Steven Givant

Advanced Topics in Relation Algebras

Relation Algebras, Volume 2

Springer

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ISBN 978-3-319-65944-2 ISBN 978-3-319-65945-9 (eBook)
DOI 10.1007/978-3-319-65945-9

Library of Congress Control Number: 2017952945

Mathematics Subject Classification: 03G15, 03B20, 03C05, 03B35

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To the memories of Alfred Tarski and Bjarni Jónsson.

Preface

This is the second volume of a two-volume textbook on relation algebras. The first volume, *Introduction to Relation Algebras*, begins with the underlying motivation, going back to the calculus of relations of De Morgan, Peirce, and Schröder, and with the basic definitions, axioms, and examples of the subject. There follows a development of the arithmetic of relation algebras in which the most important laws are derived systematically from the axioms, with special emphasis on those laws that apply to, or even characterize, specific types of elements, such as as equivalence elements, functional elements, and ideal elements. The remainder of the first volume is devoted to an exposition of the algebraic side of the subject: subalgebras, homomorphisms, ideals, quotient algebras, simple algebras, direct products, and so on.

The purpose of this second volume is to make a systematic, cohesive, and detailed presentation of a selection of more advanced topics of the subject, topics that have been active areas of research over the last few decades, more accessible to readers, with the hope of bringing them to some of the frontiers of research on relation algebras and Boolean algebras with operators.

Intended audience

This volume is aimed at, but is not limited to, graduate students and professionals in a variety of mathematical disciplines, especially various branches of logic, universal algebra, and theoretical computer science. As regards the background needed to read this volume, it is helpful to have a general understanding of the basic notions and results of the theory of relation algebras, and some familiarity with the basic notions and results of universal algebra. The background provided in the

first volume, *Introduction to Relation Algebras*, is more than sufficient. Note that this second volume contains numerous, essential references to the first. The reader is strongly encouraged to secure at least electronic access to the first book in order to make use of the second. Any reference in this volume to material in Chapters 1–13 refers to the relevant result in the first volume.

Each chapter ends with a historical section and a substantial number of exercises. The exercises vary in difficulty from routine problems that help readers understand the basic definitions and theorems presented in the text, to intermediate problems that extend or enrich the material developed in the text, to difficult problems that often present important results not covered in the text. Hints and solutions to some of the exercises are available for download from the Springer book webpage. The main topics covered in this volume are canonical extensions, completions, representation theorems, varieties and universal classes, and atom structures.

Acknowledgements

I am very much indebted to Hajnal Andréka, Robert Goldblatt, Ian Hodkinson, Peter Jipsen, Bjarni Jónsson, Richard Kramer, Roger Maddux, Ralph McKenzie, Don Monk, and István Németi for the helpful remarks and suggestions that they provided to me in correspondence during the composition of this work. Some of these remarks are referred to in the historical sections at the end of the chapters. Kexin Liu read the second draft of the entire text, caught numerous typographic errors, and made many suggestions for stylistic improvements. I am very grateful to her.

Loretta Bartolini an editor of the mathematical series *Graduate Texts in Mathematics*, *Undergraduate Texts in Mathematics*, and *Universitext* published by Springer, has served as the editor for these two volumes. She has given me a great deal of advice and guidance during the publication process, and I am very much indebted to her and her entire production team at Springer for pulling out all stops, and doing the best possible job in the fastest possible way, to produce these two companion volumes. Any errors or flaws that remain in the volumes are, of course, my own responsibility.

California, USA Steven Givant
July 2017

Introduction

A mathematical theory of binary relations, together with certain operations on and between these relations, was initiated by De Morgan [26] in 1864. It was given a proper foundation by Peirce [116], who over a ten-year period settled on a very natural set of operations and developed a kind of calculus of relations, in analogy with the Boole-Jevons calculus of classes. This calculus was further developed and extended in a very systematic way by Schröder [121].

In 1941, Tarski [132] reformulated the calculus of relations as an abstract, algebraic theory with a finite number of essentially equational axioms—later to be called the theory of relation algebras—much as Huntington had reformulated the calculus of classes as an abstract, algebraic theory with a finite number of equational axioms, later called the theory of Boolean algebras. Tarski posed several fundamental problems concerning his reformulated theory, and these problems initiated a period of sustained growth and development of the subject and the closely related, but much more general, theory of Boolean algebras with operators. A brief sketch of some of the most important developments during the period 1941–1966 is given in the introduction to the first volume, *Introduction to Relation Algebras*.

Since that time, some of the more active areas of research in the subject have been: applications to logic; applications to computer science; canonical extensions and completions of Boolean algebras with operators; connections with the theory of cylindric algebras; representation theory; decision problems; the lattice of varieties of relation algebras; axiomatizability and non-axiomatizability of classes of relation algebras, in particular, the class of representable relation algebras; free relation algebras; and generalized relativizations of relation algebras.

It would be impossible to treat all of these topics in depth in this one volume. We focus on a few of them in the main text, and briefly discuss some of the others in the Epilogue.

Description and highlights of this volume

This volume contain six chapters that deal with more advanced topics. Chapter 14 introduces the notion of the canonical, or perfect, extension of a Boolean algebra with operators $\mathfrak{A}$ as an analogue of the well-known construction of the Boolean algebra of all subsets of the set of ultrafilters in a given Boolean algebra. The canonical extension of $\mathfrak{A}$ is a complete and atomic extension of $\mathfrak{A}$ with certain additional properties, namely the compactness and atom separation properties. It is shown that every Boolean algebra with operators $\mathfrak{A}$ has a canonical extension, and this extension is unique up to isomorphisms that are the identity function on $\mathfrak{A}$. The preservation theorems for canonical extensions say that all positive equations, and certain types of positive implications, are preserved under the passage to canonical extensions. It follows, in particular, that every relation algebra has an essentially unique canonical extension that is also a relation algebra. An interesting consequence of the preservation theorems is that a relation algebra has a total direct decomposition into the product of finitely many simple factors if and only if its canonical extension has a corresponding total direct decomposition into the product of finitely many simple factors. It is also shown that every homomorphism φ between Boolean algebras with operators $\mathfrak{A}$ and $\mathfrak{B}$ can be extended in a unique way to a complete homomorphism between the canonical extensions of $\mathfrak{A}$ and $\mathfrak{B}$, and this extension is one-to-one or onto if and only if φ is one-to-one or onto. This extension theorem implies a number of algebraic preservation theorems for canonical extensions. For example, if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, then the canonical extension of $\mathfrak{A}$ is a complete subalgebra of the canonical extension of $\mathfrak{B}$. A characterization is given of when the canonical extension of a subalgebra of a full set relation algebra $\mathfrak{A}$ is itself a complete subalgebra of $\mathfrak{A}$. Finally, at the end of the chapter it is shown that every homomorphic image of a relation algebra $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to some (closed) ideal element belonging to the canonical extension of $\mathfrak{A}$. Thus, every

homomorphic image of a relation algebra $\mathfrak{A}$ is rather close to being a subalgebra of $\mathfrak{A}$ (up to isomorphisms).

A parallel development for the notion of the completion of a Boolean algebra with quasi-complete operators $\mathfrak{A}$ is given in Chapter 15. This is the analogue of the construction of the completion of a Boolean algebra via the algebra of complete ideals, or equivalently, via Dedekind cuts, and consequently it is also the analogue of the construction of the real numbers from the rational numbers via Dedekind cuts. It is shown that every Boolean algebra with quasi-complete operators $\mathfrak{A}$ has a completion that is unique up to isomorphisms that are the identity function on $\mathfrak{A}$. Just as in the case of canonical extensions, there are preservation theorems saying that positive equations and certain types of positive implications are preserved under the passage to completions. In particular, every relation algebra has an essentially unique completion that is also a relation algebra. An interesting consequence of the preservation theorems is that a relation algebra has a total direct decomposition into a product of any number of simple factors if and only if its completion has a corresponding total direct decomposition into a product of simple factors. It is also shown that every complete homomorphism φ from a Boolean algebra with quasi-complete operators $\mathfrak{A}$ into a complete Boolean algebra with quasi-complete operators $\mathfrak{B}$ can be extended in a unique way to a complete homomorphism from the completion of $\mathfrak{A}$ into $\mathfrak{B}$. Moreover, if φ is one-to-one or if $\mathfrak{B}$ is the completion of the image of $\mathfrak{A}$ under φ, then the extension of φ is one-to-one or onto respectively. As an application of this extension theorem, it is shown that the completion of $\mathfrak{A}$ can be characterized as the minimal complete, regular extension of $\mathfrak{A}$. The extension theorem also implies a number of algebraic preservation theorems for completions. For example, if $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{B}$, then the completion of $\mathfrak{A}$ is, up to isomorphism, a regular subalgebra of the completion of $\mathfrak{B}$. Finally, it is shown that every complete homomorphic image of a relation algebra $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to some ideal element belonging to the completion of $\mathfrak{A}$.

A *representation* of a relation algebra $\mathfrak{A}$ is an isomorphism from $\mathfrak{A}$ to a set relation algebra. A representation is said to be *complete* if it preserves all existing sums as unions. Representations are the subject of Chapters 16 and 17. The focus of first of these chapters is on the algebra of representations. A Löwenheim-Skolem-Tarski type theorem is proved for representations: if a relation algebra of cardinality κ is representable as a set relation algebra on an infinite set, then it is rep-

resentable as a set relation algebra on sets of every infinite cardinality greater than or equal to κ. Most of these representations are incomplete. Also, it is shown that the property of being representable is preserved under the standard algebraic constructions: subalgebras, relativizations, homomorphic images, direct products, and directed unions of representable relation algebras are all representable. Highlights of the chapter include a proof that the canonical extension of a representable relation algebra is completely representable, and a proof that the completion of a completely representable relation algebra is completely representable. As regards the relationship between representability and complete representability, it is shown that a completely representable relation algebra is necessarily atomic, and if an atomic relation algebra $\mathfrak{A}$ has the property that the relative product of any two atoms is a finite sum of atoms, then from every representation of $\mathfrak{A}$ one may construct a complete representation of $\mathfrak{A}$.

Chapter 17 contains a number of representation theorems for concrete classes of relation algebras. For example, every formula relation algebra is representable, and every representable relation algebra is isomorphic to a relativization of a formula relation algebra. Every Boolean relation algebra is representable, and every atomic Boolean relation algebra is completely representable. Every complex algebra of a group is completely representable via the Cayley representation, and every complete representation of this complex algebra is equivalent to its Cayley representation. Not every complex algebra of a projective geometry is representable; in fact, there are infinitely many complex algebras of finite projective lines that are not representable. The representability of the complex algebra of a projective geometry P may be characterized in terms of P as follows: the complex algebra of P is representable if and only if P is embeddable into a projective geometry of one higher dimension, and if this is the case, then P has a complete representation in terms of an affine geometry of which P is the geometry at infinity. A connection between the collineations of a projective geometry and the complete representations of the complex algebra of the geometry is established, and using this connection, a formula for the number of inequivalent representations of the complex algebras of finite projective lines is given. For example, the complex algebra of a projective line of order nine has 56,700 inequivalent representations. The representability of various small relation algebras is also studied, and an example of a minimal non-representable relation algebra (having four atoms) is given. An analysis of this algebra also

yields a concrete equation that is true in all representable relation algebras but not in all relation algebras.

The second part of Chapter 17 begins with a general theorem to the effect that every relation algebra has a quasi-representation, and every atomic relation has a complete quasi-representation. A quasi-representation is a bijection from a relation algebra to a set relation algebra that maps the identity element to the identity relation and preserves sums as unions, relative products as relational compositions, and converses as relational inverses; but such a mapping need not preserve products as intersections, nor complements as set-theoretic complements. With the help of this theorem, it is shown that every atomic relation algebra with functional atoms is completely representable. Similarly, it is shown that a singleton-dense relation algebra—that is to say, a relation algebra in which every non-zero element is above an element that behaves like a singleton relation—is representable.

Chapter 18 is concerned with varieties of relation algebras and universal classes of simple relation algebras, that is to say, classes of relation algebras axiomatizable by sets of equations and first-order universal sentences respectively. The initial sections contain proofs of the following theorems: the Fundamental Theorem of Ultraproducts, which says that first-order properties are preserved under the passage to ultraproducts; the closely related SP_u-Theorem, which says that a class of algebras is axiomatizable by a set of universal sentences if and only if it is closed under ultraproducts and subalgebras; and the well-known HSP-Theorem, which says that a class of algebras is axiomatizable by a set of equations if and only if it is closed under subalgebras, direct products, and homomorphic images. The Correspondence Theorem for the lattice of varieties says that the correspondence mapping every variety K of relation algebras to the universal class of simple relation algebras in K is a lattice isomorphism from the lattice of varieties of relation algebras to the lattice of universal classes of simple relation algebras. Consequently, one may gain information about the lattice of varieties by studying the lattice of universal classes of simple relation algebras.

Some of the highlights of the chapter include the following results. The class of representable relation algebras is a variety, but it is not finitely axiomatizable, and it is not even axiomatizable by an infinite set of equations that uses only finitely many variables. Above the zero element of the lattice of varieties, there are three minimal varieties of relation algebras, namely the varieties M_1, M_2, and M_3 generated

by the minimal set relation algebras on sets of cardinalities one, two, and three respectively. Just above the minimal varieties, there are the varieties that are the joins of two of the minimal varieties. There are no other varieties that are directly above M_1, and exactly one other variety that is immediately above M_2, namely the variety generated by the full set relation algebra on a set of cardinality two. As regards M_3, there are exactly three varieties generated by non-integral relation algebras that are immediately above M_3. There are seventeen known examples of varieties immediately above M_3 that are generated by finite integral relation algebras, and one that is generated by an infinite integral relation algebra. Just below the unit of the lattice of varieties of relation algebras, there are infinitely many maximal varieties. In particular, for each natural number $n \geq 1$, the variety generated by the class of simple relation algebras that are not isomorphic the full set relation algebra on a set of cardinality n is a maximal element of the lattice. The full set relation algebras on sets of finite cardinalities n generate varieties V_n that are distinct from one another for distinct natural numbers $n \geq 1$. On other hand, the full set relation algebras on infinite sets all have the same equational theory and therefore generate the same variety V_ω. The variety of representable relation algebras is the irreducible join of the varieties V_n for $1 \leq n \leq \omega$. The lattice of all subsets of the natural numbers is embeddable into the lattice of varieties of representable relation algebras. In fact, it is embeddable into the interval $[\mathsf{M}_3, \mathsf{V}_f]$ consisting of all subvarieties of V_f that include the variety M_3, where V_f is the join of the varieties V_n for positive integers n (thus, V_ω is excluded from this join). Consequently, this interval contains continuum many varieties, and it even contains a chain of the same order type as the real numbers. The interval $[\mathsf{M}_3, \mathsf{V}_{hi}]$ also contains continuum many varieties, where V_{hi} is the variety generated by the class of hereditarily infinite representable relation algebras, that is to say, the variety generated by the class of infinite, representable relation algebras all of whose non-minimal subalgebras are infinite. Thus, there are continuum many varieties of hereditarily infinite, representable relation algebras.

In Chapter 19, the duality between Boolean algebras with operators and relational structures is studied. It is shown that the complex algebra of every relational structure is a complete and atomic Boolean algebra with complete operators; and conversely, every complete and atomic Boolean algebra with complete operators is isomorphic to the complex algebra of some relational structure, namely the atom struc-

ture of the algebra, that is to say, the relational structure consisting of the atoms in the algebra under certain natural relations that are induced on the atoms by the operations of the algebra. This duality implies the following representation theorem for Boolean algebras with operators: every Boolean algebra with operators is embeddable into the complex algebra of some relational structure. In particular, every relation algebra is embeddable into the complex algebra of an appropriate relational structure. In the case of relation algebras, the class of these appropriate relational structures may be axiomatized by a very simple set of first-order formulas. More generally, if V is any variety of Boolean algebras with complete operators, then the class of atom structures of the atomic algebras in V is axiomatizable by a set of first-order formulas. In general, this axiomatization is infinite and complicated, but if the equations axiomatizing V have a certain simple form, then the class of atom structures of atomic algebras in V has a correspondingly simple form.

One of the highlights of Chapter 19 is a proof of the deep theorem that for a class of relational structures closed under ultraproducts, and in particular for a class of relational structures axiomatizable by a set of first-order formulas, the variety generated by the class L of complex algebras of these relational structures is just the class $\mathsf{SP}(\mathsf{L})$ of isomorphic copies of subalgebras of direct products of algebras in L—in other words, $\mathsf{SP}(\mathsf{L})$ is automatically closed under homomorphic images—and furthermore, $\mathsf{SP}(\mathsf{L})$ is also closed under canonical extensions in the sense that the canonical extension of every algebra in this variety also belongs to the variety. Several applications of this theorem to classes of relation algebras are given. For example, if L is the class of all complex algebras of groups, or the class of all complex algebras of projective geometries, or the class of all complex algebras of modular lattices with zero, then $\mathsf{SP}(\mathsf{L})$ is the variety generated by L, and this variety is closed under canonical extensions. As another application, an especially transparent proof is given for the theorem that the class of representable relation algebras is a variety

With one exception, the chapters in volume 2 are written to be largely independent of one another. Occasionally, a definition or theorem from one chapter is used in a later one, but in this case it suffices to read the statement of the definition or theorem. Chapter 14 (canonical extensions) and Chapter 15 (completions) are developed in parallel, but independently of one another. Chapter 17 (representation theorems) is dependent on much of the material in Chapter 16 (repre-

sentations). Chapter 18 (varieties) and Chapter 19 (atom structures) are largely independent of one another and of the earlier chapters.

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Chapter 14
Canonical extensions

The structural analysis of a relation algebra is simplified substantially when the algebra in question is complete and atomic. Of course, not all relation algebras are complete or atomic, but it is a happy state of affairs that every relation algebra can be extended to one that is. The purpose of this chapter is to study the most important of these extensions. The construction actually goes through in the context of arbitrary Boolean algebras with operators. For the sake of concreteness and to simplify notation, we always take the similarity type of the algebras under discussion to be the same as the similarity type of relation algebras; but it should be obvious from the presentation how to extend the development to Boolean algebras with operators of arbitrary ranks.

14.1 Canonical extensions of Boolean algebras

We begin with the case of Boolean algebras without any additional operators. These algebras are denoted by upper case italic letters. Here is the basic definition.

Definition 14.1. A *canonical* (or *perfect*) *extension* of a Boolean algebra A is a Boolean algebra B with the following properties.

(i) The algebra B is complete and atomic, and A is a subalgebra of B.
(ii) For any two distinct atoms a and b in B, there is an element r in A such that $a \leq r$ and $b \leq -r$.

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9_1

(iii) For any subset X of A, if 1 is the supremum of X in B, then 1 is already the supremum of some finite subset of X (in B and in A). □

A Boolean algebra B is said to have the *compactness property* with respect to a subalgebra A if condition (iii) holds. The terminology derives from the relationship of the condition to the compactness property of topological spaces that are typically associated with Boolean algebras, so-called *Boolean spaces*. As regards the parenthetical remark at the end of the condition, notice that the sum (as well as the product) of any finite subset of A is the same in A as it is in B, because of the subalgebra requirement; so it does not matter whether the sum (or product) of such a finite subset is formed in A or in B.

Condition (iii) is easily seen to be equivalent to its dual version, which asserts that for any subset X of A, if 0 is the infimum of X in B, then 0 is already the infimum of some finite subset of X. Condition (iii) also implies a more general compactness property.

Lemma 14.2. *Suppose a Boolean algebra B has the compactness property with respect to a subalgebra A. For any subsets X and Y of A, if*

$$\prod X \leq \sum Y$$

in B, then there are finite subsets X_0 of X and Y_0 of Y such that

$$\prod X_0 \leq \sum Y_0.$$

Proof. The following notation will be useful. For any subset W of A, write $-W$ for the set of complements of elements in W,

$$-W = \{-r : r \in W\},$$

and observe that $-W$ is also a subset of A. To prove the lemma, consider arbitrary subsets X and Y of A satisfying the hypothesis

$$\prod X \leq \sum Y \tag{1}$$

in B. The complement of the element $\prod X$ is the sum $\sum(-X)$, so

$$\begin{aligned} 1 = -(\textstyle\prod X) + \prod X = \sum(-X) + \prod X \\ \leq \sum(-X) + \sum Y = \sum(-X \cup Y) \end{aligned}$$

in B, by Boolean algebra and (1). The compactness property therefore implies the existence of a finite subset Z of $-X \cup Y$ such that $\sum Z = 1$. Write

$$X_0 = X \cap -Z \qquad \text{and} \qquad Y_0 = Y \cap Z,$$

and observe that $-X \cap Z = -X_0$. In more detail,

$$\begin{aligned} r \in -X_0 \quad & \text{if and only if} \quad -r \in X_0, \\ & \text{if and only if} \quad -r \in X \text{ and } -r \in -Z, \\ & \text{if and only if} \quad r \in -X \text{ and } r \in Z, \\ & \text{if and only if} \quad r \in -X \cap Z, \end{aligned}$$

by the definitions of $-X_0$, of X_0, of $-X$ and $-Z$, and of intersection. Consequently,

$$Z = (-X \cup Y) \cap Z = (-X \cap Z) \cup (Y \cap Z) = -X_0 \cup Y_0.$$

The complement of the element $\prod X_0$ is the sum $\sum(-X_0)$, so

$$1 = \sum Z = \sum(-X_0 \cup Y_0) = \sum(-X_0) + \sum Y_0 = -(\prod X_0) + \sum Y_0,$$

by Boolean algebra. Consequently, $\prod X_0 \leq \sum Y_0$. □

The general compactness property of the lemma is actually equivalent to the compactness property of condition (iii). The lemma establishes one direction of the implication. To establish the reverse direction, consider an arbitrary subset Y of A such that $\sum Y = 1$ in B, take X to be the subset $\{1\}$, and apply the general compactness property of the lemma to obtain a finite subset Y_0 of Y such that $\sum Y_0 = 1$.

There is a somewhat surprising negative consequence of compactness that should be noted. If an infinite subset Y of a Boolean algebra A has a supremum r in A, and if r is not already the supremum of some finite subset of Y, then r cannot be the supremum of Y in the canonical extension of A, by the compactness property. The proof proceeds by contraposition: if $r = \sum Y$ in the canonical extension, then $r \leq \sum Y_0$ for some finite subset Y_0 of Y, by Lemma 14.2 (with $\{r\}$ in place of X), so $r = \sum Y_0$ in both A and its canonical extension. This failure can be expressed by saying that all proper infinite sums in A are *broken* (that is to say, changed) in the passage to a canonical extension.

An element p in a Boolean algebra B is said to be *open* or *closed*, with respect to a fixed subalgebra A, according to whether p is respectively the supremum or the infimum in B of a subset of A. The

terminology derives from the fact that the corresponding sets in a certain Boolean set algebra are just the open sets and the closed sets of the associated Boolean space. Notice that every element p in A is the supremum and infimum (in both A and B) of the subset $\{p\}$ of A, so every element in A is both open and closed. Such elements are said to be *clopen*.

Corollary 14.3. *Suppose a Boolean algebra B has the compactness property with respect to a subalgebra A.*

(i) *For each closed element p in B and each subset Y of A, if $p \leq \sum Y$ in B, then there is a finite subset Y_0 of Y such that $p \leq \sum Y_0$.*
(ii) *For each open element q in B and each subset X of A, if $\prod X \leq q$ in B, then there is a finite subset X_0 of X such that $\prod X_0 \leq q$.*

Proof. To prove (i), consider a closed element p in B. There must be a subset X of A such that $p = \prod X$ in B, by the definition of a closed element. Assume $p \leq \sum Y$, and apply Lemma 14.2 to obtain finite subsets X_0 and Y_0 of X and Y respectively, such that $\prod X_0 \leq \sum Y_0$. Clearly,

$$p = \prod X \leq \prod X_0 \leq \sum Y_0,$$

by Boolean algebra. The proof of (ii) is similar. □

An atomic Boolean algebra B is said to have the *atom separation property* with respect to a subalgebra A if condition (ii) in Definition 14.12 holds for B and A, that is to say, if any two atoms in B can always be separated (in the sense of (ii)) by an element in A. The next lemma gives an equivalent formulation of this property.

Lemma 14.4. *An atomic Boolean algebra B has the atom separation property with respect to a subalgebra A if and only if*

$$a = \prod\{r : a \leq r \text{ and } r \in A\}$$

for every atom a in B.

Proof. Assume first that the atom separation property holds for B with respect to A. Consider an arbitrary atom a in B, and write

$$X = \{r \in A : a \leq r\}. \tag{1}$$

It is to be shown that a is the infimum of the set X. Certainly, a is a lower bound of X in B, by (1). Let s be any lower bound of X in B. For

each atom b different from a, the atom separation property guarantees the existence of an element r in A such that

$$a \leq r \qquad \text{and} \qquad b \leq -r. \tag{2}$$

Since r is in X, by (1) and (2), we have $s \leq r$, by the assumption that s is a lower bound of X. The atom b cannot be below s, for this would imply that $b \leq s \leq r$ and therefore $b \leq r \cdot -r = 0$, by (2), in contradiction to the assumption that b is an atom. Thus, no atom in B, except possibly a, can be below s. Since B is atomic, every element in B is the sum of the atoms below it. In particular, either $s = 0$ or $s = a$. Conclusion: a is the greatest lower bound of the set X.

To establish the reverse implication, suppose every atom a in B is the infimum of the set X defined in (1). To prove that B has the atom separation property with respect to A, consider two distinct atoms a and b in B. The sum $s = a + b$ is strictly greater than a, because a and b are distinct, so s cannot be a lower bound of the set X defined in (1), by the assumption that a is the greatest lower bound of X. Consequently, there must be an element r in X such that s is not below r. Since a is below r, by (1), it follows from the definition of s that b cannot be below r. Thus, b must be below $-r$, because b is an atom. Conclusion: the element r in A separates the atoms a and b. □

Fix a Boolean algebra A and a canonical extension B of A. One consequence of the preceding lemma is that the atoms in B are always closed elements. The closed elements in B have an interesting characterization in terms of the filters in A. Recall from Section 8.12 that a (*Boolean*) *filter* in A is a subset X of A with the following properties: first, X contains the unit element 1; second, X is closed under multiplication in the sense that if r and s are in X, then so is $r \cdot s$; and third, X is upward closed in the sense that if r is in X and $r \leq s$, then s is in X. A filter in A is said to be *proper* if it does not contain every element in A, or equivalently, if it does not contain 0.

An *ultrafilter* in A is a maximal proper filter; in other words, it is a proper filter that is not properly included in any other proper filter in A. (We shall always use the term "ultrafilter" to refer to maximal *Boolean* filters, not maximal relation algebraic filters.) The following conditions on a filter X in A are easily seen to be equivalent: (1) X is an ultrafilter; (2) for every element r in A, exactly one of r and $-r$ is in X; (3) X is a proper filter, and for every pair of elements r and s in A, the sum $r + s$ is in X if and only if r or s is in X.

A subset X of A is said to have the *finite meet property* if for each finite subset Y of X, the product $\prod Y$ is not zero. Every subset X of A with the finite meet property can be extended to an ultrafilter. Indeed, the set of elements

$$Z = \{r \in A : \prod Y \leq r \text{ for some finite set } Y \subseteq X\}$$

is a filter in A, by the Boolean version of the dual of Theorem 8.12. Moreover, Z is a proper filter because every element in Z is non-zero, by the finite meet property, which is assumed to hold for X. Consequently, Z can be extended to an ultrafilter, by the Boolean version of the dual of the Maximal Ideal Theorem 8.31.

For each element p in B, the set

$$A_p = \{r : r \in A \text{ and } p \leq r\}$$

is easily seen to be a filter in A. Indeed, $p \leq 1$, and 1 is in A, so 1 is in A_p. If r is in A_p, and if s is an element in A such that $r \leq s$, then $p \leq r \leq s$ and therefore s is in A_p. Finally, if r and s are in A_p, then each of these elements is in A, by definition, so their product is in A. Furthermore, p is below each of r and s, so p is below their product. Consequently, $r \cdot s$ is in the set A_p, by the definition of this set.

The distinct closed elements in B can be characterized as the infima of the distinct filters in A.

Lemma 14.5. *Suppose B is a canonical extension of a Boolean algebra A. Every closed element p in B is the infimum of some filter in A, namely the filter A_p. Every filter X in A has the form $X = A_p$ for some closed element p in B, namely the element p that is the infimum of X in B. Two closed elements p and q in B are equal if and only if the corresponding filters A_p and A_q are equal.*

Proof. Consider a closed element p in B. There must be a subset X of A such that p is the infimum of X in B, by the definition of a closed element. Each element in X is above p and therefore in A_p, by the definition of A_p, so X is included in A_p. Since every element in A_p is above p, we have

$$p = \prod X \geq \prod A_p \geq p,$$

by Boolean algebra. The first and last terms are the same, so equality must hold everywhere. Thus, a closed element p is always the infimum of the set A_p.

Now consider an arbitrary filter X in A, and let p be the closed element that is the infimum of X in B; the element p exists by the assumption that B is complete. It was shown in the preceding paragraph that X is included in the set A_p. To establish the reverse inclusion, observe that for any element r in A_p, we have

$$\prod X = p \leq r,$$

and therefore $\prod X_0 \leq r$ for some finite subset X_0 of X, by the compactness property in Corollary 14.3(ii). The product $\prod X_0$ belongs to X, by the closure of X under multiplication, and therefore r belongs to X, by the upward closure of X. Thus, $X = A_p$.

For closed elements p and q in B, obviously $p = q$ implies $A_p = A_q$. The reverse implication follows from the fact that p and q are the infima in B of the filters A_p and A_q respectively. □

The distinct open elements in B can be characterized in a dual fashion.

Corollary 14.6. *If B is a canonical extension of a Boolean algebra A, then the distinct open elements in B are precisely the suprema in B of the distinct ideals in A.*

For each atom a in B, the filter A_a is actually an ultrafilter. In more detail, the filter is proper because it does not contain 0, which is not above the atom a; and if r is any element in A, then the atom a must be below exactly one of r and $-r$, and therefore exactly one of r and $-r$ must be in A_a. The distinct atoms in B can be characterized as the infima of distinct ultrafilters in A.

Lemma 14.7. *Suppose B is a canonical extension of a Boolean algebra A. Every atom a in B is the infimum of some ultrafilter in A, namely the ultrafilter A_a. Every ultrafilter X in A has the form $X = A_a$ for some atom a in B, namely the atom a that is the infimum of X in B. Two atoms a and b in B are equal if and only if the corresponding ultrafilters A_a and A_b are equal.*

Proof. Every atom a in B is the infimum of the ultrafilter A_a, by Lemma 14.5 and the remarks preceding this lemma, since every atom in B is a closed element. Consider now an arbitrary ultrafilter X in A, and let a be the infimum of this ultrafilter in B. Observe that $X = A_a$, by Lemma 14.5. If a were 0, then the infimum of some finite subset

of X would be 0, by the compactness property in Corollary 14.3(ii), since 0 is an open element. It follows that 0 would belong to X, by the closure of X under multiplication, in contradiction to the assumption that X is a proper filter. Thus, a is non-zero. For every element r in A, one of r and $-r$ is in X, by the assumption that X is an ultrafilter, and therefore one of r and $-r$ is above a, by the definition of a as the infimum of X. Consequently, a is an atom in B. The final assertion of the lemma follows from the final assertion of Lemma 14.5. □

The preceding lemma suggests a method of constructing a canonical extension of an arbitrary Boolean algebra A. Atoms in a canonical extension correspond to ultrafilters in A, by the lemma. Moreover, every element in a canonical extension is a sum of atoms, so arbitrary elements in a canonical extension should correspond to arbitrary sets of ultrafilters in A. With this thought in mind, let U be the set of all ultrafilters in A, and let C be the Boolean set algebra of all subsets of U. Thus, the elements in C are the various sets of ultrafilters in A. Obviously, C is a complete and atomic Boolean algebra. The atoms in C are the singletons of ultrafilters in A, and the supremum of each subset of C is simply the union of that subset.

It turns out that the algebra C contains a copy of the algebra A. In fact, the behavior of an element r in A is mimicked in C by the set of ultrafilters that contain r.

Lemma 14.8. *The function φ defined by*

$$\varphi(r) = \{X \in U : r \in X\}$$

for each r in A is a monomorphism from A into C.

Proof. It is not difficult to check that φ is a Boolean homomorphism. Indeed, if r and s are elements in A, then

$$\begin{aligned}\varphi(r+s) &= \{X \in U : r+s \in X\} \\ &= \{X \in U : r \in X\} \cup \{X \in U : s \in X\} = \varphi(r) \cup \varphi(s)\end{aligned}$$

and

$$\varphi(-r) = \{X \in U : -r \in X\} = \{X \in U : r \notin X\} = {\sim}\varphi(r),$$

by the definition of φ and the properties of ultrafilters mentioned after the proof of Lemma 14.4.

If r is a non-zero element in A, then there is an ultrafilter Z in A that contains r, since the set $\{r\}$ has the finite meet property. The image set $\varphi(r)$ contains the ultrafilter Z, so it is not empty. Thus, every non-zero element in A is mapped by φ to a non-zero element in C, so φ must in fact be one-to-one, by Lemma 8.37. □

The Boolean algebra C is called the *canonical embedding algebra* of the Boolean algebra A, and the function φ defined in Lemma 14.8 is called the *canonical embedding* of A into C. The subalgebra of C that is the image of A under the mapping φ is an isomorphic copy of A and will be denoted by $\varphi(A)$.

Theorem 14.9. *The canonical embedding algebra of a Boolean algebra A is a canonical extension of the image of A under the canonical embedding.*

Proof. We have already seen that C is complete and atomic, and Lemma 14.8 implies that $\varphi(A)$ is a subalgebra of C. Thus, condition (i) of Definition 14.1 holds. In order to check the validity conditions (ii) and (iii) of the definition, it must be shown that C has the atom separation and compactness properties with respect to $\varphi(A)$.

To verify the atom separation property, consider distinct atoms $\{X\}$ and $\{Y\}$ in C. The sets X and Y are distinct ultrafilters in A, so there must be an element r in A such that r is in X and $-r$ is in Y. Consequently, X belongs to $\varphi(r)$, and Y to $\varphi(-r)$, so that

$$\{X\} \subseteq \varphi(r) \qquad \text{and} \qquad \{Y\} \subseteq \sim\varphi(r).$$

The two atoms are therefore separated by the element $\varphi(r)$ in $\varphi(A)$.

To verify the compactness property, it suffices to verify its dual, by the remarks preceding Lemma 14.2. The argument proceeds by contraposition. Consider an arbitrary subset X of A, and assume that no finite subset of X has the property that its image under φ has zero as its infimum (in $\varphi(A)$ or in C). It is to be shown that the image of X under φ, the set

$$\varphi(X) = \{\varphi(r) : r \in X\},$$

cannot have zero as its infimum in C. The zero element of $\mathfrak{C}$ is the empty set, and infima in $\mathfrak{C}$ are formed as intersections, so it must be shown that the intersection of the set $\varphi(X)$ is not empty. The set $\varphi(X)$ has the finite meet property in $\varphi(A)$, by assumption, and φ is a Boolean isomorphism from A to $\varphi(A)$, so the set X must have the finite meet

property in A. Consequently, X can be extended to an ultrafilter Z in A. Each element r in X belongs to Z, and therefore Z belongs to the set $\varphi(r)$ for each such r, by the definition of φ. It follows that Z belongs to the intersection

$$\bigcap \varphi(X) = \bigcap \{\varphi(r) : r \in X\},$$

so this intersection is not empty, as was to be shown. □

It has been shown that A is embeddable into a complete and atomic Boolean algebra C that is the canonical extension of the image of A under the embedding. An application of the Boolean version of the Exchange Principle (Theorem 7.15) yields a complete and atomic Boolean algebra B that is the canonical of A. The following *Existence Theorem* for canonical extensions of Boolean algebras has been proved.

Theorem 14.10. *Every Boolean algebra has a canonical extension.*

Canonical extensions of a Boolean algebra are uniquely determined up to isomorphism in a strong sense, as the following *Uniqueness Theorem* for canonical extensions of Boolean algebras asserts.

Theorem 14.11. *Any two canonical extensions of a Boolean algebra A are isomorphic via a mapping that is the identity function on A.*

Proof. Consider two canonical extensions B and C of a Boolean algebra A. Each atom b in B determines, and is uniquely determined by, an ultrafilter A_b in A, and each atom c in C determines, and is uniquely determined by, an ultrafilter A_c in A, by Lemma 14.7. Define a function φ from the set of atoms in B to the set of atoms in C by

$$\varphi(b) = c \qquad \text{if and only if} \qquad A_b = A_c. \tag{1}$$

In other words, φ maps b to c if and only if the set of elements in A that are above b in B coincides with the set of elements in A that are above c in C.

For every atom b in B, the ultrafilter A_b determines a unique atom c in C such that the right side of (1) holds, by Lemma 14.7, namely the atom c that is the infimum of the ultrafilter A_b in C. Consequently, the mapping φ is well defined and has the set of atoms in B as its domain. For every atom c in C, the ultrafilter A_c determines a unique atom b in B such that the right side of (1) holds, namely the atom b that is the infimum of the ultrafilter A_c in B. Consequently, φ is one-to-one

and has the set of atoms in C as its range. Thus, φ is a bijection from the set of atoms in B to the set of atoms in C.

Define a function ψ from B to C by

$$\psi(p) = \sum\{\varphi(b) : b \text{ is an atom in } B \text{ and } b \leq p\} \tag{2}$$

for every p in B. The version of the Atomic Isomorphism Theorem 7.11 that applies to Boolean algebras implies that ψ is a Boolean isomorphism from B to C. To see that ψ is the identity function on A, consider an element r in A. An atom b in B is below r just in case r belongs to the ultrafilter A_b determined by b in B, by the definition of the set A_b; and an atom c in C is below r just in case r belongs to the ultrafilter A_c determined by c in C, by the definition of the set A_c. The function φ maps b to c if and only if $A_b = A_c$, by (1), and therefore b is below r in B if and only if its image $\varphi(b)$ is below r in C. Since this is true for every atom b in B, it follows that φ maps the set of atoms in B that are below r onto the set of atoms in C that are below r. The algebra C is atomic, so every element in C is the sum of the atoms it dominates. In particular, this is true of r, so

$$\begin{aligned}\psi(r) &= \sum\{\varphi(b) : b \text{ is an atom in } B \text{ and } b \leq r\} \\ &= \sum\{c : c \text{ is an atom in } C \text{ and } c \leq r\} = r\end{aligned}$$

by (2) and the preceding observations. □

Theorem 14.11 justifies the common practice of speaking about *the* canonical extension of a Boolean algebra.

14.2 Canonical extensions of Boolean algebras with operators

We now turn to the study of canonical extensions of Boolean algebras with operators. The notions and notation discussed in the preceding section for Boolean algebras may be applied *mutatis mutandis* to Boolean algebras with operators $\mathfrak{B}$. For example, $\mathfrak{B}$ is said to have the atom separation property or the compactness property with respect to a subalgebra $\mathfrak{A}$ if the Boolean part of $\mathfrak{B}$ has the atom separation property or the compactness property respectively with respect to the Boolean part of $\mathfrak{A}$. An element p in $\mathfrak{B}$ is said to be open or

closed (with respect to $\mathfrak{A}$) if p is open or closed in the Boolean part of $\mathfrak{B}$ with respect to the Boolean part of $\mathfrak{A}$. For each element p in $\mathfrak{B}$, the set of all elements in $\mathfrak{A}$ that are above p (in $\mathfrak{B}$) is denoted by A_p.

It is occasionally necessary to consider together both atoms and the zero element in $\mathfrak{B}$. An element in $\mathfrak{B}$ is called a *quasi-atom* if it is either zero or an atom, and an operation in $\mathfrak{B}$ is said to be *quasi-complete* if it is quasi-completely distributive in each coordinate, that is to say, the operation is distributive in each coordinate for arbitrary non-empty sets of elements (see Section 2.2).

Definition 14.12. A *canonical* (or *perfect*) *extension* of a Boolean algebra with operators $\mathfrak{A}$ is a Boolean algebra with quasi-complete operators $\mathfrak{B}$ (of the same similarity type as $\mathfrak{A}$) that possesses the following properties.

(i) $\mathfrak{B}$ is complete and atomic, and $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.
(ii) $\mathfrak{B}$ has the atom separation property with respect to $\mathfrak{A}$.
(iii) $\mathfrak{B}$ has the compactness property with respect to $\mathfrak{A}$.
(iv) For any quasi-atoms a and b in $\mathfrak{B}$,

$$a\,;b = \prod\{r\,;s : a \leq r \text{ and } b \leq s \text{ and } r, s \in A\},$$
$$a^{\smile} = \prod\{r^{\smile} : a \leq r \text{ and } r \in A\}.$$

□

The point of formulating condition (iv) for quasi-atoms instead of for atoms is to ensure that the operators are normal in $\mathfrak{B}$ whenever they are normal in $\mathfrak{A}$. For example, suppose $a = 0$ and b is any quasi-atom in $\mathfrak{B}$. The elements 0 and 1 are in $\mathfrak{A}$, and 0 is above a while 1 is above b, so the product $0\,;1$ is one of the elements in the set on the right side of the first equation in condition (iv). Since $0\,;1 = 0$ in $\mathfrak{A}$, by the assumed normality of the operators in $\mathfrak{A}$, the infimum on the right side of the equation must be 0. Thus, in this case the equation just expresses the requirement that $0\,;b = 0$ for every quasi-atom b. Every element in $\mathfrak{B}$ is the sum of the set of quasi-atoms that are below it, and 0 is one of these quasi-atoms, so the assumed quasi-complete distributivity of the operator ; in $\mathfrak{B}$ implies that $0\,;b = 0$ for every quasi-atom b in $\mathfrak{B}$ if and only if $0\,;p = 0$ for every element p in $\mathfrak{B}$. Combine these observations to conclude that if $\mathfrak{A}$ is normal, then the first equation in condition (iv) holds for $a = 0$ if and only if $0\,;p = 0$ for every p in $\mathfrak{B}$; and similarly, the equation holds for $b = 0$ if and only if $p\,;0 = 0$ for every p in $\mathfrak{B}$.

Definition 14.12 does not require a canonical extension of a Boolean algebra with operators $\mathfrak{A}$ to satisfy the same equations as $\mathfrak{A}$. In particular, in the case of a relation algebra $\mathfrak{A}$, the definition does not require a canonical extension of $\mathfrak{A}$ to be a relation algebra. We shall see in Theorem 14.35, however, that a canonical extension of a relation algebra *is* in fact a relation algebra. Conditions (i)–(iii) in the definition immediately imply that if $\mathfrak{B}$ is a canonical extension of $\mathfrak{A}$, then the Boolean part of $\mathfrak{B}$ is a canonical extension of the Boolean part of $\mathfrak{A}$.

Before proceeding further, it is helpful to introduce some notation and make a few simple observations about this notation. Suppose $\mathfrak{B}$ is a canonical extension of $\mathfrak{A}$. For any two subsets X and Y of $\mathfrak{A}$, write

$$X + Y, \qquad X \cdot Y, \qquad X \,;\, Y, \qquad X^{\smile}, \qquad -X$$

for the complex sum, complex product, and complex relative product of the sets X and Y, and the complex converse and complex complement of the set X. Thus,

$$\begin{aligned} X + Y &= \{r + s : r \in X \text{ and } s \in Y\}, \\ X \cdot Y &= \{r \cdot s : r \in X \text{ and } s \in Y\}, \\ X \,;\, Y &= \{r \,;\, s : r \in X \text{ and } s \in Y\}, \\ X^{\smile} &= \{r^{\smile} : r \in X\}, \\ -X &= \{-r : r \in X\}. \end{aligned}$$

Observe that these sets are themselves subsets of A, since A is closed under the operations $+$, $\cdot$, $;$, $^{\smile}$, and $-$. We shall occasionally refer to both $X \cdot Y$ and $X \,;\, Y$ as *complex products*, when no confusion can arise about which product is intended. In terms of the above notation, the equations in condition (iv) of Definition 14.12 simply say that

$$a \,;\, b = \prod(A_a \,;\, A_b) \qquad \text{and} \qquad a^{\smile} = \prod(A_a^{\smile}).$$

An atom c in $\mathfrak{B}$ is the infimum of the ultrafilter A_c of all elements in $\mathfrak{A}$ that are above c, by Lemma 14.7, and 0 is obviously the infimum of the improper filter A_0 of all elements in $\mathfrak{A}$. In view of the two equations at the end of the preceding paragraph, if a and b are quasi-atoms, and c an atom, in $\mathfrak{B}$, then the inequalities $c \leq a \,;\, b$ and $c \leq a^{\smile}$ may be expressed set-theoretically via the following equivalences:

$$c \leq a \,;\, b \qquad \text{if and only if} \qquad A_a \,;\, A_b \subseteq A_c,$$

and

$$c \leq a^{\smile} \qquad \text{if and only if} \qquad A_a^{\smile} \subseteq A_c.$$

Indeed, if c is below $a\,;b$, then for every pair of elements r in A_a and s in A_b, we have

$$c \leq a\,;b \leq r\,;s,$$

by the monotony law for the operator $;$ (see Lemma 2.3), so that the element $r\,;s$ is in A_c. Consequently, the inclusion on the right side of the first equivalence holds. On the other hand, if that inclusion holds, then

$$\prod A_c \leq \prod(A_a\,;A_b),$$

by Boolean algebra. In view of Lemma 14.7, the remarks at the beginning of the paragraph, and the first of the final two equations in the preceding paragraph, we immediately obtain that c is below $a\,;b$. This establishes the first of the above equivalences. The second is established in a completely analogous fashion.

Canonical extensions are uniquely determined in the sense of the following *Uniqueness Theorem* for canonical extensions of Boolean algebras with operators.

Theorem 14.13. *Any two canonical extensions of a Boolean algebra with operators* $\mathfrak{A}$ *are isomorphic via a mapping that is the identity function on* $\mathfrak{A}$.

Proof. Consider two canonical extensions, say $\mathfrak{B}$ and $\mathfrak{C}$, of $\mathfrak{A}$. The Boolean parts of $\mathfrak{B}$ and $\mathfrak{C}$ are (Boolean) canonical extensions of the Boolean part of $\mathfrak{A}$, by Definition 14.12. Consequently, the function φ defined by

$$\varphi(b) = c \qquad \text{if and only if} \qquad A_b = A_c. \tag{1}$$

for each pair of quasi-atoms b in $\mathfrak{B}$ and c in $\mathfrak{C}$ is a bijection from the set of quasi-atoms in $\mathfrak{B}$ to the set of quasi-atoms in $\mathfrak{C}$, and φ maps 0 to 0, by (the proof of) Theorem 14.11, and by the fact that $A_c = A_0 = A$ if and only if $c = 0$.

Consider now any two quasi-atoms b_1 and b_2, and any atom b_3, in $\mathfrak{B}$, and write

$$c_1 = \varphi(b_1), \qquad c_2 = \varphi(b_2), \qquad c_3 = \varphi(b_3). \tag{2}$$

The definition of φ implies that

$$A_{b_1} = A_{c_1}, \qquad A_{b_2} = A_{c_2}, \qquad A_{b_3} = A_{c_3}. \tag{3}$$

The remarks preceding the theorem imply that

$$b_3 \leq b_1 \,;\, b_2 \qquad \text{if and only if} \qquad A_{b_1} \,;\, A_{b_2} \subseteq A_{b_3}, \tag{4}$$

and

$$c_3 \leq c_1 \,;\, c_2 \qquad \text{if and only if} \qquad A_{c_1} \,;\, A_{c_2} \subseteq A_{c_3}. \tag{5}$$

In view of the equations in (3), the right sides of the equivalences in (4) and in (5) say the same thing, so we arrive at

$$b_3 \leq b_1 \,;\, b_2 \qquad \text{if and only if} \qquad c_3 \leq c_1 \,;\, c_2.$$

Using (2), this equivalence may be rewritten as

$$b_3 \leq b_1 \,;\, b_2 \qquad \text{if and only if} \qquad \varphi(b_3) \leq \varphi(b_1) \,;\, \varphi(b_2).$$

A completely analogous argument establishes

$$b_3 \leq b_1^{\smile} \qquad \text{if and only if} \qquad \varphi(b_3) \leq \varphi(b_1)^{\smile}.$$

Finally,

$$\begin{aligned} b_3 \leq 1' \qquad & \text{if and only if} \qquad 1' \in A_{b_3}, \\ & \text{if and only if} \qquad 1' \in A_{c_3}, \\ & \text{if and only if} \qquad c_3 \leq 1', \end{aligned}$$

by the definitions of the sets A_{b_3} and A_{c_3}, and (3). Consequently,

$$b_3 \leq 1' \qquad \text{if and only if} \qquad \varphi(b_3) \leq 1',$$

by (2).

It has been shown that the bijection φ preserves the Peircean operations on quasi-atoms in the sense of the Atomic Isomorphism Theorem 7.11 (in the version applying to Boolean algebras with quasi-complete operators—see Exercise 7.41). Use this theorem to conclude

that φ has a unique extension to an isomorphism ψ from $\mathfrak{B}$ to $\mathfrak{C}$ that is defined by

$$\psi(p) = \sum\{\varphi(b) : b \text{ is an atom in } \mathfrak{B} \text{ and } b \leq p\} \tag{6}$$

for every p in $\mathfrak{B}$. The proof that ψ is the identity function on $\mathfrak{A}$ is identical to the corresponding part of the proof of Theorem 14.11. □

The preceding theorem justifies the common practice of referring to *the* canonical extension of a Boolean algebra with operators. The task of showing that such a canonical extension always exists is more involved. To avoid unnecessary complications, we focus on the special case when $\mathfrak{A}$ is a Boolean algebra with *normal* operators. (This case includes the case of relation algebras.) The general case in which the operators may not be normal is dealt with in the next section and in the exercises.

Let U be the set of all ultrafilters in $\mathfrak{A}$, and let C be the Boolean algebra of all subsets of U. The observations preceding Theorem 14.13 suggest a natural method of defining a binary operation ; and a unary operation $^\smile$ on C, and a distinguished constant 1' in C. For any two subsets V and W of U, define

$$\begin{aligned} V\,;W &= \{Z \in U : X\,;Y \subseteq Z \text{ for some } X \in V \text{ and } Y \in W\},\\ V^\smile &= \{Z \in U : X^\smile \subseteq Z \text{ for some } X \in V\},\\ 1' &= \{Z \in U : 1' \in Z\}, \end{aligned}$$

where the operations ; and $^\smile$ on the right sides of the first two equations are the complex operations defined after Definition 14.12, the distinguished constant 1' on the right side of the third equation is the distinguished constant in $\mathfrak{A}$, and the operations and the distinguished constant on the left sides of the equations are those being defined on the set C. For atoms $V = \{X\}$ and $W = \{Y\}$ in C, the definitions of the operations ; and $^\smile$ assume the form

$$\begin{aligned} \{X\}\,;\{Y\} &= \{Z \in U : X\,;Y \subseteq Z\},\\ \{X\}^\smile &= \{Z \in U : X^\smile \subseteq Z\}. \end{aligned}$$

This observation makes clear that the general definitions of the operations ; and $^\smile$ are formulated so as to ensure that the operations are completely distributive for atoms (in the sense defined before

Lemma 2.7) and therefore completely distributive, by Lemma 2.7. A direct verification of complete distributivity is also easy. For example, if

$$(V_i : i \in I) \qquad \text{and} \qquad (W_j : j \in J)$$

are systems of subsets of U, and if

$$V = \bigcup_i V_i \qquad \text{and} \qquad W = \bigcup_j W_j,$$

then

$$\begin{aligned} V\,;W &= \{Z \in U : X\,;Y \subseteq Z \text{ for some } X \in V \text{ and } Y \in W\} \\ &= \bigcup_{i,j}\{Z \in U : X\,;Y \subseteq Z \text{ for some } X \in V_i \text{ and } Y \in W_j\} \\ &= \bigcup_{i,j} V_i\,;W_j, \end{aligned}$$

by the definition of the operation $;$ in $\mathfrak{C}$ and the definition of the union of the systems under consideration. Conclusion: the algebra

$$\mathfrak{C} = (C\,,\,+\,,\,-\,,\,;\,,\,{}^{\smile}\,,\,1')$$

(where $+$ and $-$ are the set-theoretical operations of union and complement on subsets of U) is a complete and atomic Boolean algebra with complete (and therefore normal) operators. The next task is to show that $\mathfrak{C}$ contains a canonical image of the algebra $\mathfrak{A}$.

Lemma 14.14. *The function φ defined by*

$$\varphi(r) = \{Z \in U : r \in Z\}$$

for each r in $\mathfrak{A}$ is a monomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.

Proof. It was shown in Lemma 14.8 that φ is a Boolean monomorphism. It remains to prove that φ preserves the operators and the distinguished constant in $\mathfrak{A}$. For the distinguished constant 1', we have

$$\varphi(1') = \{Z \in U : 1' \in Z\},$$

by the definition of φ. The right side of this equation is, by definition, the distinguished constant in $\mathfrak{C}$. Thus, φ maps the distinguished constant in $\mathfrak{A}$ to the distinguished constant in $\mathfrak{C}$.

To prove that φ preserves the operator ${}^{\smile}$, consider an arbitrary element r in $\mathfrak{A}$, with the goal of showing that

$$\varphi(r^\smile) = \varphi(r)^\smile. \tag{1}$$

The definition of the operator $^\smile$ in $\mathfrak{C}$ and the definition of the mapping φ imply that

$$\begin{aligned}\varphi(r)^\smile &= \{Z \in U : X^\smile \subseteq Z \text{ for some } X \in \varphi(r)\} \\ &= \{Z \in U : X^\smile \subseteq Z \text{ for some } X \in U \text{ with } r \in X\}. \end{aligned} \tag{2}$$

Consequently, if Z is in $\varphi(r)^\smile$, then there is an ultrafilter X (in U) containing r such that the complex converse $X^\smile$ is included in Z, by (2). The element $r^\smile$ belongs to $X^\smile$, by the definition of $X^\smile$, so $r^\smile$ must belong to Z and therefore Z must belong to $\varphi(r^\smile)$, by the definition of the mapping φ. This proves that the right-side of (1) is included in the left-side.

To establish the reverse inclusion, suppose that Z belongs to $\varphi(r^\smile)$. Observe that the Boolean dual of the ultrafilter Z is the maximal Boolean ideal

$$-Z = \{-u : u \in Z\} = \{u \in A : u \notin Z\} = A \sim Z. \tag{3}$$

To show that Z belongs to $\varphi(r)^\smile$, an ultrafilter X containing r must be constructed with the property that $X^\smile$ is included in Z, by (2). Write

$$F_0 = \{u \in A : (u \cdot r)^\smile \in -Z\}. \tag{4}$$

The immediate goal is to show that F_0 is a proper Boolean ideal containing $-r$.

First of all,

$$(-r \cdot r)^\smile = 0^\smile = 0,$$

by Boolean algebra and the assumption that the operator $^\smile$ in $\mathfrak{A}$ is normal. The element 0 belongs to the Boolean ideal $-Z$, so $(-r \cdot r)^\smile$ belongs to $-Z$. Consequently, $-r$ is in F_0, by (4). In particular, F_0 is not empty.

Second, if u is in F_0, and if v is an element in $\mathfrak{A}$ that is below u, then

$$(v \cdot r)^\smile \leq (u \cdot r)^\smile,$$

by Boolean algebra and the monotony of the operator $^\smile$ in $\mathfrak{A}$ (see Lemma 2.3). Since $(u \cdot r)^\smile$ is in $-Z$, by (4), and since $-Z$ is a Boolean ideal (and therefore downward closed), the element $(v \cdot r)^\smile$ must also be in $-Z$. Consequently, v is in F_0, by (4), so F_0 is downward closed.

Third, if u and v are in F_0, then the elements $(u \cdot r)^\smile$ and $(v \cdot r)^\smile$ must be in $-Z$, by (4). The sum of these two elements is also in $-Z$, because $-Z$ is a Boolean ideal. Since

$$[(u + v) \cdot r]^\smile = (u \cdot r)^\smile + (v \cdot r)^\smile,$$

by Boolean algebra and the distributivity of the operator $^\smile$ in $\mathfrak{A}$, it follows that $[(u+v)\cdot r]^\smile$ is in $-Z$, and therefore $u+v$ is in F_0, by (4). Thus, F_0 is closed under addition.

Finally, F_0 does not contain the unit 1. Indeed, $r^\smile$ is in Z, by the assumption that Z is in $\varphi(r^\smile)$ and the definition of φ. Therefore, $r^\smile$ is not in $-Z$, by (3). Since

$$(1 \cdot r)^\smile = r^\smile,$$

by Boolean algebra, it follows that $(1 \cdot r)^\smile$ is not in $-Z$, so 1 is not in F_0, by (4). This completes the proof that F_0 is a proper Boolean ideal containing $-r$.

Use the Boolean version of the Maximal Ideal Theorem 8.31 to extend F_0 to a maximal Boolean ideal F. The Boolean dual of F is the ultrafilter X that is determined by

$$X = -F = \{-u : u \in F\} = \{u \in A : u \notin F\} = A \sim F. \tag{5}$$

It is not difficult to check that X has the required properties. First of all, $-r$ is in F_0 and therefore also in F, so r must be in X, by (5). Second, if u is in X, then u cannot be in F, by (5), and therefore u cannot be in F_0. It follows by (4) that the element $(u\cdot r)^\smile$ is not in $-Z$, so this element must belong to Z, by (3). Since

$$(u \cdot r)^\smile \leq u^\smile,$$

by Boolean algebra and the monotony of the operators in $\mathfrak{A}$, and since Z is an ultrafilter and therefore upward closed, it may be concluded that $u^\smile$ is in Z. Thus, $X^\smile$ is included in Z. This completes the proof of (1)

The proof that φ preserves the operator $;$ is similar in character to the preceding argument, but more complicated in details. Consider arbitrary elements r and s in $\mathfrak{A}$, with the goal of showing that

$$\varphi(r\,;s) = \varphi(r)\,;\varphi(s). \tag{6}$$

The definition of the operator ; in $\mathfrak{C}$ and the definition of the mapping φ imply that

$$\varphi(r)\,;\varphi(s) = \{Z \in U : X\,;Y \subseteq Z \text{ for some } X, Y \in U \text{ with } r \in X \text{ and } s \in Y\}. \quad (7)$$

Consequently, if Z is in $\varphi(r)\,;\varphi(s)$, then there must be ultrafilters X and Y containing r and s respectively such that the complex $X\,;Y$ is included in Z, by (7). The element $r\,;s$ belongs to $X\,;Y$, so $r\,;s$ must belong to Z and therefore Z must belong to $\varphi(r\,;s)$. This proves that the right side of (6) is included in the left side.

To establish the reverse inclusion, assume that Z belongs to $\varphi(r\,;s)$. In order to show that Z belongs to $\varphi(r)\,;\varphi(s)$, ultrafilters X and Y containing r and s respectively must be constructed with the property that $X\,;Y$ is included in Z, by (7). The Boolean dual of Z is the maximal Boolean ideal $-Z$ in (3). Write

$$G_0 = \{v \in A : r\,;(v \cdot s) \in -Z\}, \quad (8)$$

and observe that G_0 is a proper Boolean ideal containing the element $-s$. The argument is similar to the corresponding argument for the set defined in (4), so the details are left as an exercise.

Extend G_0 to a maximal Boolean ideal G. The Boolean dual of G is the ultrafilter Y that is determined by

$$Y = -G = \{-v : v \in G\} = \{w \in A : w \notin G\} = A \sim G. \quad (9)$$

It is not difficult to check that the element $-s$ is in G_0 and therefore in G. Indeed,

$$r\,;(-s \cdot s) = r\,;0 = 0,$$

by Boolean algebra and the assumption that the operator ; is normal, and 0 is in the Boolean ideal $-Z$, so $-s$ belongs to G_0, by (8) (with $-s$ in place of v). It follows that s must be in Y, by (9). Also, if v is Y, then v is not in G, by (9), and therefore v is not in G_0. Consequently, the element $r\,;(v \cdot s)$ cannot be in $-Z$, by (8), so this element must be in Z, by (3). Since

$$r\,;(v \cdot s) \leq r\,;v,$$

by Boolean algebra and the monotony of the operators in $\mathfrak{A}$, and since Z is an ultrafilter, it may be concluded that $r\,;v$ is in Z. Thus,

$$\{r\,;v : v \in Y\} \subseteq Z. \quad (10)$$

Define a subset F_0 of $\mathfrak{A}$ by

$$F_0 = \{u \in A : (u \cdot r)\,;v \in -Z \text{ for some } v \in Y\}. \tag{11}$$

As before, the set F_0 is a proper Boolean ideal in $\mathfrak{A}$ that contains the element $-r$. For example, to see that $-r$ is in F_0, observe that 1 is in the ultrafilter Y, and

$$(-r \cdot r)\,;1 = 0\,;1 = 0,$$

by Boolean algebra and the assumption that the operator $;$ is normal. Since 0 is in the Boolean ideal $-Z$, it follows that $(-r \cdot r)\,;1$ is in $-Z$, and therefore $-r$ is in F_0, by (11).

The argument that F_0 is downward closed is similar to the argument that the set in (4) is downward closed, and is left as an exercise. To see that F_0 is closed under addition, assume u and v are in F_0. There must be elements w_1 and w_2 in Y such that

$$(u \cdot r)\,;w_1 \qquad \text{and} \qquad (v \cdot r)\,;w_2 \tag{12}$$

are in $-Z$, by (11). Write $w = w_1 \cdot w_2$, and observe that w is in the filter Y, by the closure of Y under multiplication. The elements

$$(u \cdot r)\,;w \qquad \text{and} \qquad (v \cdot r)\,;w \tag{13}$$

are respectively below the elements in (12), by Boolean algebra and the monotony of the operators in $\mathfrak{A}$, so they, too, belong to the Boolean ideal $-Z$, by the downward closure of $-Z$. The sum of the elements in (13) is therefore also in $-Z$, by the closure of $-Z$ under addition. Since

$$[(u + v) \cdot r]\,;w = (u \cdot r)\,;w + (v \cdot r)\,;w,$$

by Boolean algebra and the distributivity of the operator $;$, it follows from the preceding observations that $[(u + v) \cdot r]\,;w$ is in $-Z$, and therefore $u + v$ belongs to F_0, by (11). Thus, F_0 is a Boolean ideal.

To see that the Boolean ideal F_0 is proper, observe that

$$(1 \cdot r)\,;v = r\,;v,$$

by Boolean algebra. The product $r\,;v$ belongs to Z for every v in Y, by (10), so $(1 \cdot r)\,;v$ cannot be in $-Z$ for any v in Y, by (3), and therefore 1 cannot be in F_0, by (11).

Extend F_0 to a maximal Boolean ideal F. The Boolean dual of F is the ultrafilter X determined by (5). Since the element $-r$ is in F_0, and therefore in F, the complement of this element, which is r, must be in X, by (5). If u is any element in X, then u cannot belong to F, by (5), and therefore u cannot belong to F_0. Consequently, $(u \cdot r)\,;v$ cannot belong to $-Z$ for any element v in Y, by (11). It follows from (3) that $(u \cdot r)\,;v$ must be in Z for every such v. Since Z is a filter, and therefore upward closed, and since

$$(u \cdot r)\,;v \leq u\,;v,$$

it may be concluded that $u\,;v$ belongs to Z for every v in Y. This is true for every element u in X, so the complex product $X\,;Y$ is included in Z. Conclusion: X and Y are ultrafilters containing the elements r and s respectively, and the complex product $X\,;Y$ is included in Z, so Z belongs to the set $\varphi(r)\,;\varphi(s)$, by (7). Thus, the left side of (6) is included in the right side. This completes the proof of (6) and of the theorem. □

The algebra $\mathfrak{C}$ defined before the lemma is called the *canonical embedding algebra* of the given Boolean algebra with normal operators $\mathfrak{A}$, and the mapping φ defined in the lemma is called the *canonical embedding* of $\mathfrak{A}$ into $\mathfrak{C}$. The subalgebra of $\mathfrak{C}$ that is the image of $\mathfrak{A}$ under φ is an isomorphic copy of $\mathfrak{A}$ and will be denoted by $\varphi(\mathfrak{A})$.

Theorem 14.15. *The canonical embedding algebra of a Boolean algebra with normal operators $\mathfrak{A}$ is the canonical extension of the image of $\mathfrak{A}$ under the canonical embedding.*

Proof. Let φ be the canonical embedding of $\mathfrak{A}$ into its canonical embedding algebra $\mathfrak{C}$. It was already shown in Theorem 14.9 that the Boolean part of $\mathfrak{C}$ is a complete and atomic Boolean algebra that has the atom separation and compactness properties with respect to the Boolean part of the image algebra $\varphi(\mathfrak{A})$. In the remarks leading up to Lemma 14.14, it was pointed out that the definitions of the operators $;$ and $^\smile$ in $\mathfrak{C}$ are formulated in such a way as to automatically render these operators completely distributive. In particular, the operators are normal, by Lemma 2.5, so the conditions in part (iv) of Definition 14.12 hold in those cases when one of the quasi-atoms is 0. The task remains of verifying the conditions in (iv) for atoms, with $\mathfrak{A}$ and $\mathfrak{B}$ replaced by $\varphi(\mathfrak{A})$ and $\mathfrak{C}$ respectively. Focus on the case of the binary operator $;$.

The atoms in $\mathfrak{C}$ are the singletons of elements in the set U of ultrafilters in $\mathfrak{A}$. For two such ultrafilters X and Y, the operator $;$ on the corresponding singletons is determined by the equation

$$\{X\} ; \{Y\} = \{Z \in U : X ; Y \subseteq Z\}. \tag{1}$$

This means that an ultrafilter Z in U belongs to $\{X\} ; \{Y\}$ if and only if $r ; s$ is in Z for all elements r in X and s in Y. Thus, (1) may be rewritten in the form

$$\{X\} ; \{Y\} = \bigcap\{\{Z \in U : r ; s \in Z\} : r \in X \text{ and } s \in Y\}. \tag{2}$$

The sets $\{Z \in U : r ; s \in Z\}$ in this intersection are just the images of the elements $r ; s$ under the mapping φ, since

$$\varphi(r ; s) = \{Z \in U : r ; s \in Z\},$$

by the definition of φ. Moreover,

$$\varphi(r ; s) = \varphi(r) ; \varphi(s),$$

by Lemma 14.14, so the intersection in (2) may be rewritten in the form

$$\{X\} ; \{Y\} = \bigcap\{\varphi(r) ; \varphi(s) : r \in X \text{ and } s \in Y\}. \tag{3}$$

The condition that r be in X is equivalent to the condition that X belong to $\varphi(r)$, by the definition of φ, and this second condition is equivalent to the condition that the singleton $\{X\}$ be included in $\varphi(r)$. A similar remark applies to s and Y. Consequently, (3) may be rewritten in the form

$$\{X\} ; \{Y\} = \bigcap\{\varphi(r) ; \varphi(s) : r, s \in A, \text{ and } \{X\} \subseteq \varphi(r) \text{ and } \{Y\} \subseteq \varphi(s)\}. \tag{4}$$

Since φ is an embedding of $\mathfrak{A}$ into $\mathfrak{C}$, and since inclusion is the Boolean inequality relation in $\mathfrak{C}$, the equation in (4) may be rewritten in the form

$$\{X\} ; \{Y\} = \bigcap\{\bar{r} ; \bar{s} : \bar{r}, \bar{s} \in \varphi(A), \text{ and } \{X\} \leq \bar{r} \text{ and } \{Y\} \leq \bar{s}\}.$$

Take $\{X\}$ and $\{Y\}$ for a and b respectively, take $\varphi(A)$ for A, and take $\mathfrak{C}$ for $\mathfrak{B}$, in condition (iv) of Definition 14.12 to conclude that the first equation in the condition holds for all atoms a and b in the algebra $\mathfrak{C}$ with respect to the subalgebra $\varphi(\mathfrak{A})$.

The argument that the second equation in condition (iv) also holds is similar, and is left as an exercise. □

We come finally to the *Existence Theorem* for canonical extensions of Boolean algebras with normal operators.

Theorem 14.16. *Every Boolean algebra with normal operators has a canonical extension.*

Proof. Let $\mathfrak{A}$ be a Boolean algebra with normal operators, let $\mathfrak{C}$ be the canonical embedding algebra of $\mathfrak{A}$, and let φ be the canonical embedding of $\mathfrak{A}$ into $\mathfrak{C}$. The version of the Exchange Principle (Theorem 7.15) that applies to Boolean algebras with normal operators yields a Boolean algebra with normal operators $\mathfrak{B}$ and an isomorphism from $\mathfrak{B}$ to $\mathfrak{C}$ that extends the canonical embedding φ. Using this isomorphism, it is a routine matter to prove that $\mathfrak{B}$ satisfies the conditions for being the canonical extension of $\mathfrak{A}$. □

The preceding existence Theorem is actually true for arbitrary Boolean algebras with operators; the requirement that the operators of the given algebra be normal is not needed, but in this case the definitions of the operators in the canonical embedding algebra are more complicated. Some details are presented in the exercises.

14.3 An alternative approach to existence

There is another approach to proving the existence of canonical extensions that is more abstract than the approach presented in the preceding section. It does not make any essential use of ultrafilters; rather, it defines abstract operators on an arbitrary *Boolean* canonical extension and proves that these abstract operators possess the required properties. The ideas involved in this approach play an important role in the proofs of the preservation theorems in the next section, so we go over the main points of the approach in this section.

The assumption that the operators in a given algebra $\mathfrak{A}$ are normal simplified the presentation in the previous section, but this assumption does not simplify the presentation of the abstract approach very much. The basic difference between the normal and non-normal cases in the abstract approach is that, in the former, sets of atoms play an important role, whereas in the latter, sets of atoms must often be augmented by the element 0, so that sets of quasi-atoms must be considered.

Fix a Boolean algebra with operators $\mathfrak{A}$, and let $(B, +, -)$ be a canonical extension of the Boolean part of $\mathfrak{A}$. Such an extension exists

and is unique up to isomorphisms that are the identity function on the universe of $\mathfrak{A}$, by Theorems 14.10 and 14.11. Let K be the set of closed elements in B, that is to say, the set of elements that are infima of subsets of $\mathfrak{A}$. Every element r in $\mathfrak{A}$ is obviously closed, since r is the infima of the subset $\{r\}$ of $\mathfrak{A}$; and every atom in B is closed, by Lemma 14.7. Every element p in B is the supremum of the set of atoms that it dominates, because B is atomic, so p must be the supremum of the set of closed elements that it dominates. If we write

$$K_p = \{u : u \in K \text{ and } u \leq p\},$$

then this observation may be expressed as follows.

Lemma 14.17. *For every element p in B, we have $p = \sum K_p$.*

Notice that the set K_p is never empty, since it always contains the element 0.

Define a binary operation $;$ and a unary operation $^\smile$ on the set B as follows:

$$p\,;q = \sum\{\prod\{r\,;s : r, s \in A \text{ and } u \leq r,\ v \leq s\} : \\ u, v \in K \text{ and } u \leq p,\ v \leq q\}$$

and

$$p^\smile = \sum\{\prod\{r^\smile : r \in A \text{ and } u \leq r\} : u \in K \text{ and } u \leq p\}$$

for every p and q in B (where the operations $;$ and $^\smile$ on the right are those of $\mathfrak{A}$, while the ones on the left are those being defined on B). These somewhat opaque definitions can be rendered more perspicuous by using the complex operations—defined on subsets of $\mathfrak{A}$ in terms of the operators in $\mathfrak{A}$—that were introduced in the preceding section:

$$p\,;q = \sum\{\prod(A_u\,;A_v) : u \in K_p \text{ and } v \in K_q\},$$
$$p^\smile = \sum\{\prod(A_u^\smile) : u \in K_p\}.$$

We shall gain further insight into these definitions in Lemma 14.18 and Corollary 14.20 below.

Put

$$\mathfrak{B} = (B\,,+\,,-\,,;\,,^\smile\,,1'),$$

where $+$ and $-$ are the Boolean operations of B, while $;$ and $^\smile$ are the operations defined above, and 1' is the distinguished constant of $\mathfrak{A}$.

The goal is to show that $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}$. The proof requires a number of lemmas.

For closed elements in $\mathfrak{B}$, the definitions of the operations $;$ and $^\smile$ assume a simpler form.

Lemma 14.18. *If p and q are closed elements in $\mathfrak{B}$, then*

$$p + q = \prod(A_p + A_q), \qquad p \cdot q = \prod(A_p \cdot A_q),$$
$$p\,;q = \prod(A_p\,;A_q), \qquad p^\smile = \prod(A_p^\smile).$$

Proof. The proofs of these equations are relatively easy. Since p and q are closed elements, it follows from Lemma 14.5 that

$$p = \prod A_p \qquad \text{and} \qquad q = \prod A_q. \tag{1}$$

Consequently,

$$p + q = (\prod A_p) + (\prod A_q)$$
$$= \prod\{r + s : r \in A_p \text{ and } s \in A_q\} = \prod(A_p + A_q),$$

by (1), the distributive law for addition over (possibly infinite) multiplication, and the definition of the complex sum of two subsets of $\mathfrak{A}$.

To establish the second equation, observe that each of the two sets A_p and A_q contains 1, and therefore each of these two sets is included in the complex product $A_p \cdot A_q$. Consequently, the union of the two sets is included in this complex product. It follows that

$$p \cdot q = (\prod A_p) \cdot (\prod A_q) = \prod(A_p \cup A_q) \geq \prod(A_p \cdot A_q),$$

by Boolean algebra. On the other hand, if r is in A_p, and s in A_q, then $p \leq r$ and $q \leq s$, and therefore $p \cdot q \leq r \cdot s$. Consequently,

$$p \cdot q \leq \prod\{r \cdot s : r \in A_p \text{ and } s \in A_q\} = \prod(A_p \cdot A_q).$$

Turn now to the third equation. The operation $;$ in $\mathfrak{B}$ is defined by

$$p\,;q = \sum\{\prod(A_u\,;A_v) : u \in K_p \text{ and } v \in K_q\}. \tag{2}$$

For pairs of elements u in K_p and v in K_q, we have $u \leq p$ and $v \leq q$, by the definitions of the two sets involved, and therefore

$$A_p \subseteq A_u \qquad \text{and} \qquad A_q \subseteq A_v.$$

It follows that the complex product $A_p \,;\, A_q$ is included in the complex product $A_u \,;\, A_v$, and therefore

$$\prod(A_u \,;\, A_v) \leq \prod(A_p \,;\, A_q), \tag{3}$$

by Boolean algebra. Since p and q are closed elements, they belong to the sets K_p and K_q respectively. Consequently, $\prod(A_p \,;\, A_q)$ is one of the products belonging to the set on the right side of (2), and it is the largest product in this set, by (3). Combine this observation with Boolean algebra to conclude that the right side of (2), and therefore also the left side, is equal to $\prod(A_p \,;\, A_q)$.

The proof of the fourth equation is similar to, but easier than, the preceding proof, and is left as an exercise. □

One consequence of Lemma 14.18 is that the set K of closed elements is closed under certain operations of $\mathfrak{B}$.

Corollary 14.19. *The set K of closed elements in $\mathfrak{B}$ is closed under the Boolean operations of addition and multiplication, and under the non-Boolean operations ; and $^\smile$, in $\mathfrak{B}$.*

Proof. Recall that K is defined to be the set of infima in $\mathfrak{B}$ of subsets of $\mathfrak{A}$, and an element p belongs to K just in case p is the infima in $\mathfrak{B}$ of the set A_p of elements in $\mathfrak{A}$ that are above p (see Lemma 14.5). If p and q are elements in K, then

$$p + q = \prod(A_p + A_q), \qquad p \cdot q = \prod(A_p \cdot A_q),$$
$$p \,;\, q = \prod(A_p \,;\, A_q), \qquad p^\smile = \prod(A_p^\smile),$$

by Lemma 14.18. Each of the sets

$$A_p + A_q, \qquad A_p \cdot A_q, \qquad A_p \,;\, A_q, \qquad A_p^\smile$$

is included in $\mathfrak{A}$, because $\mathfrak{A}$ is closed under the operations of addition and multiplication, and under the operations ; and $^\smile$. Consequently, the preceding equations show that

$$p + q, \qquad p \cdot q, \qquad p \,;\, q, \qquad \text{and} \qquad p^\smile$$

are all infima of subsets of $\mathfrak{A}$ and therefore all belong to K. Thus, K is closed under the operations $+$, $\cdot$, ;, and $^\smile$ of $\mathfrak{B}$. □

Here is another consequence of Lemma 14.18.

Corollary 14.20. *If p and q are arbitrary elements in $\mathfrak{B}$, then*

$$p\,;q = \sum(K_p\,;K_q) \qquad \textit{and} \qquad p^{\smile} = \sum(K_p^{\smile}).$$

Proof. According to the definition of the operation $;$,

$$p\,;q = \sum\{\prod(A_u\,;A_v) : u \in K_p \text{ and } v \in K_q\}.$$

On the other hand, Lemma 14.18 says that for any closed elements u in K_p and v in K_q,

$$u\,;v = \prod(A_u\,;A_v).$$

Combine these two equations to arrive at

$$p\,;q = \sum\{u\,;v : u \in K_p \text{ and } v \in K_q\} = \sum(K_p\,;K_q).$$

The proof for the operation $^{\smile}$ is similar. □

Lemma 14.21. $\mathfrak{A}$ *is a subalgebra of* $\mathfrak{B}$.

Proof. The Boolean part of $\mathfrak{A}$ is a subalgebra of the Boolean part of $\mathfrak{B}$, by the definition of $\mathfrak{B}$, and the element 1' is the same in both algebras, by the definition of $\mathfrak{B}$. What remains to be shown is that the restrictions of the operations $;$ and $^{\smile}$ in $\mathfrak{B}$ to the universe of $\mathfrak{A}$ coincide with the corresponding operations in $\mathfrak{A}$.

Consider any two elements p and q in $\mathfrak{A}$. Each of these element is closed, so

$$p\,;q = \prod(A_p\,;A_q), \tag{1}$$

by Lemma 14.18, where the product $p\,;q$ on the left is formed in $\mathfrak{B}$. Since the elements p and q are assumed to belong to $\mathfrak{A}$, they belong to the sets A_p and A_q respectively, and are actually the smallest elements in these sets. The product $p\,;q$ formed in $\mathfrak{A}$ is therefore one of the elements in the set $A_p\,;A_q$, and it is actually the smallest element in this set, by the monotony law for the operator $;$ in $\mathfrak{A}$ and the definition of the set $A_p\,;A_q$. Consequently,

$$\prod(A_p\,;A_q) = p\,;q, \tag{2}$$

by Boolean algebra, where the product $p\,;q$ on the right is formed in $\mathfrak{A}$. Conclusion: the left side of (1), namely the product $p\,;q$ in $\mathfrak{B}$, is equal to the right side of (2), namely product $p\,;q$ in $\mathfrak{A}$.

An entirely analogous argument leads to an analogous conclusion about the values of $p^{\smile}$ in $\mathfrak{B}$ and in $\mathfrak{A}$ whenever p is in $\mathfrak{A}$. □

The preceding lemmas and corollary imply that $\mathfrak{B}$ is normal if and only if $\mathfrak{A}$ is normal.

Lemma 14.22. *An operator in $\mathfrak{A}$ is normal if and only if the corresponding operation in $\mathfrak{B}$ is normal.*

Proof. Consider as an example the binary operation $;$. If this operation is normal in $\mathfrak{B}$, then

$$p\,;0 = 0\,;p = 0$$

for all elements p in $\mathfrak{B}$, and in particular for all elements p in $\mathfrak{A}$. Since $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, by Lemma 14.21, it follows that the operator $;$ in $\mathfrak{A}$ is normal.

Suppose now that $;$ is normal in $\mathfrak{A}$. Consider first an arbitrary closed element p in $\mathfrak{B}$. The element 1 belongs to the set A_p of elements in $\mathfrak{A}$ that are above p, and the element 0 belongs to the set A_0 of elements in $\mathfrak{A}$ that are above 0, so $1\,;0$ belongs to the complex product $A_p\,;A_0$, by the definition of this complex product. Since $;$ is normal in $\mathfrak{A}$, we have $1\,;0 = 0$, so the element 0 belongs to the complex product and is therefore the infimum of the complex product. Combine this observation with Lemma 14.18 to arrive at

$$p\,;0 = \prod(A_p\,;A_0) = 0.$$

Consider now an arbitrary element p in $\mathfrak{B}$. The set K_0 consists of the closed elements in $\mathfrak{B}$ that are below 0, by definition, so it contains only 0. Each element u in K_p is closed, by definition, so $u\,;0 = 0$, by the observations of the previous paragraph. Notice that K_p is not empty, since it contains 0. Consequently,

$$K_p\,;K_0 = \{u\,;0 : u \in K \text{ and } u \leq p\} = \{0\}.$$

Apply Corollary 14.20 to conclude that

$$p\,;0 = \sum(K_p\,;K_0) = \sum\{0\} = 0.$$

An entirely analogous argument shows that $0\,;p = 0$ for every p in $\mathfrak{B}$. The operation $;$ is therefore normal in $\mathfrak{B}$. □

The Boolean part of the algebra $\mathfrak{B}$ is complete and atomic, and has the atom separation and compactness properties with respect to $\mathfrak{A}$, by the choice of B. Consequently, $\mathfrak{B}$ automatically satisfies the first part of condition (i), and conditions (ii) and (iii), in Definition 14.12. The

second part of condition (i) holds by Lemma 14.21, and condition (iv) holds by Lemma 14.18, since quasi-atoms are always closed elements. In order to prove that $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}$, it remains to show that the operations $;$ and $^\smile$ in $\mathfrak{B}$ are quasi-completely distributive. The proof requires a number of lemmas, some of which play an important role in the next section as well. The first step is to prove that these operations are monotone.

Lemma 14.23. *The operations* $;$ *and* $^\smile$ *in* $\mathfrak{B}$ *are monotone.*

Proof. Suppose p, q, r, and s are elements in $\mathfrak{B}$ such that $p \leq r$ and $q \leq s$. These inequalities imply that

$$K_p \subseteq K_r \qquad \text{and} \qquad K_q \subseteq K_s,$$

by the definitions of the sets involved. Consequently,

$$K_p ; K_q \subseteq K_r ; K_s,$$

by the definition of the complex product of two sets (this inclusion is set theoretical in nature and does not involve any use of the monotony law for $;$). Therefore,

$$p ; q = \bigvee(K_p ; K_q) \leq \bigvee(K_r ; K_s) = r ; s,$$

by Corollary 14.20 and Boolean algebra. The proof of the inequality $p^\smile \leq r^\smile$ is similar and is left as an exercise. □

The next lemma is a rather technical consequence of compactness.

Lemma 14.24. *Let* p *and* q *be closed elements, and* a *an atom, in* $\mathfrak{B}$.

(i) *If for every quasi-atom* $b \leq p$ *there is an element* r_b *in* $\mathfrak{A}$ *such that*

$$b \leq r_b \qquad \text{and} \qquad a \cdot r_b^\smile = 0,$$

then there is an element r *in* $\mathfrak{A}$ *such that*

$$p \leq r \qquad \text{and} \qquad a \cdot r^\smile = 0.$$

(ii) *If for every quasi-atom* $b \leq p$ *and every quasi-atom* $c \leq q$ *there are elements* r_{bc} *and* s_{bc} *in* $\mathfrak{A}$ *such that*

$$b \leq r_{bc}, \qquad c \leq s_{bc}, \qquad \text{and} \qquad a \cdot (r_{bc} ; s_{bc}) = 0,$$

then there are elements r *and* s *in* $\mathfrak{A}$ *such that*

$$p \leq r, \qquad q \leq s, \qquad \text{and} \qquad a \cdot (r ; s) = 0.$$

Proof. Assume the hypothesis of (i). The closed element p is the supremum of the set W of quasi-atoms in $\mathfrak{B}$ that are below p, by the atomicity of $\mathfrak{B}$. Consequently,

$$p = \sum\{b : b \in W\} \leq \sum\{r_b : b \in W\},$$

by the hypotheses of (i). Apply compactness in the form of Corollary 14.3(i) to obtain a finite subset W_0 of W such that

$$p \leq \sum\{r_b : b \in W_0\}. \tag{1}$$

The set W is not empty (it contains the element 0), so it may be assumed that the set W_0 is not empty. The element

$$r = \sum\{r_b : b \in W_0\} \tag{2}$$

is the sum of finitely many elements in $\mathfrak{A}$, since W_0 is finite, so r belongs to $\mathfrak{A}$. Also, $p \leq r$, by (1) and (2). Finally,

$$r^{\smile} = (\sum\{r_b : b \in W_0\})^{\smile} = \sum\{r_b^{\smile} : b \in W_0\},$$

by (2), the distributive law for $^{\smile}$ in $\mathfrak{A}$, and the assumption that the set W_0 is finite and not empty. Consequently,

$$a \cdot r^{\smile} = \sum\{a \cdot r_b^{\smile} : b \in W_0\} = 0,$$

by the distributive law for multiplication in $\mathfrak{B}$, and the hypotheses of (i). This completes the proof of (i).

Turn now to the proof of (ii). The first step is to eliminate the dependence of r_{bc} and s_{bc} on the quasi-atom c by replacing each of these elements with elements r_b and s_b in $\mathfrak{A}$ such that

$$b \leq r_b, \qquad q \leq s_b, \qquad \text{and} \qquad a \cdot (r_b\,;s_b) = 0. \tag{3}$$

Fix a quasi-atom $b \leq p$. The closed element q is the supremum of the set Z of quasi-atoms in $\mathfrak{B}$ that are below q, by the atomicity of $\mathfrak{B}$. Consequently,

$$q = \sum\{c : c \in Z\} \leq \sum\{s_{bc} : c \in Z\},$$

by the hypotheses of (ii). Apply Corollary 14.3(i) to obtain a finite subset Z_b of Z (dependent on the quasi-atom b) such that

$$q \leq \sum\{s_{bc} : c \in Z_b\}. \tag{4}$$

The set Z is not empty (it contains the element 0), so it may be assumed that the set Z_b is not empty. Define

$$r_b = \prod\{r_{bc} : c \in Z_b\} \qquad \text{and} \qquad s_b = \sum\{s_{bc} : c \in Z_b\}, \tag{5}$$

and observe that each of these elements belongs to $\mathfrak{A}$, since each of them is a Boolean combination of finitely many elements in $\mathfrak{A}$. Observe also that $b \leq r_b$, by (5) and the hypotheses of (ii), and $q \leq s_b$ by (5) and (4). For each c in Z_b, we have

$$r_b \,;\, s_{bc} = (\prod\{r_{bd} : d \in Z_b\}) \,;\, s_{bc} \leq r_{bc} \,;\, s_{bc}, \tag{6}$$

by (5) and the monotony of the operator $\,;\,$ in $\mathfrak{A}$, and therefore

$$\begin{aligned} r_b \,;\, s_b = r_b \,;\, (\sum\{s_{bc} : c \in Z_b\}) \\ = \sum\{r_b \,;\, s_{bc} : c \in Z_b\} \leq \sum\{r_{bc} \,;\, s_{bc} : c \in Z_b\}, \end{aligned}$$

by (5), the distributivity of the operator $\,;\,$ in $\mathfrak{A}$, the assumption that the set Z_b is finite and not empty, (6), and Boolean algebra. Multiply the first and last terms of this inequality by the atom a, and use Boolean algebra and the hypotheses of (ii) to arrive at

$$\begin{aligned} a \cdot (r_b \,;\, s_b) \leq a \cdot (\sum\{r_{bc} \,;\, s_{bc} : c \in Z_b\}) \\ = \sum\{a \cdot (r_{bc} \,;\, s_{bc}) : c \in Z_b\} = 0. \end{aligned}$$

This establishes (3).

The closed element p is the supremum of the set W of quasi-atoms in $\mathfrak{B}$ that are below p, so

$$p = \sum\{b : b \in W\} \leq \sum\{r_b : b \in W\},$$

by the first inequality in (3). Apply Corollary 14.3(i) to obtain a finite subset W_0 of W such that

$$p \leq \sum\{r_b : b \in W_0\}. \tag{7}$$

The set W is not empty (it contains the element 0), so it may be assumed that W_0 is not empty. Define

$$r = \sum\{r_b : b \in W_0\} \qquad \text{and} \qquad s = \prod\{s_b : b \in W_0\}, \tag{8}$$

and observe that both of these elements belong to $\mathfrak{A}$, since both of them are Boolean combinations of finitely many elements in $\mathfrak{A}$. Observe also

that $p \leq r$, by (7) and (8), and $q \leq s$, by (8) and the second inequality in (3). For each b in W_0, we have

$$r_b \,;\, s = r_b \,;\, (\prod\{s_d : d \in W_0\}) \leq r_b \,;\, s_b, \tag{9}$$

by (8) and the monotony of the operator ; in $\mathfrak{A}$. Consequently,

$$\begin{aligned} r \,;\, s &= (\sum\{r_b : b \in W_0\}) \,;\, s \\ &= \sum\{r_b \,;\, s : b \in W_0\} \leq \sum\{r_b \,;\, s_b : b \in W_0\}, \end{aligned}$$

by (8), the distributivity of the operator ; in $\mathfrak{A}$, the assumption that the set W_0 is finite and not empty, (9), and Boolean algebra. Multiply the first and last terms of this inequality by the atom a, and use Boolean algebra and the final equation in (3) to arrive at

$$\begin{aligned} a \cdot (r \,;\, s) &\leq a \cdot (\sum\{r_b \,;\, s_b : b \in W_0\}) \\ &= \sum\{a \cdot (r_b \,;\, s_b) : b \in W_0\} = 0, \end{aligned}$$

as desired. □

We turn now to the proof that the operations $^\smile$ and ; in $\mathfrak{B}$ are quasi-completely distributive. The next lemma, the key step in the argument, essentially establishes this result for closed elements.

Lemma 14.25. *If p and q are closed elements in $\mathfrak{B}$, and if W and Z are the sets of quasi-atoms in $\mathfrak{B}$ that are below p and q respectively, then*

$$\begin{aligned} p \,;\, q \leq \sum(W \,;\, Z) &= \sum\{b \,;\, c : b \in W \text{ and } c \in Z\}, \\ p^\smile \leq \sum(W^\smile) &= \sum\{b^\smile : b \in W\}. \end{aligned}$$

Proof. The algebra $\mathfrak{B}$ is atomic, so every element is the sum of the set of atoms that it dominates. Consequently, in order to establish the second inequality, it suffices to show that every atom below $p^\smile$ is also below $\sum(W^\smile)$. We shall prove the contrapositive of this assertion, namely that every atom disjoint from $\sum(W^\smile)$ is also disjoint from $p^\smile$.

