



First Order Ordinary Differential Equations

By

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DIFFERENTIAL EQUATION(DE)

First Order Ordinary Differential Equations

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Definition: An equation involving one or more derivatives of a dependent variable with respect to one or more independent variables is called a differential equation(DE).

Example

$$\left. \begin{aligned} y' - xy &= 5 \\ \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} &= \cos(y) \\ \frac{d^2 u}{dx^2} + \cos(x) \frac{du}{dx} &= x \end{aligned} \right\} DEs$$



CLASSIFICATIONS OF A DE

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Definition 1

A DE involving one or more derivatives of a dependent variable with respect to only one independent variable is called an ordinary differential equation(ODE).

Definition 2

A DE involving one or more derivatives of a dependent variable with respect to two or more independent variables is called a partial differential equation(PDE).



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Example.

$$\left. \begin{array}{l} y' = x - 5 \\ y'' - xy' + y = x \end{array} \right\} ODEs$$



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Example

$$\left. \begin{aligned} \frac{\partial Z}{\partial x} - \frac{\partial Z}{\partial y} &= 0 \\ U_{zz} - U_{xx} &= x^2 \\ U_z - 4U_x &= \sin(x) \end{aligned} \right\} PDEs$$

Remark

In this course we consider only ODE.

Definition

The order of the highest derivative in a DE is called the order of the equation.



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Example

$$\left. \frac{dy}{dx} - y = x^2 + 1 \right\} \text{order 1}$$

$$\left. \begin{aligned} y''' - y'' &= x \\ (y')^{\frac{5}{2}} &= y''' + 1 \end{aligned} \right\} \text{order 3}$$



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Definition

An ODE of n^{th} order is an equation of the form

$$f(c, x, y, y', y'', \dots, y^n) = 0$$

where f is a function of $(n + 2)$ variables with $y = y(x)$ and c is a constant.

Definition

A DE of order, n , is said to be explicit if it can be expressed in the form:

$y^n = F(x, y, y', y'', y''', \dots, y^{(n-1)})$ and F is a function of $(n + 1)$ variables; otherwise it is called an implicit



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Example

$$y'' = xy' + \cos(x) \} \textit{explicit}$$

$$y'' = \cos(y'') \} \textit{implicit}$$

Definition

The degree of ODE (if it exists) is the highest exponent of the highest derivatives that occurs in the DE, after the DE is expressed as a polynomial of the dependent variable and its derivatives



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Example

1. $y' - y = 1$ } order 1, degree 1.
2. $(y'')^3 - x(y')^5 = x^3$ } order 2, degree 3.
3. $(y''')^3 - (y''')^2 + y = x$ } order 3, degree 3.



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Example

Find the degree and order of the DE

$$(y')^{\frac{5}{2}} = y''' + 1$$



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Solution.

Firstly let us write the given DE as apolynomial of the dependent variable and its derivatives, that is,

$$(y')^{\frac{5}{2}} = y''' + 1$$

$$\Leftrightarrow ((y')^{\frac{5}{2}})^2 = (y''' + 1)^2$$

$$\Leftrightarrow (y')^5 = (y''')^2 + 2y''' + 1$$

Therefore

$$(y')^{\frac{5}{2}} = y''' + 1 \Leftrightarrow (y')^5 = (y''')^2 + 2y''' + 1$$

has order 3 and degree 2.



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Example

Find the order and degree of the DE

$$y'' = y + e^{y''}$$



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Solution.

Firstly let us write the given DE as apolynomial of the dependent variable and its derivatives, that is,

$$\text{Since } e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$y'' = y + e^{y''} \Leftrightarrow y'' = y + (1 + y'' + \frac{(y'')^2}{2!} + \frac{(y'')^3}{3!} + \dots)$$

$$\Leftrightarrow 0 = 1 + y + \frac{(y'')^2}{2!} + \frac{(y'')^3}{3!} + \dots$$

Therefore

$$y'' = y + e^{y''} \Leftrightarrow 0 = 1 + y + \frac{(y'')^2}{2!} + \frac{(y'')^3}{3!} + \dots$$

has order 2 and no degree.



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Exercise

Find the degree of the following DEs

1. $y'' = \cos(y')$

2. $y'' = \cos(y'')$



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Definition

An n^{th} order DE is linear if it can be written of the form $a_n(x)y^n + a_{n-1}(x)y^{n-1} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$, where the coefficients $a_i(x)$ ($i = 0, 1, 2, \dots, n$) are function of x alone, i.e, they do not depend on y or any derivatives of y .

Note.

An equation that is not linear is called non-linear.



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Example

$$\left. \begin{aligned} 3y' + xy &= e^{-x^2} \\ e^x y'' + xy &= 2 \\ xy' + xy &= 0 \end{aligned} \right\} \text{Linear}$$

$$\left. \begin{aligned} yy'' - 2y'' &= x \\ y''' + y^2 &= 0 \\ y'' + \sqrt{y'} + y &= x^2 \end{aligned} \right\} \text{Non - Linear}$$



SOLUTIONS OF A DE

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Definition

A function that is free of derivatives and that satisfies identically a DE on some region D is called a solution of the DE, i.e., $y = y(x)$ is a solution of the DE,

$$F(x, y, y', y'', \dots, y^n) = 0$$

or satisfies the equation

$$F(x, y, y', y'', \dots, y^n) = 0$$

if

$$F(x, y(x), y'(x), y''(x), \dots, y^n(x)) = 0$$

.



