall x$

CHAPTER THREE

Introduction

In Chapter two, we studied series with constant terms. In this chapter we will consider series whose terms involve variables. Such series are of fundamental importance in many branches of Mathematics and the Physical Sciences.

3.1 Definition of power series at any x_0 and $x_0 = 0$

Power Series at any x_0

A Series of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots \quad \text{----- (1)}$$

is called a **Power Series in $(x - x_0)$** or a **power series centered at x_0** or a **Power Series about x_0**

Power Series at $x_0 = 0$

A Series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots \quad \text{----- (2)}$$

is called a **power series**, where x is a variable and the c_n 's are constants usually called the coefficients of the series.

Note: - i) A power series is a polynomial of infinite degree.

ii) All tests can be used for power series.

3.2 Convergence and divergence, Radius and interval of convergence

- i) For each fixed x , the series $\sum_{n=0}^{\infty} c_n x^n$ is a series of constants that we can test for *convergence or divergence*.
- ii) A power series may converge for some values of x and diverge for other values of x . The sum of the series is a function $f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$ whose domain is the set of all x for which the series converges.
- iii) In writing out the term corresponding to $n = 0$ in equation (1) and (2) we have the convention that $(x - a)^0 = 1$ even when $x = a$.
- iv) In Power Series centered at a , when $x = a$ all of the terms are 0 for $n \geq 1$ and so the power series in equation (1) always converges when $x = a$.

Activity

For a Power Series $f(x) = \sum_{n=0}^{\infty} c_n x^n$, take $c_n = 1$, we get a geometric series $\sum_{n=0}^{\infty} x^n$. Determine the domain of $f(x)$ and find its sum.

Example:-1) For what values of x is the Series $\sum_{n=0}^{\infty} nx^n$ converges?

Solution:- To find the values of x , let's use the Ratio test. Now let a_n denote the n^{th} term of

the series, then $a_n = nx^n$ and $a_{n+1} = (n+1)x^{n+1}$.

If $x \neq 0$, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^n \cdot x}{nx^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right| |x| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right| \\ &= |x|(1) = |x|\end{aligned}$$

Hence, by the ratio test, the series converges for $|x| < 1$.

Therefore, the given series converges for $x \in (-1, 1)$ or $-1 < x < 1$.

Example:-2) For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges.

Solution:- Let $a_n = \frac{(x-3)^n}{n}$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(x-3)}{n+1} \cdot n \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| |x-3| \\ &= |x-3| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = |x-3|\end{aligned}$$

Hence, by the Ratio test, the given series converges when $|x-3| < 1$ and diverge when $|x-3| > 1$.

Now, $|x-3| < 1 \Leftrightarrow -1 < x-3 < 1 \Leftrightarrow 2 < x < 4$

So the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

But to check the convergent at the end points

i) Let $x = 2$ in the series it becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test.

ii) Let $x = 4$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which is divergent.

Therefore, the given power series converges for $2 \leq x < 4$ or $[2, 4)$.

Theorem: - For a given Power Series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

- i. The Series converges only when $x = a$
- ii. The Series converges for all x
- iii. There is a positive number R such that the Series *converges* if $|x - a| < R$ and *diverges* if $|x - a| > R$

Radius of convergence

The number R in case (iii) above is called the **Radius of convergence** of the power Series.

By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii)

Interval of convergence

The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Note: - i. If the radius of convergence R is **zero**, then the interval of convergence of a power series consists of just a single point $x = a$

ii. If the radius of convergence R is **infinity**, then the interval is $(-\infty, \infty)$

iii. If the radius of convergence is **R**, there are four possibilities for the interval of convergence. That is, $(a - R, a + R)$, $(a - R, a + R]$, $[a - R, a + R)$ or $[a - R, a + R]$

Example:-3) Find the radius of convergence and interval of convergence of the following Series.

$$a) \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \qquad b) \sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$$

Solution:-a) Let $a_n = \frac{x^n}{\sqrt{n}}$, then $a_{n+1} = \frac{x^{n+1}}{\sqrt{n+1}}$, applying the ratio test, we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} \cdot x \right| = \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n}{n+1}} \right| |x|$$

$$= |x| \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n}{n+1}} \right| = |x|(1) = |x|$$

Now, by ratio test the series converges for $|x| < 1$.

To find the interval of convergence, check the convergence at $x = -1$ and $x = 1$

✚ At $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which is an alternating series with $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$

and $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$, hence the series converges at $x = -1$.

✚ At $x = 1$, the Series becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a P -series with $P = \frac{1}{2} < 1$, hence the series diverges.

Therefore the radius of convergence $R = 1$ and the interval of convergence is $[-1, 1)$.

a) Given the Series $\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$.

$$\text{Let } a_n = \frac{3^n x^n}{(n+1)^2} \text{ and } a_{n+1} = \frac{3^{n+1} x^{n+1}}{(n+2)^2},$$

Applying the ratio test, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} x^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{3^n x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3^n \cdot 3 \cdot x^n \cdot x}{(n+2)^2} \cdot \frac{(n+1)^2}{3^n x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| 3x \frac{(n+1)^2}{(n+2)^2} \right| \\ &= |3x| \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^2 = |3x|(1) = 3|x| \end{aligned}$$

Now, by ratio test the series converges when $3|x| < 1 \Rightarrow |x| < \frac{1}{3}$

✚ At $x = -1/3$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ and we have $\frac{1}{(n+2)^2} \leq \frac{1}{(n+1)^2}$ with

$\lim_{n \rightarrow \infty} \frac{1}{(n+1)^2} = 0$ hence by alternating series, the series $\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$ converges at $x = \frac{-1}{3}$

✚ At $x = 1/3$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ which is a p -series with $p = 2 > 1$, hence it is continuous at $x = 1/3$.

Therefore, the radius of convergence $R = \frac{1}{3}$ and the interval of convergence of the series is $\left[\frac{-1}{3}, \frac{1}{3} \right]$

Activity

Find radius and interval of convergence of the following series

$$a) \sum_{n=0}^{\infty} x^n$$

$$b) \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$c) \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2}{3}\right)^n \frac{x^n}{n}$$

3.3 Algebraic operations on convergent power series

We may also add, subtract, multiply and divide the series just like polynomials. The following theorem states the rules for algebraic operations on convergent power series.

Theorem:- The Algebra of Power Series

Assume that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ and

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots \quad \text{for } |x| < R. \text{ Then}$$

for $|x| < R$, we have

$$(i) \quad f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)x^n$$

$$(ii) \quad f(x) - g(x) = \sum_{n=0}^{\infty} (a_n - b_n)x^n$$

$$(iii) \quad f(x) \cdot g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots$$

$$(iv) \quad f(x)/g(x) \text{ is obtained by long division, if } g(x) \neq 0 \text{ for } |x| < R$$

3.4 power series Representation of functions

In this section we will see how to represent certain types of functions as a sum of a power series by manipulating geometric series or by differentiating or integrating such a series. Representation of functions as a power series is useful for integrating functions that do not have elementary anti-derivatives, for solving differential equations and for approximating functions by polynomials.

Consider the equation $1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$ with $|x| < 1$ ----- (i)

Observe that the series is a geometric series with first term $a = 1$ and common ratio $r = x$.

Since $|r| = |x| < 1$, it converges and equation (i) gives

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n \quad \text{with } |x| < 1 \quad \text{----- (ii)}$$

Hence, equation (ii) shows the function $f(x) = \frac{1}{1-x}$ as a sum of power series.

Example:-1) Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution:- We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ----- (i)

Now replacing x by $-x^2$, we get

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \\ &= 1 - x^2 + x^4 - x^6 + \dots\end{aligned}$$

Because this is a geometric series it converges when $|-x^2| < 1$, that is $x^2 < 1$ or $|x| < 1$.

Therefore, the interval of convergence is $(-1,1)$.

Example:-2) Express $\frac{1}{x-5}$ as the sum of a power series and find the interval of convergence.

Solution:- Given $f(x) = \frac{1}{x-5}$

We know that $\frac{1}{x-5} = \frac{1}{-(5-x)} = \frac{1}{-5(1-\frac{x}{5})} = \frac{-1}{5} \left(\frac{1}{1-\frac{x}{5}} \right)$

$$\begin{aligned}f(x) &= \frac{1}{x-5} = \frac{-1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5} \right)^n = - \sum_{n=0}^{\infty} \frac{1}{5} \cdot \frac{x^n}{5^n} \\ &= - \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\end{aligned}$$

This series converges for $\left| \frac{x}{5} \right| < 1$

Therefore, the power series representation of $f(x) = \frac{1}{x-5}$ is $- \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$ and the interval of convergence is $(-5,5)$.

Example:- Express $\frac{x}{4x+1}$ as the sum of a power series and find the interval of convergence.

Solution:- Given $f(x) = \frac{x}{4x+1}$

We know that $\frac{x}{4x+1} = x \cdot \frac{1}{1-(-4x)} = x \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-4)^n x^{n+1}$,

The series converges for $|-4x| < 1 \Rightarrow 4|x| < 1 \Rightarrow |x| < \frac{1}{4}$

Therefore, the power series representation of $f(x) = \frac{x}{4x+1}$ is $\sum_{n=0}^{\infty} (-4)^n x^{n+1}$, and the interval of convergence is $(-\frac{1}{4}, \frac{1}{4})$

Activity

Find a power series representation for the following functions and determine its interval of convergence

$$a) f(x) = \frac{1}{1+x^3}$$

$$b) g(x) = \frac{x}{5-x}$$

$$c) f(x) = \frac{1}{1+x^3}$$

3.5 Differentiation and integration of power series

A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ with a nonzero radius of convergence is always differentiable and the derivative is obtained from $\sum_{n=0}^{\infty} c_n(x-a)^n$ by differentiating term by term, the way we differentiate polynomials.

Theorem:- If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the

function f defined by $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$

Is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$(a) \quad f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1}$$

$$(b) \quad \int f(x) dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (a) and (b) are both R .

Note:- Equations (a) and (b) can be rewritten in the form

$$\frac{d}{dx} \left[\sum_{n=0}^{\infty} c_n(x-a)^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} [c_n(x-a)^n]$$

$$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n \right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n dx$$

Example:-1) Express $\frac{1}{(1-x)^2}$ as a power series by differentiating the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n \text{ and find radius of convergence.}$$

Solution:- Differentiating both sides of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$$

$$\text{We get } \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

Therefore, $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$

The radius of convergence $R = 1$ since *the radius of convergence of the differentiated series is the same as the radius of convergence of the original series.*

Example:-2) Find the power series representation for $f(x) = \ln(5-x)$ and determine the radius of convergence.

Solution:- Let $f(x) = \ln(5-x)$, then $-\ln(5-x) = \int \frac{1}{5-x} dx$ ----- (i)

$$\begin{aligned} \text{But } \frac{1}{5-x} &= \frac{1}{5(1-\frac{x}{5})} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} \\ &= \frac{1}{5} + \frac{x}{5^2} + \frac{x^2}{5^3} + \frac{x^3}{5^4} + \dots \end{aligned}$$

$$\begin{aligned} \text{Now } \int \frac{1}{5-x} dx &= \int \left[\frac{1}{5} + \frac{x}{5^2} + \frac{x^2}{5^3} + \frac{x^3}{5^4} + \dots \right] dx \\ &= \frac{1}{5} \int \left[1 + \frac{x}{5} + \frac{x^2}{5^2} + \frac{x^3}{5^3} + \dots \right] dx \\ &= \frac{1}{5} \left[x + \frac{x^2}{2 \cdot 5} + \frac{x^3}{3 \cdot 5^2} + \dots \right] + C \\ &= C + \frac{1}{5}x + \frac{x^2}{2 \cdot 5^2} + \frac{x^3}{3 \cdot 5^3} + \dots \end{aligned}$$

By substituting in (i), we get

$$-\ln(5-x) = C + \frac{1}{5}x + \frac{x^2}{2 \cdot 5^2} + \frac{x^3}{3 \cdot 5^3} + \dots = C + \sum_{n=1}^{\infty} \frac{x^n}{n 5^n}$$

To find the value of C we put $x = 0$, we get $-\ln(5-0) = C$

Therefore, $C = -\ln 5$

$$\text{Hence, } -\ln(5-x) = -\ln 5 + \sum_{n=1}^{\infty} \frac{x^n}{n 5^n}$$

$$\text{Then } \ln(5-x) = \ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n 5^n} \text{ and It converges for } \left| \frac{x}{5} \right| < 1 \Rightarrow |x| < 5$$

$\therefore$ The radius of convergence $R = 5$.

Example:-3) Find the power series representation for $f(x) = \tan^{-1}x$ and determine the radius of convergence.

Solution:- We know that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$$= 1 - x^2 + x^4 - x^6 + \dots$$

Observe that $\frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2}$, hence

$$\begin{aligned} \tan^{-1}x &= \int \frac{1}{1+x^2} dx = \int [1 - x^2 + x^4 - x^6 + \dots] dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \end{aligned}$$

To find C put $x = 0$ and we obtain $C = \tan^{-1}0 = 0$.

$$\text{Therefore, } \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence of the series for $\frac{1}{1+x^2}$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1.

Example:-4) Evaluate the indefinite integral $\int \frac{1}{1+x^4} dx$ as a power series

Solution:- We know that $\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}$

$$\text{Now } \int \frac{1}{1+x^4} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{4n} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1}$$

Activity

1. Find $f'(x)$, if $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and show that $f(x) = e^x$
2. Find the power series representation of $\int_0^x e^{-t^2} dt$

3.6 Taylor and Maclaurin series and application

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots \quad |x-a| < R \text{----- (i)}$$

and let us try to determine what the coefficients c_n must be in terms of f . To begin, notice that if we put $x = a$ in equation (i), then all terms after the first one are 0 and we get $f(a) = c_0$.

Differentiating the series in equation (i), term by term, we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \quad |x-a| < R \text{----- (ii)}$$

Substitute $x = a$ in equation (ii), we get $f'(a) = c_1$

Similarly, differentiating both sides of equation (ii), we obtain

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \dots \quad |x-a| < R \quad \text{----- (iii)}$$

Again, put $x = a$ in equation (iii), we get $f''(a) = 2c_2$

Let us apply the procedure one more time. Differentiate of the series in equation (iii) gives

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \dots \quad |x-a| < R \quad \text{----- (iv)}$$

and substitute $x = a$ in equation (iv), we get $f'''(a) = 2 \cdot 3c_3 = 3! c_3$

In general, $f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdots nc_n = n! c_n$

Solving this equation for the n^{th} coefficient c_n , we get $c_n = \frac{f^{(n)}(a)}{n!}$

Theorem:- If f has a power series representation (expansion) at a , that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad |x-a| < R$$

Then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$

Taylor series of the function f at a

If f has a power series expansion at a then it must be the following form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots \end{aligned}$$

Maclaurin Series

If $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

Note:-If f can be represented as a power series about a (such functions are called **analytic at a**), then f is equal to the sum of its Taylor series.

Example: -1) Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution: - If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$ for all n .

Therefore, the Taylor series for f at 0 (that is, Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\text{Then } \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0 < 1$$

So by the ratio test, the series converges for all x and the radius of convergence is $R = \infty$

Taylor Polynomial of f at a

If $f(x) = T_n(x) + R_n(x)$, where $T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$ is the n^{th} degree Taylor polynomial of f at a and $\lim_{n \rightarrow \infty} R_n(x) = 0$ (where $R_n(x)$ is remainder of

the series) for $|x-a| < R$, then f is equal to the sum its Taylor series on the interval $|x-a| < R$; that is, f is analytic at a .

Note: - In trying to show that $\lim_{n \rightarrow \infty} R_n(x) = 0$ for a specific function f , we usually use the expression in the next theorem.

Taylor's Formula

If f has $n+1$ derivatives in an interval I that contains the number a , then for x in I there is a number z strictly between x and a such that the remainder term in the Taylor series can be expressed as

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-a)^{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x$$

Example:-2) Prove that e^x is equal to the sum of its Taylor series.