Consider an atom a that is disjoint from $\sum(W^\smile)$. Since

$$0 = a \cdot \sum(W^\smile) = \sum\{a \cdot b^\smile : b \in W\},$$

by the complete distributivity of multiplication in $\mathfrak{B}$, we have

$$a \cdot b^{\smile} = 0 \tag{1}$$

for every b in W. Fix a quasi-atom b in W, with the goal of constructing an element r_b in $\mathfrak{A}$ such that

$$b \leq r_b \qquad \text{and} \qquad a \cdot r_b^{\smile} = 0. \tag{2}$$

Lemmas 14.5 and 14.18 ensure that

$$b = \prod A_b \qquad \text{and} \qquad b^{\smile} = \prod(A_b^{\smile}). \tag{3}$$

In particular,

$$0 = a \cdot b^{\smile} = a \cdot \prod(A_b^{\smile}), \tag{4}$$

by (1) and the second equation in (3). The atom a is closed, by Lemma 14.4 and the definition of a closed element, so its complement $-a$ is open. Also, the set $A_b^{\smile}$ is included in $\mathfrak{A}$, by the definition of A_b and the closure of $\mathfrak{A}$ under the operation $^{\smile}$, and

$$\prod(A_b^{\smile}) \leq -a,$$

by (4). Use compactness in the form of Corollary 14.3(ii) (with $-a$ in place of q) to obtain a finite subset X of A_b such that

$$\prod(X^{\smile}) \leq -a$$

and therefore

$$0 = a \cdot \prod(X^{\smile}). \tag{5}$$

The element

$$r_b = \prod X \tag{6}$$

is the product of finitely many elements in $\mathfrak{A}$, so it belongs to $\mathfrak{A}$. Also,

$$b = \prod A_b \leq \prod X = r_b, \tag{7}$$

by the first equation in (3), Boolean algebra, the definition of X, and (6). For each element t in X, the product $\prod X$ is below t and therefore $(\prod X)^{\smile}$ is below $t^{\smile}$, by the monotony law for $^{\smile}$ (Lemma 14.23). Consequently,

$$(\prod X)^{\smile} \leq \prod\{t^{\smile} : t \in X\} = \prod(X^{\smile}), \tag{8}$$

by Boolean algebra. Use (6) and (8) to obtain

$$r_b^{\smile} = (\prod X)^{\smile} \leq \prod(X^{\smile}). \tag{9}$$

Combine (5) and (9) to arrive at

$$a \cdot r_b^{\smile} \leq a \cdot \prod(X^{\smile}) = 0. \tag{10}$$

It follows from (7) and (10) that r_b has the properties stated in (2), so the hypotheses of Lemma 14.24(i) are satisfied. Apply the lemma to obtain an element r in $\mathfrak{A}$ such that

$$p \leq r \qquad \text{and} \qquad a \cdot r^{\smile} = 0. \tag{11}$$

Since $p^{\smile} \leq r^{\smile}$, by the first part of (11) and the monotony law for $^{\smile}$, it may be concluded from the second part of (11) that $a \cdot p^{\smile} = 0$, as desired.

Turn now to the case of the operation $;$. As in the preceding argument, it suffices to show that every atom a disjoint from $\sum(W ; Z)$ is also disjoint from $p ; q$. Since

$$0 = a \cdot \sum(W ; Z) = \sum\{a \cdot (b ; c) : b \in W \text{ and } c \in Z\},$$

by assumption and Boolean algebra, it follows that

$$a \cdot (b ; c) = 0 \tag{12}$$

for every b in W and c in Z. Fix quasi-atoms b and c in W, with the goal of constructing elements r_{bc} and s_{bc} in $\mathfrak{A}$ such that

$$b \leq r_{bc}, \qquad c \leq s_{bc}, \qquad \text{and} \qquad a \cdot (r_{bc} ; s_{bc}) = 0. \tag{13}$$

Lemmas 14.5 and 14.18 ensure that

$$b = \prod A_b, \qquad c = \prod A_c, \qquad \text{and} \qquad b ; c = \prod(A_b ; A_c). \tag{14}$$

In particular,

$$0 = a \cdot (b ; c) = a \cdot \prod(A_b ; A_c), \tag{15}$$

by (12) and the third equation in (14). The atom a is closed, by Lemma 14.4, so its complement $-a$ is open. Also, the set $A_b ; A_c$ is included in $\mathfrak{A}$, by the closure of $\mathfrak{A}$ under the operation $;$, and

$$\prod(A_b ; A_c) \leq -a,$$

by (15). Use compactness in the form of Corollary 14.3(ii) (with $-a$ in place of q) to obtain a finite subset X of $A_b\,;A_c$ such that

$$\prod X \leq -a.$$

Clearly, there must be finite subsets X_b of A_b and X_c of A_c such that X is included in $X_b\,;X_c$, by the definition of the set $A_b\,;A_c$, and therefore

$$\prod(X_b\,;X_c) \leq \prod X \leq -a,$$

by Boolean algebra. Consequently,

$$0 = a \cdot \prod(X_b\,;X_c). \tag{16}$$

The elements

$$r_{bc} = \prod X_b \qquad \text{and} \qquad s_{bc} = \prod X_c \tag{17}$$

are products of finitely many elements in $\mathfrak{A}$, so they belong to $\mathfrak{A}$. Also,

$$b = \prod A_b \leq \prod X_b = r_{bc} \qquad \text{and} \qquad c = \prod A_c \leq \prod X_c = s_{bc}, \tag{18}$$

by the first two equations in (14), the definitions of the sets X_b and X_c, and (17). For all elements t in X_b and u in X_c, the products $\prod X_b$ and $\prod X_c$ are below t and u respectively, and therefore

$$(\prod X_b)\,;(\prod X_c) \leq t\,;u,$$

by the monotony law for $;$. Consequently,

$$(\prod X_b)\,;(\prod X_c) \leq \prod\{t\,;u : t \in X_b \text{ and } u \in X_c\} = \prod(X_b\,;X_c), \tag{19}$$

by Boolean algebra. Use (17) and (19) to obtain

$$r_{bc}\,;s_{bc} = (\prod X_b)\,;(\prod X_c) \leq \prod(X_b\,;X_c). \tag{20}$$

Combine (16) and (20) to arrive at

$$a \cdot (r_{bc}\,;s_{bc}) \leq a \cdot \prod(X_b\,;X_c) = 0. \tag{21}$$

It follows from (18) and (21) that r_{bc} and s_{bc} have the properties stated in (13), so the hypotheses of Lemma 14.24(ii) are satisfied. Apply the lemma to obtain elements r and s in $\mathfrak{A}$ such that

$$p \leq r, \qquad q \leq s, \qquad \text{and} \qquad a \cdot (r\,;s) = 0. \tag{22}$$

Since $p\,;q \leq r\,;s$, by the first part of (22) and monotony law for $;$ (Lemma 14.23), it may be concluded from the second part of (22) that $a \cdot (p\,;q) = 0$, as desired. □

Lemma 14.26. *The operations* ; *and* $\smile$ *in* $\mathfrak{B}$ *are quasi-completely distributive, and completely distributive if* $\mathfrak{A}$ *is normal.*

Proof. To prove the first assertion of the lemma, it suffices to show that the operations ; and $\smile$ are quasi-completely distributive for quasi-atoms, by Lemma 2.6. Focus on the first of these operations. Let W and Z be arbitrary non-empty sets of quasi-atoms in $\mathfrak{B}$, and write

$$p = \sum W \qquad \text{and} \qquad q = \sum Z. \tag{1}$$

It is to be shown that

$$p\,;q = \sum\{b\,;c : b \in W \text{ and } c \in Z\} = \sum(W\,;Z). \tag{2}$$

If b is in W and c in Z, then $b\,;c \leq p\,;q$, by (1) and the monotony law for ; (Lemma 14.23), and therefore

$$\sum(W\,;Z) = \sum\{b\,;c : b \in W \text{ and } c \in Z\} \leq p\,;q. \tag{3}$$

To establish the reverse inequality, observe that

$$p\,;q = \sum(K_p\,;K_q) = \sum\{u\,;v : u \in K_p \text{ and } v \in K_q\}, \tag{4}$$

by Corollary 14.20. Consider fixed elements u in K_p and v in K_q, and let W_u and Z_v be the sets of quasi-atoms below u and v respectively. Since $u \leq p$ and $v \leq q$, by the definition of the sets K_p and K_q, it follows that every quasi-atom below u is below p, and every quasi-atom below v is below q. In other words, the sets W_u and Z_v are respectively included in the sets

$$W \cup \{0\} \qquad \text{and} \qquad Z \cup \{0\},$$

by (1), and consequently

$$\sum(W_u\,;Z_v) \leq \sum[(W \cup \{0\})\,;(Z \cup \{0\})]. \tag{5}$$

The sets W and Z are both non-empty, by assumption. For any fixed elements b in W and c in Z, each of the elements

$$0\,;0, \qquad b\,;0, \qquad 0\,;c$$

is below $b\,;c$, by the monotony law for ;, and consequently

$$\sum[(W \cup \{0\})\,;(Z \cup \{0\})] = \sum(W\,;Z). \tag{6}$$

Lemma 14.25 (with u and v in place of p and q respectively) implies that

$$u ; v \leq \sum(W_u ; Z_v).$$

Combine this inequality with (5) and (6) to obtain

$$u ; v \leq \sum(W ; Z). \tag{7}$$

The right side of (7) does not depend on u or v. Sum (7) over all elements u in K_p and v in K_q, and combine the result with (4), to arrive at

$$p ; q = \sum\{u ; v : u \in K_p \text{ and } v \in K_q\} \leq \sum(W ; Z). \tag{8}$$

Together, (3) and (8) yield the desired conclusion (2).

It has been shown that the operation ; in $\mathfrak{B}$ is quasi-completely distributive. If $\mathfrak{A}$ is normal, then $\mathfrak{B}$ is also normal, by Lemma 14.22, and therefore ; is completely distributive, by Lemma 2.5. □

With the proof of Lemma 14.26, the task of showing that the algebra $\mathfrak{B}$ is the canonical extension of the algebra $\mathfrak{A}$ has been completed. This yields the following alternative version of the Existence Theorem for canonical extensions of Boolean algebras with operators.

Theorem 14.27. *For a given Boolean algebra with operators* $\mathfrak{A}$*, the algebra* $\mathfrak{B}$ *constructed in the manner described before Lemma* 14.21 *is the canonical extension of* $\mathfrak{A}$*.*

14.4 First Preservation Theorem

We have seen (in two different ways) that every Boolean algebra with operators $\mathfrak{A}$ has a uniquely determined canonical extension, but we don't yet know much about the properties this canonical extension may inherit from $\mathfrak{A}$. For example, we don't yet know that the canonical extension of a relation algebra is a relation algebra. What is needed are some general theorems about the preservation of certain properties, and in particular, of certain types of equations, under the passage to canonical extensions. In establishing such preservation theorems, it is convenient to disregard the set-theoretical construction of the canonical embedding algebra given in Section 14.2, and to focus instead on the abstract construction given in Section 14.3.

We shall show that, in general, positive equations and certain forms of implications between Boolean combinations of positive equations are preserved under the passage to canonical extensions. This result will imply, in particular, that the canonical extension of a relation algebra is a relation algebra, and the canonical extension of a simple relation algebra is a simple relation algebra. Some of the key steps in the proof involve extensions of Lemma 14.18, Corollary 14.20, and Lemma 14.23 to arbitrary positive polynomials in $\mathfrak{B}$. Using these extensions, we show first that two positive polynomials which agree on $\mathfrak{A}$ must agree on K, and second that two positive polynomials which agree on K must agree on all of $\mathfrak{B}$. Conclusion: two positive polynomials which agree on $\mathfrak{A}$ must agree on $\mathfrak{B}$.

The first task is to define the notion of a positive polynomial in a Boolean algebra with operators. One approach is to define first the notion of a positive term, and then to define a *positive polynomial* of rank n to be a polynomial of rank n that is induced in the algebra by some positive term with variables among $v_0, \ldots, v_{n-1}$ (see Section 2.4). For the sake of concreteness, we restrict our attention to the language of relation algebras. First, augment this language by introducing a new binary operation symbol $\cdot$ that is intended to denote multiplication. A term in the *augmented language* is said to be *constant* if no variables occur in it. A term in the augmented language is said to be *positive* if it can be built up from variables and constant terms using the symbols for addition, multiplication, relative multiplication, and converse, but not complement. In other words, variables and constant terms are positive (this is the base clause of the definition), and if σ and τ are positive terms, then so are

$$\sigma + \tau, \qquad \sigma \cdot \tau, \qquad \sigma\,;\tau, \qquad \text{and} \qquad \sigma^{\smile}$$

(this is the induction clause of the definition), and finally, a term is positive if and only if it can be shown to be a positive by a finite number of applications of the base clause and the induction clause. Said somewhat differently, a term in the augmented language is positive just in case no variable occurs within the scope of the operation symbol for complement (but the constant terms are allowed to contain occurrences of this symbol). It follows from the preceding inductive definition and from Lemma 14.21 that if $\mathfrak{B}$ is the canonical extension of a Boolean algebra with operators $\mathfrak{A}$, then $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ and therefore the polynomial of rank n induced in $\mathfrak{A}$ by a given positive term is just

the restriction to the universe of $\mathfrak{A}$ of the polynomial of rank n induced in $\mathfrak{B}$ by the same term.

The inductive definitions of a positive term and of the polynomial of rank n induced by a term (see Section 2.4) can be combined to give an alternative, direct definition of the notion of a positive polynomial of rank n, one that does not depend on the notion of a term. From this point of view, a positive polynomial of rank n is, intuitively speaking, an operation of n arguments that is built up from the projections and constant operations using the operations $+$, $\cdot$, $;$ and $^\smile$, but not $-$. The precise definition proceeds by a type of induction.

Fix a Boolean algebra with operators $\mathfrak{A}$, and let A^n be the set of all sequences $r = (r_0, r_1, \ldots, r_{n-1})$ with coordinates r_i in $\mathfrak{A}$ for $0 \leq i < n$. Recall that the ith projection of rank n in $\mathfrak{A}$ is the operation π_i of n arguments that is defined by

$$\pi_i(r) = r_i$$

for each r in A^n. A *distinguished constant operation* of rank n on A is an operation σ of n arguments such that for some element s in the minimal subalgebra of $\mathfrak{A}$ (assuming that $\mathfrak{A}$ has distinguished constants), we have $\sigma(r) = s$ for every r in A^n. An operation of rank n on A is a *positive polynomial* in $\mathfrak{A}$ if it belongs to every set Ω of operations of rank n on A such that (1) Ω contains the projections and distinguished constant operations of rank n, and (2) if σ and τ are operations of rank n in Ω, then so are the operations $\sigma + \tau$, $\sigma \cdot \tau$, $\sigma\,;\tau$, and $\sigma^\smile$ that are respectively defined by

$$\begin{aligned} (\sigma + \tau)(r) &= \sigma(r) + \tau(r), \qquad & (\sigma \cdot \tau)(r) &= \sigma(r) \cdot \tau(r), \\ (\sigma\,;\tau)(r) &= \sigma(r)\,;\tau(r), & (\sigma^\smile)(r) &= \sigma(r)^\smile \end{aligned}$$

for every r in A^n. (The operations on the right sides of these equations are those of $\mathfrak{A}$.)

This more direct inductive definition of a positive polynomial allows us to avoid discussions of terms and induction on terms when proving theorems about positive polynomials; instead, one proves that the theorem holds for the projections and the distinguished constant operations of rank n, and that it holds for the operations

$$\sigma + \tau, \qquad \sigma \cdot \tau, \qquad \sigma\,;\tau, \qquad \text{and} \qquad \sigma^\smile$$

whenever it holds for the operations σ and τ of rank n. The definition of a positive polynomial then implies that the theorem holds for

all positive polynomials of rank n. We shall call this the *principle of induction for positive polynomials*. The next lemma, a consequence of Lemma 14.18, illustrates the method.

In the remainder of the discussion, let $\mathfrak{A}$ be a Boolean algebra with operators, and $\mathfrak{B}$ the canonical extension of $\mathfrak{A}$ defined before Lemma 14.18. A subset X of $\mathfrak{B}$ is said to be *closed under positive polynomials* if $\gamma(p)$ belongs to X whenever p is a sequence of n elements from X and γ is a positive polynomial of rank n.

Lemma 14.28. *The set K of closed elements in $\mathfrak{B}$ is closed under positive polynomials.*

Proof. The proof proceeds by induction on positive polynomials of rank n. It is obvious that K is closed under the projections, and K is closed under the distinguished constant operations because K includes the universe of $\mathfrak{A}$ (see the remarks preceding Corollary 14.3).

Assume now as the induction hypothesis that K is closed under positive polynomials σ and τ of rank n, and fix a sequence p of n elements from K. The elements $\sigma(p)$ and $\tau(p)$ belong to K, by the induction hypothesis, and therefore the elements

$$\sigma(p) + \tau(p), \qquad \sigma(p) \cdot \tau(p), \qquad \sigma(p)\,;\tau(p), \qquad \sigma(p)^{\smile}$$

all belong to K, by Corollary 14.19. It follows that if γ is one of the four polynomials

$$\sigma + \tau, \qquad \sigma \cdot \tau, \qquad \sigma\,;\tau, \qquad \sigma^{\smile},$$

then $\gamma(p)$ belongs to K. Conclusion: K is closed under all positive polynomials, by the principle of induction for positive polynomials. □

In formulating and proving the next three lemmas, it is helpful to introduce some notation. The sets B^n and A^n are Boolean algebras in their own right under the Boolean operations that are defined coordinatewise in terms of the Boolean operations of $\mathfrak{B}$ and $\mathfrak{A}$ respectively, that is to say, under the Boolean operations of the nth powers $\mathfrak{B}^n$ and $\mathfrak{A}^n$. Thus, it makes sense to speak, for example, of the sum $p + q$ and the product $p \cdot q$ when p and q are sequences in B^n; in fact, this sum and product are just the coordinatewise sum and product of p and q,

$$(p_0 + q_0, \ldots, p_{n-1} + q_{n-1}) \qquad \text{and} \qquad (p_0 \cdot q_0, \ldots, p_{n-1} \cdot q_{n-1})$$

respectively. Also, $p \leq q$ if and only if $p_i \leq q_i$ for each $i < n$. We extend the notations A_p and K_p to sequences p in B^n by writing

$$A_p = \{r \in A^n : p \leq r\} \qquad \text{and} \qquad K_p = \{r \in K^n : r \leq p\}.$$

(Technically speaking, these notations should have some reference to n, for instance A_p^n and K_p^n. The fact that p is an n-tuple, however, renders the notations A_p and K_p unambiguous.) If $p = (p_0, \ldots, p_{n-1})$, then

$$A_p = A_{p_0} \times \cdots \times A_{p_{n-1}} \qquad \text{and} \qquad K_p = K_{p_0} \times \cdots \times K_{p_{n-1}}.$$

Indeed, for a sequence $r = (r_0, \ldots, r_{n-1})$ of elements in $\mathfrak{B}$ we have

$$\begin{aligned} r \in A_p \qquad & \text{if and only if} \qquad r \in A^n \text{ and } p \leq r, \\ & \text{if and only if} \qquad r_i \in A \text{ and } p_i \leq r_i \text{ for each } i, \\ & \text{if and only if} \qquad r_i \in A_{p_i} \text{ for each } i, \end{aligned}$$

by the definition of the sets A_p and A_{p_i}, and the definition of the partial order in the Boolean algebra A^n. This argument establishes the first equality; a similar argument, with the inequalities reversed and with A replaced by K, establishes the second. The two equalities and the fact that the factor sets A_{p_j} and K_{p_j} are non-empty for all $j < n$—they contain the elements 1 and 0 respectively—imply that

$$\begin{aligned} A_{p_i} &= \{s \in B : s = r_i \text{ for some } r \in A_p\}, \\ K_{p_i} &= \{s \in B : s = r_i \text{ for some } r \in K_p\}. \end{aligned}$$

Here is the extension of Lemma 14.23 to positive polynomials.

Lemma 14.29. *Every positive polynomial γ of rank n in $\mathfrak{B}$ is monotone in the sense that $p \leq q$ implies $\gamma(p) \leq \gamma(q)$ for all p, q in B^n.*

Proof. Fix elements p and q in B^n with $p \leq q$. The proof that

$$\gamma(p) \leq \gamma(q) \tag{1}$$

proceeds by induction on positive polynomials of rank n. If γ is one of the projections π_i, then

$$\gamma(p) = \pi_i(p) = p_i \leq q_i = \pi_i(q) = \gamma(q),$$

by the definition of π_i and the assumption that $p \leq q$ (and hence $p_i \leq q_i$ for each i), so (1) holds. If γ is a distinguished constant operation, say with value s, then

$$\gamma(p) = s = \gamma(q),$$

so again (1) holds.

Assume as the induction hypothesis that positive polynomials σ and τ of rank n satisfy

$$\sigma(p) \leq \sigma(q) \qquad \text{and} \qquad \tau(p) \leq \tau(q).$$

The operations $+$, $\cdot$, $;$, and $^\smile$ in $\mathfrak{B}$ are all monotone, by Boolean algebra and Lemma 14.23, so

$$\sigma(p) + \tau(p) \leq \sigma(q) + \tau(q), \qquad \sigma(p) \cdot \tau(p) \leq \sigma(q) \cdot \tau(q),$$
$$\sigma(p) \,;\, \tau(p) \leq \sigma(q) \,;\, \tau(q), \qquad \sigma(p)^\smile \leq \sigma(q)^\smile.$$

It follows from these inequalities that (1) holds when γ is one of the four polynomials $\sigma + \tau$, $\sigma \cdot \tau$, $\sigma \,;\, \tau$, $\sigma^\smile$. Apply the principle of induction for positive polynomials to conclude that (1) holds for all positive polynomials γ. □

The task of extending Lemma 14.18 to positive polynomials is more involved.

Lemma 14.30. *If γ is a positive polynomial of rank n in $\mathfrak{B}$, then*

$$\gamma(p) = \prod\{\gamma(r) : r \in A_p\}$$

for every p in K^n.

Proof. Fix a sequence p in K^n, and observe that $p \leq r$ for every r in A_p, by the definition of the set A_p. If γ is a positive polynomial of rank n, then $\gamma(p) \leq \gamma(r)$ for every r in A_p, by the monotony law for positive polynomials established in Lemma 14.29, and therefore

$$\gamma(p) \leq \prod\{\gamma(r) : r \in A_p\},$$

by Boolean algebra.

The proof that the reverse inequality

$$\gamma(p) \geq \prod\{\gamma(r) : r \in A_p\} \tag{1}$$

also holds proceeds by induction on positive polynomials of rank n. If γ is a projection π_i, then (1) holds because

$$\gamma(p) = \pi_i(p) = p_i = \prod A_{p_i} = \prod\{s \in B : s = r_i \text{ for some } r \in A_p\}$$
$$= \prod\{\pi_i(r) : r \in A_p\} = \prod\{\gamma(r) : r \in A_p\},$$

by the assumption on γ, the definition of π_i, Lemma 14.5, and the remark preceding Lemma 14.29. If γ is a distinguished constant operation, say with value s, then $\gamma(q) = s$ for all q in B^n and therefore

$$\gamma(p) = s = \prod\{\gamma(r) : r \in A_p\},$$

so that (1) holds in this case as well. (Notice in connection with the preceding equalities that the set A_p is not empty, because it contains the unit element of A^n.)

Assume as the induction hypothesis that σ and τ are positive polynomials of rank n in $\mathfrak{B}$ satisfying the inequalities

$$\sigma(p) \geq \prod\{\sigma(r) : r \in A_p\} \qquad \text{and} \qquad \tau(p) \geq \prod\{\tau(r) : r \in A_p\}. \tag{2}$$

The proof of (1) now breaks into four cases. Consider the case

$$\gamma = \sigma \,;\, \tau \tag{3}$$

as an example. The coordinates of the sequence p belong to the set K, by assumption, and K is closed under positive polynomials, by Lemma 14.28, so the elements $\sigma(p)$ and $\tau(p)$ belong to K and are therefore closed elements. It follows that

$$\sigma(p) \,;\, \tau(p) = \prod(A_{\sigma(p)} \,;\, A_{\tau(p)})$$
$$= \prod\{s \,;\, t : s \in A_{\sigma(p)} \text{ and } t \in A_{\tau(p)}\}, \tag{4}$$

by Lemma 14.18 and the definition of the set $A_{\sigma(p)} \,;\, A_{\tau(p)}$. For every s in $A_{\sigma(p)}$ and t in $A_{\tau(p)}$, we shall construct an element r_{st} in A_p such that

$$\sigma(r_{st}) \leq s \qquad \text{and} \qquad \tau(r_{st}) \leq t. \tag{5}$$

The inequality in (1) then follows by (3), Boolean algebra (since r_{st} is in A_p), (5) and the monotony law for $\,;\,$ in $\mathfrak{A}$, and (4):

$$\begin{aligned}\prod\{\gamma(r) : r \in A_p\} &= \prod\{\sigma(r) \,;\, \tau(r) : r \in A_p\} \\ &\leq \prod\{\sigma(r_{st}) \,;\, \tau(r_{st}) : s \in A_{\sigma(p)} \text{ and } t \in A_{\tau(p)}\} \\ &\leq \prod\{s \,;\, t : s \in A_{\sigma(p)} \text{ and } t \in A_{\tau(p)}\} \\ &= \sigma(p) \,;\, \tau(p) = \gamma(p).\end{aligned}$$

Fix elements s in $A_{\sigma(p)}$ and t in $A_{\tau(p)}$, and observe that

$$\sigma(p) \leq s \qquad \text{and} \qquad \tau(p) \leq t, \tag{6}$$

by the definitions of the sets $A_{\sigma(p)}$ and $A_{\tau(p)}$. The construction of r_{st} involves a compactness argument. Both s and t are in $\mathfrak{A}$, and therefore both elements are open, by the definition of an open element (see the remarks preceding Corollary 14.3). Also, the sets

$$\{\sigma(r) : r \in A_p\} \qquad \text{and} \qquad \{\tau(r) : r \in A_p\}$$

are subsets of $\mathfrak{A}$, by the closure of $\mathfrak{A}$ under polynomials, and

$$\prod\{\sigma(r) : r \in A_p\} \leq \sigma(p) \leq s \quad \text{and} \quad \prod\{\tau(r) : r \in A_p\} \leq \tau(p) \leq t,$$

by assumption (2) and by (6). Apply compactness in the form of Corollary 14.3(ii) to these two inequalities to obtain finite subsets W and Z of A_p such that

$$\prod\{\sigma(r) : r \in W\} \leq s \qquad \text{and} \qquad \prod\{\tau(r) : r \in Z\} \leq t. \tag{7}$$

Of course, the union $W \cup Z$ is also a finite subset of A_p, so it is a finite set of elements in $\mathfrak{A}$ that are above p. The product

$$r_{st} = \prod(W \cup Z) \tag{8}$$

is again an element in $\mathfrak{A}$, by the closure of $\mathfrak{A}$ under multiplication, and it is above p, by Boolean algebra, so it must belong to the set A_p, by the definition of this set. The product r_{st} is below each element r in W, by (8) and Boolean algebra, so the monotony law for positive polynomials (Lemma 14.29) implies that $\sigma(r_{st}) \leq \sigma(r)$ for every r in W, and therefore

$$\sigma(r_{st}) \leq \prod\{\sigma(r) : r \in W\} \leq s,$$

by Boolean algebra and the first inequality in (7). This establishes the first inequality in (5). The second inequality in (5) is established by an analogous argument (with σ and W replaced by τ and Z respectively), using the second inequality in (7). Thus, r_{st} has the required properties. The proof for case (3) is complete.

The same argument, with ; replaced first by $+$ and then by $\cdot$, establishes (1) for the cases when $\gamma = \sigma + \tau$ and $\gamma = \sigma \cdot \tau$ respectively.

(There are also easier arguments that can be used to establish these cases; see the exercises.) An entirely analogous but simpler argument establishes (1) when $\gamma = \sigma^{\smile}$. This completes the induction step of the argument. The inequality in (1) now follows by the principle of induction for positive polynomials. □

Here, finally, is the extension of Corollary 14.20 to positive polynomials.

Lemma 14.31. *If γ is a positive polynomial of rank n in $\mathfrak{B}$, then*

$$\gamma(p) = \sum\{\gamma(r) : r \in K_p\}$$

for every p in B^n.

Proof. Fix an arbitrary n-tuple p in B^n, and observe that $r \leq p$ for every r in K_p, by the definition of the set K_p. If γ is a positive polynomial of rank n, then $\gamma(r) \leq \gamma(p)$ for every r in K_p, by the monotony law for positive polynomials established in Lemma 14.29, and therefore

$$\sum\{\gamma(r) : r \in K_p\} \leq \gamma(p),$$

by Boolean algebra.

The proof that the reverse inequality

$$\gamma(p) \leq \sum\{\gamma(r) : r \in K_p\} \tag{1}$$

also holds proceeds by induction on positive polynomials of rank n. If γ is a projection π_i, then (1) holds because

$$\begin{aligned}\gamma(p) = \pi_i(p) = p_i = \sum K_{p_i} &= \sum\{s \in B : s = r_i \text{ for some } r \in K_p\} \\ &= \sum\{\pi_i(r) : r \in K_p\} = \sum\{\gamma(r) : r \in K_p\},\end{aligned}$$

by the assumption on γ, the definition of π_i, Lemma 14.17, and the remark preceding Lemma 14.29. If γ is a distinguished constant operation, say with value s, then $\gamma(q) = s$ for all q in B^n and therefore

$$\gamma(p) = s = \sum\{\gamma(r) : r \in K_p\},$$

so that (1) holds in this case as well. (Notice in connection with the preceding equalities that the set K_p is not empty, because it contains the zero element of B^n.)

Before proceeding to the induction step, we make a preliminary observation. Suppose σ and τ are positive polynomials in $\mathfrak{B}$. If s and t are elements in K_p, then both elements are below $s+t$, by Boolean algebra, and therefore

$$\sigma(s) \leq \sigma(s+t) \qquad \text{and} \qquad \tau(t) \leq \tau(s+t),$$

by the monotony law for positive polynomials. It follows that

$$\sigma(s) \cdot \tau(t) \leq \sigma(s+t) \cdot \tau(s+t),$$

and hence also that

$$\sum\{\sigma(s) \cdot \tau(t) : s,t \in K_p\} \leq \sum\{\sigma(s+t) \cdot \tau(s+t) : s,t \in K_p\},$$

by Boolean algebra. The sum $s+t$ belongs to K_p, by Corollary 14.19, so

$$\sum\{\sigma(s+t) \cdot \tau(s+t) : s,t \in K_p\} \leq \sum\{\sigma(r) \cdot \tau(r) : r \in K_p\},$$

by Boolean algebra. Combine these last two inequalities to arrive at

$$\sum\{\sigma(s) \cdot \tau(t) : s,t \in K_p\} \leq \sum\{\sigma(r) \cdot \tau(r) : r \in K_p\}. \tag{2}$$

A parallel argument with $\cdot$ replaced by $;$, and using also the monotony law for the operation $;$ in $\mathfrak{B}$ (Lemma 14.23), leads to a similar conclusion for $;$, namely

$$\sum\{\sigma(s) \,;\, \tau(t) : s,t \in K_p\} \leq \sum\{\sigma(r) \,;\, \tau(r) : r \in K_p\}. \tag{3}$$

Assume now as the induction hypothesis that σ and τ are positive polynomials of rank n in $\mathfrak{B}$ satisfying the inequalities

$$\sigma(p) \leq \sum\{\sigma(r) : r \in K_p\} \qquad \text{and} \qquad \tau(p) \leq \sum\{\tau(r) : r \in K_p\}. \tag{4}$$

The proof of (1) breaks into four cases. In the case

$$\gamma = \sigma + \tau, \tag{5}$$

we have

$$\begin{aligned}\gamma(p) = \sigma(p) + \tau(p) &\leq (\sum\{\sigma(r) : r \in K_p\}) + (\sum\{\tau(r) : r \in K_p\}) \\ &= \sum\{\sigma(r) + \tau(r) : r \in K_p\} = \sum\{\gamma(r) : r \in K_p\},\end{aligned}$$

by (5), the induction hypothesis in (4), the monotony law for addition, and the associative and commutative laws for (possibly infinite) addition. Thus, (1) holds in this case.

Consider next the case when

$$\gamma = \sigma \,;\, \tau. \tag{6}$$

Use (6), the induction hypothesis in (4), the monotony law for ; in $\mathfrak{B}$ (Lemma 14.23), the quasi-complete distributivity of the operation ; (Lemma 14.26), and (3) to obtain

$$\begin{aligned}\gamma(p) = \sigma(p) \,;\, \tau(p) &\leq (\textstyle\sum\{\sigma(r) : r \in K_p\}) \,;\, (\sum\{\tau(r) : r \in K_p\}) \\ &= \textstyle\sum\{\sigma(s) \,;\, \tau(t) : s, t \in K_p\} \\ &\leq \textstyle\sum\{\sigma(r) \,;\, \tau(r) : r \in K_p\} \\ &= \textstyle\sum\{\gamma(r) : r \in K_p\}.\end{aligned}$$

(The set K_p is not empty because it contains the zero element of B^n. Consequently, the quasi-complete distributive law for the operation ; applies.) Thus, (1) holds in this case as well.

The proofs in the cases when γ is $\sigma \cdot \tau$ or $\sigma^\smile$ are quite similar to the preceding proof and are left as an exercise. (The proof when γ is $\sigma \cdot \tau$ uses (2) instead of (3).) This completes the induction step of the argument. Apply the principle of induction for positive polynomials to conclude that (1) holds for all positive polynomials γ. □

An equation between two terms in the augmented language of relation algebras is said to be *positive* if both terms are positive. The next theorem, the *First Preservation Theorem* for canonical extensions, says that positive equations are preserved under the passage to canonical extensions.

Theorem 14.32. *Any positive equation that holds in a Boolean algebra with operators $\mathfrak{A}$ continues to hold in the canonical extension of $\mathfrak{A}$.*

Proof. Consider an arbitrary positive equation ε, say with variables among $v_0, \ldots, v_{n-1}$, and assume that ε is valid in $\mathfrak{A}$. The goal is to show that ε is valid in the algebra $\mathfrak{B}$ that is the canonical extension of $\mathfrak{A}$. To this end, let σ and τ be the polynomials of rank n in $\mathfrak{B}$ that are induced by the terms on the right and left sides of ε. As was remarked at the beginning of the section, the polynomials of rank n in $\mathfrak{A}$ that are induced by the right and left sides of ε are just the restrictions of σ and τ to $\mathfrak{A}$.

The assumption that ε is valid in $\mathfrak{A}$ means that

$$\sigma(r) = \tau(r)$$

for all sequences r in A^n. In particular, for each p in K^n,

$$\{\sigma(r) : r \in A_p\} = \{\tau(r) : r \in A_p\}.$$

Form the product of both sides of this equation, and use Lemma 14.30, to arrive at

$$\sigma(p) = \prod\{\sigma(r) : r \in A_p\} = \prod\{\tau(r) : r \in A_p\} = \tau(p).$$

Thus, σ and τ agree on K^n. Consequently, for any given p in B^n,

$$\{\sigma(r) : r \in K_p\} = \{\tau(r) : r \in K_p\}.$$

Form the sum of both sides of this equation, and use Lemma 14.31, to conclude that

$$\sigma(p) = \sum\{\sigma(r) : r \in K_p\} = \sum\{\tau(r) : r \in K_p\} = \tau(p).$$

Thus, σ and τ agree on B^n, so ε is valid in $\mathfrak{B}$. □

14.5 Second Preservation Theorem

The goal of this section is to extend the First Preservation Theorem 14.32 by showing that certain implications between positive equations and inequalities are preserved under the passage to canonical extensions. The proof makes use of a new unary operation, the so-called *unary discriminator*. We need to make some preliminary remarks about this operation and about the adjunction of this operation to a given Boolean algebra with operators $\mathfrak{A}$.

As has already been mentioned, the only property of the Boolean algebra with operators $\mathfrak{A}$ that is need in the lemmas leading up to Theorem 14.32 is distributivity of the operations ; and $^\smile$. Consequently, all of the results in the preceding three sections extend without difficulty to Boolean algebras with operators of arbitrary ranks. Consider, as an example, an operator O of rank n on the universe of $\mathfrak{A}$. Let $\mathfrak{A}^*$ be

the expanded algebra obtained from $\mathfrak{A}$ by adjoining O as a new fundamental operation. The definition of a canonical extension $\mathfrak{B}^*$ of $\mathfrak{A}^*$ is obtained from Definition 14.12 by requiring the corresponding operation O in $\mathfrak{B}^*$ to be quasi-completely distributive, and by adjoining to condition (iv) the requirement that

$$O(a) = \prod\{O(r) : a \leq r \text{ and } r \in A^n\}$$

for every n-tuple a of quasi-atoms in $\mathfrak{B}^*$. The proof of the analogue of Uniqueness Theorem 14.13 in this more general setting goes through with only minor modifications. As regards the existence of a canonical extension of $\mathfrak{A}^*$, one uses the same canonical extension $\mathfrak{B}$ that was constructed in Section 14.3 for the algebra $\mathfrak{A}$, but adjoins to it an operation O of rank n on the universe of $\mathfrak{B}$ that is the natural extension of the corresponding operation in $\mathfrak{A}^*$, and is defined by

$$O(p) = \sum\{\prod\{O(r) : r \in A^n \text{ and } u \leq r\} : u \in K^n \text{ and } u \leq p\}$$

for all p in B^n (where the occurrence of O on the right side denotes the corresponding operation in $\mathfrak{A}^*$). The proofs of Lemmas 14.18–14.31 in this more general setting go through with only minor modifications, and one obtains the analogue of the First Preservation Theorem for the algebra $\mathfrak{A}^*$ and its canonical extension $\mathfrak{B}^*$.

For a quite concrete example, consider the *unary discriminator*, which is the unary operation O defined on the universe of a Boolean algebra with operators $\mathfrak{A}$ by

$$O(r) = \begin{cases} 0 & \text{if} \quad r = 0, \\ 1 & \text{if} \quad r \neq 0, \end{cases}$$

for r in $\mathfrak{A}$. It is easy to check that the operation O is completely distributive, so that the algebra $\mathfrak{A}^*$ obtained by adjoining O to $\mathfrak{A}$ as a new operation is also a Boolean algebra with operators. Indeed, let X be an arbitrary subset $\mathfrak{A}$, and suppose X has a supremum t in $\mathfrak{A}$. It is to be shown that

$$O(t) = \sum\{O(r) : r \in X\}.$$

There are two cases to consider, according to whether X does or does not contain a non-zero element. If X does contain a non-zero element, say s, then t is also non-zero, since t is above s, and therefore

$$O(t) = 1 = O(s),$$

by the definition of O. In particular, the set $\{O(r) : r \in X\}$ contains 1 (because s is one of the elements in X), so the supremum of this set is 1, by Boolean algebra. Thus, the desired equality holds. If X does not contain a non-zero element, then X is either empty or consists just of the element 0, and therefore the set $\{O(r) : r \in X\}$ is either empty or consists just of the element 0, by the definition of O. Consequently, the supremum of this set is 0. Also, the supremum t of the set X must be 0, by Boolean algebra, so $O(t) = 0$, by the definition of O. It follows that the desired equality holds in this case as well.

It turns out that the validity of certain implications in $\mathfrak{A}$ is equivalent to the validity of certain equations in $\mathfrak{A}^*$. For example, an implication of the form

$$(\varrho = 0) \to (\sigma = \tau)$$

holds in $\mathfrak{A}$ (where ϱ, σ, and τ are terms in the augmented language of $\mathfrak{A}$) if and only if the equation

$$\sigma + O(\varrho) = \tau + O(\varrho)$$

holds in $\mathfrak{A}^*$. Similarly, an implication of the form

$$[(\varrho_0 = 0) \wedge (\varrho_1 = 0) \wedge (\varrho_2 \neq 0) \wedge (\varrho_3 \neq 0)] \to (\sigma = \tau),$$

holds in $\mathfrak{A}$ if and only if the equation

$$\sigma \cdot O(\varrho_2) \cdot O(\varrho_3) + O(\varrho_0) + O(\varrho_1) \\ = \tau \cdot O(\varrho_2) \cdot O(\varrho_3) + O(\varrho_0) + O(\varrho_1)$$

holds in $\mathfrak{A}^*$. More generally, we have the following result.

Lemma 14.33. *Suppose $\varrho_0, \ldots, \varrho_{k-1}, \varrho_k, \ldots, \varrho_{\ell-1}, \sigma, \tau$ are terms with variables among $v_0, \ldots, v_{n-1}$ in the augmented language of a Boolean algebra with operators $\mathfrak{A}$. The implication*

$$\Big(\bigwedge_{i=0}^{k-1} (\varrho_i = 0)\Big) \wedge \Big(\bigwedge_{i=k}^{\ell-1} (\varrho_i \neq 0)\Big) \to (\sigma = \tau) \tag{i}$$

holds in $\mathfrak{A}$ if and only if the equation

$$\sigma \cdot \Big(\prod_{i=k}^{\ell-1} O(\varrho_i)\Big) + \sum_{i=0}^{k-1} O(\varrho_i) = \tau \cdot \Big(\prod_{i=k}^{\ell-1} O(\varrho_i)\Big) + \sum_{i=0}^{k-1} O(\varrho_i) \tag{ii}$$

holds in the expanded algebra $\mathfrak{A}^*$.

Proof. The operations induced by the terms given in the lemma are polynomials of rank n in $\mathfrak{A}$. It should not cause any confusion if we use the same symbols to refer to these polynomials as are used to refer to the terms themselves.

Consider a sequence r of n elements in $\mathfrak{A}$. If $\varrho_i(r) \neq 0$ for some i with $0 \leq i < k$, then $O(\varrho_i(r)) = 1$, by the definition of O, so the sum

$$\sum_{i=0}^{k-1} O(\varrho_i(r)) \tag{1}$$

in $\mathfrak{A}^*$ is 1, by Boolean algebra. The values of the polynomials

$$\sigma \cdot \Big(\prod_{i=k}^{\ell-1} O(\varrho_i) \Big) + \sum_{i=0}^{k-1} O(\varrho_i) \quad \text{and} \quad \tau \cdot \Big(\prod_{i=k}^{\ell-1} O(\varrho_i) \Big) + \sum_{i=0}^{k-1} O(\varrho_i) \tag{2}$$

at r (in $\mathfrak{A}^*$) are therefore 1, so equation (ii) is satisfied by r in $\mathfrak{A}^*$.

Similarly, if $\varrho_i(r) = 0$ for some i with $k \leq i < \ell$, then $O(\varrho_i(r)) = 0$, by the definition of O, so the product

$$\prod_{i=k}^{\ell-1} O(\varrho_i(r)) \tag{3}$$

in $\mathfrak{A}^*$ is 0, by Boolean algebra. The value at r of each polynomial in (2) is therefore equal to the sum in (1), so again equation (ii) is satisfied by r in $\mathfrak{A}^*$.

There remains the case when $\varrho_i(r) = 0$ for each i with $0 \leq i < k$, and $\varrho_i(r) \neq 0$ for each i with $k \leq i < \ell$. This is precisely the case in which r satisfies the hypotheses of implication (i). Under these hypotheses, we have $O(\varrho_i(r)) = 0$ for each i with $0 \leq i < k$, and $O(\varrho_i(r)) = 1$ for each i with $k \leq i < \ell$, by the definition of O, so the sum in (1) is equal to 0, and the product in (3) is equal to 1. The values at r of the polynomials in (2) are therefore $\sigma(r)$ and $\tau(r)$ respectively, by Boolean algebra. Consequently, equation (ii) is satisfied by r in $\mathfrak{A}^*$ if and only if

$$\sigma(r) = \tau(r), \tag{4}$$

that is to say, if and only if r satisfies the conclusion of implication (i).

For each sequence r of n elements in $\mathfrak{A}$, it has been shown that if r does not satisfy the hypotheses of implication (i), then r automatically satisfies implication (i) in $\mathfrak{A}$, and r trivially satisfies equation (ii) in $\mathfrak{A}^*$; and if the sequence r does satisfy the hypotheses of (i), then r satisfies

this implication in $\mathfrak{A}$ just in case (4) holds, and r satisfies equation (ii) in $\mathfrak{A}^*$ just in case (4) holds. Consequently, r satisfies implication (i) in $\mathfrak{A}$ if and only if it satisfies equation (ii) in $\mathfrak{A}^*$. □

Notice that if the terms ϱ_i, σ, and τ in the preceding lemma are all positive (in the augmented language of $\mathfrak{A}$), then equation (ii) is positive (in the augmented language of $\mathfrak{A}^*$).

Let $\mathfrak{B}$ be the canonical extension of the given Boolean algebra with operators $\mathfrak{A}$. Define a unary operation O on the universe of $\mathfrak{B}$ in the manner described at the beginning of the section:

$$O(p) = \sum\{\prod\{O(r) : r \in A \text{ and } u \leq r\} : u \in K \text{ and } u \leq p\}$$

for all p in $\mathfrak{B}$. The algebra $\mathfrak{B}^*$ obtained by adjoining this operation to $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}^*$, by Theorem 14.27 (in the version applicable to $\mathfrak{A}^*$ and $\mathfrak{B}^*$). The new operation O in $\mathfrak{B}^*$ assumes the value 0 on the element 0 (since this is true in $\mathfrak{A}^*$, and $\mathfrak{A}^*$ is a subalgebra of $\mathfrak{B}^*$), and it assumes the value 1 on non-zero elements. Indeed, a non-zero element p in $\mathfrak{B}$ is the sum of the closed elements that are below it, by Lemma 14.17, so there must be a non-zero closed element u below p. The element u, in turn, is the product of the elements in $\mathfrak{A}$ that are above it, by Lemma 14.5, and each of these elements is obviously non-zero (since u is non-zero), so the operation O in $\mathfrak{A}^*$ assumes the value 1 on each of these elements, by the definition of this operation. Use Lemma 14.18 (in the version applicable to $\mathfrak{B}^*$) to obtain

$$O(u) = \prod\{O(r) : r \in A \text{ and } u \leq r\} = 1.$$

Use Corollary 14.20 (in the form applicable to $\mathfrak{B}^*$) to arrive at

$$O(p) = \sum\{O(v) : v \in K \text{ and } v \leq p\} \geq O(u) = 1.$$

Conclusion: the canonical extension of the unary discriminator on the universe of $\mathfrak{A}$ is the the unary discriminator on the universe of $\mathfrak{B}$.

The next theorem, the *Second Preservation Theorem* for canonical extensions, concerns the preservation of certain types of implications under the passage to canonical extensions. The hypotheses of these implications are *Boolean combinations of positive equations*, that is to say, they are formulas built up from positive equations using the logical operations of conjunction, disjunction, and negation. The conclusions of the implications are positive equations. The unary discriminator plays a critical role in the proof of the theorem

Theorem 14.34. *Let ε be any Boolean combination of positive equations of the form $\varrho = 0$ (with ϱ positive), and σ and τ any positive terms, in the augmented language of a Boolean algebra with operators $\mathfrak{A}$. If the implication*

$$\varepsilon \to (\sigma = \tau)$$

holds in $\mathfrak{A}$, then it holds in the canonical extension of $\mathfrak{A}$.

Proof. Let $\mathfrak{A}^*$ be the algebra obtained from $\mathfrak{A}$ by adjoining the unary discriminator O to the fundamental operations of $\mathfrak{A}$. Let $\mathfrak{B}$ and $\mathfrak{B}^*$ be the canonical extensions of $\mathfrak{A}$ and $\mathfrak{A}^*$ respectively. The remarks preceding the theorem show that the operation O in $\mathfrak{B}^*$ is, in fact, the unary discriminator on the universe of $\mathfrak{B}$.

To prove the theorem, assume that ε is a Boolean combination of equations of the specified form and that the implication

$$\varepsilon \to (\sigma = \tau) \tag{1}$$

is valid in $\mathfrak{A}$. Consider first the case when ε is a conjunction of equations of the form $\varrho_i = 0$ and inequalities of the form $\varrho_j \neq 0$, where the terms ϱ_i and ϱ_j are all positive. The equation δ correlated with this implication in Lemma 14.33 is then valid in the algebra $\mathfrak{A}^*$, by Lemma 14.33. The equation δ is positive, so it is preserved under the passage to canonical extensions, by the First Preservation Theorem 14.32 for canonical extensions. Consequently, δ is valid in the canonical extension $\mathfrak{B}^*$ of $\mathfrak{A}^*$. Apply Lemma 14.33 again (with $\mathfrak{A}$ and $\mathfrak{A}^*$ replaced by $\mathfrak{B}$ and $\mathfrak{B}^*$ respectively) to conclude that the implication in (1) is valid in $\mathfrak{B}$.

Consider next the case when ε is a disjunction of finitely many formulas ε_i, each of which is a conjunction of equations and inequalities of the specified form. The implication in (1) is then logically equivalent to the conjunction of the implications

$$\varepsilon_i \to (\sigma = \tau). \tag{2}$$

Since (1) is assumed to be valid in $\mathfrak{A}$, it is satisfied by each sequence of elements from $\mathfrak{A}$ of the appropriate length. Consequently, each such sequence satisfies (2) for each i. In other words, each of the implications in (2) must be valid in $\mathfrak{A}$. These implications are all preserved under the passage to canonical extensions, by the observations of the preceding

paragraph, so they are all valid in $\mathfrak{B}$. Consequently, their conjunction is valid in $\mathfrak{B}$, and therefore so is the logically equivalent implication (1).

Finally, consider the general case in which ε is any Boolean combination of equations of the specified form. The formula ε is logically equivalent to a disjunction ε' of finitely many formulas each of which is a conjunction of equations and inequalities of the specified form, so the implication in (1) is logically equivalent to the implication

$$\varepsilon' \to (\sigma = \tau). \tag{3}$$

Since (1) is valid in $\mathfrak{A}$, it follows that (3) is also valid in $\mathfrak{A}$. The observations of the preceding paragraph show that (3) is preserved under the passage to canonical extensions, so it must be valid in $\mathfrak{B}$. Consequently, (1) is also valid in $\mathfrak{B}$. □

14.6 Applications to relation algebras

We have seen that every Boolean algebra with operators, and in particular every relation algebra, has a uniquely determined canonical extension. With the help of the two preservation theorems, we can now conclude that this canonical extension is itself a relation algebra.

Theorem 14.35. *Every relation algebra $\mathfrak{A}$ has a canonical extension that is uniquely determined up to isomorphisms that are the identity function on $\mathfrak{A}$, and this canonical extension is a relation algebra.*

Proof. A relation algebra $\mathfrak{A}$ has a canonical extension $\mathfrak{B}$, by Existence Theorem 14.16 (or Theorem 14.27), and this canonical extension is uniquely determined up to isomorphisms that are the identity function on $\mathfrak{A}$, by Uniqueness Theorem 14.13. It remains to show that $\mathfrak{B}$ is itself a relation algebra.

The Boolean axioms (R1)–(R3) and the relation algebraic axioms (R8) and (R9) hold automatically in $\mathfrak{B}$, because $\mathfrak{B}$ is a Boolean algebra with operators. The relation algebraic axioms (R4)–(R7) hold in $\mathfrak{B}$, because they hold in $\mathfrak{A}$ (by assumption) and they are positive equations; consequently, they are preserved under the passage to canonical extensions, by the First Preservation Theorem 14.32. The cycle law (R11) holds in $\mathfrak{B}$, because it holds in $\mathfrak{A}$ (see the discussion at the end of Section 2.1, and see the De Morgan-Tarski laws in Lemma 4.8) and it has the form of an implication

$$(\varrho = 0) \to (\sigma = 0),$$

where the terms ϱ and σ are positive terms; consequently, this law is preserved under the passage to canonical extensions, by the Second Preservation Theorem 14.34, and is therefore valid in $\mathfrak{B}$. Since Axiom (R10) is equivalent to (R11) on the basis of the other relation algebraic axioms, it, too, must hold in $\mathfrak{B}$. Conclusion: $\mathfrak{B}$ is a relation algebra. □

It is natural to inquire whether other special properties of relation algebras, or of elements in a relation algebra, are inherited by canonical extensions. For example, is the canonical extension of a set relation algebra isomorphic to a set relation algebra? What if a relation algebra is the complex algebra of a group, a lattice, or a geometry; is its canonical extension isomorphic to, or at least embeddable in, the complex algebra of a group, a lattice, or a geometry respectively? If a relation algebra is simple or integral, is its canonical extension also simple or integral? Some of these questions will be addressed in this section, and some will be addressed in Section 19.9.

Consider first the properties of elements. All properties that are defined by equations and inequalities, such as the property of being an equivalence element, a function, or an ideal element, are of course preserved under the passage to canonical extensions. A property of elements that is not of this form but that is still preserved is the property of being an atom: an atom in the original algebra $\mathfrak{A}$ remains an atom in the canonical extension $\mathfrak{B}$. For the proof, observe that if r is an atom in $\mathfrak{A}$, then the set of elements in $\mathfrak{A}$ that are above r is an ultrafilter in $\mathfrak{A}$, and in fact it is the principal ultrafilter generated by r. The infimum of this ultrafilter in the canonical extension $\mathfrak{B}$ is an atom, by Lemma 14.7, and since r is the smallest element in the ultrafilter, the infimum is obviously r. Thus, r remains an atom in the canonical extension $\mathfrak{B}$.

If $\mathfrak{A}$ is finite, then the converse direction of the preceding observation is also true: every atom in $\mathfrak{B}$ is an atom in $\mathfrak{A}$, because in this case $\mathfrak{B}$ coincides with $\mathfrak{A}$ (see below). However, if $\mathfrak{A}$ is infinite, then the converse fails in a dramatic way: the canonical extension $\mathfrak{B}$ will always have atoms that are not atoms in $\mathfrak{A}$. Indeed, in this case $\mathfrak{A}$ has non-principal ultrafilters, by the Boolean version of the dual of Corollary 8.35, and for each such ultrafilter X, the infimum of X in $\mathfrak{B}$ is an atom in $\mathfrak{B}$ (by Lemma 14.7) that is not an atom in $\mathfrak{A}$ (because the ultrafilter X is non-principal).

A finite Boolean algebra with operators $\mathfrak{A}$ is its own canonical extension. To prove this, it suffices to show that if we take $\mathfrak{B} = \mathfrak{A}$, then $\mathfrak{B}$ satisfies the conditions in Definition 14.12 with respect to the algebra $\mathfrak{A}$. Clearly, $\mathfrak{B}$ is complete and atomic, because it is finite. Also, $\mathfrak{B}$ is a Boolean algebra with operators, because $\mathfrak{A}$ is assumed to satisfy this assumption. The operators in $\mathfrak{B}$ are quasi-completely distributive because in a finite algebra, quasi-complete distributivity reduces to distributivity. The algebra $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, because $\mathfrak{A}$ is a subalgebra of itself. The compactness property holds trivially in $\mathfrak{B}$ with respect to $\mathfrak{A}$, because all subsets of $\mathfrak{A}$ are finite. The atom separation property also holds trivially in $\mathfrak{B}$ with respect to $\mathfrak{A}$, because distinct atoms a and b in $\mathfrak{B}$ are separated by the element $r = a$ in $\mathfrak{A}$. Finally, the equations in condition (iv) of Definition 14.12 hold trivially in $\mathfrak{B}$, because any quasi-atoms a and b in $\mathfrak{B}$ must belong to $\mathfrak{A}$, and therefore the elements $a \,;\, b$ and $a^\smile$ respectively belong to the sets on the right sides of the first and second equations in condition (iv); consequently, they must be the infima of these sets.

The observations of the preceding paragraph raise the question of the relative size of the canonical extension $\mathfrak{B}$ in comparison with the size of $\mathfrak{A}$, when $\mathfrak{A}$ is infinite. It can be shown that if the number of elements in $\mathfrak{A}$ is an infinite cardinal m, then the number of ultrafilters in $\mathfrak{A}$ is between m and 2^m. Each ultrafilter in $\mathfrak{A}$ determines, and is determined by, a unique atom in the canonical extension $\mathfrak{B}$, by Lemma 14.7, so $\mathfrak{B}$ has between m and 2^m atoms. Since $\mathfrak{B}$ is complete and atomic, each element in $\mathfrak{B}$ determines, and is determined by, a unique set of atoms, so $\mathfrak{B}$ has between 2^m and 2^{2^m} elements.

For relation algebras, the properties of being simple and being integral are preserved under the passage to canonical extensions.

Theorem 14.36. *A relation algebra is simple or integral if and only if its canonical extension is simple or integral.*

Proof. A non-degenerate relation algebra is simple if and only if the implication

$$(r \neq 0) \to (1 \,;\, r \,;\, 1 = 1)$$

holds in it, and integral if and only if the implication

$$(r \neq 0) \to (r \,;\, 1 = 1)$$

holds in it, by Simplicity Theorem 9.2 and Integrality Theorem 9.7 respectively. Each of these implications, being a quantifier-free formula, is preserved under the passage to subalgebras; and each of them, having the form prescribed in the Second Preservation Theorem 14.34, is preserved under the passage to canonical extensions. Consequently, each of the implications holds in a non-degenerate relation algebra $\mathfrak{A}$ if and only if it holds in the canonical extension of $\mathfrak{A}$, so $\mathfrak{A}$ is simple or integral if and only if its canonical extension is simple or integral. □

The part of the preceding theorem that concerns simplicity may be viewed as a statement about the Boolean algebra of ideal elements (see Section 8.9) in a relation algebra $\mathfrak{A}$ and its canonical extension. In this context, it says that if the Boolean algebra of ideal elements has exactly two elements in $\mathfrak{A}$, then it has exactly two elements in the canonical extension of $\mathfrak{A}$. In this form, the statement can be generalized.

Theorem 14.37. *If $\mathfrak{B}$ is the canonical extension of a relation algebra $\mathfrak{A}$, then the Boolean algebra of ideal elements in $\mathfrak{B}$ is the canonical (Boolean) extension of the Boolean algebra of ideal elements in $\mathfrak{A}$.*

Proof. Let A_0 and B_0 be the Boolean algebras of ideal elements in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively. It must be shown that conditions (i)–(iii) of Definition 14.1 hold for A_0 and B_0. Every ideal element in $\mathfrak{A}$ remains an ideal element in $\mathfrak{B}$, and the operations in $\mathfrak{A}$ are the restrictions of operations in $\mathfrak{B}$, so A_0 is certainly a Boolean subalgebra of B_0. Also, $\mathfrak{B}$ is complete and atomic, by Definition 14.12(i), and B_0 is a strongly regular Boolean subalgebra of $\mathfrak{B}$, by Lemma 8.24, so B_0 is complete and atomic, by Corollary 8.25 and Lemma 8.28. Thus, condition (i) holds.

Turn now to the verification of condition (iii), the compactness property. The algebra $\mathfrak{B}$ has the compactness property with respect to $\mathfrak{A}$, by Definition 14.12(iii). In particular, if X is a subset of A_0 with supremum 1 in $\mathfrak{B}$, then some finite subset of X already has supremum 1 in $\mathfrak{B}$. Suprema of subsets of B_0 are the same in B_0 as they are in $\mathfrak{B}$, by Lemma 8.24, so the preceding observation implies that if X is a subset of A_0 with supremum 1 in B_0, then some finite subset of X has supremum 1 in B_0. Thus, condition (iii) holds.

It remains to check condition (ii), the atom separation property. Consider two ideal element atoms in B_0, say a and b. There must be atoms c and d in $\mathfrak{B}$ such that

$$a = 1 ; c ; 1 \qquad \text{and} \qquad b = 1 ; d ; 1, \tag{1}$$

by Lemma 8.28. (One may take for c and d any atoms in $\mathfrak{B}$ that are below a and b respectively.) The atoms c and d are closed elements in $\mathfrak{B}$, by Lemma 14.7, and the unary operation γ defined by

$$\gamma(r) = 1\,;r\,;1$$

for all r in $\mathfrak{B}$ is a positive polynomial, so Lemma 14.30 implies that

$$1\,;c\,;1 = \prod\{1\,;r\,;1 : r \in A_c\},$$
$$1\,;d\,;1 = \prod\{1\,;r\,;1 : r \in A_d\}.$$

The ideal element atoms a and b are assumed to be distinct, so the preceding equations and (1) imply that there must be an element r in the set A_c such that b is not below $1\,;r\,;1$. The complement of $1\,;r\,;1$ is also an ideal element, and therefore in B_0, by Lemma 5.39(iv). Since b is an atom in B_0 that is not below $1\,;r\,;1$, it must be below $-(1\,;r\,;1)$. Consequently, $1\,;r\,;1$ is an element in A_0 that separates a and b, so condition (ii) holds. □

The preceding theorem can be used to obtain information about direct decompositions of canonical extensions. The following theorem gives an example.

Theorem 14.38. *A relation algebra $\mathfrak{A}$ has a total decomposition into finitely many simple factors*

$$\mathfrak{A} = \mathfrak{A}(a_0) \times \cdots \times \mathfrak{A}(a_{n-1})$$

if and only if its canonical extension $\mathfrak{B}$ has the total decomposition

$$\mathfrak{B} = \mathfrak{B}(a_0) \times \cdots \times \mathfrak{B}(a_{n-1}).$$

Proof. Assume first that $\mathfrak{A}$ has the given total direct decomposition. The Boolean algebra of ideal elements in $\mathfrak{A}$ is then atomic with finitely many distinct atoms $a_0, \ldots, a_{n-1}$, by Total Decomposition Theorem 11.41 applied to $\mathfrak{A}$. The Boolean algebra of ideal elements in $\mathfrak{B}$ is the canonical extension of the Boolean algebra of ideals elements in $\mathfrak{A}$, by Theorem 14.37. Since the latter is finite, it must coincide with the former, by the remarks preceding Theorem 14.36. Apply Theorem 11.41 to $\mathfrak{B}$ to conclude that $\mathfrak{B}$ has the total direct decomposition given above.

To prove the reverse implication, assume that $\mathfrak{B}$ has a finite total decomposition, as given above. The Boolean algebra of ideal elements in $\mathfrak{B}$ is then atomic with finitely many (distinct) atoms $a_0, \ldots, a_{n-1}$, by Theorem 11.41. In particular, this Boolean algebra must be finite, so it coincides with the Boolean algebra of ideal elements in $\mathfrak{A}$, by Theorem 14.37 and the remarks preceding Theorem 14.36. Apply Theorem 11.41 to $\mathfrak{A}$ to conclude that $\mathfrak{A}$ has the required total direct decomposition. □

The preceding theorem cannot be extended to a relation algebra $\mathfrak{A}$ with an infinite Boolean algebra of ideal elements. In this case, the canonical extension $\mathfrak{B}$ will always have a total direct decomposition, by Corollary 11.45, but the algebra $\mathfrak{A}$ may have no such decomposition. In fact, the Boolean algebra of ideal elements in $\mathfrak{A}$ may be atomless, so that $\mathfrak{A}$ has no simple factors whatsoever. However, even in the case when $\mathfrak{A}$ is complete with an atomic Boolean algebra of ideal elements—so that $\mathfrak{A}$ *does* have a unique total decomposition, by Theorem 11.41—the Boolean algebra of ideal elements in $\mathfrak{B}$ will have ideal element atoms that do not occur in $\mathfrak{A}$ at all, by the observations preceding Theorem 14.36. The total decomposition of $\mathfrak{B}$ will therefore contain simple factors that are not related in any way to the simple factors of $\mathfrak{A}$.

Some further insight into Theorem 14.38 will be given in Theorem 14.49 below.

14.7 Canonical extensions of homomorphisms

So far, the discussion about the preservation of properties has been limited to a single relation algebra and its canonical extension. There are, however, properties that involve more than one relation algebra, for instance the property of one relation algebra being a subalgebra or a homomorphic image of another. To deal with questions regarding the preservation of such properties, it is helpful to discuss another extension of the lemmas leading up to the First and Second Preservation Theorems. Those lemmas are concerned with distributive operations on the universe of a Boolean algebra with operators. One can also consider functions with arguments that are sequences of elements (of some fixed finite length n) in one algebra $\mathfrak{A}$ and values that are sequences of elements (of some fixed finite length m) in another algebra $\bar{\mathfrak{A}}$. If φ

is such a function, and if $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ are the canonical extensions of $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ respectively, then the *canonical extension* of φ is defined to be the function φ^+ whose value at each sequence p of n elements in $\mathfrak{B}$ is given by

$$\begin{aligned}\varphi^+(p) &= \sum\{\prod\{\varphi(r) : r \in A^n \text{ and } u \leq r\} : u \in K^n \text{ and } u \leq p\} \\ &= \sum\{\prod\{\varphi(r) : r \in A_u\} : u \in K_p\}.\end{aligned}$$

(The infima and the supremum on the right are formed coordinatewise in $\bar{\mathfrak{B}}$.) The general theorem that one obtains says, roughly speaking, that if the function φ is distributive (over addition) in each coordinate, then the canonical extension of φ is quasi-completely distributive in each coordinate and inherits all of the properties of φ that are expressible by positive equations (involving φ). Moreover, the canonical extension of the composition of such functions is equal to the composition of the canonical extensions of the individual functions.

We illustrate the ideas with an important concrete example. Fix two Boolean algebras with operators, say $\mathfrak{A}$ and $\bar{\mathfrak{A}}$, and let $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ be their respective canonical extensions. For notational convenience, we assume that the similarity type of these algebras is the same as that of relation algebras. Consider a mapping φ from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, and write

$$\varphi(A_p) = \{\varphi(r) : r \in A_p\} = \{\varphi(r) : r \in A \text{ and } p \leq r\}.$$

The canonical extension of φ is the mapping φ^+ from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$ that is defined at each p in $\mathfrak{B}$ by

$$\varphi^+(p) = \sum\{\prod \varphi(A_u) : u \in K_p\}.$$

(The infima and the supremum on the right are formed in $\bar{\mathfrak{B}}$.) The immediate goal is to prove that if φ is a homomorphism, then φ^+ is a complete homomorphism extending φ. The argument involves a series of lemmas that are the analogues of Lemmas 14.18–14.31 above. The proofs of most of these analogues involve only minor modifications of the proofs of the original lemmas, and will be left as exercises.

We begin with just the assumption that φ preserves the operation of addition in the sense that

$$\varphi(r + s) = \varphi(r) + \varphi(s)$$

for all elements r and s in $\mathfrak{A}$. The analogue of Lemma 14.18 says that if p is a closed element in $\mathfrak{B}$, then

$$\varphi^+(p) = \prod \varphi(A_p) = \{\varphi(r) : r \in A_p\}.$$

The analogue of Corollary 14.20 says that if p is an arbitrary element in $\mathfrak{B}$, then

$$\varphi^+(p) = \sum \varphi^+(K_p) = \sum\{\varphi^+(u) : u \in K_p\}.$$

The analogue of Lemma 14.21 says that φ^+ is an extension of φ, that is to say,

$$\varphi^+(r) = \varphi(r)$$

for all r in $\mathfrak{A}$. The analogue of Lemma 14.22 says that φ^+ maps 0 (in $\mathfrak{B}$) to 0 (in $\bar{\mathfrak{B}}$) if and only if φ maps 0 to 0. The analogue of Lemma 14.23 says that the function φ^+ is monotone in the sense that

$$p \leq q \qquad \text{implies} \qquad \varphi^+(p) \leq \varphi^+(q)$$

for all p and q in $\mathfrak{B}$.

The analogue of Lemma 14.24 assumes that p is a closed element, and a an atom, in $\mathfrak{B}$, and that for every quasi-atom $b \leq p$ there is an element r_b in $\mathfrak{A}$ such that

$$b \leq r_b \qquad \text{and} \qquad a \cdot \varphi(r_b) = 0.$$

It concludes that there is an element r in $\mathfrak{A}$ such that

$$p \leq r \qquad \text{and} \qquad a \cdot \varphi(r) = 0.$$

Using this lemma, one proves the analogue of Lemma 14.25, namely that if p is a closed element in $\mathfrak{B}$, and if W is the set of quasi-atoms in $\mathfrak{B}$ that are below p, then

$$\varphi^+(p) \leq \sum \varphi^+(W) = \sum\{\varphi^+(b) : b \in W\}.$$

With the help of this lemma one shows, in analogy with the proof of Lemma 14.26, that if W is an arbitrary non-empty set of quasi-atoms in $\mathfrak{B}$, and if $p = \sum W$, then

$$\varphi^+(p) = \sum \varphi^+(W).$$

This result says, in essence, that φ^+ preserves arbitrary sums of non-empty sets of quasi-atoms. One also establishes the analogue of Lemma 2.6, namely that if φ^+ preserves all sums of non-empty sets

of quasi-atoms, then φ^+ preserves all sums of non-empty sets of elements. The preceding results combine to establish the analogue of Lemma 14.26, namely that φ^+ preserves all sums of non-empty sets of elements (and not just non-empty sets of quasi-atoms) in $\mathfrak{B}$; moreover, if φ maps 0 to 0, then φ^+ preserves all sums of arbitrary sets of elements.

The next step is to establish the analogue of Lemma 14.30. In the present situation, it is not necessary to consider arbitrary positive polynomials, but rather only certain restricted "heterogeneous positive polynomials" that involve the mappings φ and φ^+, and the distributive operations of the algebras in question. In formulating the lemma, it is helpful to use some of the notation that was introduced earlier. For example, if X and Y are subsets of $\mathfrak{A}$, then

$$\begin{aligned} \varphi(X) &= \{\varphi(r) : u \in X\}, \\ X \cdot Y &= \{r \cdot s : r \in X \text{ and } s \in Y\}, \end{aligned}$$

and similarly for the sets $X\,;Y$ and $X^\smile$. These notations may be combined. For example,

$$\begin{aligned} \varphi(X \cdot Y) &= \{\varphi(r \cdot s) : r \in X \text{ and } s \in Y\}, \\ \varphi(X) \cdot \varphi(Y) &= \{\varphi(r) \cdot \varphi(s) : r \in X \text{ and } s \in Y\}. \end{aligned}$$

Lemma 14.39. *If p and q are closed elements in $\mathfrak{B}$, then*

(i) $\varphi^+(p \cdot q) = \prod \varphi(A_p \cdot A_q)$,
(ii) $\varphi^+(p) \cdot \varphi^+(q) = \prod(\varphi(A_p) \cdot \varphi(A_q))$,
(iii) $\varphi^+(p\,;q) = \prod \varphi(A_p\,;A_q)$,
(iv) $\varphi^+(p)\,;\varphi^+(q) = \prod(\varphi(A_p)\,;\varphi(A_q))$,
(v) $\varphi^+(p^\smile) = \prod \varphi(A_p^\smile)$,
(vi) $\varphi^+(p)^\smile = \prod(\varphi(A_p)^\smile)$.

Proof. As examples, here are the proofs of (iii) and (iv). We begin with (iii). If r and s are elements in A_p and A_q respectively, then $p \leq r$ and $q \leq s$, by the definitions of the sets involved (namely A_p and A_q), and therefore $p\,;q \leq r\,;s$, by Lemma 14.23. Consequently,

$$\varphi^+(p\,;q) \leq \varphi^+(r\,;s) = \varphi(r\,;s),$$

by the analogues of Lemmas 14.23 and 14.21 for φ^+. Form the product (in $\bar{\mathfrak{B}}$) over all appropriate r and s to arrive at

$$\begin{aligned}\varphi^+(p\,;q) &\leq \prod\{\varphi(r\,;s) : r \in A_p \text{ and } s \in A_q\}\\ &= \prod \varphi(A_p\,;A_q).\end{aligned}$$

To establish the reverse inequality,

$$\prod \varphi(A_p\,;A_q) \leq \varphi^+(p\,;q), \tag{1}$$

observe first that p and q are both elements in K, by assumption, and therefore $p\,;q$ also belongs to K, by Corollary 14.19. Apply the analogue of Lemma 14.18 for φ^+ to obtain

$$\varphi^+(p\,;q) = \prod \varphi(A_{p;q}). \tag{2}$$

In order to establish (1), it suffices to demonstrate that

$$\prod \varphi(A_p\,;A_q) \leq \prod \varphi(A_{p;q}), \tag{3}$$

by (2).

To prove (3), we associate with each t in $A_{p;q}$ a pair of elements r_t in A_p and s_t in A_q such that $r_t\,;s_t \leq t$. This inequality, and the assumption that φ preserves the operation of addition (and therefore also preserves inequalities—compare Lemma 2.3), together imply that

$$\varphi(r_t\,;s_t) \leq \varphi(t). \tag{4}$$

Consequently,

$$\begin{aligned}\prod \varphi(A_p\,;A_q) &= \prod\{\varphi(r\,;s) : r \in A_p \text{ and } s \in A_q\}\\ &\leq \prod\{\varphi(r_t\,;s_t) : t \in A_{p;q}\}\\ &\leq \prod\{\varphi(t) : t \in A_{p;q}\}\\ &= \prod \varphi(A_{p;q}).\end{aligned}$$

The first step follows by the definition of the set $\varphi(A_p\,;A_q)$, the second by Boolean algebra and the fact that r_t and s_t are in A_p and A_q respectively, the third by (4), and the fourth by the definition of the set $\varphi(A_{p;q})$.

The task remains of constructing r_t and s_t. Lemma 14.18, the assumption that p and q are closed elements, the assumption that t is in $A_{p;q}$, and the definition of this last set imply that

$$\prod(A_p\,;A_q) = p\,;q \leq t. \tag{5}$$

The element t is open because it belongs to $\mathfrak{A}$. Apply compactness to (5) in the form of Corollary 14.3(ii) (for $\mathfrak{B}$ with respect to $\mathfrak{A}$) to obtain a finite subset Z of $A_p \,;\, A_q$ such that $\prod Z \leq t$. Clearly, there must be finite subsets X and Y of A_p and A_q respectively such that Z is included in $X \,;\, Y$, and consequently

$$\prod(X \,;\, Y) \leq \prod Z \leq t. \tag{6}$$

Write

$$r_t = \prod X \qquad \text{and} \qquad s_t = \prod Y, \tag{7}$$

and observe that r_t is in A_p and s_t in A_q, since the sets A_p and A_q are filters (see Lemma 14.5) and are therefore closed under multiplication. Also, r_t is below each element in X, and s_t is below each element in Y, by (7), so

$$r_t \,;\, s_t \leq r \,;\, s$$

for each r in X and s in Y, by the monotony of the operator $\,;\,$ in $\mathfrak{A}$ (Lemma 2.3). Form the product (in $\mathfrak{A}$) over all appropriate r and s to obtain

$$r_t \,;\, s_t \leq \prod\{r \,;\, s : r \in X \text{ and } s \in Y\} = \prod(X \,;\, Y). \tag{8}$$

Combine (6) and (8) to arrive at the desired inequality $r_t \,;\, s_t \leq t$. This completes the proof of (1), and hence also the proof of (iii).

Turn now to the proof of (iv). If r is in A_p and s in A_q, then $p \leq r$ and $q \leq s$, by the definitions of the sets A_p and A_q, and therefore

$$\varphi^+(p) \leq \varphi^+(r) = \varphi(r) \qquad \text{and} \qquad \varphi^+(q) \leq \varphi^+(s) = \varphi(s),$$

by the analogues of Lemmas 14.23 and 14.21 for φ^+. It follows that

$$\varphi^+(p) \,;\, \varphi^+(q) \leq \varphi(r) \,;\, \varphi(s),$$

by the monotony of the operator $\,;\,$ in $\bar{\mathfrak{B}}$ (Lemma 14.23, with $\bar{\mathfrak{B}}$ in place of $\mathfrak{B}$). Form the product (in $\bar{\mathfrak{B}}$) over all appropriate r and s to arrive at

$$\begin{aligned} \varphi^+(p) \,;\, \varphi^+(q) &\leq \prod\{\varphi(r) \,;\, \varphi(s) : r \in A_p \text{ and } s \in A_q\} \\ &= \prod(\varphi(A_p) \,;\, \varphi(A_q)). \end{aligned}$$

To establish the reverse inequality,

$$\prod(\varphi(A_p) \,;\, \varphi(A_q)) \leq \varphi^+(p) \,;\, \varphi^+(q), \tag{9}$$

observe first that $\varphi(A_p)$ and $\varphi(A_q)$ are subsets of $\bar{\mathfrak{A}}$. Closed elements in $\bar{\mathfrak{B}}$ are, by definition, products of subsets of $\bar{\mathfrak{A}}$. Since

$$\varphi^+(p) = \prod \varphi(A_p) \qquad \text{and} \qquad \varphi^+(q) = \prod \varphi(A_q), \tag{10}$$

by the analogue of Lemma 14.18 for φ^+ and the assumption that p and q are closed elements, it follows that $\varphi^+(p)$ and $\varphi^+(q)$ are closed elements in $\bar{\mathfrak{B}}$. Consequently,

$$\varphi^+(p) \,;\, \varphi^+(q) = \prod(\bar{A}_{\varphi^+(p)} \,;\, \bar{A}_{\varphi^+(q)}), \tag{11}$$

by Lemma 14.18 (with $\varphi^+(p)$ and $\varphi^+(q)$ in place of p and q respectively, and with $\bar{\mathfrak{B}}$ in place of $\mathfrak{B}$; the expressions $\bar{A}_{\varphi^+(p)}$ and $\bar{A}_{\varphi^+(q)}$ in (11) denote the sets of elements in $\bar{\mathfrak{A}}$ that are above $\varphi^+(p)$ and $\varphi^+(q)$ respectively). In order to establish (9), it suffices to demonstrate that

$$\prod(\varphi(A_p) \,;\, \varphi(A_q)) \leq \prod(\bar{A}_{\varphi^+(p)} \,;\, \bar{A}_{\varphi^+(q)}), \tag{12}$$

by (11).

To prove (12), we associate with each t in $\bar{A}_{\varphi^+(p)}$ an element r_t in A_p, and with each u in $\bar{A}_{\varphi^+(q)}$ an element s_u in A_q, such that

$$\varphi(r_t) \leq t \qquad \text{and} \qquad \varphi(s_u) \leq u.$$

These inequalities, together with the monotony of the operator $\,;\,$ in $\bar{\mathfrak{A}}$ (Lemma 2.3), imply that

$$\varphi(r_t) \,;\, \varphi(s_u) \leq t \,;\, u. \tag{13}$$

Consequently,

$$\begin{aligned}
\prod(\varphi(A_p) \,;\, \varphi(A_q)) &= \prod\{\varphi(r) \,;\, \varphi(s) : r \in A_p \text{ and } s \in A_q\} \\
&\leq \prod\{\varphi(r_t) \,;\, \varphi(s_u) : t \in \bar{A}_{\varphi^+(p)} \text{ and } u \in \bar{A}_{\varphi^+(q)}\} \\
&\leq \prod\{t \,;\, u : t \in \bar{A}_{\varphi^+(p)} \text{ and } u \in \bar{A}_{\varphi^+(q)}\} \\
&= \prod(\bar{A}_{\varphi^+(p)} \,;\, \bar{A}_{\varphi^+(q)}).
\end{aligned}$$

The first step follows by the definition of the set $\varphi(A_p) \,;\, \varphi(A_q)$, the second by Boolean algebra and the fact that r_t and s_u are in A_p and A_q respectively, the third by (13), and the fourth by the definition of the set $\bar{A}_{\varphi^+(p)} \,;\, \bar{A}_{\varphi^+(q)}$.

The task remains of constructing r_t and s_u. The equations in (10), the assumption that t is in $\bar{A}_{\varphi^+(p)}$ and u in $\bar{A}_{\varphi^+(q)}$, and the definitions of the sets $\bar{A}_{\varphi^+(p)}$ and $\bar{A}_{\varphi^+(q)}$ imply that

$$\prod \varphi(A_p) = \varphi^+(p) \leq t \qquad \text{and} \qquad \prod \varphi(A_q) = \varphi^+(q) \leq u. \tag{14}$$

The elements t and u are open because they belong to $\bar{\mathfrak{A}}$. Apply compactness to (14) in the form of Corollary 14.3(ii) (for $\bar{\mathfrak{B}}$ with respect to $\bar{\mathfrak{A}}$) to obtain finite subsets X of A_p and Y of A_q such that

$$\prod \varphi(X) \leq t \qquad \text{and} \qquad \prod \varphi(Y) \leq u. \tag{15}$$

Write

$$r_t = \prod X \qquad \text{and} \qquad s_u = \prod Y, \tag{16}$$

and observe that r_t belongs to A_p and s_u to A_q, since the sets A_p and A_q are filters, by Lemma 14.5, and are therefore closed under multiplication. Also, r_t is below each element in X, and s_u is below each element in Y, by (16), so

$$\varphi(r_t) \leq \varphi(r) \qquad \text{and} \qquad \varphi(s_u) \leq \varphi(s)$$

for each r in X and s in Y, by the monotony of the function φ (which is a consequence of the fact that φ preserves addition—compare Lemma 2.3). Form the product over all appropriate r and s to obtain

$$\varphi(r_t) \leq \prod \varphi(X) \qquad \text{and} \qquad \varphi(s_u) \leq \prod \varphi(Y). \tag{17}$$

Combine (15) and (17) to arrive at the desired inequalities

$$\varphi(r_t) \leq t \qquad \text{and} \qquad \varphi(s_u) \leq u.$$

This completes the proof of (9), and hence the proof of (iv).

The proofs of (i) and (ii) are nearly identical to the proofs of (iii) and (iv) respectively, except that the operation ; must be replaced everywhere by the operation $\cdot$. The proofs of (v) and (vi) are analogous to, but easier than, the proofs of (iii) and (iv) respectively. The details are left as an exercise. □

The analogue of Lemma 14.31 extends the preceding lemma to arbitrary elements in $\mathfrak{B}$.

Lemma 14.40. *For arbitrary elements p and q in $\mathfrak{B}$,*

(i) $\varphi^+(p \cdot q) = \sum \varphi^+(K_p \cdot K_q)$,

(ii) $\varphi^+(p) \cdot \varphi^+(q) = \sum(\varphi^+(K_p) \cdot \varphi^+(K_q))$,
(iii) $\varphi^+(p\,;q) = \sum \varphi^+(K_p\,;K_q)$,
(iv) $\varphi^+(p)\,;\varphi^+(q) = \sum(\varphi^+(K_p)\,;\varphi^+(K_q))$,
(v) $\varphi^+(p^\smile) = \sum \varphi^+(K_p^\smile)$,
(vi) $\varphi^+(p)^\smile = \sum(\varphi^+(K_p)^\smile)$.

Proof. Here are the proofs of (iii) and (iv). To prove (iii), first use Lemma 14.17 to obtain

$$p = \sum K_p \qquad \text{and} \qquad q = \sum K_q, \tag{1}$$

and then use the quasi-complete distributivity of the operator ; in $\mathfrak{B}$ (Lemma 14.26) to arrive at

$$\begin{aligned} p\,;q &= (\sum K_p)\,;(\sum K_q) \\ &= \sum\{u\,;v : u \in K_p \text{ and } v \in K_q\} = \sum(K_p\,;K_q). \end{aligned} \tag{2}$$

(Notice in this connection that the sets K_p and K_q are never empty, since they contain the element 0.) Apply φ^+ to the left and right sides of (2), and use the analogue of Lemma 14.26 for the function φ^+, to arrive at

$$\varphi^+(p\,;q) = \sum\{\varphi^+(u\,;v) : u \in K_p \text{ and } v \in K_q\} = \sum \varphi^+(K_p\,;K_q).$$

This establishes (iii).

To prove (iv), apply φ^+ to each side of the two equations in (1), and use the analogue of Lemma 14.26 for φ^+, to obtain

$$\begin{aligned} \varphi^+(p) &= \sum\{\varphi^+(u) : u \in K_p\} = \sum \varphi^+(K_p), \\ \varphi^+(q) &= \sum\{\varphi^+(v) : v \in K_q\} = \sum \varphi^+(K_q). \end{aligned}$$

Use these two equations and the quasi-complete distributivity of the operator ; in $\bar{\mathfrak{B}}$ (Lemma 14.26) to arrive at

$$\begin{aligned} \varphi^+(p)\,;\varphi^+(q) &= (\sum\{\varphi^+(u) : u \in K_p\})\,;(\sum\{\varphi^+(v) : v \in K_q\}) \\ &= \sum\{\varphi^+(u)\,;\varphi^+(v) : u \in K_p \text{ and } v \in K_q\} \\ &= \sum(\varphi^+(K_p)\,;\varphi^+(K_q)). \end{aligned}$$

This establishes (iv).

The proofs of (i), (ii), (v), and (vi) are left as an exercise. □

Here, finally, is the *Existence Theorem* for canonical extensions of homomorphisms.

Theorem 14.41. *Let $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ be Boolean algebras with operators, and $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ their respective canonical extensions. If φ is a homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, then φ^+ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ. Moreover, if φ is one-to-one or onto, then so is φ^+.*

Proof. The function φ^+ is an extension of φ, by the analogue of Lemma 14.21 for φ^+, and in particular

$$\varphi^+(0) = \varphi(0) = 0, \quad \varphi^+(1) = \varphi(1) = 1, \quad \varphi^+(1') = \varphi(1') = 1'. \tag{1}$$

To check that φ^+ preserves the operation $;$, consider first the case of closed elements p and q in $\mathfrak{B}$. Since φ preserves the operation $;$, by assumption, we have

$$\begin{aligned}\varphi(A_p\,;A_q) &= \{\varphi(r\,;s) : r \in A_p \text{ and } s \in A_q\}\\ &= \{\varphi(r)\,;\varphi(s) : r \in A_p \text{ and } s \in A_q\}\\ &= \varphi(A_p)\,;\varphi(A_q).\end{aligned}$$

Consequently,

$$\prod \varphi(A_p\,;A_q) = \prod(\varphi(A_p)\,;\varphi(A_q)). \tag{2}$$

Combine (2) with parts (iii) and (iv) of Lemma 14.39 to arrive at

$$\varphi^+(p\,;q) = \prod \varphi(A_p\,;A_q) = \prod(\varphi(A_p)\,;\varphi(A_q)) = \varphi^+(p)\,;\varphi^+(q). \tag{3}$$

Consider now the case of arbitrary elements p and q in $\mathfrak{B}$. Since φ^+ preserves the operation $;$ on closed elements, by (3), we have

$$\begin{aligned}\varphi^+(K_p\,;K_q) &= \{\varphi^+(u\,;v) : u \in K_p \text{ and } v \in K_q\}\\ &= \{\varphi^+(u)\,;\varphi^+(v) : u \in K_p \text{ and } v \in K_q\}\\ &= \varphi^+(K_p)\,;\varphi(K_q).\end{aligned}$$

Consequently,

$$\sum \varphi^+(K_p\,;K_q) = \sum(\varphi^+(K_p)\,;\varphi(K_q)). \tag{4}$$

Combine (4) with parts (iii) and (iv) of Lemma 14.40 to arrive at

$$\varphi^+(p\,;q) = \sum \varphi^+(K_p\,;K_q) = \sum(\varphi^+(K_p)\,;\varphi^+(K_q)) = \varphi^+(p)\,;\varphi^+(q).$$

Thus, φ^+ preserves the operation $;$, as claimed.

The proof that φ^+ preserves the operation of multiplication is almost identical to the preceding argument. One replaces the operation ; everywhere with the operation $\cdot$ and uses parts (i) and (ii) of Lemmas 14.39 and 14.40. The proof that φ^+ preserves the operation $^\smile$ is similar, but slightly easier, and uses parts (v) and (vi) of the lemmas. The details are left as an exercise.

The function φ^+ preserves all non-empty sums, by the analogue of Lemma 14.26 for φ^+, and it preserves the empty sum, by the first equation in (1), so φ^+ preserves all sums. In particular, φ^+ preserve the operation of addition. A mapping between Boolean algebras with operators that preserves addition and multiplication, and maps zero to zero, and one to one, is a Boolean homomorphism, by the remarks following Lemma 7.6. Conclusion: φ^+ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ.

Assume now that φ is one-to-one, with the goal of showing that φ^+ is one-to-one. Consider first an arbitrary non-zero closed element p in $\mathfrak{B}$. The analogue of Lemma 14.18 for the mapping φ^+ implies that

$$\varphi^+(p) = \prod \varphi(A_p) = \prod\{\varphi(r) : r \in A \text{ and } p \leq r\}. \tag{5}$$

The set A_p is a proper Boolean filter in $\mathfrak{A}$, by Lemma 14.5, the definition of A_p, and the assumption that $p \neq 0$, so this set is closed under multiplication and does not contain zero. The Boolean monomorphism properties of the mapping φ therefore imply that the set $\varphi(A_p)$ is closed under multiplication and does not contain zero. Every set with these two properties can be extended to an ultrafilter, so the set $\varphi(A_p)$ is included in an ultrafilter Y in $\bar{\mathfrak{A}}$. The infimum of the ultrafilter Y in $\bar{\mathfrak{B}}$ is an atom, by Lemma 14.7 applied to $\bar{\mathfrak{B}}$. Since this atom is below each element in Y, it must be below each element in $\varphi(A_p)$, and consequently it must be below $\varphi^+(p)$, by (5). In particular, $\varphi^+(p)$ cannot be zero. This argument shows that φ^+ maps every non-zero closed element in $\mathfrak{B}$ to a non-zero element in $\bar{\mathfrak{B}}$.

Consider now an arbitrary non-zero element p in $\mathfrak{B}$. Since p is the supremum of the set K_p of closed elements in $\mathfrak{B}$ that are below p, by Lemma 14.17, this set must contain non-zero (closed) elements. Each of these elements is mapped by φ^+ to a non-zero element in $\bar{\mathfrak{B}}$, by the observations of the preceding paragraph. Consequently, the set

$$\varphi^+(K_p) = \{\varphi^+(u) : u \in K_p\}$$

must contain non-zero elements. The value of $\varphi^+(p)$ is the sum of this set, by the analogue of Corollary 14.20 for the mapping φ^+, so

this value must also be non-zero. Conclusion: φ^+ maps every non-zero element in $\mathfrak{B}$ to a non-zero element in $\bar{\mathfrak{B}}$, so φ^+ is one-to-one, by Lemma 8.37.

Suppose next that φ maps $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$, with the goal of showing that φ^+ maps $\mathfrak{B}$ onto $\bar{\mathfrak{B}}$. Consider first an arbitrary closed element q in $\bar{\mathfrak{B}}$. The set

$$\bar{A}_q = \{s : s \in \bar{A} \text{ and } q \leq s\}$$

is a Boolean filter in $\bar{\mathfrak{A}}$, and the infimum of this filter is q, by Lemma 14.5 applied to $\bar{\mathfrak{B}}$. The Boolean homomorphism properties of the mapping φ imply that the inverse image of $\bar{A}_q$ under φ, the set

$$X = \varphi^{-1}(\bar{A}_q) = \{r \in A : \varphi(r) \in \bar{A}_q\},$$

is a Boolean filter in $\mathfrak{A}$. The function φ is assumed to map $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$, so it must map the set X onto the set $\bar{A}_q$. In other words,

$$\varphi(X) = \bar{A}_q.$$

The infimum of X in $\mathfrak{B}$ is a closed element p, and $X = A_p$, by Lemma 14.5 applied to $\mathfrak{B}$. Combine these observations with (5) to arrive at

$$\varphi^+(p) = \prod \varphi(A_p) = \prod \varphi(X) = \prod \bar{A}_q = q.$$

Thus, φ^+ maps the set of closed elements in $\mathfrak{B}$ onto the set of closed elements in $\bar{\mathfrak{B}}$. (Notice that the image under φ^+ of a closed element in $\mathfrak{B}$ is always a closed element in $\bar{\mathfrak{B}}$, by the analogue of Lemma 14.18 for φ^+.)