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Example

The function $y = xe^x$ is a solution of the linear equation

$$y'' - 2y' + y = 0$$

on the interval $(-\infty, \infty)$. To show this, we compute

$$y' = xe^x + e^x \text{ and } y'' = xe^x + 2e^x.$$

Observe that

$$y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$$



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Example

Consider the DE

$$y' - y = 0$$

Take $y = e^x$ then $y' = e^x$, therefore $y' - y = e^x - e^x = 0$.
Thus $y = e^x$ is the solution of $y' - y = 0$. In general
 $y = ce^x$, $c \in \mathfrak{R}$ are solutions of the DE.

Exercise

1. Show that $y = ce^{x^2}$ is a solution of $y' = 2xy$.
2. Show that $y = \frac{x^4}{16}$ is a solution of $y' = xy^{\frac{1}{2}}$



SOLUTIONS OF A DE

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Definition

1. A solution of a DE of the form $y = h(x)$ is called an explicit solution.
2. A solution of the form $h(x, y) = 0$ where it is not easy to express y in terms of x is called an implicit solution.
3. A solution of a DE that is free of arbitrary parameters is called a particular solution, otherwise it is general solution.



FORMATION OF A DE

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Note

An ODE is formed in an attempt to eliminate certain arbitrary constants from a relation in the variables and constants.



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Example

Consider the simple harmonic motion given by

$$x = A \cos(nt + B), \text{ A,B are constants}$$

Since we have two constants, A and B , therefore to eliminate A and B from the equation we have to differentiate the given equation twice, i.e.,

$$x' = -An \sin(nt + B) \Leftrightarrow x'' = -An^2 \cos(nt + B) = -n^2 x.$$

, That is, $x'' = -n^2 x$ is the required DE.



FORMATION OF A DE

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Exercise

1. Form the DE from the following equation.

a. $y = ax^3 + bx^2$

b. $xy = Ae^x + Be^{-x}$

c. $y = e^x(A \cos(x) + B \sin(x))$

d. $y = (x - b)^2$

2. obtain the DE of all circles of radius a with center at (h, k) .



INITIAL AND BOUNDARY VALUE PROBLEMS

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Definition

For different applications solutions to DE may be required to satisfied certain defined conditions such conditions are called initial conditions(IC) if they are given at only one point of the independent variable, while conditions given at more than one point of the independent variable are called boundary conditions(BC)

Definition

A DE together with a set of initial conditions/boundary conditions is called initial Value problem(IVP)/boundary value problem(BVP).



INITIAL AND BOUNDARY VALUE PROBLEMS

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Example

$$\left. \begin{array}{l} y'' = \cos(x) \\ y(0) = 1 \\ y'(0) = -1 \end{array} \right\} IC \left. \vphantom{\begin{array}{l} y'' = \cos(x) \\ y(0) = 1 \\ y'(0) = -1 \end{array}} \right\} IVP$$

$$\left. \begin{array}{l} y'' = \cos(x) \\ y(0) = 1 \\ y'(1) = -1 \end{array} \right\} BC \left. \vphantom{\begin{array}{l} y'' = \cos(x) \\ y(0) = 1 \\ y'(1) = -1 \end{array}} \right\} BVP$$



SEPARABLE EQUATIONS

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Definition

An explicit FOODEs has General form:

$$y' = f(x, y) \text{ or } \frac{dy}{dx} = f(x, y)$$

Another form:

$$M(x, y)dx + N(x, y)dy = 0 \Leftrightarrow N(x, y)dy = -M(x, y)dx$$

$$\Leftrightarrow \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} = f(x, y)$$

that is, $y' = f(x, y)$.



ELEMENTARY(SEPARATED) FOODEs

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Form: $y' = f(x)$.

Method of solving: Integration, that is

$$y' = f(x) \Leftrightarrow \frac{dy}{dx} = f(x)$$

$$\Leftrightarrow dy = f(x)dx \Leftrightarrow \int dy = \int f(x)dx + c$$

$$\Leftrightarrow y = \int f(x)dx + c$$

is the general solution.



ELEMENTARY(SEPARATED) FOODEs

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Example

Solve for y such that

$$y' = \cos(x + 1)$$

Solution.

$$\frac{dy}{dx} = \cos(x + 1) \Leftrightarrow dy = \cos(x + 1)dx$$

$$\Leftrightarrow \int dy = \int \cos(x + 1)dx + c$$

$$\Leftrightarrow y = \sin(x + 1) + c$$

is the G.S.



ELEMENTARY(SEPARATED) FOODEs

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Exercise

Solve

$$\left. \begin{array}{l} 1. \quad dy - \ln(x)dx = 0 \\ 2. \quad \left. \begin{array}{l} y' = x^2 + x + 1 \\ y(1) = 2 \end{array} \right\} IVP \end{array} \right\}$$



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Definition

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side then the variables are said to be separable.

Form: $y' = f(x)g(y)$.

Method of solving: $y' = f(x)g(y) \Leftrightarrow \frac{dy}{dx} = f(x)g(y)$

$$\Leftrightarrow \frac{dy}{g(y)} = f(x)dx$$

$$\Leftrightarrow \int \frac{dy}{g(y)} = \int f(x)dx + c$$

is the general solution.



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Example

Solve for $y = y(x)$ such that

$$(1 + x)dy - ydx = 0$$



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Solution.

Dividing by $(1 + x)y$, we can write $\frac{dy}{y} = \frac{dx}{(1+x)}$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{(1+x)} \implies \ln |y| = \ln |1+x| + c_1$$

$$\implies y = e^{\ln |1+x| + c_1} = |1+x|e^{c_1}$$

$$= \pm e^{c_1}(1+x) = c(1+x)$$

$$\implies y = c(1+x) \text{ is the general solution}$$



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Example

Find the general and particular solution of the DE

$$\left. \begin{array}{l} y' = -\frac{x}{y} \\ y(4) = 2 \end{array} \right\} IVP$$



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Solution.

