



SECOND
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TIONS

By
Dagnachew
Jenber

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

By
Dagnachew Jenber

College Of Natural and Social Science
Department of Mathematics
Addis Ababa Science and Technology University
Addis Ababa, Ethiopia

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Definition.

The general form of second order ordinary differential equations (SOODE) is $F(x, y, y', y'') = 0$.

Definition.

A SOODE of the form

$$y'' + p(x)y' + f(x)y = r(x)$$

is called a linear SOODE.

If $r(x) = 0$, then $y'' + p(x)y' + f(x)y = 0$, is homogeneous SOODE otherwise non-homogeneous.



Example

$y'' + y = x^6$ and $y'' - 4xy' + 4 = 0$ are SOODE.

Definition

A solution of $F(x, y, y', y'') = 0$ on an interval I is a function φ that satisfies the differential equation at each point of I i.e., $F(x, \varphi(x), \varphi'(x), \varphi''(x)) = 0$ for all x in I .



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Example

Show that $\varphi(x) = 6 \cos(4x) - 17 \sin(4x)$ is a solution of $y'' + 16y = 0$ for all x .



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Example

Show that $\varphi(x) = x^3 \cos(\ln(x))$ is a solution of
 $x^2 y'' - 5xy' + 10y = 0, x > 0$

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Theorem(principle of super position)

If y_1 and y_2 are solutions of $y'' + p(x)y' + f(x)y = 0$ then any linear combination of y_1 and y_2 is also a solution of $y'' + p(x)y' + f(x)y = 0$.

Remark

Super position principle in general doesn't hold for non-homogeneous and non-linear.

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Example

1. $y_1 = 1 + \cos(x)$ and $y_2 = 1 + \sin(x)$ are solutions of the non-homogeneous differential equation $y'' + y = 1$ but their linear combination $y_1 + y_2 = 2 + \cos(x) + \sin(x)$ is not the solution
2. $y_1 = x^2$ and $y_2 = 1$ are the solutions of the non-linear DE $y''y - xy' = 0$ but their linear combination $y_1 + y_2 = x^2 + 1$ is not the solution.



LINEAR INDEPENDENCE AND WRONSKIAN

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Definition

A set of n functions $\{y_1, y_2, y_3, \dots, y_n\}$ defined on an interval I is said to be linearly dependent, if there are constants $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zero such that

$$\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n = 0 \text{ on } I$$

otherwise $y_1, y_2, y_3, \dots, y_n$ are linearly independent.

- For $n = 2$, $\{y_1, y_2\}$ is linearly dependent if there are constants α_1, α_2 not all zero such that

$$\alpha_1 y_1 + \alpha_2 y_2 = 0$$

$$\implies y_1 = \frac{-\alpha_2}{\alpha_1} y_2 = a y_2 \text{ where } a = -\frac{\alpha_2}{\alpha_1}, \text{ i.e., } y_1 = a y_2.$$



LINEAR INDEPENDENT AND WRONSKIAN

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Corollary(Test of Linearly Independent)

The set $\{y_1, y_2, y_3, \dots, y_n\}$ is a linearly independent set of functions on I, If

$$\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n = 0$$

then

$$\alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_n = 0.$$



LINEAR INDEPENDENT AND WRONSKIAN

Definition

The Wronskian of n functions $y_1, y_2, y_3, \dots, y_n$ at class C^{n-1} ($(n-1)$ times continuous differentiable) on an interval I is defined as

$$W(x) = W(y_1, y_2, y_3, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

- For two differentiable functions y_1 and y_2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$



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Example

$$W(\cos(x), \sin(x)) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1$$

• Suppose $\{y_1, y_2\}$ is linearly dependent on I then either $y_1 = \alpha y_2$ or $y_2 = \alpha y_1$.

Assume that $y_1 = \alpha y_2 \implies y_1' = \alpha y_2'$ then

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = \alpha y_2 y_2' - \alpha y_2 y_2' = 0$$

Hence $\{y_1, y_2\}$ is linearly dependent implies $W(y_1, y_2) = 0$



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Corollary(Test Of Linearly Independent)

If $W(y_1, y_2, \dots, y_n) \neq 0$ for some $x_0 \in I$ then $\{y_1, y_2, \dots, y_n\}$ is linearly independent on I

Example

1. Show that the set of functions, $f_1(x) = \sqrt{x} + 5$, $f_2(x) = \sqrt{x} + 5x$, $f_3(x) = x - 1$ and $f_4(x) = x^2$ is linearly dependent on $(0, \infty)$
2. Show that the set of functions, $\sin^2(x)$, $\cos^2(x)$, $\sec^2(x)$ and $\tan^2(x)$ is linearly dependent on $(-\frac{\pi}{2}, \frac{\pi}{2})$



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Exercise

1. Check the independent of $\{\cos(x), \sin(x)\}$ for every where.
2. Check the dependent of $\{\ln(x), e^x, \ln(x^2)\}$ on $(0, \infty)$.