Consider now an arbitrary element q in $\bar{\mathfrak{B}}$. This element is the supremum of the set K_q of closed elements in $\bar{\mathfrak{B}}$ that are below q, by Lemma 14.17. For each closed element v in K_q, there is a closed element in $\mathfrak{B}$ that is mapped by φ^+ to v, by the observations of the preceding paragraph. Take X to be the set of these closed elements, and observe that

$$\varphi^+(X) = K_q.$$

If p is the sum of the set X in $\mathfrak{B}$, then

$$\varphi^+(p) = \sum \varphi^+(X) = \sum K_q = q,$$

by the completeness of the homomorphism φ^+. Thus, φ^+ maps $\mathfrak{B}$ onto $\bar{\mathfrak{B}}$, as claimed. □

As it turns out, the canonical extension of a given homomorphism between Boolean algebras with operators is the only possible complete extension of the given homomorphism. A precise statement of this observation is contained in the following *Uniqueness Theorem* for canonical extensions of homomorphisms.

Theorem 14.42. *Let* $\mathfrak{A}$ *and* $\bar{\mathfrak{A}}$ *be Boolean algebras with operators, and* $\mathfrak{B}$ *and* $\bar{\mathfrak{B}}$ *their respective canonical extensions. A homomorphism from* $\mathfrak{A}$ *into* $\bar{\mathfrak{A}}$ *has only one extension to a complete homomorphism from* $\mathfrak{B}$ *into* $\bar{\mathfrak{B}}$.

Proof. A homomorphism φ from $\mathfrak{A}$ to $\bar{\mathfrak{A}}$ certainly has one extension to a complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$, namely the canonical extension φ^+. Suppose ψ is an arbitrary complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$ that agrees with φ on all elements in $\mathfrak{A}$. We shall show that ψ agrees with φ^+ on all elements in $\mathfrak{B}$.

Consider first an arbitrary closed element p in $\mathfrak{B}$. Since

$$p = \prod A_p = \{r : r \in A \text{ and } p \leq r\},$$

by Lemma 14.5, we have

$$\psi(p) = \prod\{\psi(r) : r \in A_p\} = \prod\{\varphi(r) : r \in A_p\} = \prod \varphi(A_p),$$

by the completeness of the homomorphism ψ and assumption that ψ agrees with φ on $\mathfrak{A}$. On the other hand,

$$\varphi^+(p) = \prod \varphi(A_p),$$

by the analogue of Lemma 14.18 for the mapping φ^+. Comparing these equations, we see that $\psi(p) = \varphi^+(p)$.

Consider now an arbitrary element p in $\mathfrak{B}$. Since

$$p = \sum K_p = \{u : u \in K \text{ and } u \leq p\},$$

by Lemma 14.17, we get

$$\psi(p) = \sum\{\psi(u) : u \in K_p\} = \sum\{\varphi^+(u) : u \in K_p\} = \varphi^+(p),$$

by the completeness of the homomorphism ψ, the observation in the preceding paragraph that ψ and φ^+ agree on K, and the analogue of Corollary 14.20 that applies to the mapping φ^+. Thus, ψ and φ^+ agree on all elements in $\mathfrak{B}$, as claimed. □

The preceding theorem easily implies that the canonical extension of a composition of two homomorphisms is equal to the composition of the canonical extensions of the two homomorphisms.

Corollary 14.43. *Let* $\mathfrak{A}$, $\bar{\mathfrak{A}}$, *and* $\bar{\bar{\mathfrak{A}}}$ *be Boolean algebras with operators, and* $\mathfrak{B}$, $\bar{\mathfrak{B}}$, *and* $\bar{\bar{\mathfrak{B}}}$ *their respective canonical extensions. If* φ *is homomorphism from* $\mathfrak{A}$ *into* $\bar{\mathfrak{A}}$, *and* ψ *a homomorphism from* $\bar{\mathfrak{A}}$ *into* $\bar{\bar{\mathfrak{A}}}$, *then the canonical extension of the composition* $\psi \circ \varphi$ *is just the composition* $\psi^+ \circ \varphi^+$.

Proof. The mapping φ^+ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$, and the mapping ψ^+ is a complete homomorphism from $\bar{\mathfrak{B}}$ into $\bar{\bar{\mathfrak{B}}}$, so the composition $\psi^+ \circ \varphi^+$ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\bar{\mathfrak{B}}}$. If r is any element in $\mathfrak{A}$, then $\varphi(r)$ is an element in $\bar{\mathfrak{A}}$, and therefore

$$(\psi^+ \circ \varphi^+)(r) = \psi^+(\varphi^+(r)) = \psi^+(\varphi(r)) = \psi(\varphi(r)) = (\psi \circ \varphi)(r),$$

since φ^+ agrees with φ on $\mathfrak{A}$, and ψ^+ agrees with ψ on $\bar{\mathfrak{A}}$. Thus, the composition $\psi^+ \circ \varphi^+$ agrees with the composition $\psi \circ \varphi$ on $\mathfrak{A}$. But the canonical extension of $\psi \circ \varphi$, that is to say, the mapping $(\psi \circ \varphi)^+$, is the unique complete homomorphism from $\mathfrak{B}$ to $\bar{\bar{\mathfrak{B}}}$ that agrees with $\psi \circ \varphi$ on $\mathfrak{A}$, by the preceding Uniqueness Theorem 14.42. Consequently,

$$(\psi \circ \varphi)^+ = \psi^+ \circ \varphi^+.$$

as desired. □

14.8 Applications to algebraic constructions

We now return to the question of algebraic constructions that are preserved under the passage to canonical extensions. The main results in this direction are, roughly speaking, that the canonical extension of a subalgebra is a complete subalgebra of the canonical extension, the canonical extension of a homomorphic image is a complete homomorphic image of the canonical extension, and the canonical extension of a (binary) direct product is the direct product of the canonical extensions. Here is a precise statement of the first of these preservation theorems.

Theorem 14.44. *If* $\mathfrak{A}$ *is a subalgebra of a Boolean algebra with operators* $\bar{\mathfrak{A}}$, *then the canonical extension of* $\mathfrak{A}$ *is* (*up to isomorphisms that*

are the identity function on $\mathfrak{A}$) *a complete subalgebra of the canonical extension of* $\bar{\mathfrak{A}}$, *and the universe of the canonical extension of* $\mathfrak{A}$ *consists precisely of the sums of products*, *in the canonical extension of* $\bar{\mathfrak{A}}$, *of arbitrary subsets of* $\mathfrak{A}$.

Proof. Let $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ be the canonical extensions of $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ respectively. If $\mathfrak{A}$ is a subalgebra of $\bar{\mathfrak{A}}$, then the identity function on $\mathfrak{A}$—call it φ—is a monomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, and therefore the canonical extension φ^+ is a complete monomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$, by Theorem 14.41. The image algebra of $\mathfrak{B}$ under φ^+ must therefore be a regular subalgebra of $\bar{\mathfrak{B}}$, by Lemma 7.9. Since $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ are both complete, it follows that the image algebra is a complete subalgebra of $\bar{\mathfrak{B}}$ (see the remark preceding Lemma 6.16). The image algebra is an isomorphic copy of $\mathfrak{B}$ via a function that is the identity function on $\mathfrak{A}$ (because φ^+ extends φ, and φ is the identity function on $\mathfrak{A}$), so the image algebra must also be a canonical extension of $\mathfrak{A}$. Identifying the image algebra with $\mathfrak{B}$ itself, we may conclude that $\mathfrak{B}$ is a complete subalgebra of $\bar{\mathfrak{B}}$.

The elements in $\mathfrak{B}$ are sums of closed elements, by Lemma 14.17, and each closed element is the product of a subset of $\mathfrak{A}$, by definition, so the elements in $\mathfrak{B}$ are sums of products of subsets of $\mathfrak{A}$. These sums and products may be formed either in $\mathfrak{B}$ or in $\bar{\mathfrak{B}}$, because $\mathfrak{B}$ is a complete subalgebra of $\bar{\mathfrak{B}}$. □

Lemma 14.45. *Suppose* $\mathfrak{A}$ *and* $\bar{\mathfrak{A}}$ *are Boolean algebras with operators*, *and* $\mathfrak{B}$ *and* $\bar{\mathfrak{B}}$ *their respective canonical extensions*. *If* $\mathfrak{A}$ *is a subalgebra of* $\bar{\mathfrak{A}}$, *then* $\mathfrak{A}$ *is a proper subalgebra of* $\bar{\mathfrak{A}}$ *if and only if* $\mathfrak{B}$ *is* (*up to isomorphisms that are the identity function on* $\mathfrak{A}$) *a proper subalgebra of* $\bar{\mathfrak{B}}$.

Proof. In view of Theorem 14.44, it may be assumed that $\mathfrak{B}$ is a complete subalgebra of $\bar{\mathfrak{B}}$, and in fact the complete subalgebra consisting of sums of products of subsets of $\mathfrak{A}$. Notice also that $\bar{\mathfrak{B}}$ consists of sums of products of subsets of $\bar{\mathfrak{A}}$, by the argument at the end of the proof of Theorem 14.44 (with $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ in place of $\mathfrak{A}$ and $\mathfrak{B}$ respectively).

The proof of the lemma proceeds by contraposition. If $\mathfrak{A} = \bar{\mathfrak{A}}$, then certainly $\mathfrak{B} = \bar{\mathfrak{B}}$, since in this case both algebras consist of sums of products of subsets of $\mathfrak{A}$. Since $\mathfrak{B}$ is a complete subalgebra of $\bar{\mathfrak{B}}$, it does not matter whether these sums and products are formed in $\mathfrak{B}$ or in $\mathfrak{A}$.

To establish the reverse implication, assume that $\mathfrak{B} = \bar{\mathfrak{B}}$, and consider an arbitrary element r in $\bar{\mathfrak{A}}$. Since $\bar{\mathfrak{A}}$ is a subalgebra of $\bar{\mathfrak{B}}$, the element r belongs to $\bar{\mathfrak{B}}$ and therefore also to $\mathfrak{B}$, by the assumption that the two canonical extensions are equal. It follows from Theorem 14.44 that r may be written as a sum $r = \sum Y$ (in $\bar{\mathfrak{B}}$) of some set Y such that each element p in Y is a product $p = \prod X$ (in $\bar{\mathfrak{B}}$) of some set X of elements in $\mathfrak{A}$. The element r is open in $\bar{\mathfrak{B}}$ (because r belongs to $\bar{\mathfrak{A}}$) and $\prod X \leq r$ (because $p \leq r$), so the compactness property (for $\bar{\mathfrak{B}}$ with respect to the subalgebra $\bar{\mathfrak{A}}$) in the form of Corollary 14.3(ii) may be applied to obtain a finite subset X_0 of X such that $\prod X_0 \leq r$. The element $s = \prod X_0$ belongs to $\mathfrak{A}$, because $\mathfrak{A}$ is closed under multiplication and X_0 is a subset of $\mathfrak{A}$. Also,

$$p = \prod X \leq \prod X_0 = s \leq r,$$

so the element p may be replaced in Y by the element s. Carry out this replacement for each element p in Y to arrive at a subset Z of $\mathfrak{A}$ such that $r = \sum Z$ (in $\bar{\mathfrak{B}}$). The element r is closed in $\bar{\mathfrak{B}}$ (because r belongs to $\bar{\mathfrak{A}}$), so the compactness property (for $\bar{\mathfrak{B}}$ with respect to $\bar{\mathfrak{A}}$) in the form of Corollary 14.3(i) may be applied to obtain a finite subset Z_0 of Z such that $r \leq \sum Z_0$. Since

$$r \leq \sum Z_0 \leq \sum Z = r,$$

equality must hold everywhere. Consequently, r is a sum of a finite subset of $\mathfrak{A}$ and therefore r must belong to $\mathfrak{A}$, by the closure of $\mathfrak{A}$ under addition. Conclusion: every element in $\bar{\mathfrak{A}}$ belongs $\mathfrak{A}$. Since $\mathfrak{A}$ is a subalgebra of $\bar{\mathfrak{A}}$, by assumption, it follows that $\mathfrak{A} = \bar{\mathfrak{A}}$. □

Turn now to the question of canonical extensions of homomorphic images. If φ is a homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, then the image of $\mathfrak{A}$ under φ is a subalgebra of $\bar{\mathfrak{A}}$, and one can ask what the canonical extension of this subalgebra is. The answer is given by the next theorem.

Theorem 14.46. *Suppose $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are Boolean algebras with operators, and $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ their respective canonical extensions. If φ is a homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, then the canonical extension of the image of $\mathfrak{A}$ under φ is just the image of $\mathfrak{B}$ under φ^+.*

Proof. Let $\mathfrak{C}$ be the subalgebra of $\bar{\mathfrak{A}}$ that is the image of $\mathfrak{A}$ under φ, and let $\mathfrak{D}$ be the canonical extension of $\mathfrak{C}$. Apply Theorem 14.44 (with $\mathfrak{C}$ and $\mathfrak{D}$ in place of $\mathfrak{A}$ and $\mathfrak{B}$ respectively) to see that $\mathfrak{D}$ is isomorphic to a

complete subalgebra of $\bar{\mathfrak{B}}$ via a mapping that is the identity function on $\mathfrak{C}$. Consequently, we may and do assume that $\mathfrak{D}$ is a complete subalgebra of $\bar{\mathfrak{B}}$, and in fact the complete subalgebra whose universe is the set of the sums of products of subsets of $\mathfrak{C}$, these sums and products being formed in $\bar{\mathfrak{B}}$.

The canonical extension homomorphism φ^+ is the mapping from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$ that is defined for each element p in $\mathfrak{B}$ by

$$\varphi^+(p) = \sum\{\prod \varphi(A_u) : u \in K_p\}, \tag{1}$$

where these sums and products are formed in $\bar{\mathfrak{B}}$. For each element u in K_p, the set $\varphi(A_u)$ is a subset of the subalgebra $\mathfrak{C}$, by the definitions of A_u and $\mathfrak{C}$. Consequently, each element $\varphi^+(p)$ is a sum of products (in $\bar{\mathfrak{B}}$) of subsets of $\mathfrak{C}$, by (1), and therefore each such element belongs to $\mathfrak{D}$, by the observations of the preceding paragraph. Thus, φ^+ actually maps $\mathfrak{B}$ into the complete subalgebra $\mathfrak{D}$ of $\bar{\mathfrak{B}}$.

We may now forget about the algebras $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$. The algebras $\mathfrak{B}$ and $\mathfrak{D}$ are the canonical extensions of the algebras $\mathfrak{A}$ and $\mathfrak{C}$ respectively, and the function φ maps $\mathfrak{A}$ homomorphically onto $\mathfrak{C}$, so the canonical extension homomorphism φ^+ must map $\mathfrak{B}$ onto $\mathfrak{D}$, by Existence Theorem 14.41. (Since $\mathfrak{D}$ is a complete subalgebra of $\bar{\mathfrak{B}}$, it makes no difference whether the sums and products in (1) are formed in $\mathfrak{D}$ or in $\bar{\mathfrak{B}}$. In other words, the mapping φ^+ defined by (1) with $\bar{\mathfrak{B}}$ as the target algebra, and the mapping φ^+ defined by (1) with $\mathfrak{D}$ as the target algebra are really the same mapping.) Conclusion: the canonical extension $\mathfrak{D}$ of $\mathfrak{C}$ is the image of the canonical extension $\mathfrak{B}$ of $\mathfrak{A}$ under the mapping φ^+. □

From the preceding theorem, it is not difficult to obtain the following improvement of the final assertion of Existence Theorem 14.41.

Corollary 14.47. *Under the hypotheses of Theorem* 14.41, *the canonical extension* φ^+ *maps* $\mathfrak{B}$ *onto* $\bar{\mathfrak{B}}$ *if and only if* φ *maps* $\mathfrak{A}$ *onto* $\bar{\mathfrak{A}}$.

Proof. The homomorphism φ maps $\mathfrak{A}$ onto a subalgebra $\mathfrak{C}$ of $\bar{\mathfrak{A}}$, and the canonical extension homomorphism φ^+ maps $\mathfrak{B}$ onto a subalgebra $\mathfrak{D}$ of $\bar{\mathfrak{B}}$. Theorem 14.46 implies that $\mathfrak{D}$ is the canonical extension of $\mathfrak{C}$. Apply Lemma 14.45 (with $\mathfrak{C}$ and $\mathfrak{D}$ in place of $\mathfrak{A}$ and $\mathfrak{B}$ respectively) to obtain that $\mathfrak{C} = \bar{\mathfrak{A}}$ if and only if $\mathfrak{D} = \bar{\mathfrak{B}}$. In other words, φ maps $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$ if and only if φ^+ maps $\mathfrak{B}$ onto $\bar{\mathfrak{B}}$. □

There is a slightly different version of Theorem 14.46 that is an almost immediate consequence of Theorem 14.41.

Corollary 14.48. *If $\bar{\mathfrak{A}}$ is a homomorphic image of a Boolean algebra with operators $\mathfrak{A}$, then the canonical extension of $\bar{\mathfrak{A}}$ is a complete homomorphic image of the canonical extension of $\mathfrak{A}$.*

Proof. If $\bar{\mathfrak{A}}$ is a homomorphic image of $\mathfrak{A}$, say under a homomorphism φ, then the canonical extension of φ is a complete homomorphism from the canonical extension of $\mathfrak{A}$ onto the canonical extension of $\bar{\mathfrak{A}}$, by Theorem 14.41. Consequently, the canonical extension of $\bar{\mathfrak{A}}$ is a complete homomorphic image of the canonical extension of $\mathfrak{A}$. □

The next theorem says that the canonical extension of the direct product of two Boolean algebras with operators is equal the direct product of the canonical extensions of the algebras.

Theorem 14.49. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are Boolean algebras with operators, and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ their respective canonical extensions, then $\mathfrak{B}_1 \times \mathfrak{B}_2$ is the canonical extension of $\mathfrak{A}_1 \times \mathfrak{A}_2$ (up to isomorphisms that are the identity function on $\mathfrak{A}_1 \times \mathfrak{A}_2$).*

Proof. Write

$$\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \qquad \text{and} \qquad \mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2,$$

and let $\mathfrak{C}$ be the canonical extension of $\mathfrak{A}$. The goal is to show that $\mathfrak{C}$ is isomorphic to $\mathfrak{B}$ via a mapping that is the identity function on $\mathfrak{A}$.

Take φ_1 and φ_2 to be the projections from $\mathfrak{A}$ to the factors $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, so that φ_1 maps each pair in $\mathfrak{A}$ to its first coordinate, and φ_2 maps each pair to its second coordinate. These functions are epimorphisms, so their canonical extensions to mappings φ_1^+ and φ_2^+ from $\mathfrak{C}$ to $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively are complete epimorphisms, by Existence Theorem 14.41. The (external) amalgamation of these two canonical extensions is the function ψ from $\mathfrak{C}$ into $\mathfrak{B}$ that is defined by

$$\psi(p) = (\varphi_1^+(p), \varphi_2^+(p))$$

for each p in $\mathfrak{C}$. Since each of the coordinate functions is a complete homomorphism, the external version of Lemma 11.50 ensures that the amalgamation ψ is a complete homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$.

It is easy to check that ψ is the identity function on $\mathfrak{A}$. Indeed, If r is an element in $\mathfrak{A}$, say $r = (r_1, r_2)$, then

$$\psi(r) = (\varphi_1^+(r), \varphi_2^+(r)) = (\varphi_1(r), \varphi_2(r)) = (r_1, r_2) = r,$$

by the definition of ψ, the fact that φ_1^+ and φ_2^+ are extensions of the projections φ_1 and φ_2 respectively, and the definitions of the projections.

To establish that ψ is an isomorphism, it remains to show that ψ is one-to-one and onto. Both $\mathfrak{B}$ and $\mathfrak{C}$ are complete and atomic Boolean algebras with operators, and the homomorphism ψ preserves arbitrary sums, so it suffices to show that ψ maps the set of atoms in $\mathfrak{C}$ onto the set of atoms in $\mathfrak{B}$ (see Exercises 7.16 and 7.17).

According to Lemma 14.7, each atom b_i in $\mathfrak{B}_i$ (for $i = 1, 2$) uniquely determines an ultrafilter in $\mathfrak{A}_i$ of which it is the infimum (in $\mathfrak{B}_i$), namely the ultrafilter of elements in $\mathfrak{A}_i$ that are above b_i; and each ultrafilter X_i in $\mathfrak{A}_i$ is determined in this way by a unique atom in $\mathfrak{B}_i$. The atoms in the product $\mathfrak{B}_1 \times \mathfrak{B}_2$ are the elements that have one of the two forms

$$(b_1, 0), \qquad (0, b_2), \tag{1}$$

where b_1 and b_2 are atoms in $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively, by Lemma 11.7.

Similarly, each atom in $\mathfrak{C}$ is the infimum of a uniquely determined ultrafilter in $\mathfrak{A}$, and each ultrafilter in $\mathfrak{A}$ is determined in this way by a unique atom in $\mathfrak{C}$, by Lemma 14.7. The ultrafilters in $\mathfrak{A}$ are the sets that have one of the two forms

$$X_1 \times A_2, \qquad A_1 \times X_2, \tag{2}$$

where X_1 and X_2 are ultrafilters in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively (and A_1 and A_2 are the universes of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively), by the dual of the Boolean version of Exercise 11.15.

If an atom a in $\mathfrak{C}$ is the infimum of an ultrafilter of the form $X_1 \times A_2$, and if b_1 is the atom in $\mathfrak{B}_1$ that is the infimum of the ultrafilter X_1, then

$$\begin{aligned} \psi(a) = \psi(\textstyle\prod(X_1 \times A_2)) &= \textstyle\prod \psi(X_1 \times A_2) \\ &= \textstyle\prod(X_1 \times A_2) = (b_1, 0), \end{aligned} \tag{3}$$

by assumption on a, the completeness of the homomorphism ψ, the fact that ψ is the identity function on $\mathfrak{A}$, and the fact that infima are formed coordinatewise in $\mathfrak{B}$ (by the dual of the Boolean version of Lemma 11.8). A similar computation shows that if a is the infimum of an ultrafilter of the form $A_1 \times X_2$, and if b_2 is the atom in $\mathfrak{B}_2$ that is the infimum of the ultrafilter X_2, then

$$\psi(a) = (0, b_2). \tag{4}$$

The above argument implies that ψ maps the set of atoms in $\mathfrak{C}$ onto the set of atoms in $\mathfrak{B}$. Indeed, every atom a in $\mathfrak{C}$ is mapped to an atom in $\mathfrak{B}$, by (3) and (4). Conversely, if b is an atom in $\mathfrak{B}$, then b must have one of the two forms in (1). Suppose, for example, that b has the first form. If X_1 is the ultrafilter of elements in $\mathfrak{A}_1$ that are above b_1, and if a is the atom in $\mathfrak{C}$ that is the infimum of the corresponding ultrafilter in (2), then ψ maps a to b, by (3). A similar argument applies if b has the second form in (1), but uses (4) instead of (3). □

A straightforward argument by induction on the number of algebras extends Theorem 14.49 from two algebras to any finite number of algebras. The extension fails, however, when infinitely many algebras are involved. For an example, let I be a countably infinite index set, and for each index i in I take $\mathfrak{A}_i$ to be a finite, non-degenerate relation algebra. (For instance, $\mathfrak{A}_i$ could be the two-element Boolean relation algebra for each i.) Each factor $\mathfrak{A}_i$ is its own canonical extension, by the observations preceding Theorem 14.36, but the product $\mathfrak{A} = \prod_i \mathfrak{A}_i$ is certainly not its own canonical extension. Indeed, the product $\mathfrak{A}$ has cardinality $2^{\aleph_0}$, while the canonical extension of $\mathfrak{A}$ has cardinality at least $2^{2^{\aleph_0}}$.

There are two more results concerning the preservation of algebraic constructions under the passage to canonical extensions that are worth mentioning in their application to relation algebras. The first says that if $\mathfrak{B}$ is the canonical extension of a relation algebra $\mathfrak{A}$, and if e is an equivalence element in $\mathfrak{A}$, then the relativization of $\mathfrak{B}$ to e is the canonical extension of the relativization of $\mathfrak{A}$ to e. In other words, the canonical extension of a relativization is the relativization of the canonical extension. The second says that if $\mathfrak{B}$ is the canonical extension of a relation algebra $\mathfrak{A}$, and if M is an ideal in $\mathfrak{A}$, then the set N consisting of all sums of all products of subsets of M (formed in $\mathfrak{B}$) is an ideal in $\mathfrak{B}$, and the quotient $\mathfrak{B}/N$ is isomorphic to the canonical extension of the quotient $\mathfrak{A}/M$ via a function that maps r/N to r/M for each element r in $\mathfrak{A}$. In other words, roughly speaking, the canonical extension of a quotient is a quotient of the canonical extension. The proofs of these two results are left as exercises.

14.9 Canonical extensions of set relation algebras

We now turn our attention to canonical extensions of set relation algebras. A set relation algebra $\mathfrak{A}$ with unit E is a subalgebra of the complete and atomic set relation algebra $\mathfrak{Re}(E)$ consisting of all subrelations of E. Under what conditions is the canonical extension of $\mathfrak{A}$ equal to the subalgebra of $\mathfrak{Re}(E)$ that is completely generated by (the universe of) $\mathfrak{A}$? The next theorem provides one answer to this question. In formulating and proving the theorem, we use set-theoretical versions of the notations for the complex relative product and converse of subsets of $\mathfrak{A}$ that were introduced after Definition 14.12, that is to say, for sets X and Y of relations in $\mathfrak{A}$, we write

$$X \mid Y = \{R \mid S : R \in X \text{ and } S \in Y\},$$
$$X^{-1} = \{R^{-1} : R \in X\}.$$

Notice that the relation that is the intersection of the relations in a subset X of $\mathfrak{A}$,

$$\bigcap X = \bigcap\{R : R \in X\},$$

always belongs to $\mathfrak{Re}(E)$, even though it may not belong to $\mathfrak{A}$, because $\mathfrak{Re}(E)$ is closed under arbitrary intersections.

Theorem 14.50. *Suppose $\mathfrak{A}$ is a subalgebra of $\mathfrak{Re}(E)$. The canonical extension of $\mathfrak{A}$ is the complete subalgebra of $\mathfrak{Re}(E)$ that is completely generated by $\mathfrak{A}$ if and only if $\mathfrak{A}$ satisfies the following two conditions.*

(i) *For every ultrafilter X in $\mathfrak{A}$, the intersection $\bigcap X$ is not empty.*
(ii) *If X, Y, and Z are ultrafilters in $\mathfrak{A}$ satisfying $X \mid Y \subseteq Z$, then for every pair (α, β) in $\bigcap Z$, there is an element γ such that (α, γ) belongs to $\bigcap X$ and (γ, β) to $\bigcap Y$.*

If these conditions are satisfied, then the distinct atoms in the canonical extension are the relations $\bigcap X$ for distinct ultrafilters X in $\mathfrak{A}$, and the distinct elements in the canonical extension are the unions of the distinct sets of atoms.

Proof. Assume first that $\mathfrak{A}$ satisfies conditions (i) and (ii) of the theorem. We make two preliminary observations that will be useful. First, for every pair of elements (α, β) in the unit relation E, the set

$$A_{\alpha\beta} = \{R \in A : (\alpha, \beta) \in R\} \tag{1}$$

is an ultrafilter in $\mathfrak{A}$, and obviously the pair (α, β) belongs to the intersection of the relations in this ultrafilter. Second, every ultrafilter X in $\mathfrak{A}$ is induced in this fashion by some pair (α, β) in E. For the proof, notice that the intersection $\bigcap X$ is not empty, by condition (i). Take (α, β) to be any pair in this intersection, and observe that (α, β) belongs to every relation in X, by the definition of an intersection. Therefore, X must be a subset of $A_{\alpha\beta}$, by (1). Since X and $A_{\alpha\beta}$ are both ultrafilters, it follows that $X = A_{\alpha\beta}$. Warning: distinct pairs in E may induce the same ultrafilter in $\mathfrak{A}$.

Let U be the set of ultrafilters in $\mathfrak{A}$, and for each X in U write

$$P_X = \bigcap X = \bigcap \{R : R \in X\}. \tag{2}$$

The immediate goal is to prove that these relations form the atoms of a complete subalgebra of $\mathfrak{Re}(E)$ that is atomic and includes $\mathfrak{A}$ as a subalgebra. We begin with some initial remarks. First, the relations P_X are not empty, by (2) and condition (i).

Second,

$$R = \bigcup \{P_X : X \in U \text{ and } R \in X\} \tag{3}$$

for each relation R in $\mathfrak{A}$. Indeed, if P_X is one of the relations on the right side of (3), that is to say, if X is in U and R in X, then

$$P_X = \bigcap X \subseteq R,$$

by (2), the definition of intersection, and the assumption that R is in X. The right side of (3) is therefore included in the left side. To establish the reverse inclusion, consider any pair (α, β) in R, and take X to be the set $A_{\alpha\beta}$. This set is an ultrafilter in $\mathfrak{A}$ and therefore an element in U, by the first preliminary observation, and the relation R belongs to X, by (1) and the assumption that (α, β) belongs to R. Consequently, P_X is one of the relations on the right side of (3). The pair (α, β) belongs to P_X, by (1), (2), and the first preliminary observation, so this pair belongs to the right side of (3). The left side of (3) is therefore included in the right side.

The third remark is that

$$P_X \subseteq R \qquad \text{if and only if} \qquad R \in X. \tag{4}$$

for every relation R in $\mathfrak{A}$ and every ultrafilter X in U. The implication from right to left follows immediately from (3). For the implication in the reverse direction, argue by contraposition. Suppose R is not

in X. The relation $\sim R$ is then in X, by the assumption that X is an ultrafilter, so P_X must be included in $\sim R$, by the implication from right to left that has already been established (with $\sim R$ in place of R). It follows that P_X is disjoint from R, and as P_X is not empty, by the first remark, this implies that P_X cannot be included in R.

We proceed to verify that the set

$$W = \{P_X : X \in U\} \tag{5}$$

satisfies properties (i)–(iv) of the Atomic Subalgebra Theorem 6.21 (with $\mathfrak{Re}(E)$ in place of $\mathfrak{A}$). To verify property (i), observe that if X and Y are distinct ultrafilters in U, then there is a relation R in $\mathfrak{A}$ that belongs to X, while its complement belongs to Y. Consequently, P_X and P_Y are respectively included in R and $\sim R$, by (4). Since R and $\sim R$ are disjoint, it follows that P_X and P_Y must also be disjoint. The union of the relations in W is the unit E, by (3) (with E in place of R), because the unit element E belongs to every ultrafilter in $\mathfrak{A}$.

The validity of property (ii) for W follows directly from (3) and (5) (with the identity relation from $\mathfrak{A}$ in place of R).

To verify property (iii), notice first that if X is an ultrafilter in $\mathfrak{A}$, then so is the set X^{-1} of converses of relations in X. Indeed, it is easy to check that if $X = A_{\alpha\beta}$, then $X^{-1} = A_{\beta\alpha}$. Consequently, if X is in U, then so is X^{-1}, and

$$P_X^{-1} = (\bigcap X)^{-1} = \bigcap\{R^{-1} : R \in X\} = \bigcap X^{-1} = P_{X^{-1}},$$

by (2) (for X and X^{-1}), the distributivity of relational converse over arbitrary intersections (see Lemma 4.2), and the definition of the set X^{-1}. This argument shows that the set W is actually closed under the operation of converse.

In order to verify property (iv), it suffices to show for any ultrafilters X and Y in U that

$$P_X \mid P_Y = \bigcup\{P_Z : Z \in U \text{ and } X \mid Y \subseteq Z\}. \tag{6}$$

Begin with the inclusion from right to left. Consider a relation P_Z on the right side of (6). Thus, Z is in U, and $X \mid Y$ is included in Z. If (α, β) is a pair in P_Z, then there must be an element γ such that

$$(\alpha, \gamma) \in P_X \qquad \text{and} \qquad (\gamma, \beta) \in P_Y,$$

by condition (ii) of the theorem and (2). It follows that the pair (α, β) belongs to the relation $P_X \mid P_Y$, by the definition of relational composition. This argument shows that the relation P_Z is included in the left

side of (6), and therefore the entire right side of (6) is included in the left side.

To establish the reverse inclusion, observe first that

$$P_X \mid P_Y \subseteq R \mid S \tag{7}$$

for every R in X and S in Y, by (2) and the monotony law for relational composition (see Lemma 4.5(i)). Consequently, if (α, β) is a pair in the relation $P_X \mid P_Y$, then (α, β) belongs to the relation $R \mid S$ for every R in X and S in Y, by (7). Said somewhat differently, the complex composition $X \mid Y$ is included in $A_{\alpha\beta}$. Put

$$Z = A_{\alpha\beta}. \tag{8}$$

The set Z is in U, by the first preliminary remark, and $X \mid Y$ is included in Z, so P_Z is one of the relations on the right side of (6). The pair (α, β) belongs to P_Z, by (2), (8), and the first preliminary remark, so this pair belongs to the right side of (6). Thus, the left side of (6) is included in the right side. This completes the verification of property (iv) for W.

Before proceeding further, we observe for later use that if X, Y, and Z are in U, then

$$P_Z \subseteq P_X \mid P_Y \qquad \text{if and only if} \qquad X \mid Y \subseteq Z. \tag{9}$$

The implication from right to left follows directly from (6). The reverse implication is established by contraposition. If $X \mid Y$ is not included in Z, then P_Z must be disjoint from all the relations that appear on the right side of (6), by the mutual disjointness of the relations in W (property (i) for W), and consequently P_Z must be disjoint from the composition $P_X \mid P_Y$, by (6). Hence, as a non-empty relation, P_Z cannot be included in this composition..

It has been shown that properties (i)–(iv) of Theorem 6.21 hold for the set W defined in (5). Apply that theorem to conclude that the set of unions of subsets of W is the universe of a complete subalgebra of $\mathfrak{Re}(E)$ that is atomic, the atoms being the relations P_X in W (since each of these relations is not empty). Call this subalgebra $\mathfrak{B}$.

Each relation in $\mathfrak{A}$ belongs to $\mathfrak{B}$, by (3), and both algebras are subalgebras of $\mathfrak{Re}(E)$, so $\mathfrak{A}$ must be a subalgebra of $\mathfrak{B}$. The atoms in $\mathfrak{B}$—the relations P_X—are products (that is to say, intersections) in $\mathfrak{Re}(E)$ of sets of elements (that is to say, sets of relations) in $\mathfrak{A}$, by (2), so they are completely generated by (the universe of) $\mathfrak{A}$. Arbitrary elements

in $\mathfrak{B}$ are sums (that is to say, unions) in $\mathfrak{Re}(E)$ of sets of atoms, so they, too, are completely generated by $\mathfrak{A}$. Thus, $\mathfrak{B}$ is the subalgebra of $\mathfrak{Re}(E)$ that is completely generated by $\mathfrak{A}$.

It remains to show that $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}$. There are two ways to accomplish this task. The first way is to verify directly that $\mathfrak{B}$ satisfies the conditions in Definition 14.12 for being the canonical extension of $\mathfrak{A}$. The second way, which we shall follow, is to construct an isomorphism from the canonical extension of $\mathfrak{A}$ to the algebra $\mathfrak{B}$ with the property that it is the identity function on $\mathfrak{A}$.

Let $\mathfrak{C}$ be the canonical embedding algebra of $\mathfrak{A}$, and recall that $\mathfrak{C}$ is isomorphic to the canonical extension of $\mathfrak{A}$ via a mapping that extends the canonical embedding of $\mathfrak{A}$ into $\mathfrak{C}$, by (the proof of) the Existence Theorem for canonical extensions in the form of Theorem 14.16. In particular, $\mathfrak{C}$ is a relation algebra, by Theorem 14.35. The atoms in $\mathfrak{C}$ are the singletons of ultrafilters in the algebra $\mathfrak{A}$, that is to say, they are the singletons of elements in U. The atoms in $\mathfrak{B}$ are the relations of the form P_X, that is to say, they are the elements in W, by (5). The function φ that is defined by

$$\varphi(\{X\}) = P_X \tag{10}$$

for each X in U is a bijection from the set of atoms in $\mathfrak{C}$ to the set of atoms in $\mathfrak{B}$, because the relations P_X and P_Y are disjoint (and nonempty) for distinct X and Y in U. If X, Y, and Z are elements in U, then

$$\begin{aligned} \{Z\} \subseteq \{X\} \,;\, \{Y\} \quad &\text{if and only if} \quad X \,|\, Y \subseteq Z, \\ &\text{if and only if} \quad P_Z \subseteq P_X \,|\, P_Y, \\ &\text{if and only if} \quad \varphi(\{Z\}) \subseteq \varphi(\{X\}) \,|\, \varphi(\{Y\}), \end{aligned}$$

by the definition of relative multiplication in $\mathfrak{C}$ (see the remarks preceding Lemma 14.14), (9), and (10). These equivalences imply, by the Atomic Isomorphism Theorem in the form of Corollary 7.12, that the extension of φ defined by

$$\begin{aligned} \psi(F) &= \bigcup \{\varphi(\{X\}) : X \in U \text{ and } \{X\} \subseteq F\} \\ &= \bigcup \{P_X : X \in F\} \end{aligned} \tag{11}$$

for each subset F of U is an isomorphism from $\mathfrak{C}$ to $\mathfrak{B}$. If F_R is the set of ultrafilters in U that contain a fixed relation R in $\mathfrak{A}$, then

$$\begin{aligned}\psi(F_R) &= \bigcup\{P_X : X \in F_R\} \\ &= \bigcup\{P_X : X \in U \text{ and } R \in X\} \\ &= R,\end{aligned} \tag{12}$$

by (11), the definition of F_R, and (3).

The canonical extension of $\mathfrak{A}$ is isomorphic to the canonical embedding algebra $\mathfrak{C}$ via a mapping ϑ that extends the canonical embedding of $\mathfrak{A}$ into $\mathfrak{C}$, by (the proof of) Theorem 14.16. Thus,

$$\vartheta(R) = \{X \in U : R \in X\} = F_R \tag{13}$$

for each relation R in $\mathfrak{A}$, by the definition of the canonical embedding (see Lemma 14.14). The composition $\psi \circ \vartheta$ is the desired isomorphism from the canonical extension of $\mathfrak{A}$ to $\mathfrak{B}$. It is the identity function on $\mathfrak{A}$ because it maps each relation R in $\mathfrak{A}$ to itself, by (13) and (12).

It has been shown that if $\mathfrak{A}$ satisfies conditions (i) and (ii) of the theorem, then the canonical extension of $\mathfrak{A}$ is the complete subalgebra $\mathfrak{B}$ of $\mathfrak{Re}(E)$ that is completely generated by $\mathfrak{A}$; and in this case the atoms in $\mathfrak{B}$ are the relations $\bigcap X$ for distinct ultrafilters X in $\mathfrak{A}$, and the elements in $\mathfrak{B}$ are the unions of the distinct sets of atoms. It remains to prove the reverse implication. To this end, suppose that the complete subalgebra of $\mathfrak{Re}(E)$ that is completely generated by $\mathfrak{A}$—call it $\mathfrak{B}$—is the canonical extension of $\mathfrak{A}$. The goal is to show that $\mathfrak{A}$ satisfies conditions (i) and (ii) of the theorem.

To verify condition (i), consider an arbitrary ultrafilter X in $\mathfrak{A}$. The infimum of X in $\mathfrak{Re}(E)$ is the intersection $\bigcap X$, by the definition of $\mathfrak{Re}(E)$. The algebra $\mathfrak{B}$ is assumed to be a complete subalgebra of $\mathfrak{Re}(E)$, so the infimum of X in $\mathfrak{B}$ must also be the intersection $\bigcap X$. The infimum of X is, however, an atom, by Lemma 14.7 and the assumption that $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}$, so this infimum cannot be empty.

To verify condition (ii), consider arbitrary ultrafilters X, Y, and Z in $\mathfrak{A}$ such that $X \mid Y \subseteq Z$. The atom in $\mathfrak{B}$ determined by Z must then be below the relative product of the atom determined by X and the atom determined by Y, by the remarks preceding Uniqueness Theorem 14.13 (where A_a, A_b, and A_c are the ultrafilters X, Y, and Z respectively, and a, b and c are the atoms determined by these ultrafilters) and the assumption that $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}$. The atoms determined by ultrafilters are just the intersections of the ultrafilters, by the observations of the preceding paragraph, so the intersection $\bigcap Z$ must be included in the relational composition of the

intersections $\bigcap X$ and $\bigcap Y$. This is precisely what condition (ii) says. □

It may appear that conditions (i) and (ii) in the theorem are unusual and rarely satisfied. We shall see in Chapter 16, however, that every set relation algebra is isomorphic to a set relation algebra satisfying these two conditions.

14.10 A characterization of homomorphic images

We now return to a question that was raised in Chapter 10. In general, arbitrary homomorphic images of a relation algebra (or any other abstract algebra) are rather elusive and hard to deal with. That is why the First Isomorphism Theorem is so important: it says that the concrete construction of forming a quotient by an ideal yields all homomorphic images of a given algebra (up to isomorphisms). There are, however, serious limitations to the usefulness of this theorem. For example, in the passage to a quotient, all of the basic elements and operations of the algebra are modified, and it may therefore be difficult to determine whether certain properties of the original algebra are inherited by the quotient.

As we saw in Theorem 10.3, some homomorphic images of a relation algebra $\mathfrak{A}$—namely those in which the kernel of the homomorphism is a principal ideal in $\mathfrak{A}$—are isomorphic to relativizations of $\mathfrak{A}$ modulo ideal elements. Relativizations have the property that their elements all belong to $\mathfrak{A}$, and their operations are, for the most part, just restrictions of the corresponding operations of $\mathfrak{A}$. For this reason, it is usually easier to determine whether a given property of $\mathfrak{A}$ is inherited by a relativization.

Unfortunately, not all quotients of $\mathfrak{A}$ need be isomorphic to relativizations of $\mathfrak{A}$ modulo ideal elements in $\mathfrak{A}$. It turns out, however, that if one is willing to liberalize the notion of a relativization by allowing the ideal element to belong, if not to $\mathfrak{A}$, then at least to the canonical extension of $\mathfrak{A}$, then every homomorphic image of $\mathfrak{A}$ *is* isomorphic to a relativization of $\mathfrak{A}$ to an ideal element. The proof of this theorem makes use of two lemmas. The first one characterizes the closed ideal elements in a canonical extension.

Lemma 14.51. *Suppose $\mathfrak{B}$ is the canonical extension of a relation algebra $\mathfrak{A}$. An element u in $\mathfrak{B}$ is a closed ideal element if and only if there is a relation algebraic filter N in $\mathfrak{A}$ such that $u = \prod N$ (in $\mathfrak{B}$).*

Proof. Suppose first that u is a closed ideal element in $\mathfrak{B}$. The assumption that u is closed implies that the set

$$N = \{r \in A : u \leq r\} \tag{1}$$

is a Boolean filter in $\mathfrak{A}$, and that u is the infimum of this filter in $\mathfrak{B}$, by Lemma 14.5 (with u and N in place of p and A_p respectively). In particular, this set satisfies the first three conditions in the definition of a relation algebraic filter (see Definition 8.49). The assumption that u is an ideal element implies that

$$0 \dot{+} u \dot{+} 0 = u, \tag{2}$$

by Lemma 5.38(vii). With the help of this equation, we show that N satisfies condition (iv) in Definition 8.49 and is therefore not only a Boolean filter, but also a relation algebraic filter.

For the proof, consider elements r in N and s in $\mathfrak{A}$. It is to be shown that $s \dot{+} r$ and $r \dot{+} s$ are both in N. Since $u \leq r$, by (1), we have

$$0 \dot{+} u \dot{+} 0 \leq 0 \dot{+} r \dot{+} 0, \tag{3}$$

by the monotony law for relative addition (Lemma 4.7(vii)). A similar argument, using the inequalities $0 \leq s$ and $0 \leq 0$', yields

$$0 \dot{+} r \dot{+} 0 \leq s \dot{+} r \dot{+} 0'. \tag{4}$$

Combine (2)–(4) and the identity law for relative addition (Lemma 4.7(ii)) to arrive at

$$u \leq s \dot{+} r \dot{+} 0' = s \dot{+} r. \tag{5}$$

Together, (1) and (5) show that $s \dot{+} r$ belongs to N. An analogous argument leads to the conclusion that $r \dot{+} s$ belongs to N as well. This completes the proof of the implication from left to right.

To establish the reverse implication, consider an arbitrary relation algebraic filter N in $\mathfrak{A}$, and suppose $u = \prod N$ in $\mathfrak{B}$. (This product exists because $\mathfrak{B}$ is complete.) The goal is to show that u is a closed ideal element in $\mathfrak{B}$. As the infimum of a subset of $\mathfrak{A}$, the element u is, by the definition, closed. Also, if r is an element in N, then $0 \dot{+} r \dot{+} 0$

belongs to N as well, because N is a relation algebraic filter (see condition (iv) of Definition 8.49). The assumption that u is the infimum of N implies that $u \leq 0 \dotplus r \dotplus 0$ and therefore

$$1 \,;\, u \,;\, 1 \leq 1 \,;\, (0 \dotplus r \dotplus 0) \,;\, 1 = 0 \dotplus r \dotplus 0 \leq r,$$

by Lemma 4.5(i), Corollary 4.31 and its first dual, and Lemma 4.7(viii) and its first dual. Form the product over all r in N to arrive at

$$1 \,;\, u \,;\, 1 \leq \prod\{r : r \in N\} = u.$$

The reverse inequality holds by Lemma 4.5(iii) and its first dual. Consequently, $u = 1 \,;\, u \,;\, 1$ and therefore u is an ideal element. □

The dual of the preceding lemma characterizes the open ideal elements in the canonical extension.

Corollary 14.52. *Suppose $\mathfrak{B}$ is the canonical extension of a relation algebra $\mathfrak{A}$. An element u in $\mathfrak{B}$ is an open ideal element if and only if there is a relation algebraic ideal M in $\mathfrak{A}$ such that $u = \sum M$ (in $\mathfrak{B}$).*

We turn now to the relativization homomorphism induced by a closed ideal element.

Lemma 14.53. *Suppose $\mathfrak{B}$ is the canonical extension of a relation algebra $\mathfrak{A}$. If u is the infimum in $\mathfrak{B}$ of a relation algebraic filter N in $\mathfrak{A}$, then the restriction to $\mathfrak{A}$ of the relativization homomorphism $p \longmapsto p \cdot u$ on $\mathfrak{B}$ is an epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(u)$ with cokernel N.*

Proof. Lemma 14.51 implies that u is an ideal element in $\mathfrak{B}$, so it makes sense to speak of the relativization homomorphism φ defined for p in $\mathfrak{B}$ by $\varphi(p) = p \cdot u$. The cokernel of this homomorphism is, by definition, the set M of elements in $\mathfrak{B}$ that are mapped by φ to the unit u of the relativization $\mathfrak{B}(u)$. Since $p \cdot u = u$ if and only if $u \leq p$, we may write

$$M = \{p \in B : u \leq p\}. \tag{1}$$

The restriction of φ to $\mathfrak{A}$ is an epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(u)$, by Lemma 10.12, and the cokernel of this restriction is the set $M \cap A$. It remains to show that

$$N = M \cap A. \tag{2}$$

If r is in N, then r is in $\mathfrak{A}$, because N is a subset of $\mathfrak{A}$; and r is above the infimum of N, which is u by assumption, so r belongs to M, by (1).

It follows that r belongs to the right side of (2), so the left side of (1) is included in the right side.

To establish the reverse inclusion, assume r is an element in $\mathfrak{A}$ that belongs to M. In this case, r is above u, by (1), and therefore

$$\prod N = u \leq r,$$

by the assumption on u. The element r is open, because it is assumed to belong to $\mathfrak{A}$, and the filter N is included in $\mathfrak{A}$, so we may apply compactness in the form of Corollary 14.3(ii) to obtain a finite subset X of N such that $\prod X \leq r$. The filter N is closed under multiplication, so the product $\prod X$ belongs to N, and consequently r belongs to N, by the upward closure of N (see filter condition (v) in Lemma 8.50). The right side of (2) is therefore included in the left side. This completes the verification of (2). □

Theorem 14.54. *Every homomorphic image of a relation algebra* $\mathfrak{A}$ *is isomorphic to a relativization of* $\mathfrak{A}$ *to some closed ideal element from the canonical extension of* $\mathfrak{A}$.

Proof. Every homomorphic image of $\mathfrak{A}$ is isomorphic to a quotient of $\mathfrak{A}$ modulo some relation algebraic filter in $\mathfrak{A}$, by the dual of First Isomorphism Theorem 8.39. Consequently, it suffices to show that every such quotient is isomorphic to a relativization of $\mathfrak{A}$ to some ideal element from the canonical extension of $\mathfrak{A}$. Consider a filter N in $\mathfrak{A}$, and let u be the infimum of N in the canonical extension of $\mathfrak{A}$. Lemma 14.51 ensures that u is a closed ideal element in the canonical extension, and Lemma 14.53 ensures that the restriction to $\mathfrak{A}$ of the relativization homomorphism $p \longmapsto p \cdot u$ on $\mathfrak{B}$ is an epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(u)$ with cokernel N. The quotient $\mathfrak{A}/N$ is therefore isomorphic to the relativization $\mathfrak{A}(u)$, by the dual of First Isomorphism Theorem 8.39. □

14.11 Historical remarks

Marshall Stone proved in [126] that every Boolean algebra is canonically embeddable into the Boolean algebra of all subsets of the set of maximal ideals in the algebra; and in [127], he formulated a stronger topological version of this theorem. The arguments given there are essentially just dual versions of the proof leading up to Theorem 14.10. An abstract development of the subject based on Definition 14.1 was

given by Jónsson and Tarski in [74] as part of their development of the theory of Boolean algebras with operators. They used the terminology "perfect extension" instead of "canonical extension". The definitions of open and closed elements appear in that paper, as do versions of Corollary 14.3, Theorem 14.10, and Theorem 14.11. Lemma 14.5 is due to Givant. The observations in Lemma 14.7 were probably known to Jónsson and Tarski.

The theory of Boolean algebras with operators was founded by Jónsson and Tarski in [74] (a preliminary announcement of their main results appeared in [73]), and most of the notions and results in Section 14.2 are due to them. This includes Definition 14.12, Uniqueness Theorem 14.13, and Existence Theorem 14.16. However, the particular approach used in Section 14.2 to prove the existence and uniqueness theorems is not that of [74]. Rather, it was developed by Givant under the influence of the approach taken in [50] for Boolean algebras with unary operators. In particular, versions of Lemma 14.14 and Theorem 14.15 for Boolean algebras with unary operators are given in [50].

The approach in Section 14.3 to proving the existence theorem is much closer to that of [74]. In particular, the definitions of the extension operators, presented after Lemma 14.17, are given in their general form in [74], as are Lemmas 14.18, 14.21, 14.23, and 14.26, and Theorem 14.27. (Lemmas 14.24 and 14.25 are implicit in the Jónsson-Tarski proof of Lemma 14.26.)

The First Preservation Theorem 14.32 is perhaps the main result of [74]. A simplification of the original proof was suggested by Hugo Ribeiro in [117] and incorporated into the presentation for Boolean algebras with unary operators given in [50]. This presentation, in turn, inspired the presentation given in Section 14.4 above.

Special cases of the Second Preservation Theorem 14.34 in which the antecedent formula ε has the form of an equation $\varrho = 0$ or an inequality $\varrho \neq 0$ are given in [74]. A more general version of this theorem (stated for completions—see the next chapter) is given without proof by Monk [112], and the general theorem (again, stated for completions) is given in Givant-Venema [43].

The construction of the canonical extension of a relation algebra and the proof that it is also a relation algebra (see Theorem 14.35) is the original result, obtained by Tarski around 1948, that motivated the entire study of Boolean algebras with operators. It is stated in [75], as is Theorem 14.36. The approach to proving the theorem that is outlined in Exercise 14.40 *may* have been the original method of proof employed

by Tarski. (This is just a conjecture, and is not based on any historical knowledge such as conversations with Tarski.) Theorem 14.37 is due to Andréka-Givant-Németi [7], and Theorem 14.38 is due to Givant.

Jónsson and Tarski briefly discuss in [74] the extensions of their results that are alluded to at the beginning of Section 14.7. As an example, they mention—without proof—the result that if φ is a homomorphism from $\mathfrak{A}$ to $\bar{\mathfrak{A}}$, then φ^+ is a homomorphism from the canonical extension of $\mathfrak{A}$ to the canonical extension of $\bar{\mathfrak{A}}$ (see Existence Theorem 14.41). The fact that φ^+ is actually a complete homomorphism is stated without proof in Jónsson [71].

The theorem that the canonical extension of a subalgebra is isomorphic to a subalgebra of the canonical extension is stated for cylindric algebras in [50], and attributed to Leon Henkin and Donald Monk. The stronger version of this result that is given in Theorem 14.44 is due to Andréka-Givant-Németi [7]. Lemma 14.45, Theorem 14.46, and Corollaries 14.47 and 14.48 are due to Givant. A version of Theorem 14.49 for cylindric algebras is given in [50] and attributed to Henkin and Monk. A stronger theorem that implies both results is given in Andréka-Givant-Németi [7].

Monk proved that the canonical extension of a set relation algebra is always isomorphic to a set relation algebra (see Theorem 16.22). His proof used certain connections between relation algebras and cylindric algebras (see p. 66 of [106]). Theorem 14.50 is due to Andréka and Givant, and the results in Section 14.10, and in Exercises 14.13, 14.64, and 14.65, are due to Givant.

Exercises

14.1. Prove that the compactness property in condition (iii) of Definition 14.1 is equivalent to its dual version, which asserts that for any subset X of A, if 0 is the infimum of X in B, then 0 is already the infimum of some finite subset of X.

14.2. Prove Corollary 14.3(ii).

14.3. Prove that the following conditions on a filter X in a Boolean algebra A are equivalent. (1) X is an ultrafilter. (2) For every element r in A, exactly one of r and $-r$ is X. (3) X is a proper filter, and for every pair of elements r and s in A, the sum $r + s$ is in X if and only if r or s is in X.

14.4. If $\mathfrak{B}$ is a canonical extension of a Boolean algebra with operators $\mathfrak{A}$, prove that for every quasi-atom a and every atom c in $\mathfrak{B}$,

$$c \leq a^{\smile} \qquad \text{if and only if} \qquad A_a^{\smile} \subseteq A_c.$$

14.5. Complete the proof of Theorem 14.13 by showing that for a quasi-atom b_1 and an atom b_3 in $\mathfrak{B}$,

$$b_3 \leq b_1^{\smile} \qquad \text{if and only if} \qquad \varphi(b_3) \leq \varphi(b_1)^{\smile}.$$

14.6. Prove directly (without using Theorem 14.13) the version of the Uniqueness Theorem 14.13 for canonical extensions that applies to Boolean algebras with *normal* operators.

14.7. Prove a version of the Uniqueness Theorem 14.13 for canonical extensions that applies to Boolean algebras with operators of arbitrary ranks.

14.8. Prove that the set G_0 defined in (8) of the proof of Lemma 14.14 is a proper Boolean ideal that contains $-s$.

14.9. Prove that the set F_0 defined in (11) in the proof of Lemma 14.14 is downward closed in the sense that if u is in F_0 and if $w \leq u$, then w is in F_0.

14.10. Complete the proof of Theorem 14.15 by showing that the second equation in condition (iv) of Definition 14.12 holds for the algebra $\mathfrak{C}$ with respect to the subalgebra $\varphi(\mathfrak{A})$.

14.11. Prove that the algebra $\mathfrak{B}$ in the proof of Theorem 14.16 satisfies the conditions for being the canonical extension of the given Boolean algebra with normal operators $\mathfrak{A}$.

14.12. Define the notion of the canonical embedding algebra of a Boolean algebra with normal operators of arbitrary ranks, and extend Lemma 14.14 and Theorem 14.15 to arbitrary Boolean algebras with normal operators. Conclude that a Boolean algebra with normal operators of arbitrary ranks always has a canonical extension.

14.13. This and the next exercise concern the existence of canonical extensions of Boolean algebras with operators that may not be normal. Let U be the set of all ultrafilters in a Boolean algebra with operators $\mathfrak{A}$, and let C be the set of all subsets of U. For subsets X and Y of $\mathfrak{A}$, write

$$X\,;Y = \{r\,;s : r \in X \text{ and } s \in Y\} \quad \text{and} \qquad X^{\smile} = \{r^{\smile} : r \in X\}.$$

Define a binary operation $;$ and a unary operation $^{\smile}$ on C as follows: for any two subsets V and W of U, take $V\,;W$ to be the set of ultrafilters Z in U that satisfy one of the four conditions

(1) $A\,;A \subseteq Z$,
(2) $X\,;A \subseteq Z$ for some $X \in V$,
(3) $A\,;Y \subseteq Z$ for some $Y \in W$,
(4) $X\,;Y \subseteq Z$ for some $X \in V$ and $Y \in W$,

and take $V^{\smile}$ to be the set of ultrafilters Z in U that satisfy one of the two conditions

(5) $A^{\smile} \subseteq Z$,
(6) $X^{\smile} \subseteq Z$ for some $X \in V$.

Define 1' to be the set of ultrafilters Z in U that contain the distinguished element 1' from $\mathfrak{A}$. Put

$$\mathfrak{C} = (C\,,+\,,-\,,;\,,{}^{\smile}\,,1'),$$

where $+$ and $-$ are the set-theoretic operations of union and complement on C.

(i) Prove that $\mathfrak{C}$ is a Boolean algebra with quasi-completely distributive operators.
(ii) Prove that in the case when the operators in $\mathfrak{A}$ are normal, the conditions in (1)–(3) and in (5) are never satisfied for any ultrafilter Z in U. Consequently, in this case the above definitions of the operators $;$ and $^{\smile}$ reduce to the definitions given before Lemma 14.14.
(iii) Prove that the function φ defined by

$$\varphi(r) = \{Z \in U : r \in Z\}$$

for r in $\mathfrak{A}$ is a monomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.
(iv) Prove that $\mathfrak{C}$ is a canonical extension of the subalgebra that is the image of $\mathfrak{A}$ under φ.
(v) Conclude that every Boolean algebra with operators has a canonical extension.

14.14. Extend the results in Exercise 14.13 to Boolean algebras with operators of arbitrary ranks.

14.15. Verify the equation $p^{\smile} = \prod(A_p^{\smile})$ in Lemma 14.18.

14.16. Formulate and prove a version of Lemma 14.18 that applies to operations of arbitrary ranks.

14.17. Verify the equation $p^{\smile} = \sum(K_p^{\smile})$ in Corollary 14.20.

14.18. Formulate and prove a version of Corollary 14.20 that applies to operations of arbitrary ranks.

14.19. Complete the proof of Lemma 14.21 by showing that the restriction of the operation $^{\smile}$ in $\mathfrak{B}$ to the universe of $\mathfrak{A}$ coincides with the operation $^{\smile}$ in $\mathfrak{A}$.

14.20. Prove that Lemma 14.21 continues to hold for Boolean algebras with operators of arbitrary ranks.

14.21. Prove that Lemma 14.22 continues to hold for Boolean algebras with operators of arbitrary ranks.

14.22. Complete the proof of Lemma 14.23 by showing that the operation $^{\smile}$ in $\mathfrak{B}$ is monotone.

14.23. Formulate and prove a version of Lemma 14.23 that applies to operations of arbitrary ranks.

14.24. Prove directly (without using Lemma 14.24) that the following version of Lemma 14.24 is true if $\mathfrak{A}$ is a Boolean algebra with *normal* operators. Let p and q be closed elements, and a an atom, in the extension algebra $\mathfrak{B}$.

(i) If, for every atom $b \leq p$, there is an element r_b in $\mathfrak{A}$ such that

$$b \leq r_b \qquad \text{and} \qquad a \cdot r_b^{\smile} = 0,$$

then there is an element r in $\mathfrak{A}$ such that

$$p \leq r \qquad \text{and} \qquad a \cdot r^{\smile} = 0.$$

(ii) If, for every atom $b \leq p$ and every atom $c \leq q$, there are elements r_{bc} and s_{bc} in $\mathfrak{A}$ such that

$$b \leq r_{bc}, \qquad c \leq s_{bc}, \qquad \text{and} \qquad a \cdot (r_{bc}\,;\, s_{bc}) = 0,$$

then there are elements r and s in $\mathfrak{A}$ such that

$$p \leq r, \qquad q \leq s, \qquad \text{and} \qquad a \cdot (r\,;\, s) = 0.$$

14.25. Formulate and prove a version of Lemma 14.24 that applies to operations of arbitrary ranks.

14.26. Give a direct proof of the version of Lemma 14.25 in which the operators in the algebra $\mathfrak{A}$ are normal, and the sets W and Z are the sets of *atoms* below p and q respectively. Don't use Lemma 14.25 in your proof, but rather Exercise 14.24.

14.27. Formulate and prove a version of Lemma 14.25 that applies to operations of arbitrary ranks.

14.28. Complete the proof of Lemma 14.26 by showing that the operation $^\smile$ in $\mathfrak{B}$ is quasi-completely distributive, and completely distributive if $\mathfrak{A}$ is normal.

14.29. Formulate and prove a version of Lemma 14.26 that applies to operations of arbitrary ranks.

14.30. Define the notion of a positive polynomial in a Boolean algebra with operators of arbitrary ranks.

14.31. Prove that Lemma 14.28 continues to hold when $\mathfrak{B}$ has operators of arbitrary ranks.

14.32. Prove that Lemma 14.29 continues to hold when $\mathfrak{B}$ has operators of arbitrary ranks.

14.33. Complete the proof of Lemma 14.30 by treating the induction cases

$$\gamma = \sigma + \tau, \qquad \gamma = \sigma \cdot \tau, \qquad \text{and} \qquad \gamma = \sigma^\smile.$$

14.34. Give an alternative proof of the inequality

$$\gamma(p) \geq \prod\{\gamma(r) : r \in A_p\}$$

for the induction case $\gamma = \sigma + \tau$ in the proof of Lemma 14.30, using an argument that avoids compactness and does not require Lemmas 14.18 and 14.28. Do the same for the induction case $\gamma = \sigma \cdot \tau$.

14.35. Prove that Lemma 14.30 continues to hold when $\mathfrak{B}$ has operators of arbitrary ranks.

14.36. Complete the proof of Lemma 14.31 by treating the induction steps when γ is $\sigma \cdot \tau$ and when γ is $\sigma^\smile$.

14.37. Prove that Lemma 14.31 continues to hold when $\mathfrak{B}$ has operators of arbitrary ranks.

14.38. Let O be the unary descriminator defined on the universe of a Boolean algebra with operators $\mathfrak{A}$. Verify directly the remark made before Lemma 14.33 that the implications

$$(\varrho = 0) \to (\sigma = \tau)$$

and

$$(\varrho_0 = 0) \wedge (\varrho_1 = 0) \wedge (\varrho_2 \neq 0) \wedge (\varrho_3 \neq 0)] \to (\sigma = \tau)$$

are valid in $\mathfrak{A}$ if and only if the corresponding equations

$$\sigma + O(\varrho) = \tau + O(\varrho)$$

and

$$\sigma \cdot O(\varrho_2) \cdot O(\varrho_3) + O(\varrho_0) + O(\varrho_1) \\ = \tau \cdot O(\varrho_2) \cdot O(\varrho_3) + O(\varrho_0) + O(\varrho_1)$$

are respectively valid in the expansion of $\mathfrak{A}$ obtained by adjoining the operation O. Do not use Lemma 14.33 in your proof.

14.39. Suppose $\mathfrak{A}$ is a subalgebra of a relation algebra $\mathfrak{B}$. Prove that if either $\mathfrak{A}$ or $\mathfrak{B}$ has a type, in the sense of Section 13.1, then both algebras must have a type, and in fact it must be the same type. Conclude that the canonical extension of a relation algebra of type n is again a relation algebra of type n, for $n = 1, 2, 3$.

14.40. This exercise outlines another, more direct, route to proving that every relation algebra $\mathfrak{A}$ has a canonical extension which is a relation algebra (Theorem 14.35). First, the canonical embedding algebra $\mathfrak{C}$ for the algebra $\mathfrak{A}$ is constructed in exactly the manner discussed before Lemma 14.14. This automatically guarantees the validity in $\mathfrak{C}$ of the Boolean laws (R1)–(R3) and the distributivity laws (R8) and (R9). Second, the validity in $\mathfrak{C}$ of (R4)–(R7) and (R11) for atoms, and hence for all elements, is verified directly. It *may* be that Tarski's original proof of Theorem 14.35 proceeded via this more direct route, but this is only a conjecture. The details of his original proof are not known.

(i) Verify directly that (R6) is valid for atoms in $\mathfrak{C}$.

(ii) Verify directly that (R7) is valid for atoms in $\mathfrak{C}$.
(iii) Verify directly that (R11) is valid for atoms in $\mathfrak{C}$.
(iv) Verify directly that (R5) is valid for atoms in $\mathfrak{C}$.
(v) Verify directly that (R4) is valid for atoms in $\mathfrak{C}$.

14.41. Prove directly, without using the Second Preservation Theorem 14.34, that the canonical extension of a simple relation algebra is simple.

14.42. This and the next nine exercise ask for the proofs of the various lemmas needed to establish Lemmas 14.39 and 14.40. Prove that if a mapping φ from a Boolean algebra with operators $\mathfrak{A}$ to a Boolean algebra with operators $\bar{\mathfrak{A}}$ preserves the operation of addition, then φ is monotone in the sense that $r \leq s$ implies that $\varphi(r) \leq \varphi(s)$ for all r and s in $\mathfrak{A}$.

14.43. Prove that if a mapping φ from an atomic Boolean algebra with operators $\mathfrak{A}$ to an atomic Boolean algebra with operators $\bar{\mathfrak{A}}$ preserves all existing sums of non-empty sets of quasi-atoms, then φ preserves all existing sums of non-empty sets of elements.

14.44. Prove the analogue of Lemma 14.18 for the mapping φ^+.

14.45. Prove the analogue of Corollary 14.20 for the mapping φ^+.

14.46. Prove the analogue of Lemma 14.21 for the mapping φ^+.

14.47. Prove the analogue of Lemma 14.22 for the mapping φ^+.

14.48. Prove the analogue of Lemma 14.23 for the mapping φ^+.

14.49. Prove the analogue of Lemma 14.24 for the mapping φ^+.

14.50. Prove the analogue of Lemma 14.25 for the mapping φ^+.

14.51. Prove the analogue of Lemma 14.26 for the mapping φ^+.

14.52. Complete the proof of Lemma 14.39 by proving parts (i) and (ii), and parts (v) and (vi), of the lemma.

14.53. Complete the proof of Lemma 14.40 by proving parts (i) and (ii), and parts (v) and (vi), of the lemma.

14.54. Complete the proof of Theorem 14.41 by showing that the function φ^+ preserves the operations $\cdot$ and $\smile$.

14.55. Prove the analogue of Theorem 14.41 for Boolean algebras with operators of arbitrary ranks.

14.56. Derive Uniqueness Theorem 14.13 from Theorem 14.41.

14.57. Prove the analogue of Theorem 14.42 for Boolean algebras with operators of arbitrary ranks.

14.58. Suppose $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are Boolean algebras with operators, and $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ their respective canonical extensions. Let φ be a function of two arguments from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, that is to say, $\varphi(r, s)$ is an element in $\bar{\mathfrak{A}}$ for every pair of elements r and s in $\mathfrak{A}$. The canonical extension of φ is defined to be the function φ^+ of two arguments from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that is defined by

$$\varphi^+(p, q) = \sum\{\prod\{\varphi(r, s) : r, s \in A \text{ and } u \leq r,\ v \leq s\} : u, v \in K \text{ and } u \leq p,\ v \leq q\}.$$

Prove that if φ is distributive, then φ^+ is quasi-completely distributive. Prove further that φ^+ is the unique quasi-completely distributive function of two arguments from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ.

14.59. Formulate and prove a generalization of Exercise 14.58.

14.60. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ be Boolean algebras with operators. Suppose φ and ψ are functions of two arguments from $\mathfrak{A}_1$ into $\mathfrak{A}_2$, and ϑ is a function of two arguments from $\mathfrak{A}_2$ into $\mathfrak{A}_3$. The *composition* of ϑ with φ and ψ is defined to be the function $\varrho = \vartheta(\varphi, \psi)$ of two arguments from $\mathfrak{A}_1$ into $\mathfrak{A}_3$ that is defined by

$$\varrho(r, s) = \vartheta(\varphi(r, s), \psi(r, s))$$

for all r and s in $\mathfrak{A}_1$. Prove that if the functions φ, ψ, and ϑ are all distributive, then the canonical extension of the composition of ϑ with φ and ψ is equal to the composition of the canonical extension of ϑ with the canonical extensions of φ and ψ, that is to say,

$$\varrho^+ = \vartheta^+(\varphi^+, \psi^+).$$

14.61. Formulate and prove a generalization of Exercise 14.60.

14.62. Give a direct proof of Theorem 14.49 by verifying that the direct product $\mathfrak{B}_1 \times \mathfrak{B}_2$ satisfies the conditions in Definition 14.12 for being the canonical extension of the product $\mathfrak{A}_1 \times \mathfrak{A}_2$.

14.63. Prove by induction on positive integers n that if $\mathfrak{B}_i$ is the canonical extension of a Boolean algebra with operators $\mathfrak{A}_i$ for each integer i with $0 \leq i < n$, then the direct product $\prod_{i<n} \mathfrak{B}_i$ is the canonical extension of the direct product $\prod_{i<n} \mathfrak{A}_i$ (up to isomorphisms that are the identity function on $\prod_{i<n} \mathfrak{A}_i$).

14.64. Let $\mathfrak{A}$ be a relation algebra, $\mathfrak{B}$ its canonical extension, and e an equivalence element in $\mathfrak{A}$. Prove that the relativization of $\mathfrak{B}$ to e is the canonical extension of the relativization of $\mathfrak{A}$ to e.

14.65. Let $\mathfrak{A}$ be a relation algebra, $\mathfrak{B}$ the canonical extension of $\mathfrak{A}$, and M an ideal in $\mathfrak{A}$. Prove that the set N of elements in $\mathfrak{B}$ that can be written as sums of products of subsets of M is a complete ideal in $\mathfrak{B}$, and that the quotient $\mathfrak{B}/N$ is isomorphic to the canonical extension of the quotient $\mathfrak{A}/M$ via a function that maps r/N to r/M for each element r in $\mathfrak{A}$. (An ideal in a normal Boolean algebra with operators is said to be *complete* provided that, whenever a subset of the ideal has a supremum in the algebra, then that supremum also belongs to the ideal.)

14.66. Extend the result in Exercise 14.65 to arbitrary Boolean algebras with normal operators. (See Exercise 8.78 for the appropriate notion of an ideal.)

14.67. Prove that the set relation algebra $\mathfrak{B}$ constructed in the proof of Theorem 14.50 is the canonical extension of the set relation algebra $\mathfrak{A}$ by verifying directly that $\mathfrak{B}$ satisfies the conditions in Definition 14.12 for being the canonical extension of $\mathfrak{A}$.

14.68. Prove that every set relation algebra is isomorphic to a set relation algebra satisfying the two conditions of Theorem 14.50.

14.69. Suppose $\mathfrak{B}$ is the canonical extension of a relation algebra $\mathfrak{A}$. Prove directly (without using Lemma 14.51) that an element u in $\mathfrak{B}$ is an open ideal element if and only if there is a relation algebraic ideal M in $\mathfrak{A}$ such that $u = \sum M$ in $\mathfrak{B}$. Prove further (without using Lemma 14.53) that if $u = \sum M$ for some relation algebraic ideal M in $\mathfrak{A}$, then the restriction to $\mathfrak{A}$ of the relativization homomorphism on $\mathfrak{B}$ defined by $p \longmapsto p \cdot -u$ has kernel M. Use these two observations to give a slightly different proof of Theorem 14.54.

14.70. Give an example of an incomplete homomorphism between two complete relation algebras.

14.71. Prove that the canonical extension of a countably infinite relation algebra must have cardinality at least $2^{\aleph_0}$ and at most $2^{2^{\aleph_0}}$.

14.72. Give examples of countably infinite relation algebras with canonical extensions that have cardinality $2^{\aleph_0}$ and cardinality $2^{2^{\aleph_0}}$ respectively.

Chapter 15
Completions

One can sometimes establish a certain property for a class of relation algebras by applying the following strategy. First, prove that all complete and atomic relation algebras in the class possess the property, and then show that the property is inherited by subalgebras. If the class in question is closed under the formation of canonical extensions, that is to say, if the canonical extension of every algebra in the class also belongs to the class, then every algebra in the class will possess the desired property. Canonical extensions, however, do have a serious disadvantage from this perspective, because all properly infinite sums and products in the original algebra are changed in the passage to the canonical extension, and (in the infinite case) new atoms are introduced even when the original algebra is atomic. Consequently, if the property in question involves properly infinite sums or products, or atoms, in the original algebra, then the above strategy may fail. What one would like is an extension that is complete, with the same atoms as in the original algebra, and in which all existing infinite sums and products of the original algebra are preserved. Fortunately, such an extension exists, and in fact it satisfies a very nice minimality condition.

The construction actually goes through in the context of arbitrary Boolean algebras with quasi-complete operators. As in the case of canonical extensions, in order to simplify notation we shall assume that the similarity type of the algebras under discussion is the same as the similarity type of relation algebras. The extension of the results to Boolean algebras with quasi-complete operators of arbitrary ranks is straightforward.

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9_2

15.1 Complete Boolean ideals

The construction of canonical extensions involves (explicitly or implicitly) the notion of an ultrafilter—a maximal Boolean filter. In rough analogy, the construction of a completion involves the notion of a complete Boolean ideal. We therefore begin with the study of such ideals. *In this and the next three sections, all ideals under consideration are assumed to be Boolean ideals*, so we will just refer to them as ideals. Recall that Boolean algebras are denoted with italic letters. An ideal in a Boolean algebra A is defined to be a subset X of A with following three properties: (1) 0 is in X; (2) X is closed under addition in the sense that if r and s are in X, then so is $r + s$; (3) X is closed under multiplication by elements from A in the sense that if r is in X and s in A, then $r \cdot s$ is in X. Condition (3) is equivalent to the condition that X be *downward closed* in the sense that if r is in X and $s \leq r$, then s is in X. In the presence of condition (3), condition (1) is equivalent to the condition that X be non-empty (see Lemma 8.8).

An ideal X in a Boolean algebra A is said to be *complete* if it is *closed under* (the formation of) *suprema*. This means that if a subset of X has a supremum in A, then that supremum must also belong to X. A set that is closed under formation of suprema is certainly closed under addition, so a subset X of A is a complete ideal just in case X is non-empty, downward closed, and closed under suprema. Every principal ideal is an example of a complete ideal. Indeed, if Y is a subset of the principal ideal generated by an element r in A, and if Y has a supremum s in A, then s is below r (because r is an upper bound of Y, and s is the least upper bound of Y), and therefore s is in the ideal generated by r. We shall see other examples of complete ideals in the next section.

The intersection of an arbitrary system of complete ideals in a Boolean algebra A is again a complete ideal in A. Indeed, the intersection of such a system is an ideal, by (the Boolean version of) the remarks at the beginning of Section 8.7. If Y is a subset of this intersection, and if Y has a supremum r in A, then r belongs to each of the ideals in the system, by the assumption that these ideals are complete, and therefore r belongs to the intersection of the system. Consequently, the intersection is a complete ideal.

It follows that if Y is an arbitrary subset of A, then the intersection of the set of complete ideals in A that include Y is itself a complete

ideal. (There is always at least one complete ideal that includes Y, namely the improper ideal A, so the set of complete ideals that include Y is never empty.) That intersection—call it X—is the smallest complete ideal that includes Y, In other words, X is a complete ideal in A that includes Y, and every complete ideal in A that includes Y also includes X, by the very definition of X. The set X is called the complete ideal *generated* by Y.

The definition just given is non-constructive and gives no idea which elements in A actually belong to the complete ideal generated by Y. There is another description that is quite useful. Let Y^d be the set of elements in A that are below some element in Y, so that

$$Y^d = \{s \in A : s \leq r \text{ for some } r \in Y\}.$$

The set Y^d is called the *downward closure* of Y (in A).

Lemma 15.1. *An element r in a Boolean algebra belongs to the complete ideal generated by a set Y if and only if r is the supremum of some subset of Y^d.*

Proof. Let X be the complete ideal generated by a set Y in a Boolean algebra A, and let Z be the set of elements in A that are suprema of subsets of Y^d. It is to be shown that X and Z are equal.

The first step is to verify that Z is indeed a complete ideal that includes Y. Each element r in Y is the supremum of a subset of Y^d, namely the subset $\{r\}$, so Y is certainly included in Z. The element 0 is the supremum of the empty subset, so 0 belongs to Z. To verify the closure of Z under suprema, consider a subset W of Z that has a supremum r in A. Each element s in W is the supremum of some subset Y_s of Y^d, by the definition of Z. The union $\bigcup_{s\in W} Y_s$ is a subset of Y^d, and r is the supremum of this union, since

$$r = \sum\{s : s \in W\} = \sum\{\sum Y_s : s \in W\} = \sum(\bigcup_{s\in W} Y_s),$$

by the general associative law for addition. Therefore, r belongs to Z, by the definition of Z.

It remains to check that Z is downward closed. Observe first that the set Y^d is obviously downward closed. Let r be an element in Z, and assume that $s \leq r$. The definition of Z implies that r is the supremum of some subset Y_r of Y^d. The set

$$\{t \cdot s : t \in Y_r\}$$

is also included in Y^d, by the downward closure of Y^d, and

$$s = r \cdot s = (\sum Y_r) \cdot s = \sum\{t \cdot s : t \in Y_r\},$$

so s is the supremum of a subset of Y^d and therefore belongs to Z.

So far, it has been shown that Z is a complete ideal and includes Y. The set X is the smallest complete ideal that includes Y, so X must be included in Z. On the other hand, every element in Y^d certainly belongs to X, because Y is included in X, and X is downward closed. The completeness of the ideal X therefore implies that whenever a subset of Y^d has a supremum in A, that supremum must belong to X. Thus, every element in Z belongs to X, by the definition of Z. Conclusion: $X = Z$, as claimed. □

There is another description of the complete ideal generated by a set Y in a Boolean algebra A that is worth mentioning, because it gives a different perspective on these ideals. Recall that the set U of upper bounds of Y is the set of elements in A that are above every element in Y. The set of lower bounds of U is the set L of elements in A that are below every element in U. It turns out that L is the complete ideal generated by Y.

Lemma 15.2. *The complete ideal generated by a set Y in a Boolean algebra is the set of lower bounds of the set of upper bounds of Y.*

Proof. Let X be the complete ideal generated by a set Y in a Boolean algebra A, let U be the set of upper bounds of Y, and let L be the set of lower bounds of U. It is to be shown that X and L coincide.

The first step is to verify that L is a complete ideal that includes Y. Each element in Y is a lower bound of U, by the definition of U, so Y is certainly included in the set L, by the definition of L. It is equally obvious that 0 is in L, since 0 is below every element in U. To verify that L is closed under suprema, consider an arbitrary set W of elements in L, and suppose that W has a supremum r in A. The elements in U are upper bounds of the set L, by the definition of L, so they are upper bounds of the set W, and consequently they are all above the least upper bound r of W. Thus, r is a lower bound of U, so it belongs to L, by the definition of L. The downward closure of L is equally easy to check. Any element r in L is, by definition, a lower bound of U, and therefore so is every element that is below r. It follows that all such elements are in L, by the definition of L.

The set X is, by assumption, the smallest complete ideal that includes Y. We have seen that L is also a complete ideal that includes Y, so X must be included in L.

To establish the reverse inclusion, consider an arbitrary element r in L, and let W be the set of elements in Y^d that are below r. The goal is to show that r is the supremum of the set W. Lemma 15.1 then implies that r is in X, so the set L must be included in X.

The element r is certainly an upper bound of W, by the definition of W. Consider any other upper bound of W, say s, with the aim of showing that $r \leq s$. The first step is to prove that $s + -r$ is an upper bound of Y. For each element t in Y, the product $t \cdot r$ is below r and also below t. In particular, this product must belong to the set Y^d, by the definition of Y^d and the assumption that t is in Y. Combine these observations with the definition of W to see that $t \cdot r$ must be in W. Every element in W is below s, by assumption, so $t \cdot r \leq s$. A straightforward computation yields

$$t = t \cdot 1 = t \cdot (r + -r\,) = (t \cdot r) + (t \cdot -r\,) \leq s + -r.$$

It has been shown that the sum $s + -r$ is an upper bound of Y, so this sum belongs to the set U of upper bounds of Y. The element r belongs to the set L of lower bounds of U, by assumption, so $r \leq s + -r$ and consequently

$$r = r \cdot (s + -r\,) = (r \cdot s) + (r \cdot -r\,) = (r \cdot s) + 0 = r \cdot s.$$

Thus, $r \leq s$, as was to be shown. □

There is a close connection between complete ideals and the "cuts" that play a crucial role in Dedekind's classical construction of the real numbers from the rational numbers. A *Dedekind cut* in the set of rational numbers is a pair (P, Q) of non-empty sets that partition the rational numbers and have the property that every number in P is less than every number in Q. The set P has the characteristic property that it is downward closed in the sense that if p is in P and if q is a rational number less than p, then q is in P. Similarly, the set Q is upward closed in the sense that if p is in Q and if q is a rational number greater than p, then q is in Q. The set Q can be reconstructed from P (it is just the complement of P in the set of rational numbers), so one could define a Dedekind cut to be simply a non-empty, downward closed set of rational numbers that does not coincide with the set of all rational numbers.

Consider now a subset Y of a Boolean algebra A. The set U of upper bounds of Y is upward closed, and the set L of lower bounds of U is downward closed, and the two sets L and U have at most one element in common. The pair (L, U) is therefore a kind of Dedekind cut in the partial ordering of the algebra A. (The fact that L and U may have one element in common—namely the supremum of Y, if this supremum exists—is of no real significance.) The set U can of course be reconstructed from the set L—it is just the set of upper bounds of L—so one can consider L itself to be a Dedekind cut in A, In view of Lemma 15.2, we may conclude that complete ideals are the analogues for Boolean algebras of Dedekind cuts of rational numbers.

The complete ideals in a Boolean algebra A form a complete lattice. The infimum of any system of complete ideals is the intersection of the system, and the supremum of the system is the complete ideal generated by the union of the system, or in different words, it is the intersection of the complete ideals in A that include the union of the system. Warning: the lattice of complete ideals in A is not a sublattice of the lattice of all ideals in A. The binary operation of meet is the same in both lattices, but the operation of join is not the same. In the lattice of all ideals, the join of two complete ideals X an Y is the intersection of all ideals that include $X \cup Y$; in the lattice of complete ideals, it is the intersection of all complete ideals that include $X \cup Y$.

The lattice of complete ideals in A is not only complete, it is also distributive. For the proof, consider three complete ideals X, Y, and Z in A. We shall show that meet distributes over join in the sense that

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z).$$

The ideals $X \wedge Y$ and $X \wedge Z$ are obviously included in the ideal on the left side of the equation, since Y and Z are each included in $Y \vee Z$. Consequently, the ideal on the right side of the equation—the smallest complete ideal that includes the ideals $X \wedge Y$ and $X \wedge Z$—is included in the ideal on the left side. To establish the reverse inclusion, consider an arbitrary element r in the ideal on the left side. Since r belongs to the complete ideal generated by $Y \vee Z$, and since the ideals Y and Z coincide with their downward closures, there must be a subset Y_0 of Y and a subset Z_0 of Z such that r is the supremum of the set $Y_0 \cup Z_0$, by Lemma 15.1. The element r also belongs to X, by assumption, so the set $Y_0 \cup Z_0$ must be included in X, by the downward closure of the ideal X. Consequently, the set Y_0 is a subset of $X \wedge Y$, and the set Z_0 is a subset of $X \wedge Z$, so the union $Y_0 \cup Z_0$ of these two sets is

included in the ideal on the right side of the equation. It follows that the supremum r of this union also belongs to the ideal on the right, because the ideal on the right is complete.

The dual distributive law for join over meet,

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z),$$

can be derived in a dual fashion. Alternatively, this law is lattice-theoretically derivable from the distributive law for meet over join; the proof is left as an exercise.

There is another rather surprising difference between the lattice of all ideals and the lattice of all complete ideals in a Boolean algebra A. In the former, ideals may fail to have a complement. In the latter, this never happens; every complete ideal has a complement. For the proof, consider a complete ideal X in A, and define the *annihilator* of X to be the set Y defined by

$$Y = \{r \in A : r \cdot t = 0 \text{ for all } t \in X\}.$$

It is easy to check that Y is a complete ideal in A. The element 0 belongs to Y, because $0 \cdot t = 0$ for all t in X. The set Y is downward closed, because if r is in Y and if $s \leq r$, then

$$s \cdot t \leq r \cdot t = 0$$

for all t in X, and therefore s is in Y. To see that Y is closed under suprema, let Y_0 be a subset of Y that has a supremum r in A. For each element t in X,

$$r \cdot t = (\textstyle\sum Y_0) \cdot t = \sum\{s \cdot t : s \in Y_0\} = 0,$$

by the complete distributivity of multiplication and the assumption that Y_0 is a subset of the annihilator of X. Conclusion: r belongs to Y.

We proceed to verify that Y is the complement of X in the sense that

$$X \wedge Y = \{0\} \qquad \text{and} \qquad X \vee Y = A$$

in the lattice of complete ideals in A. The first equation is almost immediate: if r is in X and also in the annihilator Y, then $r \cdot r = 0$, by the definition of the annihilator, and therefore $r = 0$. The second equation is equivalent to the assertion that the unit 1 belongs to the complete ideal generated by the set $X \cup Y$. In fact, 1 is the supremum

of this set. For the proof, observe that 1 is obviously an upper bound of the set. If r is any upper bound of the set, then every element s in X is below r, and therefore $-r \cdot s = 0$, by Boolean algebra. It follows that $-r$ belongs to the annihilator of X, which is Y. Since r is also assumed to be an upper bound of Y, we must have $-r \leq r$, which can only happen if $r = 1$. Thus, 1 is the least upper bound of the set $X \cup Y$, so 1 belongs to the complete ideal generated by this set.

A distributive, complemented lattice with zero and one is automatically a Boolean algebra, so the following theorem has been proved.

Theorem 15.3. *The set of all complete ideals in a Boolean algebra A is a complete Boolean algebra under the following operations*: *the product of two complete ideals X and Y is their intersection*, *the sum of X and Y is the complete ideal generated by their union*, *and the complement of X is the annihilator of X.*

15.2 Completions of Boolean algebras

We now turn to the study of completions, and we begin with the case of Boolean algebras without any additional operations.

Definition 15.4. A *completion* of a Boolean algebra A is a Boolean Boolean algebra B with the following properties.

(i) The algebra B is complete, and A is a subalgebra of B.
(ii) Every non-zero element in B is above a non-zero element in A.

□

A subset X of a Boolean algebra B is said to be *dense* (in B) if every non-zero element in B is above a non-zero element in X. Condition (ii) in the preceding definition requires that the (universe of the) subalgebra A be dense in B. For that reason, the condition is often called the *density property*. Density can be characterized in several different ways.

Lemma 15.5. *The following conditions on a subset X of a Boolean algebra B are equivalent.*

(i) *X is dense in B.*
(ii) *Every element in B is the supremum of some subset of X.*
(iii) *Every element p in B is the supremum of the set of all elements in X that are below p.*

Proof. Condition (iii) obviously implies condition (ii). If condition (ii) holds, then every non-zero element p in B must be above a non-zero element in X; for if the set of elements in X that are below an element p is either empty or contains only zero, then the supremum of this set—which is p, by condition (ii)—must be zero. Thus, condition (ii) implies condition (i).

Assume now that condition (i) holds, with the goal of deriving condition (iii). To this end, consider an arbitrary element p in B, and let Y be the set of elements in X that are below p. It is to be shown that p is the supremum of Y. Clearly, p is an upper bound of Y, by the definition of Y. Let q be any other upper bound of Y in B, and assume, for contradiction, that the difference $p-q$ is not zero. There is then a non-zero element r in X that is below this difference, by condition (i), so r is below both p and $-q$. Because r is in X and below p, it must belong to the set Y, by the definition of Y, and therefore it must be below q, which is assumed to be an upper bound of Y. Because r is also below $-q$, it must be below the product of q and $-q$, which is zero. This forces r to be zero, in contradiction to the assumption that r is non-zero. Conclusion: the difference $p - q$ is zero, so $p \leq q$, and therefore p is the least upper bound of Y. □

The density property is the requirement that forces all infinite sums existing in A to be left intact in the passage to B, so that A is a regular (Boolean) subalgebra of B (see Section 6.5).

Lemma 15.6. *A dense subalgebra of a Boolean algebra B is always a regular subalgebra of B.*

Proof. Let A be a dense subalgebra of B, and consider a subset Y of A with a supremum p in A. It is to be shown that p remains the supremum of Y in B. The argument is analogous to the final argument in the preceding proof.

Obviously, p is an upper bound of Y in B. Let q be any upper bound of Y in B, and suppose, for contradiction, that the difference $p-q$ is not zero. There must then be a non-zero element r in A that is below $p-q$, by the assumed density of A in B. The difference $p - r$ belongs to A, because p and r are both in A, and this difference is strictly below p, because r is non-zero and below p. We shall show that $p - r$ is also an upper bound of Y in A. This contradicts the assumption that p is the least upper bound of Y in A, so $p - q$ must in fact be zero. It follows that $p \leq q$ and therefore p is the least upper bound of Y, as claimed.

To see that $p - r$ is an upper bound of Y, fix an element s in Y. Certainly, s is below q, because q is assumed to be an upper bound of Y. Therefore, s must be disjoint from $-q$, and as a result, s must also be disjoint from the difference $p - q$. The element r is below this difference, so s must be disjoint from r, and therefore s must be below the complement $-r$. Of course, s is also below p, since p is an upper bound of Y, so s is below the difference $p - r$. Thus, $p - r$ is an upper bound of Y, as claimed. □

Lemma 15.6 implies that completions of a Boolean algebra A *preserve existing sums and products* in A in the sense that if a subset Y of A has a supremum (or an infimum) r in A, then r remains the supremum (or infimum) of Y in each completion of A.