From $ydy = -xdx$ we get

$$\int ydy = - \int xdx \implies \frac{y^2}{2} = -\frac{x^2}{2} + c.$$

From the initial condition, $c = 10$, therefore the particular solution of the IVP is

$$x^2 + y^2 = 20$$



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Example

Find the general and particular solution of the DE

$$\left. \begin{array}{l} y' = e^{(x+y)} \\ y(0) = 0 \end{array} \right\} IVP$$



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Solution.

$$y' = e^{(x+y)} = (e^x)(e^y) \Leftrightarrow e^{-y} dy = e^x dx$$

$$\Leftrightarrow \int e^{-y} dy = \int e^x dx + c$$

$$\Leftrightarrow e^{-y} + e^x = -c = C$$

$\Leftrightarrow e^{-y} + e^x = C$ is the general solution

$$\text{But } 0 = y(0) \Leftrightarrow 0 = e^{-0} + e^0 = 2 = C \Leftrightarrow C = 2$$

Therefore

$e^{-y} + e^x = 2$ is the particular solution.



SEPARABLE FOODEs

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Exercise

Solve

$$\left. \begin{aligned} x \cos(y) + (x^2 - 1)(\sin(y))y' &= 0 \\ y(0) &= \frac{\pi}{3} \end{aligned} \right\} IVP$$



EQUATIONS REDUCIBLE TO SEPARABLE FORM

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definition

The DE $y' = f(\frac{y}{x})$ is called homogeneous DE.

Note.

Any Homogeneous DE can be reduced to separable DE.



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Justification.

Any homogeneous DE has the form $y' = f(\frac{y}{x})$.

$$\text{Put } v = \frac{y}{x} \Leftrightarrow y = vx \Leftrightarrow \frac{dy}{dx} = \frac{d(xv)}{dx}$$

$$\Leftrightarrow y' = \frac{dy}{dx} = v + x \frac{dv}{dx} = v + xv'$$

$$\Leftrightarrow v + xv' = y' = f\left(\frac{y}{x}\right) = f(v)$$

$$\Leftrightarrow v + xv' = f(v)$$

$$\Leftrightarrow xv' = f(v) - v \text{ which is separable}$$

$$\Leftrightarrow \frac{dv}{f(v) - v} = \frac{dx}{x} \text{ which is separated}$$



EQUATIONS REDUCIBLE TO SEPARABLE FORM

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Example

Find $y = y(x)$ such that

$$y' = \frac{y^2 + xy}{x^2}$$



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Solution.

$$y' = \frac{y^2 + xy}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} = f\left(\frac{y}{x}\right) \text{ is homogeneous.}$$

$$\text{Let } v = \frac{y}{x} \Leftrightarrow y = xv$$

$$\Leftrightarrow y' = \frac{dy}{dx} = v + x \frac{dv}{dx} = v + xv'$$

$$\Leftrightarrow v^2 + v = \left(\frac{y}{x}\right)^2 + \frac{y}{x} = y' = v + xv'$$

$$\Leftrightarrow v^2 + v = v + xv \Leftrightarrow \frac{dv}{v^2} = \frac{dx}{x}$$



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$$\Leftrightarrow \int v^{-2} dv = \int \frac{dx}{x} \Leftrightarrow -v^{-1} = \ln |x| + c$$

$$\Leftrightarrow -\frac{x}{y} = \ln |x| + c$$

$$\Leftrightarrow y = -\frac{x}{\ln |x| + c} \text{ is the general solution}$$



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Definition

Suppose for the DE $M(x, y)dx + N(x, y)dy = 0$ there exists a continuous differentiable function $f(x, y)$ such that $df = M(x, y)dx + N(x, y)dy$, i.e., $f_x = M(x, y)$ and $f_y = N(x, y)$ then

1. The DE $M(x, y)dx + N(x, y)dy = 0$ can be written as $f_x dx + f_y dy = 0$ or $df = 0$.
2. The DE $M(x, y)dx + N(x, y)dy = 0$ is called an exact DE.
3. $f(x, y) = c$, $c \in \mathbb{R}$ defines implicitly a set of solutions of $M(x, y)dx + N(x, y)dy = 0$.



EXACT FOODEs

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Theorem

Let $M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ be continuous functions of x and y then the DE

*$M(x, y)dx + N(x, y)dy = 0$ is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
Proof: Left as exercise.*



- **Methods of solving of an exact DE:** Suppose a DE

$$M(x, y)dx + N(x, y)dy = 0$$

is exact, then there exist a function $f(x, y)$ such that

$$df = M(x, y)dx + N(x, y)dy$$

or

$$f_x = M \text{ and } f_y = N$$

$$\text{that is } df = 0 \Leftrightarrow f(x, y) = c$$

which is the general solution.

To find $f(x, y)$ we use the following methods.



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Method 1: Method of grouping.

Example

1. $ydx + xdy = 0 \Leftrightarrow d(xy) = ydx + xdy = 0 \Leftrightarrow d(xy) = 0 \Leftrightarrow xy = c$ or $y = \frac{c}{x}$ which is the G.S.

2. $xy' + y + 4 = 0 \Leftrightarrow xdy + (y + 4)dx = 0$ which is exact.