GENERAL SOLUTIONS, BASIS, INITIAL VALUE PROBLEM

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Definition

The set $\{y_1, y_2\}$ is called a fundamental system of solution (fss) or a basis of solution, if y_1 and y_2 are linearly independent.



GENERAL SOLUTIONS, BASIS, INITIAL VALUE PROBLEM

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Theorem

Let y_1 and y_2 be any two linearly independent solutions of $y'' + p(x)y' + f(x)y = 0$. If $y_3 = y_3(x)$ is any other solutions of $y'' + p(x)y' + f(x)y = 0$, then there are constants α_1 and α_2 such that $y_3(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x)$

$$\Rightarrow y = \alpha_1 y_1 + \alpha_2 y_2$$

is a general solution.



GENERAL SOLUTIONS, BASIS, INITIAL VALUE PROBLEM

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Example

Consider $(x - 1)y'' - xy' + y = 0$ on $(1, \infty)$

$y_1(x) = x$ and $y_2(x) = e^x$ are solutions.

$$W(y_1, y_2) = W(x, e^x) = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x - 1) \neq 0$$

Hence $\{y_1, y_2\}$ is linearly independent since $W(x) = e^x(x - 1) \neq 0$ on $(1, \infty)$.

Thus $\{x, e^x\}$ is fss or base and $y = ax + be^x$ is the general solution.



GIVEN ONE SOLUTION TO FIND ANOTHER

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Consider the homogeneous differential equation:

$$y'' + p(x)y' + f(x)y = 0 \quad (1)$$

Suppose a non-trivial solution $y_1 = y_1(x)$ is given.

Claim: To find $y_2 = y_2(x)$ so that $\{y_1, y_2\}$ is fss

Put $u(x) = \frac{y_2(x)}{y_1(x)}$ since y_1 and y_2 are linearly independent, $u(x)$ is not constant.



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$$\implies y_2 = uy_1 \implies y_2' = u'y_1 + uy_1'$$

$$\implies y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1'' = u''y_1 + 2u'y_1' + uy_1''$$

Substituting in 1:

$$y_2'' + p(x)y_2' + f(x)y_2 = 0$$

$$\iff u''y_1 + 2u'y_1' + uy_1'' + p(x)(u'y_1 + uy_1') + f(x)u(x)y_1(x) = 0$$



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$$\Longleftrightarrow u''y_1 + 2u'y_1' + p(x)u'y_1 + u(y_1'' + p(x)y_1' + f(x)y_1) = 0$$

$$\Longleftrightarrow u''y_1 + 2u'y_1' + p(x)u'y_1 = 0$$

Applying reduction of order: Let $z = u'$ and $z' = u''$ Then

$$z'y_1 + z(2y_1' + p(x)y_1) = 0 \Longleftrightarrow \frac{z'}{z} = \frac{-2y_1'}{y_1} - p(x)$$

$$\Longleftrightarrow \frac{1}{z}dz = \left(\frac{-2y_1'}{y_1} - p(x) \right) dx \dots \dots \dots \text{separable.}$$



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$$\Longleftrightarrow \ln |z| + 2 \ln |y_1| = - \int p(x) dx$$

$$\Longleftrightarrow zy_1^2 = e^{-\int p(x) dx}$$

or

$$z = \frac{e^{-\int p(x) dx}}{y_1^2} = u' = \frac{du}{dx}$$

Thus $u = \int \frac{e^{-\int p(x) dx}}{y_1^2} dx$ and $y_2(x) = y_1(x) \cdot \int \frac{e^{-\int p(x) dx}}{y_1^2} dx$ is the solution of (1) which is linearly independent with y_1 .



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Example

Consider $x^2y'' - xy' + y = 0$ on $I=(0, \infty)$ and $y_1(x) = x$ is a solution.



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Solution.

In this differential equation $p(x) = -\frac{1}{x}$

$$\implies \int p(x)dx = - \int \frac{1}{x}dx = -\ln|x| = -\ln(x) \text{ on } (0, \infty)$$

$$y_2(x) = y_1(x) \cdot \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$= x \int \frac{x}{x^2} dx = x \ln|x| = x \ln(x) \text{ on } I = (0, \infty), \text{ i.e., } \{x, \ln(x)\} \text{ is fss.}$$

Hence $y = c_1x + c_2x \ln(x)$ is the general solution of the given differential equation.