Solution:- If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$, so the remainder term in Taylor's Formula is

$$R_n(x) = \frac{e^z}{(n+1)!} x^{n+1}$$

Where z lies between 0 and x . (Note, however, that z depends on n)

If $x > 0$, then $0 < z < x$, so $e^z < e^x$.

$$\text{Therefore } 0 < R_n(x) = \frac{e^z}{(n+1)!} x^{n+1} < e^x \frac{x^{n+1}}{(n+1)!} \rightarrow 0$$

So $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ by the squeeze theorem

If $x < 0$, then $x < z < 0$, so $e^z < e^0 = 1$ and

$$|R_n(x)| < \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0$$

Again $R_n(x) \rightarrow 0$. Thus, by Taylor polynomial, e^x is equal to the sum of its Taylor series, that is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

Note: - In particular if we put $x = 1$, we obtain the following expression for the number e as a sum of an infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Applications of Taylor Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

We know that $T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$

$$= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Since f is the sum of its Taylor series, we know that $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ and so $T_n(x)$ can be used as an approximation to f : $f(x) \approx T_n(x)$.

Example:-3) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2.

Solution:- $f(x) = \sqrt[3]{x} = x^{1/3}$ $f(8) = 2$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad f'(8) = \frac{1}{12}$$

$$f''(x) = \frac{-2}{9}x^{-5/3} \quad f''(8) = \frac{1}{12}$$

$$f'''(x) = \frac{10}{27}x^{-8/3}$$

Thus the second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 \end{aligned}$$

Therefore, the approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

CHAPTER FOUR

Fourier series and integrals

4.1 Periodic Functions, trigonometric series

Definition:-A function f is said to be periodic if $f(x + T) = f(x)$ for all real number x in the domain of f and some positive number T . The number T is called the period of f . The least positive number T for which $f(x + T) = f(x)$ holds is called the fundamental period of f .

Example:-1) The function $f(x) = \cos(\pi x)$ and $g(x) = \sin(\pi x)$ are both periodic functions with period 2, because

$$\begin{aligned} f(x + 2) &= \cos(\pi(x + 2)) \\ &= \cos(\pi x + 2\pi) \\ &= \cos(\pi x) \cos(2\pi) - \sin(\pi x) \sin(2\pi) \\ &= \cos(\pi x) \cdot 1 - \sin(\pi x) \cdot 0 \\ &= \cos(\pi x) = f(x) \end{aligned}$$

Similarly,

$$\begin{aligned} g(x + 2) &= \sin(\pi(x + 2)) \\ &= \sin(\pi x + 2\pi) \\ &= \sin(\pi x) = g(x) \end{aligned}$$

Example:-2) The fundamental period of $\sin x$ and $\sin 2x$ are π and 2π respectively.

The function $f(x) = \sin(2\pi x)$ also has a period 2 but this is not a fundamental period. The fundamental period is 1 since it the smallest positive number for which $f(x + 1) = \sin(2\pi x + 2\pi) = \sin(2\pi x) = f(x)$

Properties

1. If T is a period of f , then nT of T is also a period of f for any positive integer n , that is, $f(x + nT) = f(x)$ for all real number x in the domain of f .
2. If the period of $f(x)$ is T , then the period of $f(ax + b)$ is $\frac{T}{a}$.
3. If $f(x)$ and $g(x)$ are periodic functions having the same period T , Then the period $h(x) = c_1f(x) + c_2g(x)$ is also T .
4. If $f(x) = c$, where c is a constant, then f is periodic and every $T > 0$ is a period. Thus there exists no smallest period, so does not have the fundamental period.
5. The sum of the number of trigonometric functions in $\sin nx$ and $\cos nx$ is also a periodic function with period the least common multiple of the periods of each term of the sum.

Example:-3) $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x$ is a periodic function.

The periods of the terms are $2\pi, \pi, \frac{2\pi}{3}$ and $\frac{\pi}{2}$ respectively. So the period of the

given function is the least common multiple of these periods which is equal to 2π

Example:-4) $f(x) = \cos(\pi x) + \frac{1}{2}\cos(2\pi x)$ is a periodic function with period 2.

Trigonometric series

Definition: - the series the form $\frac{a_0}{2} + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + \dots$

$= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ is called a trigonometric series.

A trigonometric series is a periodic function with period 2π . the convergence and divergence of the series depends on the value of x chosen and the coefficients a_k and b_k .

Trigonometric values

Let k be an integer then,

$$1. \sin k\pi = 0 = \sin(k \pm 1)\pi$$

$$4. \cos\left(k + \frac{1}{2}\right)\pi = 0$$

$$2. \cos k\pi = (-1)^k$$

$$5. \cos 2(k \pm 1)\pi = 1$$

$$3. \sin\left(k + \frac{1}{2}\right)\pi = (-1)^k$$

$$6. \cos(k \pm 1)\pi = (-1)^{k \pm 1}$$

Activity

Find the fundamental period of each of the following.

- a) $f(x) = \cos(\pi x) + \frac{1}{2}\sin(2\pi x)$
- b) $\cos(6x) - 4\sin(3x)$
- c) $a_0 + a_1\cos x + a_2\cos x$

4.2 Fourier series

Definition:- Let $f(x)$ be a periodic function with period 2π and can be expressed as a trigonometric series as $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \dots + b_1\sin x + b_2\sin 2x + \dots$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Then the series $\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ is known as the Fourier series for $f(x)$, and a_0, a_k, b_k are called Fourier coefficients of $f(x)$.

Remark:- the reason for writing the constant term $\frac{a_0}{2}$ in the Fourier series for symmetric

representation of the coefficients instead of a_0 . also $\frac{a_0}{2}$ the average value of

$f(x)$ over a cycle.

To determine the coefficients a_0, a_k and b_k assume the series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \dots\dots\dots (i)$$

can be integrable term by term in the interval $[\alpha, \alpha + 2\pi]$

To determine a_0 , integrate (i) from α to $\alpha + 2\pi$

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) dx &= \int_{\alpha}^{\alpha+2\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right) dx \\ &= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right) dx \\ &= \frac{a_0}{2} \cdot 2\pi + \sum_{k=1}^{\infty} \int_{\alpha}^{\alpha+2\pi} a_k \cos kx dx + \sum_{k=1}^{\infty} \int_{\alpha}^{\alpha+2\pi} b_k \sin kx dx \end{aligned}$$

$$= a_0\pi + 0 + 0 = a_0\pi$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx \quad \dots\dots\dots (ii)$$

To determine a_k , multiply both sides of (i) by $\cos kx$ and then integrate from α to $\alpha + 2\pi$

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \cos kx dx &= \int_{\alpha}^{\alpha+2\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right) dx \cos kx \\ &= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} \cos kx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right) \cos kx dx \\ &= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} \cos kx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{k=1}^{\infty} a_k \cos kx \right) \cos kx dx \\ &\quad + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{k=1}^{\infty} b_k \sin kx \right) \cos kx dx \\ &= 0 + a_k \int_{\alpha}^{\alpha+2\pi} \cos^2 kx dx + 0 \quad \text{(Remaining terms vanish)} \\ &= a_k \int_{\alpha}^{\alpha+2\pi} \frac{1 + \cos 2kx}{2} dx = a_k \pi \text{ for } k \neq 0 \end{aligned}$$

$$\Rightarrow a_k = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos kx dx \quad \dots\dots\dots (iii)$$

To determine b_k , multiply both sides of (i) by $\sin kx$ and then integrate from α to $\alpha + 2\pi$

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \sin kx dx &= \int_{\alpha}^{\alpha+2\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right) dx \sin kx \\ &= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} \sin kx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right) \sin kx dx \\ &= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} \sin kx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{k=1}^{\infty} a_k \cos kx \right) \sin kx dx \\ &\quad + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{k=1}^{\infty} b_k \sin kx \right) \sin kx dx \\ &= 0 + 0 + b_k \int_{\alpha}^{\alpha+2\pi} \sin^2 kx dx \quad \text{(Remaining terms vanish)} \\ &= b_k \int_{\alpha}^{\alpha+2\pi} \frac{1 - \cos 2kx}{2} dx = b_k \pi \text{ for } k \neq 0 \end{aligned}$$

$$\Rightarrow b_k = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin kx dx \quad \dots\dots\dots (iv)$$

Note:-1. If $\alpha = 0$, we obtain the Fourier series in $[0, 2\pi]$ as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

Where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx$$

2. If $\alpha = -\pi$, we obtain the Fourier series in $[-\pi, \pi]$ as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

Where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx$$

Example:-1) Determine the Fourier series for e^x in $[0, 2\pi]$

Solution:- let $f(x) = e^x$

Then the Fourier series for $f(x)$ in interval $[0, 2\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad \dots \dots \dots (*)$$

Where $a_0 = \frac{1}{\pi} \int_0^{2\pi} e^x dx = \frac{1}{\pi} [e^x]_0^{2\pi}$

$$a_0 = \frac{e^{2\pi} - 1}{\pi}$$

$$\therefore \frac{a_0}{2} = \frac{e^{2\pi} - 1}{2\pi}$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} e^x \cos kx dx$$

In order to evaluate this integral we use the following formula

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} |a \cos bx + b \sin bx|$$

$$\therefore a_k = \frac{1}{\pi} \left[\frac{e^x}{1+k^2} |\cos kx + k \sin kx| \right]_0^{2\pi}$$

$$a_k = \frac{1}{\pi(1+k^2)} [e^{2\pi} |\cos 2k\pi + k \sin 2k\pi| - e^0 |\cos 0 + k \sin 0|]$$

$$a_k = \frac{e^{2\pi} - 1}{\pi(1+k^2)} \quad \text{Since } \sin 2k\pi = \sin 0 = 0 \text{ and } \cos 2k\pi = \cos 0 = 1$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx$$

Here we apply the formula

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} |a \sin bx - b \cos bx|$$

$$\therefore b_k = \frac{1}{\pi} \left[\frac{e^x}{1+k^2} |\sin kx - k \cos kx| \right]_0^{2\pi}$$

$$b_k = \frac{1}{\pi(1+k^2)} [e^{2\pi} |\sin 2k\pi - k \cos 2k\pi| - e^0 |\sin 0 - k \cos 0|]$$

$$b_k = \frac{e^{2\pi} - 1}{\pi(1+k^2)} (-k) \quad \text{Since } \sin 2k\pi = \sin 0 = 0 \text{ and } \cos 2k\pi = \cos 0 = 1$$

Substituting the values of a_0, a_k, b_k in (*) we get

$$e^x = \frac{e^{2\pi} - 1}{2\pi} + \sum_{k=1}^{\infty} \frac{e^{2\pi} - 1}{\pi(1+k^2)} \cos kx + \sum_{k=1}^{\infty} \frac{e^{2\pi} - 1}{\pi(1+k^2)} (-k) \sin kx$$

$$\therefore e^x = \frac{e^{2\pi} - 1}{2\pi} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \frac{\cos kx}{1+k^2} - \sum_{k=1}^{\infty} \frac{k \sin kx}{1+k^2} \right]$$

Example:-2) Determine the Fourier series for $\pi + x$ in $[-\pi, \pi]$

Solution:- let $f(x) = \pi + x = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \dots (*)$

Then
$$a_0 = \frac{1}{\pi} \int_0^{2\pi} (\pi + x) dx = \frac{1}{\pi} \left[\pi x + \frac{x^2}{2} \right]_{-\pi}^{\pi} = \pi$$

$$\begin{aligned}
 a_k &= \frac{1}{\pi} \int_0^{2\pi} (\pi + x) \cos kx dx \\
 &= \frac{1}{\pi} \left[\frac{(\pi + x) \sin kx}{k} + \frac{\cos kx}{k^2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[0 + \frac{\cos k\pi}{k^2} - 0 - \frac{\cos k\pi}{k^2} \right] = 0 \\
 b_k &= \frac{1}{\pi} \int_0^{2\pi} (\pi + x) \sin kx dx \\
 &= \frac{1}{\pi} \left[-\frac{(\pi + x) \cos kx}{k} + \frac{\sin kx}{k^2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[-\frac{2\pi \cos k\pi}{k^2} + 0 + 0 + 0 \right] = -\frac{2 \cos k\pi}{k} = \frac{2(-1)^k}{k}
 \end{aligned}$$

Substituting the values of a_0 , a_k , b_k in (*) we get

$$\pi + x = \frac{\pi}{2} - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin kx$$

Example:-3) let $f(x) = \begin{cases} \frac{\pi}{4} & \text{when } 0 < x \leq \pi \\ -\frac{\pi}{4} & \text{when } -\pi \leq x \leq 0 \end{cases}$ obtain the Fourier series for $f(x)$.

Then, obtain the Fourier series for $f(x)$ in $[-\pi, \pi]$

Solution: - Then the Fourier series for $f(x)$ in interval $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad \dots \dots \dots (*)$$

Where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$$a_k = \frac{1}{\pi} \int_{-\pi}^0 -\frac{\pi}{4} dx + \frac{1}{\pi} \int_0^{\pi} \frac{\pi}{4} dx \quad k = 0, 1, 2, \dots$$

$$a_k = -\frac{1}{4} \int_{-\pi}^0 dx + \frac{1}{4} \int_0^{\pi} dx \quad k = 0, 1, 2, \dots$$

$$a_0 = -\frac{1}{4} [x]_{-\pi}^0 + \frac{1}{4} [x]_0^{\pi}$$

$$a_0 = -\frac{1}{4} [0 - (-\pi)] + \frac{1}{4} [\pi - 0] \quad k = 0, 1, 2, \dots$$

$$a_0 = -\frac{1}{4}\pi + \frac{1}{4}\pi = 0$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \quad k = 1, 2, 3 \dots$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^0 -\frac{\pi}{4} \cos kx \, dx + \frac{1}{\pi} \int_0^{\pi} \frac{\pi}{4} \cos kx \, dx$$

$$a_k = -\frac{1}{4} \int_{-\pi}^0 \cos kx \, dx + \frac{1}{4} \int_0^{\pi} \cos kx \, dx$$

$$a_k = -\frac{1}{4} \left[\frac{\sin kx}{K} \right]_{-\pi}^0 + \frac{1}{4} \left[\frac{\sin kx}{K} \right]_0^{\pi}$$

$$a_k = -\frac{1}{4k} [\sin 0 + \sin(k\pi)] + \frac{1}{4k} [\sin k\pi - \sin 0]$$

$$a_k = 0$$

And

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx \quad k = 1, 2, 3, \dots$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^0 -\frac{\pi}{4} \sin kx \, dx + \frac{1}{\pi} \int_0^{\pi} \frac{\pi}{4} \sin kx \, dx$$

$$b_k = -\frac{1}{4} \int_{-\pi}^0 \sin kx \, dx + \frac{1}{4} \int_0^{\pi} \sin kx \, dx$$

$$b_k = \frac{1}{4} \left[\frac{\cos kx}{K} \right]_{-\pi}^0 - \frac{1}{4} \left[\frac{\cos kx}{K} \right]_0^{\pi}$$

$$b_k = \frac{1}{4k} [\cos 0 - \cos(k\pi)] - \frac{1}{4k} [\cos k\pi - \cos 0]$$

$$b_k = \frac{1}{4k} [1 - \cos(k\pi)] - \frac{1}{4k} [\cos k\pi - 1]$$

$$b_k = \frac{1}{4k} [2 - 2\cos(k\pi)]$$

$$b_k = \frac{1}{2k} [1 - \cos(k\pi)]$$

By Substituting the values of a_0, a_k, b_k in (*), we get the Fourier series of the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2k} [1 - \cos(k\pi)] \sin kx = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots$$

Activity

Determine the Fourier series for each of the following functions.

- a) $f(x) = x$ in $[-\pi, \pi]$
- b) $f(x) = x \sin x$ in $[0, 2\pi]$
- c) $f(x) = e^{-x}$ in $[0, 2\pi]$
- d) $f(x) = x - x^2$ in $[-\pi, \pi]$

4.3 FUCTIONS OF ANY PERIOD

In the previous section we expanded a function $f(x)$ in a Fourier series of period 2π . In most of the Engineering problems the period to be expanded is not 2π but some other $2l$. With a suitable substitution we can represent a periodic function of any period.