Corollary 15.7. *A completion of a Boolean algebra A preserves all sums and products that exist in A.*

Proof. Suppose B is a completion of A. Condition (ii) in Definition 15.4 implies that A is a dense subalgebra of B, and therefore A is a regular subalgebra of B, by Lemma 15.6. By definition, this means that if a subset Y of A has a supremum r in A, then r remains the supremum of Y in B. Similarly, if Y has an infimum s in A, then s remains the infimum of Y in B. In more detail, since s is the infimum of Y in A, its complement $-s$ must be the supremum in A of the complementary set $\{-t : t \in Y\}$, because

$$-s = -\prod Y = \sum\{-t : t \in Y\}.$$

in A. Consequently, $-s$ must be the supremum of the complementary set in B, because A is a regular subalgebra of B. The computation

$$s = -(-s) = -\sum\{-t : t \in Y\} = \prod\{-(-t) : t \in Y\} = \prod Y$$

in B implies that s is the infimum of Y in B. □

The Boolean algebra B in Lemmas 15.5 and 15.6 is not required to be complete. In the presence of the density property, the completeness of B is equivalent to the weaker requirement that every subset of A has a supremum in B.

Lemma 15.8. *If X is a dense subset of a Boolean algebra B, then B is complete if and only if every subset of X has a supremum in B.*

Proof. If B is complete, then obviously every subset of X has a supremum in B. To establish the reverse implication, assume that every subset of X has a supremum in B, and consider an arbitrary subset Y of B. It is to be shown that Y has a supremum in B.

Each element q in Y is the supremum of the set Z_q of elements in X that are below q, by Lemma 15.5 and the assumed density of X. The union

$$Z = \bigcup \{Z_q : q \in Y\} \tag{1}$$

is also a subset of X, so it has a supremum p in B, by assumption. We now show that p is the supremum of Y in B. Certainly, p is above each element in the set Z_q, because p is the supremum of the set Z, which includes the set Z_q. Consequently, p is an upper bound of Z_q, so it must be above the least upper bound of Z_q, which is q. This is true for each q in Y, so p is an upper bound of Y. If t is any other upper bound of Y, then t is above q for each q in Y. Since q is an upper bound of the set Z_q, it follows that t is also an upper bound of Z_q. This is true for each q in Y, so t is an upper bound of the union set Z, by (1). But p is the least upper bound of Z, so $p \leq t$. This argument shows that p is below every upper bound of Y, so p is the least upper bound of Y. □

Fix a Boolean algebra A for the remainder of this section. The set of complete ideals in A is a Boolean algebra in its own right, by Theorem 15.3. Denote this algebra by C. It is not hard to check that C includes a natural copy of A as a regular subalgebra. The following lemma contains the heart of the argument.

Lemma 15.9. *For each element p in a Boolean algebra A, the set*

$$L_p = \{r \in A : r \leq p\}$$

is a complete ideal in A. The following statements are true in the Boolean algebra C of complete ideals in A, whenever p and q are in A.

(i) *The sum of L_p and L_q is L_t, where $t = p + q$.*
(ii) *The product of L_p and L_q is L_t, where $t = p \cdot q$.*
(iii) *The complement of L_p is L_t, where $t = -p$.*

Proof. It is easy to check that the set L_p is a complete ideal in A. Indeed, this set clearly contains zero and is downward closed. To check that it is closed under suprema, consider a subset X of L_p, and suppose

that X has a supremum r in A. The element p is an upper bound of the set L_p, so it is also an upper bound of the subset X. As r is the least upper bound of X, it follows that $r \leq p$ and consequently r belongs to L_p.

Turn next to the proof of (i). Write X for the complete ideal that is the sum, or join, of the ideals L_p and L_q in C. It is to be shown that

$$X = L_t, \tag{1}$$

where $t = p + q$. For the inclusion from left to right, observe that p and q are both below t, so the ideals L_p and L_q are certainly included in the ideal L_t, by the definitions of these ideals. Consequently, the join X of the two ideals is included in L_t. To establish the reverse inclusion, consider an element r in L_t, and observe that r is below t, by the definition of L_t, and therefore

$$r = r \cdot t = r \cdot (p + q) = r \cdot p + r \cdot q,$$

by Boolean algebra. The elements $r \cdot p$ and $r \cdot q$ belong to the ideals L_p and L_q respectively, by the definitions of these ideals, so the sum r of these two elements belongs to the join of the two ideals, which is X. Thus, L_t is included in X, so (1) holds.

Part (ii) follows from the equation

$$L_t = L_p \cap L_q,$$

where $t = p \cdot q$, since the intersection on the right side of this equation is just the product, or meet, of the ideals L_p and L_q in C (see Theorem 15.3). The equation is easily justified: for each element r in A,

$$\begin{array}{lll} r \in L_t & \text{if and only if} & r \leq t, \\ & \text{if and only if} & r \leq p \text{ and } r \leq q, \\ & \text{if and only if} & r \in L_p \text{ and } r \in L_q, \\ & \text{if and only if} & r \in L_p \cap L_q, \end{array}$$

by the definition of L_t, the definition of t, the definitions of L_p and L_q, and the definition of intersection.

From (i) and (ii) (with $q = -p$), and Boolean algebra, it follows that for each element p in A, the sum and product of the ideals L_p and L_{-p} in C are the ideals L_1 and L_0 respectively. These last two ideals are the unit and zero of C, so L_{-p} must be the complement of L_p in C. This proves (iii). □

Lemma 15.10. *The function φ defined by*

$$\varphi(r) = L_r$$

for each r in A is a monomorphism from A into C.

Proof. The fact that φ is a homomorphism from A into C is a direct consequence of Lemma 15.9. For example, if r is an element in A, and if t is the complement of r in A, then part (iii) of the lemma ensures that L_t is the complement of L_r in C. Consequently, $\varphi(t)$ is the complement of $\varphi(r)$, by the definition of φ, so φ preserves the operation of complement.

Similarly, if r and s are elements in A, and if t is their sum in A, then part (i) of the lemma ensures that L_t is the sum of the ideals L_r and L_s. Consequently, $\varphi(t)$ is the sum of $\varphi(r)$ and $\varphi(s)$ in C, by the definition of φ, so φ preserves the operation of addition.

If r is a non-zero element in A, then L_r is a non-trivial complete ideal, since it contains the element r. It follows that the function φ maps non-zero elements in A to non-zero elements in C, so it is monomorphism, by the Boolean analogue of Lemma 8.37. □

The Boolean algebra C of complete ideals in a Boolean algebra A is called the *completion embedding algebra* of A, and the mapping φ defined in Lemma 15.10 is called the *completion embedding* of A into C. The subalgebra of C that is the image of A under the mapping φ is an isomorphic copy of A and will be denoted by $\varphi(A)$.

Theorem 15.11. *The completion embedding algebra of a Boolean algebra A is the completion of the image of A under the completion embedding.*

Proof. The Boolean algebra C is complete, by Theorem 15.3, and Lemma 15.10 implies that $\varphi(A)$ is a subalgebra of C. Thus, condition (i) of Definition 15.4 holds. To prove that C is the completion of $\varphi(A)$, it remains to verify condition (ii) of the definition. In other words, it must be shown that $\varphi(A)$ is dense in C.

The non-zero elements in C are the complete ideals X in A that are different from the trivial ideal L_0, while the non-zero elements in $\varphi(A)$ are the principal ideals L_r for non-zero elements r in A (since r always belongs to L_r). A non-trivial complete ideal X must contain a non-zero element r, and the principal ideal L_r is obviously included in X, by the definition of L_r and the downward closure of X. Thus, every

non-zero element in C is above a non-zero element in $\varphi(A)$, so $\varphi(A)$ is dense in C. □

Corollary 15.12. *The completion embedding φ is a complete monomorphism from the Boolean algebra A into the completion embedding algebra C, and the image of A under φ is a regular subalgebra of C.*

Proof. The mapping φ is a monomorphism from A to C, and the image of A under φ is a dense subalgebra of C, by Theorem 15.11. Consequently, this image is a regular subalgebra of C, by Lemma 15.6. The monomorphism φ must therefore be complete, by the Boolean version of Lemma 7.9. □

It has been shown that A is embeddable into the algebra C, and that C is the completion of the image of A under this embedding. An application of the Exchange Principle (Theorem 7.15) yields a Boolean algebra that is the completion of A. The following *Existence Theorem* for completions of Boolean algebras has been proved.

Theorem 15.13. *Every Boolean algebra has a completion.*

The preceding completion of A was constructed as an isomorphic copy of the completion embedding algebra C. There is in fact a canonical isomorphism from each completion of A to C. The proof of this assertion begins with a generalization of Lemma 15.9 from elements in A to elements in an arbitrary completion of A.

Lemma 15.14. *Let B be a completion of a Boolean algebra A. For each element p in B, the set*

$$L_p = \{r \in A : r \leq p\}$$

is a complete ideal in A. Moreover, a subset X of A is a complete ideal in A if and only if $X = L_p$, where p is the sum of the set X in B. The following statements are true in the Boolean algebra C whenever p and q are elements in B.

(i) *The sum of L_p and L_q is L_t, where $t = p + q$.*
(ii) *The product of L_p and L_q is L_t, where $t = p \cdot q$.*
(iii) *The complement of L_p is L_t, where $t = -p$.*

Proof. The proof of the first assertion is identical to the proof of the corresponding part of Lemma 15.9, except that the element p is allowed to be in B, and not just in A. The details are left to the reader.

To prove the second assertion, consider a subset X of A, and put

$$p = \sum X \tag{1}$$

(the sum being formed in B). If $X = L_p$, then X is a complete ideal in A, by the first assertion of the lemma.

Assume now that X is a complete ideal in A. Every element r in X belongs to A, by the assumption on X, and r is below p, by (1), so r belongs to the set L_p, by the definition of this set. Consequently, X is included in L_p. To establish the reverse inclusion, consider an arbitrary element r in L_p. Certainly, r belongs to A and is below p, by the definition of the set L_p, so

$$r = r \cdot p = r \cdot (\sum X) = \sum \{r \cdot s : s \in X\}$$

in B, by Boolean algebra. The elements on the right side of this equation are all in A, as is r, so the equality of r with the sum on the right must hold in A as well (see the remark preceding the definition of a regular subalgebra in Section 6.5). Also, each product $r \cdot s$ on the right belongs to X, because X is an ideal in A, and s is in X, and r is in A. The set on the right is therefore a subset of X, and its sum exists in A and is equal to r, by the preceding observations. Since the ideal X is assumed to be complete, it follows that r must be in X, so L_p is included in X. Thus, $X = L_p$, which proves the second assertion of the lemma.

Turn now to the proof of (i). Let p and q be elements in B, and observe that

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{2}$$

by Lemma 15.5. Put

$$t = p + q \tag{3}$$

(this sum being formed in B). It is to be shown that L_t is the sum of the ideals L_p and L_q in the completion embedding algebra C. Since the elements p and q are both below t, by (3), the ideals L_p and L_q are both included in L_t, by the definitions of these ideals.

Consider now any complete ideal X in A that includes both L_p and L_q, with the goal of proving that X must also include L_t. To this end, let r be any element in L_t, and put

$$Y = \{r \cdot s : s \in L_p\} \qquad \text{and} \qquad Z = \{r \cdot s : s \in L_q\}. \tag{4}$$

The sets Y and Z are subsets of L_p and L_q respectively, since the latter are ideals and are therefore closed under multiplication by elements from A. Consequently, both sets are included in X, by the hypotheses on X. The element r is in L_t and is therefore below t, so

$$\begin{aligned} r = r \cdot t &= r \cdot (p + q) \\ &= r \cdot p + r \cdot q \\ &= r \cdot (\sum L_p) + r \cdot (\sum L_q) \\ &= \sum\{r \cdot s : s \in L_p\} + \sum\{r \cdot s : s \in L_q\} \\ &= \sum Y + \sum Z \\ &= \sum(Y \cup Z) \end{aligned}$$

in B, by Boolean algebra, (3), (2), the complete distributivity of multiplication, and (4). This computation shows that the equation

$$r = \sum(Y \cup Z)$$

holds in B. Since all of the elements involved in this equation belong to A, it follows that the equation holds in A as well. The ideal X is assumed to be complete, and it has been shown that $Y \cup Z$ is a subset of X with supremum r in A. Consequently, r must belong to X, so L_t is included in X.

The preceding argument shows that every complete ideal which includes L_p and L_q must also include L_t. In other words, L_t is the smallest complete ideal that includes the two ideals L_p and L_q. This is just the definition of the sum of these two ideals in C, by Theorem 15.3, so part (i) of the lemma has been proved.

The proofs of parts (ii) and (iii) of the lemma are identical to the proofs of the corresponding parts of Lemma 15.9, except that the elements p and q are assumed to be in B, and not just in A. The details are left to the reader. □

We can now state in a precise way the assertion that every completion of A is canonically isomorphic to the completion embedding algebra C.

Theorem 15.15. *Let B be a completion of a Boolean algebra A. The function ψ defined by*

$$\psi(p) = L_p$$

for each p in B is an isomorphism from B to the completion embedding algebra C, and ψ extends the completion embedding φ.

Proof. Lemma 15.14 implies that ψ is a well-defined bijection from B to C. For example, if X is an element in C—that is to say, if X is a complete ideal in A—then there is an element p in B (namely the sum of the set X) such that $X = L_p$, by the second assertion of the lemma, and therefore ψ maps p to X, by the definition of ψ. Thus, ψ is onto. If $\psi(p) = \psi(q)$ for some elements p and q in B, then $L_p = L_q$, by the definition of ψ, and therefore

$$p = \sum L_p = \sum L_q = q,$$

by Boolean algebra and the lemma. Thus, ψ is one-to-one.

To prove that ψ preserves the operation of addition, let p and q be elements in B, and write $t = p+q$. The sum of the two complete ideals

$$\psi(p) = L_p \qquad \text{and} \qquad \psi(q) = L_q$$

in C is the ideal $\psi(t) = L_t$, by part (i) of Lemma 15.14, so

$$\psi(p + q) = \psi(t) = L_t = \psi(p) + \psi(q).$$

A completely analogous argument, using part (iii) of the lemma, shows that ψ preserves the operation of complement. Consequently, ψ is an isomorphism from B to C. A comparison of the definitions of ψ and the completion embedding φ in Lemma 15.10 immediately leads to the conclusion that ψ is an extension of φ. □

Completions of a Boolean algebra are uniquely determined up to isomorphisms in a strong sense, as the following *Uniqueness Theorem* for completions of Boolean algebras asserts.

Theorem 15.16. *Any two completions of a Boolean algebra A are isomorphic via a mapping that is the identity function on A.*

Proof. Suppose B and D are completions of a Boolean algebra A. Let C be the completion embedding algebra of A, and φ the completion embedding of A into C. Take ψ and ϑ to be the canonical isomorphisms from B and D respectively to C that are defined in Theorem 15.15. Thus,

$$\psi(p) = L_p \qquad \text{and} \qquad \vartheta(q) = L_q \tag{1}$$

for elements p in B and q in D. Define ϱ to be the composition $\vartheta^{-1} \circ \psi$,

$$B \xrightarrow{\psi} C \xleftarrow{\vartheta} D \quad .$$

Thus,

$$\varrho(p) = q \qquad \text{if and only if} \qquad L_p = L_q,$$

by (1). A composition of isomorphisms is an isomorphism, so ϱ is an isomorphism from B to D, by Theorem 15.15. Moreover, if p is in A, then

$$\psi(p) = \varphi(p) \qquad \text{and} \qquad \vartheta(p) = \varphi(p), \tag{2}$$

again by Theorem 15.15, and therefore

$$\varrho(p) = \vartheta^{-1}(\psi(p)) = \vartheta^{-1}(\varphi(p)) = \vartheta^{-1}(\vartheta(p)) = p,$$

by the definition of ϱ and (2). Thus, ϱ is the identity function on A. □

Theorem 15.16 justifies the common practice of referring to *the* completion of a Boolean algebra.

15.3 Completions of Boolean algebras with operators

We turn now to the study of completions of Boolean algebras with operators.

Definition 15.17. A *completion* of a Boolean algebra with operators $\mathfrak{A}$ is a Boolean algebra with additional operations $\mathfrak{B}$ (of the same similarity type as $\mathfrak{A}$) that possesses the following properties.

(i) $\mathfrak{B}$ is complete, and $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.
(ii) Every non-zero element in $\mathfrak{B}$ is above a non-zero element in $\mathfrak{A}$.
(iii) For any elements p and q in $\mathfrak{B}$,

$$p\,;q = \sum\{r\,;s : r \leq p \text{ and } s \leq q \text{ and } r, s \in A\},$$
$$p^{\smile} = \sum\{r^{\smile} : r \leq p \text{ and } r \in A\}.$$

□

This definition does not require the operations $;$ and $^{\smile}$ in a completion of a Boolean algebra with operators $\mathfrak{A}$ to be distributive, much less quasi-completely distributive. The reason is that the quasi-complete distributivity of these operations is derivable from the conditions in the definition whenever the operators in $\mathfrak{A}$ are quasi-completely distributive. Also, the definition does not require a completion of $\mathfrak{A}$ to satisfy

the same equations as $\mathfrak{A}$. In particular, it does not require the completion of a relation algebra to be a relation algebra. We shall see later, however, that it is in fact a relation algebra. Conditions (i) and (ii) in the definition coincide with the same conditions in the definition of the completion of a Boolean algebra (see Definition 15.4). Consequently, if $\mathfrak{B}$ is a completion of $\mathfrak{A}$, then the Boolean part of $\mathfrak{B}$ is a completion of the Boolean part of $\mathfrak{A}$. It follows that notions such as density, and results such as Lemmas 15.5, 15.6, 15.8, and Corollary 15.7, apply automatically to Boolean algebras with operators, and we shall use them in this way without further mention.

In order to elaborate on the role of condition (iii) in the preceding definition, it is convenient to recall some notation that was already employed in the chapter on canonical extensions. For any subsets X and Y of a Boolean algebra with operators $\mathfrak{A}$, write

$$X + Y, \qquad X \cdot Y, \qquad X\,;Y, \qquad X^{\smile}$$

for the operations on complexes (subsets) of $\mathfrak{A}$ induced by the operations of $\mathfrak{A}$ (see the remarks following Definition 14.12).

The density property in condition (ii) of Definition 15.17 implies that $p = \sum L_p$ for each element p in the completion, by Lemma 15.5, where L_p is the set of elements in $\mathfrak{A}$ that are below p. In terms of this notation, condition (iii) just says that

$$p\,;q = \sum(L_p\,;L_q) \qquad \text{and} \qquad p^{\smile} = \sum(L_p^{\smile}).$$

It does not follow automatically that if $p = \sum X$ and $q = \sum Y$ (in $\mathfrak{B}$) for some arbitrary non-empty subsets X and Y of $\mathfrak{B}$, then

$$p\,;q = \sum(X\,;Y) \qquad \text{and} \qquad p^{\smile} = \sum(X^{\smile}).$$

The role of condition (iii) in Definition 15.17 is to force these equations to hold whenever the operators in $\mathfrak{A}$ are quasi-complete. The proof of this assertion requires two lemmas.

Lemma 15.18. *If $\mathfrak{B}$ is a completion of a Boolean algebra with operators $\mathfrak{A}$, then the operations $;$ and $^{\smile}$ in $\mathfrak{B}$ are monotone.*

Proof. Suppose p, q, r, and s are elements in $\mathfrak{B}$ with $r \leq p$ and $s \leq q$. These inequalities imply that

$$L_r \subseteq L_p \qquad \text{and} \qquad L_s \subseteq L_q,$$

by the definitions of the sets involved. It follows that

$$L_r \,;\, L_s \subseteq L_p \,;\, L_q,$$

by the definition of the complex product of two sets, and therefore

$$\sum(L_r \,;\, L_s) \leq \sum(L_p \,;\, L_q), \tag{1}$$

by Boolean algebra. Condition (iii) in Definition 15.17 implies that

$$r \,;\, s = \sum(L_r \,;\, L_s) \qquad \text{and} \qquad p \,;\, q = \sum(L_p \,;\, L_q). \tag{2}$$

Combine (1) and (2) to conclude that $r \,;\, s \leq p \,;\, q$. The proof of the corresponding inequality for the operation $^\smile$ is similar, but easier. □

The next lemma says that if the operators in $\mathfrak{A}$ are quasi-complete, then the operations $;$ and $^\smile$ in a completion of $\mathfrak{A}$ are quasi-complete with respect to sets of elements in $\mathfrak{A}$.

Lemma 15.19. *Suppose $\mathfrak{B}$ is a completion of a Boolean algebra with quasi-complete operators $\mathfrak{A}$. For any non-empty subsets X and Y of $\mathfrak{A}$, if $p = \sum X$ and $q = \sum Y$ (in $\mathfrak{B}$), then*

$$p \,;\, q = \sum(X \,;\, Y) \qquad \textit{and} \qquad p^\smile = \sum(X^\smile).$$

Proof. Focus on the case of the operation $;$. If r and s are elements in X and Y respectively, then $r \leq p$ and $s \leq q$, by the assumptions on p and q, and therefore $r \,;\, s \leq p \,;\, q$, by the monotony law for the operation $;$ in $\mathfrak{B}$ (Lemma 15.18). Sum this last inequality over all elements r in X and s in Y to obtain

$$\sum(X \,;\, Y) = \sum\{r \,;\, s : r \in X \text{ and } s \in Y\} \leq p \,;\, q.$$

To establish the reverse inequality, consider elements u in L_p and v in L_q. We have $u \leq p$ and $v \leq q$, by the definitions of the sets L_p and L_q, and therefore

$$u = u \cdot p = u \cdot (\sum X) = \sum\{u \cdot r : r \in X\}$$

and

$$v = v \cdot q = v \cdot (\sum Y) = \sum\{v \cdot s : s \in Y\}$$

in $\mathfrak{B}$, by Boolean algebra. The elements u and v belong to $\mathfrak{A}$, and the sets X and Y are subsets of $\mathfrak{A}$, so the equations

$$u = \sum\{u \cdot r : r \in X\} \qquad v = \sum\{v \cdot s : s \in Y\} \tag{1}$$

involve only elements in $\mathfrak{A}$. The validity of these equations in $\mathfrak{B}$ therefore implies their validity in $\mathfrak{A}$. The operator $;$ in $\mathfrak{A}$ is assumed to be quasi-complete, and the sets on the right sides of the equations in (1) are non-empty because X in Y are assumed to be non-empty, so

$$\begin{aligned} u ; v &= (\sum\{u \cdot r : r \in X\}) ; (\sum\{v \cdot s : s \in Y\}) \\ &= \sum\{(u \cdot r) ; (v \cdot s) : r \in X \text{ and } s \in Y\} \end{aligned} \tag{2}$$

in $\mathfrak{A}$. All sums in $\mathfrak{A}$ are preserved under the passage to a completion, by Corollary 15.7, so (2) must also hold in $\mathfrak{B}$. Since

$$u \cdot r \leq r \qquad \text{and} \qquad v \cdot s \leq s,$$

we may apply the monotony law for $;$ to obtain

$$(u \cdot r) ; (v \cdot s) \leq r ; s. \tag{3}$$

Sum (3) over all r in X and s in Y to arrive at

$$\begin{aligned} &\sum\{(u \cdot r) ; (v \cdot s) : r \in X \text{ and } s \in Y\} \\ &\qquad \leq \sum\{r ; s : r \in X \text{ and } s \in Y\} \end{aligned} \tag{4}$$

in $\mathfrak{B}$. Combine (2) and (4) to obtain

$$u ; v \leq \sum\{r ; s : r \in X \text{ and } s \in Y\} \tag{5}$$

in $\mathfrak{B}$. Sum the left side of (5) over all u in L_p and v in L_q to conclude that

$$\sum\{u ; v : u \in L_p \text{ and } v \in L_q\} \leq \sum\{r ; s : r \in X \text{ and } s \in Y\}.$$

In the notation of complex products, the preceding inequality may be written as

$$\sum(L_p ; L_q) \leq \sum(X ; Y). \tag{6}$$

Condition (iii) in Definition 15.17 implies that the left side of (6) is equal to $p ; q$. Consequently,

$$p\,;q \leq \sum(X\,;Y),$$

as desired. This completes the proof of the first equation in the conclusion of the lemma. The second equation is established by a similar but easier argument. □

We are now in a position to prove that for a Boolean algebra with quasi-complete operators $\mathfrak{A}$, if density condition (ii) in Definition 15.17 holds, then condition (iii) is equivalent to the requirement that the operations in $\mathfrak{B}$ are quasi-complete.

Lemma 15.20. *Let $\mathfrak{A}$ be a Boolean algebra with quasi-completely distributive operators, and $\mathfrak{B}$ a complete Boolean algebra with additional operations (of the same similarity type as $\mathfrak{A}$). If $\mathfrak{A}$ is a dense subalgebra of $\mathfrak{B}$, then the operations ; and $^\smile$ in $\mathfrak{B}$ are quasi-completely distributive if and only if condition* (iii) *in Definition 15.17 holds.*

Proof. Focus on the case of the binary operation ;. Suppose first that the operation ; in $\mathfrak{B}$ is quasi-complete. If p and q are elements in $\mathfrak{B}$, then

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{1}$$

by Lemma 15.5. Notice that the sets on the right sides of these equations are non-empty, since they always contain zero. Consequently,

$$\begin{aligned} p\,;q &= (\textstyle\sum L_p)\,;(\sum L_q) \\ &= \textstyle\sum\{r\,;s : r \in L_p \text{ and } s \in L_q\} = \sum(L_p\,;L_q), \end{aligned}$$

by (1), the assumed quasi-complete distributivity of the operation ;, and the definition of the complex product of subsets of $\mathfrak{A}$. Thus, the first equation in Definition 15.17(iii) holds.

Assume now that the first equation in Definition 15.17(iii) holds. Together with the initial hypotheses of the lemma, this assumption implies that $\mathfrak{B}$ is a completion of $\mathfrak{A}$. Let X and Y be arbitrary non-empty subsets of $\mathfrak{B}$ (and not just non-empty subsets of $\mathfrak{A}$, as was the case in Lemma 15.19), and write

$$p = \sum X \qquad \text{and} \qquad q = \sum Y. \tag{2}$$

For each element u in X, we have $u = \sum L_u$, by Lemma 15.5, and therefore

$$p = \textstyle\sum\{u : u \in X\} = \sum\{\sum L_u : u \in X\} = \sum(\bigcup_{u\in X} L_u), \tag{3}$$

by (2) and the general associative law for addition. (Warning: the set $\bigcup_{u\in X} L_u$ may be different from the set L_p of all elements that are below p.) Similarly,

$$q = \sum(\bigcup_{v\in Y} L_v). \tag{4}$$

The sets

$$\bigcup_{u\in X} L_u \qquad \text{and} \qquad \bigcup_{v\in Y} L_v \tag{5}$$

are subsets of $\mathfrak{A}$. They are non-empty because the sets X and Y are assumed to be non-empty, and because the sets L_u and L_v always contain the element zero. Apply Lemma 15.19 (with X and Y respectively replaced by the two sets in (5)), in conjunction with (3) and (4), to arrive at

$$\begin{aligned} p\,;q &= [\,\textstyle\sum(\bigcup_{u\in X} L_u)]\,;[\sum(\bigcup_{v\in Y} L_v)] \\ &= \textstyle\sum[(\bigcup_{u\in X} L_u)\,;(\bigcup_{v\in Y} L_v)] \\ &= \textstyle\sum(\bigcup\{L_u\,;L_v : u\in X \text{ and } v\in Y\}). \end{aligned} \tag{6}$$

(The final equality uses the definition of the complex product of two sets, which implies that this complex multiplication distributes over arbitrary unions.)

On the other hand,

$$u\,;v = \textstyle\sum(L_u\,;L_v), \tag{7}$$

by condition (iii), so

$$\begin{aligned} \textstyle\sum(X\,;Y) &= \textstyle\sum\{u\,;v : u\in X \text{ and } v\in Y\} \\ &= \textstyle\sum\{\sum(L_u\,;L_v) : u\in X \text{ and } v\in Y\} \\ &= \textstyle\sum(\bigcup\{L_u\,;L_v : u\in X \text{ and } v\in Y\}), \end{aligned} \tag{8}$$

by the definition of the complex product of two sets, (7), and the general associative law for addition. A comparison of (6) and (8), together with (2), leads to the conclusion that

$$(\textstyle\sum X)\,;(\sum Y) = p\,;q = \sum(X\,;Y).$$

This proves that the operation $;$ in $\mathfrak{B}$ is quasi-complete.

The proof that the second equation in Definition 15.17(iii) is equivalent to the quasi-complete distributivity of the operation $\breve{}$ in $\mathfrak{B}$ is similar but easier. □

The requirement in Lemma 15.20 that the operators of $\mathfrak{A}$ be quasi-complete is really needed. Indeed, if the operators of the completion $\mathfrak{B}$ are quasi-complete, then the operators of $\mathfrak{A}$ must also be quasi-complete, because $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{B}$, by Lemma 15.6 (see Exercise 6.28).

We now take up the question of the existence of completions. Fix a Boolean algebra with quasi-complete operators $\mathfrak{A}$ (of the same similarity type as a relation algebra). Take

$$(B\,,\,+\,,\,-)$$

be the completion of the Boolean part of $\mathfrak{A}$; such a completion exists by Theorem 15.13. Define a binary operation $;$ and a unary operation $^\smile$ on B by

$$p\,;q = \sum\{r\,;s : r, s \in A \text{ and } r \leq p,\ s \leq q\},$$
$$p^\smile = \sum\{r^\smile : r \in A \text{ and } r \leq p\},$$

for all elements p and q in B. (The operations $;$ and $^\smile$ on the right sides of these equations are those of $\mathfrak{A}$, while the ones on the left are those being defined in B. The suprema on the right are formed in B.) These operations are called the *completions* of the corresponding operations in $\mathfrak{A}$. Put

$$\mathfrak{B} = (B\,,\,+\,,\,-\,,\,;\,,\,^\smile\,,\,1'),$$

where 1' is the identity element in $\mathfrak{A}$.

Lemma 15.21. *$\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.*

Proof. The Boolean part of $\mathfrak{A}$ is certainly a subalgebra of the Boolean part of $\mathfrak{B}$, by condition (i) in Definition 15.4 and the choice of the Boolean part of $\mathfrak{B}$ as the completion of the Boolean part of $\mathfrak{A}$. The element 1' in both algebras is the same, by the definition of $\mathfrak{B}$.

To see that the operation $;$ in $\mathfrak{A}$ is the restriction of the corresponding operation in $\mathfrak{B}$, consider elements p and q in $\mathfrak{A}$. They are clearly the largest elements in $\mathfrak{A}$ that are below p and q respectively, so the relative product $p\,;q$ formed in $\mathfrak{A}$ is the largest element in the set

$$\{r\,;s : r, s \in A \text{ and } r \leq p,\ s \leq q\}, \tag{1}$$

by the monotony law for the operation $;$ (Lemma 2.3). It follows that this relative product is the supremum in $\mathfrak{A}$ of the set in (1). Since the

Boolean part of $\mathfrak{B}$ is the completion of the Boolean part of $\mathfrak{A}$, suprema must be preserved under the passage from $\mathfrak{A}$ to $\mathfrak{B}$, by Corollary 15.7. Consequently, the relative product $p\,;q$ formed in $\mathfrak{A}$ is the supremum in $\mathfrak{B}$ of the set in (1). The relative product $p\,;q$ formed in $\mathfrak{B}$ is, by definition, the supremum in $\mathfrak{B}$ of the set in (1). Consequently, the relative product $p\,;q$ in $\mathfrak{B}$ coincides with the relative product $p\,;q$ in $\mathfrak{A}$. A similar argument shows that the values of $p^\smile$ in $\mathfrak{A}$ and in $\mathfrak{B}$ coincide. Thus, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. □

We are now ready to prove the following *Existence Theorem* for completions of Boolean algebras with quasi-complete operators.

Theorem 15.22. *Every Boolean algebra with quasi-complete operators has a completion with quasi-complete operators.*

Proof. Assume $\mathfrak{A}$ is a Boolean algebra with quasi-complete operators, and let $\mathfrak{B}$ be the algebra constructed in terms of $\mathfrak{A}$ before Lemma 15.21. Thus, the Boolean part of $\mathfrak{B}$ is the completion of the Boolean part of $\mathfrak{A}$, and the additional operations in $\mathfrak{B}$ are defined according to the formulas given in Definition 15.17(iii). It must be verified that $\mathfrak{B}$ satisfies conditions (i)—(iii) of Definition 15.17. The algebra $\mathfrak{B}$ is complete by the choice of the Boolean part of $\mathfrak{B}$ (and Definition 15.4(i)), and $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, by Lemma 15.21, so condition (i) is satisfied. Every non-zero element in $\mathfrak{B}$ is above a non-zero element $\mathfrak{A}$, by the choice of $\mathfrak{B}$ (and Definition 15.4(ii)), so condition (ii) is satisfied. Condition (iii) is satisfied by the very definition of the operations $;$ and $^\smile$ in $\mathfrak{B}$. Lemma 15.20 implies that the additional operations of $\mathfrak{B}$ are quasi-complete. □

Just as in the Boolean case, completions are uniquely determined in a very strong sense, as the following *Uniqueness Theorem* for completions of Boolean algebras with quasi-complete operators asserts.

Theorem 15.23. *Any two completions of a Boolean algebra with quasi-complete operators* $\mathfrak{A}$ *are isomorphic via a mapping that is the identity function on* $\mathfrak{A}$.

Proof. Consider two completions of $\mathfrak{A}$, say $\mathfrak{B}$ and $\mathfrak{D}$. The Boolean parts of $\mathfrak{B}$ and $\mathfrak{D}$ are completions of the Boolean part of $\mathfrak{A}$, by the remarks following Definition 15.17, so there must be a Boolean isomorphism ψ from $\mathfrak{B}$ to $\mathfrak{D}$ that is the identity function on $\mathfrak{A}$, by Theorem 15.16. In particular, ψ must map the distinguished constant in $\mathfrak{B}$ to the

distinguished constant in $\mathfrak{D}$. Indeed, each of these constants coincides with the distinguished constant 1' in $\mathfrak{A}$, by the assumption that $\mathfrak{A}$ is a subalgebra of both $\mathfrak{B}$ and $\mathfrak{D}$ (condition (i) in Definition 15.17), and ψ maps each element in $\mathfrak{A}$ to itself.

To see that ψ preserves the operation $;$, consider elements p and q in $\mathfrak{B}$, and let L_p and L_q be the sets of elements in $\mathfrak{A}$ that are below p and q respectively. We have

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{1}$$

by Lemma 15.5, where these sums are formed in $\mathfrak{B}$. The sets L_p and L_q are subsets of $\mathfrak{A}$, and ψ is the identity function on $\mathfrak{A}$, so ψ is the identity function on each of these sets. In particular,

$$\psi(L_p) = L_p \qquad \text{and} \qquad \psi(L_q) = L_q, \tag{2}$$

where the sets on the left sides of these equations denote the images of the sets L_p and L_q under the mapping ψ. Similarly,

$$p\,;q = \sum(L_p\,;L_q), \tag{3}$$

by condition (iii) in Definition 15.17, where the sum on the right is formed in $\mathfrak{B}$ and

$$L_p\,;L_q = \{r\,;s : r \in L_p \text{ and } L_q\}. \tag{4}$$

The set in (4) is also a subset of $\mathfrak{A}$, so the mapping ψ is the identity function on this set as well. In particular,

$$\psi(L_p\,;L_q) = L_p\,;L_q. \tag{5}$$

From (1), the Boolean isomorphism properties of ψ, and (2) we obtain

$$\psi(p) = \psi(\textstyle\sum L_p) = \sum \psi(L_p) = \sum L_p \tag{6}$$

and

$$\psi(q) = \psi(\textstyle\sum L_q) = \sum \psi(L_q) = \sum L_q, \tag{7}$$

where the first sums in (6) and (7) are formed in $\mathfrak{B}$, and the second and third are formed in $\mathfrak{D}$. Similarly, from (3) and (5) we obtain

$$\psi(p\,;q) = \psi(\textstyle\sum(L_p\,;L_q)) = \sum \psi(L_p\,;L_q) = \sum(L_p\,;L_q). \qquad (8)$$

The operation ; in $\mathfrak{D}$ is quasi-completely distributive, by the assumption that $\mathfrak{D}$ is a completion of $\mathfrak{A}$ and by Lemma 15.20. Also, the sets L_p and L_q are non-empty, because each of them contains the element zero. Consequently,

$$\begin{aligned}(\textstyle\sum L_p)\,;(\sum L_q) &= (\sum\{r : r \in L_p\})\,;(\sum\{s : s \in L_q\}) \\ &= \textstyle\sum\{r\,;s : r \in L_p \text{ and } s \in L_q\} = \sum(L_p\,;L_q) \qquad (9)\end{aligned}$$

in $\mathfrak{D}$. Combine (6)–(9) to arrive at

$$\psi(p)\,;\psi(q) = (\textstyle\sum L_p)\,;(\sum L_q) = \sum(L_p\,;L_q) = \psi(p\,;q).$$

This shows that ψ preserves the operation ;. A similar, but easier, argument shows that ψ also preserves the operation $\smile$. Conclusion: ψ is an isomorphism from $\mathfrak{B}$ to $\mathfrak{D}$ that is the identity function on $\mathfrak{A}$. □

The preceding theorem justifies the common practice of referring to *the* completion of a Boolean algebra with quasi-complete operators.

15.4 The preservation theorems

We have seen that every Boolean algebra with quasi-complete operators $\mathfrak{A}$ has a uniquely determined completion with quasi-complete operators, but we don't yet know much about the properties that this completion inherits from $\mathfrak{A}$. In particular, we don't yet know that the completion of a relation algebra is itself a relation algebra. What we need are some general theorems about properties that are preserved under the passage to completions.

In the remainder of the discussion, let $\mathfrak{A}$ be a fixed Boolean algebra with quasi-complete operators, and $\mathfrak{B}$ the completion of $\mathfrak{A}$. Two lemmas concerning positive polynomials are needed. (The notion of a positive polynomial is defined near the beginning of Section 14.4.) The first is the analogue for completions of Lemma 14.29, and its formulation uses some of the notation introduced before that lemma. The proof is identical to the proof of that lemma, and is left as an exercise.

Lemma 15.24. *Every positive polynomial of rank n in $\mathfrak{B}$ is monotone in the sense that $p \leq q$ implies $\gamma(p) \leq \gamma(q)$ for all p, q in B^n.*

The second lemma is the analogue of Lemma 14.31. It's formulation and proof use the following extension of the notation L_p that was introduced in Lemma 15.9. For every sequence p in B^n, put

$$L_p = \{r \in A^n : r \leq p\} = \{r \in A^n : r_i \leq p_i \text{ for } 0 \leq i < n\}.$$

Lemma 15.25. *If γ is a positive polynomial of rank n in $\mathfrak{B}$, then*

$$\gamma(p) = \sum\{\gamma(r) : r \in L_p\}$$

for every p in B^n.

The proof of this lemma is very similar to the proof of Lemma 14.31, but all references in that proof to the set K_p must be replaced by references to the set L_p, and all references to Lemmas 14.29 and 14.26 must be replaced by references to Lemmas 15.24 and 15.20 respectively. The details are left as an exercise.

With the help of Lemma 15.25 it is easy to establish the *First Preservation Theorem* for completions.

Theorem 15.26. *Any positive equation that holds in a Boolean algebra with quasi-complete operators $\mathfrak{A}$ continues to hold in the completion of $\mathfrak{A}$.*

Proof. Consider an arbitrary positive equation ε, say with variables among $v_0, \ldots, v_{n-1}$, and assume that ε is valid in $\mathfrak{A}$. The goal is to show that ε is valid in the algebra $\mathfrak{B}$ that is the completion of $\mathfrak{A}$. To this end, let σ and τ be the (positive) polynomials of rank n in $\mathfrak{B}$ that are induced by the terms on the right and left sides of ε. Since $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, the polynomials of rank n in $\mathfrak{A}$ that are induced by the right and left sides of ε are just the restrictions of σ and τ to $\mathfrak{A}$.

The assumption that ε is valid in $\mathfrak{A}$ means that

$$\sigma(r) = \tau(r)$$

for all sequences r in A^n. In particular, for each sequence p in B^n,

$$\{\sigma(r) : r \in L_p\} = \{\tau(r) : r \in L_p\},$$

since L_p is a subset of A^n. Form the sums in $\mathfrak{B}$ of both sides of this equation, and use Lemma 15.25, to arrive at

$$\sigma(p) = \sum\{\sigma(r) : r \in L_p\} = \sum\{\tau(r) : r \in L_p\} = \tau(p).$$

Thus, σ and τ agree on all sequences of n elements from $\mathfrak{B}$, so ε is valid in $\mathfrak{B}$. □

The only property of the operations ; and $^\smile$ in the algebra $\mathfrak{A}$ that is needed in the lemmas leading up to Theorem 15.26 is the quasi-complete distributivity of these operations. Consequently, Lemmas 15.18–15.25 and Theorem 15.26 can be extended without difficulty to Boolean algebras with quasi-complete operators of arbitrary ranks. Consider, as an example, a quasi-complete operator O of rank n on the universe of $\mathfrak{A}$. Let $\mathfrak{A}^*$ be the expanded algebra obtained from $\mathfrak{A}$ by adjoining O as a new fundamental operation. The definition of a completion $\mathfrak{B}^*$ of $\mathfrak{A}^*$ is obtained from Definition 15.17 by adjoining to condition (iii) the requirement that the corresponding operation O in $\mathfrak{B}^*$ satisfies the condition

$$O(p) = \sum\{O(r) : r \leq p \text{ and } r \in A^n\}$$

for every sequence p of n elements in $\mathfrak{B}^*$. All the results from Lemma 15.18 through the First Preservation Theorem 15.26 continue to hold with minimal changes in their proofs.

As a concrete example, consider the unary discriminator O on the universe of $\mathfrak{A}$ that was defined and discussed after the First Preservation Theorem 14.32 for canonical extensions. As was pointed out there, this operation is completely distributive. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$, and define a unary *completion* of the operation O on the universe of $\mathfrak{B}$ by

$$O(p) = \sum\{O(r) : r \leq p \text{ and } r \in A\}$$

for every element p in $\mathfrak{B}^*$ (where the occurrence of O on the right side of this equation denotes the unary discriminator on $\mathfrak{A}$, and the occurrence on the left side denotes the completion of the unary discriminator on $\mathfrak{A}$ that is being defined on $\mathfrak{B}$). The algebra $\mathfrak{B}^*$ obtained by adjoining this operation to $\mathfrak{B}$ is the completion of $\mathfrak{A}^*$, by Theorem 15.22 (in the version applicable to $\mathfrak{A}^*$ and $\mathfrak{B}^*$). It is easy to check that the new operation O in $\mathfrak{B}$ is the unary discriminator on $\mathfrak{B}$, that is to say, it assumes the value 0 on the zero element, and it assumes the value 1 on all non-zero elements in $\mathfrak{B}$.

The remarks in the preceding paragraphs imply that every positive equation true in $\mathfrak{A}^*$ must also be true in $\mathfrak{B}^*$, by the First Preservation Theorem 15.26 applied to the expanded algebras $\mathfrak{A}^*$ and $\mathfrak{B}^*$. By imitating the proof of the Second Preservation Theorem 14.34 for canonical extensions, with all references to canonical extensions replaced by references to completions, and all references to Theorem 14.32 replaced

by references to Theorem 15.26, one arrives at a proof of the following analogue of Theorem 14.34, the *Second Preservation Theorem* for completions.

Theorem 15.27. *Let ε be any Boolean combination of positive equations of the form $\varrho = 0$ (with ϱ positive), and σ and τ any positive terms, in the augmented language of a Boolean algebra with quasi-complete operators $\mathfrak{A}$. If the implication*

$$\varepsilon \to (\sigma = \tau)$$

holds in $\mathfrak{A}$, then it holds in the completion of $\mathfrak{A}$.

15.5 Applications to relation algebras

We have seen that every Boolean algebra with quasi-complete operators has a uniquely determined completion. A relation algebra is a Boolean algebra with operators that are not only quasi-complete, but in fact complete (see Lemmas 4.2 and 4.16). It follows that every relation algebra has a uniquely determined completion. The two preservation theorems imply that this completion is itself a relation algebra.

Theorem 15.28. *Every relation algebra $\mathfrak{A}$ has a completion that is uniquely determined up to isomorphisms that are the identity function on $\mathfrak{A}$, and this completion is a relation algebra.*

The proof is very similar to the proof that every relation algebra has a uniquely determined canonical extension that is a relation algebra (Theorem 14.35). References in that proof to the Existence Theorem, the Uniqueness Theorem, and the First and Second Preservation Theorems for canonical extensions must be replaced with references to the corresponding theorems for completions. The details are left as an exercise.

As in the case of canonical extensions, it is natural to ask what other properties of elements, sets of elements, and algebras are preserved under the passage to completions. Equationally defined properties of elements are always preserved. For example, a function or an ideal element in a relation algebra $\mathfrak{A}$ remains a function or an ideal element in the completion of $\mathfrak{A}$. A property of elements that is not of this form, but that is still preserved, is the property of being an atom. In fact, a stronger result is true than in the case of canonical extensions.

Lemma 15.29. *If* $\mathfrak{B}$ *is the completion of a relation algebra* $\mathfrak{A}$, *then an element in* $\mathfrak{B}$ *is an atom if and only if it is already an atom in* $\mathfrak{A}$. *Consequently*, $\mathfrak{B}$ *is atomic if and only if* $\mathfrak{A}$ *is atomic.*

Proof. Every element p in the completion $\mathfrak{B}$ is the sum of the elements in $\mathfrak{A}$ that are below p, by Lemma 15.5. This immediately implies that an atom in $\mathfrak{A}$ must remain an atom in $\mathfrak{B}$. On the other hand, if p is an atom in $\mathfrak{B}$, then the set of elements in $\mathfrak{B}$ that are below p must contain exactly one non-zero element, namely p itself, by the definition of an atom. Since p is above a non-zero element in $\mathfrak{A}$, by Definition 15.17(ii), it follows that p must belong to $\mathfrak{A}$ and therefore must be an atom in $\mathfrak{A}$.

Assume now that $\mathfrak{A}$ is atomic. Every non-zero element p in $\mathfrak{B}$ is above a non-zero element r in $\mathfrak{A}$, by Definition 15.17(ii), and every non-zero element r in $\mathfrak{A}$ is above an atom s in $\mathfrak{A}$, by the assumption that $\mathfrak{A}$ is atomic. The atoms in $\mathfrak{A}$ remain atoms in $\mathfrak{B}$, by the remarks of the preceding paragraph, so every non-zero element p in $\mathfrak{B}$ is above an atom s in $\mathfrak{B}$. Consequently, $\mathfrak{B}$ is atomic.

In the reverse direction, if $\mathfrak{B}$ is atomic, then every non-zero element in $\mathfrak{B}$—and in particular, every non-zero element in $\mathfrak{A}$—must be above an atom in $\mathfrak{B}$. The atoms in $\mathfrak{B}$ are precisely the atoms in $\mathfrak{A}$, by the observations of the first paragraph, so every non-zero element in $\mathfrak{A}$ must be above an atom in $\mathfrak{A}$. Therefore, $\mathfrak{A}$ is atomic. □

The preceding argument goes through in part because the property of atomicity is a density property: it says that the set of atoms is dense in the algebra. Similar arguments apply in the case of other density properties. For example, a relation algebra $\mathfrak{A}$ is said to be *functionally dense* if the set of non-zero functions is dense in $\mathfrak{A}$ in the sense that every non-zero element in $\mathfrak{A}$ is above a non-zero function. An argument similar to the one given in the preceding proof shows that a relation algebra is functionally dense if and only if its completion is functionally dense.

Theorem 15.28 and Lemma 15.29 ensure that the completion of an atomic relation algebra $\mathfrak{A}$ is a complete and atomic relation algebraic extension of $\mathfrak{A}$ in which all suprema existing in $\mathfrak{A}$ are preserved. For this reason, in the case of an *atomic* relation algebra $\mathfrak{A}$, it is almost always unnecessary to pass to the canonical extension of $\mathfrak{A}$ in order to analyze $\mathfrak{A}$ itself, since the completion of $\mathfrak{A}$ possesses all of the properties that are required for this analysis.

A complete relation algebra $\mathfrak{A}$ is always its own completion, as is easily verified by checking that the conditions in Definition 15.17 are

satisfied in this case (with $\mathfrak{A}$ in place of $\mathfrak{B}$). In particular, every finite relation algebra is its own completion.

This raises the question of the relative size of the completion of a relation algebra $\mathfrak{A}$ in comparison with the size of $\mathfrak{A}$ when $\mathfrak{A}$ is infinite. If $\mathfrak{A}$ has infinite cardinality m, then the completion of $\mathfrak{A}$ has cardinality between m and 2^m. Indeed, $\mathfrak{A}$ is a subalgebra of its completion $\mathfrak{B}$, so the cardinality of $\mathfrak{B}$ is at least m (and it is exactly m when $\mathfrak{A}$ is complete). On the other hand, every element p in $\mathfrak{B}$ corresponds to a uniquely determined subset of $\mathfrak{A}$, namely the set L_p of elements in $\mathfrak{A}$ that are below p, since p is the sum of L_p in $\mathfrak{B}$. For this reason, there can be at most as many element in $\mathfrak{B}$ as their are subsets of $\mathfrak{A}$, and this number is 2^m.

For relation algebras, the properties of being simple and being integral are preserved under the passage to completions. The proof of this assertion is very similar to the proof of the corresponding result for canonical extensions (Theorem 14.36), and is left as an exercise.

Theorem 15.30. *A relation algebra is simple or integral if and only if its completion is simple or integral respectively.*

The part of the preceding theorem that concerns simplicity may be viewed as a statement about the Boolean algebra of ideal elements in a relation algebra $\mathfrak{A}$ and its completion: it says that if this Boolean algebra has exactly two elements in $\mathfrak{A}$, then it has exactly two elements in the completion of $\mathfrak{A}$. In this form, the result can be generalized.

Theorem 15.31. *If $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, then the Boolean algebra of ideal elements in $\mathfrak{B}$ is the Boolean completion of the Boolean algebra of ideal elements in $\mathfrak{A}$.*

Proof. Write A_0 and B_0 for the Boolean algebras of ideal elements in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively. It must be shown that B_0 satisfies conditions (i)–(iii) in Definition 15.4 with respect to A_0. The algebra B_0 is complete, by Corollary 8.25. Also, A_0 and B_0 are strongly regular Boolean subalgebras of the Boolean parts of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, by Lemma 8.24, and $\mathfrak{A}$ is a regular subalgebra of its completion $\mathfrak{B}$, by Lemma 15.6 (in its application to $\mathfrak{A}$ and $\mathfrak{B}$), so A_0 is a regular Boolean subalgebra of B_0. Thus, condition (i) holds.

To verify condition (ii), consider a non-zero element p in B_0. There must be a non-zero element r in $\mathfrak{A}$ that is below p, because $\mathfrak{B}$ is the completion of $\mathfrak{A}$ (see condition (ii) in Definition 15.17). The ideal

element generated by r in $\mathfrak{A}$, namely $1 ; r ; 1$, is a non-zero element in A_0 that is below p, because $0 < r \leq p$ and therefore

$$0 < r \leq 1 ; r ; 1 \leq 1 ; p ; 1 = p,$$

by Lemma 4.5(iii) and its first dual, the monotony law for relative multiplication, and the assumption that p is in B_0 and is therefore an ideal element. Thus, every non-zero element in B_0 is above a non-zero element in A_0. □

The preceding theorem can be used together with Lemma 15.29 to establish a stronger version of Theorem 14.38 for completions that applies to an arbitrary number of factors, and not just to a finite number of factors.

Theorem 15.32. *If a relation algebra* $\mathfrak{A}$ *has a total decomposition*

$$\mathfrak{A} = \prod_{i \in I} \mathfrak{A}(a_i), \tag{i}$$

then its completion $\mathfrak{B}$ *has the total decomposition*

$$\mathfrak{B} = \prod_{i \in I} \mathfrak{B}(a_i). \tag{ii}$$

Inversely, if $\mathfrak{B}$ *has a total decomposition in* (ii)*, and if the system* $(a_i : i \in I)$ *has the supremum property in* $\mathfrak{A}$*, then* $\mathfrak{A}$ *has the total decomposition in* (i)*.*

Proof. Let A_0 and B_0 be the Boolean algebras of ideal elements in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. The algebra B_0 is the completion of the algebra A_0, by Theorem 15.31, so the two algebras have the same (ideal element) atoms, and one of them is atomic if and only if the other is atomic, by the Boolean version of Lemma 15.29. Let

$$(a_i : i \in I) \tag{1}$$

be a list of the distinct atoms in these Boolean algebras.

If $\mathfrak{A}$ has the total direct decomposition given in (i), then A_0 is atomic, by the Total Decomposition Theorem 11.41; therefore, B_0 is atomic, and (1) is a list of its distinct atoms, by the remarks of the first paragraph. Apply the Atomic Decomposition Theorem 11.44 to conclude that $\mathfrak{B}$ has the total direct decomposition given in (ii). Inversely, if $\mathfrak{B}$ has the total direct decomposition given in (ii), then B_0 is atomic, by Theorem 11.41; therefore, A_0 is atomic, and (1) is a list of its distinct atoms, by the remarks of the first paragraph. If, in addition, the system in (1) has the supremum property in $\mathfrak{A}$, then $\mathfrak{A}$ has the total direct decomposition given in (i), by Theorem 11.41. □

The preceding theorem sheds light on a useful application of completions. A given relation algebra $\mathfrak{A}$ may be very close to being totally decomposable, because its Boolean algebra of ideal elements is atomic; but the supremum property may fail to hold for the system of ideal element atoms in $\mathfrak{A}$. By passing to the completion $\mathfrak{B}$, the missing suprema are filled in and we obtain a total decomposition of $\mathfrak{B}$ that is structurally very close to the total decomposition that we would like to have for $\mathfrak{A}$.

When the Boolean algebra of ideal elements is finite, the supremum property automatically holds for the system of ideal element atoms. Consequently, we at once obtain the following analogue of Theorem 14.38.

Corollary 15.33. *A relation algebra $\mathfrak{A}$ has a total decomposition into finitely many simple factors*

$$\mathfrak{A} = \mathfrak{A}(a_0) \times \cdots \times \mathfrak{A}(a_{n-1})$$

if and only if its completion $\mathfrak{B}$ has the total decomposition

$$\mathfrak{B} = \mathfrak{B}(a_0) \times \cdots \times \mathfrak{B}(a_{n-1}).$$

15.6 Completions of homomorphisms

The discussion so far about the preservation of properties has been limited to properties involving a single algebra and its completion. The properties that involve more than one algebra, for instance the property of one algebra being a subalgebra or a homomorphic image of another, require some extensions of Lemmas 15.18–15.21 in another direction. Instead of just considering quasi-complete operators, one may also consider functions with arguments that are sequences of elements (of some fixed length n) in one algebra $\mathfrak{A}$ and values that are sequences of elements (of some fixed length m) in another algebra $\bar{\mathfrak{A}}$. If φ is such a function, and if $\mathfrak{B}$ is the completion of $\mathfrak{A}$, and $\bar{\mathfrak{B}}$ any complete extension of $\mathfrak{A}$ with quasi-complete operators such that $\bar{\mathfrak{A}}$ is a regular subalgebra of $\bar{\mathfrak{B}}$—note that $\bar{\mathfrak{B}}$ need not be the completion of $\bar{\mathfrak{A}}$—then the *completion* of φ is defined to be the function φ^+ from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ whose value at each sequence p of n elements in $\mathfrak{B}$ is given by

$$\varphi^+(p) = \sum\{\varphi(r) : r \in A^n \text{ and } r \leq p\} = \sum\{\varphi(r) : r \in L_p\},$$

where the sum is formed coordinatewise in $\bar{\mathfrak{B}}$. The general theorem that one obtains says, roughly speaking, that if φ is quasi-complete, then the completion of φ is quasi-complete and inherits the positively expressible properties of φ. For example, if one composes functions that are quasi-complete, then the completion of the composition of the functions is equal to the composition of the completions of the functions.

Actually, since $\bar{\mathfrak{A}}$ is assumed to be a regular subalgebra of $\bar{\mathfrak{B}}$, all of the suprema that exist in $\bar{\mathfrak{A}}$ are preserved under the passage to $\bar{\mathfrak{B}}$. For that reason, one may dispense with the algebra $\bar{\mathfrak{A}}$ entirely and focus instead on the target algebra $\bar{\mathfrak{B}}$. In other words, one may consider φ to be a function from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$, and φ^+ a function from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$.

We illustrate the ideas with an important concrete example. Let $\mathfrak{A}$ be a Boolean algebra with quasi-complete operators, let $\mathfrak{B}$ the completion of $\mathfrak{A}$, and let $\bar{\mathfrak{B}}$ be any complete Boolean algebra with quasi-complete operators (of the same similarity type as $\mathfrak{A}$). For notational convenience, we shall assume that the similarity type of the algebras is the same as that of relation algebras. Consider a mapping φ from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$, and for each subset X of $\mathfrak{A}$ write

$$\varphi(X) = \{\varphi(r) : r \in X\}.$$

The *completion* of φ is the mapping φ^+ from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that is defined at each p in $\mathfrak{B}$ by

$$\varphi^+(p) = \sum \varphi(L_p),$$

where the sum on the right is formed in $\bar{\mathfrak{B}}$. (The completeness of $\bar{\mathfrak{B}}$ is needed in order to ensure that this sum really does exist.) The immediate goal is to prove that if φ is a complete homomorphism, then φ^+ is a complete homomorphism extending φ. The argument involves a series of lemmas that are the analogues of Lemmas 15.18–15.21 above.

We begin with just the assumption that φ is a *quasi-compete* mapping. This means that φ preserves all existing suprema of non-empty sets. In other words, if X is a non-empty subset of $\mathfrak{A}$ for which $p = \sum X$ exists (in $\mathfrak{A}$), then

$$\varphi(p) = \sum \varphi(X).$$

(Since $\bar{\mathfrak{B}}$ is complete, the supremum on the right always exists in $\bar{\mathfrak{B}}$.)

The analogue of Lemma 15.21 says that φ^+ is an extension of φ. Indeed, if p is an element in $\mathfrak{A}$, then p is the largest element in the set L_p, and consequently p is the supremum of L_p in $\mathfrak{A}$. The assumption that φ is quasi-complete therefore implies that $\varphi(p)$ is the supremum of

the set $\varphi(L_p)$ in $\bar{\mathfrak{B}}$. The element $\varphi^+(p)$ is, by definition, the supremum of the set $\varphi(L_p)$ in $\bar{\mathfrak{B}}$, so $\varphi^+(p) = \varphi(p)$.

The analogue of Lemma 15.18 says that φ^+ is monotone in the sense that $p \leq q$ implies $\varphi^+(p) \leq \varphi^+(q)$ for all elements p and q in $\mathfrak{B}$. The analogue of Lemma 15.19 says that if X is a non-empty subset of $\mathfrak{A}$, and if $p = \sum X$ in $\mathfrak{B}$, then

$$\varphi^+(p) = \sum \varphi(X) = \sum\{\varphi(r) : r \in X\},$$

where the sum is formed in $\bar{\mathfrak{B}}$. The analogue of the implication from right to left in Lemma 15.20 says that the mapping φ^+ is quasi-complete. In other words, for every non-empty subset X of $\mathfrak{B}$ (and not just of $\mathfrak{A}$), if $p = \sum X$, then

$$\varphi^+(p) = \sum\{\varphi^+(r) : r \in X\}.$$

The proofs of these lemmas are slightly modified versions of the proofs of the original lemmas, and are left as exercises.

The next step is to prove an analogue of Lemma 14.40 for completions.

Lemma 15.34. *For arbitrary elements p and q in $\mathfrak{B}$,*

(i) $\varphi^+(p \cdot q) = \sum \varphi(L_p \cdot L_q)$,
(ii) $\varphi^+(p) \cdot \varphi^+(q) = \sum(\varphi(L_p) \cdot \varphi(L_q))$,
(iii) $\varphi^+(p\,;q) = \sum \varphi(L_p\,;L_q)$,
(iv) $\varphi^+(p)\,;\varphi^+(q) = \sum(\varphi(L_p)\,;\varphi(L_q))$,
(v) $\varphi^+(p^\smile) = \sum \varphi(L_p^\smile)$,
(vi) $\varphi^+(p)^\smile = \sum(\varphi(L_p)^\smile)$.

Proof. Focus on the proofs of (iii) and (iv), and begin with (iii). From Lemma 15.5, we have

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{1}$$

and from Definition 15.17(iii), we have

$$p\,;q = \sum(L_p\,;L_q). \tag{2}$$

Observe that the sets on the right sides of these three equations are non-empty subsets of $\mathfrak{A}$, since the first two sets contain the element 0, and the last set contains the element $0\,;0$. Use (2) and the analogue

of Lemma 15.19 for φ^+ (with the set $L_p\,;L_q$ in place of X) to arrive at (iii). A similar argument, with (1) in place of (2), yields

$$\varphi^+(p) = \sum \varphi(L_p) \qquad \text{and} \qquad \varphi^+(q) = \sum \varphi(L_q). \tag{3}$$

The quasi-complete distributivity of the operation ; in $\bar{\mathfrak{B}}$, and the definitions of the sets involved, imply that

$$\begin{aligned}(\textstyle\sum \varphi(L_p))\,;(\sum \varphi(L_q)) &= (\textstyle\sum\{\varphi(r) : r \in L_p\})\,;(\sum\{\varphi(s) : s \in L_q\}) \\ &= \textstyle\sum\{\varphi(r)\,;\varphi(s) : r \in L_p \text{ and } s \in L_q\} \\ &= \textstyle\sum(\varphi(L_p)\,;\varphi(L_q)).\end{aligned} \tag{4}$$

Combine (3) and (4) to conclude that

$$\varphi^+(p)\,;\varphi^+(q) = \textstyle\sum(\varphi(L_p)\,;\varphi(L_q)).$$

This proves (iv). □

Here is the *Existence Theorem* for completions of complete homomorphisms.

Theorem 15.35. *Let $\mathfrak{A}$ be a Boolean algebra with quasi-complete operators, $\mathfrak{B}$ the completion of $\mathfrak{A}$, and $\bar{\mathfrak{B}}$ a complete Boolean algebra with quasi-complete operators. If φ is a complete homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$, then φ^+ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ. If φ is one-to-one, then so is φ^+. If $\bar{\mathfrak{B}}$ is the completion of the image of $\mathfrak{A}$ under φ, then φ^+ is onto.*

Proof. The function φ^+ is an extension of φ, by the analogue of Lemma 15.21 for φ^+, and in particular

$$\varphi^+(0) = \varphi(0) = 0, \quad \varphi^+(1) = \varphi(1) = 1, \quad \varphi^+(1') = \varphi(1') = 1'. \tag{1}$$

To see that φ^+ preserves the operation ;, consider elements p and q in $\mathfrak{B}$. The assumed homomorphism properties of φ and the definitions of the sets involved imply that

$$\begin{aligned}\varphi(L_p\,;L_q) &= \{\varphi(r\,;s) : r \in L_p \text{ and } s \in L_q\} \\ &= \{\varphi(r)\,;\varphi(s) : r \in L_p \text{ and } s \in L_q\} \\ &= \varphi(L_p)\,;\varphi(L_q).\end{aligned}$$

Consequently,

$$\sum \varphi(L_p\,;L_q) = \sum(\varphi(L_p)\,;\varphi(L_q)), \tag{2}$$

by Boolean algebra. Combine (2) with parts (iii) and (iv) of Lemma 15.34 to arrive at

$$\varphi^+(p\,;q) = \sum \varphi(L_p\,;L_q) = \sum(\varphi(L_p))\,;(\varphi(L_q)) = \varphi^+(p)\,;\varphi^+(q).$$

The proof that φ^+ preserves the operation of multiplication is almost identical to the preceding argument. One replaces the operation ; everywhere with the operation $\cdot$ and uses parts (i) and (ii) of Lemma 15.34. The proof that φ^+ preserves the operation $^\smile$ is similar, but easier, and uses parts (v) and (vi) of Lemma 15.34. The details are left as an exercise.

The function φ^+ preserves arbitrary non-empty sums, by the analogue of Lemma 15.20 for φ^+, and it preserves the empty sum, by the first equation in (1), so it preserves arbitrary sums. In particular, it preserves the operation of addition. A mapping between Boolean algebras with operators that preserves addition and multiplication, and maps zero to zero, and one to one, is necessarily a Boolean homomorphism (see the remarks following the proof of Lemma 7.6). Conclusion: φ^+ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ.

Assume now that φ is one-to-one, with the goal of showing that φ^+ is one-to-one. Consider a non-zero element p in $\mathfrak{B}$. There must be a non-zero element r in $\mathfrak{A}$ that is below p, by Definition 15.17(ii). The homomorphism φ is assumed to be one-to-one, so $\varphi(r)$ is not zero. The homomorphism φ^+ is monotone, by the analogue of Lemma 15.18 for φ^+, and it extends φ, by the analogue of Lemma 15.21 for φ^+, so

$$0 < \varphi(r) = \varphi^+(r) \leq \varphi^+(p).$$

Thus, every non-zero element in $\mathfrak{B}$ is mapped by φ^+ to a non-zero element in $\bar{\mathfrak{B}}$. It follows that φ^+ is one-to-one, by Lemma 8.37.

Turn now to the final assertion of the theorem. Write $\bar{\mathfrak{A}}$ for the image of the algebra $\mathfrak{A}$ under the homomorphism φ, and suppose that $\bar{\mathfrak{B}}$ is the completion of $\bar{\mathfrak{A}}$. To prove that φ^+ maps $\mathfrak{B}$ onto $\bar{\mathfrak{B}}$, consider an element q in $\bar{\mathfrak{B}}$, and let $\bar{L}_q$ be the set of elements in $\bar{\mathfrak{A}}$ that are below q. Observe that $q = \sum \bar{L}_q$, by Lemma 15.5 applied to $\bar{\mathfrak{B}}$. For each element s in $\bar{L}_q$, there must be an element r_s in $\mathfrak{A}$ that is mapped to s by φ, because φ maps $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$. The set

$$X = \{r_s : s \in \bar{L}_q\}$$

is a subset of $\mathfrak{A}$ that is mapped onto $\bar{L}_q$ by the homomorphism φ. Also, X has a supremum p in $\mathfrak{B}$, because $\mathfrak{B}$ is the completion of $\mathfrak{A}$. Use the analogue of Lemma 15.19 for φ^+, and the preceding observations, to arrive at

$$\varphi^+(p) = \sum \varphi(X) = \sum \bar{L}_q = q.$$

Thus, every element in $\bar{\mathfrak{B}}$ is the image of an element in $\mathfrak{B}$, so φ^+ is onto.

□

It turns out that the completion of a complete homomorphism φ is the only possible complete extension of φ, as the following *Uniqueness Theorem* for completions of complete homomorphisms makes clear.

Theorem 15.36. *Let $\mathfrak{A}$ be a Boolean algebra with quasi-complete operators, $\mathfrak{B}$ the completion of $\mathfrak{A}$, and $\bar{\mathfrak{B}}$ a complete Boolean algebra with quasi-complete operators. A complete homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$ has only one extension to a complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$.*

Proof. A complete homomorphism φ from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$ certainly has one extension to a complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$, namely the completion φ^+. Let ψ be any complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$ that extends φ. Each element p in $\mathfrak{B}$ is the sum of the set L_p of elements in $\mathfrak{A}$ that are below p, so

$$\psi(p) = \sum \psi(L_p) = \sum \varphi(L_p) = \varphi^+(p),$$

by the completeness of the homomorphism ψ, the assumption that ψ extends φ, and the definition of the homomorphism φ^+. It follows that ψ and φ^+ agree on all elements in $\mathfrak{B}$, so the two homomorphisms coincide. □

The preceding theorem implies that the completion of the composition of two complete homomorphisms is equal to the composition of the completions of the two homomorphisms.

Corollary 15.37. *Let $\mathfrak{A}$, $\bar{\mathfrak{A}}$, and $\bar{\bar{\mathfrak{A}}}$ be Boolean algebras with quasi-complete operators, and $\mathfrak{B}$, $\bar{\mathfrak{B}}$, and $\bar{\bar{\mathfrak{B}}}$ their respective completions. If φ is a complete homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, and ψ a complete homomorphism from $\bar{\mathfrak{A}}$ into $\bar{\bar{\mathfrak{A}}}$, then the completion of the composition $\psi \circ \varphi$ is just the composition of the completions $\psi^+ \circ \varphi^+$.*

The proof is very similar to the proof of Corollary 14.43, and is left as an exercise.

15.7 Minimality

A Boolean algebra with quasi-complete operations $\mathfrak{A}$ has many complete extensions with quasi-complete operators, that is to say, there are many complete Boolean algebras with quasi-complete operators that include $\mathfrak{A}$ as a subalgebra. For instance, when $\mathfrak{A}$ is infinite, the canonical extension of $\mathfrak{A}$, the canonical extension of the canonical extension of $\mathfrak{A}$, and so on are all distinct complete extensions of $\mathfrak{A}$ with quasi-complete operators. The completion of $\mathfrak{A}$ distinguishes itself from these other complete extensions by its minimality: it is the smallest complete extension of $\mathfrak{A}$ with quasi-complete operators that preserves all suprema existing in $\mathfrak{A}$. The following precise statement of this result is known as the *Minimality Theorem* for completions.

Theorem 15.38. *The completion of a Boolean algebra with quasi-complete operators* $\mathfrak{A}$ *can be completely embedded, via a mapping that is the identity function on* $\mathfrak{A}$*, into any regular extension of* $\mathfrak{A}$ *that is complete with quasi-complete operators.*

Proof. Suppose $\mathfrak{B}$ is the completion of $\mathfrak{A}$, and let $\mathfrak{C}$ be any regular extension of $\mathfrak{A}$ that is complete with quasi-complete operators. Take φ to be the identity automorphism of $\mathfrak{A}$, and observe that φ is a complete monomorphism of $\mathfrak{A}$ into $\mathfrak{C}$, by Lemma 7.9 and the assumption that $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{C}$. The completion of φ is a complete monomorphism from $\mathfrak{B}$ into $\mathfrak{C}$ that extends φ, by the Existence Theorem 15.35 for completions of complete homomorphisms. Since φ is the identity function on $\mathfrak{A}$, it follows that the restriction of φ^+ must also be the identity function on $\mathfrak{A}$. □

Here is another way to think about the minimality of completions.

Lemma 15.39. *If* $\mathfrak{B}$ *is the completion of a Boolean algebra with quasi-complete operators* $\mathfrak{A}$*, then the only complete homomorphism of* $\mathfrak{B}$ *into itself that is the identity function on* $\mathfrak{A}$ *is the identity automorphism of* $\mathfrak{B}$*.*

Proof. Let φ be the identity automorphism of $\mathfrak{A}$, and φ^+ the completion of φ. Existence Theorem 15.35 (with $\mathfrak{B}$ in place of $\mathfrak{C}$) implies that φ^+ is an automorphism of $\mathfrak{B}$ extending φ. The identity function on $\mathfrak{B}$ is also an automorphism of $\mathfrak{B}$ extending φ. Consequently, φ^+ must be the identity automorphism of $\mathfrak{B}$, by Uniqueness Theorem 15.36. Any other complete homomorphism from $\mathfrak{B}$ into itself

that agrees with φ on $\mathfrak{A}$ must coincide with φ^+, by Theorem 15.36, and must therefore be the identity automorphism of $\mathfrak{B}$, by the preceding observation. □

The minimality of the completion may be characterized in yet another way.

Theorem 15.40. *If $\mathfrak{A}$ and $\mathfrak{C}$ are Boolean algebra with quasi-complete operators, then $\mathfrak{C}$ is the completion of $\mathfrak{A}$ if and only if $\mathfrak{C}$ is a regular extension of $\mathfrak{A}$ that is complete, and no proper subalgebra of $\mathfrak{C}$ that extends $\mathfrak{A}$ is complete with quasi-complete operators.*

Proof. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$, which exists by the Existence Theorem 15.22 for completions. The first step is to show that $\mathfrak{B}$ has the properties stated in the theorem (with $\mathfrak{B}$ in place of $\mathfrak{C}$). To this end, let $\mathfrak{D}$ be any subalgebra of $\mathfrak{B}$ that extends $\mathfrak{A}$ and is complete with quasi-complete operators. It is to be shown that $\mathfrak{D}$ coincides with $\mathfrak{B}$.

The algebra $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{B}$, because $\mathfrak{B}$ is the completion of $\mathfrak{A}$ (see Lemma 15.6), and $\mathfrak{A}$ is a subalgebra of $\mathfrak{D}$, which in turn is a subalgebra of $\mathfrak{B}$, by assumption. It follows that $\mathfrak{A}$ must be a regular subalgebra of $\mathfrak{D}$, by Lemma 6.16(ii) (with $\mathfrak{A}$, $\mathfrak{D}$, and $\mathfrak{B}$ in place of $\mathfrak{C}$, $\mathfrak{B}$, and $\mathfrak{A}$ respectively). Also, $\mathfrak{D}$ is assumed to be complete with quasi-complete operators. Apply Minimality Theorem 15.38 to obtain a complete embedding ψ of $\mathfrak{B}$ into $\mathfrak{D}$ that is the identity function on $\mathfrak{A}$.

It is not hard to see that ψ is the identity function on $\mathfrak{D}$ as well. Indeed, every element p in $\mathfrak{D}$ belongs to $\mathfrak{B}$ (since $\mathfrak{D}$ is a subalgebra of $\mathfrak{B}$) and is therefore the supremum in $\mathfrak{B}$ of some subset X of $\mathfrak{A}$, by Lemma 15.5 and the assumption that $\mathfrak{B}$ is the completion of $\mathfrak{A}$. Since p belongs to $\mathfrak{D}$, it must also be the supremum in $\mathfrak{D}$ of the set X, by Lemma 6.15. Use this observation, the completeness of the mapping ψ, and the fact that ψ is the identity function on the elements in $\mathfrak{A}$ (and therefore on the elements in X) to arrive at

$$\psi(p) = \psi(\textstyle\sum X) = \sum\{\psi(r) : r \in X\} = \sum X = p,$$

where the first sum is formed in $\mathfrak{B}$, and the second and third sums are formed in $\mathfrak{D}$.

The desired conclusion now follows easily. The function ψ maps $\mathfrak{B}$ into $\mathfrak{D}$, and it maps each element in $\mathfrak{D}$ (which is a subalgebra of $\mathfrak{B}$) to itself. The one-to-oneness of ψ therefore forces $\mathfrak{B}$ to coincide with $\mathfrak{D}$. This establishes the implication from left to right in the theorem.

To establish the reverse implication, assume that $\mathfrak{C}$ has the stated property—namely, it is a regular extension of $\mathfrak{A}$ that is complete, and no proper subalgebra of $\mathfrak{C}$ that extends $\mathfrak{A}$ is complete with quasi-complete operators. The goal is to show that $\mathfrak{B}$ is isomorphic to $\mathfrak{C}$ via a mapping that is the identity function on $\mathfrak{A}$, so that $\mathfrak{C}$ is also a completion of $\mathfrak{A}$. There is a complete embedding ψ of $\mathfrak{B}$ into $\mathfrak{C}$ that is the identity function on $\mathfrak{A}$, by Minimality Theorem 15.38. The image of $\mathfrak{B}$ under ψ is of course a subalgebra of $\mathfrak{C}$ that extends $\mathfrak{A}$, and this image algebra is complete with quasi-complete operators, because $\mathfrak{B}$ has this property. Consequently, this image algebra must coincide with $\mathfrak{C}$, by the assumptions on $\mathfrak{C}$, so ψ is an isomorphism from $\mathfrak{B}$ to $\mathfrak{C}$ that is the identity function on $\mathfrak{A}$. □

A relation algebra is a Boolean algebra with complete (and not just quasi-complete) operators, so for relation algebras the preceding theorem may be formulated in a somewhat simpler way.

Corollary 15.41. *A relation algebra $\mathfrak{C}$ is the completion of a relation algebra $\mathfrak{A}$ if and only if $\mathfrak{C}$ is a regular extension of $\mathfrak{A}$ that is complete, and no proper subalgebra of $\mathfrak{C}$ that extends $\mathfrak{A}$ is complete.*

15.8 Applications to algebraic constructions

The main results concerning the preservation of algebraic constructions under the passage to completions say, roughly speaking, that in the class of Boolean algebras with quasi-complete operators (of a fixed similarity type), the completion of a regular subalgebra is a regular subalgebra (and therefore a complete subalgebra) of the completion, the completion of a complete homomorphic image is a complete homomorphic image of the completion, and the completion of a direct product is the direct product of the completions. Here is a precise statement of the first of these results.

Theorem 15.42. *Let $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ be Boolean algebras with quasi-complete operators. If $\mathfrak{A}$ is a regular subalgebra of $\bar{\mathfrak{A}}$, then the completion of $\mathfrak{A}$ is (up to isomorphisms that are the identity function on $\mathfrak{A}$) a complete subalgebra of the completion of $\bar{\mathfrak{A}}$.*

Proof. Let $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ be the completions of $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ respectively. The relation of being a regular subalgebra is transitive on the class

of all Boolean algebras with operators of a fixed similarity type, by Lemma 6.16(i). Since $\mathfrak{A}$ is a regular subalgebra of $\bar{\mathfrak{A}}$, by assumption, and $\bar{\mathfrak{A}}$ is a regular subalgebra of $\bar{\mathfrak{B}}$, by Lemma 15.6, it follows that $\mathfrak{A}$ is a regular subalgebra of $\bar{\mathfrak{B}}$. Also, $\bar{\mathfrak{B}}$ is complete with quasi-complete operators, by Definition 15.17 and Lemma 15.20. Apply Minimality Theorem 15.38 to obtain a complete embedding of $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that is the identity function on $\mathfrak{A}$. The image of $\mathfrak{B}$ under this complete embedding is complete (because $\mathfrak{B}$ is complete) and a regular subalgebra of $\bar{\mathfrak{B}}$, by Lemma 7.9. Consequently, this image algebra is a complete subalgebra of $\bar{\mathfrak{B}}$ (see the remark preceding Lemma 6.16). Identify $\mathfrak{B}$ with its image in $\bar{\mathfrak{B}}$ under this embedding to obtain the desired conclusion. □

The analogue of Lemma 14.45 does not hold for completions, that is to say, the equality of the completions in Theorem 15.42 does not imply the equality of the algebras $\mathfrak{A}$ and $\bar{\mathfrak{A}}$, even in the case of relation algebras. For example, take $\mathfrak{A}$ to be any incomplete relation algebra, and take $\bar{\mathfrak{A}}$ to be the completion of $\mathfrak{A}$. The two algebras have the same completion, namely $\bar{\mathfrak{A}}$, but $\mathfrak{A}$ is not equal to $\bar{\mathfrak{A}}$. The same example shows that the analogue of Corollary 14.47 does not hold for completions. Indeed, the identity function φ on $\mathfrak{A}$ is a complete embedding of $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, by Lemma 7.9. The completion of φ is a complete monomorphism φ^+ from the completion of $\mathfrak{A}$ into the completion of $\bar{\mathfrak{A}}$ that extends φ, by Existence Theorem 15.35. In other words, φ^+ is a complete monomorphism from $\bar{\mathfrak{A}}$ into $\bar{\mathfrak{A}}$ that extends φ. The identity function on $\bar{\mathfrak{A}}$ is also a complete monomorphism from $\bar{\mathfrak{A}}$ into $\bar{\mathfrak{A}}$ that extends φ, so φ^+ must be the identity function on $\bar{\mathfrak{A}}$, by Uniqueness Theorem 15.36. Thus, φ^+ maps the completion of $\mathfrak{A}$ onto the completion of $\bar{\mathfrak{A}}$, but φ does not map $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$.

The preservation theorem for complete homomorphic images says, roughly speaking, that the completion of a complete homomorphic image is a complete homomorphic image of the completion. It is an almost immediate consequence of Existence Theorem 15.35.

Theorem 15.43. *Let $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ be Boolean algebras with quasi-complete operators. If $\bar{\mathfrak{A}}$ is the image of $\mathfrak{A}$ under a complete homomorphism φ, then the completion of $\bar{\mathfrak{A}}$ is the image of the completion of $\mathfrak{A}$ under a complete homomorphism that extends φ.*

Proof. The function φ is a complete homomorphism from $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$, by assumption. Since $\bar{\mathfrak{A}}$ is a regular subalgebra of its completion, it

follows that φ is a complete homomorphism from $\mathfrak{A}$ into the completion of $\bar{\mathfrak{A}}$. The completion of φ is a complete homomorphism φ^+ from the completion of $\mathfrak{A}$ onto the completion of $\bar{\mathfrak{A}}$ that extends φ, by Theorem 15.35. □

There is one more notable difference between canonical extensions and completions: arbitrary direct products are preserved under the passage to completions, not just products that involve only finitely many factors.

Theorem 15.44. *Let $(\mathfrak{A}_i : i \in I)$ be a system of Boolean algebras with quasi-complete operators, and $(\mathfrak{B}_i : i \in I)$ the corresponding system of completions. The completion of the product $\prod_i \mathfrak{A}_i$ is just the product $\prod_i \mathfrak{B}_i$ of the completions (up to isomorphisms that are the identity function on $\prod_i \mathfrak{A}_i$).*

Proof. One can verify directly that $\prod_i \mathfrak{B}_i$ satisfies the conditions in Definition 15.17 for being the completion of $\prod_i \mathfrak{A}_i$. Instead of taking this direct approach, however, we take an indirect approach by showing that the completion of $\prod_i \mathfrak{A}_i$ is isomorphic to $\prod_i \mathfrak{B}_i$ via a mapping that is the identity function on $\prod_i \mathfrak{A}_i$. It seems notationally easier to use internal products instead of external products. Accordingly, let $\mathfrak{A}$ be the internal product of the algebras $\mathfrak{A}_i$, let $\mathfrak{B}$ be the internal product of the completions $\mathfrak{B}_i$, and let $\mathfrak{C}$ be the completion of $\mathfrak{A}$. The goal is to show that $\mathfrak{C}$ is isomorphic to $\mathfrak{B}$ via a mapping that is the identity function on $\mathfrak{A}$.

The product $\mathfrak{B}$ is complete, by Corollary 11.38. The operators in $\mathfrak{B}$ are performed componentwise, so they inherit the quasi-completeness of the operators in the factor algebras (see Exercise 11.54). Also, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, by the internal version of Corollary 11.20. To see that $\mathfrak{A}$ is dense in $\mathfrak{B}$, and therefore a regular subalgebra of $\mathfrak{B}$ (Lemma 15.6), consider a non-zero element p in $\mathfrak{B}$, say $p = \sum p_i$ with p_i in $\mathfrak{B}_i$ for each i. There must be an index i such that p_i is not zero. The algebra $\mathfrak{A}_i$ is dense in $\mathfrak{B}_i$, because $\mathfrak{B}_i$ is assumed to be the completion of $\mathfrak{A}_i$, so there must be a non-zero element s in $\mathfrak{A}_i$ that is below p_i. The definition of an internal product ensures that s is a non-zero element in $\mathfrak{A}$ that is below p. Thus, every non-zero element in $\mathfrak{B}$ is above a non-zero element in $\mathfrak{A}$, so $\mathfrak{A}$ is dense in $\mathfrak{B}$. Conclusion: $\mathfrak{B}$ is a regular extension of $\mathfrak{A}$ that is complete with quasi-complete operators.

Apply Minimality Theorem 15.38 to obtain a complete embedding ψ of $\mathfrak{C}$ into $\mathfrak{B}$ that is the identity function on $\mathfrak{A}$. It remains to prove

that ψ is onto and therefore an isomorphism. Consider an arbitrary element q in $\mathfrak{B}$, say $q = \sum q_i$ with q_i in $\mathfrak{B}_i$ for each i. Since $\mathfrak{B}_i$ is the completion of $\mathfrak{A}_i$, there must be a subset X_i of $\mathfrak{A}_i$ such that q_i is the supremum of X_i in $\mathfrak{B}_i$, by Lemma 15.5 applied to $\mathfrak{B}_i$. Observe that q_i is also the supremum of X_i in $\mathfrak{B}$, by Corollary 11.38 (with X_i in place of X). The union $X = \bigcup_i X_i$ is a subset of $\mathfrak{A}$, so it must have a supremum p in $\mathfrak{C}$. Use the completeness of the embedding ψ, the fact that ψ is the identity function on $\mathfrak{A}$, the definition of the set X, the general associative law for addition, and the assumptions about q_i and q, to obtain

$$\psi(p) = \psi(\sum X) = \sum \psi(X) = \sum X$$
$$= \sum(\bigcup_i X_i) = \sum_i \sum X_i = \sum q_i = q.$$

Thus, ψ maps $\mathfrak{C}$ onto $\mathfrak{B}$. □

There are two more results concerning the preservation of algebraic constructions under the passage to completions that should be mentioned. The first says that if $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and if e is an equivalence element in $\mathfrak{A}$, then the relativization $\mathfrak{B}(e)$ is the completion of the relativization $\mathfrak{A}(e)$. In other words, the completion of a relativization is the relativization of the completion.

To formulate the second result, we need a definition. A relation algebraic ideal M in $\mathfrak{A}$ is said to be *complete* if it is complete as a Boolean ideal (see Section 15.1). This means that if a subset of M has a supremum in $\mathfrak{A}$, then that supremum belongs to M. The second result says that if $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and if M is a complete relation algebraic ideal in $\mathfrak{A}$, then the set of all suprema in $\mathfrak{B}$ of subsets of M is a complete relation algebraic ideal N in $\mathfrak{B}$, and the quotient $\mathfrak{B}/N$ is isomorphic to the completion of the quotient $\mathfrak{A}/M$ via a function that maps r/N to r/M for each element r in $\mathfrak{A}$. In other words, roughly speaking, the completion of a quotient is the quotient of the completion. The proofs of these two results are left as exercises.

15.9 A characterization of complete homomorphic images

It was shown in Theorem 14.54 that every homomorphic image of a relation algebra $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to an ideal el-

ement from the canonical extension of $\mathfrak{A}$. The corresponding theorem for complete homomorphic images says that every complete homomorphic image of $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to an ideal element from the completion of $\mathfrak{A}$. There are two possible approaches to proving this theorem, one using complete ideals and the other using complete filters. In the present setting, the first approach seems more natural.

Here are two preliminary observations about complete ideals. Their proofs are easy and are left as exercises. First, every complete ideal in a relation algebra is the kernel of a complete epimorphism, namely the quotient homomorphism onto the corresponding quotient algebra. Second, if φ is a complete homomorphism from a relation algebra $\mathfrak{A}$ onto a relation algebra $\mathfrak{B}$, then the kernel of φ is a complete ideal M in $\mathfrak{A}$, and the quotient $\mathfrak{A}/M$ is isomorphic to $\mathfrak{B}$ via the function that maps r/M to $\varphi(r)$ for every r in $\mathfrak{A}$, by the First Isomorphism Theorem 8.39.

We begin with a characterization of the complete relation algebraic ideals in $\mathfrak{A}$.

Lemma 15.45. *A subset M of a relation algebra $\mathfrak{A}$ is a complete relation algebraic ideal if and only if the supremum of M in the completion of $\mathfrak{A}$ is an ideal element u and $M = L_u$.*

Proof. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. Consider a subset M of $\mathfrak{A}$, and let u be the supremum of M in $\mathfrak{B}$. Assume first that u is an ideal element in $\mathfrak{B}$ and $M = L_u$, that is to say, M is the set of elements in $\mathfrak{A}$ that are below u. Lemma 15.14 (with u in place of p) implies that M is a complete Boolean ideal in $\mathfrak{A}$. To show that M is closed under relative multiplication by arbitrary elements from $\mathfrak{A}$, and is therefore a complete relation algebraic ideal (see Definition 8.7), consider an element r in M and an element s in $\mathfrak{A}$. The definition of M, the monotony law for relative multiplication, and the assumption that u is an ideal element imply that

$$r \mathbin{;} s \leq u \mathbin{;} 1 = u,$$

so $r \mathbin{;} s$ is in M. A symmetric argument shows that $s \mathbin{;} r$ is in M.

Assume now that M is a complete relation algebraic ideal in $\mathfrak{A}$. In particular, M is a complete Boolean ideal in $\mathfrak{A}$, so $M = L_u$, by the definition of u and Lemma 15.14 (with u in place of p). Use the definition of u, the complete distributivity of relative multiplication,

and the closure of M under relative multiplication by elements from $\mathfrak{A}$, to obtain

$$1 ; u ; 1 = 1 ; (\sum M) ; 1 = \sum\{1 ; r ; 1 : r \in M\} \leq \sum M = u.$$

The reverse inequality holds by Lemma 4.5(iii) and its first dual, so u is an ideal element in $\mathfrak{B}$. □

The lemma easily implies the following analogue of Corollary 14.52, giving a characterization of arbitrary ideal elements in the completion of a relation algebra.

Corollary 15.46. *Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$. An element u in $\mathfrak{B}$ is an ideal element if and only if there is a complete relation algebraic ideal M in $\mathfrak{A}$ such that $u = \sum M$ in $\mathfrak{B}$.*

Proof. Consider an arbitrary element u in $\mathfrak{B}$. If M is a complete relation algebraic ideal in $\mathfrak{A}$ such that

$$u = \sum M, \tag{1}$$

then u is an ideal element, by Lemma 15.45. On the other hand, if u is an ideal element, then the set $M = L_u$ is a complete relation algebraic ideal in $\mathfrak{A}$, by Lemma 15.45, and (1) holds, by Lemma 15.5 and the definition of the set L_u.

The next lemma is the analogue of Lemma 14.53.

Lemma 15.47. *Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$. If u is the supremum in $\mathfrak{B}$ of a complete relation algebraic ideal M in $\mathfrak{A}$, then the restriction to $\mathfrak{A}$ of the relativization homomorphism $p \longmapsto p \cdot -u$ on $\mathfrak{B}$ is a complete epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(-u)$ with kernel M.*

Proof. Lemma 15.45 implies that u is an ideal element in $\mathfrak{B}$, so the complement $-u$ is also an ideal element, by Lemma 5.39(iv). It therefore makes sense to speak of the relativization homomorphism φ from $\mathfrak{B}$ to $\mathfrak{B}(u)$ that is defined by

$$\varphi(p) = p \cdot -u$$

for p in $\mathfrak{B}$. The kernel of this homomorphism is, by definition, the set K of elements in $\mathfrak{B}$ that are mapped to 0 by φ. Since $p \cdot -u = 0$ if and only if $p \leq u$, we may write

$$K = \{p \in B : p \leq u\}. \tag{1}$$

The restriction of φ to $\mathfrak{A}$ is an epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(u)$, by Lemma 10.12, and in fact it is a complete epimorphism. For the proof, suppose that r is the supremum in $\mathfrak{A}$ of a subset X of $\mathfrak{A}$. Since $\mathfrak{B}$ is the completion of $\mathfrak{A}$, the element r remains the supremum of X in $\mathfrak{B}$, by Corollary 15.7, and therefore

$$\begin{aligned}\varphi(\textstyle\sum X) = \varphi(r) = r \cdot -u = (\textstyle\sum X) \cdot -u \\ = \textstyle\sum\{s \cdot -u : s \in X\} = \sum\{\varphi(s) : s \in X\}.\end{aligned}$$

Thus, the restriction of φ to $\mathfrak{A}$ preserves all sums that happen to exist in $\mathfrak{A}$.