GIVEN ONE SOLUTION TO FIND ANOTHER

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Example

Find the general solution to

$$x^2 y'' + 2xy' - 2y = 0$$

given that $y_1(x) = x$ is a solution.



DIFFERENTIAL OPERATOR

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Definition

Let D stands for the operator $\frac{d}{dx}$ i.e., $D = \frac{d}{dx}$. Then
 $Dy = \frac{dy}{dx} = y'$ and $(D - C)y = (\frac{d}{dx} - C)y = \frac{dy}{dx} - Cy$ where C is a constant.

Note

$\frac{d}{dx}$, $\frac{d^2}{dx^2}$, $\frac{d^3}{dx^3}$, ... etc denoted by $D, D^2, D^3, \dots$ respectively.



DIFFERENTIAL OPERATOR

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Remark

1. Two operators A and B are equal if and only if $Ay = By$ for all functions y

2. The linear DE of the n^{th} order with constant coefficient,

$$c_n \frac{d^n y}{dx^n} + c_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + c_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + c_1 \frac{dy}{dx} + c_0 y = r(x) \quad [*]$$

can be written using the differential operator as:

$$c_n D^n y + c_{n-1} D^{n-1} y + c_{n-2} D^{n-2} y + \dots + c_1 D y + c_0 y = r(x)$$

$$\iff (c_n D^n + c_{n-1} D^{n-1} + c_{n-2} D^{n-2} + \dots + c_1 D + c_0) y = r(x)$$

$$\iff f(D) y = r(x)$$

$$\text{where } f(D) = c_n D^n + c_{n-1} D^{n-1} + c_{n-2} D^{n-2} + \dots + c_1 D + c_0$$



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Note

$$f(D) = 0$$

$$\iff c_n D^n + c_{n-1} D^{n-1} + c_{n-2} D^{n-2} + \dots + c_1 D + c_0 = 0$$

is called the auxiliary equation of $[*]$.

Example

The DE $\frac{d^2 y}{dx^2} + \frac{5dy}{dx} + 6y = 0$ has symbolic form $(D^2 + 5D + 6)y = 0$ and $D^2 + 5D + 6 = 0$ is its auxiliary equation.



DIFFERENTIAL OPERATOR

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Theorem

If $f(D)$ is a polynomial in D and $f(m) = 0$, where $m \in \mathbb{R}$ then $f(D)e^{mx} = 0$

Theorem

For $a \in \mathbb{R}$, $n \in \mathbb{N}$ then $(D - a)^n e^{ax} y = e^{ax} D^n y$.

Corollary

If $f(D)$ is a polynomial in D with constant coefficients then $e^{ax} f(D)y = f(D - a)e^{ax} y$.



REAL ROOTS, COMPLEX ROOTS AND DOUBLE ROOTS OF THE CHARACTERISTIC EQUATION

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Second Order Homogeneous Linear DEs with Constant Coefficients

Form: $y'' + ay' + by = 0$ (1)

Differential operator form: $(D^2 + aD + b)y = 0$

Claim: To solve (1)

Trial solution: $y = e^{mx} \implies y' = me^{mx}$ and $y'' = m^2e^{mx}$

Substituting in (1): $m^2e^{mx} + ame^{mx} + be^{mx} = 0$

$$\implies (m^2 + am + b)e^{mx} = 0$$

$$\implies m^2 + am + b = 0 \text{ because } e^{mx} \neq 0$$



REAL ROOTS, COMPLEX ROOTS AND DOUBLE ROOTS OF THE CHARACTERISTIC EQUATION

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The equation $m^2 + am + b = 0$ is called the characteristics equation (CE) of (1) and the characteristics root (CR) are

$$m = m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Case-1. $a^2 - 4b > 0$

The characteristics equation has two distinct real roots:

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

$\Rightarrow y_1(x) = e^{m_1 x}$ and $y_2(x) = e^{m_2 x}$ are a linearly independent solutions of (1).

Therefore $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution of (1).



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Example

$$\text{Solve } 2y'' - 3y' = 0$$



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Solution.

Trial solution. $y(x) = e^{mx} \implies y' = me^{mx} \implies y'' = m^2 e^{mx}$

Substituting. $2m^2 e^{mx} - 3m e^{mx} = 0$ or $(2m^2 - 3m)e^{mx} = 0$

CE. $2m^2 - 3m = 0$

CR. $m_1 = 0, m_2 = \frac{3}{2}$

Hence $\{y_1(x), y_2(x)\} = \left\{1, e^{\frac{3}{2}x}\right\}$ is fss and $y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 + c_2 e^{\frac{3}{2}x}$ is the general solution.