Now $xdy + (y + 4)dx = 0 \Leftrightarrow xdy + ydx + 4dx = 0 \Leftrightarrow d(xy) + d(4x) = 0 \Leftrightarrow d(xy + 4x) = 0 \Leftrightarrow xy + 4x = c$ is the G.S.



Method 2: Using either of the two options given below.

● **Option 1.**

Since $\frac{\partial f}{\partial x} = M(x, y)$

$$\Rightarrow \int \frac{\partial f}{\partial x} dx = \int M(x, y) dx + g(y)$$

$$\Rightarrow f(x, y) = \int M(x, y) dx + g(y) \quad (1)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(\int M(x, y) dx + g(y) \right)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x, y) dx + g'(y)$$



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Since $\frac{\partial f}{\partial y} = N(x, y)$ then

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x, y) dx + g'(y)$$

$$\implies N(x, y) = \frac{\partial}{\partial y} \int M(x, y) dx + g'(y)$$

$$\implies g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx$$

$$\implies g(y) = \int \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right) dy$$



But from equation (1), we've

$$f(x, y) = \int M(x, y) dx + g(y)$$

$$\Rightarrow f(x, y) = \int M(x, y) dx + \int \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right) dy$$

Since $f(x, y) = c$, hence

$$c = \int M(x, y) dx + \int \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right) dy$$

is the general solution of the given exact DE.



• **Option 2.**

Since $\frac{\partial f}{\partial y} = N(x, y)$

$$\implies \int \frac{\partial f}{\partial y} dy = \int N(x, y) dy + g(x)$$

$$\implies f(x, y) = \int N(x, y) dy + g(x) \quad (2)$$

$$\implies \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(\int N(x, y) dy + g(x) \right)$$

$$\implies \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int N(x, y) dy + g'(x)$$



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Since $\frac{\partial f}{\partial x} = M(x, y)$ then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int N(x, y) dy + g'(x)$$

$$\implies M(x, y) = \frac{\partial}{\partial x} \int N(x, y) dy + g'(x)$$

$$\implies g'(x) = M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy$$

$$\implies g(x) = \int \left(M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy \right) dx$$



But from equation (2), we've

$$f(x, y) = \int N(x, y) dy + g(x)$$

$$\Rightarrow f(x, y) = \int N(x, y) dy + \int \left(M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy \right) dx$$

Since $f(x, y) = c$, hence

$$c = \int N(x, y) dy + \int \left(M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy \right) dx$$

is the G.S of the given exact DE.



EXACT FOODEs

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Example

Solve

$$2xydx + (x^2 - 1)dy = 0$$



Solution.

With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact, there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = xy \text{ and } \frac{\partial f}{\partial y} = x^2 - 1$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2y + g(y)$$



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Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

it follows that $g'(y) = -1$ and $g(y) = -y$.
Therefore $x^2y - y = c$ is the general solution.



EXACT FOODEs

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Example

Solve

$$2x + y^3 + (3xy^2 - e^{-2y})y' = 0$$



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Solution.

$$2x + y^3 + (3xy^2 - e^{-2y})y' = 0$$

$$\iff (2x + y^3)dx + (3xy^2 - e^{-2y})dy = 0$$

that is $M(x, y) = 2x + y^3$, $N(x, y) = 3xy^2 - e^{-2y}$

$$\iff M_y = 3y^2 = N_x \text{ i.e., the DE is exact}$$



EXACT FOODEs

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Suppose $f(x, y) = c_2$, $c_2 \in \mathbb{R}$ be the solution of the given DE.

$$\Longleftrightarrow \frac{\partial f}{\partial x} = M(x, y) = 2x + y^3$$

$$\Longleftrightarrow f(x, y) = x^2 + y^3 x + g(y)$$

$$\Longleftrightarrow \frac{\partial f}{\partial y} = 3y^2 x + g'(y) = N = 3xy^2 - e^{-2y}$$

$$\Longleftrightarrow g'(y) = -e^{-2y}$$



EXACT FOODEs

First Order Ordinary Differential Equations

By

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$$\Longleftrightarrow g(y) = \frac{e^{-2y}}{2} + c_1$$

Therefore

$$f(x, y) = x^2 + y^3x + \frac{e^{-2y}}{2} + c_1$$

$$\Longleftrightarrow x^2 + y^3x + \frac{e^{-2y}}{2} + c_1 = c_2$$

or

$$x^2 + y^3x + \frac{e^{-2y}}{2} + c = 0$$

is an implicit solution of the DE.



First Order
Ordinary
Differential
Equations

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Exercise

Solve

$$1. \left. \begin{aligned} (\sin(x) \cosh(y))dx - (\cos(x) \sinh(y))dy &= 0 \\ y(0) &= 0 \end{aligned} \right\} IVP$$

$$2. (x^2 - y^2)dx - 2xydy = 0$$



INTEGRATING FACTORS

First Order Ordinary Differential Equations

By
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Definition

Suppose the DE $M(x, y)dx + N(x, y)dy = 0$ is not exact but after multiplying it by a suitable function say $I(x, y)$, the new equation $IM(x, y)dx + IN(x, y)dy = 0$ is exact. In this case such a multiplier function $I(x, y)$ is called an integrating factor of the DE.



INTEGRATING FACTORS

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Thus if $I(x, y)$ is an integrating factor of

$$M(x, y)dx + N(x, y)dy = 0$$

then

$$IM(x, y)dx + IN(x, y)dy = 0$$

is exact DE, i.e.,

$$(IM)_y = (IN)_x \Leftrightarrow I_y M + M_y I = I_x N + N_x I$$

Remark

Generally finding an integrating factor is difficult, however for some DEs the following theorem can be used.