The kernel of the restriction of φ to $\mathfrak{A}$ is the set $K \cap A$ (see Exercise 8.23). It is easy to see that

$$M = K \cap A. \tag{2}$$

Indeed, M is a complete relation algebraic ideal in $\mathfrak{A}$, and u is its supremum in $\mathfrak{B}$, by assumption, so M must be the set of all elements in $\mathfrak{A}$ that are below u, by Lemma 15.45. But $K \cap A$ is also the set of all elements in $\mathfrak{A}$ that are below u, by (1) and the definition of intersection, so (2) holds. This completes the proof of the lemma. □

Theorem 15.48. *Every complete homomorphic image of a relation algebra $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to some ideal element from the completion of $\mathfrak{A}$.*

Proof. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. The kernel of a complete epimorphism on $\mathfrak{A}$ is a complete ideal in $\mathfrak{A}$, and the quotient of $\mathfrak{A}$ modulo that complete ideal is isomorphic to the image of $\mathfrak{A}$ under the epimorphism, by the preliminary observations made before Lemma 15.45. Consequently, it suffices to show that every quotient of $\mathfrak{A}$ modulo a complete ideal is isomorphic to the relativization of $\mathfrak{A}$ to some ideal element in $\mathfrak{B}$.

Consider a complete relation algebraic ideal M in $\mathfrak{A}$, and let u be the supremum of M in $\mathfrak{B}$. Lemma 15.45 ensures that u is an ideal element in $\mathfrak{B}$, and Lemma 15.47 ensures that the restriction to $\mathfrak{A}$ of the relativization homomorphism $p \longmapsto p \cdot -u$ on $\mathfrak{B}$ is a complete epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(-u)$ with kernel M. The quotient $\mathfrak{A}/M$ is therefore isomorphic to the relativization $\mathfrak{A}(-u)$, by the First Isomorphism Theorem 8.39. □

15.10 Historical remarks

Stone [127], [128] and Tarski [131] proved that the class of complete ideals in a Boolean algebra is itself a complete Boolean algebra (Theorem 15.3). Dedekind [27] constructed the real numbers as a kind of order completion of the rational numbers, using sets of rational numbers called Dedekind cuts (see the remarks following Lemma 15.2). MacNeille [91] extended Dedekind's method to construct completions of partial orderings, and in particular, completions of Boolean algebras. Existence Theorem 15.13 for completions of Boolean algebras is due to MacNeille [91] and Tarski [131], who also showed that the completion of a Boolean algebra A can be embedded into any complete extension of A via a mapping that is the identity function on A. This implies the uniqueness of the completion up to isomorphisms over A. Tarski [131] also pointed out that a Boolean algebra is atomic if and only if its completion is atomic (see Lemma 15.29).

Piero Mangani [104] constructed the completion of an arbitrary cylindric algebra A, and showed that every complete monomorphism from A into a complete cylindric algebra C can be extended to a monomorphism from the completion of A into C. Monk [112] developed the general theory of completions of Boolean algebras with quasi-complete operators. Definition 15.17 and the results in Section 15.3 are essentially due to him, as are the First Preservation Theorem 15.26 and a somewhat weaker form of the Second Preservation Theorem 15.27 in Section 15.4. He used these results to show that the properties of being a relation algebra, a simple relation algebra, and an integral relation algebra are preserved under the passage to completions (see Theorems 15.28 and 15.30). He also proved a slightly weaker version of Minimality Theorem 15.38.

The Second Preservation Theorem 15.27 in its present form is due to Givant and Yde Venema [43]. The results in Section 15.6 are due to Givant. So, too, are Theorems 15.31 and 15.32, Corollary 15.33, Lemma 15.39, and Theorem 15.40 in Section 15.7, as well as the results in Section 15.8 (including the theorems mentioned at the end of the section and given in Exercises 15.58 and 15.60), and in Section 15.9 (including the result in Exercise 15.65).

Hodkinson [62] (see also [59]) proved the important and difficult theorem that the completion of a set relation algebra need not be isomorphic to a set relation algebra.

Exercises

15.1. Prove that if M is a complete relation algebraic ideal in a relation algebra $\mathfrak{A}$, then the quotient homomorphism from $\mathfrak{A}$ to $\mathfrak{A}/M$ is a complete epimorphism. Conclude that every complete relation algebraic ideal is the kernel of a complete homomorphism.

15.2. Prove that the kernel of a complete homomorphism on a relation algebra is a complete ideal. Conclude that every complete homomorphic image of a relation algebra is isomorphic to a quotient of the algebra modulo a complete ideal.

15.3. By imitating the argument used to establish the distributive law for meet over join in the Boolean algebra of all complete ideals in a Boolean algebra, prove the distributive law for join over meet.

15.4. Derive the distributive law for join over meet from the distributive law for meet over join, using the standard axioms of lattice theory.

15.5. Verify directly, without appealing to Theorem 15.3, that the laws

$$-(X \vee Y) = -X \wedge -Y \qquad \text{and} \qquad -(X \wedge Y) = -X \vee -Y$$

hold in the Boolean algebra of complete ideals in a Boolean algebra (where the complement of a complete ideal is defined to be the annihilator of the ideal).

15.6. Prove that an ideal X in a Boolean algebra is complete if and only if it is the annihilator of the annihilator of X.

15.7. Give an example to show that the operation of join in the Boolean algebra of complete ideals in a Boolean algebra A is not always the same as the operation of join in the lattice of ideals in A.

15.8. Give an example to show that an ideal may not have a complement in the lattice of ideals of a Boolean algebra.

15.9. Fill in the missing details in the proof of Lemma 15.14.

15.10. Complete the proof of Theorem 15.15 by showing that the function ψ preserves the operation of complement.

15.11. Give a direct proof of Uniqueness Theorem 15.16 without making use of Lemma 15.14.

15.12. If two Boolean algebras are isomorphic via a mapping φ prove that their completions are isomorphic via a uniquely determined mapping that extends φ.

15.13. Prove that condition (i) in Definition 15.17 may be weakened to say that $\mathfrak{B}$ is complete, the Boolean part of $\mathfrak{A}$ is a subalgebra of the Boolean part of $\mathfrak{B}$, and the distinguished constant 1' is the same in both algebras.

15.14. Formulate a version of Definition 15.17 that applies to Boolean algebras with operators of arbitrary ranks.

15.15. Complete the proof of Lemma 15.18 by showing that the operation $^\smile$ in $\mathfrak{B}$ is monotone.

15.16. Formulate and prove a version of Lemma 15.18 for Boolean algebras with operators of arbitrary ranks.

15.17. Complete the proof of Lemma 15.19 by showing that for every non-empty subset X of $\mathfrak{A}$, if $p = \sum X$, then $p^\smile = \sum(X^\smile)$.

15.18. Formulate and prove a version of Lemma 15.19 for Boolean algebras with quasi-complete operators of arbitrary ranks.

15.19. Complete the proof of Lemma 15.20 by treating the case of the operation $^\smile$.

15.20. Formulate and prove a version of Lemma 15.20 for Boolean algebras with quasi-complete operators of arbitrary ranks.

15.21. Complete the proof of Lemma 15.21 by treating the case of the operation $^\smile$.

15.22. Formulate a version of the definition preceding Lemma 15.21 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks, and prove the analogues of Lemma 15.21 and Existence Theorem 15.22 for those algebras.

15.23. Complete the proof of Uniqueness Theorem 15.23 by showing that the function ψ preserves the operation $^\smile$.

15.24. Prove a version of Uniqueness Theorem 15.23 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.25. For a Boolean algebra with quasi-complete operators $\mathfrak{A}$, prove that there is at most one isomorphism between two completions of $\mathfrak{A}$ that is the identity function on $\mathfrak{A}$.

15.26. If two Boolean algebras with quasi-complete operators are isomorphic via a mapping φ prove that their completions are isomorphic via a uniquely determined mapping that extends φ.

15.27. Prove Lemma 15.24.

15.28. Prove a version of Lemma 15.24 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.29. Prove Lemma 15.25.

15.30. Prove a version of Lemma 15.25 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.31. For relation algebras, the existence of completions (in the sense of Definition 15.17) with quasi-complete operators is a consequence of Theorem 15.22. Assuming this existence, prove directly (without using the lemmas or theorems of Section 15.4) that the relation algebraic axioms (R4)–(R7) and the cycle law (R11) are all valid in a completion.

15.32. Suppose $\mathfrak{A}$ is a Boolean algebra with quasi-complete operators, and $\mathfrak{B}$ its completion. Prove that the completion of the unary discriminator on the universe of $\mathfrak{A}$ is the unary discriminator on the universe of $\mathfrak{B}$.

15.33. Prove the Second Preservation Theorem 15.27.

15.34. Prove Theorem 15.28.

15.35. Formulate and prove a version of Lemma 15.29 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.36. Prove that the completion of a functionally dense relation algebra is functionally dense.

15.37. Prove that a complete Boolean algebra with quasi-complete operators is always its own completion.

15.38. Prove that the completion of an atomless Boolean algebra with quasi-complete operators is atomless.

15.39. Prove that the completion of a countably infinite relation algebra must have cardinality $2^{\aleph_0}$.

15.40. Assuming Theorem 15.28, prove directly—without using the preservation theorems—that a relation algebra is simple or integral if and only if its completion is simple or integral respectively.

15.41. Prove Theorem 15.30 by imitating the proof of Theorem 14.36.

15.42. Prove that the mapping φ^+ defined in Section 15.6 is monotone.

15.43. Prove the analogue of Lemma 15.19 for the mapping φ^+. In other words, prove that for every non-empty subset X of $\mathfrak{A}$, if $p = \sum X$, then $\varphi^+(p) = \sum\{\varphi(r) : r \in X\}$.

15.44. Prove that the mapping φ^+ defined in Section 15.6 is quasi-complete.

15.45. Prove parts (i), (ii), (v), and (vi) of Lemma 15.34.

15.46. Complete the proof of Theorem 15.35 by showing that the function φ^+ preserves the operations $\cdot$ and $\breve{}$.

15.47. Let $\mathfrak{B}$ be the completion of a Boolean algebra with quasi-complete operators $\mathfrak{A}$, and suppose φ is a mapping from $\mathfrak{A}$ into a Boolean algebra with operators $\bar{\mathfrak{B}}$. If the completion φ^+ extends φ and is quasi-complete, prove that φ must be quasi-complete.

15.48. Given an example to demonstrate the necessity, even for relation algebras, of the hypothesis in Theorem 15.35 that φ be a complete homomorphism. In other words, give an example of an incomplete homomorphism between two relation algebras that cannot be extended to a complete homomorphism between the completions.

15.49. How would one define the notion of a quasi-complete homomorphism between Boolean algebras with operators? Prove that a quasi-complete homomorphism is necessarily a complete homomorphism. (This explains why the notion of a quasi-complete homomorphism was not introduced in the text, for example, for Theorem 15.35.)

15.50. Prove Corollary 15.37

15.51. Suppose $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are Boolean algebras with quasi-complete operators, and $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ are their completions. Let φ be a function of two arguments from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, so that $\varphi(r,s)$ is an element in $\bar{\mathfrak{A}}$ for every pair of elements r and s in $\mathfrak{A}$. The completion of φ is defined to be the function φ^+ of two arguments from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that is defined by

$$\begin{aligned}\varphi^+(p,q) &= \sum\{\varphi(r,s) : r,s \in A \text{ and } r \leq p,\ s \leq q\}\\ &= \sum\{\varphi(r,s) : r \in L_p \text{ and } s \in L_q\}.\end{aligned}$$

Prove that if φ is quasi-complete, then so is φ^+. Prove further that φ^+ is the unique quasi-completely distributive function of two arguments from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ.

15.52. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ be Boolean algebras with quasi-complete operators. Suppose φ and ψ are functions of two arguments from $\mathfrak{A}_1$ into $\mathfrak{A}_2$, and ϑ is a function of two arguments from $\mathfrak{A}_2$ into $\mathfrak{A}_3$. Prove that if these three functions are quasi-complete, then the completion of the composition $\rho = \vartheta(\varphi,\psi)$ (see Exercise 14.60) is equal to the composition of the completion of ϑ with the completions of φ and ψ, that is to say,

$$\rho^+ = \vartheta^+(\varphi^+,\psi^+).$$

15.53. Derive Uniqueness Theorem 15.23 as a corollary of the Existence Theorem 15.35 for completions of complete homomorphisms.

15.54. Derive Uniqueness Theorem 15.23 as a corollary of Minimality Theorem 15.38 and Lemma 15.39.

15.55. Prove the following alternative version of Theorem 15.40 without using that theorem in the proof. If $\mathfrak{A}$ and $\mathfrak{C}$ are Boolean algebra with quasi-complete operators, then $\mathfrak{C}$ is the completion of $\mathfrak{A}$ if and only if $\mathfrak{C}$ is a regular extension of $\mathfrak{A}$ that is complete, and no extension of $\mathfrak{A}$ that is complete is a proper regular subalgebra of $\mathfrak{C}$.

15.56. Give an example to demonstrate the necessity, even for relation algebras, of the hypothesis in Theorem 15.42 that $\mathfrak{A}$ be a regular subalgebra of $\bar{\mathfrak{A}}$. In other words, give an example of relation algebras $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ such that $\mathfrak{A}$ is a subalgebra, but not a regular subalgebra, of $\bar{\mathfrak{A}}$, and the completion of $\mathfrak{A}$ is not a complete subalgebra of the completion of $\bar{\mathfrak{A}}$.

15.57. Prove Theorem 15.44 directly, by showing that the product $\prod_i \mathfrak{B}_i$ of the completions satisfies the conditions in Definition 15.17 for being the completion of the product $\prod_i \mathfrak{A}_i$.

15.58. Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and e is an equivalence element in $\mathfrak{A}$. Prove that the relativization $\mathfrak{B}(e)$ is the completion of the relativization $\mathfrak{A}(e)$.

15.59. Here is the sketch of another proof of Theorem 15.44 for the case of relation algebras. Fill in the missing details. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. The system of units u_i of the relation algebras $\mathfrak{A}_i$ is a partition of unity in $\mathfrak{A}$, so it must also be a partition of unity in $\mathfrak{B}$. Consequently, $\mathfrak{B}$ is the internal product of the relativizations $\mathfrak{B}(u_i)$, for i in I. The relativization $\mathfrak{B}(u_i)$ is the completion of the relativization $\mathfrak{A}(u_i)$, by Exercise 15.58, and $\mathfrak{A}(u_i)$ coincides with $\mathfrak{A}_i$, so $\mathfrak{B}(u_i)$ is the completion of $\mathfrak{A}_i$ for each i.

15.60. Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and M is a complete relation algebraic ideal in $\mathfrak{A}$. Prove that the set N of all elements in $\mathfrak{B}$ that can be written as sums of subsets of M is a complete ideal in $\mathfrak{B}$, and the quotient $\mathfrak{B}/N$ is isomorphic to the completion of the quotient $\mathfrak{A}/M$ via a function that maps r/N to r/M for each element r in $\mathfrak{A}$.

15.61. Two Boolean algebras with quasi-complete operators are said to be *essentially isomorphic* if their completions are isomorphic. Suppose $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are atomic Boolean algebras with quasi-complete operators (of the same similarity type as relation algebras), and φ is a bijection from the set of atoms in $\mathfrak{A}$ onto the set of atoms in $\bar{\mathfrak{A}}$ with the property that

$$\begin{aligned} t \leq r\,;s \quad &\text{if and only if} \quad \varphi(t) \leq \varphi(r)\,;\varphi(s), \\ t \leq r^{\smile} \quad &\text{if and only if} \quad \varphi(t) \leq \varphi(r)^{\smile}, \\ t \leq 1' \quad &\text{if and only if} \quad \varphi(t) \leq 1', \end{aligned}$$

for all atoms r, s, and t in $\mathfrak{A}$. Prove that $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are essentially isomorphic, and in fact φ can be extended to an isomorphism from the completion of $\mathfrak{A}$ to the completion of $\bar{\mathfrak{A}}$.

15.62. Formulate and prove a version of Exercise 15.61 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.63. Formulate and prove a stronger version of Exercise 15.61 that applies to relation algebras.

15.64. Let $\mathfrak{B}$ be the completion of a relation algebra $\mathfrak{A}$. Prove that an element u in $\mathfrak{B}$ is an ideal element if and only if there is a complete relation algebraic filter N in $\mathfrak{A}$ such that $u = \prod N$. Moreover, if such a filter N exists, then $N = \{r \in A : u \leq r\}$, and the restriction of the relativization homomorphism $p \longmapsto p \cdot u$ (from $\mathfrak{B}$ to $\mathfrak{B}(u)$) to the subalgebra $\mathfrak{A}$ has cokernel N. Use these two observations to give a somewhat different proof of Theorem 15.48.

15.65. Prove that the set of complete relation algebraic ideals in a relation algebra $\mathfrak{A}$ is a complete Boolean subalgebra of the Boolean algebra of all complete Boolean ideals in $\mathfrak{A}$.

Chapter 16
Representations

When Tarski first created the abstract theory of relation algebras by proposing in [132] a finite set of axioms for the calculus of relations (essentially equivalent to the axioms presented in Definition 2.1), he immediately raised two questions that were not settle for some time. First, he asked whether every abstract relation algebra—that is to say, is every model of his finite set of axioms—is isomorphic to a concrete algebra of binary relations. This is known as the *representation problem*, and it has been the inspiration for a good deal of the subsequent research in the theory of relation algebras. Second, he asked whether his finite set of axioms is adequate to the task of deriving all laws that are true in every algebra of binary relations. This is called the *completeness problem*. We shall see in Section 18.5 that the two problems are essentially equivalent, although this was not realized at the time Tarski wrote his paper. The focus of the present chapter is on what one might call the *algebra of representations*, namely how one goes about constructing new representations from given ones. The next chapter focuses on concrete examples of representation theorems for specific classes of relation algebras and on examples of the analyses of the different kinds of representations that are possible for a given relation algebra.

16.1 Alternative views of representations

A relation algebra $\mathfrak{A}$ is said to be *representable* if it is isomorphic to a set relation algebra, that is to say, to an algebra of binary relations. An isomorphism φ from $\mathfrak{A}$ to a set relation algebra $\mathfrak{B}$ is called a *represen-*

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9_3

tation of $\mathfrak{A}$, or sometimes a *representation* of $\mathfrak{A}$ *over* U, *with unit* E, when U is the base set and E is the unit relation of the algebra $\mathfrak{B}$. In this last case, U and E are also referred to as the *base set* and the *unit* of the representation φ respectively. The *cardinality* of a representation is defined to be the cardinality of its base set. In particular, a representation is said to be *finite* or *infinite* according to whether the base set of the representation is finite or infinite. Occasionally, the terminology "a representation of $\mathfrak{A}$" is also used to refer to the set relation algebra $\mathfrak{B}$ that is the image of $\mathfrak{A}$ under a representation φ, but this terminology is somewhat misleading: the algebra $\mathfrak{A}$ may well be isomorphic to $\mathfrak{B}$ via several different representations that are not equivalent. We shall refer to $\mathfrak{B}$ as an algebra *representing* $\mathfrak{A}$. If $\mathfrak{B}$ is an algebra representing $\mathfrak{A}$, then the unit E of $\mathfrak{B}$ is always an equivalence relation on the base set U of $\mathfrak{B}$, and $\mathfrak{B}$ is a subalgebra of the full set relation algebra algebra $\mathfrak{Re}(E)$ of all subrelations of E (see the relevant remarks in Section 3.1). Thus, a representation of $\mathfrak{A}$ may be thought of as nothing more than an embedding φ of $\mathfrak{A}$ into the set relation algebra $\mathfrak{Re}(E)$ for some equivalence relation E on a set U.

If the unit relation E is in fact the universal relation on U, then φ is called a *square representation* of $\mathfrak{A}$ over U, and in this case φ is just an embedding of $\mathfrak{A}$ into the algebra $\mathfrak{Re}(U)$ of all binary relations on the set U. The algebra $\mathfrak{Re}(U)$ is simple when U is non-empty (see the remarks following Corollary 9.3), and every subalgebra of a simple relation algebra is simple, by Corollary 9.3, so a relation algebra with a square representation over a non-empty set is necessarily simple.

The converse is also true: a representable relation algebra that is simple has a square representation. For the proof, suppose φ is an embedding of a simple relation algebra $\mathfrak{A}$ into $\mathfrak{Re}(E)$ for some non-empty equivalence relation E. To construct a square representation of $\mathfrak{A}$, let V be any equivalence class of E, and take ψ to be the complete homomorphism from $\mathfrak{Re}(E)$ to $\mathfrak{Re}(V)$ that is defined by

$$\psi(R) = R \cap (V \times V)$$

for every R in $\mathfrak{Re}(E)$ (see Section 7.3 and the remarks at the beginning of Section 7.4). The composition $\psi \circ \varphi$ is a homomorphism from $\mathfrak{A}$ into the full set relation algebra $\mathfrak{Re}(V)$, and the latter is a non-degenerate algebra because its unit $V \times V$ is different from its zero, the empty relation. The assumption that $\mathfrak{A}$ is simple therefore implies that this composition is one-to-one and hence an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(V)$.

Conclusion: the composition is a square representation of $\mathfrak{A}$. These observations may be summarized as follows.

Lemma 16.1. *A representable relation algebra has a square representation over a non-empty set if and only if the algebra is simple.*

A different perspective on representations may be gained by viewing them in terms of relational structures instead of set relation algebras. If φ is a mapping from a relation algebra $\mathfrak{A}$ into the set of relations on some set U, then we may form the relational structure

$$\mathfrak{U}_\varphi = (U\ ,\ \varphi(r))_{r\in A}$$

with U as the universe, and with the binary relations $\varphi(r)$, for r in $\mathfrak{A}$, as the fundamental relations of the structure. The property of φ being a representation of $\mathfrak{A}$ may then be expressed by a collection of sentences in the first-order language of this structure.

To this end, consider a version of the elementary language of relations $\mathcal{L}^*$ that was discussed in Section 2.5. The non-logical symbols of the language include a binary relation symbol for each element in $\mathfrak{A}$. If we use lower case lightface letters r, s, and t to denote elements in $\mathfrak{A}$, and the corresponding upper case lightface letters R, S, and T to denote the relations representing these elements under the mapping φ, then we shall use the corresponding upper case boldface letters $\boldsymbol{R}$, $\boldsymbol{S}$, and $\boldsymbol{T}$ to denote the corresponding relation symbols in $\mathcal{L}^*$. In other words, the relational symbols $\boldsymbol{R}$, $\boldsymbol{S}$, and $\boldsymbol{T}$ in $\mathcal{L}^*$ are interpreted in the structure $\mathfrak{U}_\varphi$ by the relations

$$\varphi(r) = R, \qquad \varphi(s) = S, \qquad \varphi(t) = T,$$

and so on. There are two exceptions to this rule: the relation symbols corresponding to the unit and the identity element in $\mathfrak{A}$ will be denoted by $\boldsymbol{E}$ and $\boldsymbol{I}$ respectively.

Let $\mathcal{T}$ be the set of all sentences of the following forms, where r, s, and t range over the elements in the relation algebra $\mathfrak{A}$:

$\forall \boldsymbol{x}\forall \boldsymbol{y}(\boldsymbol{T}\boldsymbol{x}\boldsymbol{y} \leftrightarrow (\boldsymbol{R}\boldsymbol{x}\boldsymbol{y} \vee \boldsymbol{S}\boldsymbol{x}\boldsymbol{y}))$	whenever	$t = r + s,$	(T1)
$\forall \boldsymbol{x}\forall \boldsymbol{y}(\boldsymbol{T}\boldsymbol{x}\boldsymbol{y} \leftrightarrow (\boldsymbol{E}\boldsymbol{x}\boldsymbol{y} \wedge \neg \boldsymbol{R}\boldsymbol{x}\boldsymbol{y}))$	whenever	$t = -r,$	(T2)
$\forall \boldsymbol{x}\forall \boldsymbol{y}(\boldsymbol{T}\boldsymbol{x}\boldsymbol{y} \leftrightarrow \exists \boldsymbol{z}(\boldsymbol{R}\boldsymbol{x}\boldsymbol{z} \wedge \boldsymbol{S}\boldsymbol{z}\boldsymbol{y}))$	whenever	$t = r\,;s,$	(T3)
$\forall \boldsymbol{x}\forall \boldsymbol{y}(\boldsymbol{T}\boldsymbol{x}\boldsymbol{y} \leftrightarrow \boldsymbol{R}\boldsymbol{y}\boldsymbol{x})$	whenever	$t = r^\smile,$	(T4)
$\forall \boldsymbol{x}\forall \boldsymbol{y}(\boldsymbol{I}\boldsymbol{x}\boldsymbol{y} \leftrightarrow \boldsymbol{x} = \boldsymbol{y}),$			(T5)
$\exists \boldsymbol{x}\exists \boldsymbol{y}\boldsymbol{T}\boldsymbol{x}\boldsymbol{y}$	whenever	$t \neq 0.$	(T6)

When $\mathfrak{A}$ is simple, we may consider one more sentence, namely

$$\forall \boldsymbol{x} \forall \boldsymbol{y} \boldsymbol{E} \boldsymbol{x} \boldsymbol{y}. \tag{T7}$$

The following lemma establishes the connection between representations of $\mathfrak{A}$ and models of $\mathcal{T}$.

Lemma 16.2. *A mapping φ from a relation algebra $\mathfrak{A}$ into the set of all binary relations on a set U is a representation of $\mathfrak{A}$ over U if and only if the set of sentences $\mathcal{T}$ is true in the relational structure $\mathfrak{U}_\varphi$. If φ is such a representation, then it is a square representation over U if and only if the sentence* (T7) *is true in $\mathfrak{U}_\varphi$.*

Proof. The underlying assumption of the lemma is that φ maps the elements in the relation algebra $\mathfrak{A}$ to elements in the set relation algebra $\mathfrak{Re}(U)$. Let r, s, and t be arbitrary elements in $\mathfrak{A}$, and write

$$\varphi(r) = R, \qquad \varphi(s) = S, \qquad \varphi(t) = T. \tag{1}$$

To say that φ preserves the operation of relative multiplication means, in the present context, that

$$t = r \,;\, s \qquad \text{implies} \qquad \varphi(t) = \varphi(r) \,|\, \varphi(s).$$

In view of (1), this implication is equivalent to the implication

$$t = r \,;\, s \qquad \text{implies} \qquad T = R \,|\, S.$$

The final equation holds just in case, for all elements α and β in U, the pair (α, β) belongs to the relation T if and only if there is an element γ in U such that (α, γ) is in R and (γ, β) is in S, by the definition of relational composition, and this is just what the sentence on the left side of (T3) expresses. Consequently, φ preserve relative multiplication if and only if (T3) is true in $\mathfrak{U}_\varphi$.

Entirely analogous arguments show that φ preserves the identity element and the operations of addition, complementation with respect to the relation

$$\varphi(1) = E, \tag{2}$$

and converse if and only if (T5), (T1), (T2), and (T4) are respectively true in $\mathfrak{U}_\varphi$.

The argument so far establishes that φ is a homomorphism from $\mathfrak{A}$ into $\mathfrak{Re}(E)$ (where E is defined by (2)) just in case (T1)–(T5) are

true in $\mathfrak{U}_\varphi$. Assume that φ is such a homomorphism. In this case, φ is one-to-one if and only if

$$t \neq 0 \qquad \text{implies} \qquad \varphi(t) \neq \varnothing,$$

by Lemma 8.37. In view of (1), this implication is equivalent to the implication

$$t \neq 0 \qquad \text{implies} \qquad T \neq \varnothing,$$

The final inequality holds just in case there are elements α and β in U such that the pair (α, β) belongs to T, and this is just what the sentence on the left side of (T6) expresses. Consequently, φ is one-to-one if and only if (T6) is true in $\mathfrak{U}_\varphi$. Conclusion: φ is an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$, and therefore a representation of $\mathfrak{A}$ over U with unit E, if and only if (T1)–(T6) are true in $\mathfrak{U}_\varphi$.

Suppose now that φ is such a representation. To say that φ is square means that the unit E of the representation contains the pair (α, β) for all α and β in U (since the representation is assumed to be over the set U). This is just what the sentence in (T7) asserts, so φ is a square representation if and only if (T7) is true in $\mathfrak{U}_\varphi$. □

One advantage of the alternative view of representations is that it enables one to use well-known techniques from first-order logic, and specifically from model theory, to obtain new representations from given ones. As an example, the next theorem illustrates how the Löwenheim-Skolem-Tarski Theorems—discussed in Section 6.8 in the context of relation algebras—may be used in this way. More sophisticated examples will be given later on.

Theorem 16.3. *If a relation algebra of cardinality κ has a representation—respectively a square representation—over an infinite base set, then it has representations—respectively square representations—over base sets of every infinite cardinality greater than or equal to κ.*

Proof. Let $\mathfrak{A}$ be a relation algebra of cardinality κ, and suppose φ is a representation of $\mathfrak{A}$ over an infinite base set U. The set $\mathcal{T}$ of first-order sentences described in (T1)–(T6) above, with (T7) adjoined to $\mathcal{T}$ when the representation φ is square, is true in the relational structure $\mathfrak{U}_\varphi$ defined in terms of φ (with the base set U as its universe and with the relations $\varphi(r)$, for r in $\mathfrak{A}$, as its fundamental relations), by Lemma 16.2.

Let λ be an infinite cardinal greater than or equal to κ. Apply the downward or the upward Löwenheim-Skolem-Tarski Theorem (see Theorems 6.26 and 6.28) in its relational form—according to whether λ is less than or equal to, or greater than, the cardinality of U—to obtain an elementary substructure or an elementary extension $\mathfrak{V}$ of $\mathfrak{U}_\varphi$ such that the universe of $\mathfrak{V}$ has cardinality λ. The sentences in $\mathcal{T}$ are all preserved under the passage to elementary substructures and elementary extensions, so they are all true in $\mathfrak{V}$. Consequently, the function that maps each element r in $\mathfrak{A}$ to the fundamental relation of $\mathfrak{V}$ that is the interpretation of the relation symbol $\boldsymbol{R}$ associated with r is a representation of $\mathfrak{A}$ (and a square representation when φ is square), by Lemma 16.2. The base set of this representation is the universe of $\mathfrak{V}$, which has cardinality λ. □

The proof of this theorem establishes more than the theorem states. The relational structure $\mathfrak{V}$ in the proof is actually a model of every first-order sentence that is true in the structure $\mathfrak{U}_\varphi$. Consequently, any property of the representation φ that is expressible as a first-order sentence about the fundamental relations of the structure $\mathfrak{U}_\varphi$ is true in $\mathfrak{V}$, and therefore this property is inherited by the representation ψ that corresponds to the structure $\mathfrak{V}$. For example, if the equivalence relation that is the unit of the representation φ has a finite number of equivalence classes, then the equivalence relation that is the unit of the representation ψ has the same finite number of equivalence classes. For another example, if λ is less than or equal to the cardinality of U, then each fundamental relation of $\mathfrak{V}$ is an elementary subrelation of the corresponding relation of $\mathfrak{U}_\varphi$, and consequently ψ is a "contraction" of the representation φ in the sense that the relation $\psi(r)$ is an elementary subrelation of the relation $\varphi(r)$ for every r in $\mathfrak{A}$. In particular, $\psi(r)$ is just the restriction of $\varphi(r)$ to the universe of $\mathfrak{V}$. Similarly, if λ is greater than the cardinality of U, then ψ is an "expansion" of φ in the sense that $\varphi(r)$ is an elementary subrelation of $\psi(r)$ for every r in $\mathfrak{A}$.

16.2 Equivalent representations

A representable relation algebra has, in general, an infinite number of representations. Many of them differ from one another only in the shape of the elements that belong to the base set, and for that reason

they may be viewed as essentially the same, just as isomorphic algebras are generally viewed as essentially the same.

To be more precise, suppose φ is a representation of a relation algebra $\mathfrak{A}$ over a base set U, with unit E, that is to say, φ is an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$. Let V be any set of the same cardinality as U, and δ any bijection from U to V. The function ϑ defined by

$$\vartheta(R) = \{(\delta(\alpha), \delta(\beta)) : (\alpha, \beta) \in R\}$$

for every relation R on U is a base isomorphism from $\mathfrak{Re}(U)$ to $\mathfrak{Re}(V)$, by the remarks in Section 7.5. If F is the image of E under ϑ, then an appropriate restriction of ϑ is a base isomorphism from $\mathfrak{Re}(E)$ to $\mathfrak{Re}(F)$. The composition $\psi = \vartheta \circ \varphi$ is therefore an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(F)$, so it is a representation of $\mathfrak{A}$ over V, with unit F.

Two representations φ and ψ of a relation algebra are said to be *equivalent*, or *base isomorphic*, if there is a bijection δ from the base set of φ to the base set of ψ such that $\psi = \vartheta \circ \varphi$, where ϑ is the base isomorphism induced by δ. In the study of representations, equivalent representations are considered to be essentially the same. Nevertheless, it is often useful to pass from a given representation to an equivalent one that possesses certain desirable properties.

For example, suppose that φ_i is a representation of a relation algebra $\mathfrak{A}_i$ over a base set U_i for each index i in some index set. It is often helpful to assume that the sets U_i are mutually disjoint. This can be accomplished by first selecting, for each index i, a new set V_i of the same cardinality as U_i in such a way that these new sets are mutually disjoint, and then selecting any bijection δ_i from U_i to V_i. If ϑ_i is the base isomorphism from $\mathfrak{Re}(U_i)$ to $\mathfrak{Re}(V_i)$ that is induced by δ_i, then the composition

$$\psi_i = \vartheta_i \circ \varphi_i$$

is a representation of $\mathfrak{A}_i$ over V_i that is equivalent to (and therefore structurally the same as) the original representation φ_i, but the system of these new representations has the additional property that their base sets are disjoint. A system of representations will be said to be *disjoint* if the base sets of the representations are non-empty and mutually disjoint

There is an analogy with abstract relation algebras. Isomorphic relation algebras are considered to be structurally the same. Nevertheless, we occasionally pass from a given relation algebra to an isomorphic copy that has certain additional properties. This is just what the Exchange Principle (Theorem 7.15) and its variants allow us to do. Thus,

the ability to pass to equivalent representations may be viewed as a version of the Exchange Principle for representations.

Representations of a relation algebra that are not equivalent are said to be *inequivalent*. For example, the representations guaranteed by Theorem 16.3 for different infinite cardinals λ are inequivalent because the base sets in question have different cardinalities. Somewhat surprisingly, a relation algebra $\mathfrak{A}$ may have inequivalent representations with respect to the same set relation algebra $\mathfrak{B}$ in the sense that there may exist different isomorphisms φ and ψ from $\mathfrak{A}$ to $\mathfrak{B}$ such that there is no base automorphism ϑ of $\mathfrak{B}$ for which the equation $\psi = \vartheta \circ \varphi$ holds. The next lemma gives a method for computing the number of such inequivalent representations. The matter will be studied in more depth in Section 17.4.

Lemma 16.4. *If a relation algebra $\mathfrak{A}$ is isomorphic to a set relation algebra $\mathfrak{B}$, then the number of inequivalent representations of $\mathfrak{A}$ with respect to $\mathfrak{B}$ is equal to the number of right cosets of the subgroup of base automorphisms of $\mathfrak{B}$ in the group of automorphisms of $\mathfrak{B}$.*

Proof. Let φ be an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Every isomorphism ψ from $\mathfrak{A}$ to $\mathfrak{B}$ can be written in one and only one way as a composition

$$\psi = \varrho \circ \varphi \tag{1}$$

for some automorphism ϱ of $\mathfrak{B}$. In more detail, if ϱ is an automorphism of $\mathfrak{B}$, then (1) is certainly an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and conversely, if ψ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then $\varrho = \psi \circ \varphi^{-1}$ is an automorphism of $\mathfrak{B}$ such that (1) holds (see the diagram below).

$$\mathfrak{B} \xleftarrow{\psi} \mathfrak{A} \xrightarrow{\varphi} \mathfrak{B}$$

Since ϱ and ψ determine each other, in terms of the given isomorphism φ, different choices for the automorphism ϱ lead to different isomorphisms ψ. Two such isomorphisms, say

$$\psi_1 = \varrho_1 \circ \varphi \qquad \text{and} \qquad \psi_2 = \varrho_2 \circ \varphi,$$

are, by definition, equivalent if and only if there is a base automorphism ϑ of $\mathfrak{B}$ such that $\psi_2 = \vartheta \circ \psi_1$, or what amounts to the same thing,

$$\varrho_2 \circ \varphi = \vartheta \circ \varrho_1 \circ \varphi.$$

Compose both sides of this equation on the right with φ^{-1} to see that ψ_1 and ψ_2 are equivalent if and only if there is a base automorphism ϑ of $\mathfrak{B}$ such that $\varrho_2 = \vartheta \circ \varrho_1$. This last equation just expresses that ϱ_1 and ϱ_2 belong to the same right coset of the subgroup of base automorphisms of $\mathfrak{B}$ in the group of automorphisms of $\mathfrak{B}$. Conclusion: two representations of $\mathfrak{A}$ that have $\mathfrak{B}$ as the representing algebra are equivalent if and only if they belong to the same right coset of the subgroup of base automorphisms of $\mathfrak{B}$ in the group of automorphisms of $\mathfrak{B}$. Put a different way, the number of inequivalent representations of $\mathfrak{A}$ that have $\mathfrak{B}$ as the representing algebra is equal to the number of distinct right cosets of the subgroup of base automorphisms of $\mathfrak{B}$ in the group of automorphisms of $\mathfrak{B}$. □

16.3 Complete representations

Every representation φ preserves sums of finite sets of elements as unions of the image relations of those elements in the sense that

$$r = s_0 + s_1 + \cdots + s_{n-1}$$

implies

$$\varphi(r) = \varphi(s_0) \cup \varphi(s_1) \cup \cdots \cup \varphi(s_{n-1}).$$

Some representations φ possess a stronger property: they preserve all existing suprema as unions (and hence all existing infima as intersections) in the sense that if X is an arbitrary subset of the domain algebra for which a supremum r exists, then

$$\varphi(r) = \bigcup\{\varphi(s) : s \in X\}.$$

A representation with this property is said to be *complete*, and a representation that fails to have this property is said to be *incomplete*.

A complete representation of a relation algebra $\mathfrak{A}$ is nothing more than a complete embedding φ from $\mathfrak{A}$ into a full set relation algebra $\mathfrak{Re}(E)$ for some equivalence relation E. If E is the universal relation on some set, then φ is called a *complete square representation* of $\mathfrak{A}$. A relation algebra with a complete representation is said to be *completely representable*. Here is the analogue of Lemma 16.1 for complete representations.

Lemma 16.5. *A completely representable relation algebra has a complete square representation over a non-empty set if and only if the algebra is simple.*

Proof. The proof of the implication from left to right is the same as for Lemma 16.1. To establish the reverse implication, assume that φ is a complete embedding of a simple relation algebra $\mathfrak{A}$ into $\mathfrak{Re}(E)$. To construct a complete square representation of $\mathfrak{A}$, let V be any equivalence class of E, and take ψ to be the complete homomorphism from $\mathfrak{Re}(E)$ to $\mathfrak{Re}(V)$ that is defined by

$$\psi(R) = R \cap (V \times V)$$

for every R in $\mathfrak{Re}(E)$ (see the proof of Lemma 16.1). The composition $\psi \circ \varphi$ is a complete homomorphism from $\mathfrak{A}$ into $\mathfrak{Re}(V)$, by Lemma 7.8. The algebra $\mathfrak{Re}(V)$ is non-degenerate (because the equivalence class V is non-empty), so the assumption that $\mathfrak{A}$ is simple implies that the composition homomorphism $\psi \circ \varphi$ is one-to-one and therefore a complete embedding of $\mathfrak{A}$ into $\mathfrak{Re}(V)$. Conclusion: the composition is a complete square representation of $\mathfrak{A}$. □

As the preceding lemma demonstrates, many results about representations have versions that apply to complete representations. We shall usually refer to the latter by speaking of the *complete version* of the result in question.

The fact that the composition of two complete homomorphisms is a complete homomorphism has other implications as well. For example, any representation equivalent to a complete representation must be complete. In more detail, a base isomorphism preserves all unions that happen to exist. Consequently, if φ is a complete embedding of $\mathfrak{A}$ into a set relation algebra $\mathfrak{Re}(E)$, and if ϑ is a base isomorphism from $\mathfrak{Re}(E)$ to $\mathfrak{Re}(F)$ for some equivalence relation F, then the composition $\vartheta \circ \varphi$ is a complete embedding of $\mathfrak{A}$ into $\mathfrak{Re}(F)$ and therefore a complete representation of $\mathfrak{A}$.

The next lemma contains several useful characterizations of the property of being complete, for representations. We shall say that *suprema are unions* in an algebra of relations $\mathfrak{B}$ if whenever a relation R is the supremum of a set X of relations in $\mathfrak{B}$, then R is actually the union of the relations in X.

Lemma 16.6. *If φ is an isomorphism from a relation algebra $\mathfrak{A}$ to a subalgebra $\mathfrak{B}$ of $\mathfrak{Re}(E)$, then the following conditions are equivalent.*

(i) φ *is a complete representation.*
(ii) $\mathfrak{B}$ *is a regular subalgebra of* $\mathfrak{Re}(E)$.
(iii) *Suprema are unions in* $\mathfrak{B}$.
(iv) *The unit* E *of* $\mathfrak{B}$ *is the union of the set of atoms in* $\mathfrak{B}$.

Proof. The equivalence of (i) and (ii) follows from Lemma 7.9. Condition (iii) is just a rephrasing of condition (ii), since suprema are unions in $\mathfrak{Re}(E)$. The equivalence of (ii) and (iv) follows from Corollary 6.20, again because suprema are unions in $\mathfrak{Re}(E)$. □

If φ is a complete embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$, then the image algebra $\mathfrak{B}$ is a regular subalgebra of $\mathfrak{Re}(E)$, by the preceding lemma. The algebra $\mathfrak{Re}(E)$ is clearly atomic (its atoms are the singletons of pairs in E), and a regular subalgebra of an atomic algebra is atomic, by Theorem 6.18, so $\mathfrak{B}$ must be atomic. Since $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, it follows that $\mathfrak{A}$ is atomic. The following theorem has been proved.

Theorem 16.7. *A completely representable relation algebra is atomic.*

This theorem easily implies the existence of representations that are incomplete. In fact, every representation of a non-atomic relation algebra, and in particular every representation of a non-degenerate atomless relation algebra, is necessarily incomplete, as such an algebra can have no complete representation. The converse of the theorem is false: there exist atomic relation algebras that are representable, but not completely representable. The construction of examples of such algebras is unfortunately rather involved. The first example was given in [89]. In fact, as we shall see, the class of atomic representable relation algebras is elementary in the sense that it is axiomatizable by a set of first-order sentences. However, it was shown in [52] (see also [54] and [59]) that the class of completely representable relation algebras (all of which are necessarily atomic, by Theorem 16.7) is not elementary.

Despite the disparity between representable and completely representable atomic relation algebras, there is an interesting class of relation algebras for which the existence of a representation does imply the existence of a complete representation. Call a relation algebra *locally small* if it is atomic and if the relative product of any two atoms is always the sum of a finite set of atoms.

Theorem 16.8. *A representable relation algebra that is locally small always has a complete representation.*

Proof. Let $\mathfrak{A}$ be a locally small relation algebra, and B the set of atoms in $\mathfrak{A}$. For each element r in $\mathfrak{A}$, write B_r for the set of atoms in $\mathfrak{A}$ that are below r. We shall need the following easily proved facts about these sets of atoms. For all elements r, s, and t in $\mathfrak{A}$, and all subsets X of $\mathfrak{A}$,

$$\begin{aligned} r = -s \quad & \text{implies} \quad & B_r = B \sim B_s, & \quad (1)\\ r = s^\smile \quad & \text{implies} \quad & B_r = \{b^\smile : b \in B_s\}, & \quad (2)\\ r = \textstyle\sum X \quad & \text{implies} \quad & B_r = \bigcup\{B_s : s \in X\}, & \quad (3)\\ r = s \quad & \text{if and only if} \quad & B_r = B_s, & \quad (4) \end{aligned}$$

and $B_1 = B$, where 1 is the unit of $\mathfrak{A}$.

Here, as an example, is the proof of (2). Write

$$C = \{b^\smile : b \in B_s\}$$

and observe that C is a set of atoms in $\mathfrak{A}$, because the converse of an atom is always an atom (see Lemma 4.1(vii)). Since $\mathfrak{A}$ is atomic, the element s is the sum of the set B_s of atoms that are below s. Use this fact, the assumption that r is the converse of s, the complete distributivity of converse, and the definition of C to obtain

$$r = s^\smile = (\textstyle\sum B_s)^\smile = \sum\{b^\smile : b \in B_s\} = \sum C.$$

In a Boolean algebra, if an element is the supremum of a set of atoms, then that set must be the set of all atoms that are below the element. Consequently, the equality $r = \sum C$ implies that C is the set of all atoms below r. In other words, $C = B_r$, as was to be shown.

Assume that φ is a representation of $\mathfrak{A}$ over some base set U. Define a function ψ from $\mathfrak{A}$ into the set of all relations on U by

$$\psi(r) = \bigcup\{\varphi(a) : a \in B_r\} \tag{5}$$

for r in $\mathfrak{A}$. In particular,

$$\psi(1) = \bigcup\{\varphi(a) : a \in B_1\} = \bigcup\{\varphi(a) : a \in B\}. \tag{6}$$

If $r = -s$, then

$$\begin{aligned} \psi(r) &= \bigcup\{\varphi(a) : a \in B_r\}\\ &= \bigcup\{\varphi(a) : a \in B \sim B_s\}\\ &= \bigcup\{\varphi(a) : a \in B\} \sim \bigcup\{\varphi(a) : a \in B_s\}\\ &= \psi(1) \sim \psi(s), \end{aligned}$$

by (5), (1), the disjointness of the representations of distinct atoms in $\mathfrak{A}$ under the mapping φ, and (6). Similarly, if $r = \sum X$, then

$$\begin{aligned}\psi(r) &= \bigcup\{\varphi(a) : a \in B_r\}\\ &= \bigcup\{\varphi(a) : a \in \bigcup_{s\in X} B_s\}\\ &= \bigcup\{\varphi(a) : a \in B_s \text{ for some } s \in X\}\\ &= \bigcup_{s\in X}\bigcup\{\varphi(a) : a \in B_s\}\\ &= \bigcup_{s\in X}\psi(s),\end{aligned}$$

by (5), (3), the definition of the union of a system of sets, and the general associative law for unions. Thus, ψ preserves all existing sums in $\mathfrak{A}$ as unions. These computations imply that ψ is a complete Boolean homomorphism from the Boolean part of $\mathfrak{A}$ into the Boolean set algebra of all subrelations of $\psi(1)$.

Suppose now that an element r is in the kernel of ψ. In this case,

$$\bigcup\{\varphi(a) : a \in B_r\} = \psi(r) = \varnothing, \tag{7}$$

by (5) and the definition of the kernel of a Boolean homomorphism. Thus, $\varphi(a) = \varnothing$ for each atom a in B_r. The image of an atom under a representation is always a non-empty relation, so it follows from (7) that the set B_r must be empty. Consequently,

$$r = \sum B_r = \sum \varnothing = 0.$$

Conclusion: the kernel of ψ contains just zero, so ψ is one-to-one and therefore a complete Boolean monomorphism.

Turn now to the Peircean operations. If $r = s^\smile$, then

$$\begin{aligned}\psi(r) &= \bigcup\{\varphi(a) : a \in B_r\}\\ &= \bigcup\{\varphi(b^\smile) : b \in B_s\}\\ &= \bigcup\{\varphi(b)^{-1} : b \in B_s\}\\ &= (\bigcup\{\varphi(b) : b \in B_s\})^{-1}\\ &= \psi(s)^{-1},\end{aligned}$$

by (5), (2), the preservation properties of the representation φ, and the distributive law for set-theoretic converse over arbitrary unions. Thus, ψ preserves the operation of converse.

If r is the relative product of two atoms, say $r = b\,;c$, then the set B_r of atoms below r is finite, by the assumption that $\mathfrak{A}$ is locally

small, and r is the sum of this finite set, by the assumption that $\mathfrak{A}$ is atomic. Consequently,

$$\psi(b\,;c) = \psi(r) = \bigcup\{\varphi(a) : a \in B_r\} = \varphi(\sum B_r) \\ = \varphi(r) = \varphi(b\,;c) = \varphi(b) \,|\, \varphi(c), \quad (8)$$

by (5), the assumption on r, and the preservation properties of the representation φ. If r is the relative product of two arbitrary elements in $\mathfrak{A}$, say $r = s\,;t$, then

$$s = \sum B_s \qquad \text{and} \qquad t = \sum B_t,$$

by the assumed atomicity of $\mathfrak{A}$, and therefore

$$r = s\,;t = (\sum B_s)\,;(\sum B_t) = \sum\{b\,;c : b \in B_s \text{ and } c \in B_t\}, \quad (9)$$

by the complete distributivity of relative multiplication (see Corollary 4.18). Use (9), the completeness of ψ as a Boolean monomorphism, (8), the distributive law for relational composition over arbitrary unions, and (5) to arrive at

$$\begin{aligned} \psi(r) &= \bigcup\{\psi(b\,;c) : b \in B_s \text{ and } c \in B_t\} \\ &= \bigcup\{\varphi(b) \,|\, \varphi(c) : b \in B_s \text{ and } c \in B_t\} \\ &= (\bigcup\{\varphi(b) : b \in B_s\}) \,|\, (\bigcup\{\varphi(c) : c \in B_t\}) \\ &= \psi(s) \,|\, \psi(t). \end{aligned}$$

Thus, ψ preserves relative multiplication.

The preservation properties established in the preceding paragraphs imply that the relation

$$E = \psi(1) \quad (10)$$

is an equivalence relation on some subset V of the base set U, since

$$E^{-1} = \psi(1)^{-1} = \psi(1^{\smile}) = \psi(1) = E$$

and

$$E \,|\, E = \psi(1) \,|\, \psi(1) = \psi(1\,;1) = \psi(1) = E$$

(see Section 5.3). Indeed, V is just the field of E.

The proof that ψ maps the identity element 1' in $\mathfrak{A}$ to the identity relation id_V is more subtle. For each atom a below 1', we have

$$\varphi(a) \subseteq \varphi(1') = id_U, \tag{11}$$

by the preservation properties of φ, and therefore

$$\psi(1') = \bigcup\{\varphi(a) : a \in B_{1'}\} \subseteq id_U, \tag{12}$$

by (5) and (11). Since $1' \leq 1$ in $\mathfrak{A}$, we have

$$\psi(1') \subseteq \psi(1) = E,$$

by (10) and the Boolean preservation properties of ψ, so the coordinate α of a pair (α, α) in $\psi(1')$ must belong to the field V of the relation E. It follows from this observation and (12) that $\psi(1')$ is actually included in the relation id_V.

To establish the reverse inclusion, consider an element α in V. Because E is an equivalence relation on the set V, the pair (α, α) certainly belongs to E. Consequently, there must be an atom a in B such that (α, α) belongs to $\varphi(a)$, by (6) and (10). The preservation properties of the representation φ imply that

$$\varphi(a) \cap id_U = \varphi(a) \cap \varphi(1') = \varphi(a \cdot 1').$$

Since (α, α) belongs to relation on the left, it must also belong to the relation on the right, so this last relation is not empty and therefore the product $a \cdot 1'$ is not zero, by the preservation properties of φ. It follows that a, being an atom, must be below $1'$ and must therefore belong to the set $B_{1'}$, by the definition of this set. Thus, $\varphi(a)$ is one of the relations that make up union in (12). The pair (α, α) belongs to $\varphi(a)$, so it must belong to $\psi(1')$ as well. Consequently, $\psi(1') = id_V$.

Conclusion: ψ is a complete monomorphism from $\mathfrak{A}$ into $\mathfrak{Re}(E)$. □

A representable relation algebra $\mathfrak{A}$ that is simple and locally small always has a square representation φ over a non-empty base set, by Lemma 16.1. However, the complete representation ψ that is constructed from φ in the preceding proof need not be square, because the unit relation $\psi(1)$ is in general different from the relation $\varphi(1)$. Nevertheless, there always does exist a complete square representation of $\mathfrak{A}$, by Lemma 16.5.

Corollary 16.9. *A representable relation algebra that is simple and locally small always has a complete square representation.*

It is natural to turn around the question about the existence of complete representations, and ask about the existence of *incomplete* representations, that is to say, representations that do not preserve all existing sums as union. A finite relation algebra can have only complete representations, because all sums in such an algebra are finite. An infinite relation algebra, however, if it is representable at all, must have a plethora of incomplete representations, at least one of every cardinality greater than or equal to the cardinality of the algebra. The proof of this assertion is similar to, but somewhat more sophisticated than, the proof of Theorem 16.3.

Theorem 16.10. *An infinite, representable relation algebra always has incomplete representations. In fact, if* $\mathfrak{A}$ *is a representable relation algebra of infinite cardinality* κ*, then* $\mathfrak{A}$ *has incomplete representations over base sets of cardinality* λ *for every cardinal* $\lambda \geq \kappa$*. If* $\mathfrak{A}$ *is also simple, then these incomplete representations may be taken to be square.*

Proof. Let $\mathfrak{A}$ be representable relation algebra of infinite cardinality κ. Each of the representations of $\mathfrak{A}$ is over an infinite base set, because only finite relation algebras can have representations over finite base sets. Also, if $\mathfrak{A}$ is simple, then it must have square representations over infinite base sets, by Lemma 16.1 and the preceding remark.

Consider first the case when $\mathfrak{A}$ is not atomic. The algebra $\mathfrak{A}$ has representations over base sets of all cardinalities $\lambda \geq \kappa$, and these representations may be taken to be square when $\mathfrak{A}$ is simple, by Theorem 16.3. Moreover, each of these representations is incomplete, by Theorem 16.7 and the assumption that $\mathfrak{A}$ is not atomic,.

The case when $\mathfrak{A}$ is atomic is more involved. In contrast to the non-atomic case, one cannot just apply Theorem 16.3, because there is now no assurance that the representations guaranteed by that theorem are incomplete. Consequently, it is necessary to prove a stronger version of that theorem. To this end, fix some representation φ of $\mathfrak{A}$ over an infinite base set U, and assume that the representation φ is square when $\mathfrak{A}$ is simple (see Lemma 16.1). Write $\mathfrak{U}$ for the relational structure associated with φ, so that

$$\mathfrak{U} = (U\ ,\ \varphi(r))_{r \in A}.$$

Let $\mathcal{L}^*$ be the language of first-order logic associated with the structure $\mathfrak{U}$, and let $\mathcal{T}$ be the set of sentences (T1)–(T6), with (T7) adjoined

when $\mathfrak{A}$ is simple (see the remarks preceding Lemma 16.2). Adjoin to the language $\mathcal{L}^*$ two new individual constant symbols, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Take $\mathcal{R}$ to be the set of all sentences of the form $\neg \boldsymbol{R}\boldsymbol{\alpha}\boldsymbol{\beta}$, where $\boldsymbol{R}$ ranges over the relation symbols in $\mathcal{L}^*$ that are associated with the relations $\varphi(r)$ that interpret atoms r in $\mathfrak{A}$.

For every finite subset $\mathcal{F}$ of $\mathcal{R}$, there is an expansion of the structure $\mathfrak{U}$, obtained by adding two new distinguished constants interpreting the symbols $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively, such that the set of sentences

$$\mathcal{T} \cup \mathcal{F} \cup \{\boldsymbol{E}\boldsymbol{\alpha}\boldsymbol{\beta}\} \tag{1}$$

is true in that expansion. For the proof, observe that there are only finitely many relation symbols $\boldsymbol{R}$ from $\mathcal{L}^*$ that can occur in the set of sentences $\mathcal{F}$, because this set is assumed to be finite. Choose an atom s in $\mathfrak{A}$ with the property that $\varphi(s)$ is not the interpretation of any of the relation symbols in $\mathcal{F}$. Such an atom certainly exists, because $\mathfrak{A}$ is assumed to be infinite and atomic, so it has infinitely many atoms, while there are only finitely many atoms r such that $\varphi(r)$ interprets a relation in $\mathcal{F}$. Take (α, β) to be any pair in the relation $\varphi(s)$; such a pair exists because the representation properties of φ imply that $\varphi(s)$ is not empty. Take $\bar{\mathfrak{U}}$ to be the expansion of $\mathfrak{U}$ obtained by adjoining the distinguished constants α and β to the system of fundamental relations in $\mathfrak{U}$ as interpretations of the individual constant symbols $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

It is not difficult to check that the sentences in (1) are true in $\bar{\mathfrak{U}}$. Indeed, the sentences in $\mathcal{T}$ are true in $\bar{\mathfrak{U}}$, because they are true in $\mathfrak{U}$, by Lemma 16.2 and the assumption that φ is a representation of $\mathfrak{A}$. The sentences in $\mathcal{F}$ are true in $\bar{\mathfrak{U}}$ because the pair (α, β) cannot belong to any of the relations $\varphi(r)$ interpreting relation symbols that occur in $\mathcal{F}$. In more detail, if r is an atom in $\mathfrak{A}$ such that $\varphi(r)$ interprets a relation symbol in $\mathcal{F}$, then r must be different from s, by the choice of s, and consequently the relation $\varphi(r)$ must be disjoint from the relation $\varphi(s)$, because φ, as a representation, maps distinct atoms to disjoint relations. The pair (α, β) belongs to $\varphi(s)$, so it cannot belong to $\varphi(r)$. Finally, the last sentence in (1) is true in $\bar{\mathfrak{U}}$ because the predicate $\boldsymbol{E}$ is interpreted in $\bar{\mathfrak{U}}$ by the relation $\varphi(1)$; the inequality $s \leq 1$ and the representation properties of φ imply that the relation $\varphi(s)$ must be included in $\varphi(1)$, so the pair (α, β) must belong to $\varphi(1)$.

It has been shown that every finite subset of the set of sentences

$$\mathcal{T} \cup \mathcal{R} \cup \{\boldsymbol{E}\boldsymbol{\alpha}\boldsymbol{\beta}\} \tag{2}$$

is true in some expansion of the structure $\mathfrak{U}$. Apply the Compactness Theorem for first-order logic (see Section 2.4), in the version that applies to $\mathcal{L}^*$, to conclude that (2) has a model. This model is necessarily infinite because it has infinitely many distinct fundamental relations (one for each element r in $\mathfrak{A}$). The language $\mathcal{L}^*$ has κ non-logical relation symbols, so the upward and downward Löwenheim-Skolem-Tarski Theorems (see Theorems 6.26 and 6.28), in the versions that apply to $\mathcal{L}^*$, ensure that for any cardinal $\lambda \geq \kappa$, the set of sentences in (2) has a model $\bar{\mathfrak{V}}$ in which the universe has cardinality λ (see the proof of Theorem 16.3).

Let $\mathfrak{V}$ be the structure obtained from $\bar{\mathfrak{V}}$ by deleting from the system of fundamental relations the elements, say $\bar{\alpha}$ and $\bar{\beta}$, that interpret the constant symbols $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. Warning: these elements are not deleted from the universe of the structure, but only from the system of fundamental relations. Let ψ be the function that maps each element r in $\mathfrak{A}$ to the interpretation in $\mathfrak{V}$ of the relation symbol $\boldsymbol{R}$ associated with r. Since $\mathfrak{V}$ is a model of $\mathcal{T}$, the function ψ must be a representation of $\mathfrak{A}$ that is square when $\mathfrak{A}$ is simple, by Lemma 16.2. The base set of the representation ψ has cardinality λ, by the choice of the model $\bar{\mathfrak{V}}$. The pair $(\bar{\alpha}, \bar{\beta})$ does not belong to the relation $\psi(r)$ for any atom r in $\mathfrak{A}$, because $\bar{\mathfrak{V}}$ is a model of every sentence in $\mathcal{R}$. On the other hand, $(\bar{\alpha}, \bar{\beta})$ does belong to the unit relation

$$E = \psi(1),$$

because $\bar{\mathfrak{V}}$ is a model of the sentence $\boldsymbol{E\alpha\beta}$. It follows that the unit relation E is not the union of the atomic relations $\psi(r)$, where r ranges over the atoms in $\mathfrak{A}$. Consequently, the representation ψ cannot be complete, by Lemma 16.6. □

16.4 Subalgebras and relativizations

The next task is the determination of those algebraic operations under which the class of representable relation algebras is closed. We begin with the operation of forming subalgebras. Suppose φ is an embedding of a relation algebra $\mathfrak{A}$ into a full set relation algebra $\mathfrak{Re}(E)$. If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then the restriction of φ to $\mathfrak{B}$ is an embedding of $\mathfrak{B}$ into $\mathfrak{Re}(E)$. This observation may be summarized as follows.

Lemma 16.11. *Every subalgebra of a representable relation algebra* $\mathfrak{A}$ *is representable. In fact, every representation of* $\mathfrak{A}$ *restricts to a representation of any subalgebra of* $\mathfrak{A}$.

The complete version of the lemma is not true: not every subalgebra of a completely representable relation algebra is completely representable. In fact, every representable relation algebra is embeddable into a complete set relation algebra in which suprema are unions, namely $\mathfrak{Re}(E)$ for some equivalence relation E, and consequently every representable relation algebra is isomorphic to a subalgebra of a completely representable relation algebra. However, not every representable relation algebra is completely representable, by Theorem 16.7. The basic problem is that, in the passage to a subalgebra, some of the existing suprema may be changed, and other suprema that previously did not exist may appear. The problem can be overcome by considering only subalgebras that preserve suprema.

Lemma 16.12. *Every regular subalgebra of a completely representable relation algebra* $\mathfrak{A}$ *is completely representable. In fact, every complete representation of* $\mathfrak{A}$ *restricts to a complete representation of each regular subalgebra of* $\mathfrak{A}$.

Proof. Let $\mathfrak{A}$ be a completely representable relation algebra, say φ is a complete embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$ for some equivalence relation E. Consider a regular subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. The identity function δ on $\mathfrak{B}$ must be a complete embedding of $\mathfrak{B}$ into $\mathfrak{A}$, by Lemma 7.9. Consequently, the composition $\varphi \circ \delta$ of these two complete embeddings is a complete embedding of $\mathfrak{B}$ into $\mathfrak{Re}(E)$, by Lemma 7.8, and therefore a complete representation of $\mathfrak{B}$. Since δ is the identity function on $\mathfrak{B}$, the composition $\varphi \circ \delta$ is just the restriction of φ to $\mathfrak{B}$. □

Turn next to the operation of forming relativizations.

Lemma 16.13. *Every relativization of a representable relation algebra* $\mathfrak{A}$ *is representable. In fact, every representation of* $\mathfrak{A}$ *restricts to a representation of each relativization of* $\mathfrak{A}$.

Proof. Suppose φ is a representation of $\mathfrak{A}$, say φ maps $\mathfrak{A}$ isomorphically to a subalgebra $\mathfrak{B}$ of $\mathfrak{Re}(F)$, for some equivalence relation F. Let e be an equivalence element in $\mathfrak{A}$, and write

$$E = \varphi(e) \qquad \text{and} \qquad \mathfrak{C} = \mathfrak{Re}(E). \tag{1}$$

The restriction of φ to the relativization $\mathfrak{A}(e)$ maps this relativization isomorphically to the relativization $\mathfrak{B}(E)$, by Lemma 10.4(iii). The relativization $\mathfrak{B}(E)$ is a subalgebra of $\mathfrak{C}(E)$, by Lemma 10.4(i). Finally, the latter algebra is just $\mathfrak{Re}(E)$, by (1) and the remarks in Section 10.1. Combine these observations to conclude that the appropriate restriction of φ is an embedding of $\mathfrak{A}(e)$ into $\mathfrak{Re}(E)$ and therefore a representation of $\mathfrak{A}(e)$. □

The preceding proof establishes more than is actually stated in the lemma, namely, if φ embeds $\mathfrak{A}$ into $\mathfrak{Re}(F)$ and if $E = \varphi(e)$ for some equivalence element e in $\mathfrak{A}$, then the appropriate restriction of φ embeds $\mathfrak{A}(e)$ into $\mathfrak{Re}(E)$. Notice that even when F is the universal relation on some set U, so that φ is a square representation of $\mathfrak{A}$ over the set U, the restriction of φ to $\mathfrak{A}(e)$ will not in general be a square representation of the relativization $\mathfrak{A}(e)$.

Quite surprisingly, there is a kind of converse to Lemma 16.13 that is true: if a relativization $\mathfrak{A}(e)$ of a relation algebra $\mathfrak{A}$ is representable, then the subalgebra of $\mathfrak{A}$ generated by this relativization is also representable. The proof is rather involved and will not be given here; see [35] or [40] for details.

The complete version of Lemma 16.13 is also true. It says that every relativization of a completely representable relation algebra $\mathfrak{A}$ is completely representable, and in fact every complete representation of $\mathfrak{A}$ restricts to a complete representation of any relativization of $\mathfrak{A}$. The proof is similar to the proof of Lemma 16.13. All references to representations and subalgebras in that proof must be replaced by references to complete representations and regular subalgebras, the reference to part (i) of Lemma 10.6 must be replaced by a reference to part (iii) of that lemma, and Lemma 7.9 must be used.

The next corollary extends Lemma 16.13 to (generalized) relativizations that involve ideal elements belonging to extension algebras (see Section 10.5).

Corollary 16.14. *If $\mathfrak{B}$ is a subalgebra of a representable relation algebra $\mathfrak{A}$, and e an ideal element in $\mathfrak{A}$, then the relativization $\mathfrak{B}(e)$ is representable. In fact, every representation of $\mathfrak{A}$ restricts to a representation of $\mathfrak{B}(e)$.*

Proof. Let φ be a representation of $\mathfrak{A}$. An appropriate restriction of φ—call it ψ—is a representation of $\mathfrak{A}(e)$, by Lemma 16.13. The relativization $\mathfrak{B}(e)$ is a subalgebra of $\mathfrak{A}(e)$, by Lemma 10.12, so

Lemma 16.11 may be applied to conclude that a restriction of ψ is a representation of $\mathfrak{B}(e)$. □

As with Lemma 16.11, the complete version of this corollary is not true. The hypotheses of the corollary must be strengthened to obtain a valid result. We begin with a lemma.

Lemma 16.15. *If $\mathfrak{B}$ is a dense subalgebra of a relation algebra $\mathfrak{A}$, and e an ideal element in $\mathfrak{A}$, then the relativization $\mathfrak{B}(e)$ is a dense subalgebra, and hence a regular subalgebra, of the relativization $\mathfrak{A}(e)$.*

Proof. Consider an arbitrary non-zero element s in $\mathfrak{A}(e)$, and observe that s is below e, by the definition of $\mathfrak{A}(e)$. The subalgebra $\mathfrak{B}$ is assumed to be dense in $\mathfrak{A}$, so there must be a non-zero element r in $\mathfrak{B}$ that is below s. In particular, r is also below e, so $r = r \cdot e$, and consequently r belongs to the relativization $\mathfrak{B}(e)$. Conclusion: for every non-zero element s in $\mathfrak{A}(e)$, there is a non-zero element r in $\mathfrak{B}(e)$ that is below s, so $\mathfrak{B}(e)$ is dense in $\mathfrak{A}(e)$, as claimed. Apply Lemma 10.12 and the version of Lemma 15.6 that applies to relation algebras to conclude that $\mathfrak{B}(e)$ is a regular subalgebra of $\mathfrak{A}(e)$. □

Corollary 16.16. *If $\mathfrak{B}$ is a dense subalgebra of a completely representable relation algebra $\mathfrak{A}$, and e an ideal element in $\mathfrak{A}$, then the relativization $\mathfrak{B}(e)$ is completely representable.*

Proof. The algebra $\mathfrak{A}$ is assumed to be completely representable, so its relativization $\mathfrak{A}(e)$ must also be completely representable, by the complete version of Lemma 16.13. Because $\mathfrak{B}$ is assumed to be dense in $\mathfrak{A}$, the relativization $\mathfrak{B}(e)$ is a regular subalgebra of the relativization $\mathfrak{A}(e)$, by Lemma 16.15. Apply Lemma 16.12 (with $\mathfrak{A}(e)$ and $\mathfrak{B}(e)$ in place of $\mathfrak{A}$ and $\mathfrak{B}$ respectively) to conclude that $\mathfrak{B}(e)$ is completely representable. □

16.5 Simple algebras

Many of the most important basic examples of relation algebras are simple relation algebras. Moreover, every relation algebra is a subdirect product—and every finite relation algebra is a direct product—of simple relation algebras (see the Semi-simplicity Theorem 12.10 and Corollary 11.45). Consequently, a key step in the analysis of representations of relation algebras is the analysis of representations of simple

relation algebras. That is the goal of this section. We shall need the following easy consequence of Lemma 11.50 and Theorem 11.40.

Lemma 16.17. *Let $\mathfrak{A}$ be a relation algebra, and $(E_i : i \in I)$ a non-empty system of disjoint equivalence relations with union E. For each index i, let φ_i be a homomorphism from $\mathfrak{A}$ into $\mathfrak{Re}(E_i)$ with kernel M_i. If $\bigcap_i M_i = \{0\}$, then the internal amalgamation φ defined by*

$$\varphi(r) = \bigcup_i \varphi_i(r)$$

for r in $\mathfrak{A}$ is an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$, and therefore a representation of $\mathfrak{A}$. This representation is complete if and only if each homomorphism φ_i is complete.

It was shown Lemma 16.1 that a simple representable relation algebra $\mathfrak{A}$ must have a square representation. It turns out that the arbitrary representations of $\mathfrak{A}$ are precisely the amalgamations of systems of disjoint square representation of $\mathfrak{A}$.

Theorem 16.18. *The representations of a simple relation algebra $\mathfrak{A}$ are exactly the internal amalgamations of non-empty systems of disjoint square representations of $\mathfrak{A}$. More precisely, every internal amalgamation of a non-empty system of disjoint square representations of $\mathfrak{A}$ is a representation of $\mathfrak{A}$, and every representation of $\mathfrak{A}$ can be written in one and only one way as an internal amalgamation of a non-empty system of disjoint square representations of $\mathfrak{A}$.*

Proof. Assume first that $(\varphi_i : i \in I)$ is a non-empty system of square representations of $\mathfrak{A}$ over disjoint base sets U_i. (Notice that these base sets are automatically non-empty, because $\mathfrak{A}$ is simple and each representation φ_i is one-to-one.) The kernel of each of these representations is the trivial ideal (in $\mathfrak{A}$), by Lemma 8.37, because representations are monomorphisms. Consequently, the intersection of the kernels is the trivial ideal. Apply Lemma 16.17 (with $E_i = U_i \times U_i$) to conclude that the amalgamation of the given system of representations is a representation of $\mathfrak{A}$.

Consider now an arbitrary representation φ of $\mathfrak{A}$. Since $\mathfrak{A}$ is non-degenerate, the representation φ must embed $\mathfrak{A}$ into $\mathfrak{Re}(E)$ for some equivalence relation E on a non-empty set. Let $(U_i : i \in I)$ be a listing of the distinct equivalence classes of E, and notice that the index set I is non-empty, and the equivalence classes U_i are non-empty and mutually disjoint. The set relation algebra $\mathfrak{Re}(E)$ is the internal

product of the system of set relation algebras $\mathfrak{Re}(U_i)$, for i in I, by Decomposition Theorem 11.43. Use the assumed simplicity of $\mathfrak{A}$ and Corollary 11.52 to conclude that the ith projection of φ, that is to say, the function φ_i from $\mathfrak{A}$ into $\mathfrak{Re}(U_i)$ that is defined by

$$\varphi_i(r) = \varphi(r) \cap (U_i \times U_i)$$

for r in $\mathfrak{A}$, is an embedding, and therefore a square representation of $\mathfrak{A}$, and that φ is the amalgamation of these disjoint square representations.

To prove the uniqueness assertion of the theorem, consider a non-empty system $(\psi_j : j \in J)$ of disjoint square representations of $\mathfrak{A}$, say ψ_j embeds $\mathfrak{A}$ into $\mathfrak{Re}(V_j)$. The sets V_j are non-empty (because $\mathfrak{A}$ is non-degenerate and ψ_j is one-to-one) and mutually disjoint (by assumption). Suppose that the representation φ from the preceding paragraph is the amalgamation of this system, so that

$$\varphi(r) = \bigcup_j \psi_j(r) \tag{1}$$

for r in $\mathfrak{A}$. In particular,

$$E = \varphi(1) = \bigcup_j \psi_j(1) = \bigcup_j (V_j \times V_j), \tag{2}$$

by the assumption that φ embeds $\mathfrak{A}$ into $\mathfrak{Re}(E)$, (1), and the assumption that ψ_j embeds $\mathfrak{A}$ into $\mathfrak{Re}(V_j)$. The equivalence relation E is the union of the squares of its equivalence classes U_i, so it follows from (2) and the fact that the sets V_j are non-empty and mutually disjoint that $(V_j : j \in J)$ is a listing of the distinct equivalence classes of E. By reindexing these equivalence classes, it may be assumed that $J = I$, and $V_i = U_i$ for each i. Consequently, ψ_i embeds $\mathfrak{A}$ into $\mathfrak{Re}(U_i)$ for each i. Apply Corollary 11.52 to conclude that $\varphi_i = \psi_i$ for each i. □

The complete version of the preceding theorem holds with the word "representation" replaced everywhere by "complete representation". The proof is entirely analogous to the preceding proof.

16.6 Products

Representations of direct products can be obtained from representations of the factor algebras by forming the product of the factor representations, just as one forms the product of any system of homomorphisms (see Section 11.13). The converse is also true, modulo a

condition on the product representation. We formulate the condition and the resulting theorem in terms of the notion of an internal product.

A representation φ of the internal product of a system $(\mathfrak{A}_i : i \in I)$ of relation algebras is said to *preserve the supremum property* if

$$\varphi(r) = \bigcup_i \varphi(r_i)$$

whenever $r = \sum_i r_i$ is an element in the product (with r_i in $\mathfrak{A}_i$ for each i). Preserving the supremum property is weaker than being a complete representation, because a complete representation preserves all existing suprema as unions, and not just the suprema of systems of elements from distinct factor algebras. If I is a finite set, then every representation φ of the product will automatically preserve the supremum property, since φ preserves all finite sums as unions; however, φ may not preserve all existing suprema as unions.

Theorem 16.19. *If $\mathfrak{A}$ is the internal product of a system $(\mathfrak{A}_i : i \in I)$ of non-degenerate relation algebras, then the representations of $\mathfrak{A}$ that preserve the supremum property are exactly the products of systems of disjoint representations of the factor algebras. More precisely, every product of a system of disjoint representations of the factor algebras is a representation of $\mathfrak{A}$ that preserves the supremum property, and every representation of $\mathfrak{A}$ that preserves the supremum property can be written in one and only one way as the product of a system of disjoint representations of the factor algebras.*

Proof. Suppose $(\varphi_i : i \in I)$ is a system of disjoint representations of the given factor algebras; say φ_i embeds $\mathfrak{A}_i$ into $\mathfrak{Re}(E_i)$ for each i, and the equivalence relations E_i and E_j are disjoint when $i \neq j$. Write

$$E = \bigcup_i E_i.$$

The internal product of the system of set relation algebras

$$(\mathfrak{Re}(E_i) : i \in I)$$

is the set relation algebra $\mathfrak{Re}(E)$, by the Relativization Decomposition Theorem 11.40, and the product of the given system of representations is, by definition, the function φ from $\mathfrak{A}$ into $\mathfrak{Re}(E)$ that is defined by

$$\varphi(r) = \bigcup_i \varphi_i(r_i)$$

for each $r = \sum_i r_i$ in $\mathfrak{A}$ (with r_i in $\mathfrak{A}_i$ for each i). The function φ is an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$ that extends each of the factor representations φ_i, by Lemma 11.48. In particular, $\varphi(r_i) = \varphi_i(r_i)$ for each r_i in $\mathfrak{A}_i$, and therefore

$$\varphi(r) = \bigcup_i \varphi_i(r_i) = \bigcup_i \varphi(r_i)$$

for each $r = \sum_i r_i$ in $\mathfrak{A}$. Thus, φ preserves the supremum property.

Consider now any representation φ of $\mathfrak{A}$ that preserves the supremum property, say φ maps $\mathfrak{A}$ isomorphically to a subalgebra $\mathfrak{B}$ of $\mathfrak{Re}(E)$ for some equivalence relation E. Let u_i be the unit of $\mathfrak{A}_i$, and observe that $(u_i : i \in I)$ is an orthogonal system of ideal elements in $\mathfrak{A}$ with the property that

$$\mathfrak{A}_i = \mathfrak{A}(u_i) \tag{1}$$

for each i, by Product Decomposition Theorem 11.39. Put $E_i = \varphi(u_i)$, and observe that

$$(E_i : i \in I)$$

must be an orthogonal system of ideal elements in $\mathfrak{B}$, by the isomorphism properties of φ. In particular, the relations in this system must be non-empty and mutually disjoint, and $\mathfrak{B}$ must be the internal product of the relativizations $\mathfrak{B}(E_i)$ for i in I (see Theorem 11.49 and its proof).

Apply Homomorphism Decomposition Theorem 11.49 to arrive at the following conclusions. First, the restriction of φ to the relativization $\mathfrak{A}(u_i)$—call it φ_i—is an isomorphism from $\mathfrak{A}(u_i)$ to $\mathfrak{B}(E_i)$, and therefore φ_i is a representation of $\mathfrak{A}_i$, by (1) and the fact that $\mathfrak{B}(E_i)$ is a subalgebra of $\mathfrak{Re}(E_i)$ (see Lemma 10.4(i)). Second, φ is the product of the system of representations $(\varphi_i : i \in I)$ in the sense that

$$\varphi(r) = \sum_i \varphi_i(r_i)$$

(in $\mathfrak{B}$) for $r = \sum_i r_i$ in $\mathfrak{A}$ (with r_i in $\mathfrak{A}_i$ for each i). Since φ is assumed to preserve the supremum property, and since φ_i is the restriction of φ to $\mathfrak{A}_i$, the preceding equation may rewritten as

$$\varphi(r) = \bigcup_i \varphi(r_i) = \bigcup_i \varphi_i(r_i).$$

Third, if φ is the product of any system $(\psi_i : i \in I)$ of disjoint representations of the given factor algebras $\mathfrak{A}_i$, then $\psi_i = \varphi_i$ for each i. Consequently, φ can be written in one and only one way as the product of a system of disjoint representations of the given factor algebras. □

The complete version of the preceding theorem is also true. It says that the complete representations of the internal product $\mathfrak{A}$ are exactly the products of systems of disjoint complete representations of the given factor algebras $\mathfrak{A}_i$. Furthermore, every complete representation of $\mathfrak{A}$ can be written in just one way as a product of such a system of complete representations of the factor algebras. The proviso regarding the preservation of the supremum property is not needed in this case.

Corollary 16.20. *A product of relation algebras is representable if and only if each factor algebra is representable.*

Proof. Assume that $\mathfrak{A}$ is the internal product of a system of relation algebras $(\mathfrak{A}_i : i \in I)$. If each factor algebra $\mathfrak{A}_i$ is representable, then by passing to equivalent representations, it may be assumed that there is a system of disjoint representations of the factor algebras (see the relevant remarks in Section 16.2). The product of this system of disjoint representations is a representation of $\mathfrak{A}$, by Theorem 16.19.

In the reverse direction, suppose φ is a representation of $\mathfrak{A}$. The unit, say u_i, of the factor algebra $\mathfrak{A}_i$ is an ideal element in $\mathfrak{A}$, and $\mathfrak{A}_i$ is just the relativization $\mathfrak{A}$ to u_i, by Product Decomposition Theorem 11.39. Consequently, the restriction of φ to $\mathfrak{A}_i$ is a representation of $\mathfrak{A}_i$, by Lemma 16.13. □

Notice that Theorem 16.19 cannot be applied directly to prove the reverse direction of the preceding corollary, because it is not assumed that the representation φ preserves the supremum property.

The complete version of the corollary is also true. It says that a product of relation algebras is completely representable if and only if each factor algebra is completely representable. The proof is analogous to the proof of Corollary 16.20.

Theorems 16.18 and 16.19 may be applied, in particular, to representable relation algebras that are totally decomposable. They reduce the problem of describing all possible representations (or complete representations) of such algebras to the problem of describing all square representations (or all complete square representations) of the simple factor algebras.

16.7 Canonical extensions

We have seen that subalgebras of representable relation algebras are representable. It is natural to ask about the converse: under what circumstances does the representability of a relation algebra $\mathfrak{A}$ imply the representability of some relation algebraic extension of $\mathfrak{A}$? There are relation algebras that are not representable at all (examples will be given in the next chapter), but every relation algebra has at least one representable subalgebra, namely its minimal subalgebra, (see Theorem 13.6). Turn this observation around, by taking $\mathfrak{A}$ to be a minimal subalgebra of a non-representable relation algebra, to arrive at the conclusion that the representability of a relation algebra $\mathfrak{A}$ does not automatically imply the representability of a relation algebraic extension of $\mathfrak{A}$.

The question assumes a sharper focus if we narrow our attention to two specific types of extensions, namely canonical extensions and completions. The situation regarding the representability of canonical extensions is somewhat involved, but still quite satisfactory. Not every representation of a relation algebra $\mathfrak{A}$ can be extended to a representation of the canonical extension of $\mathfrak{A}$; but if $\mathfrak{A}$ is representable at all, then there does exist at least one representation of $\mathfrak{A}$ that *can* be extended to a representation—and in fact to a complete representation—of the canonical extension of $\mathfrak{A}$. The proof makes use of the alternative view of representations that was discussed in Section 16.1. Here is the key lemma.

Lemma 16.21. *Every set relation algebra $\mathfrak{A}$ is isomorphic to a set relation algebra satisfying the conditions of Theorem 14.50. This isomorphic copy of $\mathfrak{A}$ may be taken to be a square set relation algebra if $\mathfrak{A}$ is a square set relation algebra.*

Proof. Fix a set relation algebra $\mathfrak{A}$ with base set U and unit E, and take $\mathfrak{U}$ to be the relational structure

$$\mathfrak{U} = (U\,,\, R)_{R\in A}. \tag{1}$$

Let $\mathcal{L}^*$ be the language of first-order logic associated with the structure in (1), and let $\mathcal{T}$ be the set of sentences (T1)–(T6) given before Lemma 16.2, with the sentence (T7) adjoined when $E = U \times U$. Adjoin to $\mathcal{L}^*$ a large supply of new individual constant symbols, and in particular a new constant symbol $\boldsymbol{\alpha}$ for every element α in U. (The exact

number of new constant symbols is not important, but there should be enough so that new, as yet unused constant symbols are always available when needed.)

We proceed to construct, by induction on natural numbers n, a sequence of relational structures

$$\mathfrak{U}_n = (U_n\,,\, R_n)_{R\in A} \tag{2}$$

of the same similarity type as the structure in (1), with the following properties. First, $\mathfrak{U}_0$ coincides with $\mathfrak{U}$. Second, $\mathfrak{U}_n$ is a model of $\mathcal{T}$ for each n. Third, $\mathfrak{U}_n$ is a substructure of $\mathfrak{U}_{n+1}$ in the sense that

$$U_n \subseteq U_{n+1}, \qquad \text{and} \qquad R_n = R_{n+1} \cap (U_n \times U_n) \tag{3}$$

for each n and each relation R in $\mathfrak{A}$. Fourth, if X is any ultrafilter in $\mathfrak{A}$, then

$$\bigcap\{R_1 : R \in X\} \neq \varnothing \tag{4}$$

(where R_1 is the fundamental relation in $\mathfrak{U}_1$ that corresponds to the relation R in $\mathfrak{A}$). Fifth, for all triples of ultrafilters X, Y, and Z in $\mathfrak{A}$ satisfying the condition

$$X \,|\, Y \subseteq Z \tag{5}$$

from Theorem 14.50, and for all positive integers n, if a pair of elements (α, β) belongs to the intersection

$$\bigcap\{R_n : R \in Z\}, \tag{6}$$

then there is an element γ in U_{n+1} such that the pairs (α, γ) and (γ, β) belong to the intersections

$$\bigcap\{R_{n+1} : R \in X\} \qquad \text{and} \qquad \bigcap\{R_{n+1} : R \in Y\} \tag{7}$$

respectively. The purpose of the fourth and fifth properties is to ensure that conditions (i) and (ii) in Theorem 14.50 are respectively satisfied.

We complete the proof under the assumption that such a sequence of relational structures $\mathfrak{U}_n$ has been constructed. An easy argument by induction on positive integers k, using (3), shows that $\mathfrak{U}_n$ is a substructure of $\mathfrak{U}_{n+k}$ for all positive integers k, that is to say,

$$U_n \subseteq U_{n+k}, \qquad \text{and} \qquad R_n = R_{n+k} \cap (U_n \times U_n). \tag{8}$$

Indeed, when $k = 1$ the formulas in (8) hold by the assumed properties in (3). Suppose now as the induction hypothesis that (8) holds for a given positive integer k. First,

$$U_n \subseteq U_{n+k} \subseteq U_{n+k+1},$$

by the induction hypothesis on the left side of (8) and the assumed property on the left side of (3) (with $n+k$ in place of n). Second,

$$\begin{aligned} R_n &= R_{n+k} \cap (U_n \times U_n) \\ &= [R_{n+k+1} \cap (U_{n+k} \times U_{n+k})] \cap (U_n \times U_n) \\ &= R_{n+k+1} \cap [(U_{n+k} \times U_{n+k}) \cap (U_n \times U_n)] \\ &= R_{n+k+1} \cap (U_n \times U_n), \end{aligned}$$

by the induction hypotheses in (8) and the assumed properties in (3) (with $n+k$ in place of n).

Write

$$U_\omega = \bigcup_n U_n, \qquad \text{and} \qquad R_\omega = \bigcup_n R_n \tag{9}$$

for every relation R in $\mathfrak{A}$, and observe that

$$R_\omega \cap (U_m \times U_m) = \bigcup_n [R_n \cap (U_m \times U_m)] = R_m \tag{10}$$

for each m, by (8) and (9). It is not difficult to verify the following implications:

$$T = R \cup S \quad \text{implies} \quad T_\omega = R_\omega \cup S_\omega, \tag{11}$$
$$T = E \sim R \quad \text{implies} \quad T_\omega = E_\omega \sim R_\omega, \tag{12}$$
$$T = R \,|\, S \quad \text{implies} \quad T_\omega = R_\omega \,|\, S_\omega, \tag{13}$$
$$T = R^{-1} \quad \text{implies} \quad T_\omega = (R_\omega)^{-1}, \tag{14}$$
$$T = id_U \quad \text{implies} \quad T_\omega = id_{U_\omega}, \tag{15}$$
$$T = U \times U \quad \text{implies} \quad T_\omega = U_\omega \times U_\omega. \tag{16}$$

As an example, here is the verification of (13).

Suppose $T = R \,|\, S$. Each relational structure in (2) is assumed to be a model of $\mathcal{T}$, so

$$T_n = R_n \,|\, S_n \tag{17}$$

for every natural number n, by (T3) (with $R \,|\, S$ and T in place of $r\,;s$ and t respectively). If a pair (α, β) belongs to T_ω, then there must be a natural number n such that (α, β) is in T_n, by the definition of T_ω in (9) (with T in place of R). In view of (17), there must be an element γ such that (α, γ) belongs to R_n and (γ, β) to S_n. Since R_n and S_n are included in R_ω and S_ω respectively, by (9), the pairs (α, γ) and (γ, β) must belong to R_ω and S_ω respectively. Consequently, (α, β) belongs

to the relation $R_\omega \mid S_\omega$, by the definition of relational composition. This proves that T_ω is included in $R_\omega \mid S_\omega$. To establish the reverse inclusion, assume that (α, β) belongs to $R_\omega \mid S_\omega$. There must then be an element γ such that (α, γ) is in R_ω and (γ, β) in S_ω, by the definition of relational composition. Consequently, there exist natural numbers m and n such that (α, γ) is in R_m and (γ, β) in S_n, by (9). Without loss of generality, it may be assumed that $m \leq n$. In this case, the pair (α, γ) is also in R_n, by (8), so (α, β) belongs to $R_n \mid S_n$, by the definition of relational composition. It follows from (17) that (α, β) is in T_n, and therefore also in T_ω, by (9) (with T in place of R). This completes the verification of (13). The other implications are established in a similar fashion.

The point of the preceding argument is that the structures in (2) form a chain of substructures, by the third property, and the sentences in the set $\mathcal{T}$ are universal-existential in form. All such sentences are preserved under unions of chains of substructures, as the argument in the preceding paragraph indicates. Since the sentences in $\mathcal{T}$ are true in the structures in (2), by the second property, they must be true in the structure

$$\mathfrak{U}_\omega = (U_\omega\,,\, R_\omega)_{R \in A}$$

that is the union of the structures in (2).

The implications (11)–(16) imply that the function φ defined by

$$\varphi(R) = R_\omega \tag{18}$$

for R in $\mathfrak{A}$ is a homomorphism from $\mathfrak{A}$ into the set relation algebra $\mathfrak{Re}(E_\omega)$. For instance, if $T = R \mid S$, then

$$\varphi(R \mid S) = \varphi(T) = T_\omega = R_\omega \mid S_\omega = \varphi(R) \mid \varphi(S),$$

so that φ preserves the operation of relative multiplication. The remaining preservation properties of φ are established in a completely analogous fashion. If R is not empty, then $\varphi(R)$ cannot be not empty, since $R = R_0 \subseteq R_\omega$, by (9). Consequently, the kernel of φ contains just the empty relation, so φ is a monomorphism, by Lemma 8.37.

Let $\mathfrak{A}_\omega$ be the subalgebra of the set relation algebra $\mathfrak{Re}(E_\omega)$ that is the isomorphic image of $\mathfrak{A}$ under the mapping φ. Notice that the unit relation E_ω is the universal relation on the set U_ω if E is the universal relation on the set U, by (16). The next step is to prove that $\mathfrak{A}_\omega$ satisfies conditions (i) and (ii) of Theorem 14.50. The mapping φ carries each ultrafilter X in $\mathfrak{A}$ to an ultrafilter

$$X_\omega = \{R_\omega : R \in X\} \tag{19}$$

in $\mathfrak{A}_\omega$, and every ultrafilter in $\mathfrak{A}_\omega$ is the image under φ of a uniquely determined ultrafilter in $\mathfrak{A}$, by the isomorphism properties of φ. Consequently, when considering images under φ of ultrafilters in $\mathfrak{A}$, we are really considering ultrafilters in $\mathfrak{A}_\omega$.