Let $f(x)$ be a periodic function of period $2l$ defined on $[\alpha, \alpha + 2l]$. we convert x into t such that the period becomes 2π .

$$\begin{aligned} \text{Let } \frac{2l}{x} = \frac{2\pi}{t} &\Rightarrow \frac{x}{l} = \frac{t}{\pi} \\ &\Rightarrow x = \frac{lt}{\pi} \quad \text{or} \quad t = \frac{\pi x}{l} \\ &dt = \frac{\pi}{l} dx \end{aligned}$$

$$\text{Where} \quad x = \alpha, \quad t = \frac{\pi\alpha}{l} = d(\text{say})$$

$$\text{When} \quad x = \alpha + 2l, \quad t = \frac{\pi}{l}(\alpha + 2l) = \frac{\pi\alpha}{l} + 2\pi$$

$$\therefore \quad t = d + 2\pi$$

$$\therefore f(x) \text{ defined in } [\alpha, \alpha + 2l] \text{ is equivalent to } f\left(\frac{lt}{\pi}\right) = F(t) \text{ in } [d, d + 2\pi]$$

Hence we now write the Fourier series for $F(t)$ in $[d, d + 2\pi]$ as

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \quad \dots \dots \dots (1)$$

Where the coefficients a_0, a_k, b_k are given by

$$a_0 = \frac{1}{\pi} \int_d^{d+2\pi} F(t) dt \quad \dots \dots \dots (2)$$

$$a_k = \frac{1}{\pi} \int_d^{d+2\pi} F(t) \cos kt dt \quad \dots \dots \dots (3)$$

$$b_k = \frac{1}{\pi} \int_d^{d+2\pi} F(t) \sin kt dt \quad \dots \dots \dots (4)$$

We now make the inverse substitutions $t = \frac{\pi x}{l}$

$$F(t) = f\left(\frac{lt}{\pi}\right) = f(x)$$

$$t = d + 2\pi$$

$$\Rightarrow x = x + 2l$$

$\therefore$ The Fourier series for $f(x)$ in $[\alpha, \alpha + 2l]$ is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right)$$

Where $a_0 = \frac{1}{\pi} \int_d^{d+2\pi} F(t) dt$ from (2)

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2l} f(x) \frac{\pi}{l} dx$$

$$\therefore a_0 = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_d^{d+2\pi} F(t) \cos kt dt \quad \text{from (3)}$$

$$a_k = \frac{1}{\pi} \int_{\alpha}^{\alpha+2l} f(x) \cos \frac{k\pi x}{l} \cdot \frac{\pi}{l} dx$$

$$\therefore a_k = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(x) \cos \frac{k\pi x}{l} dx$$

$$b_k = \frac{1}{\pi} \int_d^{d+2\pi} F(t) \sin kt dt \quad \text{from (4)}$$

$$b_k = \frac{1}{\pi} \int_{\alpha}^{\alpha+2l} f(x) \sin \frac{k\pi x}{l} \cdot \frac{\pi}{l} dx$$

$$\therefore b_k = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(x) \sin \frac{k\pi x}{l} dx$$

Hence the Fourier series for $f(x)$ in $[\alpha, \alpha + 2l]$ is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right)$$

Where
$$a_0 = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(x) dx$$

$$a_k = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(x) \cos \frac{k\pi x}{l} dx$$

$$b_k = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(x) \sin \frac{k\pi x}{l} dx$$

Example:-1) obtain the Fourier series for $f(x) = x^2$ in $[-l, l]$

Solution:- Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right) \dots \dots \dots (1)$$

Where
$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$a_0 = \frac{1}{l} \int_{-l}^l x^2 dx$$

$$a_0 = \frac{2}{l} \int_0^l x^2 dx \text{ Since } x^2 \text{ is even.}$$

$$a_0 = \frac{2}{l} \left[\frac{x^3}{3} \right]_0^l = \frac{2l^2}{3}$$

$$\therefore \frac{a_0}{2} = \frac{l^2}{3}$$

$$a_k = \frac{1}{l} \int_{-l}^l x^2 \cos \frac{k\pi x}{l} dx$$

$$a_k = \frac{2}{l} \int_0^l x^2 \cos \frac{k\pi x}{l} dx \text{ Since } x^2 \cos \frac{k\pi x}{l} \text{ is even}$$

$$a_k = \frac{2}{l} \left[x^2 \sin \frac{k\pi x}{l} \cdot \frac{l}{k\pi} - 2x \left(-\cos \frac{k\pi x}{l} \right) \frac{l^2}{k^2 \pi^2} + 2 \left(-\sin \frac{k\pi x}{l} \right) \frac{l^3}{k^3 \pi^3} \right]$$

$$= \frac{2}{l} \cdot \frac{2l^2}{k^2 \pi^2} \left[x \cos \frac{k\pi x}{l} \right]_0^l$$

$$= \frac{4l}{k^2 \pi^2} [l \cos k\pi - 0 \cdot \cos 0]_0^l$$

$$\therefore a_k = \frac{4l^2}{k^2 \pi^2} (-1)^k$$

$$b_k = \frac{1}{l} \int_{-l}^l x^2 \sin \frac{k\pi x}{l} dx$$

$$\text{Since } x^2 \sin \frac{k\pi x}{l} \text{ is odd } b_k = 0$$

By Substituting the values of a_0, a_k, b_k in (1), we get the Fourier series of the function

$$\begin{aligned} f(x) &= \frac{l^2}{3} + \sum_{k=1}^{\infty} \left(\frac{4l^2(-1)^k}{k^2\pi^2} \cos \frac{k\pi x}{l} \right) \\ &= \frac{l^2}{3} - \frac{4l^2}{\pi^2} \sum_{k=1}^{\infty} \left(\frac{(-1)^{k-1}}{k^2} \cos \frac{k\pi x}{l} \right) \end{aligned}$$

Activity

Determine the Fourier series for each of the following functions.

- Find the Fourier series period 3 to represent $f(x) = 2x - x^2$ in $(0,3)$
- Find the Fourier series period 2 to represent $f(x) = 2x - x^2$ in $(0,2)$

4.4 EVEN AND ODD FUNCTIONS, HALF RANGE EXPANTION

Definition: - A function $f(x)$ is said to be even if $f(-x) = f(x)$ for all x .

Example:-1) a) $f(x) = \sec x$ is even function because

$$f(-x) = \sec(-x) = \sec x = f(x) \text{ for all } x.$$

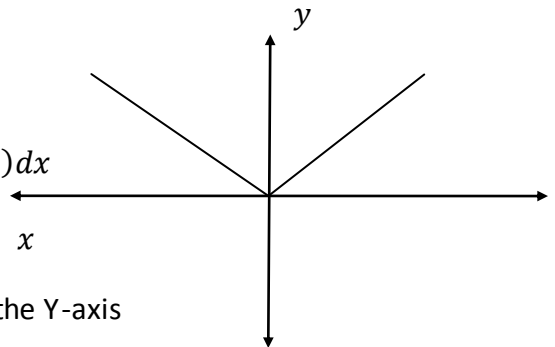
b) $f(x) = x^4 + x^2$ is an even function since $f(-x) = f(x)$ for all x .

Note: - 1) The sum of two even functions is even.

2) The product of two even functions is even

3) If $f(x)$ is an even function $\int_{-\alpha}^{\alpha} f(x) dx = 2 \int_0^{\alpha} f(x) dx$

4) the graph of an even function is symmetric about the Y-axis



Definition: - A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for all x

Example:-2) a) $f(x) = \sin x$ is odd function because

$$f(-x) = \sin(-x) = -\sin x = -f(x) \text{ for all } x.$$

$$\text{b) } f(x) = x^3 + x \text{ is an odd function since } f(-x) = -f(x) \text{ for all } x.$$

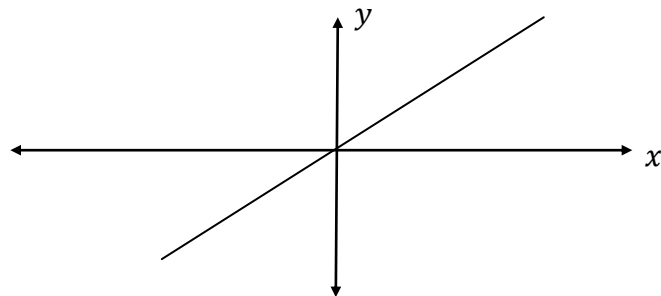
Note: - 1) The sum of two odd functions is odd.

2) The product of two odd functions is an even function

3) The product of an odd function and an even function is an odd function.

4) If $f(x)$ is an odd function $\int_{-\alpha}^{\alpha} f(x) dx = 0$

5) The graph of an even function is symmetric about the origin.



However there are functions which are neither odd nor even. In this case the functions can be uniquely decomposed as even and odd functions say f_e and f_o for even and odd function respectively.

$$\text{Then} \quad f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = f_e(x) + f_o(x)$$

$$\text{Where} \quad f_e(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x)-f(-x)}{2}$$

$$\text{Since} \quad f_e(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(-x)+f(x)}{2} = \frac{f(x)+f(-x)}{2} = f_e(x)$$

$$\text{And} \quad f_o(-x) = \frac{f(-x)-f(-(-x))}{2} = \frac{f(-x)-f(x)}{2} = -\frac{f(x)-f(-x)}{2} = -f_o(x)$$

Example:-3) $f(x) = e^x$ is neither odd nor even but by putting

$$f(x) = e^x \text{ and } f(-x) = e^{-x} \text{ we get } f_e(x) = \frac{e^x + e^{-x}}{2} \text{ and } f_o(x) = \frac{e^x - e^{-x}}{2}$$

Fourier series for even and odd function in the interval $[-\pi, \pi]$

1. If $f(x)$ is an even function in the interval $[-\pi, \pi]$, we get the Fourier series given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

Where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx dx, \quad b_k = 0$$

Here we need not calculate b_k .

2. If $f(x)$ is an odd function in the interval $[-\pi, \pi]$ we get the Fourier series given by

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

Where

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx dx, \quad a_0 = 0, \quad a_k = 0$$

Here we also need not calculate a_0 and a_k .

Note: - If $f(x)$ is neither an even nor an odd function in the interval $[-\pi, \pi]$, then calculate a_0 , a_k and b_k using the formula

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx$$

Example:-4) Find the Fourier series for x^2 in $[-\pi, \pi]$

Solution: - let $f(x) = x^2$

$\therefore$ $f(x)$ is an even function

Hence the Fourier series in $[-\pi, \pi]$ is $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \dots\dots\dots (*)$

Where
$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \cdot \frac{\pi^3}{3}$$

$$\therefore a_0 = \frac{2\pi^2}{3} \Rightarrow \frac{a_0}{2} = \frac{\pi^2}{3}$$

$$\begin{aligned} a_k &= \frac{2}{\pi} \int_0^\pi f(x) \cos kx \, dx = \frac{2}{\pi} \int_0^\pi x^2 \cos kx \, dx \\ &= \frac{2}{\pi} \left[\frac{x^2}{k} \sin kx - 2x \left(-\frac{1}{k^2} \cos kx \right) + 2 \left(-\frac{1}{k^3} \sin kx \right) \right]_0^\pi \\ &= \frac{2}{\pi} \cdot \frac{2}{k^2} [\pi \cos kx] \end{aligned}$$

$$\therefore a_k = \frac{4}{k^2} (-1)^k$$

Substituting for a_0 and a_k in (*), we get

$$f(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k \cos kx$$

$$f(x) = \frac{\pi^2}{3} - 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos kx$$

Example:-5) Expand $f(x) = x$ in Fourier series in $[-l, l]$

Solution:- let $f(x) = x$

$$\therefore f(x) \text{ is an odd function}$$

$$\therefore \text{The Fourier series is } f(x) = \sum_{k=1}^{\infty} b_k \sin kx \quad \dots\dots\dots (*)$$

$$\begin{aligned} \text{Where } b_k &= \frac{2}{\pi} \int_0^l x \sin kx \, dx \\ &= \frac{2}{\pi} \left[\frac{-x \cos kx}{k} + \frac{\sin kx}{k^2} \right]_0^\pi = \frac{2}{\pi} [\pi \cos k\pi] = -\frac{2}{k} (-1)^k \end{aligned}$$

$$\therefore b_k = \frac{-2(-1)^k}{k} = \frac{2(-1)^{k+1}}{k}$$

$$\text{Substituting for } a_k \text{ in } (*), \text{ we get } f(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

Activity

Find the Fourier series for each of the following functions in the interval $[-\pi, \pi]$.

a) $f(x) = x^3$

c) $f(x) = x^2 - 2$

e) $f(x) = 2x^2 - 1$

b) $f(x) = \sin x$

d) $f(x) = x^2 + x$

f) $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$

Half-Range series

Some times we come across the Fourier expansion of a function in a range which is half the period of the Fourier series i.e., a function $f(x)$ is defined in the interval $[0, l]$ in a Fourier series of period $2l$. such a series is known as Half-Range Fourier series.

Suppose a periodic function $f(x)$ of period $2l$ is defined only on half interval $[0, l]$. It is possible to extend the definition of $f(x)$ on the other half $[-l, 0]$ so that we obtain either an even or an odd function in $[-l, l]$

Half-Range cosine series

Let $f(x)$ be defined in the range $[0, l]$. in order to obtain Fourier series we construct a new function in such a way that the function $f(x)$ plus its extension yield an even function.

Define

$$F(x) = f(x) \text{ for } 0 < x < l$$

$$= f(x) \text{ for } -l < x < 0$$

So that $F(x)$ is an even function in $[-l, l]$. Hence the Fourier series of $F(x)$ contains only cosine terms.

$$\therefore F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l}$$

Where

$$b_k = \frac{1}{l} \int_{-l}^l F(x) \sin \frac{k\pi x}{l} dx = 0$$

$$a_0 = \frac{1}{l} \int_{-l}^l F(x) dx = \frac{2}{l} \int_0^l F(x) dx = \frac{2}{l} \int_0^l f(x) dx$$

$$a_k = \frac{2}{l} \int_0^l F(x) \cos \frac{k\pi x}{l} dx = \frac{2}{l} \int_0^l f(x) \cos \frac{k\pi x}{l} dx$$

$$\text{Since } F(x) = f(x) \text{ in } [0, l]$$

In this case the Fourier series is called **Half-Range Fourier Cosine series** which is defined

as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l}$$

Where

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_k = \frac{2}{l} \int_0^l f(x) \cos \frac{k\pi x}{l} dx$$

Note: - The Half-Range cosine series in $[0, \pi]$ is obtained by putting $l = \pi$.

Half-Range sine series

Let $f(x)$ be defined in the range $[0, l]$. in order to obtain Fourier series we construct a new function in such a way that the function $f(x)$ plus its extension yield an odd function.

$$\begin{aligned} \text{Define } F(x) &= f(x) \text{ for } 0 < x < l \\ &= -f(-x) \text{ for } -l < x < 0 \end{aligned}$$

So that $F(x)$ is an odd function in $[-l, l]$. Hence the Fourier series of $F(x)$ contains only sine terms.

$$\therefore F(x) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l}$$

$$\text{Where } a_0 = \frac{1}{l} \int_{-l}^l F(x) dx = 0, \quad b_k = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{k\pi x}{l} dx = 0$$

$$b_k = \frac{1}{l} \int_{-l}^l F(x) \sin \frac{k\pi x}{l} dx = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l} dx = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l} dx$$

$$\text{Since } F(x) = f(x) \text{ in } [0, l]$$

In this case the Fourier series is called **Half-Range Fourier Sine series** which is defined

$$\text{as } f(x) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l}$$

$$\text{Where } a_k = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l} dx$$

Note: - The Half-Range sine series in $[0, \pi]$ is obtained by putting $l = \pi$.