INTEGRATING FACTORS

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Theorem

Consider the DE $M(x, y)dx + N(x, y)dy = 0$ which is not exact

but $IM(x, y)dx + IN(x, y)dy = 0$ is exact, then

• *If I and $\frac{1}{N}(M_y - N_x)$ are independent of y and say*

$\frac{1}{N}(M_y - N_x) = g(x)$ then the integrating factor of

$M(x, y)dx + N(x, y)dy = 0$ is given by $I(x) = e^{\int g(x)dx}$.



INTEGRATING FACTORS

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Theorem

Consider the DE $M(x, y)dx + N(x, y)dy = 0$ which is not exact

but $IM(x, y)dx + IN(x, y)dy = 0$ is exact, then

• *If I and $\frac{1}{N}(M_y - N_x)$ are independent of y and say*

$\frac{1}{N}(M_y - N_x) = g(x)$ then the integrating factor of $M(x, y)dx + N(x, y)dy = 0$ is given by $I(x) = e^{\int g(x)dx}$.

• *If I and $\frac{1}{M}(M_y - N_x)$ are independent of x and say*

$\frac{1}{M}(M_y - N_x) = h(y)$ then the integrating factor of $M(x, y)dx + N(x, y)dy = 0$ is given by $I(y) = e^{-\int h(y)dy}$.

Proof: Left as exercise.



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Example

Find an integrating factor for the DE

$$2 \sin(y^2) dx + xy \cos(y^2) dy = 0$$



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Solution.

$$M(x, y) = 2 \sin(y^2), \quad N(x, y) = xy \cos(y^2)$$



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Solution.

$$M(x, y) = 2 \sin(y^2), \quad N(x, y) = xy \cos(y^2)$$

$$\implies M_y = 4y \cos(y^2) \neq y \cos(y^2) = N_x$$



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Solution.

$$M(x, y) = 2 \sin(y^2), \quad N(x, y) = xy \cos(y^2)$$

$$\implies M_y = 4y \cos(y^2) \neq y \cos(y^2) = N_x$$

Hence the DE is not exact.



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Solution.

$$M(x, y) = 2 \sin(y^2), \quad N(x, y) = xy \cos(y^2)$$

$$\implies M_y = 4y \cos(y^2) \neq y \cos(y^2) = N_x$$

Hence the DE is not exact. But

$$\frac{1}{N}(M_y - N_x) = \frac{1}{xy \cos(y^2)}(4y \cos(y^2) - y \cos(y^2)) = \frac{3}{x}$$

which is independent of y .

Therefore

$$I(x) = e^{\int \frac{3}{x} dx} = x^3$$



INTEGRATING FACTORS

First Order
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Equations

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Solution.

$$M(x, y) = 2 \sin(y^2), N(x, y) = xy \cos(y^2)$$

$$\implies M_y = 4y \cos(y^2) \neq y \cos(y^2) = N_x$$

Hence the DE is not exact. But

$$\frac{1}{N}(M_y - N_x) = \frac{1}{xy \cos(y^2)}(4y \cos(y^2) - y \cos(y^2)) = \frac{3}{x}$$

which is independent of y .

Therefore

$$I(x) = e^{\int \frac{3}{x} dx} = x^3$$

Hence

$$2x^3 \sin(y^2) dx + x^4 y \cos(y^2) dy = 0$$

is exact DE.



INTEGRATING FACTORS

First Order Ordinary Differential Equations

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Exercise

1. Find an integrating factor of the DE

$$(3x^2 - y^2)dy - 2xydx = 0$$

2. Determine whether $-\frac{1}{x^2}$ is an integrating factor for the DE $ydx - xdy = 0$ or not.

Remark

The solution of exact DE $IM(x, y)dx + IN(x, y)dy = 0$ is the solution of non-exact DE $M(x, y)dx + N(x, y)dy = 0$, where I is an integrating factor of $M(x, y)dx + N(x, y)dy = 0$



Definition

A DE that can be expressed of the form

$$y' + p(x)y = f(x) \quad (3)$$

is called first order linear differential equation(FOLDE).

If $f(x) = 0$, then $y' + p(x)y = 0$ is homogeneous otherwise non-homogeneous.

Example

$y' = \sin(x) \rightsquigarrow$ is non-homogeneous LDE.

$y' + xy^2 = 0 \rightsquigarrow$ is non-Linear.

$3y' + 4xy = 0 \Leftrightarrow y' + \frac{4}{3}xy = 0 \rightsquigarrow$ is homogeneous LDE.



First Order
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Method of Solving.

Case 1. If $f(x) = 0$, then equation (3) becomes,



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Method of Solving.

Case 1. If $f(x) = 0$, then equation (3) becomes,

$$y' + p(x)y = 0 \iff \frac{dy}{y} = -p(x)dx \rightsquigarrow \text{Separated DE.}$$



Method of Solving.

Case 1. If $f(x) = 0$, then equation (3) becomes,

$$y' + p(x)y = 0 \iff \frac{dy}{y} = -p(x)dx \rightsquigarrow \text{Separated DE.}$$

$$\iff \ln |y| = - \int p(x)dx + c_1$$



Method of Solving.

Case 1. If $f(x) = 0$, then equation (3) becomes,

$$y' + p(x)y = 0 \iff \frac{dy}{y} = -p(x)dx \rightsquigarrow \text{Separated DE.}$$

$$\iff \ln |y| = - \int p(x)dx + c_1$$

$$\iff |y| = e^{c_1} e^{-\int p(x)dx} = ce^{-\int p(x)dx}$$



Method of Solving.