To verify condition (i), assume that X is a ultrafilter in $\mathfrak{A}$. Apply (19), (9), and (4) to obtain

$$\bigcap X_\omega = \bigcap\{R_\omega : R \in X\} \supseteq \bigcap\{R_1 : R \in X\} \neq \varnothing.$$

To verify condition (ii), assume that X, Y, and Z are ultrafilters in $\mathfrak{A}$ satisfying (5). Observe that (5) is equivalent to the inclusion

$$X_\omega \,|\, Y_\omega \subseteq Z_\omega,$$

by the isomorphism properties of φ. Consider an arbitrary pair (α, β) in $\bigcap Z_\omega$. Thus, (α, β) belongs to the relation R_ω for every R in Z, by (19) (with Z in place of X). The elements α and β belong to the set U_n for some positive integer n, by (8) and (9), so the pair (α, β) belongs to the relation R_n for every R in Z, by (10). Consequently, (α, β) belongs to the intersection in (6). Apply the fifth property of the sequence of structures in (2) to obtain an element γ in U_{n+1} such that the pairs (α, γ) and (γ, β) respectively belong to the sets on the left and right in (7). Since R_{n+1} is included in R_ω for every relation R in $\mathfrak{A}$, by (9), it follows

$$(\alpha, \gamma) \in \bigcap X_\omega \qquad \text{and} \qquad (\gamma, \beta) \in \bigcap Y_\omega,$$

as required in condition (ii).

The task remains of constructing the sequence of structures in (2). Take $\mathfrak{U}_0$ to be the structure in (1). The first property, and the second property for the case $n = 0$, hold, by the choice of $\mathfrak{U}_0$ and by Lemma 16.2 (since the identity function on $\mathfrak{A}$ is clearly a representation of $\mathfrak{A}$).

The next step is the construction of the structure $\mathfrak{U}_1$. Define two sets of sentences, $\mathcal{R}_1$ and $\mathcal{S}_1$, in $\mathcal{L}^*$. The set $\mathcal{R}_1$ consists of all sentences of the form

$$\boldsymbol{R\alpha\beta} \qquad \text{whenever} \qquad (\alpha, \beta) \in R_0,$$

and all sentences of the form

$$\neg \boldsymbol{R\alpha\beta} \qquad \text{whenever} \qquad (\alpha, \beta) \notin R_0,$$

where R ranges over the relations in $\mathfrak{A}$, and α and β over the elements in $\mathfrak{U}_0$. (Keep in mind that $R_0 = R$.) Notice, in particular, that the sentence $\neg \boldsymbol{I\alpha\beta}$ is in $\mathcal{R}_1$ whenever $\alpha \neq \beta$. Let $\bar{\mathfrak{U}}_0$ be the expansion of $\mathfrak{U}_0$ obtained by adding to its system of fundamental relations the distinguished constants α in $\mathfrak{U}_0$ interpreting the individual constant symbols $\boldsymbol{\alpha}$ used in the definition of the set $\mathcal{R}_1$. This expansion is obviously a model of the set of sentences

$$\mathcal{T} \cup \mathcal{R}_1, \tag{20}$$

by the definition of $\mathcal{R}_1$ and the fact that $\mathfrak{U}_0$ is a model of $\mathcal{T}$. The purpose of the sentences in $\mathcal{R}_1$ is to ensure that the structure $\mathfrak{U}_1$ (to be constructed) is an extension of $\mathfrak{U}_0$.

The set $\mathcal{S}_1$ consists of all sentences that are determined in the following way. For each ultrafilter X in $\mathfrak{A}$, choose two new (distinct) individual constant symbols in $\mathcal{L}^*$, say $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, that do not occur in the sentences in $\mathcal{R}_1$ and that have not been used for the other ultrafilters in $\mathfrak{A}$, and put the sentence $\boldsymbol{R\gamma\delta}$ into the set $\mathcal{S}_1$ for each relation R in X. The purpose of the sentences in $\mathcal{S}_1$ is to ensure that the structure $\mathfrak{U}_1$ possesses the fourth property.

For every finite subset $\mathcal{F}$ of $\mathcal{S}_1$, there is an expansion of the relational structure $\bar{\mathfrak{U}}_0$, obtained by adding new distinguished constants to the system of fundamental relations, that is a model of the set of sentences

$$\mathcal{T} \cup \mathcal{R}_1 \cup \mathcal{F}. \tag{21}$$

Indeed, for any ultrafilter X in $\mathfrak{A}$, there can be only finitely many relations $R^0, \ldots, R^{\ell-1}$ in X such that the sentences $\boldsymbol{R}^i\boldsymbol{\gamma\delta}$ occur in $\mathcal{F}$, by the assumed finiteness of the set $\mathcal{F}$. The intersection $\bigcap_{i=0}^{\ell-1} R^i$ of these relations belongs to X, and is therefore not empty, because X is an ultrafilter. Choose any pair of elements (γ, δ) in this intersection, and use γ and δ as the interpretations of the individual constant symbols $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ respectively. Expand $\bar{\mathfrak{U}}_0$ by adding as distinguished constants the chosen interpretations of the individual constant symbols used for each of the ultrafilters in $\mathfrak{A}$. The definitions of the interpretations of the individual constant symbols occurring in $\mathcal{F}$, and the observation made above that $\bar{\mathfrak{U}}_0$ is a model of (20), together imply that this expansion is a model of (21).

It has been shown that, in particular, every finite subset of the set of sentences

$$\mathcal{T} \cup \mathcal{R}_1 \cup \mathcal{S}_1 \tag{22}$$

has a model. Apply the Compactness Theorem for the first-order logic $\mathcal{L}^*$ (see Section 2.4) to conclude that the set of sentences (22) has a model $\bar{\mathfrak{U}}_1$. Let $\mathfrak{U}_1$ be the structure obtained from $\bar{\mathfrak{U}}_1$ by deleting all of the distinguished constants from the system of fundamental relations (but not from the universe).

The sentences in $\mathcal{R}_1$ ensure that the correspondence mapping each element α in $\mathfrak{U}_0$ to the interpretation of the individual constant symbol $\boldsymbol{\alpha}$ in $\bar{\mathfrak{U}}_1$ is an embedding of $\mathfrak{U}_0$ into $\mathfrak{U}_1$ in the sense that a pair (α, β) of elements from $\mathfrak{U}_0$ belongs to a relation R_0 in $\mathfrak{U}_0$ if and only if the image of this pair under the embedding belongs to the corresponding relation R_1 in $\mathfrak{U}_1$, for each R in $\mathfrak{A}$. By applying an appropriate version of the Exchange Principle (see Theorem 7.15), it may be assumed that $\mathfrak{U}_0$ is actually a substructure of $\mathfrak{U}_1$. Thus, (3) holds for the case $n = 0$. The sentences in $\mathcal{S}_1$ ensure that for every ultrafilter X in $\mathfrak{A}$, the intersection of the relations R_1, for R in X, is not empty. Consequently, (4) holds.

Assume now that n is any positive integer and that relational structures $\mathfrak{U}_k$ have been defined for $k = 0, \ldots, n$ so that the five properties stated at the beginning of the proof hold for these structures (the third and fifth properties holding for $k = 0, \ldots, n-1$ in place of n). The construction of the relational structure $\mathfrak{U}_{n+1}$ from $\mathfrak{U}_n$ is similar to the construction $\mathfrak{U}_1$ from $\mathfrak{U}_0$, but uses sets of sentences $\mathcal{R}_{n+1}$ and $\mathcal{S}_{n+1}$ that are different from the sets $\mathcal{R}_1$ and $\mathcal{S}_1$ that were defined above. The set $\mathcal{R}_{n+1}$ consists of all sentences of the form

$$\boldsymbol{R\alpha\beta} \qquad \text{whenever} \qquad (\alpha, \beta) \in R_n,$$

and all sentences of the form

$$\boldsymbol{\neg R\alpha\beta} \qquad \text{whenever} \qquad (\alpha, \beta) \notin R_n,$$

where R ranges over the relations in $\mathfrak{A}$, and α and β over the elements in $\mathfrak{U}_n$, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are distinct new individual constant symbols intended to denote α and β respectively. Let $\bar{\mathfrak{U}}_n$ be the expansion of $\mathfrak{U}_n$ obtained by adding to the system of fundamental relations of $\mathfrak{U}_n$ the elements α in $\mathfrak{U}_n$ as distinguished constants interpreting the individual constant symbols $\boldsymbol{\alpha}$ used in the definition of the set $\mathcal{R}_{n+1}$. This expansion is a model of the set of sentences

$$\mathcal{T} \cup \mathcal{R}_{n+1}, \tag{23}$$

by the definition of the set $\mathcal{R}_{n+1}$ and by the induction hypothesis, which assumes that $\mathfrak{U}_n$ is a model of $\mathcal{T}$. The purpose of the sentences in $\mathcal{R}_{n+1}$ is to ensure that the structure $\mathfrak{U}_{n+1}$ (to be constructed) is an extension of $\mathfrak{U}_n$.

The set $\mathcal{S}_{n+1}$ consists of all sentences determined in the following way. For every triple of ultrafilters X, Y, and Z in $\mathfrak{A}$ satisfying (5), and for every pair of elements (α, β) in the intersection (6), choose a new individual constant symbol $\boldsymbol{\gamma}$ in $\mathcal{L}^*$ that does not occur in the sentences in $\mathcal{R}_{n+1}$ and that has not been used for any other pairs in the intersection (6) or for any other triples of ultrafilters in $\mathfrak{A}$. For each pair of relations R in X and S in Y, put the sentence

$$\boldsymbol{R\alpha\gamma} \wedge \boldsymbol{S\gamma\beta}$$

into the set $\mathcal{S}_{n+1}$. The purpose of the sentences in $\mathcal{S}_{n+1}$ is to ensure that the structure $\mathfrak{U}_{n+1}$ possesses the fifth property with respect to the structure $\mathfrak{U}_n$.

For every finite subset $\mathcal{F}$ of $\mathcal{S}_{n+1}$, there is an expansion of the relational structure $\bar{\mathfrak{U}}_n$, obtained by adding new distinguished constants to the system of fundamental relations, that is a model of the set of sentences

$$\mathcal{T} \cup \mathcal{R}_{n+1} \cup \mathcal{F}. \tag{24}$$

Indeed, for any triple X, Y, and Z of ultrafilter in $\mathfrak{A}$ satisfying (5), and for any pair (α, β) belonging to the intersection (6), there are only finitely many relations $R^0, \ldots, R^{\ell-1}$ in X and $S^0, \ldots, S^{\ell-1}$ in Y such that the sentences

$$\boldsymbol{R}^i\boldsymbol{\alpha\gamma} \wedge \boldsymbol{S}^i\boldsymbol{\gamma\beta}$$

occur in $\mathcal{F}$ for $i = 0, \ldots, \ell - 1$, by the assumed finiteness of the set $\mathcal{F}$. The intersections

$$R = \bigcap_{i=0}^{\ell-1} R^i \qquad \text{and} \qquad S = \bigcap_{i=0}^{\ell-1} S^i \tag{25}$$

of these relations belong to the ultrafilters X and Y respectively, because ultrafilters are closed under finite intersections. The relational composition $T = R \,|\, S$ must belong to Z, by the hypothesis that (5) holds. Therefore, the pair (α, β), which is assumed to be in (6), must belong to the relation T_n, because T_n is one of the relations used to form the intersection in (6). The relation T_n coincides with $R_n \,|\, S_n$, because $\mathfrak{U}_n$ is assumed to be a model of the sentences in (T3), so (α, β) must belong to $R_n \,|\, S_n$. Consequently, there must exist an element γ

such that (α, γ) is in R_n and (γ, β) in S_n. The relations R and S are included in the relations R^i and S^i respectively for each $i = 0, \ldots, \ell - 1$, by (25), so the relations R_n and S_n are included in the relations R_n^i and S_n^i respectively for each $i = 0, \ldots, \ell - 1$, because $\mathfrak{U}_n$ is a model of the sentences in (T1) (and the relation of set-theoretic inclusion is definable in terms of union). The pairs (α, γ) and (γ, β) therefore belong to R_n^i and S_n^i respectively for each $i = 0, \ldots, \ell - 1$. Use γ as the interpretation of the individual constant symbol $\boldsymbol{\gamma}$. Expand $\bar{\mathfrak{U}}_n$ by adding to the system of fundamental relations the distinguished constants γ that are the chosen interpretations of the individual constant symbols $\boldsymbol{\gamma}$ used for the triples of ultrafilters X, Y, and Z satisfying (5) and the pairs (α, β) in (6). The definitions of the interpretations of the individual constant symbols occurring in $\mathcal{F}$, and the observation made above that $\bar{\mathfrak{U}}_n$ is a model of (23), together imply that this expansion of $\bar{\mathfrak{U}}_n$ is a model of (24).

It has been shown, in particular, that every finite subset of the set of sentences

$$\mathcal{T} \cup \mathcal{R}_{n+1} \cup \mathcal{S}_{n+1} \tag{26}$$

has a model. Apply the Compactness Theorem for the first-order logic $\mathcal{L}^*$ to conclude that the set of sentences (26) has a model $\bar{\mathfrak{U}}_{n+1}$. Let $\mathfrak{U}_{n+1}$ be the structure obtained from $\bar{\mathfrak{U}}_{n+1}$ by deleting all of the distinguished constants from the system of fundamental relations (but not from the universe).

As in the case when $n = 1$, the sentences in $\mathcal{R}_{n+1}$ ensure that the correspondence mapping each element α in $\mathfrak{U}_n$ to the interpretation of the individual constant symbol $\boldsymbol{\alpha}$ in $\bar{\mathfrak{U}}_{n+1}$ is an embedding of $\mathfrak{U}_n$ into $\mathfrak{U}_{n+1}$ in the sense that a pair (α, β) of elements from $\mathfrak{U}_n$ belongs to a relation R_n of $\mathfrak{U}_n$ if and only if the image of this pair under the embedding belongs to the corresponding relation R_{n+1} of $\mathfrak{U}_{n+1}$, for each R in $\mathfrak{A}$. By applying an appropriate version of the Exchange Principle, it may be assumed that $\mathfrak{U}_n$ is actually a substructure of $\mathfrak{U}_{n+1}$.

The sentences in $\mathcal{S}_{n+1}$ ensure that if X, Y, and Z are ultrafilters in $\mathfrak{A}$ satisfying (5), then for every pair of elements (α, β) in the intersection (6), there is an element γ in U_{n+1} such that the pairs (α, γ) and (γ, β) belong to the left-hand and right-hand intersections in (7) respectively. In other words, the structure $\mathfrak{U}_{n+1}$ possesses the fifth property with respect to the substructure $\mathfrak{U}_n$. Conclusion: the five properties stated at the beginning of the proof all hold for the sequence of relational structures $\mathfrak{U}_k$ with $k = 0, \ldots, n+1$ (the third and

fifth properties holding for $k = 0, \ldots, n$ in place of n). This completes the induction step of the construction of the structures in (2), and therefore completes the proof of the lemma. □

It may be of interest to note that, by applying the upward and downward Löwenheim-Skolem-Tarski Theorems in the preceding proof, one can actually exert some control over the size of base set of the constructed algebra $\mathfrak{A}_\omega$ that is isomorphic to the given set relation algebra $\mathfrak{A}$ and satisfies the conditions of Theorem 14.50. As this size does not play a role in our application of the lemma, the details are left as an exercise.

Here, finally, is the theorem regarding the representability of canonical extensions toward which we have been working.

Theorem 16.22. *A relation algebra is representable if and only if its canonical extension is completely representable.*

Proof. Let $\mathfrak{A}$ be a relation algebra, and $\mathfrak{B}$ its canonical extension. If $\mathfrak{B}$ is completely representable, then its subalgebra $\mathfrak{A}$ is certainly representable, by Lemma 16.11. In fact, any representation of $\mathfrak{B}$, and in particular, any complete representation of $\mathfrak{B}$, restricts to a representation of $\mathfrak{A}$.

Assume now that $\mathfrak{A}$ is representable, say it is isomorphic to a set relation algebra $\mathfrak{A}'$. Lemma 16.21 ensures that $\mathfrak{A}'$—and therefore also $\mathfrak{A}$—is isomorphic to a set relation algebra $\bar{\mathfrak{A}}$, say with unit E, that satisfies the conditions of Theorem 14.50. The canonical extension of $\bar{\mathfrak{A}}$—call it $\bar{\mathfrak{B}}$—is the complete subalgebra of $\mathfrak{Re}(E)$ that is completely generated by $\bar{\mathfrak{A}}$, by Theorem 14.50. If φ is an isomorphism from $\mathfrak{A}$ to $\bar{\mathfrak{A}}$, then the canonical extension of φ—call it ψ—is an isomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$, by the Existence Theorem 14.41 for canonical extensions of homomorphisms. The algebra $\bar{\mathfrak{B}}$ is a complete subalgebra, and therefore a regular subalgebra, of $\mathfrak{Re}(E)$, so the isomorphism ψ is a complete representation of $\mathfrak{B}$, by Lemma 16.6. □

Corollary 16.23. *If the canonical extension of a relation algebra is representable at all, then it has a complete representation.*

Proof. If the canonical extension, say $\mathfrak{B}$, of a relation algebra $\mathfrak{A}$ is representable at all, then $\mathfrak{A}$ must be representable, by Lemma 16.11 or Theorem 16.22. Consequently, $\mathfrak{B}$ has a completely representation, by Theorem 16.22. □

One can easily modify the proof of Theorem 16.22 to conclude that a simple relation algebra $\mathfrak{A}$ has a square representation if and only if its canonical extension has a complete square representation. The same conclusion may also be reached by applying Theorem 14.36 and the complete version of Theorem 16.18.

16.8 Completions

In view of the representability of canonical extensions of representable relation algebras, it is natural to expect that an analogous result for completions is also true. Unfortunately, this turns out not to be the case: there exist representable relation algebras whose completions are not representable. The proof involves a rather lengthy and intricate construction; see [62] and [59]. On the other hand, a complete representation of a relation algebra can always be extended in a unique way to a complete representation of the completion.

Theorem 16.24. *If φ is a complete embedding of a relation algebra $\mathfrak{A}$ into a full set relation algebra $\mathfrak{Re}(E)$, then the completion of φ is the unique complete embedding of the completion of $\mathfrak{A}$ into $\mathfrak{Re}(E)$ that extends φ.*

Proof. Let $\mathfrak{B}$ be the completion of the given relation algebra $\mathfrak{A}$, and assume that φ is a complete embedding of $\mathfrak{A}$ into a full set relation algebra $\mathfrak{Re}(E)$. The completion of φ—call it ψ—is a complete embedding of $\mathfrak{B}$ into $\mathfrak{Re}(E)$, by Existence Theorem 15.35 for completions of complete homomorphisms, and ψ is the only extension of φ to a complete homomorphism from $\mathfrak{B}$ to $\mathfrak{Re}(E)$, by Uniqueness Theorem 15.36 for completions of complete homomorphisms. □

Corollary 16.25. *If $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, then the complete representations of $\mathfrak{B}$ are precisely the completions of complete representations of $\mathfrak{A}$.*

Proof. Suppose first that φ is a complete representation of $\mathfrak{A}$, say φ completely embeds $\mathfrak{A}$ into a full set relation algebra $\mathfrak{Re}(E)$. The completion of φ is a complete embedding of $\mathfrak{B}$ into $\mathfrak{Re}(E)$, by Theorem 16.24, so it is a complete representation of $\mathfrak{B}$.

Assume now that ψ is a complete representation of $\mathfrak{B}$, say ψ completely embeds $\mathfrak{B}$ into a full set relation algebra $\mathfrak{Re}(E)$. The restriction

of ψ to $\mathfrak{A}$—call it φ—is then a complete embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$, by Lemma 16.12 and the fact that $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{B}$ (see Lemmas 15.6 and 15.21). The mapping φ has a unique extension to a complete embedding of $\mathfrak{B}$ into $\mathfrak{Re}(E)$, by Theorem 16.24, and of course ψ is also such an extension, so ψ must be the completion of φ. □

In particular, there is a bijective correspondence between the complete representations of a relation algebra $\mathfrak{A}$ and the complete representations of the completion of $\mathfrak{A}$.

Corollary 16.26. *A relation algebra is completely representable if and only if its completion is completely representable.*

16.9 Homomorphic images and directed unions

We have seen that subalgebras, direct products, and canonical extensions of representable relation algebras are representable. Happily, the same is true of homomorphic images. There are a number of very different proofs of this result. The following one depends heavily on Theorem 16.22 and has a more algebraic flavor than the original proof.

Theorem 16.27. *Every homomorphic image of a representable relation algebra is representable.*

Proof. Let $\mathfrak{A}$ be a relation algebra, and $\mathfrak{C}$ any homomorphic image of $\mathfrak{A}$. Assume that $\mathfrak{A}$ is representable. The canonical extension of $\mathfrak{A}$ is then completely representable, by Theorem 16.22. Every homomorphic image of $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to some closed ideal element from the canonical extension, by Theorem 14.54. In particular, there is a closed ideal element u in the canonical extension such that $\mathfrak{C}$ is isomorphic to $\mathfrak{A}(u)$. The relativization $\mathfrak{A}(u)$ is, in turn, representable, by Corollary 16.14 (with $\mathfrak{A}$ and $\mathfrak{B}$ interchanged), so the isomorphic algebra $\mathfrak{C}$ must also be representable. □

The complete version of the preceding theorem is not true: not every homomorphic image of a completely representable relation algebra is completely representable. As in the case of the corresponding result for subalgebras (Lemma 16.11), the basic problem is that some of the existing suprema may be changed, and other suprema that previously did not exist may appear, in the passage to a homomorphic image.

The difficulty may be overcome by considering complete homomorphic images, that is to say, images under complete homomorphisms.

Theorem 16.28. *Every complete homomorphic image of a completely representable relation algebra is completely representable.*

Proof. Let $\mathfrak{A}$ be a relation algebra, and $\mathfrak{C}$ any complete homomorphic image of $\mathfrak{A}$. Assume that $\mathfrak{A}$ is complete representable. The completion of $\mathfrak{A}$—call it $\mathfrak{B}$—is then completely representable, by Corollary 16.26. Since $\mathfrak{A}$ is a dense subalgebra of $\mathfrak{B}$, by Definition 15.17, it follows from Corollary 16.16 (with $\mathfrak{A}$ and $\mathfrak{B}$ interchanged) that the relativization of $\mathfrak{A}$ to any ideal element from $\mathfrak{B}$ is also completely representable.

Every complete homomorphic image of $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to some ideal element from $\mathfrak{B}$, by Theorem 15.48. In particular, there is an ideal element u in $\mathfrak{B}$ such that $\mathfrak{C}$ is isomorphic to the relativization $\mathfrak{A}(u)$. The algebra $\mathfrak{A}(u)$ is completely representable, by the observations of the previous paragraph, so $\mathfrak{C}$ is completely representable. □

We next address the problem of the preservation of representability under the passage to directed unions.

Theorem 16.29. *The union of a directed system of representable relation algebras is representable.*

Proof. A well-known theorem of general algebra says that if a class of (similar) algebras is closed under the formation of homomorphic images, subalgebras, and direct products, then the class must also be closed under the formation of unions of directed systems of algebras. The class of representable relation algebras is closed under the formation of homomorphic images, by Theorem 16.27, under the formation of subalgebras, by Lemma 16.11, and under formation of direct products, by Corollary 16.20. Consequently, the class is closed under the formation of unions of directed systems of algebras. □

Corollary 16.30. *A relation algebra is representable if and only if every countable elementary subalgebra is representable.*

Proof. A relation algebra $\mathfrak{A}$ is the union of the directed system of its countable elementary subalgebras, by Lemma 6.31. If each of these subalgebras is representable, then $\mathfrak{A}$ is representable, by Theorem 16.29. On the other hand, if $\mathfrak{A}$ is representable, then every subalgebra, and in particular every countable elementary subalgebra, of $\mathfrak{A}$ is representable, by Lemma 16.11. □

This corollary provides a strategy for proving the representability of every algebra in an elementary class K of relation algebras. One first shows that every countable algebra in K is representable. The assumption of countability means that the elements of the algebra can be enumerated in a denumerable sequence, and this enumeration gives a certain measure of control when trying to carry out various constructions. For an arbitrary relation algebra $\mathfrak{A}$ in K, every countable elementary subalgebra of $\mathfrak{A}$ belongs to K, by the assumption that K is elementary. Each of these countable elementary subalgebras has already been shown to be representable, so $\mathfrak{A}$ must be representable, by Corollary 16.30.

An argument analogous to the proof of Corollary 16.30, but using Corollary 6.10 instead of Lemma 6.31 establishes another quite useful result.

Corollary 16.31. *A relation algebra is representable if and only if every finitely generated subalgebra is representable.*

The complete version of Corollary 16.31 is not true. Indeed, let $\mathfrak{A}$ be any relation algebra whose finitely generated subalgebras are all finite. If $\mathfrak{A}$ is representable, but not completely representable, then every finitely generated subalgebra of $\mathfrak{A}$, being finite, is completely representable, but $\mathfrak{A}$—which is the directed union of these finitely generated subalgebras—is not completely representable. A concrete example is obtained by taking $\mathfrak{A}$ to be an infinite, atomless Boolean relation algebra.

16.10 Historical remarks

The representation problem was first raised by Tarski in [132], in connection with his proposed axiomatization of the calculus of relations. In analogy with the result of Stone [127] that every Boolean algebra is isomorphic to a Boolean algebra of sets, Tarski asked whether every model of his axioms is isomorphic to an algebra of binary relations. This problem remained open for almost a decade, but several partial positive results were obtain in the intervening period (see the historical remarks in the next chapter). Perhaps as a result, there has been historically a strong focus on representation questions.

The connection between square representations and simple relation algebras was noted already by Jónsson and Tarski in [75], which

contains an equivalent form of Lemma 16.1. The connection between square representations and the set of first-order sentences (T1)–(T7) is implicit already in Tarski's original paper [132]. An equivalent form of Lemma 16.2 and the special case in Theorem 16.3 for square representations are mentioned by Jónsson in the proof of Theorem 8.1 in [68]; see also Theorem 3.31 in [59].

The notion of equivalent representations was introduced by Lyndon in [90]. The alternative terminology *base isomorphic representations* is a natural extension of the terminology *base isomorphism* that was discussed in Section 7.5 (see also the historical notes in Section 7.10). Lemma 16.4 is due to Givant and first appeared in a more concrete form in [37].

Complete representations first came to the fore due to a mistake in [88]. In attempting to prove that the class of representable relation algebras is not axiomatizable by any set of equations, Lyndon neglected to distinguish between representations and complete representations. Tarski [143] subsequently showed that the class of representable relation algebras is in fact axiomatizable by a set of equations. It was Dana Scott who tracked down the error in Lyndon's argument. He suggested the notion of a *strong representation*—his terminology for a complete representation—to give Lyndon's result a correct formulation, namely that the class of completely representable, atomic relation algebras is not axiomatizable by any set of universal sentences; see Lyndon [89]. The fact that the class of completely representable relation algebras is not elementary was established much later by Hirsch in [52]; see also [54] and [59]. Lemma 16.5 is due to Givant. Theorem 16.7 is due to Hirsch-Hodkinson [54], who proved this theorem in the context of arbitrary Boolean algebras with additional operations. In the process, they also proved a form of the equivalence of (i) and (iv) in Lemma 16.6; see also [59]. The remaining equivalences in the lemma are due to Givant (see [37]). Theorem 16.8 and its generalization in Exercise 16.5 are due to Maddux [99]; the observation in Corollary 16.9 is due to Givant. Theorem 16.10 is due independently to Andréka-Givant and to Hirsch-Hodkinson (see [59], pp. 129).

Jónsson and Tarski showed in [75] that subalgebras of representable relation algebras are representable, and Tarski certainly knew the content of Lemma 16.11, and mentioned it in his lectures. It is also very probably that he knew the contents of Lemma 16.13. The extensions of these results to complete representations, given in Lemma 16.12 and Corollary 16.16 are due to Givant. The results on representations

of simple relation algebras in Section 16.5 and on representations of products of relation algebras in Section 16.6, are due to Givant. However, Tarski was aware very early on that direct products and, more generally, subdirect products of representable relation algebras are representable, and in fact this result follows from Theorem 2.4 in [143].

The theorem that canonical extensions of representable relation algebras are representable is due to Monk. He never published this result, but it is alluded to in McKenzie's thesis [106], with an attribution to Monk. According to McKenzie, Monk's proof was based on certain connections between relation algebras and cylindric algebras. The first published proof appeared in Maddux [96]. Other proofs of Monk's theorem were later published by Andréka-Givant-Németi [7], Hirsch-Hodkinson [59], and Maddux [101]. The stronger result in Theorem 16.22 that a relation algebra is representable if and only if its canonical extension is *completely* representable is due independently to Hirsch-Hodkinson [59] and Andréka-Givant; the proof given here, based on Theorem 14.50 and Lemma 16.21, is due to Andréka-Givant.

The theorem that there exist representable relation algebras whose completions are not representable is due to Hodkinson [62]; see also [59]. Theorem 16.24 may be viewed as a special case of Minimality Theorem 15.38; see Section 15.7 and the historical remarks in Section 15.10. Corollaries 16.25 and 16.26 are due to Givant.

As regards the results in Section 16.9, Theorems 16.27 and 16.29, and Corollary 16.31 are due to Tarski [143]. His proof of Theorem 16.27 is metamathematical in character, and in his later courses on relation algebras he raised the problem of finding an algebraic proof. Such proofs were given by Maddux [93] (see also [101]), Jónsson [68] (following a suggestion of McKenzie), and Hirsch-Hodkinson [59]. The proof given here may be considered another algebraic proof of the theorem. Theorem 16.28 is due to Givant, as are the observations in Exercises 16.17, 16.18, 16.19, and 16.20.

Exercises

16.1. Fill in the missing details in the proof of Lemma 16.2 by showing that the mapping φ preserves the identity element and the operations of addition, complementation with respect to the relation E, and con-

verse if and only if the implications in (T5), (T1), (T2), and (T4) are respectively valid in $\mathfrak{U}_\varphi$.

16.2. Prove that the relation of two representations being equivalent is an equivalence relation on the class of all representations of a relation algebra.

16.3. Prove that a complete representation of a relation algebra preserves all existing infima as intersections.

16.4. Prove (1), (3), and (4) in Theorem 16.8.

16.5. If a locally small relation algebra $\mathfrak{A}$ is embeddable in a complete and atomic relation algebra $\mathfrak{B}$, prove that there is a complete embedding of $\mathfrak{A}$ into some relativization of $\mathfrak{B}$ to an equivalence element.

16.6. Prove the complete version of Lemma 16.13.

16.7. Prove Lemma 16.17.

16.8. Prove the complete version of Theorem 16.18.

16.9. Give a direct proof of Theorem 16.18 that does not make use of Lemma 16.17 or the results in Chapter 11 such as Corollary 11.52.

16.10. Prove the complete version of Theorem 16.19.

16.11. Give a direct proof of Theorem 16.19 that does not make use of the results in Chapter 11 such as Lemma 11.48 and Theorem 11.49.

16.12. Prove the complete version of Corollary 16.20.

16.13. Fill in the missing details in the proof of Lemma 16.21 by verifying (11), (12), (14), (15), and (16), and then showing that the mapping φ defined in (18) preserves the identity element and the operations of addition, complement, and converse.

16.14. If an infinite set relation algebra of cardinality κ_1 has a base set of cardinality κ_2, prove that $\mathfrak{A}$ is isomorphic to a set relation algebra satisfying the conditions of Theorem 14.50 and having a base set of cardinality at most 2^κ, where $\kappa = \max\{\kappa_1, \kappa_2\}$.

16.15. Use Theorem 14.50 and the second part of Lemma 16.21 to prove that a simple relation algebra has a square representation if and only if its canonical extension has a complete square representation.

16.16. Use Theorem 14.36, the complete version of Theorem 16.18, and Theorem 16.22 to give an alternative proof of the result in Exercise 16.15 that a relation algebra has a square representation if and only if its canonical extension has a complete square representation.

16.17. Is the analogue of Corollary 16.23 for completions true? In other words, if the completion of a relation algebra is representable at all, must it have a complete representation?

16.18. Give an example to show that not every homomorphic image of a completely representable relation algebra is completely representable.

16.19. Give an example to show that the complete version of Theorem 16.29 is not true. In other words, give an example of a directed system of completely representable relation algebras such that the union of the system is not completely representable.

16.20. Give an example to show that the complete version of Theorem 16.29 is not true even for chains of simple relation algebras. In other words, give an example of a chain of completely representable simple relation algebras such that the union of the chain is not completely representable.

16.21. Prove Corollary 16.31.

16.22. Give an example of a relation algebra $\mathfrak{A}$ such that every finitely generated subalgebra of $\mathfrak{A}$ is completely representable, but $\mathfrak{A}$ itself is not completely representable.

Chapter 17
Representation theorems

The discussion of representations in the preceding chapter focused on techniques for constructing new representations from given ones. In particular, we saw that the representations of simple relation algebras are precisely the amalgamations of disjoint systems of square representations; the study of representations of simple relation algebras therefore reduces to the study of square representations. We have not yet looked at questions of representability for concrete classes of relation algebras. That is the task of the present chapter.

There are two types of representation theorems that are commonly found in the literature. The first type concerns itself with classes of relation algebras that are algebraically constructed—for example, the class of formula relation algebras or the class of geometric complex algebras—and shows either that all members in the class are representable, or else that all members of some well-determined subclass are representable. The second type focuses instead on abstract properties that a relation algebra may possess—for example functional density—and shows that all relation algebras satisfying this property are representable.

In Sections 17.1–17.5 we focus on the first type of representation theorem by examining questions of representability for five of the concrete classes of relation algebras that are discussed in Chapter 3, namely formula relation algebras, Boolean relation algebras, group complex algebras, geometric complex algebras, and small relation algebras. (The representability of all matrix relation algebras was already established in Section 7.5.) In Sections 17.6–17.8, we turn our attention to the second type of representation theorem and establish, among other things, a quasi-representation theorem for all relation algebras and full rep-

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9_4

resentation theorems for all atomic relation algebras with functional atoms and for all singleton dense relation algebras.

17.1 Formula relation algebras

We begin by considering the class of formula relation algebras, defined in Section 3.3. It turns out that every formula relation algebra is representable, and in fact a representation can be constructed from the class of models of the first-order theory underlying the construction of the formula relation algebra.

Theorem 17.1. *Every formula relation algebra is representable.*

Proof. Consider a first-order language $\mathcal{L}^*$ and a set $\mathcal{S}$ of formulas in $\mathcal{L}^*$. The elements of the formula relation algebra $\mathfrak{A}$ constructed from $\mathcal{S}$ are the equivalence classes $[\Gamma]$, modulo provable equivalence in $\mathcal{S}$, of formulas Γ in $\mathcal{L}^*$ that have at most two free variables, $\boldsymbol{x}$ and $\boldsymbol{y}$. We shall construct a representation of $\mathfrak{A}$ from the class of models of $\mathcal{S}$.

If the set of formulas $\mathcal{S}$ is inconsistent in the sense that every sentence in $\mathcal{L}^*$ is derivable from $\mathcal{S}$, then of course $\mathcal{S}$ has no models. In this case, all two-variable formulas in $\mathcal{L}^*$ are equivalent modulo $\mathcal{S}$, and consequently $\mathfrak{A}$ has just one element, namely the equivalence class of the formula $\neg(\boldsymbol{x} = \boldsymbol{x})$. Since all degenerate relation algebras are isomorphic to one another, it follows that $\mathfrak{A}$ must be isomorphic to the degenerate set relation algebra, that is to say, to the full set relation algebra on the empty set.

Assume now that $\mathcal{S}$ is consistent. In this case, $\mathcal{S}$ has at least one model, by the Completeness Theorem for first-order logic. Let $\mathfrak{U}$ be any such model. Each element $[\Gamma]$ in $\mathfrak{A}$ determines a binary relation on the universe of the model $\mathfrak{U}$, namely the relation R that is defined in $\mathfrak{U}$ by the formula Γ. In more detail, a pair (α, β) belongs to the relation R if and only if α and β are elements in the model $\mathfrak{U}$, and the pair (α, β) satisfies the formula Γ in $\mathfrak{U}$ in the sense that if the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are assigned the values α and β respectively, then Γ expresses a true statement about α and β in $\mathfrak{U}$. If Δ is any other formula in the equivalence class of Γ, then the sentence

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\Gamma \leftrightarrow \Delta),$$

is derivable from the set of formulas $\mathcal{S}$, by the definition of equivalence modulo $\mathcal{S}$, and therefore this sentence holds in all models of $\mathcal{S}$, includ-

ing $\mathfrak{U}$. It follows that the formulas Δ and Γ define the same relation in $\mathfrak{U}$, so that it makes sense to speak of the relation defined by the equivalence class $[\Gamma]$.

Let

$$r = [\Gamma], \qquad s = [\Delta], \qquad t = [\Omega] \tag{1}$$

be elements in $\mathfrak{A}$, and let R, S, T be the relations defined by these elements in $\mathfrak{U}$. It is not difficult to check that the following implications hold:

$$\begin{aligned}
&t = r + s && \text{implies} && T = R \cup S, && (2)\\
&t = -r && \text{implies} && T = {\sim}R, && (3)\\
&t = r\,;s && \text{implies} && T = R \mid S, && (4)\\
&t = r^{\smile} && \text{implies} && T = R^{-1}, && (5)\\
&t = 1' && \text{implies} && T = id_U, && (6)
\end{aligned}$$

where U is the universe of $\mathfrak{U}$, and ${\sim}R$ is the complement of R with respect to the universal relation on U, and 1' is the identity element in $\mathfrak{A}$, that is the say, 1' is the equivalence class modulo $\mathcal{S}$ of the formula $\boldsymbol{x} = \boldsymbol{y}$. For instance, to verify (4), recall from Section 3.3 that relative multiplication is defined in $\mathfrak{A}$ by

$$r\,;s = [\exists \boldsymbol{z}(\Gamma(\boldsymbol{x}, \boldsymbol{z}) \wedge \Delta(\boldsymbol{z}, \boldsymbol{y}))].$$

It follows from this definition, and from (1) and the definition of relational composition, that the element $r\,;s$ defines the relation $R \mid S$ in the model $\mathfrak{U}$. If $t = r\,;s$, then the formulas

$$\Omega \qquad \text{and} \qquad \exists \boldsymbol{z}(\Gamma(\boldsymbol{x}, \boldsymbol{z}) \wedge \Delta(\boldsymbol{z}, \boldsymbol{y}))$$

must be equivalent modulo $\mathcal{S}$, by (1), and therefore they must define the same relations in $\mathfrak{U}$. Consequently, $T = R \mid S$.

Define a function φ from $\mathfrak{A}$ into the full relation algebra $\mathfrak{Re}(U)$ by putting

$$\varphi(r) = R$$

for each element r in $\mathfrak{A}$, where R is the relation defined by r in $\mathfrak{U}$. The observations of the previous paragraph imply that φ is a homomorphism. For example, if $t = r\,;s$, then (continuing with the notation of the previous paragraph)

$$\varphi(r\,;s) = \varphi(t) = T = R \mid S = \varphi(r) \mid \varphi(s),$$

by the definition of φ and (4). Thus, φ preserves the operation of relative multiplication. Analogous arguments, using (2), (3), (5), and (6), show that φ preserves the operations of addition, complement, and converse, and maps the identity element in $\mathfrak{A}$ to the identity relation in $\mathfrak{Re}(U)$. Thus, φ is a homomorphism. We shall say that φ is the homomorphism on $\mathfrak{A}$ *induced* by the model $\mathfrak{U}$.

In general, the homomorphism on $\mathfrak{A}$ induced by a model of $\mathcal{S}$ may not be one-to-one, and may therefore fail to be a representation of $\mathfrak{A}$. For example, it may happen that a non-zero element $r = [\Gamma]$ in $\mathfrak{A}$ defines the empty relation in a model $\mathfrak{U}$, even though the formula Γ is not universally false in $\mathcal{S}$ in the sense that the negation of the existential sentence

$$\exists \boldsymbol{x} \exists \boldsymbol{y} \Gamma(\boldsymbol{x}, \boldsymbol{y}) \tag{7}$$

is derivable from $\mathcal{S}$. In this situation, the element r belongs to the kernel of the homomorphism on $\mathfrak{A}$ induced by $\mathfrak{U}$, even though r is not the zero element in $\mathfrak{A}$, so this homomorphism cannot be one-to-one, by Lemma 8.37.

To overcome this hurdle, we again invoke the Completeness Theorem. If $r = [\Gamma]$ is a non-zero element in $\mathfrak{A}$, then the existential sentence in (7) must be consistent with $\mathcal{S}$, and therefore there must exist at least one model $\mathfrak{U}_r$ of $\mathcal{S}$ in which the existential sentence is true, by the Completeness Theorem. The element r defines a non-empty relation in $\mathfrak{U}_r$, by the assumed truth of the above existential sentence in $\mathfrak{U}_r$, so r does not belong to the kernel of the homomorphism on $\mathfrak{A}$ induced by $\mathfrak{U}_r$.

For each non-zero element r in $\mathfrak{A}$, select a model $\mathfrak{U}_r$ of $\mathcal{S}$ such that r is not in the kernel of the induced homomorphism φ_r. This is possible by the observations of the preceding paragraph. Carry out this selection in such a way that the universes of the various selected models are mutually disjoint (see the relevant remarks in Section 16.2). Let E be the equivalence relation whose equivalence classes are the universes of these models; in other words, put

$$E = \bigcup \{U_r \times U_r : r \in A \text{ and } r \neq 0\}.$$

Apply Lemma 16.17 to conclude that the amalgamation of the system of homomorphisms

$$(\varphi_r : r \in A \text{ and } r \neq 0)$$

is an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$, and therefore a representation of $\mathfrak{A}$. □

Since relativizations of representable relation algebras are representable, by Lemma 16.13, we at once obtain the following corollary.

Corollary 17.2. *Every relativization of a formula relation algebra is representable.*

In general, the representation of the formula relation algebra $\mathfrak{A}$ constructed in Theorem 17.1 is neither square nor complete. If, however, the given set of formulas $\mathcal{S}$ is complete in the sense that every sentence or its negation is derivable from $\mathcal{S}$, and consistent in the sense that not all sentences are derivable from $\mathcal{S}$, then the formula relation algebra $\mathfrak{A}$ is simple (see Exercise 9.6), and the homomorphism induced by any given model of $\mathcal{S}$ is already one-to-one and therefore a square representation of $\mathfrak{A}$.

As it turns out, the converse of Corollary 17.2 is also true.

Theorem 17.3. *Every representable relation algebra is embeddable into the relativization of some formula relation algebra to a reflexive equivalence element.*

Proof. Clearly, it suffices to prove that every set relation algebra is embeddable into a relativization of some formula relation algebra to a reflexive equivalence element. Suppose, therefore, that $\mathfrak{B}$ is a set relation algebra with base set U. Let $\mathcal{L}^*$ be the language of relations that has a binary relation symbol $\boldsymbol{R}$ for each relation R in $\mathfrak{B}$, and let $\mathcal{T}$ be the set of sentences of $\mathcal{L}^*$ that occur on the left sides of the schemata (T1)–(T6) in Section 16.1 (with the elements r, s, and t on the right sides of the schemata interpreted as relations R, S, and T in $\mathfrak{B}$, and the operations on the right sides of the schemata interpreted as the set-theoretic operations of $\mathfrak{B}$). Take $\mathfrak{A}$ to be the formula relation algebra of $\mathcal{L}^*$ modulo $\mathcal{T}$. The goal is to construct an embedding of $\mathfrak{B}$ into a relativization of $\mathfrak{A}$.

To this end, consider arbitrary relations R, S, and T in $\mathfrak{B}$. Write E for the unit relation, and I for the identity relation, in $\mathfrak{B}$. Denote the corresponding relation symbols in $\mathcal{L}^*$ by $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{E}$, and $\boldsymbol{I}$ respectively. Let $\boldsymbol{Rxy}$, $\boldsymbol{Sxy}$, $\boldsymbol{Txy}$, $\boldsymbol{Exy}$, and $\boldsymbol{Ixy}$ be the corresponding atomic formulas with free variables (in order) $\boldsymbol{x}$ and $\boldsymbol{y}$, and let

$$r = [\boldsymbol{Rxy}], \qquad s = [\boldsymbol{Sxy}], \qquad t = [\boldsymbol{Txy}],$$
$$e = [\boldsymbol{Exy}], \qquad 1' = [\boldsymbol{Ixy}] \quad (1)$$

be the equivalence classes modulo $\mathcal{T}$ of these atomic formulas. The following implications regarding the connections between the operations in $\mathfrak{B}$ and those in $\mathfrak{A}$ are rather straightforward consequences of schemata (T1)–(T6) in $\mathcal{T}$:

$$\begin{aligned}
&T = R \cup S && \text{implies} && t = r + s, && (2)\\
&T = {\sim}R && \text{implies} && t = e \cdot {-}r, && (3)\\
&T = R \,|\, S && \text{implies} && t = r \,;\, s, && (4)\\
&T = R^{-1} && \text{implies} && t = r^{\smile}, && (5)\\
&T = I && \text{implies} && t = 1', && (6)\\
&T \neq \varnothing && \text{implies} && t \neq 0. && (7)
\end{aligned}$$

For example, if the hypothesis of (3) holds, that is to say, if T is the complement of R in $\mathfrak{B}$ with respect to the unit relation E, then the hypothesis of (T2) holds in $\mathfrak{B}$ (with R and T in place of r and t respectively, and with $\mathfrak{B}$ in place of $\mathfrak{A}$), so the formulas

$$\boldsymbol{Txy} \qquad \text{and} \qquad \boldsymbol{Exy} \wedge \neg \boldsymbol{Rxy}$$

are equivalent modulo $\mathcal{T}$, by schema (T2). Consequently,

$$t = [\boldsymbol{Txy}] = [\boldsymbol{Exy} \wedge \neg \boldsymbol{Rxy}] = [\boldsymbol{Exy}] \cdot {-}[\boldsymbol{Rxy}] = e \cdot {-}r,$$

by (1), the definition of the equivalence classes modulo $\mathcal{T}$, and the definitions of the operations $\cdot$ and $-$ in $\mathfrak{A}$. This proves (3).

If the hypothesis of (4) holds, that is to say, if T is the relational composition of R and S in $\mathfrak{B}$, then the hypothesis of (T3) holds in $\mathfrak{B}$ (with R, S, and T in place of r, s, and t respectively, and with $\mathfrak{B}$ in place of $\mathfrak{A}$), so the formulas

$$\boldsymbol{Txy} \qquad \text{and} \qquad \exists \boldsymbol{z}(\boldsymbol{Rxz} \wedge \boldsymbol{Szy})$$

are equivalent modulo $\mathcal{T}$, by schema (T3). Consequently,

$$t = [\boldsymbol{Txy}] = [\exists \boldsymbol{z}(\boldsymbol{Rxz} \wedge \boldsymbol{Szy})] = [\boldsymbol{Rxy}] \,;\, [\boldsymbol{Sxy}] = r \,;\, s,$$

by (1), the definition of the equivalence classes modulo $\mathcal{T}$, and the definition of the operation $;$ in $\mathfrak{A}$. This proves (4).

As a final example, assume that the hypothesis of (7) holds, that is to say, assume that the relation T is not empty. Observe that the sentences

$$\exists \boldsymbol{x}\exists \boldsymbol{y}(\boldsymbol{Txy}) \qquad \text{and} \qquad \neg\exists \boldsymbol{x}\exists \boldsymbol{y}(\boldsymbol{Exy} \wedge \neg \boldsymbol{Exy})$$

are derivable from $\mathcal{T}$. Indeed, the hypothesis of (7) implies that the hypothesis of (T6) holds in $\mathfrak{B}$ (with T in place of t, and with $\mathfrak{B}$ in place of $\mathfrak{A}$), so the first sentence above is in $\mathcal{T}$, by schema (T6). The second sentence is a logical validity. Consequently, the formulas

$$\boldsymbol{Txy} \qquad \text{and} \qquad \boldsymbol{Exy} \wedge \neg \boldsymbol{Exy},$$

cannot be equivalent modulo $\mathcal{T}$, because the set of sentences $\mathcal{T}$ is consistent. In more detail, the relational structure

$$\mathfrak{U} = (U\,,\, R)_{R\in B}$$

correlated with the set relation algebra $\mathfrak{B}$ (see the remarks after Lemma 16.1, with $\mathfrak{B}$ in place of $\mathfrak{A}$, and with φ taken to be the identity function on the universe of $\mathfrak{B}$) is a model of $\mathcal{T}$, by Lemma 16.2, because the the identity function φ is obviously a representation of $\mathfrak{B}$, and the base set U is not empty (since the given relation T on U is assumed to be non-empty). The preceding observations imply that

$$t = [\boldsymbol{Txy}] \neq [\boldsymbol{Exy} \wedge \neg \boldsymbol{Exy}] = 0,$$

by (1), the definition of the equivalence classes modulo $\mathcal{T}$, and the definition of zero in $\mathfrak{A}$. This proves (7).

Several conclusions may be drawn from the preceding remarks. First, the element e must be a reflexive equivalence element in $\mathfrak{A}$, since the unit relation E in $\mathfrak{B}$ is a reflexive equivalence element in $\mathfrak{B}$. In more detail,

$$I \subseteq E, \qquad E^{-1} = E, \qquad E \,|\, E = E$$

in $\mathfrak{B}$, because $\mathfrak{B}$ is a relation algebra with unit E (see, for example, Lemmas 4.1(vi) and 4.5(iv)), so (1), (2) and (4)–(6) directly imply that

$$1' \leq e, \qquad e^{\smile} = e, \qquad e\,;e = e$$

in $\mathfrak{A}$. It therefore makes sense to speak of the relativization of $\mathfrak{A}$ to the element e.

Second, the function ψ that maps each relation R in $\mathfrak{B}$ to the element $r = [\boldsymbol{Rxy}]$ in $\mathfrak{A}$ is a monomorphism from $\mathfrak{B}$ into $\mathfrak{A}(e)$. To see this, let R, S, and T be arbitrary relations in $\mathfrak{B}$, and in agreement with (1), write

$$\psi(R) = r, \qquad \psi(S) = s, \qquad \psi(T) = t. \tag{8}$$

If $T = {\sim}R$, then

$$\psi({\sim}R) = \psi(T) = t = e \cdot {-}r,$$

by (3) and (8). The term on the right is, by definition, the complement of r in the relativization of $\mathfrak{A}(e)$, so ψ preserves the operation of complement. Similarly, if $T = R \mid S$ in $\mathfrak{B}$, then

$$\psi(R \mid S) = \psi(T) = t = r \,;\, s,$$

by (4) and (8), so ψ preserves the operation of relative multiplication. Similar arguments show that ψ preserves the operations of addition and converse, and maps the identity relation in $\mathfrak{B}$ to the identity element in $\mathfrak{A}(e)$ (which is the same as the identity element in $\mathfrak{A}$, by the reflexivity of e). Finally, if T is a non-empty relation in $\mathfrak{B}$, then

$$\psi(T) = t \neq 0,$$

by (7) and (8), so the kernel of ψ contains only the empty relation, and therefore ψ is one-to-one. □

If $\mathfrak{B}$ is a square set relation algebra, then its unit E is the universal relation on the base set U, and one can actually construct an embedding of $\mathfrak{B}$ into a formula relation algebra. In other words, in this case the passage to a relativization in Theorem 17.3 is unnecessary. To prove this assertion, it suffices to include the sentence (T7) from Section 16.1 in $\mathcal{T}$ and to show that in this case, the equivalence element e defined in (1) of the preceding proof is actually the unit of the formula relation algebra of $\mathcal{L}^*$ modulo $\mathcal{T}$. The details are left as an exercise.

Corollary 17.4. *Every simple representable relation algebra is embeddable into a formula relation algebra.*

17.2 Boolean relation algebras

We turn now to the class of Boolean relation algebras, which was introduced in Section 3.4. It turns out that these algebras are always representable. The argument is a relational variant of the well-known proof

that every Boolean algebra is isomorphic to a Boolean set algebra, and it uses the following facts about maximal filters N in a Boolean algebra or a Boolean relation algebra. (Recall that in a Boolean relation algebra, relation algebraic filters are the same things as Boolean filters, and maximal relation algebraic filters are the same thing as ultrafilters.) First, the complement $-r$ of an element r belongs to N if and only if r is not in N. Second, the sum $r+s$ of elements r and s belongs to N if and only if at least one of r and s is in N. Third, every nonzero element r belongs to some maximal filter (this is an immediate consequence of the dual of Theorem 8.31, applied to Boolean relation algebras).

Theorem 17.5. *Every Boolean relation algebra is representable.*

Proof. Suppose $\mathfrak{A}$ is a Boolean relation algebra. Let U be the set of all maximal filters in $\mathfrak{A}$, and let $\mathfrak{B}$ be the set relation algebra consisting of all subrelations of the identity relation id_U. Thus, $\mathfrak{B}$ coincides with the relativization of the full set relation algebra $\mathfrak{Re}(U)$ to the identity relation id_U, that is to say, $\mathfrak{B} = \mathfrak{Re}(id_U)$. Define a function φ from $\mathfrak{A}$ into $\mathfrak{B}$ by putting

$$\varphi(r) = \{(N, N) : N \in U \text{ and } r \in N\}$$

for all r in $\mathfrak{A}$. It is not difficult to check that φ preserves all Boolean operations. For instance, the second property of maximal filters mentioned above implies that

$$\begin{aligned}\{(N, N) : N \in U \text{ and } r + s \in N\}&\\ = \{(N, N) : N \in U \text{ and either } r \in N \text{ or } s \in N\}&\\ = \{(N, N) : N \in U \text{ and } r \in N\}&\\ \cup \{(N, N) : N \in U \text{ and } s \in N\}.&\end{aligned}$$

The first set is just $\varphi(r + s)$, while the last set is the union of $\varphi(r)$ and $\varphi(s)$, so the mapping φ preserves addition:

$$\varphi(r + s) = \varphi(r) \cup \varphi(s).$$

Similarly, the first property of maximal filters mentioned above implies that

$$\begin{aligned}\{(N, N) : N \in U \text{ and } -r \in N\}&\\ = \{(N, N) : N \in U \text{ and } r \notin N\}&\\ = id_U \sim \{(N, N) : N \in U \text{ and } r \in N\}.&\end{aligned}$$

The first set is just $\varphi(-r)$, while the last set is the difference of id_U and $\varphi(r)$, so the mapping φ preserves complement:

$$\varphi(-r) = id_U \sim \varphi(r).$$

Every Boolean operation is definable by a polynomial that involves only the operations of addition and complement. Since φ preserves these last two operations, it must preserve all Boolean operations, and in particular it must preserve multiplication and the unit element. Each of the algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a Boolean relation algebra—$\mathfrak{A}$ by assumption, and $\mathfrak{B}$ by Lemma 3.1—so in each algebra the operations of relative multiplication and converse, and the identity element, coincide respectively with multiplication, the identity function, and the unit element. It follows that

$$\begin{aligned} \varphi(r\,;s) = \varphi(r \cdot s) = \varphi(r) \cap \varphi(s) = \varphi(r) \mid \varphi(s), \\ \varphi(r^\smile) = \varphi(r) = \varphi(r)^{-1}, \\ \varphi(1') = \varphi(1) = id_U. \end{aligned}$$

In other words, φ also preserves the Peircean operations and is therefore a homomorphism.

To show that φ is actually an embedding, it suffices to check that non-zero elements in $\mathfrak{A}$ are mapped to non-zero elements in $\mathfrak{B}$, by Lemma 8.37. For each non-zero element r in $\mathfrak{A}$, there exists a maximal filter N in $\mathfrak{A}$ that contains r, by the third property of maximal filters mentioned above. The pair (N, N) belongs to the relation $\varphi(r)$, by the definition of φ, so this relation is not empty. □

If a Boolean relation algebra $\mathfrak{A}$ is finite, then the representation of $\mathfrak{A}$ defined above is of course complete; but if $\mathfrak{A}$ is infinite, then that representation is necessarily incomplete (the proof is left as an exercise). It is therefore natural to ask under what circumstances $\mathfrak{A}$ possesses a complete representation. Certainly, $\mathfrak{A}$ must be atomic, by Theorem 16.7, and it turns out that this condition is also sufficient. The proof is a relational variant of the argument that every atomic Boolean algebra is isomorphic to a Boolean set relation algebra. It uses the following facts about atoms in an atomic Boolean algebra or Boolean relation algebra. First, an atom a is below a complement $-r$ if and only if a is not below r. Second, if an atom a is below the sum $\sum X$ of some subset X, then a is below at least one of the elements in X. Third, every non-zero element in the algebra is above an atom.

Theorem 17.6. *Every atomic Boolean relation algebra is completely representable.*

Proof. Let $\mathfrak{A}$ be an atomic Boolean relation algebra, let U be the set of atoms in $\mathfrak{A}$, and let $\mathfrak{B}$ be the set relation algebra consisting of all subrelations of the identity relation id_U. Thus, $\mathfrak{B}$ coincides with the relativization of the full set relation algebra $\mathfrak{Re}(U)$ to the identity relation id_U, that is to say, $\mathfrak{B} = \mathfrak{Re}(id_U)$. Define a function φ from $\mathfrak{A}$ into $\mathfrak{B}$ by

$$\varphi(r) = \{(a, a) : a \in U \text{ and } a \leq r\}$$

for each r in $\mathfrak{A}$. The argument showing that φ is a complete representation is similar in spirit to the proof of Theorem 17.5. For instance, the second property of atoms mentioned above implies that for any subset X of $\mathfrak{A}$, if the supremum $r = \sum X$ exists in $\mathfrak{A}$, then

$$\begin{aligned}\{(a, a) : a \in U \text{ and } a \leq r\} &= \{(a, a) : a \in U \text{ and } a \leq s \text{ for some } s \in X\} \\ &= \textstyle\bigcup_{s \in X} \{(a, a) : a \in U \text{ and } a \leq s\}.\end{aligned}$$

The first set is just $\varphi(r)$, and the last is the union of the set $\varphi(s)$ for s in X, so

$$\varphi(r) = \textstyle\bigcup_{s \in X} \varphi(s).$$

Thus, φ preserves all sums that exist in $\mathfrak{A}$.

The proofs that φ preserves the operations of complement, relative multiplication, and converse, and maps the identity element in $\mathfrak{A}$ to the identity relation in $\mathfrak{B}$ are all very similar to the corresponding arguments in the proof of Theorem 17.5, and are left as an exercise. Together, these arguments show that φ is a complete homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

To see that φ is actually a complete embedding, consider a non-zero element r in $\mathfrak{A}$. There must be an atom a in $\mathfrak{A}$ that is below r, by the third property of atoms mentioned above. The pair (a, a) belongs to the relation $\varphi(r)$, by the definition of φ, so this relation is not empty. Consequently, φ is one-to-one, by Lemma 8.37. □

If the Boolean relation algebra $\mathfrak{A}$ in the preceding proof is also complete, then the function φ defined in the proof is easily seen to map $\mathfrak{A}$ onto $\mathfrak{B}$. Indeed, consider an arbitrary relation R in $\mathfrak{B}$. It has the form

$$R = \{(a, a) : a \in X\}$$

for some subset X of U, that is to say, for some set X of atoms in $\mathfrak{A}$. The assumptions that $\mathfrak{A}$ is both complete and atomic imply that the sum $r = \sum X$ exists in $\mathfrak{A}$, and that X is the set of all atoms below r. Consequently,

$$\varphi(r) = \sum\{(a, a) : a \in U \text{ and } a \leq r\} = \{(a, a) : a \in X\} = R.$$

Conclusion: the range of φ is all of $\mathfrak{B}$, and therefore φ is an isomorphism. The following corollary has been proved.

Corollary 17.7. *If $\mathfrak{A}$ is a complete and atomic Boolean relation algebra, and U its set of atoms, then $\mathfrak{A}$ is isomorphic to the set relation algebra $\mathfrak{Re}(id_U)$.*

The proof of Theorem 17.6 can be modified to give a complete description of the complete representations of an atomic Boolean relation algebra $\mathfrak{A}$. To this end, let U be the set of atoms in $\mathfrak{A}$. Consider any system $(V_a : a \in U)$ of non-empty, mutually disjoint sets indexed by the atoms of $\mathfrak{A}$, and write V for the union of this system. By following the lines of the proof of Theorem 17.6, it is not difficult to show that the function ψ from $\mathfrak{A}$ into $\mathfrak{Re}(id_V)$ defined by

$$\psi(r) = \bigcup\{id_{V_a} : a \in U \text{ and } a \leq r\}$$

for r in $\mathfrak{A}$ is actually a complete embedding, and therefore a complete representation, of $\mathfrak{A}$. Conversely, every complete representation of $\mathfrak{A}$ has the above form for some disjoint system of non-empty sets indexed by the atoms of $\mathfrak{A}$. The details are left as an exercise.

17.3 Group complex algebras

Group complex algebras—introduced in Section 3.5—are always representable and in fact they are completely representable. The argument is a variant of the well-known proof that every group is isomorphic to a group of permutations.

Consider a group

$$(G\,,\ \circ\,,\ ^{-1}\,,\ \iota).$$

For each element f in G, define a relation R_f on the set G by

$$R_f = \{(g, g \circ f) : g \in G\}.$$

It is easy to check that the equations

$$R_f \,|\, R_g = R_{f \circ g}, \qquad R_f^{-1} = R_{f^{-1}}, \qquad R_\iota = id_G$$

always hold. For instance, a pair (h, k) belongs to the relational composition $R_f \,|\, R_g$ if and only if there is an element ℓ in G such that the pair (h, ℓ) is in R_f and (ℓ, k) is in R_g, by the definition of composition. The definitions of the relations R_f and R_g imply that such an element ℓ exists if and only if there are elements g_0 and g_1 in G such that

$$(h, \ell) = (g_0, g_0 \circ f) \qquad \text{and} \qquad (\ell, k) = (g_1, g_1 \circ g).$$

These equations hold exactly when

$$h = g_0, \qquad \ell = g_0 \circ f = h \circ f, \qquad \ell = g_1, \qquad k = g_1 \circ g = \ell \circ g,$$

from which it follows that

$$k = h \circ f \circ g.$$

On the other hand, if this last equation holds, then take $\ell = h \circ f$ to obtain pairs (h, ℓ) in R_f and (ℓ, k) in R_g. The existence of the required element ℓ is therefore equivalent to the condition that $k = h \circ f \circ g$, which in turn is equivalent to the condition that the pair (h, k) belong to the relation $R_{f \circ g}$. It follows that the relations $R_f \,|\, R_g$ and $R_{f \circ g}$ contain precisely the same pairs, so they are equal. The verifications of the other two equations above are similar, but easier, and are left as an exercise.

The set

$$W = \{R_f : f \in G\}$$

contains the identity relation on G and is closed under relational composition and converse, by the observations of the previous paragraph. Moreover, the relations in W are mutually disjoint and have $G \times G$ as their union. In more detail, if two relations R_f and R_g in W have a non-empty intersection, then there must be elements h and k in G such that

$$(h, h \circ f) = (k, k \circ g),$$

by the definitions of R_f and R_g. This equation implies that

$$h = k \qquad \text{and} \qquad h \circ f = k \circ g,$$

from which it follows that $f = g$. Thus, the relations R_f and R_g are disjoint whenever $f \neq g$.

For an arbitrary pair (f, g) in the set $G \times G$, write $h = f^{-1} \circ g$, and observe that

$$(f, f \circ h) = (f, g).$$

Consequently, (f, g) belongs to the relation R_h, by the definition of this relation. It follows that the union of the relations in W is $G \times G$.

The preceding observations imply that the set W satisfies the hypotheses of the Atomic Subalgebra Theorem in the form of Corollary 6.22 (with $\mathfrak{Re}(G)$ in place of $\mathfrak{A}$). Apply the corollary to conclude that the set of all possible unions of relations in W—that is to say, the set of all relations of the form

$$R_X = \bigcup \{R_f : f \in X\},$$

where X ranges over the subsets of G —is the universe of a complete subalgebra of $\mathfrak{Re}(G)$ that is atomic. Moreover, the atoms of this subalgebra are just the relations in W. Call this subalgebra $\mathfrak{B}$.

The function φ defined by $\varphi(f) = R_f$ for f in G, is a bijection from the set G of atoms in the group complex algebra $\mathfrak{Cm}(G)$ to the set W of atoms in $\mathfrak{B}$. (Keep in mind that the elements in G are identified with the atoms in $\mathfrak{Cm}(G)$, that is to say, each element f in G is identified with its singleton $\{f\}$.) Furthermore, this bijection preserves the operation of relative multiplication on atoms. In more detail,

$$\varphi(f \,;\, g) = \varphi(f \circ g) = R_{f \circ g} = R_f \,|\, R_g = \varphi(f) \,|\, \varphi(g),$$

by the definition of relative multiplication in $\mathfrak{Cm}(G)$ as group complex multiplication (see Section 3.5), the definition of φ, and the observations made at the beginning of the section. Apply the Atomic Isomorphism Theorem in the form of Corollary 7.12 to conclude that φ can be extended to an isomorphism ψ from $\mathfrak{Cm}(G)$ to $\mathfrak{B}$ that is defined by

$$\psi(X) = R_X$$

for subsets X of G. Because ψ is an isomorphism from $\mathfrak{Cm}(G)$ to a complete subalgebra of $\mathfrak{Re}(G)$, it follows from Lemma 7.9 that ψ must be a complete monomorphism from $\mathfrak{Cm}(G)$ into $\mathfrak{Re}(G)$ and therefore a complete representation of $\mathfrak{Cm}(G)$. We shall call it the *Cayley representation* of $\mathfrak{Cm}(G)$.

Theorem 17.8. *The complex algebra of every group is completely representable via its Cayley representation.*

For a specific example of a Cayley representation, consider the cyclic group $\mathbb{Z}_4$ of order four. The elements of $\mathbb{Z}_4$ are the integers 0, 1, 2, and 3. The group operations of composition and inverse are addition modulo four and negation modulo four respectively (see Table 17.1). The identity element is 0. The Cayley representations of the elements

$\circ$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

	$^{-1}$
0	0
1	3
2	2
3	1

Table 17.1 Composition and inverse tables for the group $\mathbb{Z}_4$.

in $\mathbb{Z}_4$, that is to say, of the atoms in $\mathfrak{Cm}(\mathbb{Z}_4)$, are the relations

$$R_0 = \{(0,0),(1,1),(2,2),(3,3)\}, \quad R_1 = \{(0,1),(1,2),(2,3),(3,0)\},$$
$$R_2 = \{(0,2),(1,3),(2,0),(3,1)\}, \quad R_3 = \{(0,3),(1,0),(2,1),(3,2)\}.$$

For example, the relation R_3 consists of the pairs (m,n) such that

$$m = 0,1,2,3 \qquad \text{and} \qquad n = m + 3 \mod 4.$$

These relations may be visualized by means of the diagram in Figure 17.1. The elements of the group complex algebra $\mathfrak{Cm}(\mathbb{Z}_4)$ are the

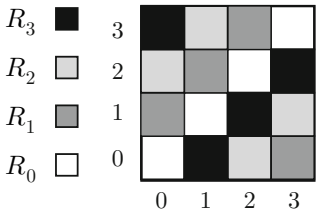


Fig. 17.1 The Cayley representation of $\mathfrak{Cm}(\mathbb{Z}_4)$.

subsets of $\mathbb{Z}_4$, and each subset X is represented by the relation R_X that is the union of the atomic relations R_f for f in X. By visualizing

the unions of the various atomic relations in Figure 17.1, one arrives at a visualization of the Cayley representation of $\mathfrak{Cm}(\mathbb{Z}_4)$.

It turns out that the Cayley representation of a group complex algebra is the unique complete square representation of the algebra in the sense that every complete square representation of the algebra is equivalent to the Cayley representation. The proof is based on a group-theoretic observation that was already used above: the permutations that make up the Cayley representation of a group G form a partition of the set $G \times G$ in the sense that they are non-empty, mutually disjoint, and have $G \times G$ as their union. The next lemma says that this property is characteristic of Cayley representations.

Lemma 17.9. *If H is a group of permutations on a set U under the operation of relational composition, and if the permutations in H form a partition of $U \times U$, then it is possible to define group operations on U in such a way that H becomes the Cayley representation of the group U.*

Proof. Fix an element ξ in U for the duration of the proof. For each element α in U, there is a unique permutation R in H that contains the pair (ξ, α). Indeed, the permutation R exists because (ξ, α) belongs to the relation $U \times U$, which is assumed to be the union of the permutations in H; and the permutation R is unique because distinct permutations in H are assumed to be disjoint. Write R_α for the unique permutation in H that contains the pair (ξ, α).

Since H is assumed to be a group of permutations of U under the operation of relational composition, it must contain the identity permutation id_U of U. Of course, the pair (ξ, ξ) belongs to the identity relation, and it also belongs to the permutation R_ξ, by the definition of that permutation. Consequently, we must have

$$R_\xi = id_U, \tag{1}$$

because distinct relations in H are disjoint.

The function ψ from U to H defined by

$$\psi(\alpha) = R_\alpha \tag{2}$$

for each element α in U is a bijection. To see that ψ is one-to-one, assume that $\psi(\alpha) = \psi(\beta)$, that is to say, assume

$$R_\alpha = R_\beta.$$

The pairs (ξ, α) and (ξ, β) must then both belong to R_α, by the definitions of R_α and R_β. Since R_α is a permutation and therefore a function, it follows that $\alpha = \beta$. To see that ψ is onto, consider an arbitrary element R in H. Since R is assumed to be a permutation of the set U, it must map the element ξ to some element α in U. Consequently, the pair (ξ, α) belongs to R. This means that R is equal to the relation R_α, by the definition of R_α, so ψ maps α to R.

Define a binary operation $\circ$ and a unary operation $^{-1}$ on U by putting

$$\alpha \circ \beta = \gamma \qquad \text{if and only if} \qquad R_\alpha \,|\, R_\beta = R_\gamma, \tag{3}$$

$$\alpha^{-1} = \gamma \qquad \text{if and only if} \qquad R_\alpha^{-1} = R_\gamma. \tag{4}$$

Each element α in U corresponds under ψ to a unique permutation R_α in H, and conversely, by the observations of the preceding paragraph, so the operations just defined are in fact well defined and turn ψ into an isomorphism from the algebra

$$(U\,,\ \circ\,,\ ^{-1}\,,\ \xi) \tag{5}$$

to the group

$$(H\,,\ |\,,\ ^{-1}\,,\ id_U). \tag{6}$$

For example, if $\alpha \circ \beta = \gamma$, , then

$$\psi(\alpha \circ \beta) = \psi(\gamma) = R_\gamma = R_\alpha \,|\, R_\beta = \psi(\alpha) \,|\, \psi(\beta),$$

by (2) and (3). Furthermore,

$$\psi(\xi) = R_\xi = id_U,$$

by (1) and (2). It follows that the algebra in (5) must be a group.

It remains to show that H—or more precisely, the function ψ—is the Cayley representation of the group in (5), that is to say,

$$\psi(\alpha) = R_\alpha = \{(\gamma, \gamma \circ \alpha) : \gamma \in U\} \tag{7}$$

for each α in U. Consider a pair of the form $(\gamma, \gamma \circ \alpha)$ with γ in U. The pairs (ξ, γ) and $(\xi, \gamma \circ \alpha)$ belong to the permutations R_γ and $R_{\gamma \circ \alpha}$ respectively, by the definitions of these permutations. It follows that the pair (γ, ξ) belongs to the converse permutation R_γ^{-1}, and the pair $(\xi, \gamma \circ \alpha)$ belongs to the composite permutation $R_\gamma \,|\, R_\alpha$, because

$$R_{\gamma \circ \alpha} = R_\gamma \mid R_\alpha,$$

by (3). Consequently, the pair $(\gamma, \gamma \circ \alpha)$ belongs to the composition

$$R_\gamma^{-1} \mid R_\gamma \mid R_\alpha,$$

by the definition of relational composition. The group properties of (6) imply that this composition coincides with the relation R_α, so the pair $(\gamma, \gamma \circ \alpha)$ must belong to R_α. This holds for each γ in U. No other pair of the form (γ, δ), for some δ in U, can belong to R_α, because this relation is assumed to be a permutation of U, and therefore a function. Combine these observations with (2) to arrive at (7). □

The point of the preceding lemma is that any complete square representation of the complex algebra of a group G, when restricted to the atoms, is the Cayley representation of a group that is isomorphic to G. Moreover, the isomorphism between the two groups lifts to a base isomorphism between the two representations. The proof of the next theorem makes these ideas clear.

Theorem 17.10. *Every complete square representation of a group complex algebra is equivalent to the Cayley representation of the algebra.*

Proof. Suppose $(G, \circ, {}^{-1}, \iota)$ is a group, and φ the Cayley representation of this group, so that

$$\varphi(f) = R_f = \{(g, g \circ f) : g \in G\} \tag{1}$$

for elements f in G. It was shown in Theorem 17.8 that φ can be extended to a complete square representation $\hat{\varphi}$ of the group complex algebra $\mathfrak{Cm}(G)$ over the set G, where $\hat{\varphi}$ is defined by

$$\hat{\varphi}(X) = \bigcup\{\varphi(f) : f \in X\} = \bigcup\{R_f : f \in X\} \tag{2}$$

for subsets X of G. The image of $\mathfrak{Cm}(G)$ under $\hat{\varphi}$ is the complete subalgebra $\mathfrak{A}$ of $\mathfrak{Re}(G)$ that is completely generated by the set

$$W = \{R_f : f \in G\}$$

of atoms in $\mathfrak{A}$ (see the proof of Theorem 17.8).

Consider now an arbitrary complete square representation ϑ of the complex algebra $\mathfrak{Cm}(G)$ over a base set U, that is to say, a complete

embedding ϑ of $\mathfrak{Cm}(G)$ into $\mathfrak{Re}(U)$. The set U cannot be empty, because the group G has at least one element and therefore the complex algebra of the group is not degenerate. Let $\mathfrak{B}$ be the image of $\mathfrak{Cm}(G)$ under ϑ, and notice that $\mathfrak{B}$ is a regular subalgebra, and therefore a complete subalgebra, of $\mathfrak{Re}(U)$, by Lemma 7.9. Also, $\mathfrak{B}$ is atomic, since $\mathfrak{Cm}(G)$ is atomic, and $\mathfrak{B}$ is completely generated by the set

$$H = \{\vartheta(f) : f \in G\}$$

of images of the atoms in $\mathfrak{Cm}(G)$, since ϑ is a complete square representation of $\mathfrak{Cm}(G)$. Each atom f in $\mathfrak{Cm}(G)$ is a permutation in the sense that

$$f^{\smile} ; f = f ; f^{\smile} = 1',$$

so each image relation $\vartheta(f)$ is a (set-theoretic) permutation of the set U. (Recall that atoms in $\mathfrak{Cm}(G)$ are really singletons of elements in G, but for notational purposes they are being identified with the elements in G.) The set G of atoms in $\mathfrak{Cm}(G)$ is a group under the restricted Peircean operations of $\mathfrak{Cm}(G)$, namely complex multiplication and inverse (with ι—or, more precisely, $\{\iota\}$—as the identity element), so the set H of the images of these atoms is a group of permutations of the set U under the restricted Peircean operations of $\mathfrak{B}$, namely relational composition and converse, with the identity relation as the identity element. Finally, the atoms in $\mathfrak{Cm}(G)$, namely the (singletons of) elements in G, are non-zero, mutually disjoint, and sum to the unit element in $\mathfrak{Cm}(G)$, so the permutations in H are non-empty, mutually disjoint, and have the unit relation $U \times U$ in $\mathfrak{B}$ as their union, again because ϑ is a complete representation.

Apply Lemma 17.9 to conclude that for any fixed element ξ in U there is a binary operation $\circ$ and a unary operation $^{-1}$ on U (defined in terms of ξ) such that

$$(U\,,\ \circ\,,\ ^{-1}\,,\ \xi)$$

is a group, and the Cayley representation of this group, namely the function ψ defined by

$$\psi(\alpha) = R_\alpha = \{(\gamma, \gamma \circ \alpha) : \gamma \in U\} \tag{3}$$

for elements α in U, is an isomorphism from the group U to the group H. The Cayley representation ψ can be extended to a complete square representation $\hat{\psi}$ of the group complex algebra $\mathfrak{Cm}(U)$ over the set U, where $\hat{\psi}$ is defined by

$$\hat{\psi}(X) = \bigcup\{\psi(\alpha) : \alpha \in X\} = \bigcup\{R_\alpha : \alpha \in X\}$$

for subsets X of U, by Theorem 17.8 and (3). The image of $\mathfrak{Cm}(U)$ under the complete embedding $\hat{\psi}$ is the complete subalgebra of $\mathfrak{Re}(U)$ that is completely generated by the set of images under $\hat{\psi}$ of the atoms in $\mathfrak{Cm}(U)$, that is to say, it is the complete subalgebra of $\mathfrak{Re}(U)$ that is completely generated by the set of relations

$$H = \{R_\alpha : \alpha \in U\} = \{\psi(\alpha) : \alpha \in U\}.$$

In other words, the image algebra is the algebra $\mathfrak{B}$ defined above, so $\hat{\psi}$ maps $\mathfrak{Cm}(U)$ isomorphically to $\mathfrak{B}$.

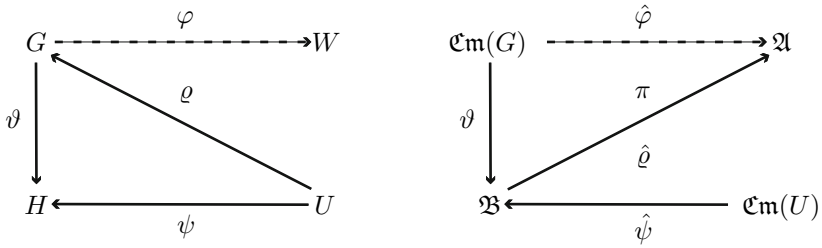


Fig. 17.2 The uniqueness of the Cayley representation.

The function ϱ that is the composition of an appropriate restriction of ϑ^{-1} and ψ is an isomorphism from the group U to the group G (see Figure 17.2). The definitions of these functions imply that

$$\begin{aligned} \varrho(\alpha) = f \qquad & \text{if and only if} \qquad \vartheta^{-1}(\psi(\alpha)) = f, \\ & \text{if and only if} \qquad \vartheta^{-1}(R_\alpha) = f, \\ & \text{if and only if} \qquad \vartheta(f) = R_\alpha. \end{aligned} \tag{4}$$

The base isomorphism from $\mathfrak{Re}(U)$ to $\mathfrak{Re}(G)$ that is induced by ϱ is the function $\hat{\varrho}$ defined by

$$\hat{\varrho}(R) = \{(\varrho(\alpha), \varrho(\beta)) : (\alpha, \beta) \in R\} \tag{5}$$

for every relation R in $\mathfrak{Re}(U)$ (see Section 7.5). If

$$\varrho(\alpha) = f \qquad \text{and} \qquad \varrho(\gamma) = g,$$

then

$$\varrho(\gamma \circ \alpha) = \varrho(\gamma) \circ \varrho(\alpha) = g \circ f,$$

by the isomorphism properties of ϱ, and therefore

$$(\varrho(\gamma), \varrho(\gamma \circ \alpha)) = (g, g \circ f). \tag{6}$$

Consequently,

$$\begin{aligned} \hat{\varrho}(R_\alpha) &= \hat{\varrho}(\{(\gamma, \gamma \circ \alpha) : \gamma \in U\}) \\ &= \{(\varrho(\gamma), \varrho(\gamma \circ \alpha)) : \gamma \in U\} \\ &= \{(g, g \circ f) : g \in G\} \\ &= R_f, \end{aligned} \tag{7}$$

by (3), (5), (6), and (1), where $\vartheta(f) = R_\alpha$, by (4). In particular, the base isomorphism $\hat{\varrho}$ maps the set of atoms in $\mathfrak{B}$ onto the set of atoms in $\mathfrak{A}$, by the remarks above, and therefore it maps the set of relations in $\mathfrak{B}$ onto the set of relations in $\mathfrak{A}$, since each relation in each algebra is the union of a set of atoms in the algebra. An appropriate restriction of $\hat{\varrho}$ is therefore a base isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ (see the remarks at the end of Section 7.5).

Because ϑ^{-1} maps $\mathfrak{B}$ isomorphically to $\mathfrak{Cm}(G)$, and $\hat{\varphi}$ maps $\mathfrak{Cm}(G)$ isomorphically to $\mathfrak{A}$, the composition

$$\pi = \hat{\varphi} \circ \vartheta^{-1} \tag{8}$$

maps $\mathfrak{B}$ isomorphically to $\mathfrak{A}$ (see Figure 17.2). If $\vartheta(f) = R_\alpha$, then

$$\pi(R_\alpha) = \hat{\varphi}(\vartheta^{-1}(R_\alpha)) = \hat{\varphi}(f) = \varphi(f) = R_f,$$

by (8), (2), and (1). Compare this with (7) to see that π coincides with the appropriate restriction of $\hat{\varrho}$, so this restriction of $\hat{\varrho}$ is also equal to the composition $\hat{\varphi} \circ \vartheta^{-1}$, by (8). Form the composition of both functions on the right with ϑ to arrive at the conclusion that the Cayley representation $\hat{\varphi}$ is equal to the composition of the base isomorphism $\hat{\varrho}$ with the representation ϑ, and therefore the representations $\hat{\varphi}$ and ϑ are equivalent. □

The preceding theorem and Theorem 16.18 together yield a complete description of the representations of finite group complex algebras and the complete representations of infinite group complex algebras.

Corollary 17.11. (i) *The complex algebra of a finite group G has, up to equivalence, a unique square representation, namely the Cayley representation. The representations of $\mathfrak{Cm}(G)$ are precisely the amalgamations of non-empty systems of disjoint copies of the Cayley representation.*

(ii) *The complex algebra of an infinite group G has, up to equivalence, a unique complete square representation, namely the Cayley representation. The complete representations of $\mathfrak{Cm}(G)$ are precisely the amalgamations of non-empty systems of disjoint copies of the Cayley representation.*

17.4 Geometric complex algebras

The situation regarding the representability of the geometric complex algebras that were introduced in Section 3.6 is more complicated than for Boolean relation algebras and group complex algebras. It turns out that the following conditions on a geometric complex algebra $\mathfrak{Cm}(P)$ are equivalent: $\mathfrak{Cm}(P)$ is representable; $\mathfrak{Cm}(P)$ is completely representable; P is embeddable into a projective geometry of one higher dimension. Since there exist projective lines and projective planes that are not embeddable into projective geometries of higher dimension, there must exist geometric complex algebras that are not representable at all. An important consequence of this observation is that not all relation algebras are representable.

To prove the results just mentioned, as well as related results concerning the number of inequivalent complete representations of a geometric complex algebra, it is necessary to introduce a class of geometries that are closely related to projective geometries. An *affine geometry* is defined to be a set of points, a set of lines, and a set of planes, together with a relation of incidence between points and lines, points and planes, and lines and planes, such that the following axioms hold. (1) Through any two points there passes one and only one line. (2) Through a given point not on a given line ℓ, there passes one and only one line that is parallel to ℓ. (3) Every line has at least two points. (4) Through three non-collinear points there passes one and only one plane. (5) Two lines parallel to a third line are parallel to each other. Recall in this connection that three distinct points are said to be *collinear* if they all lie on a common line; and two lines

are said to be *parallel* if they are *coplanar* in the sense that they lie in a common plane, and if they are either equal or do not intersect. The unique affine line through two distinct points a and b is usually denoted by ab. It is not difficult to prove that in a given affine geometry D, all lines have the same number of points, and this number is called the *order* of D. The relation of parallelism between lines is an equivalence relation and the equivalence classes are called the *parallel classes* of the lines. In an affine plane of order n, there are $n+1$ parallel classes of lines, each parallel class has n lines, and altogether there are n^2 points.

Recall from Section 3.6 that a projective geometry consists of a set of points, a set of lines, and a relation of incidence between the points and the lines, such that the following axioms hold. (1) Through any two distinct points, there passes one and only one line. (2) Every line has at least three points. (3) For distinct points p, q, s, and t, if the line through p and q intersects the line through s and t, then the line through p and s intersects the line through q and t (see Figure 17.3). This third axiom is called the *Pasch Axiom*, and it expresses the requirement that two coplanar lines always intersect.

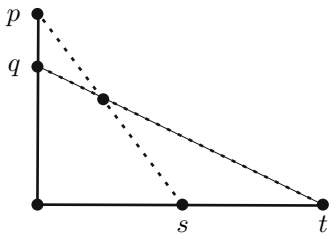


Fig. 17.3 The Pasch Axiom of projective geometry.

A classic result of projective geometry says that every affine geometry D can be extended to a projective geometry Q of the same order and dimension. For each parallel class of lines in D one adds a new *point at infinity*, or *ideal point*, that is defined to be incident with all the lines of the parallel class. For each parallel class of planes in D, one adds a new *line at infinity*, or *ideal line*, that is defined to lie in each of the planes of the parallel class; the line consists of the points at infinity that are incident with some line lying in one of the planes of the given class. The projective geometry Q constructed in this way

is unique up to isomorphisms that leave D pointwise fixed. It is called the *projective extension* of the affine geometry D. The set of points and lines at infinity form a projective geometry P, called the *geometry at infinity* of D (in Q). The dimension of Q is one higher than the dimension of P, so P is in fact a maximal projective subspace of Q. It is clear from the construction of the projective extension Q, that the affine geometry D can be recovered from Q by deleting all points and lines at infinity, that is to say, all points and lines in P; in particular, the point at infinity is removed from each line in Q that is not a line at infinity.

An *isomorphism*, or a *collineation*, from a projective geometry P to a projective geometry Q is a bijection from the sets of points in P to the set of points in Q that preserves the relation of collinearity in the sense that three distinct points p, q, and r in P are collinear if and only if their images in Q under the bijection are collinear. An *autocollineation* of P is a collineation from P to itself. An *embedding* of P into Q is a collineation from P to a subspace of Q. A similar terminology applies to affine geometries.

We return now to the results mentioned at the start of the section, and we begin by showing that if a projective geometry P is embeddable into a projective geometry Q of one higher dimension, then the complex algebra of the geometry P has a complete representation. By applying a version of the Exchange Principle (Theorem 7.15) that is formulated for projective geometries, it may be assumed that the embedding of P into Q is the identity function on P, so that P is actually a maximal subspace of Q. Let D be the affine geometry obtained from Q by deleting the points and lines of P. In other words, Q is the projective extension of D in which P is the geometry at infinity of D (or the set of ideal points of lines in D, together with the set of ideal lines of planes in D). We construct a complete representation of $\mathfrak{Cm}(P)$ using the set of (affine) points in D as the base set, and we use this same letter D to denote this set of points.

For each point p in P, define a relation R_p on D by putting

$$R_p = \{(a, b) : a, b \text{ are distinct points in } D, \text{ and } p \text{ is the ideal point of the line } ab\};$$

and take R_ι to be the identity relation on D,

$$R_\iota = id_D = \{(a, a) : a \in D\}.$$

The relation R_p consists of all the pairs of distinct affine points that lie on one of the lines in the class $\mathit{\Pi}_p$ of parallel affine lines having p as their point at infinity, so R_p may be written in the form

$$R_p = \bigcup\{\ell \times \ell : \ell \in \mathit{\Pi}_p\} \sim \{(a, a) : a \in D\}.$$

(See Figure 17.4 for an illustration in the case when D is a plane and P is the unique line at infinity of D. The oblique lines in the figure all have the same point p in P as their point at infinity. The relation R_p consists of the pairs of distinct points on each of the oblique lines.) We shall say a bit more about the intuition behind this definition in a

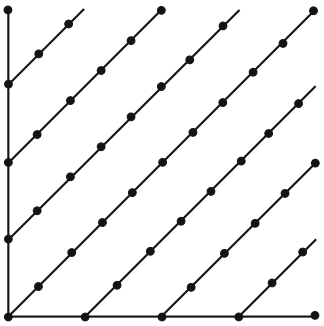


Fig. 17.4 The relation R_p consists of the pairs of distinct points on each of the lines in a class of parallel lines.

moment.

If an affine line ab has p as its point at infinity, then so does the line ba (they are the same line), so the relation R_p is always symmetric; in other words, if (a, b) is in R_p, then so is (b, a), and therefore

$$R_p^{-1} = R_p.$$

The operation of composition on these relations is determined by the formulas

$$R_p \,|\, R_p = \begin{cases} R_\iota & \text{if} \quad D \text{ has order } 2\,, \\ R_\iota \cup R_p & \text{if} \quad D \text{ has order } > 2 \end{cases}$$

$$R_p \,|\, R_q = \bigcup\{R_r : r \text{ is collinear with, and distinct from, } p \text{ and } q\},$$

$$R_p \,|\, R_\iota = R_\iota \,|\, R_p = R_p,$$

$$R_\iota \,|\, R_\iota = R_\iota,$$

where p and q are distinct points in P. When D is an affine plane, and P the line at infinity, the second formula assumes a somewhat simpler form, namely

$$R_p \,|\, R_q = D \times D \sim (R_p \cup R_q \cup R_\iota).$$

Here, as an example, is the verification of the formula for computing the composition of relations R_p and R_q when $p \neq q$. Suppose first that a pair (a, b) of affine points belongs to the relation R_r for some projective point r that is collinear with, but different from, p and q. The five points a, b, p, q, and r are then all distinct. They all lie in the projective plane in Q determined by the point a and the projective line pq, because r lies on the line pq, by assumption, and also on the projective line ab, by the assumption that the pair (a, b) is in the relation R_r (see Figure 17.5(a)). The projective lines ab and pq intersect in the point r, so the Pasch Axiom (see Figure 17.3, with a and b in place of s and t respectively) may be applied to conclude that the projective lines ap and bq in Q intersect in a point c. The point c cannot be collinear with p and q, for that would force at least one of the points a and b to be collinear with p and q, and therefore a projective point in P, in contradiction to the assumption that both a and b are affine points in D (the sets D and P being disjoint). Nor can c coincide with a, for that would force a, b, and q to be collinear, and therefore q would equal r, in contradiction to the assumption that r is distinct from q (see Figure 17.5(b)). An analogous argument shows that the equality of c with b would force the equality of p and r, in contradiction to the assumption that r is distinct from p (see Figure 17.5(d)). Thus, c is an affine point that is different from a and b, and that lies on the projective lines ap and bq, so the pairs (a, c) and (c, b) belong to the relations R_p and R_q respectively, by the definition of these relations. Consequently, the pair (a, b) belongs to the relational composition $R_p \,|\, R_q$. Conclusion: the relation R_r is included in the composition $R_p \,|\, R_q$ whenever r is a projective point that is collinear with, but different from, p and q.

Now suppose that a pair (a, b) belongs to the composition of the relations R_p and R_q. There must then be an affine point c different from a and b such that (a, c) is in R_p and (c, b) in R_q, by the definition of relational composition and the definition of the relations R_p and R_q. In particular, c is the unique point of intersection of the projective lines ap and bq. The five points a, b, c, p, and q are all distinct.