REAL ROOTS, COMPLEX ROOTS AND DOUBLE ROOTS OF THE CHARACTERISTIC EQUATION

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Example

Solve the following IVP

$$4y'' - 5y' = 0, y(-2) = 0, y'(-2) = 7$$



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Case (2). $a^2 - 4b = 0$ then the characteristic equation has only one double real root i.e., $m = -\frac{a}{2}$ (double)

Therefore $y_1(x) = e^{-\frac{a}{2}x}$ is one solution. To find $y_2(x)$ apply given one solution to find another i.e.,

$$y'' + ay' + by = 0 \implies p(x) = a$$

$$\text{Therefore } y_2(x) = y_1(x) \cdot \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} = e^{-\frac{a}{2}x} \cdot \int \frac{e^{-\int a dx}}{e^{-\frac{a}{2}x^2}} = xe^{-\frac{a}{2}x}$$

Hence $\{e^{-\frac{a}{2}x}, xe^{-\frac{a}{2}x}\}$ is fss and

$y(x) = c_1 e^{-\frac{a}{2}x} + c_2 x e^{-\frac{a}{2}x}$ is the general solution.



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Example

1. Solve $y'' + 4y' + 4y = 0$ 2. Solve the following IVP

$$y'' + 14y' + 49y = 0, y(-4) = -1, y'(-4) = 5$$



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Case(3). $a^2 - 4b < 0$

The characteristic equation has complex solutions:

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a \pm \sqrt{4b - a^2}i}{2} \implies m = m_{1,2} = \alpha \pm \beta i, \text{ where } \alpha = -\frac{a}{2} \text{ and } \beta = \frac{\sqrt{4b - a^2}}{2}$$

Note. From Euler's formula:

$e^{i\theta} = \cos(\theta) + i \sin(\theta)$, $\theta \in \mathfrak{R}$ and $e^{-i\theta} = \cos \theta - i \sin \theta$ then

$$e^{(\alpha + \beta i)x} = e^{\alpha x} (\cos(\beta x) + i \sin(\beta x)) = y_1(x) \text{ and}$$

$$e^{(\alpha - \beta i)x} = e^{\alpha x} (\cos(\beta x) - i \sin(\beta x)) = y_2(x)$$

$$y_1^*(x) = \frac{(y_1 + y_2)}{2} = e^{\alpha x} \cos(\beta x)$$

$$y_2^*(x) = -\frac{1}{2}i(y_1 - y_2) = e^{\alpha x} \sin(\beta x)$$



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Therefore $\{y_1^*(x), y_2^*(x)\} = \{e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)\}$ is fss.

Hence $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$ is general solution.

Example

Solve $y'' - 2y' + 10y = 0$



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Solution.

CE. $m^2 - 2m + 10 = 0$

CR. $m_1 = 1 + 3i, m_2 = 1 - 3i$

this implies $\alpha = 1, \beta = 3$

$$y = e^x (c_1 \cos(3x) + c_2 \sin(3x))$$



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Exercise

Solve

$$1. \begin{cases} 2y'' + 9y' - 5y = 0 \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$

$$2. y'' + 5y' + 6y = 0$$

$$3. y'' = 0$$

$$4. y'' - 2y' - 5y = 0$$

$$5. 3y'' + y' - 10y = 0.$$



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Non-homogeneous equations with constant coefficients

Definition

Let $D = \frac{d}{dx}$ (differential operator), if a differential operator, say A , applied to a function, say U , results zero (i.e., $AU = 0$), the operator A is said to be the annihilator U . If $AU = 0$ we say A annihilator of U .

Example

$f(x) = x$ annihilated by D^2 i.e., $D^2x = 0$, $g(x) = 8x^4 - 3x$ annihilated by D^5 .



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Annihilator's of Some Common Functions

Function	Annihilator
x^{m-1}	D^m
$e^{\alpha x}$	$D - \alpha$
$\cos(\beta x)$	$D^2 + \beta^2$
$\sin(\beta x)$	$D^2 + \beta^2$
$x^{m-1} e^{\alpha x}$	$(D - \alpha)^m$
$x^{m-1} \cos(\beta x)$	$(D^2 + \beta^2)^m$
$x^{m-1} \sin(\beta x)$	$(D^2 + \beta^2)^m$
$e^{\alpha x} \cos(\beta x)$	$D^2 - 2\alpha D + (\alpha^2 + \beta^2)$
$e^{\alpha x} \sin(\beta x)$	$D^2 - 2\alpha D + (\alpha^2 + \beta^2)$
$x^{m-1} e^{\alpha x} \cos(\beta x)$	$(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^m$
$x^{m-1} e^{\alpha x} \sin(\beta x)$	$(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^m$



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Note.

It is difficult to find the annihilator for the functions like e^{x^2} , $\log(x)$, $\tan(x)$.