Case 1. If $f(x) = 0$, then equation (3) becomes,

$$y' + p(x)y = 0 \iff \frac{dy}{y} = -p(x)dx \rightsquigarrow \text{Separated DE.}$$

$$\iff \ln |y| = - \int p(x)dx + c_1$$

$$\iff |y| = e^{c_1} e^{-\int p(x)dx} = ce^{-\int p(x)dx}$$

$$\iff y = ce^{-\int p(x)dx} \rightsquigarrow \text{is the general solution}$$



LINEAR FOODEs

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Case 2. If $f(x) \neq 0$, then equation (3) becomes,

$$y' + p(x)y = f(x) \quad (4)$$



Case 2. If $f(x) \neq 0$, then equation (3) becomes,

$$y' + p(x)y = f(x) \quad (4)$$

$$\Longleftrightarrow (p(x)y - f(x))dx + dy = 0$$



Case 2. If $f(x) \neq 0$, then equation (3) becomes,

$$y' + p(x)y = f(x) \quad (4)$$

$$\iff (p(x)y - f(x))dx + dy = 0$$

$$\iff M(x, y) = (p(x)y - f(x)), N(x, y) = 1$$



Case 2. If $f(x) \neq 0$, then equation (3) becomes,

$$y' + p(x)y = f(x) \quad (4)$$

$$\iff (p(x)y - f(x))dx + dy = 0$$

$$\iff M(x, y) = (p(x)y - f(x)), N(x, y) = 1$$

$$\iff M_y = p(x) \neq N_x = 0 \rightsquigarrow \text{the DE is not exact}$$



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Since

$$\frac{1}{N}(M_y - N_x) = p(x) \iff I(x) = e^{\int p(x)dx}$$

Multiplying equation (4) by $I(x)$ we get,



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Since

$$\frac{1}{N}(M_y - N_x) = p(x) \iff I(x) = e^{\int p(x)dx}$$

Multiplying equation (4) by $I(x)$ we get,

$$y' e^{\int p(x)dx} + p(x) y e^{\int p(x)dx} = f(x) e^{\int p(x)dx}$$



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Since

$$\frac{1}{N}(M_y - N_x) = p(x) \iff I(x) = e^{\int p(x) dx}$$

Multiplying equation (4) by $I(x)$ we get,

$$y' e^{\int p(x) dx} + p(x) y e^{\int p(x) dx} = f(x) e^{\int p(x) dx}$$

$$\iff e^{\int p(x) dx} dy + \left(y p(x) e^{\int p(x) dx} \right) dx = \left(f(x) e^{\int p(x) dx} \right) dx$$



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Since

$$\frac{1}{N}(M_y - N_x) = p(x) \iff I(x) = e^{\int p(x)dx}$$

Multiplying equation (4) by $I(x)$ we get,

$$y' e^{\int p(x)dx} + p(x) y e^{\int p(x)dx} = f(x) e^{\int p(x)dx}$$

$$\iff e^{\int p(x)dx} dy + \left(y p(x) e^{\int p(x)dx} \right) dx = \left(f(x) e^{\int p(x)dx} \right) dx$$

$$\iff d \left(y e^{\int p(x)dx} \right) = \left(f(x) e^{\int p(x)dx} \right) dx$$



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$$\Leftrightarrow ye^{\int p(x)dx} = \int \left(f(x)e^{\int p(x)dx} \right) dx + c$$



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$$\Leftrightarrow ye^{\int p(x)dx} = \int \left(f(x)e^{\int p(x)dx} \right) dx + c$$

$$\Leftrightarrow y = e^{-\int p(x)dx} \left[\int \left(f(x)e^{\int p(x)dx} \right) dx + c \right]$$



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$$\Leftrightarrow ye^{\int p(x)dx} = \int \left(f(x)e^{\int p(x)dx} \right) dx + c$$

$$\Leftrightarrow y = e^{-\int p(x)dx} \left[\int \left(f(x)e^{\int p(x)dx} \right) dx + c \right]$$

$$\Leftrightarrow y = \frac{1}{I(x)} \int \left(f(x)e^{\int p(x)dx} \right) dx + \frac{c}{I(x)}$$

is the general solution



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Remark

Sometimes FOODE can't be put in the standard form $y' + p(x)y = f(x)$ for such DEs we regard y as an independent variable and x a dependent variable and may write the DE of the form, $\frac{dx}{dy} + p_1(y)x = q_1(y)$.



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Remark

Sometimes FOODE can't be put in the standard form $y' + p(x)y = f(x)$ for such DEs we regard y as an independent variable and x a dependent variable and may write the DE of the form, $\frac{dx}{dy} + p_1(y)x = q_1(y)$.

Example

$$(x + 2y^3) \frac{dy}{dx} = y \rightsquigarrow \text{Non-Linear}$$

But



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Remark

Sometimes FOODE can't be put in the standard form $y' + p(x)y = f(x)$ for such DEs we regard y as an independent variable and x a dependent variable and may write the DE of the form, $\frac{dx}{dy} + p_1(y)x = q_1(y)$.