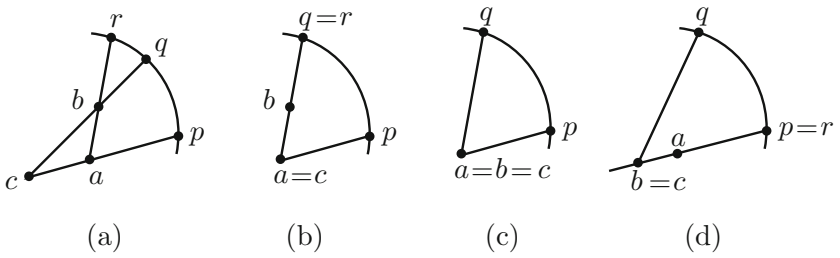


Fig. 17.5

In more detail, the equality of the points a and b would force the intersection point of the lines ap and bq to be $a = b$, and therefore c would coincide with a and b, in contradiction to the stated inequality of c with a and b (see Figure 17.5(c)). Consequently, a, b, and c are distinct affine points in D, and p and q are distinct projective points in P, and the sets of points D and P are disjoint, so all five points are distinct from one another. The five distinct points all lie in the projective plane in Q determined by the point c and the projective line pq, since a and b lie on the projective lines cp and cq respectively. The projective lines ap and qb in Q intersect in the point c, so the projective lines ab and pq must intersect in a point r, by the Pasch Axiom (see Figure 17.3, with a, q, and b in place of q, s and t respectively). Obviously, r is collinear with p and q. The equality of r with p would force the point b to be collinear with a and p; therefore, the intersection point of the lines ap and bq would be b, so that b would coincide with c, contradicting the stated inequality of c with b (see Figure 17.5(d)). Similarly, the equality of r with q would force a to be collinear with b and q; therefore, the point of intersection of the lines ap and bq would be a, so that a would coincide with c, contradicting the stated inequality of c and a (see Figure 17.5(b)). Thus, r is a point that is collinear with, but different from p and q, and the pair (a, b) obviously belongs to the relation R_r, by the definition of this relation. Conclusion: each pair (a, b) in the composition $R_p \,|\, R_q$ belongs to the relation R_r for some point r collinear with but different from p and q, so the composition $R_p \,|\, R_q$ is included in the union of the relations R_r over all such r . This completes the verification of the formula for computing $R_p \,|\, R_q$ for distinct p and q.

As another example, here is the verification of the formula for computing the composition of relations R_p and R_q when $p = q$ and the

affine geometry D has order greater than 2. Consider a pair of affine points (a, b). Suppose first that this pair belongs to the relation R_ι, that is to say, suppose $a = b$. Take c to be any point on the line ap that is different from a and p. Such a point must exist, because every projective line has at least three points. The line ap contains an affine point, namely a, and therefore it contains just one projective point, namely p. In more detail, if the line contained two projective points, it would be a line in the geometry at infinity, and therefore it would contain no affine points. The point c is different from p, by assumption, so it must be an affine point. The line ac coincides with the line ap, because the three points a, c, and p are distinct and collinear, so the line ac has p as its point at infinity. It follows that the pair (a, c) belongs to the relation R_p, by the definition of this relation, and therefore so does the pair (c, b), since $b = a$. Consequently, the pair (a, b) belongs to the composition $R_p \mid R_p$, by the definition of relational composition. Conclusion: the relation R_ι is included in the composition of R_p with itself.

Suppose next that the pair (a, b) belongs to the relation R_p. The definition of this relation implies that a and b are distinct affine points, and that p is the (unique) point at infinity on the line ab. The affine geometry D is assumed to have order greater than 2, so there must be an affine point c on the line ab that is different from both a and b. The pairs (a, c) and (c, b) are then both in the relation R_p, by the definition of this relation, so the pair (a, b) belongs to the composition $R_p \mid R_p$, by the definition of relational composition. Conclusion: the relation R_p is included in the composition of R_p with itself.

It remains to show that the composition of R_p with itself is included in the union of R_ι and R_p. To this end, assume that the pair (a, b) belongs to the composition $R_p \mid R_p$. There must then be a point c such that (a, c) and (c, b) are both in R_p, by the definition of relational composition. The definition of the relation R_p implies that a and c are distinct affine points, and that p is the point at infinity of the line ac. Similarly, b and c are distinct affine points, and p is the point at infinity of the line bc. It follows that the points a and b are both on the line cp. If the points a and b are equal, then of course the pair (a, b) belongs to the identity relation R_ι. If the points are distinct, then the projective line ab must coincide with the line cp, and therefore p must be the point at infinity of the line ab. Consequently, the pair (a, b) belongs to the relation R_p, by the definition of this relation. Conclusion: $R_p \mid R_p$ is included in the union $R_\iota \cup R_p$. This completes the verification of the

formula for computing $R_p \mid R_p$ in the case when D has order greater than 2.

Every pair of distinct affine points (a, b) lies on one and only one affine line, by the first axiom of affine geometry, and this line has a unique point p as its point at infinity in the projective geometry P, so the pair (a, b) belongs to the relation R_p for a unique point p in P; and every pair of equal affine points (a, a) belongs to the identity relation R_ι, and to none of the relations R_p for points p in P. Consequently, the set of relations

$$W = \{R_p : p \in P^+\},$$

where $P^+ = P \cup \{\iota\}$, is a partition of the unit relation $D \times D$ of the full set relation algebra $\mathfrak{Re}(D)$. The identity relation R_ι belongs to W by definition, and the set W is closed under the formation of converses because each relation in W is symmetric and therefore equal to its own converse. The composition of any two relations in W is a union of relations in W, by the formulas for relational composition given above. Thus, the set W satisfies the conditions of the Atomic Subalgebra Theorem in the form of Corollary 6.22, with respect to the algebra $\mathfrak{Re}(D)$. Apply that corollary to conclude that the set of all unions of relations in W, that is to say, the set of all relations of the form

$$R_X = \bigcup\{R_p : p \in X\},$$

where X ranges over the subsets of P^+, is the universe of a complete subalgebra of $\mathfrak{Re}(D)$ that is atomic, and the atoms of this subalgebra are just the relations in W. Call this subalgebra $\mathfrak{B}$.

The function φ defined by $\varphi(p) = R_p$ for p in P^+ is a bijection from the set P^+ of atoms in the geometric complex algebra $\mathfrak{Cm}(P)$ to the set W of atoms in $\mathfrak{B}$. (Keep in mind that the elements in P^+ are identified with their singletons, the atoms in $\mathfrak{Cm}(P)$.) It is not difficult to check that this bijection preserves the operation of relative multiplication on atoms. For example, if p and q are distinct points in P, then an atom r is below the relative product $p\,;q$ if and only if r is a point in P that is collinear with, but different from, p and q, by the definition of relative multiplication in $\mathfrak{Cm}(P)$. This is precisely the criterion for the relation R_r to be included in the composition $R_p \mid R_q$, so

$$\begin{array}{lll} r \leq p\,;q & \text{if and only if} & R_r \subseteq R_p \mid R_q, \\ & \text{if and only if} & \varphi(r) \subseteq \varphi(p) \mid \varphi(q). \end{array}$$

Similar arguments yield analogous results in the remaining cases, namely the cases when p and q are equal, and when at least one of p and q is the identity element. Apply the Atomic Isomorphism Theorem in the form of Corollary 7.12 to conclude that φ can be extended to an isomorphism ψ from $\mathfrak{Cm}(P)$ to $\mathfrak{B}$ that is defined by

$$\psi(X) = R_X$$

for subsets X of P^+. The function ψ is a complete representation of the algebra $\mathfrak{Cm}(P)$, by Lemma 16.6, because $\mathfrak{B}$ is a complete subalgebra of $\mathfrak{Re}(D)$. We shall call it an *affine representation* of $\mathfrak{Cm}(P)$. Notice that the definition of φ (and therefore of ψ) depends on three things: the projective geometry P, the affine geometry D, and the connection between the lines in D and the points in P.

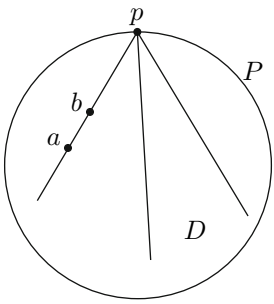


Fig. 17.6 The intuition behind affine representations.

A word about the intuition behind affine representations may be helpful. In plane projective geometry there is a duality between points and lines: every line may be identified with the set of points lying on it, and every point may be identified with the set of lines passing through it. Consider any projective plane Q and any line P in Q. Let D be the affine plane obtained from Q by deleting the points and lines of P, and view P as the line at infinity of the plane D. Under the identification of points with sets of lines, every point p in P may be identified with the set of lines in Q that pass through p. This is tantamount to identifying the point p with the set of pairs (a, b) such that a and b are distinct points in D, and the line ab passes through p. (Here, one line is being disregarded, namely the line P at infinity; it passes through every point in P and therefore does not help to distinguish the points in P from

one another.) This identification is in fact just the affine representation of the atoms (points) in $\mathfrak{Cm}(P)$ (see Figure 17.6).

A concrete example may help to illuminate further the ideas behind affine representations. Let P be the projective line of order three; it consists of four points, say

$$(0), \qquad (1), \qquad (2), \qquad (\infty),$$

which are assumed to be collinear. The affine plane D of order three is constructed from the three-element field,

$$(\mathbb{Z}_3\,,\,+\,,\,\cdot\,,\,0\,,\,1),$$

where $\mathbb{Z}_3 = \{0, 1, 2\}$, and $+$ and $\cdot$ are the operations of addition and multiplication modulo 3. The points of D are pairs of numbers from the field, that is to say, they are elements in the Cartesian square $\mathbb{Z}_3 \times \mathbb{Z}_3$. The lines of D are the sets of points (pairs) satisfying the standard linear equations with coefficients from the field. The set of points satisfying the equation

$$y = m \cdot x + b$$

will be denoted by ℓ_{mb} (where x and y are variables, and m and b are constants denoting elements of the field). For instance, the lines

$$\ell_{12} = \{(0, 2), (1, 0), (2, 1)\} \qquad \text{and} \qquad \ell_{02} = \{(0, 2), (1, 2), (2, 2)\}$$

are defined by the equations $y = x + 2$ and $y = 2$ respectively. The set of points satisfying the equation $x = b$ is denoted by $\ell_{\infty b}$. For instance, the lines

$$\ell_{\infty 0} = \{(0, 0), (0, 1), (0, 2)\} \qquad \text{and} \qquad \ell_{\infty 1} = \{(1, 0), (1, 1), (1, 2)\}$$

are defined by the equations $x = 0$ and $x = 1$ respectively. The projective extension of D with respect to P is the projective plane Q of order three that consists of the points of the affine plane D and the points of P, which are the points at infinity. The lines in Q are just the single line at infinity, namely P, and the affine lines augmented by the appropriate point at infinity; the point at infinity of each of the lines ℓ_{mb} with $b = 0, 1, 2$ is (m), for $m = 0, 1, 2, \infty$. (See Figure 17.7.)

The base set of the affine representation of $\mathfrak{Cm}(P)$ is the set D with nine elements, each of which is a pair (a, b) with coordinates a and b in the field $\mathbb{Z}_3$. We shall use the shorthand a_b to denote this pair; for

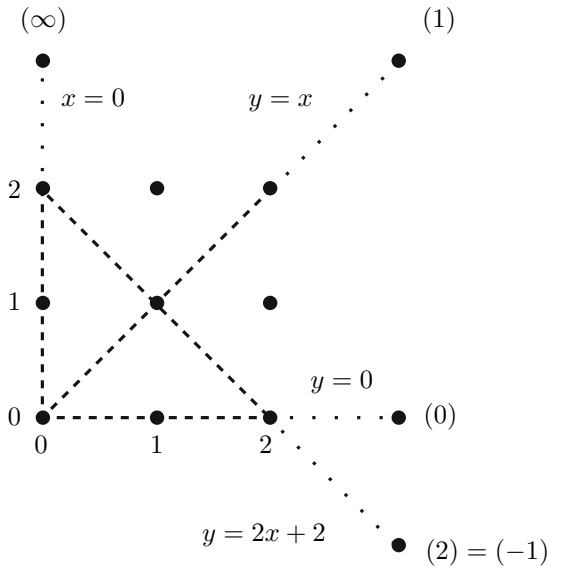


Fig. 17.7 The projective plane of order three, with some sample lines. The projective line at infinity consists of the points (0), (1), (2), and (∞).

example, 2_1 denotes the pair $(2, 1)$. The complex algebra $\mathfrak{Cm}(P)$ has five atoms: the identity element ι and the four points in P. The affine representation of ι is the identity relation $R_\iota = id_D$, and the affine representation of each point $p = (m)$ in P is the linear relation

$$R_p = R_{(m)} = (\bigcup_b \ell_{mb} \times \ell_{mb}) - id_D.$$

In more detail, the relation $R_{(\infty)}$ consists of the pairs of distinct affine points lying on the lines for which (∞) is the point at infinity; these lines are defined by equations of the form $x = c$ (for $c = 0, 1, 2$), so

$$R_{(\infty)} = \{(0_a, 0_b), (1_a, 1_b), (2_a, 2_b) : a, b \in \mathbb{Z}_3 \text{ and } a \neq b\}.$$

The relation $R_{(0)}$ consists of the pairs of distinct affine points lying on the lines for which (0) is the point at infinity; these lines are defined by equations of the form $y = c$, so

$$R_{(0)} = \{(a_0, b_0), (a_1, b_1), (a_2, b_2) : a, b \in \mathbb{Z}_3 \text{ and } a \neq b\}.$$

The relation $R_{(1)}$ consists of the pairs of distinct affine points lying on the lines for which (1) is the point at infinity; these lines are defined by equations of the form $y = x + c$, so

$$R_{(1)} = \{(a_a, b_b), (a_{a+1}, b_{b+1}), (a_{a+2}, b_{b+2}) : a, b \in \mathbb{Z}_3 \text{ and } a \neq b\}.$$

Finally, the relation $R_{(2)}$ consists of the pairs of distinct affine points lying on the lines for which $(2) = (-1)$ is the point at infinity; these lines defined by equations of the form $y = -x + c = 2x + c$, so

$$R_{(2)} = \{(a_{-a}, b_{-b}), (a_{-a+1}, b_{-b+1}), (a_{-a+2}, b_{-b+2}) : a, b \in \mathbb{Z}_3 \text{ and } a \neq b\}.$$

These distinct atomic relations are illustrated in Figure 17.8.

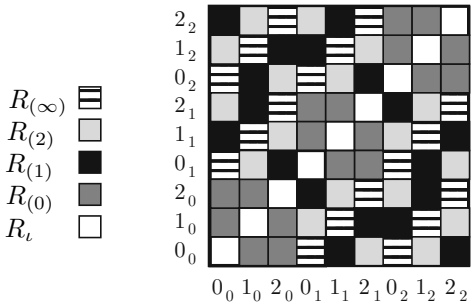


Fig. 17.8 The affine representation of $\mathfrak{Cm}(P)$, where P is the projective line of order three.

The elements of the geometric complex algebra $\mathfrak{Cm}(P)$ are the subsets of P^+, and each such subset X is represented by the relation R_X that is the union of the atomic relations R_p for p in X. By visualizing the unions of the various atomic relations in Figure 17.8, one arrives at an illustration of the affine representation of $\mathfrak{Cm}(P)$.

We now turn to the task of showing that if the complex algebra of a projective geometry P has a complete square representation φ, then P is embeddable into a projective geometry Q of one higher dimension, and φ is equivalent to an affine representation. Each line in a projective geometry will be identified with the set of points on the line, so that the relation of incidence is implicit in the definitions of the notions of point and line. By passing to an equivalent representation, it may be assumed that the base set of the given representation φ is a set D that is disjoint from P. (This is the only place in the proof where the passage to an equivalent representation is needed.) The points in Q are defined to be the elements of the set $P \cup D$, and the lines in Q are defined to be the lines in P and the sets of points of the form

$$\ell_{ap} = \{b \in D : (a, b) \in \varphi(p)\} \cup \{a\} \cup \{p\}$$

for a in D and p in P. The elements in D are called *affine points* and those in P are called *ideal points* or *points at infinity*. The first step is to show that the defined structure Q satisfies the axioms of projective geometry. The verification of each axiom breaks down into a series of case arguments that mirror the case definition of Q.

The first axiom says that through two distinct points there passes one and only one line. Before verifying this axiom, it is helpful to make a few observations. First, each line in Q that comes from P contains no affine points, because the sets P and D are assumed to be disjoint. Second, each line in Q of the form ℓ_{ap} contains a unique ideal point, namely the point p. Third, the affine portions of two lines ℓ_{ap} and ℓ_{bp} are either equal or disjoint, and the union of the affine portions of the lines ℓ_{ap}, where a varies over D and p is fixed, is all of D. To prove the third observation, notice that the sum $p + \iota$ in $\mathfrak{Cm}(P)$ is a reflexive equivalence element, by the definitions of relative multiplication and converse in the complex algebra, and the distributive laws for relative multiplication and converse. In more detail,

$$(p + \iota) \,;\, (p + \iota) = p \,;\, p + p \,;\, \iota + \iota \,;\, p + \iota \,;\, \iota = p + \iota,$$

and

$$(p + \iota)^{\smile} = p^{\smile} + \iota^{\smile} = p + \iota.$$

The representation of $p + \iota$ under φ, that is to say, the relation

$$E = \varphi(p + \iota) = \varphi(p) \cup \varphi(\iota),$$

must therefore be an equivalence relation on the set D. Of course, $\varphi(\iota)$ is the identity relation on D, because φ is a representation over the set D, so two points a and b in D are equivalent modulo E if and only if the pair (a, b) is in $\varphi(p)$ or else $a = b$. The equivalence classes of E are therefore just the sets ℓ_{ap} with the point p removed, for various points a in D, by the definition of the lines ℓ_{ap}. Since two equivalence classes of E are either equal or disjoint, and since the union of the equivalence classes of E is all of D, the desired conclusion follows at once.

To verify the first axiom, let s and t be distinct points in Q. There are three cases to consider. If s and t are both ideal points, then there

is a unique line in P that passes through both of them, because P is assumed to be a projective geometry. None of the lines in Q that are not already lines in P can pass through both s and t, because each line of the form ℓ_{ap} contains only one ideal point. Consequently, there is exactly one line in Q that passes through both points. Now assume that one of the two points, say s, is affine and the other, say t, is ideal. No line in Q that comes from P can pass through both points, because the lines in P contain no affine points. If a line ℓ_{ap} passes through both points, then we must have $p = t$, since t is assumed to be an ideal point and p is the unique ideal point on the line. The affine point s belongs to exactly one line of the form ℓ_{ap}, because the affine portions of these lines form a partition of D, by the observations of the preceding paragraph. Thus, there is again just one line in Q that passes through both s and t. Assume finally that both s and t are affine points. There is a unique point p in P such that the pair (s, t) belongs to the relation $\varphi(p)$. Indeed, the elements in P^+ are (identified with) the atoms in $\mathfrak{Cm}(P)$, which are mutually disjoint and sum to the unit. As φ is a complete square representation of $\mathfrak{Cm}(P)$, it follows that the relations $\varphi(p)$, for p in P^+, must be mutually disjoint and have $D \times D$ as their union, and therefore the pair (s, t) must belong to exactly one of these relations. The element p cannot coincide with the identity element ι, because the points s and t are assumed to be distinct, and $\varphi(\iota)$ is the identity relation on D. Consequently, p belongs to P. Exactly one line of the form ℓ_{ap} contains the point s, because the affine portions of these lines (for a fixed point p) partition D, by the observations of the preceding paragraph; and the line of this form that does contain s must also contain t, by the definition of the line ℓ_{ap} and the fact that, as we have just seen, the pair (s, t) belongs to the relation $\varphi(p)$. Furthermore, no line in Q that comes from P can contain s or t, because the lines in P do not contain any affine points. Thus, in this case as well there is a unique line in Q that passes through both points. This completes the verification of the first axiom in Q.

The second axiom says that every line has at least three points. Certainly, each line in Q that comes from P has at least three points, because P is assumed to be a projective geometry. To see that each line in Q of the form ℓ_{ap}, with a in D and p in P, has at least three points, recall that $\mathfrak{Cm}(P)$ is an integral relation algebra, and therefore the domain and range of each atom in this algebra must be the identity element, by Theorem 9.7. In particular, the domain and range of

the atom p must be the identity element ι. Consequently, the domain and range of the relation $\varphi(p)$ representing p must respectively coincide with the domain and range of the relation $\varphi(\iota)$ representing ι. Since $\varphi(\iota) = id_D$, this domain and range must be the set D. It follows that there is an element b in D such that the pair (a, b) belongs to the relation $\varphi(p)$. The points b and a are distinct, because the atoms p and ι are distinct and consequently the relations $\varphi(p)$ and $\varphi(\iota)$ are disjoint. The line ℓ_{ap} therefore contains the three points a, b, and p. This completes the verification of the second axiom in Q.

Before verifying the third axiom in Q, we prove two lemmas. A set of points in Q is said to be *collinear* if there is a line that contains all of the points. Recall that the line through two points s and t is denoted by st.

Lemma 17.12. *Let a, b, and c be distinct affine points, and p, q, and r the ideal points of the lines ab, ac, and bc respectively. The following conditions are equivalent.*

(i) *The points p, q, and r are not all equal.*
(ii) *The points p, q, and r are all distinct from one another and are collinear.*
(iii) *The points a, b, and c are not collinear.*

Proof. The implication from (ii) to (i) is trivial. To establish the implication from (i) to (iii), argue by contraposition. Every line with affine points has a unique ideal point, by the definition of the lines in Q. Consequently, if the affine points a, b, and c are collinear, then the ideal points p, q, and r must be equal to one another.

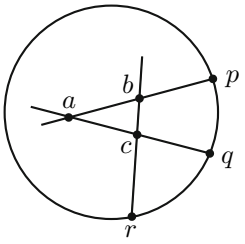


Fig. 17.9

To establish the implication from (iii) to (ii), assume that the affine points a, b, and c are not collinear. The ideal points p and q must

then be distinct, because the equality of p and q would force the affine points all to lie on one line, namely the line ℓ_{ap}, in contradiction to the stated assumption (see Figure 17.9). The pairs (b, a) and (a, c) belong to the relations $\varphi(p)$ and $\varphi(q)$ respectively, by the definitions of the projective lines

$$ab = \ell_{ap} \qquad \text{and} \qquad ac = \ell_{aq}.$$

Consequently, the pair (b, c) belongs to the composition of the relations $\varphi(p)$ and $\varphi(q)$, by the definition of relational composition. This pair also belongs to the relation $\varphi(r)$, by definition of the line ℓ_{br}, so the pair belongs to the intersection

$$\varphi(r) \cap [\varphi(p) \,|\, \varphi(q)].$$

Since this intersection coincides with the relation $\varphi(r \cdot (p\,;q))$, by the isomorphism properties of φ, it follows that this relation is non-empty and therefore the element $r \cdot (p\,;q)$ is not zero. The ideal point r is an atom in $\mathfrak{Cm}(P)$, so r must be below the relative product $p\,;q$, and therefore r must belong to this relative product, by the definition of relative multiplication in $\mathfrak{Cm}(P)$. It follows from this definition that r is collinear with, but distinct from, the points p and q. □

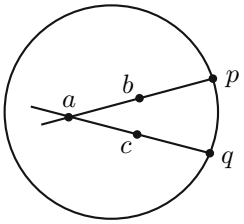


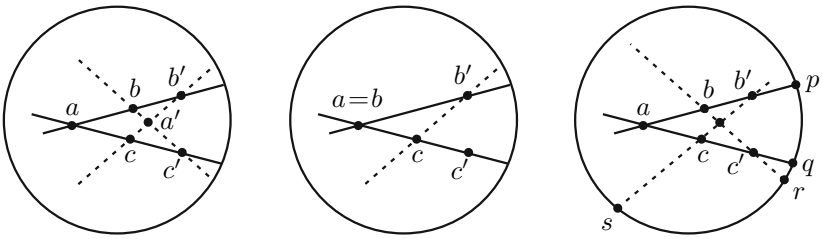
Fig. 17.10

Lemma 17.13. *Suppose b and c are affine points, and p and q ideal points. If the pair (b, c) belongs to the composition $\varphi(p) \,|\, \varphi(q)$, then the lines ℓ_{bp} and ℓ_{cq} intersect in some affine point.*

Proof. The assumption that the pair (b, c) belongs to the composition $\varphi(p) \,|\, \varphi(q)$ implies the existence of an affine point a such that (b, a) is in $\varphi(p)$, and (a, c) in $\varphi(q)$, by the definition of relational composition.

The point a belongs to each of the lines ℓ_{bp} and ℓ_{cq}, by the definitions of these lines, so the two lines intersect in the point a (see Figure 17.10). □

We turn now to the verification of the third axiom, the Pasch Axiom, in Q. Let b, b', c, and c' be arbitrary distinct points in Q, and assume that the projective lines bb' and cc' intersect in a point a. Under this assumption, the Pasch Axiom asserts that the projective lines bc' and $b'c$ must intersect in a point a' (see Figure 17.11(a)). If the intersection point a coincides with the given point b, then the points b, c, and c' are collinear, and therefore the lines bc' and $b'c$ intersect in the point c (see Figure 17.11(b)). Similar arguments apply if a coincides with one of the points b', c, or c'. Consequently, from now on we may and do assume that a is different from the four given points, so that all five points are distinct.



(a) (b) (c)

Fig. 17.11

Let p, q, r, s, t, and u be the respective ideal points of those of the lines bb', cc', bc', $b'c$, bc, and $b'c'$ that do not lie in P. The verification of the Pasch Axiom splits into a number of cases. We begin with a preliminary observation. For most of the cases, it suffices to prove that one of the points p, q, and u is collinear with r and s, and that the three collinear points are distinct. For example, if p, r, and s are distinct and collinear, then the definition of relative multiplication in $\mathfrak{Cm}(P)$ implies that $p \leq r \,;\, s$, and therefore

$$\varphi(p) \subseteq \varphi(r) \,|\, \varphi(s),$$

by the isomorphism properties of φ. Assuming that the points b and b' are affine, we obtain that the pair (b, b') belongs to the relation $\varphi(p)$

and therefore also to the composition $\varphi(r) \,|\, \varphi(s)$. Apply Lemma 17.13 to conclude that the lines

$$\ell_{br} = bc' \qquad \text{and} \qquad \ell_{b's} = b'c$$

intersect (see Figure 17.11(c)). The argument when the points q, r, and s are distinct and collinear is similar and is based on the assumption that the points of the pair (c', c) are affine (see Figure 17.11(c)). Analogously, the argument when the points u, r, and s are distinct and collinear is similar and is based on the assumption that the points of the pair (c', b') are affine.

To order to verify the Pasch Axiom, we consider seven cases. For the first case, assume that the five given points a, b, b', c, and c' are all affine (see Figure 17.11(c)). If $p = q$, then the five given points are collinear (they lie on the line ℓ_{ap}) and the desired conclusion is trivial because $bc' = b'c$. If $r = s$, then again the five given points are collinear (they lie on the line ℓ_{br}) and again $bc' = b'c$. Suppose $p \neq q$ and $r \neq s$. In this case, the points p, q, and r are not all equal, so they are distinct and collinear, by Lemma 17.12 (with a, b, and c' as the affine points). Similarly, the points p, q, and s are not all equal, so they are distinct and collinear, by Lemma 17.12 (with a, b', and c as the affine points). Consequently, the points p, r, and s are distinct and collinear, so the Pasch Axiom holds in this case by the preliminary observation made in the preceding paragraph.

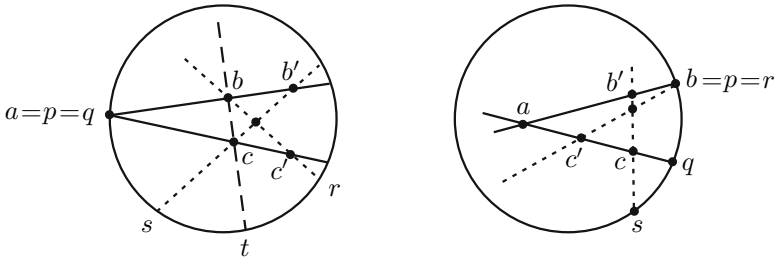


Fig. 17.12

For the second case, assume that the points b, b', c, and c' are affine, and a is ideal (see Figure 17.12(a)). In this case, p and q must be equal to a, since a has to be the unique ideal point on each of the lines bb'

and cc'. If $r = s$, then the lines bc' and $b'c$ obviously intersect in the point r. If $p = r$, then the line bb' passes through the points b and r and therefore contains the point c', since c' lies on the line br. Similarly, if $p = s$, then the line bb' passes through the points b' and s and therefore contains the point c, since c lies on the line $b's$. In either case, the lines bb' and cc' have two points in common, namely the ideal point a and one of the affine points c' or c, so the two lines are equal and therefore so are the lines bc' and $b'c$. In particular, the last two lines intersect. Suppose now that $p \neq r$ and $p \neq s$ and $r \neq s$. The points p, r, and t are not all equal, so they are distinct and collinear, by Lemma 17.12 (with c', c, and b as the affine points, and using the fact that $p = q$). Similarly, the points p, s, and t are not all equal, so they are distinct and collinear, by Lemma 17.12 (with b', b, and c as the affine points). Consequently, the points p, r, and s are distinct and collinear, as required by the preliminary observation made above, so the desired conclusion follows from that observation.

For the third case, assume that the points a, b', c, and c' are affine, and b is ideal (see Figure 17.12(b)). In this case, b is the unique ideal point on each of the lines bb' and bc', so the points b, p, and r must be equal, by the definitions of p and r. If $p = q$, then the lines bb' and cc' both pass through the points a and b, so the two lines are equal, and therefore the five given affine points are collinear. It follows that the lines bc' and $b'c$ are equal, so the desired conclusion holds trivially. If $p \neq q$, then the points p, q, and s are not all equal, so they must be distinct and collinear, by Lemma 17.12 (with a, b', and c as the affine points). Since $p = r$, the points r, q, and s are distinct and collinear, as required by the preliminary observation. The desired conclusion therefore follows from that observation. The cases when one of b', c, and c' is the unique ideal point among the five given points are handled in a completely analogous fashion.

For the fourth case, assume that the points a, b', and c' are affine, while b and c are ideal (see Figure 17.13(a)). The point b is the unique ideal point on each of the lines bb' and bc', so the points b, p, and r must be equal, by the definitions of p and r. Similarly, c is the unique ideal point on each of the lines $b'c$ and cc', so the points c, q, and s must be equal, by the definitions of q and s. From the assumed inequality of b and c, it follows that $r \neq s$. The points u, r, and s are therefore not all equal, so they must be distinct and collinear, by Lemma 17.12 (with b', c', and a as the affine points), as required by the preliminary observation made above. The desired conclusion therefore follows

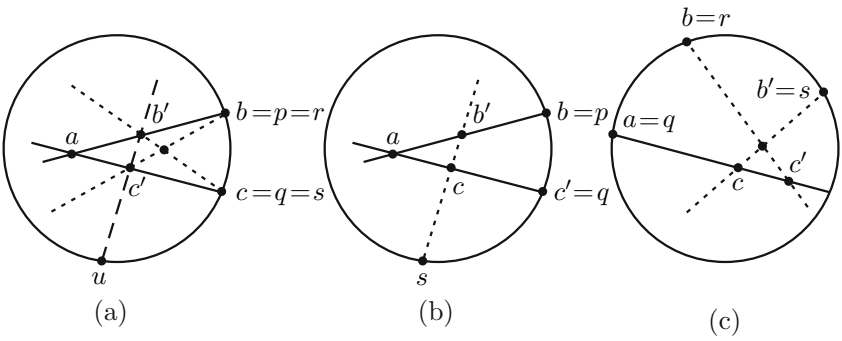


Fig. 17.13

from that observations. The case when the points b' and c' are ideal, while a, b, and c are affine, is handled in an analogous fashion.

For the fifth case, assume that the points a, b', and c are affine, while b and c' are ideal (see Figure 17.13(b)). In this case, b coincides with p, and c' with q, by the definitions of p and q, so the assumed inequality of b and c' implies that of p and q. The points p, q, and s are therefore not all equal, so they must be distinct and collinear, by Lemma 17.12 (with a, b', and c as the affine points). The lines pq and bc' coincide, because $b = p$ and $c' = q$, so bc' must contain the point s. It follows that the lines bc' and $b'c$ intersect in the point s. The case when the points b' and c are ideal, while a, b, and c' are affine, is handled in an analogous fashion.

For the sixth case, assume that the points c and c' are affine, while a, b, and b' are ideal (see Figure 17.13(c)). These three ideal points are then the unique ideal points on the lines cc', bc', and $b'c$ respectively, so they must coincide with the ideal points q, r, and s respectively, by the definitions of q, r, and s. Since a, b, and b' are assumed to be distinct and collinear, it follows that q, r, and s are distinct and collinear, as required by the preliminary observation made above. The desired conclusion therefore follows from that observation. The case when the points b and b' are affine, while a, c, and c' are ideal, is handled in an analogous fashion.

In all other cases, the given five points must all be ideal, so the intersection of the lines bc' and $b'c$ follows from the validity of the Pasch Axiom in the projective geometry P. For example, if the points a, b, and c are ideal, then the lines ab and ac are projective lines in the

geometry P. The points b' and c' lie on these lines respectively, by the definition of a, so b' and c' must also be ideal points. The remaining cases are handled in a completely analogous fashion. This completes the verification of the Pasch Axiom in Q, and the proof that Q is a projective geometry.

The geometry Q includes the geometry P as a subspace, by the very definitions of the notions of point and line in Q. The next task is to show that the dimension of Q over P is one. The complex algebra $\mathfrak{Cm}(P)$ is not degenerate, so the base set D of the given complete representation cannot be empty. Consequently, Q contains points that do not belong to P, namely the points in the set D (which is assumed to be disjoint from P). It follows that P is a proper subspace of Q, and the dimension of Q over P is at least one.

Consider now two cases. If the geometry P is empty, then it has dimension -1, by convention. In this case, the identity element ι is also the unit element of the complex algebra $\mathfrak{Cm}(P)$, and therefore $\mathfrak{Cm}(P)$ is isomorphic to the minimal simple relation algebra $\mathfrak{M}_1$, by Lemma 13.1 and Theorem 13.10. The algebra $\mathfrak{M}_1$ has a unique square representation, up to equivalence, and the base set of this representation has just one element, since the unit and identity relations can never be equal in a set relation algebra on a set of cardinality greater than one. Consequently, the set D must have exactly one element, and therefore the geometry Q, which is the union of P and D, must have exactly one element. It follows that Q has dimension 0, which is one dimension higher than the dimension of P.

Turn now to the case when the geometry P is not empty. Fix an arbitrary affine point a in D, and let b be any affine point in D that is different from a. There is a unique point p in P such that the pair (a, b) belongs to the relation $\varphi(p)$, because φ is assumed to be a complete square representation of $\mathfrak{Cm}(P)$ over D, and therefore every pair of distinct points from D must belong to the representation of some atom different from ι. The point b belongs to the line ℓ_{ap}, by the definition of this line. Consequently, every point in Q that is not already in P lies on some line through a and a point in P. A basis for Q may therefore be obtained by adjoining the point a to a basis for P. Consequently, the dimension of Q over P is exactly one.

A final task remains, namely to check that φ is the affine representation of $\mathfrak{Cm}(P)$ with respect to the projective geometry Q. The affine representation of $\mathfrak{Cm}(P)$ maps each point p in P to the relation R_p that is defined by

$$R_p = \{(a, b) : a, b \in D,\ a \neq b,\ \text{ and } ab \text{ has the ideal point } p\}.$$

For a given affine point a in D, the line ℓ_{ap} is the unique line in Q that passes through the two points a and p, and p is the only ideal point on this line. Consequently, the preceding equation may be rewritten as

$$R_p = \{(a, b) : a \in D,\ b \in \ell_{ap},\ \text{ and } b \neq a, p\}.$$

A comparison of this last equation with the definition of ℓ_{ap} leads to the conclusion that

$$R_p = \{(a, b) : (a, b) \in \varphi(p)\} = \varphi(p).$$

Thus, φ agrees with the affine representation on all atoms in $\mathfrak{Cm}(P)$ different from the identity element ι. Of course, both representations map ι to the identity relation id_D, since both representations are over the base set D, so φ agrees with the affine representation on all atoms in $\mathfrak{Cm}(P)$. It was shown above that the affine representation of $\mathfrak{Cm}(P)$ with respect to Q is complete. Two complete representations that agree on the atoms of $\mathfrak{Cm}(P)$ must agree on all elements in $\mathfrak{Cm}(P)$, so φ is the affine representation of $\mathfrak{Cm}(P)$ with respect to Q.

The the following theorem summarizes the observations made so far in this section.

Theorem 17.14. *The complex algebra of a projective geometry P has a complete representation if and only if P is embeddable into a projective geometry of one higher dimension. If P is so embeddable, then every complete square representation of the complex algebra of P is equivalent to an affine representation.*

This theorem actually characterizes the representability—and not just the complete representability—of finite geometric complex algebra, because every representation of a finite relation algebra is complete. When combined with the Bruck-Ryser Theorem, it yields a powerful tool for constructing finite relation algebras that are not representable. The Bruck-Ryser Theorem is a statement about the nonexistence of projective planes of finite order n. It says that if a positive integer n is congruent to 1 or 2 modulo 4—that is to say, if n has the form $4k + 1$ or $4k + 2$ for some non-negative integer k—then the existence of a projective plane of order n implies that n must be the sum of the squares of two integers. Put somewhat differently, the theorem says that if a positive integer n is congruent to 1 or 2 modulo 4, but

is not the sum of two square integers, then there can be no projective plane of order n.

Every sequence of four consecutive positive integers contains two integers that are congruent to 1 or 2 modulo 4; it is more difficult for a positive integer to be the sum of two square integers. In fact, there are infinitely many positive integers that are congruent to 1 or 2 modulo 4, but that are not the sum of two squares of integers, and for each of these integers there can be no projective plane of that order, by the Bruck-Ryser Theorem. For instance, the first ten positive integers that are congruent to 1 or 2 modulo 4 are

$$1, \quad 2, \quad 5, \quad 6, \quad 9, \quad 10, \quad 13, \quad 14, \quad 17, \quad 18.$$

Eight of these are sums of squares, namely

$$1, \quad 2, \quad 5, \quad 9, \quad 10, \quad 13, \quad 17, \quad 18,$$

and in fact

$$1 = 0^2 + 1^2, \qquad 2 = 1^2 + 1^2, \qquad 5 = 1^2 + 2^2, \qquad 9 = 0^2 + 3^2,$$
$$10 = 1^2 + 3^2, \qquad 13 = 2^2 + 3^2, \qquad 17 = 1^2 + 4^2, \qquad 18 = 3^2 + 3^2.$$

The theorem gives no information about the existence of projective planes of these eight orders. On the other hand, neither 6 nor 14 can be written as the sum of two square integers, so there are no projective planes of orders 6 or 14. This conclusion implies that the projective lines of orders 6 and 14, which are just sets of seven and fifteen points respectively, cannot be embedded into a projective plane, since such a plane would have to have the same order as the line. Consequently, the complex algebras of the projective lines of orders 6 and 14 are examples of relation algebras that are not representable, by Theorem 17.14.

As a direct consequence of the preceding remarks, we have the following theorem.

Theorem 17.15. *For infinitely many positive integers n, the complex algebra of the projective line of order n is a relation algebra that is not representable.*

Theorem 17.14 does not characterize the representability (in contrast to the complete representability) of infinite geometric complex algebras. To obtain such a characterization, some connection between representability and complete representability must be established.

The next theorem, which concerns Desarguesian geometries, is the key. A projective geometry is said to be *Desarguesian* if the theorem of Desargues holds in it. This theorem says that if the lines pp', qq' and rr' through the corresponding vertices of two triangles pqr and $p'q'r'$ intersect in a point, then the intersection points of the corresponding sides of the triangles—pq and $p'q'$, pr and $p'r'$, and qr and $q'r'$—are collinear (see Figure 17.14, in which s is the intersection point of the three lines through the corresponding vertices, and u, v, and w are the intersection points of the lines through the corresponding sides). It is known that every projective geometry of dimension at least three is Desarguesian, and also that there exist projective planes that are not Desarguesian.

Theorem 17.16. *For the complex algebra of a projective plane to be representable at all, the plane must be Desarguesian.*

Proof. Suppose that the complex algebra $\mathfrak{Cm}(P)$ of a projective plane P is representable, say via a mapping φ. Consider two triangles pqr and $p'q'r'$ in P such that the lines pp', qq' and rr' all intersect in a point s (see Figure 17.14). In the complex algebra $\mathfrak{Cm}(P)$, the

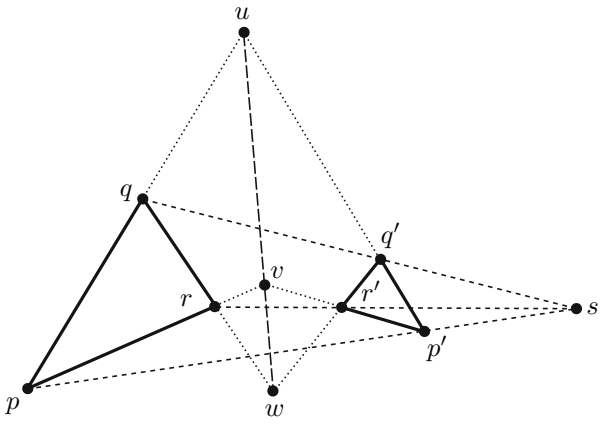


Fig. 17.14 Desarguesian configuration.

non-trivial case of this geometric assumption may be expressed algebraically by the equations

$$s = (p\,;p') \cdot (q\,;q') = (p\,;p') \cdot (r\,;r') = (q\,;q') \cdot (r\,;r'). \qquad (1)$$

Denote the intersection point of the sides pq and $p'q'$ by u, the intersection point of the sides pr and $p'r'$ by v, and the intersection point of the sides qr and $q'r'$ by w. The assumption that these three points are the points of intersection of the given sides can be expressed algebraically in $\mathfrak{Cm}(P)$ by the equations

$$u = (p\,;q)\cdot(p'\,;q'), \quad v = (p\,;r)\cdot(p'\,;r'), \quad w = (q\,;r)\cdot(q'\,;r'). \tag{2}$$

It is to be shown that u, v, and w are collinear, or what is essentially the same thing, that u is below the relative product $v\,;w$.

Consider an arbitrary pair (a,b) in the relation $\varphi(u)$, with the goal of showing that (a,b) must belong to the relation $\varphi(v\,;w)$. The first equation in (2) and the assumption that φ is a representation of $\mathfrak{Cm}(P)$ imply that

$$\varphi(u) = (\varphi(p)\,|\,\varphi(q))\cap(\varphi(p')\,|\,\varphi(q')),$$

so the pair (a,b) must belong to the relation on the right side of this equation. Consequently, there are elements c and d such that

$$(a,c)\in\varphi(p), \quad (c,b)\in\varphi(q), \quad (a,d)\in\varphi(p'), \quad (d,b)\in\varphi(q'), \tag{3}$$

by the definitions of intersection and relational composition. It follows from these equations that the pair (c,d) belongs to the relations

$$\varphi(p)\,|\,\varphi(p') \qquad \text{and} \qquad \varphi(q)\,|\,\varphi(q'),$$

by the definition of relational composition and the symmetry of $\mathfrak{Cm}(P)$ (according to which, every element in $\mathfrak{Cm}(P)$ is its own converse); consequently, the pair (c,d) belongs to the intersection of these two relations and hence also to the relation

$$\varphi((p\,;p')\cdot(q\,;q')),$$

by the isomorphism properties of φ.

The product $(p\,;p')\cdot(q\,;q')$ coincides with the point s, and s is below the relative product $r\,;r'$, by the equations in (1), so the pair (c,d) must belong to the relation $\varphi(s)$ and therefore also to the relation $\varphi(r)\,|\,\varphi(r')$, by the observations of the previous paragraph and the isomorphism properties of φ. Apply the definition of relational composition again to obtain an element e such that

$$(c,e)\in\varphi(r) \qquad \text{and} \qquad (e,d)\in\varphi(r'). \tag{4}$$

The pair (a, e) belongs to the relations

$$\varphi(p) \mid \varphi(r) \qquad \text{and} \qquad \varphi(p') \mid \varphi(r'),$$

by (3), (4), the symmetry of $\mathfrak{Cm}(P)$, and the definition of relational composition; consequently, the pair (a, e) belongs to the intersection of these relations and hence also to the relation

$$\varphi((p\,;r) \cdot (p'\,;r')),$$

by the isomorphism properties of φ. The product $(p\,;r) \cdot (p'\,;r')$ coincides with v, by the second equation in (2), so the pair (a, e) must belong to the relation $\varphi(v)$.

A completely analogous argument shows that the pair (e, b) belongs to the relations

$$\varphi(q) \mid \varphi(r) \qquad \text{and} \qquad \varphi(q') \mid \varphi(r'),$$

so it belongs to the intersection of these two relations and hence also to the relation

$$\varphi((q\,;r) \cdot (q'\,;r')).$$

The product $(q\,;r) \cdot (q'\,;r')$ coincides with w, by (2), so the pair (e, b) must belong to the relation $\varphi(w)$. Combine these observations with those of the preceding paragraph, and apply the definition of relational composition and the isomorphism properties of φ, to conclude that the pair (a, b) belongs to the relation $\varphi(v) \mid \varphi(w)$ and therefore also to the relation $\varphi(v\,;w)$.

It has been shown that every pair in the relation $\varphi(u)$ belongs to the relation $\varphi(v\,;w)$, so $\varphi(u)$ must be included in $\varphi(v\,;w)$. Consequently, the element u must be below the relative product $v\,;w$, by the isomorphism properties of φ. This just means that u is collinear with, but distinct from v and w, by the definition of relative multiplication in $\mathfrak{Cm}(P)$. In particular, the points u, v, and w are collinear, as was to be shown. □

Theorems 17.14 and 17.16 together yield the conclusion that representability implies complete representability for geometric complex algebras.

Theorem 17.17. *A geometric complex algebra is representable if and only if it is completely representable.*

Proof. Obviously, a completely representable geometric complex algebra is representable. To prove the converse, assume that the complex algebra of a projective geometry P is representable. If P has dimension at least three, then it is automatically Desarguesian, and if P has dimension two, then it is Desarguesian, by Theorem 17.16. Since a Desarguesian projective geometry is always embeddable into a projective geometry of one higher dimension, it follows that if P has dimension at least two, then $\mathfrak{Cm}(P)$ is completely representable, by Theorem 17.14.

Consider now the case when P has dimension one. In this case, P is essentially a set of points, all of which are assumed to be collinear. If P is infinite, then there are certainly projective planes in which the lines have the same number of points as P, and any bijection from P to the points on one of these lines is a projective embedding of P into the plane. Consequently, $\mathfrak{Cm}(P)$ is completely representable, by Theorem 17.14. On the other hand, if P is finite, then any representation of $\mathfrak{Cm}(P)$ is automatically a complete representation.

Finally, if P has dimension 0 or -1, then $\mathfrak{Cm}(P)$ is finite, so all representations of $\mathfrak{Cm}(P)$ are complete □

The preceding theorem shows that the characterization given in Theorem 17.14 applies more generally to representable geometric complex algebras.

Corollary 17.18. *The complex algebra of a projective geometry P is representable if and only if P is embeddable into a projective geometry of one higher dimension.*

Theorem 17.17 does not say that all representations of a geometric complex algebra are necessarily complete. In the infinite case, the vast majority of the representations are incomplete. For example, as we shall see in Theorem 17.27 below, the complex algebra of an infinite Desarguesian geometry has an essentially unique complete square representation, but it has incomplete square representations over sets of arbitrarily large cardinality, by Theorem 16.3.

The proof that an infinite Desarguesian geometry has an essentially unique complete square representation requires some lemmas. The first lemma reduces the problem of constructing isomorphisms from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$ to the problem of constructing collineations from P to P'. Write $\mathrm{Col}\,(p, q, r)$ to express that three points p, q, and r in a projective geometry are distinct and collinear.

Lemma 17.19. *Every collineation from a projective geometry P to a projective geometry P' extends in a unique way to an isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$, and every isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$ is the extension of a uniquely determined collineation from P to P'.*

Proof. Assume first that σ is a collineation from P to P'. Under the identification of points in geometries with subdiversity atoms (singletons of points) in the corresponding complex algebras, the mapping σ may be viewed as a bijection from the set of subdiversity atoms in $\mathfrak{Cm}(P)$ to the set of subdiversity atoms in $\mathfrak{Cm}(P')$. Extend σ to a bijection from the set of atoms in $\mathfrak{Cm}(P)$ to the set of atoms in $\mathfrak{Cm}(P')$ by putting $\sigma(\iota) = \iota'$, where ι and ι' are the identity elements in P and P' respectively. It is not difficult to check that for all atoms p, q, and r in $\mathfrak{Cm}(P)$,

$$r \leq p\,;q \qquad \text{if and only if} \qquad \sigma(r) \leq \sigma(p)\,;\sigma(q). \tag{1}$$

For example, if p and q are distinct points in P, then

$$\begin{aligned} r \leq p\,;q \qquad & \text{if and only if} \qquad \mathrm{Col}\,(p,q,r), \\ & \text{if and only if} \qquad \mathrm{Col}\,(\sigma(p),\sigma(q),\sigma(r)), \\ & \text{if and only if} \qquad \sigma(r) \leq \sigma(p)\,;\sigma(q), \end{aligned}$$

by the definition of relative multiplication in $\mathfrak{Cm}(P)$ and in $\mathfrak{Cm}(P')$, and the assumption that σ is a collineation. If p and q are the same point in P, and if P has order greater than 2, then

$$\begin{aligned} r \leq p\,;q \qquad & \text{if and only if} \qquad r = \iota \text{ or } r = p, \\ & \text{if and only if} \qquad \sigma(r) = \iota' \text{ or } \sigma(r) = \sigma(p), \\ & \text{if and only if} \qquad \sigma(r) \leq \sigma(p)\,;\sigma(q), \end{aligned}$$

by the definition of relative multiplication in $\mathfrak{Cm}(P)$ and in $\mathfrak{Cm}(P')$, and the definition of σ. When P has order 2, the proof is almost the same except that the possibility $r = p$ is excluded. The arguments for the remaining cases in the proof of (1) are similar, but easier. Apply the Atomic Isomorphism Theorem in the form of Corollary 7.12 to conclude that σ can be extended in one and only one way to an isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$.

Consider now an arbitrary isomorphism φ from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$. Certainly, φ maps the set of subdiversity atoms in $\mathfrak{Cm}(P)$ bijectively to

the set of subdiversity atoms in $\mathfrak{Cm}(P')$, by the isomorphism properties of φ, so the restriction of φ to P is a bijection σ from P to P'. To check that σ must be a collineation, let p, q, and r be any three distinct points in P. The definition of relative multiplication in $\mathfrak{Cm}(P)$ and in $\mathfrak{Cm}(P')$, the isomorphism properties of φ, and the definition of σ imply that

$$\begin{array}{lll} \mathrm{Col}\,(p,q,r) & \text{if and only if} & r \leq p\,;q, \\ & \text{if and only if} & \varphi(r) \leq \varphi(p)\,;\varphi(q), \\ & \text{if and only if} & \sigma(r) \leq \sigma(p)\,;\sigma(q), \\ & \text{if and only if} & \mathrm{Col}\,(\sigma(p),\sigma(q),\sigma(r)). \end{array}$$

Consequently, σ is a collineation from P to P'. There is a unique isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$ that extends σ, by the first part of the theorem. That isomorphism must be φ, because φ is an isomorphism that extends σ, by the very definition of σ. □

The lemma implies that the group of automorphisms of a geometric complex algebra $\mathfrak{Cm}(P)$ is isomorphic to the group of autocollineations of the geometry P. Indeed, the function mapping each autocollineation σ of P to the unique automorphism $\hat{\sigma}$ of $\mathfrak{Cm}(P)$ that extends σ is a bijection, by the lemma. Furthermore, the composition of two autocollineations σ and τ of P is an autocollineation ϱ of P. Let $\hat{\sigma}$, $\hat{\tau}$, and $\hat{\varrho}$ be the corresponding extensions of these autocollineations. The composition of $\hat{\sigma}$ and $\hat{\tau}$ is an automorphism of $\mathfrak{Cm}(P)$ that, when restricted to P, coincides with the composition of σ and τ. In other words, this restriction of the composition is equal to ϱ. Since ϱ has a unique extension to an automorphism of $\mathfrak{Cm}(P)$, namely the function $\hat{\varrho}$, by the lemma, it follows that $\hat{\varrho}$ must be equal to the composition of $\hat{\sigma}$ and $\hat{\tau}$. Thus, the bijection mapping each autocollineation σ of P to the induced automorphism $\hat{\sigma}$ of $\mathfrak{Cm}(P)$ preserves the group operation of composition and is therefore a group isomorphism.

Corollary 17.20. *The group of autocollineations of a projective geometry P is isomorphic to the group of automorphisms of the complex algebra $\mathfrak{Cm}(P)$ via the correspondence that maps each autocollineation σ to the induced automorphism $\hat{\sigma}$.*

In what follows, let ψ be a fixed affine representation of a geometric complex algebra $\mathfrak{Cm}(P)$ with respect to a projective geometry Q that includes P as a maximal subspace. Write D for the affine geometry obtained from Q by deleting the points and lines of P, and write $\mathfrak{Af}(D)$

for the complete subalgebra of $\mathfrak{Re}(D)$ (the full set relation algebra over the set D) that is the image of $\mathfrak{Cm}(P)$ under the mapping ψ. We shall refer to $\mathfrak{Af}(D)$ as an *affine algebra*. Similarly, let ϑ be an affine representation of a geometric complex algebra $\mathfrak{Cm}(P')$ with respect to a projective geometry Q' that includes P' as a maximal subspace. Write D' for the affine geometry obtained from Q' by deleting the points and lines of P', and write $\mathfrak{Af}(D')$ for the complete subalgebra of $\mathfrak{Re}(D')$ that is the image of $\mathfrak{Cm}(P')$ under the mapping ϑ. The representations ψ and ϑ are determined by the equations

$$\psi(p) = R_p \qquad \text{and} \qquad \vartheta(p') = R_{p'}$$

for p in P and p' in P'. Lemma 17.19 can equally well be formulated as a result about isomorphisms from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$.

Lemma 17.21. *A collineation σ from P to P' induces a uniquely determined isomorphism φ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ via the condition*

$$\varphi(R_p) = R_{\sigma(p)},$$

and every isomorphism from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ is induced in this way by a collineation from P to P'.

Proof. If σ is a collineation from P to P', and if $\hat{\sigma}$ is the extension of σ to an isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$ guaranteed by Lemma 17.19, then the composition

$$\varphi = \vartheta \circ \hat{\sigma} \circ \psi^{-1} \tag{1}$$

is an isomorphism from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$, and φ is completely determined by its value on relations R_p for p in P (see the diagram below).

$$\begin{array}{ccccc} P & & \mathfrak{Cm}(P) & \xrightarrow{\psi} & \mathfrak{Af}(D) \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\hat{\sigma}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ P' & & \mathfrak{Cm}(P') & \xrightarrow{\vartheta} & \mathfrak{Af}(D') \end{array}$$

That value is just

$$\varphi(R_p) = \vartheta(\hat{\sigma}(\psi^{-1}(R_p))) = \vartheta(\hat{\sigma}(p)) = \vartheta(\sigma(p)) = R_{\sigma(p)}. \tag{2}$$

Consider now an arbitrary isomorphism φ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$. The composition

$$\vartheta^{-1} \circ \varphi \circ \psi \tag{3}$$

is an isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$ (see the diagram above). The restriction of this isomorphism to P is a collineation—call it σ—from P to P', and the composition in (3) is the unique extension of σ to an isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Cm}(P')$, by Lemma 17.19. Writing $\hat{\sigma}$ for the unique extension of σ to $\mathfrak{Cm}(P)$, we have

$$\hat{\sigma} = \vartheta^{-1} \circ \varphi \circ \psi,$$

by the preceding remarks, so equation (1) holds. Consequently, φ must be the isomorphism determined by (2), by the observations in the first paragraph of the proof. □

The complex algebra $\mathfrak{Cm}(P)$ and the affine algebra $\mathfrak{Af}(D)$ are isomorphic via the representation ψ, by the assumptions made after Corollary 17.20. The groups of automorphisms of these two algebras are therefore also isomorphic via the function that maps each automorphism χ of $\mathfrak{Cm}(P)$ to the automorphism $\psi \circ \chi \circ \psi^{-1}$ of $\mathfrak{Af}(D)$. The group of autocollineations of P is isomorphic to the group of automorphisms of $\mathfrak{Cm}(P)$ via the function that maps each autocollineation σ of P to the induced automorphism $\hat{\sigma}$ of $\mathfrak{Cm}(P)$, by Corollary 17.20. Combine these observations to conclude that the group of autocollineations of P is isomorphic to the group of automorphisms of $\mathfrak{Af}(D)$ via the function that maps each autocollineation σ of P to the automorphism

$$\varphi = \psi \circ \hat{\sigma} \circ \psi^{-1}$$

of $\mathfrak{Af}(D)$. The automorphism φ is determined by the condition

$$\varphi(p) = R_{\sigma(p)}$$

for points p in P, by Lemma 17.21 (with P and D in place of P' and D', and with ψ in place of ϑ in the proof). We shall call φ the automorphism of $\mathfrak{Af}(D)$ *induced* by σ.

Corollary 17.22. *The group of autocollineations of P is isomorphic to the group of automorphisms of $\mathfrak{Af}(D)$ via the function that maps each autocollineation σ to the automorphism induced by σ.*

The next lemma characterizes the base isomorphisms from the affine algebra $\mathfrak{Af}(D)$ to the affine algebra $\mathfrak{Af}(D')$ in terms of affine collineations. The proof of the lemma and the proofs of Lemma 17.24 and Theorem 17.27 require some results from projective geometry. The first result says that every affine collineation σ from D to D' can be

extended in a unique way to a projective collineation $\overline{\sigma}$ from Q to Q' as follows: if p is the ideal point in Q of an affine line ab in D, then $\overline{\sigma}(p)$ is defined to be the ideal point in Q' of the affine line $\sigma(a)\sigma(b)$ in D' (see, for example, Theorem 1 in [37]). The second result says that if the dimension of Q is at least three, then every collineation from the maximal subspace P to the maximal subspace P' can be extended to a collineation from Q to Q' (see, for example, Theorem 61 in [37]). The second result is needed only for the proof of Theorem 17.27 below.

Lemma 17.23. (i) *If σ is an affine collineation from D to D', and if $\overline{\sigma}$ is the extension of σ to a projective collineation from Q to Q', then the mapping φ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ determined by*

$$\varphi(R_p) = R_{\overline{\sigma}(p)}$$

is a base isomorphism that is induced by σ.

(ii) *If σ is a bijection from D to D' that induces a base isomorphism φ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$, then σ must be an affine collineation, and φ must be the base isomorphism determined in* (i).

Proof. Suppose σ is an affine collineation from D to D'. Extend σ to a projective collineation $\overline{\sigma}$ from Q to Q', using the first result from projective geometry mentioned before the lemma. Of course, an appropriate restriction of $\overline{\sigma}$ is a projective collineation from P to P'. As such, $\overline{\sigma}$ induces a uniquely determined isomorphism φ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ via the condition

$$\varphi(R_p) = R_{\overline{\sigma}(p)},$$

by Lemma 17.21. We shall show that φ is a base isomorphism induced by σ.

Each of the following statements is easily seen to be equivalent to its neighbor: (1) the pair (a, b) belongs to the relation R_p; (2) a and b are distinct points in D that are collinear (in Q) with p; (3) $\sigma(a)$ and $\sigma(b)$ are distinct points in D' that are collinear (in Q') with $\overline{\sigma}(p)$; (4) the pair $(\sigma(a), \sigma(b))$ belongs to the relation $R_{\overline{\sigma}(p)}$. Indeed, the equivalence of (1) with (2), and of (3) with (4), uses the definitions of the relations R_p and $R_{\overline{\sigma}(p)}$, while the equivalence of (2) with (3) uses the fact that $\overline{\sigma}$ is a collineation extending σ. The equivalence of (1) and (4) implies that

$$\varphi(R_p) = R_{\overline{\sigma}(p)} = \{(\sigma(a), \sigma(b)) : (a, b) \in R_p\},$$

so φ is a base isomorphism induced by σ. This proves (i).

Assume now that σ is a bijection from D to D' inducing a base isomorphism φ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$. We shall show that σ must be an affine collineation. If a, b, and c are distinct collinear points in D, and if p is the point at infinity of the affine line that passes through these three points, then the pairs (a, b) and (a, c) are both in R_p, by the definition of the relation R_p. Let $R_{p'}$ be the atom in $\mathfrak{Af}(D')$ that is the image of the atom R_p in $\mathfrak{Af}(D)$ under the isomorphism φ. The pairs $(\sigma(a), \sigma(b))$ and $(\sigma(a), \sigma(c))$ must both be in $R_{p'}$, by the assumption that φ is a base isomorphism induced by σ. Consequently, the projective line in Q' that passes through the points $\sigma(a)$ and p' must also contain the points $\sigma(b)$ and $\sigma(c)$, by the definition of the relation $R_{p'}$. In particular, the three points $\sigma(a)$, $\sigma(b)$, and $\sigma(c)$ are collinear in Q' and therefore they are also collinear in D'. Run this argument backwards to see that if $\sigma(a)$, $\sigma(b)$, and $\sigma(c)$ are distinct collinear points in D', then a, b, and c are distinct collinear points in D. Thus, a, b, and c are distinct collinear points in D if and only if $\sigma(a)$, $\sigma(b)$, and $\sigma(c)$ are distinct collinear points in D', so σ is a collineation from D to D'. Apply part (i) of the lemma to conclude that the base isomorphism φ induced by σ must satisfy the equation in (i). □

The preceding observations may be applied to analyze the base automorphisms of the affine algebra $\mathfrak{Af}(D)$. These base automorphisms clearly form a group under the operation of functional composition, as do the autocollineations of the affine geometry D. It turns out that the first group is a homomorphic image of the second.

Lemma 17.24. *The function mapping each autocollineation of the affine geometry D to the induced base automorphism of $\mathfrak{Af}(D)$ is a group epimorphism.*

Proof. First of all, observe that the correspondence mapping each (affine) autocollineation σ of D to its extension $\overline{\sigma}$ to a (projective) autocollineation of Q is a group embedding. Indeed, this correspondence is well defined and one-to-one, by the first result from projective geometry mentioned prior to Lemma 17.23. If σ and τ are arbitrary autocollineations of D, then the composition $\overline{\sigma} \circ \overline{\tau}$ of the corresponding extensions $\overline{\sigma}$ and $\overline{\tau}$ is an autocollineation of Q that extends $\sigma \circ \tau$. Since $\overline{\sigma \circ \tau}$ is the unique autocollineation of Q that extends $\sigma \circ \tau$, by the first result from projective geometry mentioned above, it follows

that

$$\overline{\sigma \circ \tau} = \overline{\sigma} \circ \overline{\tau}. \tag{1}$$

For each autocollineation σ of D, write φ_σ for the base automorphism of $\mathfrak{Af}(D)$ induced by σ. As pointed out in Lemma 17.13, this base automorphism is completely determined by the condition

$$\varphi_\sigma(R_p) = R_{\overline{\sigma}(p)} \tag{2}$$

for points p in P. The correspondence mapping each autocollineation σ to the base automorphism φ_σ is a well-defined mapping from the group of autocollineations of D into the group of base automorphisms of $\mathfrak{Af}(D)$, by Lemma 17.23(i), and this mapping is onto by Lemma 17.23(ii). To see that the mapping preserves the group operation of functional composition and is therefore a group epimorphism, consider two autocollineations σ and τ of D. It follows from (1), (2) (applied first to $\sigma \circ \tau$, then to σ, and then to τ), and the definition of functional composition that

$$\varphi_{\sigma \circ \tau}(R_p) = R_{\overline{\sigma \circ \tau}(p)} = R_{\overline{\sigma} \circ \overline{\tau}(p)} = R_{\overline{\sigma}(\overline{\tau}(p))}$$
$$\varphi_\sigma(R_{\overline{\tau}(p)}) = \varphi_\sigma(\varphi_\tau(R_p)) = \varphi_\sigma \circ \varphi_\tau(R_p)$$

for all points p in P. Consequently,

$$\varphi_{\sigma \circ \tau} = \varphi_\sigma \circ \varphi_\tau,$$

so the mapping does preserve the operation of functional composition, as claimed. □

Different affine collineations may induce the same base isomorphism. A characterization of when this occurs uses the geometric notion of a dilation. A *dilation* of an affine geometry D is an autocollineation of D that maps every line in D to a parallel line; in other words, for every line ℓ in D, if k is the line consisting of the images of points on ℓ under the autocollineation, then ℓ and k are parallel in D.

Lemma 17.25. *Two affine collineations σ and τ from D to D' induce the same base isomorphism from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ if and only if the composition $\tau^{-1} \circ \sigma$ is a dilation of D.*

Proof. Let $\overline{\sigma}$ and $\overline{\tau}$ be the extensions of σ and τ to projective collineations from the projective geometry Q to the projective geometry Q'. Thus, if p is a point in P, then $\overline{\sigma}(p) = p'$ just in case each line

in D that has p as its ideal point is mapped by σ to a line in D' that has p' as its ideal point,by the definition of $\overline{\sigma}$, and similarly for τ. The base isomorphisms φ and ψ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ that are induced by σ and τ respectively are completely determined by the actions of $\overline{\sigma}$ and $\overline{\tau}$ on points in P, since

$$\varphi(R_p) = R_{\overline{\sigma}(p)} \qquad \text{and} \qquad \psi(R_p) = R_{\overline{\tau}(p)}$$

for p in P, by Lemma 17.13(i). In particular, $\varphi = \psi$ if and only if the collineations $\overline{\sigma}$ and $\overline{\tau}$ agree on P, or put another way, if and only if the composition $\overline{\tau}^{-1} \circ \overline{\sigma}$ is the identity function on P. This last condition just means that if a line ℓ in D has p as its ideal point, then the image of ℓ under $\tau^{-1} \circ \sigma$ (which is the restriction of $\overline{\tau}^{-1} \circ \overline{\sigma}$ to D) also has p as its ideal point. In other words, every line though p is mapped by the affine collineation $\tau^{-1} \circ \sigma$ to another line through p. Since the affine lines that have p as their ideal point form a class of parallel lines, it follows that $\varphi = \psi$ if and only if $\tau^{-1} \circ \sigma$ is a dilation. □

Lemma 17.26. *The group of base automorphisms of the affine algebra $\mathfrak{Af}(D)$ is isomorphic to the group of autocollineations of D modulo the subgroup of dilations of D.*

Proof. Each autocollineation σ of the affine geometry D determines a base automorphism φ_σ of the affine algebra $\mathfrak{Af}(D)$ by means of the formula

$$\varphi_\sigma(R_p) = R_{\overline{\sigma}(p)}$$

for each point p in P, by Lemma 17.23(i). The function mapping σ to φ_σ for each autocollineation σ is an epimorphism from the group of autocollineations of D to the group of base automorphisms of $\mathfrak{Af}(D)$, by Lemma 17.24. The kernel of this epimorphism consists of those autocollineations of D that are mapped to the identity base automorphism of $\mathfrak{Af}(D)$, that is to say, it consists of those autocollineations σ such that

$$\varphi_\sigma = \varphi_\tau, \tag{1}$$

where τ is the identity autocollineation of D. Lemma 17.25 implies that (1) holds if and only if the composition $\tau^{-1} \circ \sigma$ is a dilation of D. In view of the definition of τ, this means that (1) holds if and only if σ is a dilation of D. Thus, the kernel of the epimorphism consists precisely of the dilations of D. In particular, this kernel is a normal subgroup of the group of all autocollineations of D. Apply the First Isomorphism Theorem for groups to obtain the desired conclusion. □

Theorem 17.27. *If a projective geometry P has dimension at least three or is a Desarguesian plane, then $\mathfrak{Cm}(P)$ has, up to equivalence, a unique complete square representation.*

Proof. It is well known that projective geometries of dimension at least three are Desarguesian, and that every Desarguesian geometry is a maximal subspace of a projective geometry of one higher dimension (or, put another way, every Desarguesian geometry is embeddable into a projective geometry of one higher dimension). The given assumption on P therefore implies that P is a maximal subspace of some projective geometry Q of dimension one more than that of P. Notice in passing that the dimension of Q must be at least three, since the dimension of P is assumed to be at least two. Let D be the affine geometry obtained from Q by deleting the points and lines of P. Apply Theorem 17.14 to conclude that the complex algebra $\mathfrak{Cm}(P)$ has a complete square representation, namely the affine representation of $\mathfrak{Cm}(P)$ over the affine geometry D. Call this representation ψ.

Consider now an arbitrary complete square representation ϱ of the complex algebra $\mathfrak{Cm}(P)$. In view of the second part of Theorem 17.14, there must be an affine representation ϑ of $\mathfrak{Cm}(P)$ that is equivalent to ϱ. The very definition of an affine representation implies that there is a projective geometry Q' which includes P as a maximal subspace and such that ϑ is the affine representation of $\mathfrak{Cm}(P)$ over the affine geometry D' obtained from Q' by deleting the points and lines of P. The composition $\varphi = \vartheta \circ \psi^{-1}$ is an isomorphism from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ (see Figure 17.15). The isomorphism φ is induced by an autocollinea-

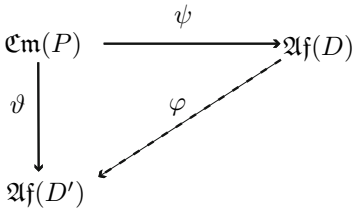


Fig. 17.15 The uniqueness of the affine representation.

tion σ of P and is determined by the equation

$$\varphi(R_p) = R_{\sigma(p)} \tag{1}$$

for p in P, by Lemma 17.21 (with P in place of P', since P is the geometry at infinity in Q' of the affine geometry D').

As noted earlier, the dimension of the projective geometry Q is at least three. Therefore, the collineation σ can be extended to a collineation τ from Q to Q', by the second result from projective geometry mentioned before Lemma 17.23. The geometries Q and Q' are the projective extensions of the affine geometries D and D', obtained by adjoining P as the geometry at infinity; consequently, the points in Q and Q' are the disjoint unions of the points in P with the points in D and D' respectively. The projective collineation τ is an extension of σ, and σ maps the set P bijectively to itself, so τ must map the set D bijectively to the set D', and therefore τ, restricted to D, must be an affine collineation from D to D'. This restriction induces a base isomorphism φ_τ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$ that is determined by

$$\varphi_\tau(R_p) = R_{\tau(p)}$$

for points p in P, by Lemma 17.23(i). Since τ and σ agree on P, the preceding equation may be written as

$$\varphi_\tau(R_p) = R_{\sigma(p)} \tag{2}$$

for p in P. Compare (1) and (2) to conclude that φ coincides with φ_τ, and therefore φ is a base isomorphism from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D')$.

It follows from the definition of φ that $\varphi \circ \psi = \vartheta$. Since φ is a base isomorphism, the preceding equation shows that the affine representation ϑ is equivalent to the affine representation ψ. The given complete representation ϱ is equivalent to ϑ, by assumption, so ϱ must be equivalent to ψ. Thus, every complete square representation of $\mathfrak{Cm}(P)$ is equivalent to the affine representation ψ. □

For each projective geometry P of dimension at least two, Theorems 17.16 and 17.27 entirely settle the questions of when the complex algebra $\mathfrak{Cm}(P)$ is representable and what complete square representations it may have: if P has dimension at least three or is a Desarguesian plane, then $\mathfrak{Cm}(P)$ is representable, and in fact it has a complete square representation that is unique up to equivalence and is an affine representation of $\mathfrak{Cm}(P)$. If P is a non-Desarguesian plane, then $\mathfrak{Cm}(P)$ is not representable at all.

For projective geometries P of dimension one, that is to say, for projective lines, the situation is more complicated. There may be a

multitude of inequivalent complete square representations of $\mathfrak{Cm}(P)$. Fortunately, for finite projective lines P, if $\mathfrak{Cm}(P)$ is representable at all, then there is an explicit formula that calculates the number of inequivalent square representations of $\mathfrak{Cm}(P)$. (Keep in mind that for finite relation algebras, every representation is complete.) Write

$$\mathrm{Aut}\,(D) \qquad \text{and} \qquad \mathrm{Dil}\,(D)$$

for the group of autocollineations of an affine geometry D and the subgroup of dilations of D. Denote the cardinality of a group G by $|G|$.

Lemma 17.28. *If a projective line P is the line at infinity of an affine plane D of finite order n, then $\mathfrak{Cm}(P)$ has*

$$\frac{(n+1)!}{|\mathrm{Aut}\,(D)|/|\mathrm{Dil}\,(D)|}$$

inequivalent affine representations over D.

Proof. The points in P are all collinear with one another because P is assumed to be a line. Consequently, every permutation of P is an autocollineation. The given affine plane D is assumed to have order n, and P is the line at infinity of D, so P must have $n+1$ points and therefore $(n+1)!$ permutations, that is to say, autocollineations.

The number of inequivalent isomorphisms from $\mathfrak{Cm}(P)$ to $\mathfrak{Af}(D)$ is equal to the number of automorphisms of $\mathfrak{Af}(D)$ divided by the number of base automorphisms of $\mathfrak{Af}(D)$, by Lemma 16.4. The group of automorphisms of $\mathfrak{Af}(D)$ is isomorphic to the group of autocollineations of P, by Corollary 17.22, so there must be $(n+1)!$ automorphisms of $\mathfrak{Af}(D)$, by the observations of the preceding paragraph. The group of base automorphisms of $\mathfrak{Af}(D)$ is isomorphic to the quotient of the group of autocollineations of D modulo the normal subgroup of dilations of D, by Lemma 17.26, so there are

$$|\mathrm{Aut}\,(D)|/|\mathrm{Dil}\,(D)|$$

base automorphisms of $\mathfrak{Af}(D)$. Combine these observations to arrive at the conclusion of the lemma. □

The lemma implies the following formula for computing the number of inequivalent square representations of the complex algebra of a finite projective line.

Theorem 17.29. *Suppose that P is a projective line of finite order n. If $(D_i : i \in I)$ is a list of the non-isomorphic affine planes of order n, then the number of inequivalent square representations of $\mathfrak{Cm}(P)$ is the sum (over i in I) of the integers*

$$\frac{(n+1)!}{|\mathrm{Aut}\,(D_i)|/|\mathrm{Dil}\,(D_i)|} \ .$$

Proof. We begin with a preliminary remark. Without loss of generality, it may be assumed that the given line P is disjoint from each of the finitely many given affine planes D_i in the list. For each index i in I, it is then possible to construct a projective plane Q_i of order n extending D_i such that P is the line at infinity of D_i in Q_i. In more detail, the line P has $n+1$ points and the plane D_i has $n+1$ equivalence classes of parallel lines. In a one-to-one fashion, assign to every equivalence class of lines some point in P to serve as the point at infinity of each line in the equivalence class. The geometry Q_i consists of the points in D_i, the points in P, the lines in D_i augmented by the appropriate point at infinity, and the line P itself, which serves as the line at infinity of the plane D_i.

Turn now to the proof of the theorem. Every representation of the algebra $\mathfrak{Cm}(P)$ is complete, because P is finite, and therefore every square representation of $\mathfrak{Cm}(P)$ is equivalent to an affine representation of $\mathfrak{Cm}(P)$, by Theorem 17.14. Consequently, it suffices to count the number of inequivalent affine representations. The first step is to show that every affine representation of $\mathfrak{Cm}(P)$ is equivalent to an affine representation over one of the planes in the given list.

To this end, consider an arbitrary affine representation ψ of $\mathfrak{Cm}(P)$ over some affine plane D. Thus, there is a projective plane Q extending D such that P is the line at infinity of D in Q, and ψ is the function from $\mathfrak{Cm}(P)$ to $\mathfrak{Af}(D)$ that is determined by

$$\psi(p) = R_p \tag{1}$$

for each point p in P. Here, R_p is the relation on D that is defined by

$$R_p = \{(a,b) : a, b \in D,\ a \neq b, \\ \text{and } ab \text{ has the ideal point } p \text{ in } Q\}. \tag{2}$$

The plane D is isomorphic to one of the planes D_i in the given list, by assumption, so there is a collineation σ from D to D_i. This

collineation can be extended to a collineation $\overline{\sigma}$ from the projective plane Q to the projective plane Q_i by requiring that if p is the ideal point in Q of an affine line ab in D, then $\overline{\sigma}(p)$ is the ideal point in Q_i of the affine line $\sigma(a)\sigma(b)$ in D_i (see the remarks preceding Lemma 17.23). The collineation σ induces a base isomorphism φ from $\mathfrak{Af}(D)$ to $\mathfrak{Af}(D_i)$ that is determined by

$$\varphi(R_p) = R_{\overline{\sigma}(p)} \tag{3}$$

for points p in P, by Lemma 17.23(i), where the relation $R_{\overline{\sigma}(p)}$ in (3) is defined just as in (2), except that D, Q and p are replaced by D_i, Q_i, and $\overline{\sigma}(p)$ respectively. The composition $\vartheta = \varphi \circ \psi$ is an isomorphism from $\mathfrak{Cm}(P)$ to $\mathfrak{Af}(D_i)$ that is equivalent to the given representation ψ, because φ is a base isomorphism (see Figure 17.15, with D_i in place of D').

It turns out that ϑ is an affine representation of $\mathfrak{Cm}(P)$ over the plane D_i, but this is not immediately apparent. The reason it is not apparent is that

$$\vartheta(p) = \varphi(\psi(p)) = \varphi(R_p) = R_{\overline{\sigma}(p)} \tag{4}$$

for points p in P, by (1), (2), and the definition of ϑ, and therefore ϑ does not map p to the relation R_p on D_i that is defined by (2) (with D_i and Q_i in place of D and Q respectively), but rather to the relation $R_{\overline{\sigma}(p)}$ on D_i.

To overcome this obstacle, define another projective plane Q_i' in which P is the line at infinity of the affine plane D_i, just as in Q_i, but the assignment of points in P to lines in D_i is different than it is in Q_i: if an affine line ab in D_i has p as its point at infinity in Q_i, then the point at infinity of ab in Q_i' is defined to be $\overline{\sigma}^{-1}(p)$. In particular, if a line ab has $\overline{\sigma}(p)$ as its point at infinity in Q_i, then that line has

$$p = \overline{\sigma}^{-1}(\overline{\sigma}(p))$$

as its point and infinity in Q_i', and therefore the relation $R_{\overline{\sigma}(p)}$ in (3) and (4) coincides with the relation R_p on D_i that is defined as in (2), but with D and Q replaced by D_i and Q_i' respectively. Thus, for this last relation R_p, we do have

$$\vartheta(p) = R_p,$$

by (1) and (4), so that ϑ is, after all, an affine representation of $\mathfrak{Cm}(P)$ over the affine plane D_i, but with respect to the geometry Q_i', not the

geometry Q_i. This completes the proof that every affine representation, and hence every square representation, of $\mathfrak{Cm}(P)$ is equivalent to a representation of $\mathfrak{Cm}(P)$ over one of the affine planes in the given list.

The next step is to show that an affine representation of $\mathfrak{Cm}(P)$ cannot be equivalent simultaneously to affine representations over non-isomorphic planes D_i and D_j. Indeed, if some affine representation of $\mathfrak{Cm}(P)$ were equivalent simultaneously to affine representations ϑ_i and ϑ_j over the planes D_i and D_j respectively, then ϑ_i and ϑ_j would be equivalent to one another. Consequently, there would have to be a bijection τ from D_i to D_j inducing a base isomorphism, say ϱ, from $\mathfrak{Af}(D_i)$ to $\mathfrak{Af}(D_j)$ such that $\vartheta_j = \varrho \circ \vartheta_i$. But then τ would be a collineation from D_i to D_j, by Lemma 17.23(ii), contracting the assumption that D_i and D_j are not isomorphic.

It has been shown that every square representation of $\mathfrak{Cm}(P)$ is equivalent to an affine representation of $\mathfrak{Cm}(P)$ over exactly one of the given planes D_i. As a result, in order to determine the number of inequivalent square representations of $\mathfrak{Cm}(P)$, it suffices to determine the number of inequivalent affine representations of $\mathfrak{Cm}(P)$ over the plane D_i for each index i, and then sum these integers over all indices i in I. The conclusion of the theorem now follows immediately from Lemma 17.28. □

As an illustration of a calculation using the preceding formula, we compute the number of inequivalent square representations of the complex algebra of a projective line of order nine. There are (up to isomorphism) four projective planes of order nine: the plane Δ constructed from the nine-element field, a plane Ω constructed from the unique non-commutative near-field of order nine, its dual plane Ω^d, and the Hughes plane Ψ of order nine (see [118] and [80]). There are seven affine planes of order nine: the plane Δ_0 obtained from Δ by deleting an arbitrary line, the planes Ω_0 and Ω_1 obtained from Ω by respectively deleting the unique translation line and any other line, the planes Ω_0^d and Ω_1^d obtained from Ω^d by respectively deleting a line through the translation point and any other line, and the planes Ψ_0 and Ψ_1 obtained from Ψ by respectively deleting a real line and a complex line. The number of autocollineations and dilations of each of these affine planes is given in Table 17.2, along with the quotient of the two numbers (see [37] for the details of these calculations).

The projective line of order nine has ten points and therefore 10! autocollineations. The number of inequivalent square representations

affine planes	Δ_0	Ω_0	Ω_1	Ω_0^d	Ω_1^d	Ψ_0	Ψ_1
autocollineations	933120	311040	3456	31104	3840	2592	432
dilations	648	162	8	72	2	18	1
quotient	1440	1920	432	432	1920	144	432

Table 17.2 Autocollineations and dilations of affine planes of order nine.

of the complex algebra of a projective line of order nine is therefore the sum of the quotients

$$10!/1440, \quad 10!/1920, \quad 10!/432, \quad 10!/432,$$
$$10!/1920, \quad 10!/144, \quad 10!/432,$$

by Theorem 17.29. In other words, it is

$$2520 + 1890 + 8400 + 8400 + 1890 + 25200 + 8400 = 56700.$$

It is worth emphasizing that only seven set relation algebras are used as image algebras for all of these inequivalent representations.

As this example makes clear, a great deal of geometric information is often needed to carry through the computation of the number of inequivalent square representations of the complex algebra of a finite projective line. In many cases, this information is in fact as yet unknown. Nevertheless, one can still use the formula in Theorem 17.29 to compute a lower bound for the number of inequivalent square representations. For example, when n is the kth power of a prime, then the affine plane over a field of cardinality n has

$$k \cdot n^3 \cdot (n^2 - 1) \cdot (n - 1)$$

autocollineations and $n^2 \cdot (n - 1)$ dilations (see [37] for the details of this computation). The quotient of these two numbers is

$$k \cdot n \cdot (n^2 - 1) = k \cdot (n + 1) \cdot n \cdot (n - 1).$$

Consider the concrete case of a projective line of order 29. The line has 30! autocollineations, because it has 30 points. The formulas in the preceding paragraph, with $n = 29$ and $k = 1$, imply that the affine plane constructed from a field with 29 elements has

$$29^3 \cdot (29^2 - 1) \cdot (29 - 1)$$

autocollineations and $29^2 \cdot (29-1)$ dilations, and the quotient of these two numbers is $30 \cdot 29 \cdot 28$. Divide 30! by $30 \cdot 29 \cdot 28$ to arrive at

$$27! = 1.0888869 \times 10^{28}$$

as a lower bound for the number of inequivalent square representations of the complex algebra of a projective line of order 29. All of these inequivalent representations involve only one set relation algebra as the image algebra, namely the affine algebra over the affine plane constructed from a field with 29 elements.

As regards infinite projective lines, the corresponding complex algebras are always completely representable. Indeed, any infinite projective line can be embedded into the projective plane over a field of the same cardinality as the line: just map the set of points on the line bijectively to the set of points on some (any) fixed line in the projective plane over the field. For example, a countably infinite projective line is embeddable into the projective plane over the rational numbers via any bijection of the set of points on the line to the set of points on the x-axis of the plane, that is to say, to the set of points of the form $(r, 0)$, where r varies over the set of extended rational numbers (the set of rational numbers and a point ∞ at infinity). Such a bijection certainly exists because both sets are countably infinite.

17.5 Small relation algebras

Small relation algebras are finite relation algebras with a very limited number of atoms, usually seven or less (see Section 3.8). Problems regarding the representability of such algebras are traditionally treated on a case-by-case basis, often with the aid of a computer. We give a small sample of the known results in this area by presenting five concrete examples.

Consider first an eight-element relation algebra $\mathfrak{A}$ with three atoms, say 1', s, and t, that satisfy the rules for relative multiplication and converse given in Table 17.3. This algebra is integral and therefore simple, by Integrality Theorem 9.7 and Corollary 9.10, so if it has a representation at all, then it must have a square representation over some non-empty base set U, by Lemma 16.1. In fact, such a base set U must have at least two elements, because $\mathfrak{A}$ has two subdiversity atoms. Imagine what properties the relations representing the atoms of $\mathfrak{A}$ in

;	1'	s	t
1'	1'	s	t
s	s	s	1
t	t	1	t

	$\smile$
1'	1'
s	t
t	s

Table 17.3 Relative multiplication and converse tables for the atoms in the first example.

such a representation—call them id_U, S, and T—must possess. They must be disjoint, because the corresponding atoms 1', s, and t in $\mathfrak{A}$ are disjoint; and they must satisfy the equations

$$T = S^{-1} \qquad \text{and} \qquad S \mid S \subseteq S,$$

because the atoms s and t satisfy the corresponding equations in $\mathfrak{A}$. Consequently, the relation S must be irreflexive, asymmetric, and transitive, that is to say, it must be a strict partial ordering of the base set U. Furthermore, the sum of the atoms 1', s, and $s^\smile$ in $\mathfrak{A}$ is 1, so the union of the relations id_U, S, and S^{-1} must be the unit $U \times U$ of the representing algebra. This union is just the relation of comparability with respect to S, so any two elements in U must be comparable with respect to S, that is to say, S must be a strict linear (or total) ordering of U. The equations

$$S \mid S^{-1} = S \mid T = U \times U \qquad \text{and} \qquad S^{-1} \mid S = T \mid S = U \times U$$

must also hold, because s and t satisfy the corresponding equations in $\mathfrak{A}$. These equations express that any two elements in the base set U have a strict upper bound and a strict lower bound with respect to the linear order S, so S can have no greatest or least element. Finally, the inclusion $S \subseteq S \mid S$ must hold, because the corresponding equation holds for the atom s in $\mathfrak{A}$; and this equation expresses that the linear order S must be dense in the sense that for any two elements α and β that stand in the relation S to one another, there is another element γ that is between α and β in the sense of the relation S. Summarizing, S must be a dense linear ordering without endpoints.

Let $<$ (less than) be the standard linear ordering of the set $\mathbb{Q}$ of rational numbers (or, more generally, any dense linear ordering without endpoints of an arbitrary set of cardinality at least two), and write $>$ (greater than) for the relation that is the converse of $<$. Easy calculations show that the three relations $id_{\mathbb{Q}}$, $<$, and $>$ form a partition

of $\mathbb{Q} \times \mathbb{Q}$ and have the relational composition and converse tables set forth in Table 17.4. The Atomic Subalgebra Theorem in the form of

$\mid$	$id_{\mathbb{Q}}$	$<$	$>$
$id_{\mathbb{Q}}$	$id_{\mathbb{Q}}$	$<$	$>$
$<$	$<$	$<$	$\mathbb{Q} \times \mathbb{Q}$
$>$	$>$	$\mathbb{Q} \times \mathbb{Q}$	$>$

	$^{-1}$
$id_{\mathbb{Q}}$	$id_{\mathbb{Q}}$
$<$	$>$
$>$	$<$

Table 17.4 Relational composition and converse tables for a dense linear order without endpoints.

Corollary 6.22 ensures that these three relations are the atoms of a subalgebra $\mathfrak{B}$ of the full set relation algebra $\mathfrak{Re}(\mathbb{Q})$, and the elements in $\mathfrak{B}$ are just the eight possible relations that are unions of various combinations of the three atoms. A comparison of the relative multi-

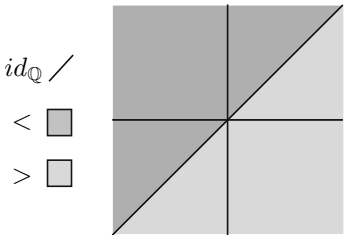


Fig. 17.16 The dense linear order representation of the first example.

plication and converse tables for the atoms in $\mathfrak{A}$ and in $\mathfrak{B}$, together with the Atomic Isomorphism Theorem in the form of Corollary 7.12, shows that the function mapping the atoms 1', s, and t respectively to the relations $id_{\mathbb{Q}}$, $<$, and $>$ extends to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. This extension is a square representation of $\mathfrak{A}$ (see Figure 17.16).

The preceding argument establishes more than the representability of $\mathfrak{A}$. It actually describes all square representations of $\mathfrak{A}$ in terms of (non-empty) dense linear orderings without endpoints. Only an infinite set can have such an ordering, so the base set of every square representation—and hence every representation—of $\mathfrak{A}$ must be infinite (see Theorem 16.18). Said another way, $\mathfrak{A}$ has no representations over finite sets. A well-known theorem due to Cantor says that up to order isomorphism there is only one dense linear ordering without endpoints

on a countably infinite set, namely the standard strict linear ordering of the rational numbers. Consequently, the algebra $\mathfrak{A}$ has, up to equivalence, exactly one square representation over a countable base set.

As the second example, consider an eight-element symmetric relation algebra $\mathfrak{A}$ with three atoms, say 1', s, and t, that satisfy the rules for relative multiplication given in Table 17.5. This algebra is also

;	1'	s	t
1'	1'	s	t
s	s	1' + s	t
t	t	t	1' + s

Table 17.5 Relative multiplication table for the atoms in the second example.

integral and therefore simple, by Integrality Theorem 9.7 and Corollary 9.10, so if it has a representation at all, then it must have a square representation over some non-empty base set U, by Lemma 16.1. Imagine such a representation in which the atoms 1', s, and t are represented by relations id_U, S, and T respectively, and write $R = id_U \cup S$. An easy calculation using Table 17.5 shows that R must be an equivalence relation on the set U. For example,

$$\begin{aligned} R \mid R &= (id_U \cup S) \mid (id_U \cup S) \\ &= (id_U \mid id_U) \cup (id_U \mid S) \cup (S \mid id_U) \cup (S \mid S) \\ &= id_U \cup S = R, \end{aligned}$$

because an analogous calculation using 1', s, and $r = 1' + s$ is valid in $\mathfrak{A}$. Also, the complement of R is T, and therefore

$$(\sim R) \mid (\sim R) = T \mid T = id_U \cup S = R,$$

again because an analogous calculation is valid in $\mathfrak{A}$. The equality of the first and last relations expresses that the equivalence relation R has exactly two equivalence classes. On the other hand, a similar argument shows that

$$(di_U \cap R) \mid (di_U \cap R) = S \mid S = id_U \cup S = R.$$

The equality of the first and last relations expresses (in the relativization of the representing relation algebra to the equivalence relation R)

that each of the two equivalence classes of R has at least three elements. Summarizing, R is an equivalence relation on U with exactly two equivalence classes, each of which has at least three elements.