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Non-homogeneous Second Order Linear DE with constant coefficients

Form. $y'' + ay' + by = r(x)$

HDE form. $y'' + ay' + by = 0$

Method of solution. Undetermined coefficient(UC)



REAL ROOTS, COMPLEX ROOTS AND DOUBLE ROOTS OF THE CHARACTERISTIC EQUATION

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

By

Dagnachew
Jenber

Steps to Solve $f(D)y = r(x)$ Using Undetermined Coefficient(UC)

Consider $y'' + ay' + by = r(x)$ or $(D^2 + aD + b)y = r(x)$

$\iff f(D)y = r(x)$ where $f(D) = D^2 + aD + b$.

The differential equation $f(D)y = r(x)$ has constant coefficients and the function $r(x)$ consists of finite sums and products of constants, polynomials, exponential functions(e^{ax}), sines and cosines.



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Step-1. Find the complementary solution y_c for the homogeneous DE $f(D)y = 0$.

Step-2. Operate both sides of the non-homogeneous DE $f(D)y = r(x)$ with the annihilator of $r(x)$ (say A).

Step-3. Find the general solution of the homogeneous DE $Af(D)y = 0$.

Step-4. Delete all those terms from the solution in (Step-3) that are duplicated in the complementary solution y_c found in (Step-1).

Step-5. Substitute y_p found in Step-4 in to $f(D)y = r(x)$. Match coefficients of various function on each side of the equality and solve the resulting system of equations for the unknown coefficients in y_p .

Step-6. With the particular solution (y_p) found in step-5 form the general solution $y = y_c + y_p$ of the DE.



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Example

Solve $y'' + 8y = 5x + 2e^{-x}$ with UC



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Solution. $y'' + 8y = 5x + 2e^{-x} \dots (*)$

Hom.form: $y'' + 8y = 0$

CE: $m^2 + 8 = 0$

CR: $m_{1,2} = \pm i2\sqrt{2}$

Therefore $y_c = c_1 \cos(2\sqrt{2}x) + c_2 \sin(2\sqrt{2}x)$

Since $D^2 5x = 0 = (D + 1)2e^{-x}$, the annihilator of $5x + 2e^{-x}$ is $D^2(D + 1)$

apply $D^2(D + 1)$ to $(*)$, we have

$$y'' + 8y = 5x + 2e^{-x}$$

$$\iff (D^2 + 8)y = 5x + 2e^{-x}$$

$$\iff D^2(D + 1)(D^2 + 8)y = D^2(D + 1)[5x + e^{-x}] = 0$$

$$\iff D^2(D + 1)(D^2 + 8)y = 0$$



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CE: $m^2(m+1)(m^2+8)=0$

CR: $0, 0, -1, \pm 2\sqrt{2}$

$$y_p = c_1 \cos(2\sqrt{2}x) + c_2 \sin(2\sqrt{2}x) + A + Bx + ce^{-x}$$

delete the duplicate terms in y_c , then

$$y_p = A + Bx + Ce^{-x} \iff y'_p = B - Ce^{-x} \iff y''_p = Ce^{-x}$$

Substituting y_p, y''_p in (*).

$$Ce^{-x} + 8(A + Bx + Ce^{-x}) = 5x + 2e^{-x}$$

$$\iff 9Ce^{-x} + 8A + 8Bx = 5x + 2e^{-x}$$

$$\iff A = 0, B = \frac{5}{8}, C = \frac{2}{9}$$

Therefore $y = y_c + y_p = c_1 \cos(2\sqrt{2}) + c_2 \sin(2\sqrt{2}) + \frac{5}{8}x + \frac{2}{8}x$ is the general solution.



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SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Example

Find a particular solution for the following DE

$$y'' - 4y' - 12y = 3e^{5x} + \sin(2x) + xe^{4x}$$



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Exercise

Solve the following Differential Equation using the methods of undetermined coefficient

1. $y'' - y' - 12y = e^{4x}$

2. $y'' + 25y = 6\sin x$ [2mm] 3. $y'' + 6y' + 9y = -xe^{4x}$

4. $y'' - 2y' + 5y = e^x \sin x$



REAL ROOTS, COMPLEX ROOTS AND DOUBLE ROOTS OF THE CHARACTERISTIC EQUATION

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Solving $f(D)y = r(x)$ Using Variation of Parameters (VP)

Let a set $\{y_1, y_2\}$ be a fss of $f(D)y = r(x)$.