Example

$$(x + 2y^3)\frac{dy}{dx} = y \rightsquigarrow \text{Non-Linear}$$

$$\text{But } x' - \frac{1}{y}x = 2y^2 \rightsquigarrow \text{Linear}$$



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Example

Solve the IVP

$$xy' + y = 2x, y(1) = 0$$



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Solution. Write the given equation as

$$y' + \frac{1}{x}y = 2$$



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Solution. Write the given equation as

$$y' + \frac{1}{x}y = 2$$

here $p(x) =$



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Solution. Write the given equation as

$$y' + \frac{1}{x}y = 2$$

here $p(x) = \frac{1}{x}$, then I.F is



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Solution. Write the given equation as

$$y' + \frac{1}{x}y = 2$$

here $p(x) = \frac{1}{x}$, then I.F is $e^{\ln(x)} = x$, and so



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Solution. Write the given equation as

$$y' + \frac{1}{x}y = 2$$

here $p(x) = \frac{1}{x}$, then I.F is $e^{\ln(x)} = x$, and so

$$\frac{d}{dx}(xy) = 2x$$

gives $xy = x^2 + c$. Solving for y yields the general solution



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Solution. Write the given equation as

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$$y = x + \frac{c}{x}$$



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Solution. Write the given equation as

$$y' + \frac{1}{x}y = 2$$

here $p(x) = \frac{1}{x}$, then I.F is $e^{\ln(x)} = x$, and so

$$\frac{d}{dx}(xy) = 2x$$

gives $xy = x^2 + c$. Solving for y yields the general solution

$$y = x + \frac{c}{x}$$

But $y(1) = 0$ implies



Solution. Write the given equation as

$$y' + \frac{1}{x}y = 2$$

here $p(x) = \frac{1}{x}$, then I.F is $e^{\ln(x)} = x$, and so

$$\frac{d}{dx}(xy) = 2x$$

gives $xy = x^2 + c$. Solving for y yields the general solution

$$y = x + \frac{c}{x}$$

But $y(1) = 0$ implies $c = -1$. Hence the particular solution is

$$y = x - \frac{1}{x}, \quad 0 < x < \infty$$



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Example

Solve

$$y' + 5y = x$$



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Solution.

In this DE $p(x) =$



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Solution.

In this DE $p(x) = 5$.



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Solution.

In this DE $p(x) = 5$. Hence

$$I(x) = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x} \iff I(x) = e^{5x}$$



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Solution.

In this DE $p(x) = 5$. Hence

$$I(x) = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x} \iff I(x) = e^{5x}$$

Therefore

$$y'e^{5x} + 5ye^{5x} = xe^{5x}$$

is exact DE.



Solution.

In this DE $p(x) = 5$. Hence

$$I(x) = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x} \iff I(x) = e^{5x}$$

Therefore

$$y'e^{5x} + 5ye^{5x} = xe^{5x}$$

is exact DE.

$$\iff \int d(ye^{5x}) = \int xe^{5x} dx + c$$



Solution.

In this DE $p(x) = 5$. Hence

$$I(x) = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x} \iff I(x) = e^{5x}$$

Therefore

$$y'e^{5x} + 5ye^{5x} = xe^{5x}$$

is exact DE.

$$\iff \int d(ye^{5x}) = \int xe^{5x} dx + c$$

$$\iff ye^{5x} = \frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} + c$$



Solution.

In this DE $p(x) = 5$. Hence

$$I(x) = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x} \iff I(x) = e^{5x}$$

Therefore

$$y'e^{5x} + 5ye^{5x} = xe^{5x}$$

is exact DE.

$$\iff \int d(ye^{5x}) = \int xe^{5x} dx + c$$

$$\iff ye^{5x} = \frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} + c$$

$$\iff y = \frac{x}{5} - \frac{1}{25} + ce^{-5x}$$

is the General Solution.



First Order
Ordinary
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Equations

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Exercise

Solve

1. $y' + y \cos(x) = \sin(x).$

2. $y' + y = \frac{1}{1+e^{2x}}.$

3. $\left. \begin{array}{l} y' = x^3 - 2xy \\ y(1) = 1 \end{array} \right\} IVP.$

4. $(x + 2y^3)y' = y.$



NON-LINEAR FOODEs

First Order
Ordinary
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Equations

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Bernoulli's DE

Form.

$$y' + p(x)y = f(x)y^r, r \in \mathbb{R} \quad (5)$$

Method of Solving.

1. If $r = 0$ then (5) reduced to FOLDE.



NON-LINEAR FOODEs

First Order Ordinary Differential Equations

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Bernoulli's DE

Form.

$$y' + p(x)y = f(x)y^r, r \in \mathbb{R} \quad (5)$$

Method of Solving.

1. If $r = 0$ then (5) reduced to FOLDE.
2. If $r = 1$ then (5) is separable.



First Order Ordinary Differential Equations

By

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Bernoulli's DE

Form.

$$y' + p(x)y = f(x)y^r, r \in \mathbb{R} \quad (5)$$

Method of Solving.

1. If $r = 0$ then (5) reduced to FOLDE.
2. If $r = 1$ then (5) is separable.
3. For $r \neq 0$,



LINEAR FOODEs

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$$y' + p(x)y = f(x)y^r \iff y'y^{-r} + p(x)y^{1-r} = f(x) \quad (6)$$



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$$y' + p(x)y = f(x)y^r \iff y'y^{-r} + p(x)y^{1-r} = f(x) \quad (6)$$

Let $v = y^{1-r}$, then $\frac{dv}{dx} = v' = (1-r)y^{-r}y'$, i.e., $v = y^{(1-r)}$
and $y^{-r}y' = \frac{v'}{1-r}$.



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$$y' + p(x)y = f(x)y^r \iff y'y^{-r} + p(x)y^{1-r} = f(x) \quad (6)$$

Let $v = y^{1-r}$, then $\frac{dv}{dx} = v' = (1-r)y^{-r}y'$, i.e., $v = y^{(1-r)}$
and $y^{-r}y' = \frac{v'}{1-r}$.