Example:-6) Find the Half-Range cosine and Sine series for the function

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \end{cases}$$

Solution: - Here $l = 2 - 0 = 2$

i) the Half-Range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l}$$

Where

$$\begin{aligned}
 a_0 &= \frac{2}{l} \int_0^l f(x) dx = \frac{2}{2} \left[\int_0^1 x dx + \int_1^2 (2-x) dx \right] \\
 &= \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 \\
 &= \left[\frac{1}{2} - 0 \right] + \left[(4-2) - \left(2 - \frac{1}{2} \right) \right] = 1
 \end{aligned}$$

$$\therefore a_0 = 1$$

$$\begin{aligned}
 a_k &= \frac{2}{2} \left[\int_0^1 x \cos \frac{k\pi x}{2} dx + \int_1^2 (2-x) \cos \frac{k\pi x}{2} dx \right] \\
 &= \left\{ \left(x \sin \frac{k\pi x}{2} \right) \frac{2}{k\pi} + \left(\cos \frac{k\pi x}{2} \right) \frac{4}{k^2 \pi^2} \right\}_0^1 + \left\{ (2-x) \left(\sin \frac{k\pi x}{2} \right) \frac{2}{k\pi} + \left(-\cos \frac{k\pi x}{2} \right) \frac{4}{k^2 \pi^2} \right\}_1^2 \\
 &= \frac{2}{k\pi} \left[\sin \frac{k\pi}{2} - 0 \right] + \frac{4}{k^2 \pi^2} \left[\cos \frac{k\pi}{2} - 1 \right] + \frac{2}{k\pi} \left[0 - \sin \frac{k\pi}{2} \right] - \frac{4}{k^2 \pi^2} \left[\cos k\pi - \cos \frac{k\pi}{2} \right] \\
 &= \frac{2}{k\pi} \left[\sin \frac{k\pi}{2} \right] + \frac{4}{k^2 \pi^2} \left[\cos \frac{k\pi}{2} - 1 \right] - \frac{2}{k\pi} \left[\sin \frac{k\pi}{2} \right] - \frac{4}{k^2 \pi^2} \left[\cos k\pi - \cos \frac{k\pi}{2} \right] \\
 &= \frac{4}{k^2 \pi^2} \left[\cos \frac{k\pi}{2} - 1 - \cos k\pi + \cos \frac{k\pi}{2} \right] \\
 &= \frac{4}{k^2 \pi^2} \left[2 \cos \frac{k\pi}{2} - 1 - \cos k\pi \right]
 \end{aligned}$$

$$\therefore a_k = \frac{4}{k^2 \pi^2} \left[2 \cos \frac{k\pi}{2} - 1 - (-1)^k \right]$$

Hence substituting for a_0 and a_k gives

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{k^2 \pi^2} \left[2 \cos \frac{k\pi}{2} - 1 - (-1)^k \right] \cos \frac{k\pi x}{l}$$

ii) the Half-Range Sine series

$$f(x) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l}$$

Where

$$b_k = \frac{2}{2} \left[\int_0^1 x \sin \frac{k\pi x}{2} dx + \int_1^2 (2-x) \sin \frac{k\pi x}{2} dx \right]$$

$$\begin{aligned}
 &= \left\{ x \left(-\cos \frac{k\pi x}{2} \right) \frac{2}{k\pi} + \left(\sin \frac{k\pi x}{2} \right) \frac{4}{k^2 \pi^2} \right\}_0^1 + \left\{ (2-x) \left(-\cos \frac{k\pi x}{2} \right) \frac{2}{k\pi} - \left(\sin \frac{k\pi x}{2} \right) \frac{4}{k^2 \pi^2} \right\}_1^2 \\
 &= \frac{2}{k\pi} \left[-\cos \frac{k\pi}{2} - 0 \right] + \frac{4}{k^2 \pi^2} \left[\sin \frac{k\pi}{2} - 0 \right] + \frac{2}{k\pi} \left[0 + \cos \frac{k\pi}{2} \right] - \frac{4}{k^2 \pi^2} \left[\sin k\pi - \sin \frac{k\pi}{2} \right]
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{k\pi} \left[\cos \frac{k\pi}{2} \right] + \frac{4}{k^2\pi^2} \left[\sin \frac{k\pi}{2} \right] + \frac{2}{k\pi} \left[\cos \frac{k\pi}{2} \right] - \frac{4}{k^2\pi^2} \left[\sin k\pi - \sin \frac{k\pi}{2} \right] \\
&= \frac{4}{k^2\pi^2} \left[\sin \frac{k\pi}{2} \right] + -\frac{4}{k^2\pi^2} \left[\sin k\pi - \sin \frac{k\pi}{2} \right] \\
&= \frac{4}{k^2\pi^2} \left[\sin \frac{k\pi}{2} \right] + -\frac{4}{k^2\pi^2} \left[0 - \sin \frac{k\pi}{2} \right] = \frac{8}{k^2\pi^2} \sin \frac{k\pi}{2}
\end{aligned}$$

$$\therefore b_k = \frac{8}{k^2\pi^2} \sin \frac{k\pi}{2}$$

Substituting for b_k in $f(x) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l}$ we get

$$f(x) = \sum_{k=1}^{\infty} \frac{8}{k^2\pi^2} \sin \frac{k\pi}{2} \sin \frac{k\pi x}{2}$$

$$\therefore f(x) = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{k\pi}{2} \sin \frac{k\pi x}{2}$$

Activity

Find the Half-Range sine and cosine series for the function

i) $f(x) = x$ in $0 < x < \pi$

ii) $f(x) = x^2$ in $0 < x < \pi$

iii) $f(x) = (x-1)^2$ in $0 < x < 1$

iv) $f(x) = l - x$ in $0 < x < l$

1.5. FOURIER INTEGRALS

Definition:-the Fourier integral of the function $f(x)$ defined on $-\infty < x < \infty$ is

given by

$$f(x) = \int_0^{\infty} [a(y) \cos yx + b(y) \sin yx] dy$$

Where

$$a(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos yx \, dx$$

$$b(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin yx \, dx$$

Example:-1) Find the Fourier integral for the function $f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x < -1, x > 1 \end{cases}$

Solution:- $a(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos yx \, dx$

$$a(y) = \frac{1}{\pi} \left[\int_{-1}^1 1 \cdot \cos yx \, dx \right] = \frac{2}{\pi} \left[\int_0^1 \cos yx \, dx \right]$$

$$= \frac{2}{\pi} \left[\frac{\sin yx}{y} \right]_0^1 = \frac{2 \sin y}{\pi y}$$

$$b(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin yx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-1}^1 \sin yx \, dx \right] = 0 \quad (\because \sin yx \text{ is an odd function.})$$

$\therefore$ Substituting for $a(y)$ and $b(y)$ in $f(x) = \int_0^{\infty} [a(y) \cos yx + b(y) \sin yx] dy$ gives

$$f(x) = \int_0^{\infty} \left[\frac{2 \sin y}{\pi y} \cos yx + 0 \cdot \sin yx \right] dy$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin y \cos yx \frac{dy}{y}$$

Example:-2) Find the Fourier integral for the function $f(x) = \begin{cases} -x & \text{if } -5 \leq x \leq 0 \\ 0 & \text{if } x < -5, x > 0 \end{cases}$

Solution:-

$$a(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos yx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{-5} f(x) \cos yx \, dx + \int_{-5}^0 f(x) \cos yx \, dx + \int_0^{\infty} f(x) \cos yx \, dx \right]$$

$$= \frac{1}{\pi} \left[0 + \int_{-5}^0 f(x) \cos yx \, dx + 0 \right]$$

$$= \frac{1}{\pi} \left[\int_{-5}^0 f(x) \cos yx \, dx \right] = \frac{1}{\pi} \left[\int_{-5}^0 -x \cos yx \, dx \right]$$

$$= \frac{-1}{\pi} \left[\int_{-5}^0 x \cos yx \, dx \right]$$

$$= \frac{-1}{\pi} \left[\frac{x}{y} \sin yx + \frac{1}{y^2} \cos yx \right]_{-5}^0$$

$$= \frac{-1}{\pi} \left[\frac{1}{y^2} - \left(\frac{-5}{y} \sin(-5y) + \frac{1}{y^2} \cos(-5y) \right) \right]$$

$$= \frac{-1}{\pi} \left[\frac{1}{y^2} - \left(\frac{5}{y} \sin 5y + \frac{1}{y^2} \cos 5y \right) \right]$$

$$= \frac{-1}{\pi} \left[\frac{1 - \cos 5y}{y^2} - \left(\frac{5}{y} \sin 5y \right) \right]$$

$$= \frac{\cos 5y - 1}{\pi y^2} + \frac{5}{\pi y} \sin 5y$$

$$\begin{aligned}
b(y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin yx \, dx \\
&= \frac{1}{\pi} \left[\int_{-\infty}^{-5} f(x) \sin yx \, dx + \int_{-5}^0 f(x) \sin yx \, dx + \int_0^{\infty} f(x) \sin yx \, dx \right] \\
&= \frac{1}{\pi} \left[0 + \int_{-5}^0 f(x) \sin yx \, dx + 0 \right] \\
&= \frac{1}{\pi} \left[\int_{-5}^0 f(x) \sin yx \, dx \right] = \frac{1}{\pi} \left[\int_{-5}^0 -x \sin yx \, dx \right] \\
&= \frac{-1}{\pi} \left[\int_{-5}^0 x \sin yx \, dx \right] \\
&= \frac{-1}{\pi} \left[\frac{-x}{y} \cos yx + \frac{1}{y^2} \sin yx \right]_{-5}^0 \\
&= \frac{-1}{\pi} \left[0 - \left(\frac{5}{y} \cos 5y + \frac{1}{y^2} \sin(-5y) \right) \right] \\
&= \frac{-1}{\pi} \left[-\frac{5}{y} \cos 5y + \frac{1}{y^2} \sin 5y \right] \\
&= \frac{5}{\pi y} \cos 5y - \frac{1}{\pi y^2} \sin 5y
\end{aligned}$$

Substituting for $a(y)$ and $b(y)$ in $f(x) = \int_0^{\infty} [a(y) \cos yx + b(y) \sin yx] dy$ gives

$$\begin{aligned}
f(x) &= \int_0^{\infty} \left[\left(\frac{\cos 5y - 1}{\pi y^2} + \frac{5}{\pi y} \sin 5y \right) \cos yx + \left(\frac{5}{\pi y} \cos 5y - \frac{1}{\pi y^2} \sin 5y \right) \sin yx \right] dy \\
&= \frac{1}{\pi} \int_0^{\infty} [(\cos 5y - 1 + 5y \sin 5y) \cos yx + (5y \cos 5y - \sin 5y) \sin yx] \frac{dy}{y^2} \\
\therefore f(x) &= \frac{1}{\pi} \int_0^{\infty} [(\cos 5y + 5y \sin 5y - 1) \cos yx + (5y \cos 5y - \sin 5y) \sin yx] \frac{dy}{y^2}
\end{aligned}$$

Activity

Find the Fourier integral for each of the following functions

$$a) f(x) = \begin{cases} 100 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x < 0, x > 2 \end{cases}$$

$$b) f(x) = \begin{cases} e^x & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x < -1, x > 1 \end{cases}$$

CHAPTER 5

Introduction

So far we have dealt with the calculus of functions of a single variable. But, in the real world, physical quantities usually depend on two or more variables, so in this chapter we turn our attention to functions of several variables and extend the basic ideas of differential calculus to such functions.

Functions of Several Variables

Example

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y , or as a function of the pair (x, y) .

We indicate this functional dependence by writing $T = f(x, y)$.

Example:- The volume V of a circular cylinder depends on its radius r and height h . In fact, we know that

$$V = \pi r^2 h. \text{ We say that } V \text{ is a function of } r \text{ and } h, \text{ and we write } V(r, h) = \pi r^2 h.$$

Definition

Let $D \subset \mathbb{R}^2$. A **function f of two variables** is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) | (x, y) \in D\}$.

Note

- i. We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y) . The variables x and y are independent variables and z is the dependent variable. [Compare this with the notation $y = f(x)$ for functions of a single variable]
- ii. The situation described in above definition is indicated by the notation $f: D \rightarrow \mathbb{R}$. A function of two variables is just a special case of the general idea of a function $f: X \rightarrow Y$, where $X = D \subset \mathbb{R}^2$ and $Y \subset \mathbb{R}$.
- iii. If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well defined real number.

Example:- Find the domain of each of the following functions and evaluate $f(3,2)$.

$$(a) f(x, y) = \frac{\sqrt{x+y+1}}{x-1} \quad (b) f(x, y) = x \ln(y^2 - x)$$

Solution:- a) Given that $f(x, y) = \frac{\sqrt{x+y+1}}{x-1} \Rightarrow f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) | x + y + 1 \geq 0, x \neq 1\}$$

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain.

b. Given that $f(x, y) = x \ln(y^2 - x) \Rightarrow f(3,2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is

$$D = \{(x, y) | x < y^2\}. \text{ This is the set of points to the left of the parabola } x = y^2.$$

5.1 Notations, Examples, level Curves and graphs

Graph of a function of two variables

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can see the graph S of f as lying directly above or below its domain D in the xy -plane.

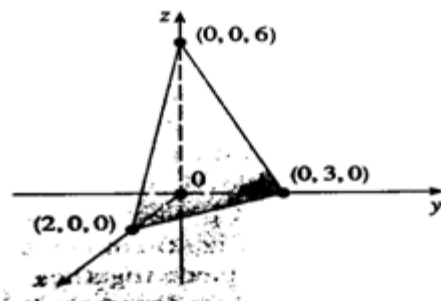
If f is a function of two variables with domain D , the graph of f is the set

$$S = \{(x, y, z) \in \mathbb{R}^3 | z = f(x, y), (x, y) \in D\}$$

Example:- Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$

Solution:- The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$,

which is a plane. The portion of this graph that lies in the first octant is sketched.



$$f(x, y) = 6 - 3x - 2y$$

Example:- Find the domain and range and sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution:- The domain of g is

$$D = \{(x, y) | 9 - x^2 - y^2 \geq 0\} = \{(x, y) | x^2 + y^2 \leq 9\}$$

Which is the disk with center $(0, 0)$ and radius 3.

The range of g is $\{z | z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$

Since z is positive square root, $z \geq 0$. also

$$9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$$

So the range is $\{z | 0 \leq z \leq 3\} = [0, 3]$.

The graph has the equation $z = \sqrt{9 - x^2 - y^2}$,

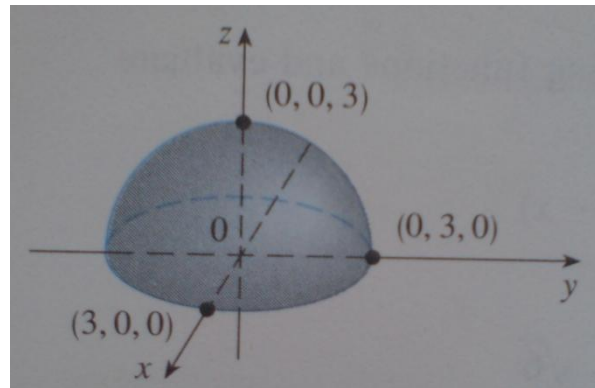
We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$,

Which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere.

Level Curves

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f)



Note

- ✚ A level curve $f(x, y) = k$ is the locus of all points at which f takes on a given value k . In other words, it shows where the graph of f has height k .

Example

Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Solution:-The level curves are

$$6 - 3x - 2y = k \quad \text{or} \quad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$.

The four particular level curves with $k = -6, 0, 6, \text{ and } 12$ are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and

$3x + 2y + 6 = 0$. They are sketched in above figure.

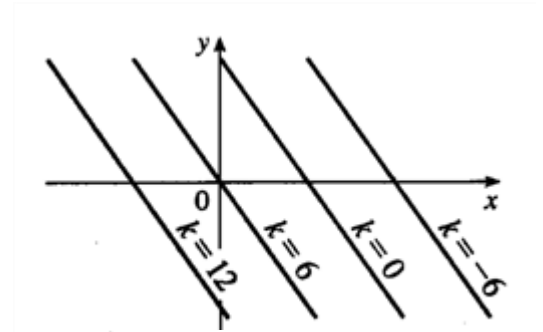
Example:-Sketch some level curves of the function $h(x, y) = 4x^2 + y^2$.