The complementary solution of $f(D)y = r(x)$ is

$$y_c = c_1 y_1(x) + c_2 y_2(x), c_1 = c_2 = \text{constants}$$

To determine y_p vary the constant c_1 and c_2 to functions of x , that is,

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$$

where $c_1(x), c_2(x)$ are functions of x . Then

$$y'_p = c'_1(x)y_1(x) + c_1(x)y'_1(x) + c'_2(x)y_2(x) + c_2(x)y'_2(x)$$

$$\iff y''_p(x) = c''_1(x)y'_1(x) + 2c'_1(x)y'_1(x) + c_1(x)y''_1(x) + c''_2(x)y_2(x) + 2c'_2(x)y_2(x) + c_2(x)y''_2(x)$$



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Substituting in (1):

$$c_1''y_1' + 2c_1'y_1' + c_1y_1'' + c_2''y_2 + 2c_2'y_2' + c_2y_2'' + p(x)(c_1'y_1 + c_1y_1' + c_2'y_2 + c_2y_2'') + f(x)(c_1y_1 + c_2y_2) = r(x)$$

$$\iff c_1''y_1 + 2c_1'y_1' + c_2''y_2 + 2c_2'y_2' + p(x)(c_1'y_1 + c_2'y_2) + c_1(y_1'' + p(x)y_1' + f(x)y_1) + c_2(y_2'' + p(x)y_2' + f(x)y_2) = r(x)$$

$$\iff (c_1''y_1 + c_1'y_1' + c_2''y_2 + c_2'y_2') + (c_1'y_1' + c_2'y_2') + p(x)(c_1'y_1 + c_2'y_2) = r(x)$$

$$\iff (c_1'y_1 + c_2'y_2)' + (c_1'y_1' + c_2'y_2') + p(x)(c_1'y_1 + c_2'y_2) = r(x)$$

Choose $c_1(x)$ and $c_2(x)$ so that $c_1'y_1 + c_2'y_2 = 0$, then $c_1'y_1' + c_2'y_2' = r(x)$



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$$\text{Solve } \begin{cases} c_1' y_1 + c_2' y_2 = 0 \\ c_1' y_1' + c_2' y_2' = r(x) \end{cases}$$

$$\text{Since } W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0$$

From Cramer's rule $c_1'(x)$ and $c_2'(x)$ are uniquely determined

$$c_1'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2'(x) \end{vmatrix}}{W(x)} \text{ and } c_2'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(x)}$$



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$$\text{Then } c_1'(x) = -\frac{(y_2(x))(r(x))}{W(x)} \iff c_1(x) = -\int \frac{(y_2(x))(r(x))}{W(x)} dx$$

$$\text{and } c_2'(x) = \frac{(y_1(x))(r(x))}{W(x)} \iff c_2(x) = \int \frac{y_1(x)r(x)}{W(x)} dx$$

Therefore $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$

Hence $y = y_c + y_p$ is the general solution.

Example

Solve $y'' - 2y' + y = \frac{1}{x^2}e^x, x > 0$ using VP



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Solution. $y'' - 2y' + y = \frac{1}{x^2}e^x$

Hom.form. $y'' - 2y' + y = 0$

CE. $m^2 - 2m + 1 = 0$

CR. $m = 1(\text{double.})$

$$y_c(x) = c_1 e^x + c_2 x e^x$$

To find y_p we vary c_1 and c_2 as a function of x .

i.e., $y_p = c_1(x)e^x + c_2(x)xe^x$

Since $y_1(x) = e^x \iff y_1'(x) = e^x$ and $y_2(x) = xe^x \iff y_2'(x) = e^x + xe^x$



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$$\text{Now } \begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = r(x) \end{cases}$$

$$\iff \begin{cases} c_1'(x)e^x + c_2'(x)xe^x = 0 \\ c_1'(x)e^x + c_2'(x)e^x(1+x) = \frac{1}{x^2}e^x \end{cases}$$

$$\iff \begin{cases} c_1'(x) + c_2'(x)x = 0 \\ c_1'(x) + c_2'(x)(1+x) = x^{-2} \end{cases}$$

$$\iff c_2'(x) = x^{-2} \iff c_2(x) = \int x^{-2} dx = -\frac{1}{x} \text{ and}$$

$$c_1'(x) = c_2'(x)x = -\frac{1}{x} \iff c_1(x) = -\int \frac{1}{x} dx = -\ln|x|$$

$$\text{Thus } y_p(x) = -e^x \ln(x) - \frac{1}{x}xe^x = -e^x \ln|x| - e^x$$

$y(x) = y_c + y_p = c_1e^x + c_2xe^x - e^x \ln|x| - e^x$ is the general solution.