Substituting in equation (6) we get,



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$$y' + p(x)y = f(x)y^r \iff y'y^{-r} + p(x)y^{1-r} = f(x) \quad (6)$$

Let $v = y^{1-r}$, then $\frac{dv}{dx} = v' = (1-r)y^{-r}y'$, i.e., $v = y^{(1-r)}$
and $y^{-r}y' = \frac{v'}{1-r}$.

Substituting in equation (6) we get,

$$\frac{v'}{1-r} + p(x)v = q(x) \iff v' + (1-r)p(x)v = (1-r)q(x).$$

Which is FOLDE. Therefore the Bernoulli's DE with $v = y^{(1-r)}$ reduces to a linear FODE.



LINEAR FOODEs

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Example

Solve

$$xy' - y = x^2y^2, x > 0$$



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Solution.

The DE is Bernoulli's with $r = 2$.

$$xy' - y = x^2y^2 \iff y'y^{(-2)} - \frac{1}{x}y^{-1} = x.$$



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Solution.

The DE is Bernoulli's with $r = 2$.

$$xy' - y = x^2y^2 \iff y'y^{(-2)} - \frac{1}{x}y^{-1} = x.$$

Put $v = y^{-1}$, then $v' = -y^{-2}y'$.



LINEAR FOODEs

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Solution.

The DE is Bernoulli's with $r = 2$.

$$xy' - y = x^2y^2 \iff y'y^{(-2)} - \frac{1}{x}y^{-1} = x.$$

Put $v = y^{-1}$, then $v' = -y^{-2}y'$.

Substituting: $-v' - \frac{1}{x}v = x \iff v' + \frac{1}{x}v = -x \rightsquigarrow \text{FOLDE.}$



Solution.

The DE is Bernoulli's with $r = 2$.

$$xy' - y = x^2y^2 \iff y'y^{(-2)} - \frac{1}{x}y^{-1} = x.$$

Put $v = y^{-1}$, then $v' = -y^{-2}y'$.

Substituting: $-v' - \frac{1}{x}v = x \iff v' + \frac{1}{x}v = -x \rightsquigarrow$ FOLDE.

$$I(x) = e^{\int \frac{1}{x} dx} = x.$$



LINEAR FOODEs

First Order
Ordinary
Differential
Equations

By

Dagnachew
Jenber

Solution.

The DE is Bernoulli's with $r = 2$.

$$xy' - y = x^2y^2 \iff y'y^{(-2)} - \frac{1}{x}y^{-1} = x.$$

Put $v = y^{-1}$, then $v' = -y^{-2}y'$.

Substituting: $-v' - \frac{1}{x}v = x \iff v' + \frac{1}{x}v = -x \rightsquigarrow$ FOLDE.

$$I(x) = e^{\int \frac{1}{x} dx} = x.$$

$$\implies xv' + v = -x^2 \iff \frac{d(xv)}{dx} = -x^2$$



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$$\iff y = \frac{3}{3cx^{-1} - x^2} \text{ is the required G.S.}$$



LINEAR FOODEs

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Exercise

Solve

1. $y' + y = x^2 y^{-2}.$

2. $2xydy - (x^2 + y^2 + 1)dx = 0$



The Riccati's DE

Form.

$$y' + p(x)y + f(x)y^2 = r(x) \quad (7)$$

Method of solving.

1. If $f(x) = 0$, then (7) is FOLDE.

2. **Form.**

$$y' + p(x)y + f(x)y^2 = r(x) \quad (8)$$

If $r(x) = 0$, then (8) is Bernoulli's DE.

3. In general Riccati's DE can't be solved by elementary methods. However, it can be solved if at least one non-trivial particular solution is known.



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Suppose $y_1 = y_1(x)$ is a non-trivial solution of (8).



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Suppose $y_1 = y_1(x)$ is a non-trivial solution of (8).
Let $y = v + y_1$ be the general solution of (8) then
 $y' = v' + y_1'$ and if we substitute these equations in (8) we
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$$\iff y_1' + p(x)y_1 + f(x)y_1^2 + v' + p(x)v + f(x)v^2 + 2f(x)vy_1 = r(x)$$



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which is Bernoulli's DE.



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which is Bernoulli's DE. Therefore if v is the general solution of (9) then the general solution of (8) is $y = v + y_1$, where y_1 is the given non-trivial solution of (8).



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Example

Solve $y' + xy^2 = x$ with $y_1 = 1$ a given solution.



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Solution.

The given DE is Riccat's DE.



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Solution.

The given DE is Riccat's DE.

Let $y = v + y_1 = v + 1$ is the general solution.



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Solution.

The given DE is Riccat's DE.

Let $y = v + y_1 = v + 1$ is the general solution.

$$\leadsto y' = v'.$$



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Solution.

The given DE is Riccat's DE.

Let $y = v + y_1 = v + 1$ is the general solution.

$\leadsto y' = v'$.

Substituting: $v' + x(v + 1)^2 = x$



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$$\text{Substituting: } v' + x(v + 1)^2 = x$$

$$\iff v' + xv^2 + 2xv + x = x$$



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$$\text{After some calculation we get, } v = \frac{1}{-\frac{1}{2} + ce^{x^2}}.$$



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$$\text{After some calculation we get, } v = \frac{1}{-\frac{1}{2} + ce^{x^2}}.$$

$$\text{Thus } y = \frac{1}{-\frac{1}{2} + ce^{x^2}} + 1 \text{ is the required general solution.}$$