Solution

The level curves are

$$4x^2 + y^2 = k \quad \text{or} \quad \frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

Which, for $k > 0$, describes a family of ellipses with Semi-axis $\sqrt{k}/2$ and $\sqrt{k}$. They are sketched in above Figure.



Contour map of

$$h(x, y) = 4x^2 + y^2$$

5.2 Limit and Continuity**Limit**

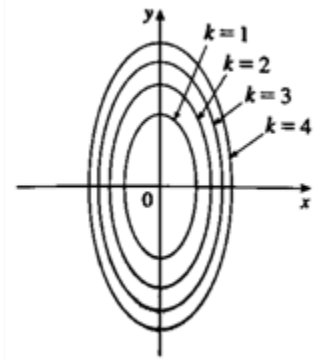
Let f be a function of two variables defined on a disk with center (a, b) , except possibly at (a, b) . Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b) is L** and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

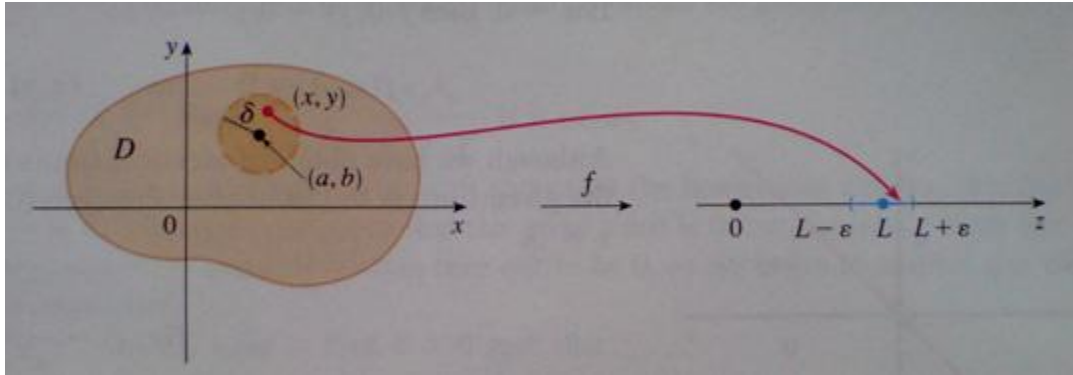
If for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

The above definition says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) and (a, b) sufficiently small (but not 0). The following figure illustrates limit by means of



an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L , then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.



Note

The above limit definition refers only to the distance between (x, y) and (a, b) . It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b) . Thus, if we can find two different paths of approach along which $f(x, y)$ has different limits, then it follows that $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ does not exist.

If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2 , where $L_1 \neq L_2$, then $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ does not exist.

Example:- Find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ if it exists.

Solution:- Let $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$.

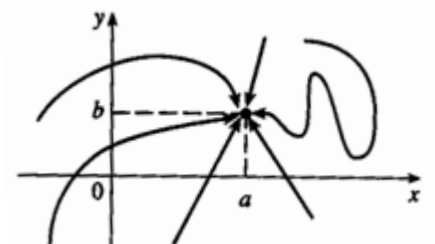
First let us approach $(0, 0)$ along the x -axis. Then $y = 0$ gives $f(x, 0) = \frac{x^2}{x^2} = 1 \forall x \neq 0$, so

$$f(x, y) \rightarrow 1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y -axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1 \forall y \neq 0$, so

$$f(x, y) \rightarrow -1 \quad \text{as} \quad (x, y) \rightarrow (0, 0) \text{ along the } y\text{-axis}$$

Since f has two different limits along two different lines, the given limit doesn't exist.



Example:- Find $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists. (by definition)

Solution

Step 1: Choosing a value for δ

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

that is,
$$\frac{3x^2|y|}{x^2+y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so

$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3(x^2 + y^2) \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

Thus, choose $\delta = \varepsilon/3$

Step 2: Proof

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} \leq 3\delta = 3(\varepsilon/3) = \varepsilon$$

Hence, by limit definition

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

Continuity

Let f be a function of two variables defined on a disk with center (a, b) . Then f is called **continuous at (a, b)** if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

Example:- Evaluate $\lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

Solution

Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so the limit can be found by direct substitution.

$$\lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

Example: - Let
$$f(x, y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there, also we have

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0 = f(0,0)$$

Therefore, f is continuous at $(0, 0)$, and so it is continuous on $\mathbb{R}^2$.

Theorem

If f is continuous at (a, b) and g is a function of a single variable that is continuous at $f(a, b)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (a, b) .

Example: - On what set is the function $h(x, y) = \ln(x^2 + y^2 - 1)$ continuous?

Solution:- Let $f(x, y) = x^2 + y^2 - 1$ and $g(t) = \ln t$. Then

$$g(f(x, y)) = \ln(x^2 + y^2 - 1) = h(x, y)$$

So $h = g \circ f$. Now f is continuous everywhere since it is a polynomial, and g is continuous on its domain $\{t | t > 0\}$. Thus by above theorem, h is continuous on its domain

$$D = \{(x, y) | x^2 + y^2 - 1 > 0\} = \{(x, y) | x^2 + y^2 > 1\}$$

This consists of all points outside the circle $x^2 + y^2 = 1$.

5.3 Partial Derivatives, Higher order partial derivatives

If f is a function of two variables x and y , suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we really considering a function of a single variable x , namely, $g(x) = f(x, b)$. If g has a derivative at a , then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

By definition, we know that

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$$

Therefore the above equation becomes

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of f with respect to y at (a, b)** , denoted by $f_y(a, b)$ is obtained by keeping x fixed ($x = a$) and finding the ordinary derivatives at b of the function $G(y) = f(a, y)$:

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b + h) - f(a, b)}{h}$$

Definition

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notations for partial derivatives

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rules for finding partial derivatives of $z = f(x, y)$

1. To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .
2. To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .

Example:- If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$

Solution:-Holding y as a constant and differentiating with respect to x , we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

holding x constant and differentiating with respect to y , we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Example:-Find $\partial z / \partial x$ and $\partial z / \partial y$, if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Solution:-To find $\partial z / \partial x$, we differentiate implicitly with respect to x , by treating y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z / \partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

5.4 Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f . If $z = f(x, y)$, we use the following notations:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\frac{\partial^2 f}{\partial y \partial x}$) means that we first differentiate with respect to x and then with respect to y , whereas in computing f_{yx} the order is reversed.

Example: - Find the second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$

Solution: We know that

$$f_x(x, y) = 3x^2 + 2xy^3$$

$$f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then $f_{xy}(a, b) = f_{yx}(a, b)$

5.5 Tangent planes and Total differential

Total Differential

The **differentials** dx and dy are independent variables; that is, they can be given any values. Then the **differential** dz , also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Example:- If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz .

Solution:- $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (2x + 3y)dx + (3x - 2y)dy$

Tangent planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S . Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S . Then the point P lies on both C_1 and C_2 . Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P . Then the tangent plane to the surface S at the point P is defined to be the plane that contains both of the tangent lines T_1 and T_2 .

The following table shows the equation of the tangent plane to the surface S

An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Application: tangent plane approximation of values of a function

Example:- Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution:- Let $f(x, y) = 2x^2 + y^2$. Then

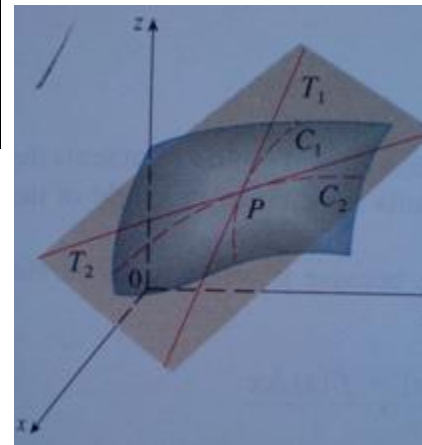
$$f_x(x, y) = 4x \quad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \quad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

$$4x + 2y - z = 3$$



5.6 The Chain Rule and Implicit differentiation

The Chain Rule

Case (i)

Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Note:- We often write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

Example:- If $z = x^2y + 3xy^4$, where $x = e^t$ and $y = \sin t$, find $\frac{dz}{dt}$.

Solution: The Chain Rule gives

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (2xy + 3y^4)e^t + (x^2 + 12xy^3) \cos t \\ &= (2e^t \sin t + 3\sin^4 t)e^t + (e^{2t} + 12e^t \sin^3 t) \cos t \end{aligned}$$

The Chain Rule

Case (ii)

Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$, $y = h(s, t)$, and the partial derivatives g_s, g_t, h_s , and h_t exist. Then

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \end{aligned}$$

Example:- If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\frac{\partial z}{\partial s}$ and $\partial z/\partial t$.

Solution:- Applying Case 2 of the Chain Rule, we get

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t) \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)\end{aligned}$$

Implicit Differentiation

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Example:- Find y' if $x^3 + y^3 = 6xy$.

Solution:- The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

So by Implicit differentiation, we get

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Definition

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example:- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution:- Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz = 1$.

Then, from the above definition, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$$

5.7 Directional derivatives and gradients

Definition

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$$

If this limit exists.

Theorem

If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Note:-If the unit vector $\mathbf{u}$ makes an angle θ with the positive x – axis, then we can write

$\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and the formula becomes

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

Example:-Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. what is $D_{\mathbf{u}}f(1, 2)$?

Solution:We know that

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6}$$

$$= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \frac{1}{2}$$

$$= \frac{1}{2} [3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y]$$

Therefore

$$D_u f(1,2) = \frac{1}{2} [3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)] = \frac{13-3\sqrt{3}}{2}$$

Gradient

If f is a function of two variables x and y , then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

Example:- If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

$$\nabla f(0,1) = \langle 2, 0 \rangle$$

Note:- $D_u f(x, y) = \nabla f(x, y) \cdot u$

Example:- Find the directional derivative of the function $f(x, y) = x^2 y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Solution:- We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3 i + (3x^2 y^2 - 4)j$$

$$\nabla f(2, -1) = -4i + 8j$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$u = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j$$

Therefore, by above note, we have $D_u f(2, -1) = \nabla f(2, -1) \cdot u = (-4i + 8j) \cdot \left(\frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j\right)$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

Definition

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Definition

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot u$$

Example:- If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:- (a) The gradient of f is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle\end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$.

Then unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\begin{aligned}D_u f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot u \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}} \mathbf{i} + \frac{2}{\sqrt{6}} \mathbf{j} - \frac{1}{\sqrt{6}} \mathbf{k} \right) \\ &= 3 \left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}}\end{aligned}$$

5.8 Relative extreme of functions of two variables**Definition**

A function of two variables has a **relative (local) maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) . The number $f(a, b)$ is called a **relative (local) maximum value**. If $f(x, y) \geq f(a, b)$ for all (x, y) in such a disk, $f(a, b)$ is a **relative (local) minimum value**.

Theorem

If f has a relative (local) extremum (that is, a relative (local) maximum or minimum) at (a, b) the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Example:- Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then $f_x(x, y) = 2x - 2$ $f_y(x, y) = 2y - 6$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$,

so the only critical point is $(1, 3)$.

By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y . Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f . This can be confirmed geometrically from the graph of f , which is the elliptic paraboloid with vertex $(1, 3, 4)$.

5.9 Largest and smallest values of a function on a given set**Second Derivatives Test**

Suppose the second partial derivatives of f are continuous in a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$ then $f(a, b)$ is not a local extremum.

Example:- Find the local extrema of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution:- We first locate the critical points:

$$f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 + 1)(x^4 + 1)$$

So there are three real roots; $x = 0, 1, -1$. The three critical points are $(0,0)$, $(1,1)$, and $(-1, -1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0,0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local extremum at $(0,0)$. Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, we see from case (a) of the test that $f(1,1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .

2. Find the extreme values of f on the boundary of D .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example:- Find the absolute maximum and minimum values of the function

$$f(x, y) = x^2 - 2xy + 2y \text{ on the rectangle } D = \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 2\}.$$

Solution:- Since f is a polynomial, it is continuous on the closed, bounded rectangle D , so there is both an absolute maximum and an absolute minimum. According to step 1 we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \quad f_y = -2x + 2 = 0$$

So the only critical point is $(1, 1)$, and the values of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D , which consists of the four line segments L_1, L_2, L_3, L_4 . On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \quad 0 \leq x \leq 3$$

This is an increasing function of x , so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \quad 0 \leq y \leq 2$$

This is a decreasing function of y , so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \quad 0 \leq x \leq 3$$

By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$.

Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \quad 0 \leq y \leq 2$$

With maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude

that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$.

5.10 Extreme values under constraint conditions: Lagrange's multiplier

Let f and g differentiable at (x_0, y_0) . Let c be the curve $g(x, y) = c$ that contains (x_0, y_0) . Assume that c is smooth and that (x_0, y_0) is not an end point of the curve. If $\text{grad } g(x_0, y_0) \neq 0$ and f has an extreme value on c at (x_0, y_0) then there is a number λ such that $\text{grad } f(x_0, y_0) = \lambda \text{grad } g(x_0, y_0)$. The number λ is called Lagrange's Multiplier.

Example:- Let $f(x, y) = x^2 + 4y^3$. Find the extreme values of f on the ellipse $x^2 + 2y^2 = 1$ and points at which they occur.

Solution:- Let $g(x, y) = x^2 + 2y^2$ so that the constraint is $g(x, y) = x^2 + 2y^2 = 1$

$$\text{Since grad } f(x, y) = 2xi + 12y^2j$$

$$\text{Grad } g(x, y) = 2xi + 4yj$$

Use these equations to find x, y constraint $x^2 + 2y^2 = 1$ ----- (1)

$$\text{grad } f(x, y) = \lambda \text{grad } g(x, y)$$

$$2xi + 12y^2j = \lambda(2xi + 4yj)$$

$$2x = \lambda(2x) \text{ ----- (2)}$$

$$12y^2 = 4y\lambda \text{ ----- (3)}$$

By equation (2), either $x = 0$ or $\lambda = 1$

$$\text{If } x = 0 \text{ from (1) } y = \pm \frac{1}{\sqrt{2}} \Rightarrow \left(0, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}\right)$$

$$\text{If } \lambda = 1 \text{ from (3) } 12y^2 = 4y \Rightarrow y = 0 \text{ or } y = \frac{1}{3}$$

$$\text{If } y = 0 \Rightarrow \text{from (1) } x = \pm 1 \Rightarrow (1, 0), (-1, 0)$$

$$\text{If } y = \frac{1}{3} \Rightarrow \text{from (1) } x = \pm \frac{\sqrt{7}}{3} \Rightarrow \left(\frac{\sqrt{7}}{3}, \frac{1}{3}\right), \left(-\frac{\sqrt{7}}{3}, \frac{1}{3}\right)$$

$\therefore$ The extreme values of f occurs at

$$\left(0, \frac{1}{\sqrt{2}}\right) \left(0, -\frac{1}{\sqrt{2}}\right) (1, 0) (-1, 0) \left(\frac{\sqrt{7}}{3}, \frac{1}{3}\right) \left(-\frac{\sqrt{7}}{3}, \frac{1}{3}\right)$$

$$\text{Now, } f\left(0, \frac{1}{\sqrt{2}}\right) = \sqrt{2}, \quad f\left(0, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$$

$$f(1, 0) = 1 = f(-1, 0); \quad f\left(\frac{\sqrt{7}}{3}, \frac{1}{3}\right) = \frac{25}{27} = f\left(-\frac{\sqrt{7}}{3}, \frac{1}{3}\right)$$

∴ The maximum value $\sqrt{2}$ occur at $(0, \frac{1}{\sqrt{2}})$

and minimum value occur at $-\sqrt{2}$ at $(0, -\frac{1}{\sqrt{2}})$

Example:- Let $f(x, y) = 3x^2 + 2y^2 - 4y + 1$ Find the extreme values of f as $x^2 + y^2 = 16$.