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Example

Find the general solution to

$$ty'' - (t + 1)y' + y = t^2$$

given that

$$y_1(t) = e^t, y_2(t) = t + 1$$



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SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Exercise

Solve the following using VP.

$$1. \begin{cases} y'' + 4y = \sec(2x) \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

$$2. y'' - 2y' + y = e^x \ln(x), x > 0$$



Euler's-Cauchy DEs

SECOND ORDER ORDI- NARY DIFFER- ENTIAL EQUA- TIONS

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Definition

Any DE of the form

$a_n x^n D^n y + a_{n-1} x^{n-1} D^{n-1} y + \dots + a_1 x Dy + a_0 y = g(x)$ where
 $a_n, a_{n-1}, \dots, a_1, a_0$ are constants is called Euler's DE.



Second Order Euler's Differential Equation(SOEDE)

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Form:

$$a_2x^2D^2y + a_1xDy + a_0y = g(x) \quad (1)$$

Hom.form:

$$a_2x^2D^2y + a_1xDy + a_0y = 0 \quad (2)$$

Trial solution: $y = x^m$ where m to be determined.

$$\implies y' = mx^{m-1}, y'' = m(m-1)x^{m-2}$$

Substituting in (2):

$$a_2x^2D^2y + a_1xDy + a_0y = 0$$

$$\implies a_2x^2(m(m-1))x^{m-2} + a_1xm x^{m-1} + a_0x^m = 0$$



Second Order Euler's Differential Equation(SOEDE)

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$$\implies a_2m(m-1)x^m + a_1mx^m + a_0x^m = 0$$

$$\implies a_2m(m-1) + a_1m + a_0 = 0 \text{ since } x^m \neq 0$$

$$\implies a_2m^2 + (a_1 - a_2)m + a_0 = 0 \quad (3)$$

which is the CE of (2)

Case-1: If m_1 and m_2 are distinct real roots of (3).

Then $\{y_1(x), y_2(x)\} = \{x^{m_1}, x^{m_2}\}$ is fss.

Hence $y = c_1x^{m_1} + c_2x^{m_2}$ is the general solution.



Second Order Euler's Differential Equation(SOEDE)

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Example

$$\text{Solve } x^2 D^2 y - 2x Dy - 4y = 0$$



Second Order Euler's Differential Equation(SOEDE)

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Solution:

Trial solution: $y = x^m$

CE: $m^2 - 3m - 4 = 0$

CR: $m_{1,2} = -1, 4.$

$y = c_1x^{-1} + c_2x^4$ is the general solution.



Second Order Euler's Differential Equation(SOEDE)

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Case-2: $m_1 = m_2$ (double roots of (3)), i.e.,

$$m = m_1 = m_2 = \frac{(a_2 - a_1)}{2a_2}$$

Hence $y_1(x) = x^{(\frac{a_2 - a_1}{2a_2})}$ is one solution. To find the other solution apply given one solution to find another, i.e.,

$$a_2 x^2 D^2 y + a_1 x D y + a_0 y = 0$$

$$\iff D^2 y + \frac{a_1}{a_2 x} D y + \frac{a_0}{a_2 x^2} y = 0$$



Second Order Euler's Differential Equation(SOEDE)

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$$\Longleftrightarrow y_2 = x^m \int \frac{e^{-\int p(x)dx}}{x^{m2}} dx$$

$$= x^m \int \frac{e^{-\int \frac{a_1}{a_2} dx}}{x^{(\frac{a_2-a_1}{2a_2})^2}} dx$$

$$= x^m \int \frac{e^{-\frac{a_1}{a_2} \ln(x)}}{x^{\frac{(a_2-a_1)}{a_2}}} dx = x^m \ln(x)$$

Therefore $y(x) = c_1 y_1 + c_2 y_2 = c_1 x^{\frac{a_2-a_1}{2a_2}} + c_2 \ln(x) x^{\frac{a_2-a_1}{2a_2}}$ is the general solution.



Second Order Euler's Differential Equation(SOEDE)

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Example

$$\text{Solve } 4x^2 D^2 y + 8x Dy + y = 0$$



Second Order Euler's Differential Equation(SOEDE)

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Solution.

Trial solution: $y = x^m$

CE: $4m^2 + (8 - 4)m + 1 = 0 \iff 4m^2 + 4m + 1 = 0$

CR: $m_{1,2} = -\frac{1}{2}$ (double).

Therefore $y_1(x) = x^{-\frac{1}{2}}$ and $y_2(x) = x^{-\frac{1}{2}} \ln(x)$

Hence $y(x) = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln(x)$ is the general solution.



Second Order Euler's Differential Equation(SOEDE)

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Note

For higher order Euler's DE. If m is a root of multiplicity $k > 1$, then it can be shown that, $x^m, x^m \ln(x), x^m (\ln(x))^2, \dots, x^m (\ln(x))^{k-1}$ are k linearly independent solutions.