Solution:- Let $g(x, y) = x^2 + y^2 \Rightarrow g(x, y) = 16$

$$\text{grad } f(x, y) = 6xi + (4y - 4)j$$

$$\text{grad } g(x, y) = 2xi + 2yj$$

Write the equation to find x, y constraint $x^2 + y^2 = 16$ ----- (1)

$$\text{grad } f(x, y) = \lambda \text{ grad } g(x, y) \quad 6x = 2x\lambda \quad \text{----- (2)}$$

$$4y - 4 = 2y\lambda \quad \text{----- (3)}$$

from (2) $\Rightarrow x = 0$ or $\lambda = 3$

If $x = 0 \Rightarrow$ from (1) $y = \pm 4 \Rightarrow (0, 4), (0, -4)$

If $\lambda = 3$ from (3) $4y - 4 = 6y \Rightarrow y = -2$

If $y = -2 \Rightarrow$ from (1) $x = \pm\sqrt{12} \Rightarrow (\sqrt{12}, -2), (-\sqrt{12}, -2)$

∴ The extreme values on the circle are $(0, 4)$ $(0, -4)$ $(\sqrt{12}, -2)$ $(-\sqrt{12}, -2)$

$$f(0, 4) = 17, \quad f(0, -4) = 49$$

$$f(\sqrt{12}, -2) = 53 = f(-\sqrt{12}, -2)$$

The maximum value is 53 and the minimum value is 17

CHAPTER 6

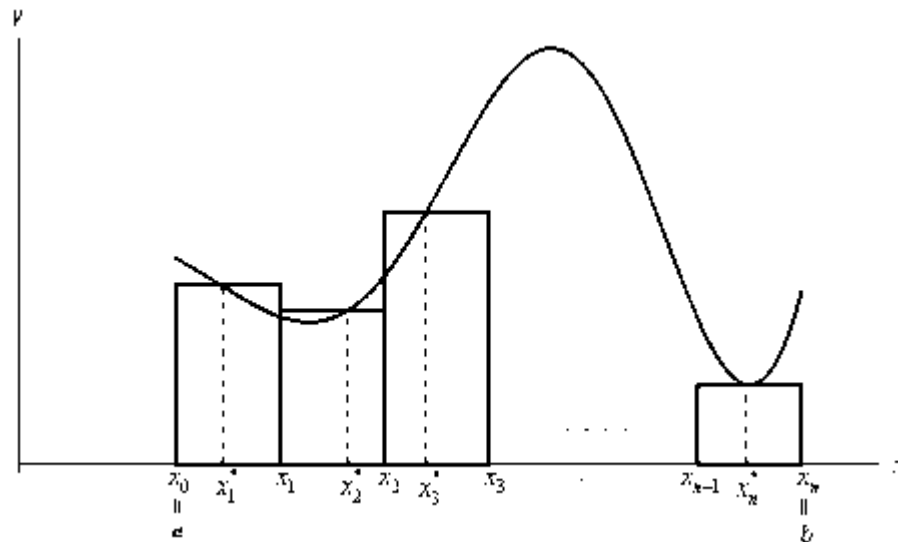
6.1 Double Integrals

Before starting on double integrals let's do a quick review of the definition of a definite integrals for functions of single variables. First, when working with the integral,

$$\int_a^b f(x) dx$$

We think of x 's as coming from the interval $a \leq x \leq b$. For these integrals we can say that we are integrating over the interval $a \leq x \leq b$. Note that this does assume that $a < b$, however, if we have $b < a$ then we can just use the interval $b \leq x \leq a$.

Now, when we derived the definition of the definite integral we first thought of this as an area problem. To find the area under the curve, break up the interval $a \leq x \leq b$ into n subintervals of width Δx and choose a point, x_i^* , from each interval as shown below,



Each of the rectangles has height of $f(x_i^*)$ and we could then use the area of each of these rectangles to approximate the area as follows.

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_i^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

To get the exact area we then take the limit as n goes to infinity and this is also the definition of the definite integral. $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$

In this section we want to integrate a function of two variables, $f(x, y)$. With functions of one variable we integrated over an interval (*i.e.* a one-dimensional space) and so it makes some

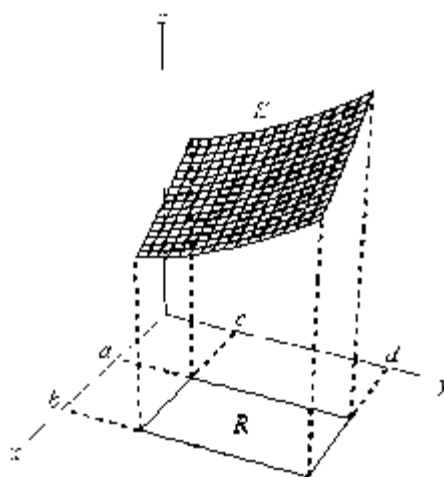
sense then that when integrating a function of two variables we will integrate over a region of $\mathbb{R}^2$ (two dimensional space).

We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

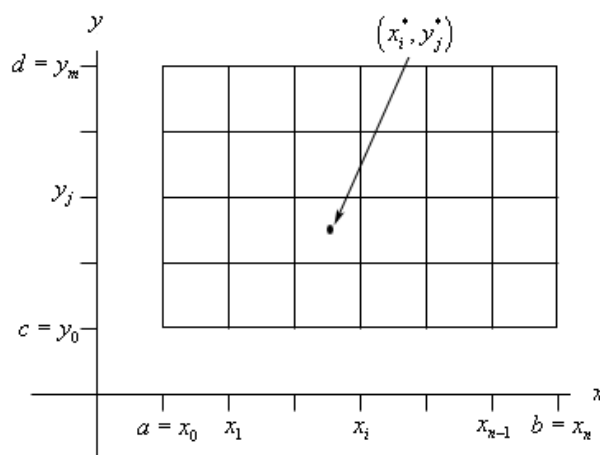
$$R = [a, b] \times [c, d]$$

This means that the ranges for x and y are $a \leq x \leq b$ and $c \leq y \leq d$.

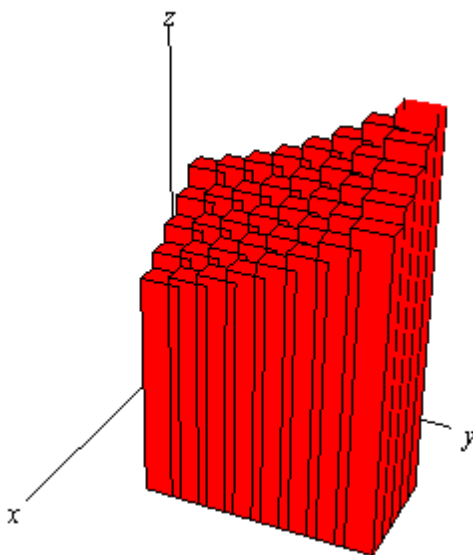
Also, we will initially assume that $f(x, y) \geq 0$ although this doesn't really have to be the case. Let's start with the graph of the surface S given by graphing $f(x, y)$ over the rectangle R .



Now, just like with functions of one variable let's not worry about integrals quite yet. Let's first find the volume of the region under S (and above the xy – plane of course). We will first approximate the volume much as we approximated the area above. We will first divide $a \leq x \leq b$ into n subintervals and divide $c \leq y \leq d$ into m subintervals. This will divide R into a series of smaller rectangles and from each of these we will choose a point (x_i^*, y_j^*) . Here is sketch of this set up.



Now, over each of these smaller rectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$. Here is a sketch of that.



Each of the rectangles has a base area of ΔA and a height of $f(x_i^*, y_j^*)$ so the volume of each of these boxes is $f(x_i^*, y_j^*) \Delta A$. The volume under the surface S is then approximately,

$$V \approx \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

We will have a *double* sum since we will need to add up volumes in both the x and y directions.

To get a better estimation of the volume we will take n and m larger and larger and to get the exact volume we will need to take the limit as both n and m go to infinity. In other words,

$$V = \lim_{n,m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

Now, this should look familiar. This looks a lot like the definition of the integral of a function of single variable. In fact this is also the definition of a double integral, or more exactly an integral of a function of two variables over a rectangle.

Here is the official definition of a double integral of a function of two variables over a rectangular region R as well as the notation that we'll use for it.

$$\iint_R f(x,y) dA = \lim_{n,m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

Note the similarities and differences in the notation to single integrals. We have two integrals to denote the fact that we are dealing with a two dimensional region and we have a differential here as well. Note that the differential is dA instead of the dx and dy that we're used to seeing.

Note as well that we don't have limits on the integrals in this notation. Instead we have the R written below the two integrals to denote the region that we are integrating over.

Note that one interpretation of the double integral of $f(x, y)$ over the rectangle R is the volume under the function $f(x, y)$ (and above the xy – plane). Or,

$$\text{Volume} = \iint_R f(x, y) dA$$

We can use this double sum in the definition to estimate the value of a double integral if we need to. We can do this by choosing (x_i^*, y_j^*) to be the midpoint of each rectangle. When we do this we usually denote the point as $(\bar{x}_i, \bar{y}_j)$. This leads to the **Midpoint Rule**,

$$\iint_R f(x, y) dA \approx \sum_{i=1}^n \sum_{j=1}^m f(\bar{x}_i, \bar{y}_j) \Delta A$$

Iterated Integrals

In the previous section we gave the definition of the double integral. However, just like with the definition of a single integral the definition is very difficult to use in practice and so we need to start looking into how we actually compute double integrals. We will continue to assume that we are integrating over the rectangle.

$$R = [a, b] \times [c, d]$$

We will look at more general regions in the next section.

The following theorem tells us how to compute a double integral over a rectangle.

Fubini's Theorem

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

These integrals are called **iterated integrals**.

Note that there are in fact two ways of computing a double integral and also notice that the inner differential matches up with the limits on the inner integral and similarly for the outer differential and limits. In other words, if the inner differential is dy then the limits on the inner integral must be y limits of integration and if the outer differential is dx then the limits on the outer integral must be x limits of integration.

Now, on some level this is just notation and doesn't really tell us how to compute the double integral. Let's just take the first possibility above and change the notation a little.

$$\iint_R f(x, y) dA = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

We will compute the double integral by first computing $\int_c^d f(x, y) dy$ and we compute this by holding x constant and integrating with respect to y as if this were a single integral. This will give a function involving only x 's which we can in turn integrate.

We've done a similar process with partial derivatives. To take the derivative of a function with respect to y we treated the x 's as constants and differentiated with respect to y as if it was a function of a single variable.

Double integrals work in the same manner. We think of all the x 's as constants and integrate with respect to y or we think of all y 's as constants and integrate with respect to x .

Let's take a look at some examples.

6.2 Double integrals over rectangular regions

Example 1. Compute each of the following double integrals over the indicated rectangles

$$(a) \iint_R 6xy^2 dA, \quad R = [2, 4] \times [1, 2]$$

$$(b) \iint_R \{x^2y^2 + \cos(\pi x) + \sin(\pi y)\} dA, \quad R = [-2, -1] \times [0, 1]$$

$$(c) \iint_R \frac{1}{(2x+3y)^2} dA, \quad R = [0, 1] \times [1, 2]$$

$$(d) \iint_R [2x - 4y^3] dA, \quad R = [-5, 4] \times [0, 3]$$

Solutions:- a) $\iint_R 6xy^2 dA, \quad R = [2, 4] \times [1, 2]$

It doesn't matter which variable we integrate with respect to first, we will get the same answer regardless of the order of integration. To prove that let's work this one with each order to make sure that we do get the same answer.

Method 1:- In this case we will integrate with respect to y first. So, the iterated integral that we

$$\text{need to compute is, } \iint_R 6xy^2 dA = \int_2^4 \int_1^2 6xy^2 dy dx$$

When setting these up make sure the limits match up to the differentials. Since the dy is the inner differential (*i.e.* we are integrating with respect to y first) the inner integral needs to have y limits. To compute this we will do the inner integral first and we typically keep the outer integral around as follows,

$$\begin{aligned} \iint_R 6xy^2 dA &= \int_2^4 (2xy^3) \Big|_1^2 dx \\ &= \int_2^4 (16x - 2x) dx \\ &= \int_2^4 14x dx \end{aligned}$$

Remember that we treat the x as a constant when doing the first integral and we don't do any integration with it yet. Now, we have a normal single integral so let's finish the integral by computing this.

$$\iint_R 6xy^2 dA = 7x^2 \Big|_2^4 = 84$$

Method 2:- In this case we'll integrate with respect to x first and then y . Here is the work for this solution.

$$\begin{aligned}\iint_R 6xy^2 dA &= \int_1^2 \int_2^4 6xy^2 dx dy \\ &= \int_1^2 3x^2 y^2 \Big|_2^4 dy \\ &= \int_1^2 36y^2 dy \\ &= 12y^3 \Big|_1^2 \\ &= 84\end{aligned}$$

Sure enough the same answer as the first solution.

So, remember that we can do the integration in any order.

(a) $\iint_R \{x^2 y^2 + \cos(\pi x) + \sin(\pi y)\} dA, \quad R = [-2, -1] \times [0, 1]$

In this case we will integrate with respect to x first.

$$\begin{aligned}\iint_R \{x^2 y^2 + \cos(\pi x) + \sin(\pi y)\} dA &= \int_0^1 \int_{-2}^{-1} \{x^2 y^2 + \cos(\pi x) + \sin(\pi y)\} dx dy \\ &= \int_0^1 \left[\frac{1}{3} x^3 y^2 + \frac{1}{\pi} \sin(\pi x) + x \sin(\pi y) \right] \Big|_{-2}^{-1} dy \\ &= \int_0^1 \left[\frac{7}{3} y^2 + \sin(\pi y) \right] dy \\ &= \left[\frac{7}{9} y^3 - \frac{1}{\pi} \cos(\pi y) \right] \Big|_0^1 \\ &= \frac{7}{9} + \frac{2}{\pi}\end{aligned}$$

(b) $\iint_R \frac{1}{(2x+3y)^2} dA, \quad R = [0, 1] \times [1, 2]$

In this case because the limits for x are kind of nice (*i.e.* they are zero and one which are often nice for evaluation) let's integrate with respect to x first. We'll also rewrite the integrand to help with the first integration.

$$\iint_R \frac{1}{(2x+3y)^2} dA = \iint_R (2x+3y)^{-2} dA$$

$$\begin{aligned}
&= \int_1^2 \int_0^1 (2x + 3y)^{-2} dx dy \\
&= \int_1^2 \left[-\frac{1}{2} (2x + 3y)^{-1} \right] \Big|_0^1 dy \\
&= -\frac{1}{2} \int_1^2 \left[\frac{1}{2+3y} - \frac{1}{3y} \right] dy \\
&= -\frac{1}{2} \left[\frac{1}{3} \ln(2 + 3y) - \frac{1}{3} \ln|y| \right] \Big|_1^2 \\
&= -\frac{1}{6} (\ln 8 - \ln 2 - \ln 5)
\end{aligned}$$

(c) $\iint_R [2x - 4y^3] dA$, $R = [-5, 4] \times [0, 3]$

For this integral we'll integrate with respect to y first.

$$\begin{aligned}
\iint_R [2x - 4y^3] dA &= \int_{-5}^4 \int_0^3 [2x - 4y^3] dy dx \\
&= \int_{-5}^4 [2xy - y^4] \Big|_0^3 dx \\
&= \int_{-5}^4 [6x - 81] dx \\
&= [3x^2 - 81x] \Big|_{-5}^4 \\
&= -756
\end{aligned}$$

Remember that when integrating with respect to y all x's are treated as constants and so as far as the inner integral is concerned the 2x is a constant and we know that when we integrate constants with respect to y we just tack on a y and so we get 2xy from the first term.

As we saw in the previous set of examples we can do the integral in either direction. However, sometimes one direction of integration is significantly easier than the other so make sure that you think about which one you should do first before actually doing the integral.

The next topic of this section is a quick fact that can be used to make some iterated integrals somewhat easier to compute on occasion.

Fact

If $f(x, y) = g(x)h(y)$ and we are integrating over the rectangle $R = [a, b] \times [c, d]$ then

$$\iint_R f(x, y) dA = \iint_R [g(x)h(y)] dA = \left[\int_a^b g(x) dx \right] \left[\int_c^d h(y) dy \right]$$

So, if we can break up the function into a function only of x times a function of y then we can do the two integrals individually and multiply them together.

Let's do a quick example using this integral.

Example 2 :- Evaluate $\iint_R x \cos^2(y) dA$, $R = [-2, 3] \times [0, \frac{\pi}{2}]$

Solution:- Since the integrand is a function of x times a function of y we can use the fact.

$$\begin{aligned} \iint_R x \cos^2(y) dA &= \left[\int_{-2}^3 x dx \right] \left[\int_0^{\pi/2} \cos^2(y) dy \right] \\ &= \left\{ \left[\frac{1}{2} x^2 \right]_{-2}^3 \right\} \left\{ \frac{1}{2} \int_0^{\pi/2} [1 + \cos(2y)] dy \right\} \\ &= \left[\frac{5}{2} \right] \left\{ \frac{1}{2} \left[y + \frac{1}{2} \sin(2y) \right] \right\} \Big|_0^{\pi/2} \\ &= \frac{5\pi}{8} \end{aligned}$$

Exercise

- Let R be the rectangular region bounded by the lines $x = -1$, $x = 2$, $y = 0$ and $y = 2$ then find $\iint_R x^2 y dA$
- Evaluate $\iint_R x dA$, $R = [1, 2] \times [0, 3]$
- Evaluate $\iint_R 7x^2 y dA$, $R = [2, 3] \times [1, 2]$
- Evaluate $\iint_R 3x^2 y^3 dA$, $R = [1, 3] \times [0, 2]$
- Evaluate $\iint_R \frac{1}{(3x-4y)^2} dA$, $R = [2, 3] \times [0, 1]$
- Evaluate $\iint_R x^2 \sin^2(y) dA$, $R = [-1, 2] \times [0, \pi]$
- Evaluate $\iint_R (4x^2 - 2y) dA$, $R = [0, 1] \times [2, 3]$

Answers

$$a. 6 \quad b. \frac{9}{2} \quad c. \frac{133}{2} \quad d. 104 \quad e. \frac{-1}{12} (\ln 5 - \ln 3) \quad f. \frac{3\pi}{2} \quad g. \frac{-11}{3}$$

6.3 Double Integrals in Polar Coordinates

In this section we want to look at some regions that are much easier to describe in terms of polar coordinates. For instance, we might have a region that is a disk, ring, or a portion of a disk or ring. In these cases using Cartesian coordinates could be somewhat cumbersome. For instance let's suppose we wanted to do the following integral,

$$\iint_D f(x, y) dA, \quad D \text{ is the disk of radius } 2$$

To this we would have to determine a set of inequalities for x and y that describe this region.

These would be,

$$\begin{aligned} -2 \leq x \leq 2 \\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} \end{aligned}$$

With these limits the integral would become

$$\iint_D f(x,y) dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y) dy dx$$

Due to the limits on the inner integral this is liable to be an unpleasant integral to compute.

However, a disk of radius 2 can be defined in polar coordinates by the following inequalities,

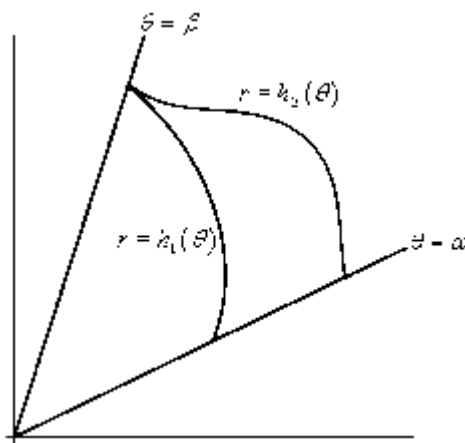
$$\begin{aligned} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 \end{aligned}$$

These are very simple limits and, in fact, are constant limits of integration which almost always makes integrals somewhat easier.

So, if we could convert our double integral formula into one involving polar coordinates we would be in pretty good shape. The problem is that we can't just convert the dx and the dy into a dr and a $d\theta$. In computing double integrals to this point we have been using the fact that $dA = dx dy$ and this really does require Cartesian coordinates to use. Once we've moved into polar coordinates

$dA \neq dr d\theta$ and so we're going to need to determine just what dA is under polar coordinates.

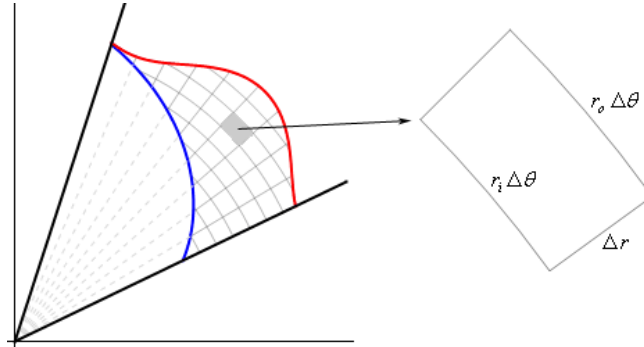
Here is a sketch of some region using polar coordinates.



So, our general region will be defined by inequalities, $\alpha \leq \theta \leq \beta$

$$h_1(\theta) \leq r \leq h_2(\theta)$$

Now, to find dA let's redo the figure above as follows,



As shown, we'll break up the region into a mesh of radial lines and arcs. Now, if we pull one of the pieces of the mesh out as shown we have something that is almost, but not quite a rectangle.

The area of this piece is ΔA . The two sides of this piece both have length $\Delta r = r_o - r_i$ where r_o is the radius of the outer arc and r_i is the radius of the inner arc. Basic geometry then tells us that the length of the inner edge is $r_i \Delta \theta$ while the length of the out edge is $r_o \Delta \theta$ where $\Delta \theta$ is the angle between the two radial lines that form the sides of this piece.

Now, let's assume that we've taken the mesh so small that we can assume that $r_i \approx r_o = r$ and with this assumption we can also assume that our piece is close enough to a rectangle that we can also then assume that,

$$\Delta A \approx r \Delta \theta \Delta r$$

Also, if we assume that the mesh is small enough then we can also assume that,

$$dA \approx \Delta A \quad d\theta \approx \Delta \theta \quad dr \approx \Delta r$$

With these assumes we then get $dA \approx r dr d\theta$

In order to arrive at this we had to make the assumption that the mesh was very small. This is not an unreasonable assumption. *Recall* that the definition of a double integral is in terms of two limits and as limits go to infinity the mesh size of the region will get smaller and smaller. In fact, as the mesh size gets smaller and smaller the formula above becomes more and more accurate and so we can say that,

$$dA = r dr d\theta$$

Now, if we're going to be converting an integral in Cartesian coordinates into an integral in polar coordinates we are going to have to make sure that we've also converted all the x 's and y 's into polar coordinates as well. To do this we'll need to remember the following conversion formulas,

$$x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2$$

We are ready to write down a formula for the double integral in terms of polar coordinates.

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

It is important to not forget the added r and don't forget to convert the Cartesian coordinates in the function over to polar coordinates.

Let's look at a couple of examples of these kinds of integrals.

Example 1 Evaluate the following integrals by converting them into polar coordinates.

- $\iint_D 2xy \, dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.
- $\iint_D e^{x^2+y^2} \, dA$, D is the unit circle centered at the origin.

Solution

(a). $\iint_D 2xy \, dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

First let's get D in terms of polar coordinates. The circle of radius 2 is given by $r = 2$ and the circle of radius 5 is given by $r = 5$. We want the region between them so we will have the following inequality for r .

$$2 \leq r \leq 5$$

Also, since we only want the portion that is in the first quadrant we get the following range of θ 's.

$$0 \leq \theta \leq \frac{\pi}{2}$$

Now that we've got these we can do the integral.

$$\iint_D 2xy \, dA = \int_0^{\pi/2} \int_2^5 2(r\cos\theta)(r\sin\theta)r \, dr \, d\theta$$

Don't forget to do the conversions and to add in the extra r . Now, let's simplify and make use of the double angle formula for sine to make the integral a little easier.

$$\begin{aligned}\iint_D 2xy \, dA &= \int_0^{\pi/2} \int_2^5 r^3 \sin(2\theta) \, dr \, d\theta \\ &= \int_0^{\pi/2} \left[\frac{1}{4} r^4 \sin(2\theta) \right]_2^5 d\theta \\ &= \int_0^{\pi/2} \frac{609}{4} \sin(2\theta) \, d\theta \\ &= \left[-\frac{609}{8} \cos(2\theta) \right]_0^{\pi/2} \\ &= \frac{609}{4}\end{aligned}$$

(b). $\iint_D e^{x^2+y^2} \, dA$, D is the unit circle centered at the origin.

In this case we can't do this integral in terms of Cartesian coordinates. We will however be able to do it in polar coordinates. First, the region D is defined by,

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

In terms of polar coordinates the integral is then,

$$\iint_D e^{x^2+y^2} \, dA = \int_0^{2\pi} \int_0^1 r e^{r^2} \, dr \, d\theta$$

Notice that the addition of the r gives us an integral that we can now do. Here is the work for this integral.

$$\begin{aligned}\iint_D e^{x^2+y^2} \, dA &= \int_0^{2\pi} \int_0^1 r e^{r^2} \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{2} e^{r^2} \right]_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{2} (e - 1) \, d\theta \\ &= \pi(e - 1)\end{aligned}$$

There is one more type of example that we need to look at before moving on to the next section.

Sometimes we are given an iterated integral that is already in terms of x and y and we need to convert this over to polar so that we can actually do the integral. We need to see an example of how to do this kind of conversion.

Example2 Evaluate the following integral by first converting to polar coordinates.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2) dx dy$$

Solution

First, notice that we cannot do this integral in Cartesian coordinates and so converting to polar coordinates may be the only option we have for actually doing the integral. Notice that the function will convert to polar coordinates nicely and so shouldn't be a problem.

Let's first determine the region that we're integrating over and see if it's a region that can be easily converted into polar coordinates. Here are the inequalities that define the region in terms of Cartesian coordinates.

$$\begin{aligned} 0 &\leq y \leq 1 \\ 0 &\leq x \leq \sqrt{1-y^2} \end{aligned}$$

Now, the upper limit for the x 's is,

$$x = \sqrt{1-y^2}$$

and this looks like the right side of the circle of radius 1 centered at the origin. Since the lower limit for the x 's is $x = 0$ it looks like we are going to have a portion (or all) of the right side of the disk of radius 1 centered at the origin.

The range for the y 's however, tells us that we are only going to have positive y 's. This means that we are only going to have the portion of the disk of radius 1 centered at the origin that is in the first quadrant.

So, we know that the inequalities that will define this region in terms of polar coordinates are then,

$$0 \leq \theta \leq \frac{\pi}{2} \quad 0 \leq r \leq 1$$

Finally, we just need to remember that,

$$dx dy = dA = r dr d\theta$$

and so the integral becomes,

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^1 r \cos(r^2) dr d\theta$$

Note that this is an integral that we can do. So, here is the rest of the work for this integral.

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2) dx dy &= \int_0^{\pi/2} \frac{1}{2} \sin(r^2) \Big|_0^1 d\theta \\ &= \int_0^{\pi/2} \frac{1}{2} \sin(1) d\theta \\ &= \frac{\pi}{4} \sin(1) \end{aligned}$$

Exercise

- Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.
- Evaluate $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2 + y^2) dx dy$ by changing into polar coordinates.
- If D is the part of the annulus $0 < a^2 \leq x^2 + y^2 \leq b^2$ lying in the first quadrant and below the line $y = x$, evaluate $\iint_D \frac{y^2}{x^2} dA$
- Find the volume of the solid lying in the first octant inside the cylinder $x^2 + y^2 = a^2$ and under the plane $z = y$.
- Suppose D is the region bounded by the circles $r = 1$, $r = 2$ and $0 \leq \theta \leq 2\pi$, Find $\iint_D (3x + 8y^2) dA$
- Suppose D is the region bounded by the circles $r = 0$, $r = 2$ and $0 \leq \theta \leq 2\pi$ then evaluate $\iint_D (4 - x^2 - y^2) dA$

Answers

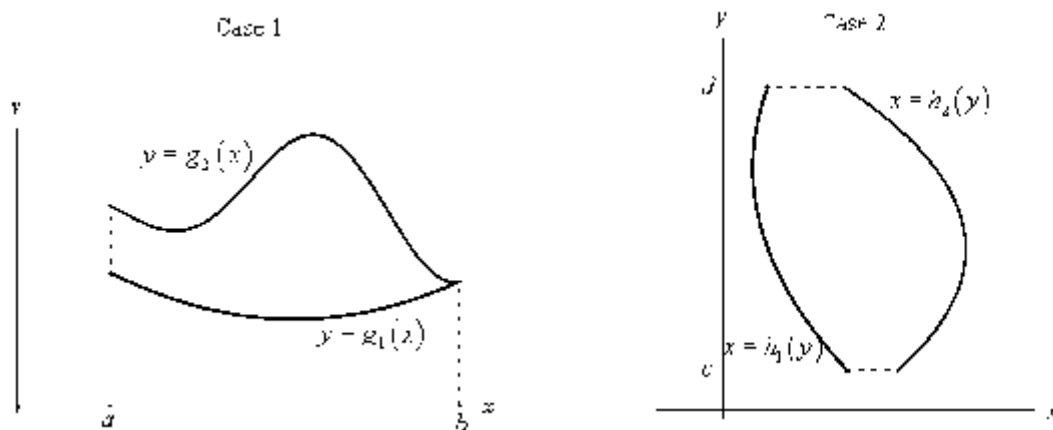
- (a) $\frac{38\pi}{3}$ (b) $\frac{3}{4}a^4\pi$ (c) $\frac{4-\pi}{8}(b^2 - a^2)$ (d) $\frac{1}{3}a^3$
 (e) 30π (f) 8π

6.4 Double Integrals over General Regions

In the previous section we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint_D f(x,y) dA \text{ Where } D \text{ is any region}$$

There are two types of regions that we need to look at. Here is a sketch of both of them.



We will often use set builder notation to describe these regions. Here is the definition for the region in Case 1

$$D = \{(x,y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

and here is the definition for the region in Case 2.

$$D = \{(x,y) | h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$$

This notation is really just a fancy way of saying we are going to use all the points, (x, y) , in which both of the coordinates satisfy the two given inequalities.

The double integral for both of these cases are defined in terms of iterated integrals as follows.

In Case 1 where $D = \{(x,y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ the integral is defined to be

$$\iint_D f(x,y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$

In Case 2 where $D = \{(x,y) | h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$ the integral is defined to be

$$\iint_D f(x,y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy$$

Here are some properties of the double integral that we should go over before we actually do some examples. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Properties

1. $\iint_D \{f(x,y) + g(x,y)\} dA = \iint_D f(x,y) dA + \iint_D g(x,y) dA$
2. $\iint_D c f(x,y) dA = c \iint_D f(x,y) dA$, where c is any constant.
3. If the region D can be split in to two separate regions D_1 and D_2 then the integral written as $\iint_D f(x,y) dA = \iint_{D_1} f(x,y) dA + \iint_{D_2} f(x,y) dA$

Let us take a look at some examples of double integrals over general regions

Example 1 Evaluate each of the following integrals over the given region D .

- a. $\iint_D e^{x/y} dA$, $D = \{(x,y) | 1 \leq y \leq 2, y \leq x \leq y^3\}$
- b. $\iint_D (4xy - y^3) dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$
- c. $\iint_D (6x^2 - 40y) dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

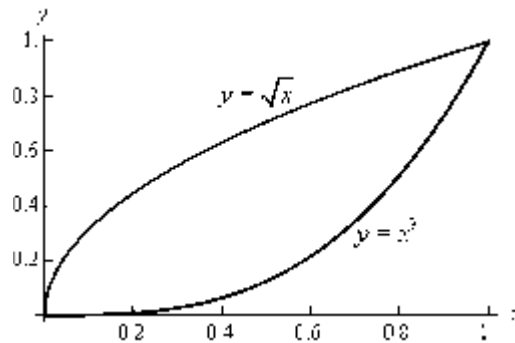
Solutions:- a. $\iint_D e^{x/y} dA$, $D = \{(x,y) | 1 \leq y \leq 2, y \leq x \leq y^3\}$

Okay, this first one is set up to just use the formula above so let's do that.

$$\begin{aligned}\iint_D e^{x/y} dA &= \int_1^2 \int_y^{y^3} e^{x/y} dx dy = \int_1^2 y e^{x/y} \Big|_y^{y^3} dy \\ &= \int_1^2 [y e^{y^2} - y e] dy \\ &= \left(\frac{1}{2} e^{y^2} - \frac{1}{2} e y^2 \right) \Big|_1^2 = \frac{1}{2} e^4 - 2e\end{aligned}$$

b. $\iint_D (4xy - y^3) dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$

In this case we need to determine the two inequalities for x and y that we need to do the integral. The best way to do this is the graph the two curves. Here is a sketch.