Second Order Euler's Differential Equation(SOEDE)

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Case-3: If m_1 and m_2 are complex conjugate roots, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ then $y = c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta}$ is a solution.
But $x^{i\beta} = (e^{\ln(x)})^{i\beta} = e^{i\beta \ln(x)} = \cos(\beta \ln(x)) + i \sin(\beta \ln(x))$
and $x^{-i\beta} = \cos(\beta \ln(x)) - i \sin(\beta \ln(x))$



Second Order Euler's Differential Equation(SOEDE)

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$$\begin{aligned}\text{Hence } y &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} = x^{\alpha}(c_1 x^{i\beta} + c_2 x^{-i\beta}) \\ &= x^{\alpha}(c_1(\cos(\beta \ln(x)) + i \sin(\beta \ln(x))) + c_2(\cos(\beta \ln(x)) - i \sin(\beta \ln(x)))) \\ &= x^{\alpha}((c_1 + c_2) \cos(\beta \ln(x)) + (c_1 - c_2)i \sin(\beta \ln(x))) \\ &= x^{\alpha}(k_1 \cos(\beta \ln(x)) + k_2 \sin(\beta \ln(x))) \text{ is the general solution where} \\ &k_1 = c_1 + c_2 \text{ and } k_2 = (c_1 - c_2)i\end{aligned}$$



Second Order Euler's Differential Equation(SOEDE)

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Example

$$\text{Solve } x^2 y'' + 3xy' + 3y = 0$$



Second Order Euler's Differential Equation(SOEDE)

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Solution:

CE: $m^2 + 2m + 3 = 0$

CR: $-1 \pm \sqrt{2}i$

Thus $y = x^{-1}(k_1 \cos(\sqrt{2} \ln(x)) + k_2 \sin(\sqrt{2} \ln(x)))$ is the general solution.



Second Order Euler's Differential Equation(SOEDE)

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Exercise

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$ (Hint: use Euler's and VP method.)



SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS(SOFOODEs)

SECOND ORDER ORDI- NARY DIFFER- ENTIAL EQUA- TIONS

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Definition

1. The system of differential equations is a collection of equations in several unknown functions and their derivatives.
2. The system is said to be linear if the unknown functions and their derivatives involve in each equations only to the first power.

Note.

1. Each equation has at least one derivative.
2. A linear system is of normal form when it is solved for all derivatives.



SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS(SOFOODEs)

SECOND ORDER ORDI- NARY DIFFER- ENTIAL EQUA- TIONS

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Example

The system of differential equation

$$\left\{ \begin{array}{l} \frac{dx}{dt} = x - y + \frac{3}{2}t^2 \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \frac{dy}{dt} = -4x - 2y + 4t + 1 \end{array} \right. \quad (2)$$

is linear and is of normal form.



SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS(SOFOODEs)

SECOND ORDER ORDI- NARY DIFFER- ENTIAL EQUA- TIONS

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Example

Solve the linear system of the above example.

Solution. From equation (1) we have $\frac{d^2x}{dt^2} = \frac{dx}{dt} - \frac{dy}{dt} + 3t$ (3)

From equation (1) again we have $y = x - \frac{dx}{dt} + \frac{3}{2}t^2$ substituting this equation in to (2) we get

$\frac{dy}{dt} = -4x - 2(x - \frac{dx}{dt} + \frac{3}{2}t^2) + 4t + 1 = -6x + 2\frac{dx}{dt} - 3t^2 + 4t + 1$
substituting this equation in to (3) we get a second order linear equation:



SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS(SOFOODEs)

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$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 3t^2 - 4t - 1$$

$$\iff x = c_1 e^{2t} + c_2 e^{-3t} - \frac{1}{2}t^2$$

is the general solution for x and if we substitute this in to equation (1) we get the other solution, that is,

$$y = -\frac{dx}{dt} + x + \frac{3}{2}t^2 = -c_1 e^{2t} + 4c_2 e^{-3t} + t^2 + t$$

Therefore the solution of the system is

$$x = c_1 e^{2t} + c_2 e^{-3t} - \frac{1}{2}t^2$$

$$y = -c_1 e^{2t} + 4c_2 e^{-3t} + t^2 + t$$



SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS(SOFOODEs)

SECOND ORDER ORDI- NARY DIFFER- ENTIAL EQUA- TIONS

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Exercise

Solve

$$1. \begin{cases} \frac{dx}{dt} = 4x - y \\ \frac{dy}{dt} = 2x + y \end{cases}$$

$$2. \begin{cases} \frac{dx}{dt} = x + 2y + t - 1 \\ \frac{dy}{dt} = 3x + 2y - 5t - 2 \end{cases}$$