So, from the sketch we can see that the two inequalities are

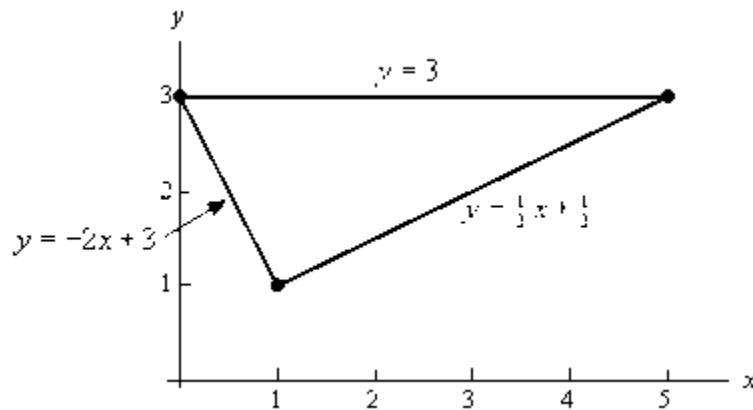
$$0 \leq x \leq 1 \quad x^3 \leq y \leq \sqrt{x}$$

We can now do the integral

$$\begin{aligned}\iint_D (4xy - y^3) dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3) dy dx \\ &= \int_0^1 \left(2xy^2 - \frac{1}{4} y^4 \right) \Big|_{x^3}^{\sqrt{x}} dx \\ &= \int_0^1 \left(\frac{7}{4} x^2 - 2x^7 + \frac{1}{4} x^{12} \right) dx \\ &= \left(\frac{7}{12} x^3 - \frac{1}{4} x^8 + \frac{1}{52} x^{13} \right) \Big|_0^1 = \frac{55}{156}\end{aligned}$$

c. $\iint_D (6x^2 - 40y) dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

We got even less information about the region this time. Let's us start this off by sketching the triangle.



Since we have two points on each edge it is easy to get the equations for each edge and so we'll leave it to you to verify the equations.

Now, there are two ways to describe this region. If we use functions of x , as shown in the image we will have to break the region up into two different pieces since the lower function is different depending upon the value of x . In this case the region would be given by $D = D_1 \cup D_2$ where

$$D_1 = \{(x, y) \mid 0 \leq x \leq 1, -2x + 3 \leq y \leq 3\}$$

$$D_2 = \{(x, y) \mid 1 \leq x \leq 5, \frac{1}{2}x + \frac{1}{2} \leq y \leq 3\}$$

Note the $\cup$ is the "union" symbol just means that D is the region we get by combining the two regions. If we do this then we'll need to do two separate integrals, one for each of the regions.

To avoid this we could turn things around and solve the two equations for x to get,

$$y = -2x + 3 \quad \Rightarrow \quad x = \frac{-1}{2}y + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2} \quad \Rightarrow \quad x = 2y - 1$$

If we do this we can notice that the same function is always on the right and the same function is always on the left and so the region is,

$$D = \{(x, y) \mid \frac{-1}{2}y + \frac{3}{2} \leq x \leq 2y - 1, 1 \leq y \leq 3\}$$

Writing the region in this form means doing a single integral instead of the two integrals we'd have to do otherwise.

Either way should give the same answer and so we can get an example in the notes of splitting a region up let's do both integrals.

Solution 1:- $\iint_D (6x^2 - 40y) dA = \iint_{D_1} (6x^2 - 40y) dA + \iint_{D_2} (6x^2 - 40y) dA$

$$= \int_0^1 \int_{-2x+3}^3 (6x^2 - 40y) dy dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^3 (6x^2 - 40y) dy dx$$

$$= \int_0^1 (6x^2y - 20y^2) \Big|_{-2x+3}^3 dx + \int_1^5 (6x^2y - 20y^2) \Big|_{\frac{1}{2}x+\frac{1}{2}}^3 dx$$

$$= \int_0^1 [12x^3 - 180 + 20(3 - 2x)^2] dx + \int_1^5 [-3x^3 + 15x^2 - 180 + 20(\frac{1}{2}x + \frac{1}{2})^2] dx$$

$$= \left[3x^4 - 180x - \frac{10}{3}(3 - 2x)^3 \right] \Big|_0^1 + \left[-\frac{3}{4}x^4 + 5x^3 - 180x + \frac{40}{3}(\frac{1}{2}x + \frac{1}{2})^3 \right] \Big|_1^5$$

$$= -\frac{935}{3}$$

Solution 2: This solution will be a lot less work since we are only going to do a single integral.

$$\iint_D (6x^2 - 40y) dA = \int_1^3 \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} (6x^2 - 40y) dx dy$$

$$= \int_1^3 (2x^3 - 40xy) \Big|_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} dy$$

$$= \int_1^3 [100y - 100y^2 + 2(2y - 1)^3 - 2(-\frac{1}{2}y + \frac{3}{2})^3] dy$$

$$= [50y^2 - \frac{100}{3}y^3 + \frac{1}{4}(2y - 1)^4 + (-\frac{1}{2}y + \frac{3}{2})^4] \Big|_1^3$$

$$= -\frac{935}{3}$$

Exercise

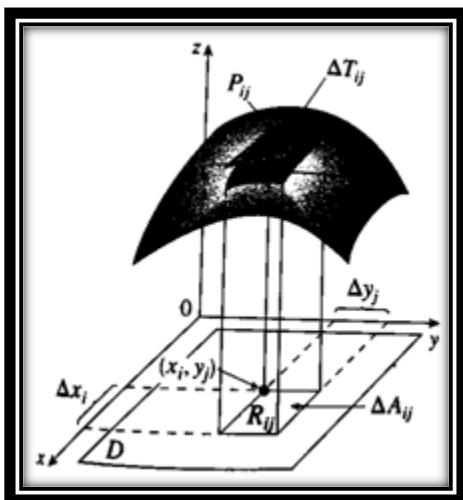
- a. Evaluate $\iint_D xy \, dA$, $D = \{(x,y) | 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$
- b. Evaluate $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$
- c. Evaluate $\iint_D x \cos y \, dA$, where D is bounded by $y = 0$, $y = x^2$, and $x = 1$

Answers

a. $\frac{1}{12}$ b. $\frac{32}{15}$ c. $\frac{1 - \cos 1}{2}$

6.5 Surface Areas

In this section we apply double integral to the problem of computing the area of a surface area. We found the area of a very special type of surface – a surface of revolution – by the method of single-variable calculus. Here we compute the area of a surface with equation $z = f(x, y)$, the graph of a function of two variables.



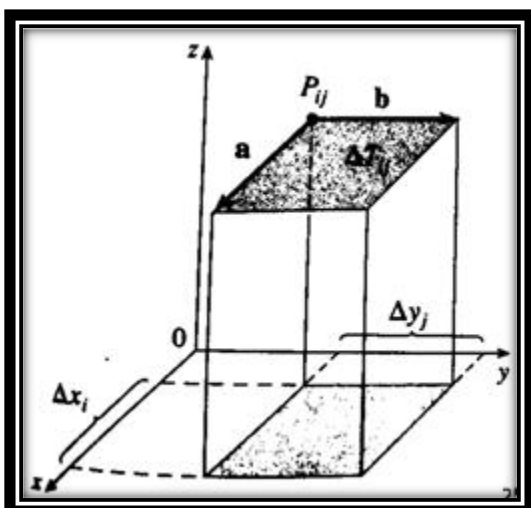
Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. For simplicity in deriving the surface area formula, we assume that $f(x, y) \geq 0$ and the domain D of f is a rectangle. We consider a partition P of D into small rectangles R_{ij} with areas $\Delta A_{ij} = \Delta x_i \Delta y_j$. If (x_i, y_j) is the corner of R_{ij} closest to the origin, let $P_{ij}[x_i, y_j, f(x_i, y_j)]$ be the point on S directly above it. The tangent plane to S at P_{ij} is an approximation to S near P_{ij} . So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij} . Thus the sum $\sum \sum \Delta T_{ij}$ is an

approximation to the total area of S , and this approximation appears to improve as $\|P\| \rightarrow 0$. Therefore, we define the **surface area** of S to be

(1)

$$A(S) = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n \Delta T_{ij}$$

To find a formula that is more convenient than Equation 1 for computational purpose, we let $\mathbf{a}$ and $\mathbf{b}$ be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij} (see in following figure). Then $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$ and $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the direction of $\mathbf{a}$ and $\mathbf{b}$. Therefore



$$\mathbf{a} = \Delta x_i \mathbf{i} + f_x(x_i, y_j) \Delta x_i \mathbf{k}$$

$$\mathbf{b} = \Delta y_j \mathbf{j} + f_y(x_i, y_j) \Delta y_j \mathbf{k}$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_j) \Delta x_i \\ 0 & \Delta y_j & f_y(x_i, y_j) \Delta y_j \end{vmatrix}$$

$$= -f_x(x_i, y_j) \Delta x_i \Delta y_j \mathbf{i} - f_y(x_i, y_j) \Delta x_i \Delta y_j \mathbf{j} + \Delta x_i \Delta y_j \mathbf{k}$$

$$= [-f_x(x_i, y_j) \mathbf{i} - f_y(x_i, y_j) \mathbf{j} + \mathbf{k}] \Delta A_{ij}$$

Thus
$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1} \Delta A_{ij}$$

From definition 1 we then have

$$\begin{aligned} A(S) &= \lim_{\|P\| \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n \Delta T_{ij} \\ &= \lim_{\|P\| \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1} \Delta A_{ij} \end{aligned}$$

and by the definition of a double integral we get the following formula

The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA$$

Example 1 Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy -plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Solution

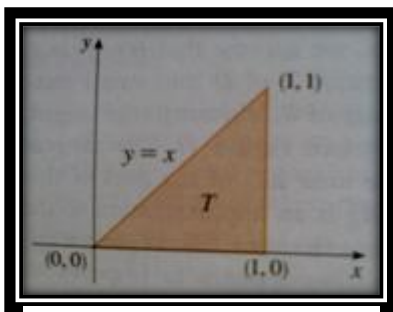


Figure 1



Figure 2

The region shown in figure 1 and is described by $T = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$

Using surface area formula with $f(x, y) = x^2 + 2y$, we get

$$\begin{aligned} A &= \iint_T \sqrt{(2x)^2 + (2)^2 + 1} \, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5} \, dy \, dx \\ &= \int_0^1 x \sqrt{4x^2 + 5} \, dx = \frac{1}{8} \cdot \frac{2}{3} (4x^2 + 5)^{3/2} \Big|_0^1 = \frac{1}{12} (27 - 5\sqrt{5}) \end{aligned}$$

Figure 2 shows the portion of the surface whose area we have just computed.

Example 2:-

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

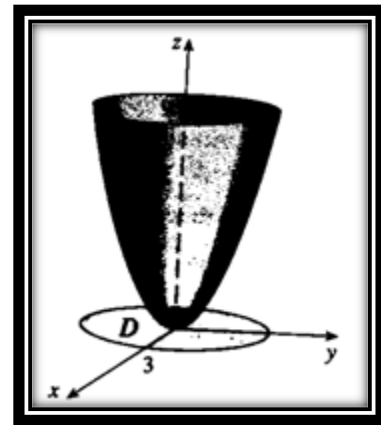
Solution:- The plane intersects the paraboloid in the circle $x^2 + y^2 = 9, z = 9$. Therefore, the given surface lies above the disk D with center the origin and radius 3.

Using the surface area formula, we obtain

$$A = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1} \, dA = \iint_D \sqrt{4(x^2 + y^2) + 1} \, dA$$

Converting to polar coordinates, we obtain

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} \, r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^3 \frac{1}{8} \sqrt{4r^2 + 1} (8r) \, dr \\ &= 2\pi \left(\frac{1}{8} \right) \frac{2}{3} (4r^2 + 1)^{3/2} \Big|_0^3 = \frac{\pi}{6} (37\sqrt{37} - 1) \end{aligned}$$



Exercise

- Let R be the rectangular region bounded by the lines $x = 0, x = 3, y = 0$, and $y = 2$ and let $f(x, y) = \frac{2}{3}x^{3/2}$, find surface area over the region R .
- Find surface area S of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above xy plane.
- Find the surface area S of the portion of the paraboloid $z = 9 - x^2 - y^2$ above xy plane.
- Find the surface area S of the portion of the paraboloid $z = 9 - x^2 - y^2$ above the plane $z = 5$

Answers

- i. $\frac{28}{3}$ ii. $\frac{1}{6}\pi(17^{3/2} - 1)$ iv. $\frac{\pi}{6}(17^{3/2} - 1)$

6.6 TRIPLE INTEGRALS

Fubini's Theorem for Triple Integrals

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z .

Example 1 Evaluate the triple integral $\iiint_B xyz^2 dV$ where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$$

Solution: We first integrate with respect x , then y , and then z , we obtain

$$\begin{aligned} \iiint_B xyz^2 dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dx dy dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2 y z^2}{2} \right]_{x=0}^{x=1} dy dz \\ &= \int_0^3 \int_{-1}^2 \left[\frac{y z^2}{2} \right] dy dz = \int_0^3 \left[\frac{y^2 z^2}{4} \right]_{y=-1}^{y=2} dz \\ &= \int_0^3 \left[\frac{3z^2}{4} \right] dz = \left[\frac{z^3}{4} \right]_{z=0}^{z=3} = \frac{27}{4} \end{aligned}$$

Exercise

Evaluate $\int_0^1 \int_0^z \int_0^y xyz dx dy dz$

- Evaluate $\int_0^\pi \int_0^2 \int_0^{\sqrt{4-z^2}} z \sin y dx dz dy$
- Evaluate $\int_{-1}^1 \int_0^x \int_{x-y}^{x+y} (z - 2x - y) dz dy dx$
- Evaluate $\iiint_E yz dV$, where $E = \{(x, y, z) \mid 0 \leq z \leq 1, 0 \leq y \leq 2z, 0 \leq x \leq z + 2\}$

Answers

- $\frac{1}{48}$
- $\frac{16}{3}$
- 0
- $\frac{7}{5}$

6.7 Applications of Triple Integrals

Volume

We know that if $f(x) \geq 0$, then the single integral $\int_a^b f(x) dx$ represents the area under the curve $y = f(x)$ from a to b , and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y) dA$ represents the volume under the surface $z = f(x, y)$ and above D . The corresponding interpretation of a triple integral $\iiint_E f(x, y, z) dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be the “hypervolume” of a four dimensional object and, of course that is impossible to visualize. (Remember E is just the *domain* of the function f ; the graph of f lies in four dimensional space). Nonetheless, the triple integral $\iiint_E f(x, y, z) dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x , y , z and $f(x, y, z)$.

Let us begin with the special case where $f(x, y, z) = 1$ for all points in E . Then the triple integral does represents the volume of E :

$$V(E) = \iiint_E dV$$

Example 1 Use a triple integral to the volume of the tetrahedron T bounded by the planes

$$x + 2y + z = 2, x = 2y, x = 0, \text{ and } z = 0.$$

Solution

The tetrahedron T and its projection D on the xy -plane are shown in the following figures. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$. Therefore, we have

Figures 12, 13 page 851

$$\begin{aligned} V(T) &= \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz dy dx \\ &= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) dy dx = \frac{1}{3} \end{aligned}$$