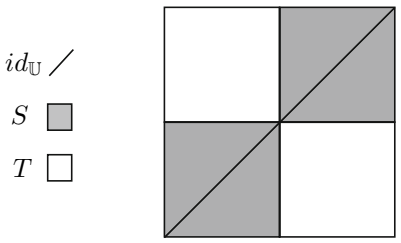


Fig. 17.17 The equivalence relation representation of the second example.

Consider now an arbitrary equivalence relation E on a set U with at least six elements, and suppose that E has exactly two equivalence classes, each of which has at least three elements. Write

$$S = E \sim id_U \qquad \text{and} \qquad T = \sim E = U \times U \sim E.$$

It is not difficult to check that the three relations id_U, S, and T are symmetric and form a partition of $U \times U$. Moreover, the relational composition of any two of them is determined by Table 17.6. The three

$\mid$	id_U	S	T
id_U	id_U	S	T
S	S	$id_U \cup S$	T
T	T	T	$id_U \cup S$

Table 17.6 Relational composition table for the atoms generated by an equivalence relation with two equivalence classes, each of which has at least three elements.

relations are therefore the atoms of a subalgebra $\mathfrak{B}$ of the full set relation algebra $\mathfrak{Re}(U)$, and the elements of $\mathfrak{B}$ are just the eight possible relations that are unions of various combinations of these three atoms, by the Atomic Subalgebra Theorem in the form of Corollary 6.22. A comparison of the relative multiplication tables for the atoms of $\mathfrak{A}$ and $\mathfrak{B}$, together with the Atomic Isomorphism Theorem in the form of Corollary 7.12, shows that the function mapping the atoms 1', s,

and t respectively to the relations id_U, S, and T extends to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. This extension is a square representation of $\mathfrak{A}$ (see Figure 17.17).

As in the preceding example, the argument just given proves more than the representability of $\mathfrak{A}$. It gives a complete description of the square representations of $\mathfrak{A}$ in terms of equivalence relations. In particular, $\mathfrak{A}$ has square representations over every set of cardinality at least six, and no representations over sets of any smaller size.

For the third example, consider the sixteen-element relation algebra $\mathfrak{A}$ with four atoms, say 1', d, s, and t, that satisfy the rules for relative multiplication and converse given in Table 17.7.

;	1'	d	s	t
1'	1'	d	s	t
d	d	1'	t	s
s	s	t	d	1'
t	t	s	1'	d

	$\smile$
1'	1'
d	d
s	t
t	s

Table 17.7 Relative multiplication and converse tables for the atoms in the third example.

These tables show that the set of atoms in $\mathfrak{A}$ is closed under the operations of relative multiplication and converse, and that each atom has an inverse with respect to the operation of relative multiplication and the identity element 1', namely the converse of the atom. The set of atoms is therefore a group under the operations of relative multiplication and converse, with 1' as the identity element. In fact, the function that maps 1', d, s, and t to 0, 2, 1, and 3 respectively is an isomorphism from the group of atoms of $\mathfrak{A}$ to the additive cyclic group $\mathbb{Z}_4$ of order four. It follows that $\mathfrak{A}$ is isomorphic to the complex algebra of the group $\mathbb{Z}_4$, so it has, up to equivalence, a unique square representation, by Corollary 17.11(i), namely the Cayley representation of its group of atoms (see Figure 17.1).

For the fourth example, consider the sixteen-element integral relation algebra $\mathfrak{A}$ with four atoms, say 1', d, s, and t, that satisfy the rules for relative multiplication and converse given in Table 17.8 (see Section 3.8, where this example is discussed.) This algebra has no representations at all. In order to prove this assertion, it is necessary to exhibit a property that holds in all set relation algebras but fails to

;	1'	d	s	t
1'	1'	d	s	t
d	d	$1' + s + t$	$d + s$	$d + t$
s	s	$d + s$	s	1
t	t	$d + t$	1	t

	$\smile$
1'	1'
d	d
s	t
t	s

.

Table 17.8 Relative multiplication and converse tables for the atoms in the fourth example.

hold in $\mathfrak{A}$. As we shall eventually see, one can actually select a property that is expressible by means of an equation.

The argument proceeds by contradiction. Assume, to the contrary, that $\mathfrak{A}$ is representable, and therefore has a square representation over some base set U. Let id_U, D, S, and T be the relations on U that represent the atoms 1', d, s, and t respectively. Just as in the first example above, the relative multiplication table for $\mathfrak{A}$ implies that the relation S must be a strict partial ordering of the base set U, and under this partial ordering any two elements have a strict upper bound and a strict lower bound. The relation $E = id_U \cup S$ is the corresponding non-strict partial ordering of the set U. The converse table for $\mathfrak{A}$ implies that the relation T is the converse of S, so the union

$$id_U \cup S \cup T = E \cup E^{-1}$$

is the relation of comparability under the partial ordering. The complement

$$D = \sim(id_U \cup S \cup T) = \sim(E \cup E^{-1})$$

must therefore be the relation of incomparability under the partial ordering. The *incomparability equation*

$$D \mid D = id_U \cup S \cup T = E \cup E^{-1},$$

which follows directly from Table 17.8, expresses that any two elements are comparable if and only if they are incomparable to a third element.

The incomparability relation D is an atom in the algebra representing $\mathfrak{A}$, since d is an atom in $\mathfrak{A}$, and therefore D is non-empty. Consider two incomparable elements, say α and β, in U. They must have a common upper bound γ, by the remarks at the beginning of the preceding paragraph. Since α and γ are comparable, there is an element ξ that is incomparable with both α and γ, by the incomparability equation.

The incomparability of ξ with α, and of α with β, implies that ξ and β must be comparable, by the incomparability equation. If the pair (ξ, β) were in E, then because (β, γ) is also in E, the pair (ξ, γ) would have to be in E, by the transitivity of E, and this would contradict the incomparability of ξ with γ. It follows that (ξ, β) is not in E, so it must be in E^{-1} (because ξ and β are comparable) and therefore (β, ξ) must be in E.

A similar argument shows that there is an element η that is incomparable with both β and γ. Since β is incomparable with α, the elements η and α must be comparable, by the incomparability equation. The assumption that (η, α) is in E leads to the contradiction that η is comparable with γ (because (α, γ) is in E, and E is transitive). Consequently, (η, α) is in E^{-1}, and therefore (α, η) is in E.

The elements ξ and η are both incomparable with γ, so they must be comparable with each other, by the incomparability equation. If the pair (ξ, η) is in E, then (β, η) is in E, because (β, ξ) is in E and E is transitive; this contradicts the incomparability of β with η. On the other hand, if (η, ξ) is in E, then (α, ξ) is in E, because (α, η) is in E, and E is transitive; this contradicts the incomparability of α with ξ. Thus, ξ and η cannot be comparable with one another. The desired contradiction has arrived and leads to the conclusion that the algebra $\mathfrak{A}$ is not representable.

The preceding argument uses only the following properties of the relation E to arrive at a contradiction: E is a partial ordering, that is to say, it is a reflexive, anti-symmetric, and transitive relation; any two elements in the universe have a common upper bound; and two elements are comparable to each other if and only if they are incomparable with a third element. These properties are respectively expressed by the three equations

$$e \cdot e^{\smile} = 1', \qquad e \,;\, e \leq e, \qquad e \,;\, e^{\smile} = 1,$$

and the incomparability equation

$$-(e + e^{\smile}) \,;\, -(e + e^{\smile}) = e + e^{\smile}.$$

The argument given above shows that the conjunction of the first three equations implies the negation of the fourth equation in all square set relation algebras. This implication fails to hold in the algebra $\mathfrak{A}$, however, because the element $e = 1' + s$ in $\mathfrak{A}$ satisfies the hypothesis of

the implication (namely the first three equations above) and also the conclusion of the implication (namely, the incomparability equation).

Every quantifier-free formula Γ in the language of relation algebras is equivalent in all simple relation algebras to an equation ε that can be constructed in a recursive fashion from Γ, by Theorem 9.5 and its proof. By applying the procedure outlined in that proof to the implication discussed in the preceding paragraph, and then introducing some simplifications in the resulting equation, one arrives at the equation

$$1 ; r ; 1 = 1,$$

where r is the term

$$(e \dotplus e) + [e ; e + 1' + (-(e + e^\smile) ; -(e + e^\smile)) \cdot -e^\smile] \cdot -e + -(e ; e^\smile).$$

This equation—call it ε—is true in all square set relation algebras, and therefore in all set relation algebras, by Theorem 12.11, but it fails to be true in the relation algebra $\mathfrak{A}$.

To verify directly the failure of the equation ε in $\mathfrak{A}$, take e to be the element $1' + s$ in $\mathfrak{A}$, and observe that

$$e ; e + 1' = (1' + s) ; (1' + s) + 1' = 1' + s + s + s ; s + 1' = e$$

and

$$-(e + e^\smile) ; -(e + e^\smile) = d ; d = 1' + s + t = e + e^\smile$$

in $\mathfrak{A}$, by Table 17.8. These two computations imply that the element

$$[e ; e + 1' + (-(e + e^\smile) ; -(e + e^\smile)) \cdot -e^\smile] \cdot -e$$

coincides with the element

$$[e + (e + e^\smile) \cdot -e^\smile] \cdot -e$$

in $\mathfrak{A}$, and this last element is just 0, since

$$[e + (e + e^\smile) \cdot -e^\smile] \cdot -e = e \cdot -e + e \cdot -e^\smile \cdot -e + e^\smile \cdot -e^\smile \cdot -e = 0,$$

by Boolean algebra. Analogous computations, using the definition of relative addition, the definition of e, and Table 17.8, yield

$$e \dotplus e = 0 \qquad \text{and} \qquad -(e ; e^\smile) = 0.$$

In more detail,

$$\begin{aligned} e \dagger e &= -(-e\,;-e) \\ &= -(-(1' + s)\,;-(1' + s)) \\ &= -((d + t)\,;(d + t)) \\ &= -(d\,;d + d\,;t + t\,;d + t\,;t) \\ &= -(1' + s + t + d) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} -(e\,;e^\smile) &= -((1' + s)\,;(1' + s)^\smile) \\ &= -((1' + s)\,;(1' + t)) \\ &= -(1' + s + t + s\,;t) \\ &= 0. \end{aligned}$$

It follows that the term r defined above evaluates to 0 in $\mathfrak{A}$, so

$$1\,;r\,;1 = 1\,;0\,;1 = 0$$

in $\mathfrak{A}$. Conclusion: the equation ε fails in $\mathfrak{A}$.

Theorem 17.30. *There are equations, and even equations with a single variable, that are true in all set relation algebras, but not in all relation algebras and not even in all finite relation algebras. The equation* $1\,;r\,;1 = 1$, *where* r *is the term*

$$(e \dagger e) + [e\,;e + 1' + (-(e + e^\smile)\,;-(e + e^\smile)) \cdot -e^\smile] \cdot -e + -(e\,;e^\smile)$$

is an example.

Every equation that is true in all relation algebras is obviously true in all set relation algebras. Theorem 17.30 implies that the converse of this statement is false. Consequently, the set of equations true in all set relation algebras includes the set of equations true in all relation algebras as a proper subset. Another way of stating this is that there exist equations which are true in all set relation algebras but which are not derivable from the ten axioms of relation algebra. Thus, the ten axioms of relation algebra are incomplete in the sense that they are not strong enough to derive all equations that are true in every set relation algebra.

The equation in Theorem 17.30 was not the first known example of an equation that is true in all set relation algebras but not in all relation algebras. That honor goes to the six-variable inequality

$$(p\,;q)\cdot(r\,;u)\cdot(v\,;w)\leq p\,;[(p^{\smile}\,;r)\cdot(q\,;u^{\smile})\cdot \\ ([(p^{\smile}\,;v)\cdot(q\,;w^{\smile})]\,;[(v^{\smile}\,;r)\cdot(w\,;u^{\smile})])]\,;u.$$

(Recall from Section 2.1 that all inequalities may be viewed as equations.) This inequality—call it ε'—may be viewed as a six-variable analogue of the four-variable inequality given in Lemma 4.27, an inequality that *is* true in all relation algebras. The advantage of the equation given in Theorem 17.30 over ε' is that it has only one variable and is somewhat shorter in length.

The original finite relation algebra in which ε' was shown to fail is rather large and complicated in its structure: it has 56 atoms and therefore 2^{56} elements. It turns out, however, that there are substantially smaller relation algebras in which ε' fails. One of these is the relation algebra $\mathfrak{A}$ with four atoms, say 1', d, s, and t, that satisfy the rules for relative multiplication and converse given in Table 17.9 (where $0' = d + s + t$).

;	1'	d	s	t
1'	1'	d	s	t
d	d	$1' + s + t$	0'	0'
s	s	0'	$d + t$	$1' + d$
t	t	0'	$1' + d$	$d + s$

	$\smile$
1'	1'
d	d
s	t
t	s

Table 17.9 Relative multiplication and converse tables for the atoms in the fifth example.

To see that ε' fails in $\mathfrak{A}$, assign to the variables p, q, r, u, v, and w in the inequality the elements d, s, s, s, t, and s in $\mathfrak{A}$, respectively. The left side of ε' evaluates to the atom d in $\mathfrak{A}$, because

$$(p\,;q)\cdot(r\,;u)\cdot(v\,;w) = (d\,;s)\cdot(s\,;s)\cdot(t\,;s) \\ = 0'\cdot(d+t)\cdot(1'+d) = d.$$

The right side of ε' evaluates to zero in $\mathfrak{A}$. To see this, observe first that

$$[(p^\smile ; v) \cdot (q ; w^\smile)] ; [(v^\smile ; r) \cdot (w ; u^\smile)] \\ = [(d^\smile ; t) \cdot (s ; s^\smile)] ; [(t^\smile ; s) \cdot (s ; s^\smile)],$$

and the right side of this equality reduces stepwise to $1' + s + t$ as follows:

$$\begin{aligned}[(d^\smile ; t) \cdot (s ; s^\smile)] ; [(t^\smile ; s) \cdot (s ; s^\smile) &= [(d ; t) \cdot (s ; t)] ; [(s ; s) \cdot (s ; t)] \\ &= [0' \cdot (1' + d)] ; [(d + t) \cdot (1' + d)] \\ &= d ; d = 1' + s + t,\end{aligned}$$

by the entries in Table 17.9. Observe also that

$$(p^\smile ; r) \cdot (q ; u^\smile) = (d^\smile ; s) \cdot (s ; s^\smile) \\ = (d ; s) \cdot (s ; t) = 0' \cdot (1' + d) = d.$$

Combine these observations to conclude that the expression

$$(p^\smile ; r) \cdot (q ; u^\smile) \cdot ([(p^\smile ; v) \cdot (q ; w^\smile)] ; [(v^\smile ; r) \cdot (w ; u^\smile)])$$

occurring in the right side of ε' evaluates to the element $d \cdot (1' + s + t)$, which is 0 in $\mathfrak{A}$. As a result, the right side of ε' must also evaluate to 0, by Corollary 4.17 and its first dual. Since d is not below 0, it follows that ε' is not valid in $\mathfrak{A}$.

One more observation is of interest: the algebra $\mathfrak{A}$ is generated by a single element, namely the atom s, so there must actually be a one-variable instance of ε' that fails to hold in $\mathfrak{A}$.

17.6 Quasi-representations

The problem of whether every relation algebra is representable as a set relation algebra led to an early emphasis on representation theorems. The existence of non-representable relation algebras serves to reinforce this emphasis because it precludes the possibility of anyone finding a general representation theorem that supersedes and renders obsolete the various specialized representation results that have preceded it.

Quite surprisingly, however, a version of the technique employed to represent group complex algebras comes very close to representing all

relation algebras. To make this statement precise, let us define a *quasi-representation* of a relation algebra $\mathfrak{A}$ to be a one-to-one function φ from $\mathfrak{A}$ into a full set relation algebra $\mathfrak{Re}(U)$ that preserves zero, the identity element, and the operations of converse, addition, and relative multiplication in the sense that

$$\begin{aligned} \varphi(0) = \varnothing, \qquad \varphi(1') = id_U, \qquad \varphi(r^\smile) = \varphi(r)^{-1}, \\ \varphi(r + s) = \varphi(r) \cup \varphi(s), \qquad \varphi(r\,;s) = \varphi(r) \mid \varphi(s) \end{aligned}$$

for all elements r and s in $\mathfrak{A}$. It is not required that φ preserve the unit element and the operations of complement, multiplication, and relative addition. The set U is called the *base set* of the quasi-representation. A quasi-representation φ of $\mathfrak{A}$ is said to be *complete* if it preserves all existing suprema as unions, that is to say, if the existence of a supremum $r = \sum X$ in $\mathfrak{A}$ of some subset X always implies that

$$\varphi(r) = \bigcup\{\varphi(s) : s \in X\}.$$

A relation algebra that has a quasi-representation or a complete quasi-representation is said to be *quasi-representable* or *completely quasi-representable* respectively.

Suppose φ is a quasi-representation of $\mathfrak{A}$ over a base set U. Take B to be the subset of $\mathfrak{Re}(U)$ consisting of those relations that are images of elements in $\mathfrak{A}$, so that

$$B = \{\varphi(r) : r \in A\}.$$

The definition of a quasi-representation implies that the set B contains the empty relation $\varnothing$ and the identity relation id_U, and is closed under the set-theoretic operations of union, relational composition, and relational converse. Thus, B is close to satisfying the conditions for being a subuniverse of $\mathfrak{Re}(U)$—and therefore the universe of a set relation algebra—except that it may not be closed under the set-theoretic operation of complement. Nevertheless, it is possible to turn B into the universe of a relation algebra $\mathfrak{B}$ by defining a unary operation $-$ on B as follows:

$$-R = \varphi(-\varphi^{-1}(R))$$

for every relation R in B, where the operation $-$ on the right side of this equation is that of forming complements in $\mathfrak{A}$. In other words,

$$\varphi(r) = R \qquad \text{implies} \qquad -R = \varphi(-r).$$

This definition implies that the set B is closed under the operation $-$, so the algebra

$$\mathfrak{B} = (B\,,\cup,\,-,\,|\,,\;^{-1}\,,\,id_U\,)$$

is well-defined, and φ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. In particular, $\mathfrak{B}$ must also be a relation algebra. In general, $\mathfrak{B}$ is not a set relation algebra, but it is very close to being one: its universe consists of binary relations on the set U, and its operations of addition, relative multiplication, and converse coincide the the set-theoretic operations of union, relational composition, and relational converse. Moreover, its zero and identity element are the empty relation and the identity relation on U respectively, and its unit relation $E = \varphi(1)$ is an equivalence relation on the set U, because

$$id_U = \varphi(1') \subseteq \varphi(1) = E,$$

and

$$E^{-1} \,|\, E = \varphi(1)^{-1} \,|\, \varphi(1) = \varphi(1^{\smile}\,;1) = \varphi(1\,;1) = \varphi(1) = E,$$

by the definition of E, the isomorphism properties of φ, Boolean algebra, and Lemmas 4.1(vi), 4.5(iv), and 5.8(iv) (with E in place of r). Only the operation of complement in $\mathfrak{B}$, and consequently also the defined operations of multiplication and relative addition, do not coincide with the set-theoretic versions of these operations.

The most important result concerning quasi-representations is that every atomic relation algebra has one.

Theorem 17.31. *Every atomic relation algebra has a complete quasi-representation.*

Proof. Let $\mathfrak{A}$ be an atomic relation algebra, and let U be the set of atoms in $\mathfrak{A}$. Define a function φ from the universe of $\mathfrak{A}$ into the full set relation algebra $\mathfrak{Re}(U)$ by putting

$$\varphi(r) = \{(a,b) : a,b \in U \text{ and } b \leq a\,;r\}.$$

for each r in $\mathfrak{A}$. The goal is to show that φ is a quasi-representation. Clearly,

$$\varphi(0) = \varnothing \qquad \text{and} \qquad \varphi(1') = id_U. \tag{1}$$

Indeed, $a\,;0 = 0$ for every atom a, by Corollary 4.17, so there are no atoms a and b with the property that $b \leq a\,;0$. Thus, the first equation

in (1) holds. Also, $a\,;1' = a$ for every atom a, by the identity law for relative multiplication, so for all atoms a and b we have $b \leq a\,;1'$ if and only if $a = b$. Thus, the second equation in (1) holds.

To see that φ preserves all existing suprema as unions, suppose r is the supremum of a set X of elements in $\mathfrak{A}$. The distributive law for relative multiplication over arbitrary sums (Lemma 4.16) implies that

$$a\,;r = \sum\{a\,;s : s \in X\}.$$

Consequently, for atoms a and b we have

$$b \leq a\,;r \qquad \text{if and only if} \qquad b \leq a\,;s$$

for some s in X, by Boolean algebra, and therefore

$$\begin{aligned}
(a,b) \in \varphi(r) \qquad & \text{if and only if} \qquad b \leq a\,;r, \\
& \text{if and only if} \qquad b \leq a\,;s \text{ for some } s \text{ in } X, \\
& \text{if and only if} \qquad (a,b) \in \varphi(s) \text{ for some } s \text{ in } X, \\
& \text{if and only if} \qquad (a,b) \in \textstyle\bigcup_{s\in X} \varphi(s),
\end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of the union of a system of elements. Thus,

$$\varphi(r) = \textstyle\bigcup_{s\in X} \varphi(s).$$

To check that φ preserves the operation of converse, observe that for atoms a and b,

$$\begin{aligned}
a \leq b\,;r \qquad & \text{if and only if} \qquad a \cdot (b\,;r) \neq 0, \\
& \text{if and only if} \qquad b \cdot (a\,;r^{\smile}) \neq 0, \\
& \text{if and only if} \qquad b \leq a\,;r^{\smile},
\end{aligned}$$

by Boolean algebra, the assumption that a and b are atoms, and the equivalence of (i) and (iii) in Lemma 4.8 (with a, b, and r in place of t, r, and s respectively). Consequently,

$$\begin{aligned}
(a,b) \in \varphi(r^{\smile}) \qquad & \text{if and only if} \qquad b \leq a\,;r^{\smile}, \\
& \text{if and only if} \qquad a \leq b\,;r, \\
& \text{if and only if} \qquad (b,a) \in \varphi(r), \\
& \text{if and only if} \qquad (a,b) \in \varphi(r)^{-1},
\end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of the converse of a relation. Thus,

$$\varphi(r^{\smile}) = \varphi(r)^{-1}.$$

The proof that φ preserves the operation of relative multiplication is somewhat more involved. Consider an atom a and an arbitrary element r, and let X be the set of atoms below $a\,;r$. Since $\mathfrak{A}$ is assumed to be atomic, every element is the sum of the atoms that it dominates, so $a\,;r = \sum X$. The distributive law for relative multiplication over arbitrary sums implies that

$$\begin{aligned} a\,;r\,;s = (\textstyle\sum X)\,;s &= \textstyle\sum\{c\,;s : c \in X\} \\ &= \textstyle\sum\{c\,;s : c \in U \text{ and } c \leq a\,;r\}. \end{aligned}$$

For every atom b, it follows that

$$b \leq a\,;r\,;s \qquad \text{if and only if} \qquad b \leq c\,;s \quad \text{and} \quad c \leq a\,;r$$

for some c in U. Consequently, letting c vary over the atoms in U, we have

$$\begin{aligned} &(a,b) \in \varphi(r\,;s) \\ &\qquad \text{if and only if} \quad b \leq a\,;r\,;s, \\ &\qquad \text{if and only if} \quad b \leq c\,;s \text{ and } c \leq a\,;r \text{ for some } c, \\ &\qquad \text{if and only if} \quad (c,b) \in \varphi(s) \text{ and } (a,c) \in \varphi(r) \text{ for some } c, \\ &\qquad \text{if and only if} \quad (a,b) \in \varphi(r) \,|\, \varphi(s), \end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of relational composition. Thus,

$$\varphi(r\,;s) = \varphi(r) \,|\, \varphi(s).$$

It remains to check that φ is one-to-one. Consider elements r and s in $\mathfrak{A}$. Let X be the set of atoms below 1', and observe that $1' = \sum X$, by the assumption that $\mathfrak{A}$ is atomic. Consequently,

$$r = 1'\,;r = (\textstyle\sum X)\,;r = \textstyle\sum\{a\,;r : a \in U \text{ and } a \leq 1'\},$$

by the identity law for relative multiplication and the distributive law for relative multiplication over arbitrary sums. For each atom b, we therefore have

$$b \leq r \qquad \text{if and only if} \qquad b \leq a\,;r$$

for some atom $a \leq 1'$. A similar argument leads to the conclusion that

$$b \leq s \qquad \text{if and only if} \qquad b \leq a\,;s$$

for some atom $a \leq 1'$. Suppose now that $\varphi(r) = \varphi(s)$. From the definition of φ we see that for any two atoms a and b,

$$b \leq a\,;r \qquad \text{if and only if} \qquad b \leq a\,;s.$$

Combine these three equivalences to arrive at the conclusion that for any atom b,

$$b \leq r \qquad \text{if and only if} \qquad b \leq s.$$

The elements r and s are the sums of the atoms that they dominate, by the assumed atomicity of $\mathfrak{A}$, so the last equivalence implies that $r = s$. □

The definition of the quasi-representation in the preceding proof may be viewed as a natural extension to atomic relation algebras of the definition of the Cayley representation of a group complex algebra $\mathfrak{Cm}(G)$. Recall that the atoms in $\mathfrak{Cm}(G)$ may be identified with the group elements in G, and the elements in $\mathfrak{Cm}(G)$ are arbitrary subsets of G. The Cayley representation of $\mathfrak{Cm}(G)$ is the function φ that is defined on elements f in G by

$$\varphi(f) = \{(g, g \circ f) : g \in G\} = \{(g, h) : g \in G \text{ and } h = g \circ f\},$$

and extended to arbitrary elements X in $\mathfrak{Cm}(G)$ by requiring

$$\begin{aligned}\varphi(X) = \{(g, g \circ f) : g \in G \text{ and } f \in X\} \\ = \{(g, h) : g, h \in G \text{ and } h \leq g\,;X\},\end{aligned}$$

where $g\,;X = \{g \circ f : f \in X\}$. By respectively replacing the atoms g and h, and the element X, in the complex algebra $\mathfrak{Cm}(G)$ with the atoms a and b, and the element r, in the atomic relation algebra $\mathfrak{A}$, we arrive at the definition of the quasi-representation given in the preceding proof.

Every relation algebra $\mathfrak{A}$ is a subalgebra of an atomic relation algebra, namely its canonical extension $\mathfrak{B}$, by Theorem 14.35. The atomic

algebra $\mathfrak{B}$ has a complete quasi-representation φ, by the preceding theorem, and the restriction of φ to $\mathfrak{A}$ is a quasi-representation (but not necessarily a complete quasi-representation) of $\mathfrak{A}$, by the analogue of Lemma 16.11 for quasi-representations. Thus, we obtain the following *Quasi-representation Theorem* for relation algebras.

Theorem 17.32. *Every relation algebra is quasi-representable.*

This theorem implies a partial positive solution to the fundamental problem of whether an equation that is true in all set relation algebras is necessarily true in all relation algebras. We saw in Section 17.5 that the general solution to this problem is negative. It is positive, however, when the equation does not contain certain symbols.

Corollary 17.33. *If a relation algebraic equation contains no occurrences of the symbols for multiplication, complement, relative addition, and diversity, and if it is valid in all full algebras of relations on sets, then it is valid in all relation algebras.*

Proof. Suppose Γ is an equation of the specified form, and assume that Γ is valid in all full algebras of relations on sets. The equation must then be valid in all full algebras of relations on equivalence relations, because all such algebras are isomorphic to direct products of full algebras of relations on sets, by Theorem 11.43, and the validity of an equation is always preserved under the passage to direct products, by Lemma 11.17(ii). An arbitrary relation algebra $\mathfrak{A}$ is always quasi-representable as a subalgebra, with respect to the elements 0, 1, 1', and the operations $+$, $;$, and $^\smile$, of a full algebra of relations on some equivalence relation E, by Theorem 17.32. Since Γ is valid in $\mathfrak{Re}(E)$ and contains, besides variables, only the symbols for elements and operations with respect to which $\mathfrak{A}$ is (up to isomorphism) a subalgebra of $\mathfrak{Re}(E)$, it follows that Γ must be valid in $\mathfrak{A}$. □

17.7 Atomic relation algebras with functional atoms

Theorem 17.31 implies full representation theorems for certain specialized classes of relation algebras. One of these is the class of atomic relation algebras with functional atoms, that is to say, the class of atomic relation algebras in which all atoms are functions.

Theorem 17.34. *Every atomic relation algebra with functional atoms is completely representable.*

Proof. Assume that $\mathfrak{A}$ is an atomic relation algebra with functional atoms. Let U be the set of atoms in $\mathfrak{A}$, and let φ be the complete quasi-representation of $\mathfrak{A}$ that is defined in the proof of Theorem 17.31. Thus,

$$\varphi(r) = \{(a, b) : a, b \in U \text{ and } b \leq a\,;\, r\}$$

for every element r in $\mathfrak{A}$. Write E for the equivalence relation that is the image of the unit under the mapping φ. The proof of Theorem 17.31 and the remarks preceding that theorem show that φ is a one-to-one function from $\mathfrak{A}$ into the set relation algebra $\mathfrak{Re}(E)$ with the following properties: first, φ preserves zero, one, and the identity element in the sense that

$$\varphi(0) = \varnothing, \qquad \varphi(1) = E, \qquad \text{and} \qquad \varphi(1') = id_U;$$

second, φ preserves all existing sums as unions; and third, φ preserves the operations of relative multiplication and converse as relational composition and relational converse respectively.

The assumed functional nature of the atoms in $\mathfrak{A}$ implies that φ also preserves the operation of multiplication as intersection. To see this, let a and b be atoms, and r and s arbitrary elements, in $\mathfrak{A}$. Apply the distributive law for functions in Lemma 5.75 (with a, r, and s in place of r, s, and t respectively) to obtain

$$a\,;\,(r \cdot s) = (a\,;\,r) \cdot (a\,;\,s),$$

and consequently,

$$\begin{aligned} b \leq a\,;\,(r \cdot s) \quad & \text{if and only if} \quad b \leq (a\,;\,r) \cdot (a\,;\,s), \\ & \text{if and only if} \quad b \leq a\,;\,r \text{ and } b \leq a\,;\,s. \end{aligned}$$

It follows that

$$\begin{aligned} (a, b) \in \varphi(r \cdot s) \quad & \text{if and only if} \quad b \leq a\,;\,(r \cdot s), \\ & \text{if and only if} \quad b \leq a\,;\,r \text{ and } b \leq a\,;\,s, \\ & \text{if and only if} \quad (a, b) \in \varphi(r) \text{ and } (a, b) \in \varphi(s), \\ & \text{if and only if} \quad (a, b) \in \varphi(r) \cap \varphi(s), \end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of intersection. Thus,

$$\varphi(r \cdot s) = \varphi(r) \cap \varphi(s).$$

A function that preserves the operations of addition and multiplication, and maps zero to zero, and one to one, must preserve the operation of complement, by the remarks following the proof of Lemma 7.6. Consequently, the function φ is a complete embedding of $\mathfrak{A}$ into the set relation algebra $\mathfrak{Re}(E)$ and is therefore a complete representation of $\mathfrak{A}$. □

The full set relation algebras over equivalence relations E form a natural class of examples of atomic relation algebras with functional atoms. The atoms are just the singletons of pairs (α, β) in E. Of course, it is obvious that the algebras in this class are completely representable (in fact, the identity function on each algebra in the class is a complete representation of the algebra). The complex algebras of groups G form another class of examples. In this case, the atoms are just the singletons of elements in the group G. These algebras are completely representable, by Theorem 17.8. A complete description of the class of atomic relation algebras with functional atoms is given in the exercises.

Theorem 17.34 can be used to establish a very useful characterization of the relation algebras that are representable.

Corollary 17.35. *A relation algebra is representable if and only if it is embeddable into an atomic relation algebra with functional atoms.*

Proof. Consider a relation algebra $\mathfrak{A}$, and suppose first that $\mathfrak{A}$ is embeddable into an atomic relation algebra $\mathfrak{B}$ with functional atoms. The algebra $\mathfrak{B}$ is completely representable, by Theorem 17.34, and the restriction to $\mathfrak{A}$ of any complete representation of $\mathfrak{B}$ is a representation (though not necessarily a complete representation) of $\mathfrak{A}$, by Lemma 16.11. Consequently, $\mathfrak{A}$ is representable.

Assume now that $\mathfrak{A}$ is representable. By definition, this means that $\mathfrak{A}$ is embeddable into a full set relation algebra $\mathfrak{Re}(E)$ on some equivalence relation E. The algebra $\mathfrak{Re}(E)$ is atomic with functional atoms, by the remarks preceding the corollary, so $\mathfrak{A}$ is embeddable into an atomic relation algebra with functional atoms. □

The preceding corollary applies in particular to relation algebras in which the unit is the sum of finitely many functions.

Corollary 17.36. *If the unit of a relation algebra $\mathfrak{A}$ is the sum of a finite set of functions, then $\mathfrak{A}$ is representable.*

Proof. Assume $\mathfrak{A}$ satisfies the hypotheses of the corollary, and let $\mathfrak{B}$ be the canonical extension of $\mathfrak{A}$, the existence of which is guaranteed

by Theorem 14.35. Finite sums are preserved under the passage to canonical extensions (since $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$), so the unit of $\mathfrak{B}$ is the sum of the same finite set of functions. The algebra $\mathfrak{B}$ is atomic, by construction, and each atom in $\mathfrak{B}$, being below the unit, must be below a function, by the Boolean properties of atoms. Since subelements of functions are functions, by Lemma 5.66(ii), it follows that $\mathfrak{B}$ is an atomic relation algebra with functional atoms. Consequently, $\mathfrak{A}$ is representable, by Corollary 17.35. □

17.8 Singleton dense relation algebras

A second class of relation algebras to which Theorem 17.34 may be applied is the class of atomic relation algebras in which the atoms satisfy a slightly stronger condition than functionality. An element r in a relation algebra is said to be a *constant function* if $r \neq 0$ and

$$r^\smile ; 1 ; r \leq 1'.$$

Such an element is automatically a function, because

$$r^\smile ; r \leq r^\smile ; 1 ; r \leq 1',$$

by the first dual of Lemma 4.5(iii), the monotony law for relative multiplication, and the assumption on r. In a square set relation algebra, a non-zero element satisfies the constant function inequality if and only if it really is a constant function in the set-theoretical sense that it is a function with exactly one element in its range. A relation algebra is said to be *constant function dense* if every non-zero element is above a constant function. It is easy to see that an atomic relation algebra $\mathfrak{A}$ is constant function dense if and only if each atom in $\mathfrak{A}$ is a constant function. Indeed, every non-zero element in $\mathfrak{A}$ is above an atom, so if every atom is a constant function, then $\mathfrak{A}$ is constant function dense. Conversely, if $\mathfrak{A}$ is constant function dense, then every atom must be above a constant function and therefore every atom must in fact be a constant function.

Closely related to the notion of a constant function is the notion of a singleton. An element r is called a *singleton* if both r and its converse are constant functions, that is to say, if r is non-zero and satisfies the two inequalities

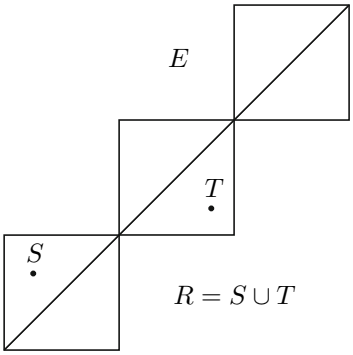


Fig. 17.18 Singleton relations S and T in a full set relation algebra $\mathfrak{Re}(E)$, and a singleton $R = S \cup T$ that is not a singleton relation.

$$r^\smile ; 1 ; r \leq 1' \qquad \text{and} \qquad r ; 1 ; r^\smile \leq 1'.$$

In square set relation algebras, singletons are just singleton relations in the sense that they consist of a single pair of elements (α, β). In set relation algebras on equivalence relations, singletons are just the non-empty unions of singleton relations over mutually distinct equivalence classes (see Figure 17.18). In a Boolean relation algebra, every non-zero element is a singleton.

A relation algebra is said to be *singleton dense* if every non-zero element is above a singleton. Examples of such algebras include the full set relation algebras on equivalence relations and Boolean relation algebras. An argument similar to the one at the end of the first paragraph of this section shows that an atomic relation algebra is singleton dense if and only if every atom is a singleton.

It is not difficult to see that the notions of constant function density and singleton density are actually equivalent. In one direction, this assertion is trivial: a singleton dense relation algebra is always constant function dense, because every singleton is a constant function. To establish the reverse implication, assume that a relation algebra $\mathfrak{A}$ is constant function dense, and consider an arbitrary non-zero element r in $\mathfrak{A}$, with the goal of showing that there must be a singleton below r. The element $r^\smile$ is also non-zero, so the assumption of constant function density implies the existence of a constant function s below $r^\smile$. The element $s^\smile$ is non-zero as well, so there is a constant function t below $s^\smile$. Of course, $t^\smile ; 1 ; t \leq 1'$, because t is a constant function; and

$$t\,;1\,;t^{\smile} \leq s^{\smile}\,;1\,;s^{\smile\smile} = s^{\smile}\,;1\,;s \leq 1',$$

by the assumption that t is below $s^{\smile}$, the monotony laws for relative multiplication and converse, the first involution law, and the assumption that s is a constant function. Since t is non-zero, it follows that t is a singleton. Moreover, t is below r because

$$t \leq s^{\smile} \leq r^{\smile\smile} = r.$$

Thus, every non-zero element in $\mathfrak{A}$ is above a singleton, so $\mathfrak{A}$ is singleton dense.

Every singleton is automatically a function, so Theorem 17.34 immediately yields the following corollary.

Corollary 17.37. *Every atomic relation algebra with singleton atoms is completely representable.*

This corollary implies a somewhat different characterization of representable relation algebras than the one given in Corollary 17.35.

Corollary 17.38. *A relation algebra is representable if and only if it is embeddable into an atomic relation algebra with singleton atoms.*

The proof is similar to that of Corollary 17.35. The reference to Theorem 17.34 must be replaced by a reference to Corollary 17.37, and references to functional atoms must be replaced by references to singleton atoms.

The hypothesis of atomicity in Corollary 17.37 is superfluous in the case of simple relation algebras.

Lemma 17.39. *In a simple relation algebra, every singleton is an atom. Consequently, a singleton dense relation algebra that is simple is necessarily atomic with singleton atoms and is therefore completely representable.*

Proof. Let $\mathfrak{A}$ be a simple, singleton dense relation algebra, and consider an arbitrary singleton r in $\mathfrak{A}$. The first step is to show that the domain of r—call it x—is also a singleton. Indeed,

$$x\,;1 = r\,;1 \qquad \text{and} \qquad 1\,;x^{\smile} = 1\,;r^{\smile}, \tag{1}$$

by Lemma 5.48, the second involution law, and Lemma 4.1(vi), and consequently

$$x\,;1\,;x^\smile = x\,;1\,;1\,;x^\smile = r\,;1\,;1\,;r^\smile = r\,;1\,;r^\smile \leq 1', \qquad (2)$$

by Lemma 4.5(iv), (1), and the assumption that r is a singleton. Every subidentity element is its own converse, by Lemma 5.20(i), so $x = x^\smile$ and therefore

$$x\,;1\,;x = x^\smile\,;1\,;x = x\,;1\,;x^\smile \leq 1',$$

by (2). In particular, x is a singleton.

According to Corollary 13.8, in simple relation algebras, singletons below the identity element must be atoms, so the observations of the preceding paragraph imply that x is an atom. Use the functionality of the singleton r, and apply Lemma 5.86(i), to conclude that r itself is an atom.

It has been shown that every singleton in $\mathfrak{A}$ is an atom. Since $\mathfrak{A}$ is assumed to be singleton dense, every non-zero element is above a singleton, and therefore every non-zero element is above an atom. Consequently, $\mathfrak{A}$ is atomic, and therefore completely representable, by Corollary 17.37. □

It is possible to give a complete description of all atomic relation algebras with singleton atoms. The phrase "complete description" needs a word of explanation. The intention is that the description be complete up to isomorphism. However, two relation algebras may fail to be isomorphic simply because certain sums of infinite sets exist in one algebra, but not in the other. Such "incompleteness" is a relatively minor defect that may be overcome by passing to the completions of the algebras. We shall say that two relation algebras are *essentially isomorphic* if their completions are isomorphic. The description of all atomic relation algebras with singleton atoms will be complete up to essential isomorphism, in the sense that it will not distinguish between essentially isomorphic algebras.

Before taking up the task of giving a description of atomic relation algebras with singleton atoms, it is helpful to prove a series of rather easy but useful lemmas. The first lemma says that the property of being a singleton is closed under formation of subelements, subalgebras, non-zero homomorphic images, and direct products.

Lemma 17.40. *Let $\mathfrak{A}$ be a relation algebra.*

(i) *If r is a singleton in $\mathfrak{A}$, then every non-zero element s that is below r is a singleton in every subalgebra of $\mathfrak{A}$ that contains s.*

(ii) *If r is a singleton in $\mathfrak{A}$, and φ a homomorphism on $\mathfrak{A}$, then $\varphi(r)$ is either a singleton or zero in the image algebra.*

(iii) *If $\mathfrak{A}$ is the product of a system of relation algebras, then r is a singleton in $\mathfrak{A}$ if and only if each component of r in each factor algebra is either zero or a singleton, and at least one component is a singleton.*

Proof. To prove part (i), assume that s is a non-zero element below a singleton r in $\mathfrak{A}$. The monotony laws for converse and relative multiplication, and the assumption on r imply that

$$s^\smile ; 1 ; s \leq r^\smile ; 1 ; r \leq 1' \qquad \text{and} \qquad s ; 1 ; s^\smile \leq r ; 1 ; r^\smile \leq 1',$$

so s satisfies the conditions for being a singleton. These conditions continue to hold in all subalgebras of $\mathfrak{A}$ to which s belongs, because quantifier-free statements are preserved under the passage to subalgebras, by Lemma 6.2.

Turn now to the proof of part (ii). Suppose φ is a homomorphism on $\mathfrak{A}$, and s is the image under φ of a singleton r in $\mathfrak{A}$. Since

$$r^\smile ; 1 ; r \leq 1' \qquad \text{and} \qquad r ; 1 ; r^\smile \leq 1', \tag{1}$$

we have

$$\begin{aligned} s^\smile ; 1 ; s &= \varphi(r)^\smile ; \varphi(1) ; \varphi(r) = \varphi(r^\smile ; 1 ; r) \leq \varphi(1') = 1', \\ s ; 1 ; s^\smile &= \varphi(r) ; \varphi(1) ; \varphi(r)^\smile = \varphi(r ; 1 ; r^\smile) \leq \varphi(1') = 1', \end{aligned}$$

by the homomorphism properties of φ. Therefore, either s is zero or these inequalities imply that s is a singleton.

For the proof of (iii), recall that an element $r = (r_i : i \in I)$ in a product satisfies a set of equations—and in particular the equations in (1)—in $\mathfrak{A}$ if and only if each component r_i satisfies the same equations in the corresponding factor algebra, by Lemma 11.17(i). Also, r is non-zero if and only if at least one of its components is non-zero. These two observations and the definition of a singleton immediately yield (iii). □

Lemma 17.41. *If a relation algebra $\mathfrak{A}$ is the product of a system of relation algebras, then $\mathfrak{A}$ is singleton dense if and only if each factor algebra is singleton dense.*

Proof. Let $\mathfrak{A}$ be the internal product of a system $(\mathfrak{A}_i : i \in I)$ of relation algebras. There must be an orthogonal system $(u_i : i \in I)$ of ideal elements in $\mathfrak{A}$ such that

$$\mathfrak{A}_i = \mathfrak{A}(u_i) \tag{1}$$

for each index i, by the Product Decomposition Theorem 11.39.

Assume first that $\mathfrak{A}$ is singleton dense, and consider an arbitrary non-zero element r in one of the factor algebras $\mathfrak{A}_i$. The element r belongs to $\mathfrak{A}$ as well, by (1), so there must be a singleton s in $\mathfrak{A}$ that is below r, by the assumed singleton density of $\mathfrak{A}$. Since $s \leq r \leq u$, the element s belongs to $\mathfrak{A}_i$, by (1), and s remains a singleton in $\mathfrak{A}_i$, by Lemma 17.40(iii). Conclusion: each non-zero element in a factor algebra $\mathfrak{A}_i$ is above a singleton, so each factor algebra is singleton dense.

Assume now that each factor algebra $\mathfrak{A}_i$ is singleton dense, and consider a non-zero element r in $\mathfrak{A}$. At least one of the components of r, say r_i, must be non-zero, so there must be a singleton s in the corresponding factor algebra $\mathfrak{A}_i$ that is below r_i, by the assumed singleton density of the factor algebras. The element s remains a singleton in $\mathfrak{A}$, by Lemma 17.40(iii), and $s \leq r_i \leq r$, so s is below r in $\mathfrak{A}$. Conclusion: each non-zero element in $\mathfrak{A}$ is above a singleton, so $\mathfrak{A}$ is singleton dense. □

Lemma 17.42. *A relation algebra is singleton dense if and only if its completion is singleton dense.*

Proof. Let $\mathfrak{A}$ be a relation algebra, and $\mathfrak{B}$ the completion of $\mathfrak{A}$. Suppose first that $\mathfrak{A}$ is singleton dense. Consider an arbitrary non-zero element p in $\mathfrak{B}$. This element is above some non-zero element r in $\mathfrak{A}$, by the definition of a completion, and r is above a singleton s in $\mathfrak{A}$, by the assumption that $\mathfrak{A}$ is singleton dense. The element s remains a singleton in $\mathfrak{B}$, by Lemma 17.40(i), and $s \leq r \leq p$, so p is above the singleton s in $\mathfrak{B}$. Consequently, $\mathfrak{B}$ is singleton dense.

Suppose now that $\mathfrak{B}$ is singleton dense. Consider an arbitrary non-zero element r in $\mathfrak{A}$. The element r belongs to $\mathfrak{B}$, and is therefore above some singleton s in $\mathfrak{B}$, by the assumption that $\mathfrak{B}$ is singleton dense. Because $\mathfrak{B}$ is the completion of $\mathfrak{A}$, there must be a non-zero element t in $\mathfrak{A}$ that is below s. It follows from Lemma 17.40(i) that t is a singleton in $\mathfrak{A}$, and obviously $t \leq s \leq r$, so the element r is above the singleton t in $\mathfrak{A}$. Thus, $\mathfrak{A}$ is singleton dense. □

The next lemma characterizes when a relation algebra is singleton dense.

Lemma 17.43. *The following conditions on a relation algebra* $\mathfrak{A}$ *are equivalent.*

(i) $\mathfrak{A}$ *is singleton dense.*
(ii) *The unit is a sum of singletons.*
(iii) *The unit is the sum of the set of all singletons.*
(iv) *Every element is a sum of singletons.*
(v) *Every element is the sum of the set of all singletons that are below it.*

Proof. To establish the implication from (i) to (v), assume that $\mathfrak{A}$ is singleton dense, let r be an arbitrary element in $\mathfrak{A}$, and write X for the set of all singletons in $\mathfrak{A}$ that are below r. The element r is certainly an upper bound for the set X. If s is any other upper bound for X, then $r \cdot -s$ must be 0; for otherwise there would be a singleton below this product, by the assumed singleton density of $\mathfrak{A}$, and consequently there would be a singleton in X that is not below s, in contradiction to the assumption that s is an upper bound of X. It follows that $r \leq s$, so r is the least upper bound of the set X.

The implications from (v) to (iv), from (iv) to (iii), and from (iii) to (ii) are obviously true. To establish the implication from (ii) to (i), argue by contraposition. Assume $\mathfrak{A}$ is not singleton dense. There must then be a non-zero element r in $\mathfrak{A}$ that is not above a singleton. Consider an arbitrary singleton s in $\mathfrak{A}$. The product $s \cdot r$ is below the singleton s, so it is either a singleton or 0, by Lemma 17.40(i). Since there are no singletons below r, by assumption, it follows that $s \cdot r = 0$, and therefore $s \leq -r$. Conclusion: $-r$ is above every singleton in $\mathfrak{A}$, so $-r$ is an upper bound of the set of all singletons in $\mathfrak{A}$. Since $-r$ is strictly below 1, because r is non-zero, it follows that 1 cannot be the least upper bound of some set of singletons. □

We return to the problem of describing the atomic relation algebras with singleton atoms, and we begin with the special case of simple algebras.

Theorem 17.44. *A relation algebra is simple and singleton dense if and only it is essentially isomorphic to a full set relation algebra on some non-empty set.*

Proof. Consider a relation algebra $\mathfrak{A}$, and let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. Assume first that $\mathfrak{B}$ is isomorphic to a full set relation algebra on some non-empty set. In this case, $\mathfrak{B}$ must be a simple and atomic with singleton atoms, because this is true of all full relation algebras on non-empty sets. Subalgebras of simple algebras are simple, by Corollary 9.3, so $\mathfrak{A}$, as a subalgebra of $\mathfrak{B}$, is certainly simple. Furthermore, $\mathfrak{A}$ and $\mathfrak{B}$ have the same atoms, by Lemma 15.29, so $\mathfrak{A}$ must also be atomic with singleton atoms, by Lemma 17.40(i). Conclusion: $\mathfrak{A}$ is simple and singleton dense.

To establish the reverse implication, assume that $\mathfrak{A}$ is simple and singleton dense, and observe that its completion $\mathfrak{B}$ must also be simple and singleton dense, by Theorem 15.30 and Lemma 17.42. Consequently, $\mathfrak{B}$ is completely representable, by Lemma 17.39, so there must in fact be a complete square representation of $\mathfrak{B}$ over some non-empty set U, by Lemma 16.5. In other words, there must be a complete embedding φ of $\mathfrak{B}$ into $\mathfrak{Re}(U)$. Monomorphisms preserve the property of being a singleton, by Lemma 17.40(ii), so φ must map the set of singletons in $\mathfrak{B}$ into the set of singletons in $\mathfrak{Re}(U)$.

The unit in $\mathfrak{B}$ is the sum of the set of singletons, by Lemma 17.43 and the fact that $\mathfrak{B}$ is singleton dense. Notice that the singletons in $\mathfrak{B}$ are just the atoms in $\mathfrak{B}$, by Lemma 17.39. Since φ is a complete monomorphism, it follows that the unit in $\mathfrak{Re}(U)$ must be the union of the set of singleton relations that are images under φ of the singletons in $\mathfrak{B}$. The singleton relations in $\mathfrak{Re}(U)$ are just the atoms of $\mathfrak{Re}(U)$, and the unit in $\mathfrak{Re}(U)$ is the disjoint union of the set of these atoms, so φ must map the set of singletons in $\mathfrak{B}$ onto the set of singleton relations in $\mathfrak{Re}(U)$. Put another way, φ must map the set of atoms in $\mathfrak{B}$ onto the set of atoms in $\mathfrak{Re}(U)$.

Here is a summary of what has been established. There is a complete monomorphism φ from $\mathfrak{B}$ to $\mathfrak{Re}(U)$ that maps the set of atoms in $\mathfrak{B}$ bijectively to the set of atoms in $\mathfrak{Re}(U)$. Both algebras are complete, so the suprema of arbitrary subsets exist in both algebras. Apply the Atomic Isomorphism Theorem 7.11 to conclude that φ must map $\mathfrak{B}$ isomorphically onto $\mathfrak{Re}(U)$. Since $\mathfrak{B}$ is the completion of $\mathfrak{A}$ (and $\mathfrak{Re}(U)$ is its own completion), it follows that $\mathfrak{A}$ is essentially isomorphic to $\mathfrak{Re}(U)$. □

With the preceding theorem in hand, the description of atomic relation algebras with singleton atoms is not difficult.

Theorem 17.45. *A relation algebra is atomic with singleton atoms if and only if it is essentially isomorphic to a full set relation algebra on some equivalence relation.*

Proof. Consider a relation algebra $\mathfrak{A}$, and let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. Assume first that $\mathfrak{B}$ is isomorphic to a full set relation algebra on some equivalence relation. In this case, $\mathfrak{B}$ must be atomic with singleton atoms, because this is true of every full relation algebra on an equivalence relation. Since $\mathfrak{B}$ is the completion of $\mathfrak{A}$, it follows from Lemma 15.29 that $\mathfrak{A}$ is atomic and has the same atoms as $\mathfrak{B}$. Consequently, $\mathfrak{A}$ is atomic with singleton atoms, by Lemma 17.40(i).

To establish the reverse implication, assume that $\mathfrak{A}$ is atomic with singleton atoms, and observe that its completion $\mathfrak{B}$ must also be atomic with singleton atoms, by Lemmas 15.29 and 17.40(i). As a complete and atomic relation algebra, $\mathfrak{B}$ has a unique internal decomposition into an internal product of simple factors, by Corollary 11.45. Each of these simple factors is complete, by Corollary 11.38, and each of them is singleton dense, by Lemma 17.41. Apply Theorem 17.44 to see that each factor is isomorphic to a full relation algebra on a non-empty set. Without loss of generality, it may be assumed that the base sets of these full set relation algebras are mutually disjoint (see the relevant remarks in Section 16.2). Apply Lemma 11.48 to conclude that $\mathfrak{B}$ is isomorphic to the internal product of a (possibly empty) system

$$(\mathfrak{Re}(U_i) : i \in I) \tag{1}$$

of full set relation algebras on non-empty, mutually disjoint sets.

Let E be the equivalence relation whose equivalence classes are the base sets U_i, so that

$$E = \bigcup_{i \in I} U_i \times U_i.$$

The full set relation algebra $\mathfrak{Re}(E)$ is equal to the internal product of the system (1), by Decomposition Theorem 11.43, and therefore $\mathfrak{B}$ is isomorphic to $\mathfrak{Re}(E)$, by the observations of the preceding paragraph. Conclusion: the completion of $\mathfrak{A}$ is isomorphic to $\mathfrak{Re}(E)$ (and $\mathfrak{Re}(E)$ is its own completion), so $\mathfrak{A}$ is essentially isomorphic to a full set relation algebra on some equivalence relation. □

It is natural to ask whether every singleton dense relation algebra, atomic or not, is representable. The answer turns out to be positive,

but in general the representation is not complete because of the restriction imposed by Theorem 16.7. The next lemma contains the key idea of the proof.

Lemma 17.46. *If a singleton dense relation algebra is complete, then every non-zero element r is above a singleton that generates the same ideal element as r.*

Proof. Let $\mathfrak{A}$ be a complete, singleton dense relation algebra, and r a fixed non-zero element in $\mathfrak{A}$. Consider the collection $\mathcal{X}$ of all those subsets X of $\mathfrak{A}$ with the the following properties: (1) each element in X is a singleton that is below r; and (2) distinct elements in X generate disjoint ideal elements in $\mathfrak{A}$. Such sets certainly exist. Indeed, the algebra $\mathfrak{A}$ is assumed to be singleton dense, so there is at least one singleton s below r; the set $X = \{s\}$ satisfies (1) and (2), and is therefore in $\mathcal{X}$. Also, it is clear that $\mathcal{X}$ is partially ordered by the relation of set-theoretic inclusion. Finally, $\mathcal{X}$ is closed under unions of chains. To see this, consider an arbitrary chain in $\mathcal{X}$. The union of this chain clearly satisfies (1). To check that it also satisfies (2), suppose s and t are distinct elements in the union. There must then exists two sets in the chain that contain s and t respectively, by the definition of union. The larger of these two sets contains both s and t, and since this larger set is assumed to satisfy (2), it follows that s and t must generate disjoint ideal elements.

The preceding argument shows that the hypothesis of Zorn's Lemma is satisfied. Apply the lemma to conclude that $\mathcal{X}$ has a maximal set X_0. The supremum r_0 of this set exists in $\mathfrak{A}$, because $\mathfrak{A}$ is assumed to be complete. Notice that r_0 is below r, because r is an upper bound of X_0, by (1), and r_0 is the least upper bound of X_0.

The next step is to show that r_0 is a singleton. Observe first that if s and t are distinct elements in X_0, then

$$(1 ; s ; 1) \cdot t \leq (1 ; s ; 1) \cdot (1 ; t ; 1) = 0,$$

by Lemma 4.5(iii) and its first dual, and property (2) of the set X_0. Consequently,

$$0 = [(1 ; s)^{\smile} ; t] \cdot 1 = (1 ; s)^{\smile} ; t = s^{\smile} ; 1^{\smile} ; t = s^{\smile} ; 1 ; t,$$

by the De Morgan-Tarski laws (Lemma 4.8, with $1 ; s$ and 1 in place of r and s respectively), Boolean algebra, the second involution law, and Lemma 4.1(vi).

Use the equality of the first and last terms, the definition of r_0, the distributive laws for converse and relative multiplication over arbitrary sums, and the fact that X_0 consists of singletons to arrive at

$$\begin{aligned} r_0^\smile ; 1 ; r_0 = (\sum X_0)^\smile ; 1 ; (\sum X_0) &= \sum\{s^\smile ; 1 ; t : s, t \in X_0\} \\ &= \sum\{s^\smile ; 1 ; s : s \in X_0\} \leq 1'. \end{aligned}$$

A completely analogous argument shows that

$$r_0 ; 1 ; r_0^\smile \leq 1'.$$

Since r_0 is not zero (because $\mathcal{X}$ contains non-empty sets), it follows that r_0 is a singleton, as claimed.

It remains to show that r_0 and r generate the same ideal element. Certainly,

$$1 ; r_0 ; 1 \leq 1 ; r ; 1,$$

by the monotony law for relative multiplication and the fact that r_0 is below r. Assume, for a contradiction, that this inequality is strict, so that

$$(1 ; r ; 1) \cdot -(1 ; r_0 ; 1) \neq 0,$$

by Boolean algebra. The element $-(1 ; r_0 ; 1)$ is an ideal element, by Lemma 5.39(iv), so the preceding inequality implies that

$$r \cdot -(1 ; r_0 ; 1) \neq 0,$$

by Lemma 5.46 (with r replaced by $-(1 ; r_0 ; 1)$, and s by r). Invoke the assumed singleton density of $\mathfrak{A}$ to obtain a singleton s that is below the product on the left side of this last inequality. Since the singleton s is below $-(1 ; r_0 ; 1)$, it is disjoint from the ideal element $1 ; r_0 ; 1$, and therefore $1 ; s ; 1$ is also disjoint from $1 ; r_0 ; 1$, by Lemma 5.46 (with $1 ; r_0 ; 1$ in place of r). Consequently, $1 ; s ; 1$ is disjoint from all of the ideal elements generated by elements in the set X_0, because these ideal elements are all below $1 ; r_0 ; 1$. Since the singleton s is also below r, the set $X_0 \cup \{s\}$ satisfies both properties (1) and (2) above, and therefore belongs to the collection $\mathcal{X}$. The maximality of X_0 in $\mathcal{X}$ now implies that s is in X_0. But then s must be below the supremum of X_0, which is r_0. This is impossible because $1 ; s ; 1$ is disjoint from $1 ; r_0 ; 1$. The desired contradiction has arrived. The strict inequality that was assumed above must be false, so the ideal elements $1 ; r_0 ; 1$ and $1 ; r ; 1$ are equal, as claimed. □

Theorem 17.47. *Every singleton dense relation algebra is representable.*

Proof. Consider a singleton dense relation algebra $\mathfrak{A}$, and let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. The algebra $\mathfrak{B}$ is also singleton dense, by Lemma 17.42. We proceed to use the completeness of $\mathfrak{B}$ to show that $\mathfrak{B}$ is representable. Since $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, it may then be concluded that $\mathfrak{A}$ is representable as well, by Lemma 16.11.

The algebra $\mathfrak{B}$ is a subdirect product of a system $(\mathfrak{B}_i : i \in I)$ of simple relation algebras, by the Semi-simplicity Theorem 12.10. Each factor algebra $\mathfrak{B}_i$ is singleton dense. To see this, let s be an arbitrary non-zero element in one of the factors $\mathfrak{B}_i$. The factor $\mathfrak{B}_i$ is a homomorphic image of $\mathfrak{B}$ via some epimorphism φ_i, by the definition of a subdirect product, so there must be a non-zero element r in $\mathfrak{B}$ that is mapped to s by φ_i. The algebra $\mathfrak{B}$ is singleton dense and complete, so there is a singleton r_0 in $\mathfrak{B}$ that is below r and that generates the same ideal element as r, by Lemma 17.46. The image element $s_0 = \varphi_i(r_0)$ is a singleton below s. Indeed, s_0 is non-zero because it generates a non-zero ideal element, namely the same ideal element that is generated by s:

$$\begin{aligned} 1 \,;\, s_0 \,;\, 1 &= \varphi_i(1) \,;\, \varphi_i(r_0) \,;\, \varphi_i(1) = \varphi_i(1 \,;\, r_0 \,;\, 1) \\ &= \varphi_i(1 \,;\, r \,;\, 1) = \varphi_i(1) \,;\, \varphi_i(r) \,;\, \varphi_i(1) = 1 \,;\, s \,;\, 1 \geq s > 0, \end{aligned}$$

by the definitions of the elements s_0, r_0, and r, the homomorphism properties of φ_i, and the assumption on s. (Notice how the property that r_0 generates the same ideal element as r plays a key role in ensuring that the image of r_0 under φ_i is nonzero.) A non-zero homomorphic image of a singleton is again a singleton, by Lemma 17.40(ii). Since r_0 is a singleton below r, and its image s_0 under the homomorphism φ_i is non-zero, it follows that s_0 is a singleton below the image s of r under φ_i.

It has been shown that each factor algebra $\mathfrak{B}_i$ is simple and singleton dense. Apply Lemma 17.39 to conclude that each factor algebra is completely representable. The direct product of these factor algebras is also completely representable, by the complete version of Corollary 16.20. Since $\mathfrak{B}$ is isomorphic to a subalgebra of this direct product, it follows that $\mathfrak{B}$ must be representable as well (but not necessarily completely representable), by Lemma 16.11. □

A complete description of all singleton dense relation algebras is currently not known, but the following *Structure Theorem* for singleton

dense relation algebras reduces the problem to one of describing all singleton dense relation algebras with an atomless Boolean algebra of ideal elements.

Theorem 17.48. *Every singleton dense relation algebra is essentially isomorphic to the direct product of two complete, singleton dense relation algebras, one of which is atomic and the other is atomless and has an atomless Boolean algebra of ideal elements.*

Proof. Consider a singleton dense relation algebra $\mathfrak{A}$, and let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. It suffices to show that $\mathfrak{B}$ has the desired decomposition. The algebra $\mathfrak{B}$ is singleton dense, by Lemma 17.42. According to the Complete Decomposition Theorem 11.46, $\mathfrak{B}$ is the internal product of two complete relation algebras $\mathfrak{B}_0$ and $\mathfrak{B}_1$ with the property that $\mathfrak{B}_0$ is an internal product of simple relation algebras, and $\mathfrak{B}_1$ is atomless with an atomless Boolean algebra of ideal elements (and hence no simple factors at all). Both of these factors are singleton dense, by Lemma 17.41 (with $\mathfrak{B}$ in place of $\mathfrak{A}$). Each of the internal simple factors of $\mathfrak{B}_0$ is also singleton dense, by Lemma 17.41 (with $\mathfrak{B}_0$ in place of $\mathfrak{A}$). A simple, singleton dense relation algebra is atomic, by Lemma 17.39, so $\mathfrak{B}_0$ is the internal product of atomic relation algebras, and therefore $\mathfrak{B}_0$ is itself atomic, by Corollary 11.37. □

It may happen that in the decomposition of the preceding theorem, one or more of the factors is degenerate. For example, if the given relation algebra $\mathfrak{A}$ is atomic, then its completion $\mathfrak{B}$ is atomic and therefore has no non-degenerate atomless factors. Consequently, the atomless factor $\mathfrak{B}_1$ in the proof of the theorem is degenerate. On the other hand, if $\mathfrak{A}$ has an atomless Boolean algebra of ideal elements, then the same is true of the completion $\mathfrak{B}$, so the factor $\mathfrak{B}_0$ in the proof of the theorem is degenerate.

The theorem divides the problem of describing all singleton dense relation algebras into two tasks: describe all such algebras that are complete and atomic, and describe all such algebras that are complete and have an atomless Boolean algebra of ideal elements. The first task is accomplished in Theorem 17.45: up to essential isomorphisms, the algebras are just the full relation algebras on equivalence relations. The second task remains open. Singleton dense relation algebras with atomless Boolean algebras of ideal elements are not hard to find, however. Every atomless Boolean algebra is an example. To obtain more interesting examples, take $\mathfrak{A}$ to be any non-degenerate atomic relation algebra with singleton atoms—for example, $\mathfrak{A}$ could be the full

set relation algebra on some non-empty equivalence relation. The construction in Section 11.12 produces a subdirect power of $\mathfrak{A}$ that is singleton dense but has no ideal element atoms—see Theorem 11.47.

17.9 Historical remarks

When Tarski first proposed his axiomatization of the theory of relation algebras in [132], he raised two problems concerning the axiomatization. His *completeness problem* asked whether the axiomatization is complete in the sense that every equation true in all set relation algebras is derivable from the set of axioms. His *representation problem* asked whether every model of the set of axioms is isomorphic to an algebra of binary relations. A number of the results in this chapter grew out of the original efforts that were made to understand and solve these two problems.

The theorem that every formula relation algebra is representable (Theorem 17.1) is due to Tarski and was presented in his lectures [146]. The theorem that every representable relation algebra is embeddable into a relativization of some formula relation algebra (Theorem 17.3) is due to Tarski and Givant, and grew out of some observations made by Tarski in [146].

The theorem that every Boolean algebra is representable as an algebra of sets is due to Stone [127], and the result given in Theorem 17.5 that every Boolean relation algebra is representable is an adaptation by Tarski of Stone's theorem to the class of Boolean relation algebras. Similarly, the theorem that every atomic Boolean algebra is completely representable is essentially due to Tarski [129], and the result given in Theorem 17.6 that every atomic Boolean relation algebra is completely representable is an adaptation by Givant of Tarski's theorem to the class of atomic Boolean relation algebras. The observations made at the end of Section 17.2 and in Exercises 17.7 and 17.10 are due to Givant.

The theorem that every group is representable via a version of the mapping given in Section 17.3 is due to Cayley [**?**], and the result contained in Theorem 17.8 that every group complex algebra is representable is an adaptation (probably by McKinsey and Tarski—see [73] and also [146]) of a slightly modified version of Cayley's theorem to the class of group complex algebras. The observation that this representa-

tion is complete and that, up to equivalence, it is the unique complete square representation of a group complex algebra (Theorem 17.10), is due independently to Hirsch and Hodkinson [59] and to Andréka and Givant (see [59], p. 135). The results in Exercises 17.18–17.20 are due to Andréka and Givant.

The idea of using projective geometries to construct representable and non-representable relation algebras is due to Jónsson [67]. The construction was further developed and refined by Lyndon [90]. In particular, the notion of an affine representation is introduced in Lyndon's paper, and Theorems 17.14 and 17.27 are proved there (see his Theorems 1 and Corollary 1.1) In the same paper, Lyndon observed in Corollary 1.3 that the construction, when applied to projective lines, yields examples of non-representable relation algebras. The use of the Bruck-Ryser Theorem to obtain infinitely many non-representable relation algebras as complex algebras of projective lines (Theorem 17.15) is due to Monk [110]. The theorem that a non-Desarguesian projective plane yields a non-representable complex algebra (Theorem 17.16) is stated without proof and in a somewhat misleading way by Lyndon as a corollary to his Theorem 1 (see his Corollary 1.2). What Lyndon probably meant to say (and what follows from his Theorem 1) is that the complex algebra of a non-Desarguesian plane has no complete representation (he omits the qualifier *complete*—see footnote 7 in Givant [37] for more details). Givant [37] adapted a proof due to Jónsson [67] to give a correct proof of Theorem 17.16. Using this result, Givant showed that a geometric complex algebra is completely representable if and only if it is representable (Theorem 17.17). The results in Lemma 17.19 through Lemma 17.26, Lemma 17.28, and Theorem 17.29 are due (in a somewhat different form) to Givant [37], and were used by him to determine the number of inequivalent representations of relation algebras constructed as complex algebras of projective lines of order at most 10 (see Exercise 17.29), and to give lower bounds on the number of inequivalent complete representations for projective lines of higher orders, for example of order 29 (see the computations at the end of Section 17.4). The results in Exercise 17.26 are due to Lyndon [90].

Small relation algebras were first investigated by Lyndon, who was motivated by Tarski's original problems. Lyndon's first example in [88] of a non-representable relation algebra is relatively large and complicated in structure, with 56 atoms; and he mentions in a footnote that he has also found an example with 52 atoms. He demonstrated the

non-representability of his algebra by showing that a certain first-order implication which is true in all set relation algebras fails to hold in his algebra. Lyndon also analyzed the relation algebras with at most three atoms. The results of this analysis were never published by him, but they are mentioned in a footnote at the end of [89]. The observation in the first example of Section 17.5, namely that the relation algebra given there has only infinite representations arising from dense linear orderings without endpoints is due to Tarski [146]. Andréka-Maddux [11] contains a careful study of all relation algebras with at most three atoms, and in particular of the various possible representations that these algebras can have. The discussion of the possible representations of the second example in Section 17.5 is due to them, as are the results implicit in Exercises 17.32 and 17.33. The material in Exercises 17.34–17.36 is also from that paper, but as is mentioned above, the results implicit in these exercises were known to Lyndon.

McKenzie, in his doctoral dissertation [106], repeated Lyndon's earlier analysis of the relation algebras with at most three atoms, and he went beyond it by studying the relation algebras with exactly four atoms. His dissertation gives a partial description of the small relation algebras with at most three atoms, stating among other things that they are all representable; and he proves that the algebra given in the fourth example of Section 17.5 is not representable. In connection with this last result, see also McKenzie [107]. Comer used a computer to study the integral relation algebras with four atoms—see [24] and [25]. The integral relation algebras with five and six atoms were studied with the aid of computers by Jipsen, Lukács, and Maddux; see [101], which contains a survey of these results including questions regarding the representability of these algebras. The representability of the algebras involved in (ii)–(iv) from Exercise 17.38 and in Exercise 17.40 is pointed out in Maddux [101], as is the non-representability of the algebras involved in Exercises 17.39 and 17.41. (The representability of the algebra involved in (i) from Exercise 17.38 is easy and has been known for a long time.) McKenzie was almost certainly aware of these representability results, but he did not publish anything; the non-representability results were apparently first observed by Maddux.

In his 1970 seminar on relation algebras (see [146]), Tarski discussed the known negative results concerning his completeness problem; and, as an example of an equation that holds in all set relation algebras but not in all relation algebras, he gave the equation with six variables (formulated as an inequality) that is discussed in Section 17.5

after Theorem 17.30, and that first appeared in Chin-Tarski [23]. The proof that this equation in fact fails in the relation algebra with four atoms given at the end of Section 17.5 is due to Maddux [101]. Tarski, in his seminar, raised the question of whether there exist equations with a single variable that are true in all set relation algebras, but not in all relation algebras. This question was answered positively not long afterwards by Givant, who constructed an example of an equation with just one variable that is true in all set relation algebras but fails to be true in McKenzie's four-atom non-representable relation algebra. Later, Tarski and George McNulty simplified Givant's equation somewhat, and the result—presented in Theorem 17.30 above—was published in [147] in a different but related context.

All of the results in Sections 17.6 and 17.7, except for the statements of the completeness of the representations involved, are due to Jónsson-Tarski [75]. This includes, in particular, the quasi-representation theorems (Theorems 17.31 and 17.32), the representation theorem for atomic relation algebras with functional atoms (Theorem 17.34), and Corollary 17.36. The definition of the quasi-representation given in the original proof of Theorem 17.31 is different from the one given here. The version given here, and the remarks concerning its connection to the Cayley representation of group complex algebras, come from Andréka-Givant [3]. The proof that some of the representations in the Jónsson-Tarski paper are in fact complete is also from [3]. The Jónsson-Tarski paper also gives a description of all complete and atomic relation algebras with functional atoms in terms of Brandt groupoids. A more direct, set-theoretical description of these algebras is given in [3] (see Exercise 17.63). In particular, Jónsson and Tarski prove that every integral, complete and atomic relation algebra with functional atoms is isomorphic to the complex algebra of a group (see Exercise 17.62).

As regards the results in Section 17.8, Corollary 17.37 (without the statement regarding the completeness of the representation) and Corollary 17.38 are due to Jónsson-Tarski [75], as are the following versions of Theorems 17.44 and 17.45: a relation algebra $\mathfrak{A}$ is simple, complete, and atomic with singleton atoms if and only if $\mathfrak{A}$ is isomorphic to a full set relation algebra on some set; and $\mathfrak{A}$ is complete and atomic with singleton atoms if and only if it is isomorphic to a full set relation algebra on some equivalence relation. Jónsson and Tarski cite an earlier and rather different axiomatic characterization of the full relation algebras on sets due to McKinsey [108]. Lemma 17.39 is due to Maddux [99], and Lemma 17.46 and the representation theorem for single-

ton dense relation algebras, Theorem 17.47, are special cases of more general results in Maddux [99] that concern relation algebras that are singleton dense or pair dense below the identity element (see Exercises 17.54–17.61 and the further remarks below). In particular, the results in Exercises 17.54–17.58, and the implication from left to right in Exercise 17.60 are due to Maddux. Lemmas 17.40 through 17.43, the Structure Theorem 17.48, and the non-Boolean examples following that theorem are due to Givant, as are the results in Exercises 17.59 and 17.61, and the implication from right to left in Exercise 17.60.

A number of interesting representation theorems concerning specific classes of relation algebras have appeared in the literature. Tarski [133], [137] proved that every relation algebra in which there are two functions r and s satisfying the equation $r^\smile ; s = 1$ must be representable (see [147] for a full proof of this result). Such a pair of functions is called a pair of *quasi-projections*. Maddux and Tarski [103] proved that every functionally dense relation algebra, that is to say, every relation algebra in which each non-zero element is above a non-zero function (or equivalently, in which the unit is the sum of a set of functions), is representable, thereby answering positively a question originally posed in Jónsson-Tarski [75]. A different proof of this theorem, together with a structure theorem for all functionally dense relation algebras, and examples of simple, functionally dense relation algebras that are atomless, and functionally dense relation algebras that have atomless Boolean algebras of ideal elements, are given in Andréka-Givant [3]. In this connection, the equivalence of (i) and (iii) in Exercise 17.66, and the results in Exercises 17.75 and 17.76 are due to Maddux-Tarski [103], while those in Exercises 17.63–17.65, 17.67–17.70, and 17.72–17.74 are due to Andréka-Givant [3]. As regards the result in Exercise 17.71, it was announced in Maddux–Tarski [103], but was never published with proof. The result was rediscovered by Andréka and Givant, and published with proof in [3].

Generalizing the theorems of Tarski [147] and Maddux-Tarski [103], Maddux [93] proved that a relation algebra in which the unit is the sum of elements of the form $r^\smile ; s$, where r and s range over all functions, is representable. Németi [114] proved that every relation algebra generated by a set of right ideal elements and left ideal elements is representable, and Jónsson [69] proved that every relation algebra generated by an equivalence element is finite and representable. Generalizing Jónsson's theorem, Givant [35] proved that every relation algebra generated by a chain of equivalence elements or even by a tree

of equivalence elements is representable, and finite if the chain or tree is finite.

Maddux [99] proved that every relation algebra in which the identity element is the sum of pairs, that is to say, the sum of non-zero elements r satisfying the equation

$$r\,;0'\,;r\,;0'\,;r \leq 1',$$

is representable, and completely representable if the algebra is also simple. This result includes, as a special case, the theorem that a relation algebra in which the identity element is a sum of singletons is representable, and completely representable if the algebra is also simple (see Exercise 17.58). The results in Exercises 17.78–17.82, the equivalence of (i) and (iii) in Exercise 17.84, the implication from left to right in Exercise 17.86, and the results in Exercises 17.87–17.90 are all from Maddux [99]. The results in Exercises 17.91 and 17.92 are due to Givant.

The observation in Exercise 17.46 that a relation algebra with a complete quasi-representation is necessarily atomic follows from a more general result in Hirsch-Hodkinson [54] to the effect that a completely representable Boolean algebra must always be atomic. The notion of a pseudo-atom in Exercise 17.51 and the notion of rectangle density in Exercise 17.77 are taken from Andréka-Givant-Mikulás-Németi-Simon [4] (where pseudo-atoms are called quasi-atoms).

Exercises

17.1. Prove the implications (2), (3), (5), and (6) in the proof of Theorem 17.1.

17.2. Prove that the function φ defined in the proof of Theorem 17.1 preserves operations of addition, complement, and converse, and maps the identity element in $\mathfrak{A}$ to the identity relation in $\mathfrak{Re}(U)$. Conclude that φ is a homomorphism.

17.3. Prove that if a set $\mathcal{S}$ of formulas in a first-order language $\mathcal{L}^*$ is complete and consistent, then the resulting formula relation algebra $\mathfrak{A}$ has a square representation, and in fact the homomorphism induced by any model of $\mathcal{S}$ is one-to-one and therefore a square representation of $\mathfrak{A}$.

17.4. Prove implications (2), (5), and (6) in the proof of Theorem 17.3.

17.5. Prove that the function ψ defined in the proof of Theorem 17.3 preserves the operations of addition and converse, and maps the identity relation in $\mathfrak{B}$ to the identity element in $\mathfrak{A}$.

17.6. Prove Corollary 17.4.

17.7. Prove that the representation φ of a Boolean relation algebra $\mathfrak{A}$ defined by

$$\varphi(r) = \{(N, N) : N \text{ is a maximal filter in } \mathfrak{A} \text{ and } r \in N\}$$

for r in $\mathfrak{A}$ is necessarily incomplete when $\mathfrak{A}$ is infinite.

17.8. Show that the function φ defined in the proof of Theorem 17.6 preserves the operations of complement, relative multiplication, and converse, and maps the identity element in $\mathfrak{A}$ to the identity element in $\mathfrak{B}$.

17.9. Prove Corollary 17.7.

17.10. Prove the assertion made at the end of Section 17.2 that the complete representations of an atomic Boolean relation algebra $\mathfrak{A}$ are precisely the functions ψ defined by

$$\psi(r) = \bigcup\{id_{V_a} : a \in U \text{ and } a \leq r\}$$

for r in $\mathfrak{A}$, where U is the set of atoms in $\mathfrak{A}$ and $(V_a : a \in U)$ is an arbitrary disjoint system of non-empty sets indexed by the atoms of $\mathfrak{A}$.

17.11. There is, up to isomorphism, only one simple Boolean relation algebra, namely the two-element relation algebra. Prove that any two square representations of this algebra are equivalent.

17.12. Let $\mathfrak{A}$ be a four element Boolean relation algebra with atoms r and s. Give explicitly the representation of each element in $\mathfrak{A}$ in two different ways: first by using the representation from Theorem 17.5, and then by using the representation from Theorem 17.6. Illustrate each of these representations of $\mathfrak{A}$ by a diagram similar in spirit to the one in Figure 17.1, with the base set of the representing algebra on each of two perpendicular coordinate axes, and small squares representing the ordered pairs of elements from this base set.

17.13. If R_f is the Cayley representation of an element f in a group

$$(G\,,\ \circ\,,\ ^{-1}\,,\ \iota),$$

prove that

$$R_f^{-1} = R_{f^{-1}} \qquad \text{and} \qquad R_\iota = id_G.$$

17.14. Construct a relative multiplication table and a converse table for the atoms in the complex algebra of the Klein group of order four (the direct product of the two-element group $\mathbb{Z}_2$ with itself). List the Cayley representations of the atoms as sets of ordered pairs, and illustrate these representations in a diagram similar to Figure 17.1.

17.15. Construct a relative multiplication table and a converse table for the atoms in the complex algebra of the symmetric group of order three (the group of permutations of the set $\{0, 1, 2\}$). List the Cayley representations of the atoms as sets of ordered pairs, and illustrate these representations in a diagram similar to Figure 17.1.

17.16. Give a direct proof of Theorem 17.10 that does not use Lemma 17.9.

17.17. If a subalgebra of the complex algebra of a group G contains all of the atoms of $\mathfrak{Cm}(G)$, prove that the subalgebra has a unique complete representation up to equivalence, namely the restriction of the Cayley representation of $\mathfrak{Cm}(G)$.

17.18. Let $\mathfrak{A}$ be the finite-cofinite subalgebra of the complex algebra of an infinite group, that is to say, the subalgebra of all finite subsets and all cofinite subsets of the group (see Exercise 6.1).

(i) Suppose φ_i is a complete square representations of $\mathfrak{A}$ over a base set U_i for each index i in an index set I of cardinality at least two, and suppose that the base sets of these representations of $\mathfrak{A}$ are mutually disjoint. Let φ be the amalgamation of the system, so that

$$\varphi(r) = \bigcup\{\varphi_i(r) : i \in I\}$$

for each r in $\mathfrak{A}$. Prove that the function ψ defined by $\psi(r) = \varphi(r)$ if r is finite, and by

$$\varphi(r) \cup \bigcup\{U_i \times U_j : i, j \in I \text{ and } i \neq j\}$$

if r is cofinite, is an incomplete square representation of $\mathfrak{A}$.

(ii) Prove that every incomplete square representation of $\mathfrak{A}$ is of the form in (i) for some disjoint system $(\varphi_i : i \in I)$ of complete square representations of $\mathfrak{A}$, with I of cardinality at least two.

17.19. If a group G is countably infinite, prove that every square representation of the group complex algebra $\mathfrak{Cm}(G)$ over a set of countably infinite cardinality must be complete and therefore equivalent to the Cayley representation.

17.20. Let κ be an infinite cardinal such that $\kappa^\omega = \kappa$. If a group G has cardinality κ, prove that the group complex algebra $\mathfrak{Cm}(G)$ has an incomplete square representation over a set of cardinality κ. (Thus, the group complex algebra has incomplete representations over sets that have the same cardinality as the group. It has incomplete representations over sets of all cardinalities greater than or equal to 2^κ, by Theorem 16.10.)

17.21. Give the details of the proof of Corollary 17.11.

17.22. Verify the formula $R_p \mid R_q = R_\iota$ for computing the composition of two atoms R_p and R_q in a geometric complex algebra in the case when when $p = q$ and the affine plane D has order two.

17.23. Let P be the projective line of order five, consisting of the six points (0), (1), (2), (3), (4), and (∞).

(i) Describe the points and lines of the affine plane D of order five.
(ii) Describe the points and lines of the projective plane Q of order five that is the projective extension of D with respect to P. Draw a picture of this plane that is similar in spirit to Figure 17.7.
(iii) Describe the set of points in the affine representation of each of the seven atoms in $\mathfrak{Cm}(P)$, and illustrate the representations of these atoms in a diagram similar to Figure 17.8.

17.24. Repeat Exercise 17.23 for the projective line P of order two, consisting of the three points (0), (1), and (∞).

17.25. Repeat Exercise 17.23 for the projective plane P of order three, which has thirteen points. In this case, D will be the three-dimensional affine geometry of order three, and Q will be the corresponding three-dimensional projective geometry of order three.

17.26. Suppose φ is the affine representation of a geometric complex algebra $\mathfrak{Cm}(P)$ with respect to a projective geometry Q that includes P as a maximal subspace. Let D be the affine geometry obtained from Q by deleting the points and lines of P. Under the assumption that the group G of translations of D is transitive, prove the following statements.

(i) The function that maps each point in p to the set of translations in G in the direction of p extends to an embedding ψ of $\mathfrak{Cm}(P)$ into the complex algebra $\mathfrak{Cm}(G)$ of the group G.
(ii) If $\mathfrak{A}$ is the subgroup of $\mathfrak{Cm}(G)$ that is the image of $\mathfrak{Cm}(P)$ under ψ, then the given affine representation φ is equivalent to the Cayley representation of $\mathfrak{Cm}(G)$ restricted to $\mathfrak{A}$.

(A *translation* of D is defined to be a dilation f of D with the property that, for some ideal point p, every affine line ℓ that has p as its point at infinity is mapped bijectively to itself by f, that is to say, a point a lies on ℓ if and only if its image $f(a)$ lies on ℓ. The point p is called the *center*, or the *direction*, of the translation, and if f is not the identity function, then the direction of f is uniquely determined. The group G is called *transitive* if for every pair of points a and b in D, there is a translation f such that $f(a) = b$.)

17.27. Complete the verification of equivalence (1) in the proof of Lemma 17.19 by treating the cases when at least one of p and q is ι.

17.28. Prove Lemma 17.21 directly, without using Lemma 17.19.

17.29. For each of the projective lines P of order $n = 2, 3, 4, 5, 7, 8, 10$, determine the number of inequivalent square representations of $\mathfrak{Cm}(P)$.

17.30. Let P be the projective line of order five. Find two inequivalent representations of $\mathfrak{Cm}(P)$ and illustrate these representations as is done in Figure 17.8.

17.31. Prove that all square representations of the minimal relation algebra $\mathfrak{M}_2$ are equivalent. Is the same true of $\mathfrak{M}_3$?

17.32. Up to equivalence, describe all of the square representations of the relation algebras in parts (i) and (v) of Exercise 3.36.

17.33. Up to equivalence, describe all of the square representations of the relation algebras in parts (ii), (iii), and (iv) of Exercise 3.36.

17.34. Show that the relation algebras in parts (vi) and (vii) of Exercise 3.36 are representable.

17.35. Prove that the two relation algebras discussed in Exercise 3.37 are representable.

17.36. Prove that the three relation algebras described in Exercise 3.38 are representable.

17.37. Prove that all relation algebras with at most eight elements are representable.

17.38. Prove that the relation algebras described in parts (i)–(iv) of Exercise 3.40 are representable.

17.39. Prove that the relation algebra described in part (v) of Exercise 3.40 is not representable.

17.40. Prove that the relation algebra described in part (ii) of Exercise 3.41 is representable.

17.41. Prove that the relation algebra described in part (iii) of Exercise 3.41 is not representable.

17.42. Prove that the relation algebra with four atoms determined by Table 17.9 is generated by the single element s.

17.43. Find an instance of the inequality ε'—mentioned at the end of Section 17.5—that has just one variable and is true in all set relation algebras, but not in all relation algebras.

17.44. In the proof of Theorem 17.31, show that

$$\varphi(1) = \{(a, b) : a, b \in U \text{ and } a\,;1 = b\,;1\}.$$

In other words, show that $\varphi(1)$ consists of the pairs of atoms that have the same domain. Conclude from this, without using the argument preceding Theorem 17.31, that $\varphi(1)$ is an equivalence relation on the set of atoms.

17.45. If the relation algebra in the proof of Theorem 17.31 is integral, prove that $\varphi(1) = U \times U$.

17.46. Prove that the converse of Theorem 17.31 is also true. In other words, prove that a relation algebra with a complete quasi-representation is necessarily atomic.

17.47. Prove that an atomic relation algebra is singleton dense if and only if every atom is a singleton.

17.48. Prove directly, without using Theorems 17.31 or 17.34, that an atomic relation algebra with singleton atoms is representable.

17.49. Prove Corollary 17.38.

17.50. Is a singleton in a non-simple relation algebra necessarily an atom?

17.51. An element r in a relation algebra is called a *pseudo-atom* if

$$s = r \cdot (1 \,;\, s \,;\, 1)$$

for every element $s \leq r$ (see Figure 17.19). A relation algebra is called *pseudo-atomic* if every non-zero element is above a non-zero quasi-atom. Prove that every singleton dense relation algebra (not necessarily simple) is pseudo-atomic.

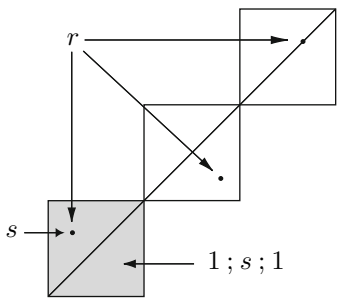


Fig. 17.19 Example of a quasi-atom r.

17.52. Is a subalgebra of a singleton dense relation algebra necessarily singleton dense? How about a regular subalgebra?

17.53. Is a regular subalgebra of an atomic relation algebra with singleton atoms necessarily atomic with singleton atoms?

17.54. An element r is called an *identity singleton* if it is a singleton below the identity element. Prove that the following conditions on an element r are equivalent.

(i) r is an identity singleton.
(ii) $r \neq 0$ and $r\,;1\,;r \leq 1'$.

17.55. If an element r satisfies the inequality $r\,;1\,;r \leq 1'$ in a relation algebra, prove that $r\,;1\,;r = r$ and $r\,;0'\,;r = 0$.

17.56. A relation algebra $\mathfrak{A}$ is said to be *singleton dense below the identity*, or *identity singleton dense*, if every non-zero subidentity element is above an identity singleton. This and the next five exercises concern these relation algebras. Prove that the following conditions on $\mathfrak{A}$ are equivalent.

(i) $\mathfrak{A}$ is singleton dense below the identity.
(ii) The identity element is a sum of singletons.
(iii) The identity element is the sum of the set of all identity singletons.
(iv) Every subidentity element is a sum of singletons.
(v) Every subidentity element is the sum of the set of all identity singletons that are below it.

17.57. If r and s are identity singletons, prove that $r\,;1\,;s = 0$ or else $r\,;1\,;s$ is a singleton. Conclude that if the relation algebra is simple, then $r\,;1\,;s$ must be a singleton.

17.58. Prove that a relation algebra is identity singleton dense if and only if it is singleton dense. Conclude that every identity singleton dense relation algebra is representable, and every identity singleton dense relation algebra that is simple or atomic is completely representable.

17.59. The relationship between singletons and identity singletons is a close one. Prove that the following conditions on an element r are equivalent.

(i) r is a singleton.
(ii) $0 < r = [(r\,;1)\cdot 1']\,;1\,;[(1\,;r)\cdot 1']$, and $(r\,;1)\cdot 1'$ and $(1\,;r)\cdot 1'$ are identity singletons.
(iii) There are identity singletons x and y such that

$$r = x\,;1\,;y \qquad \text{and} \qquad (1\,;x\,;1)\cdot(1\,;y\,;1) \neq 0.$$

17.60. Prove directly (without using Exercise 17.58) that a relation algebra is identity singleton dense if and only if its completion is identity singleton dense.

17.61. Let $\mathfrak{A}$ be an identity singleton dense relation algebra that is simple, and let U be the set of identity singletons in $\mathfrak{A}$. Prove directly that the function φ from $\mathfrak{A}$ into $\mathfrak{Re}(U)$ defined by

$$\varphi(r) = \{(x, y) : x, y \in U \text{ and } x \,;1\,; y \leq r\}$$

for each r in $\mathfrak{A}$ is a complete representation of $\mathfrak{A}$, and in fact, if $\mathfrak{A}$ is complete, then φ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{Re}(U)$. Conclude that a relation algebra is simple and identity singleton dense if and only if it is essentially isomorphic to a full set relation algebra on some non-empty set U.

17.62. Prove that every integral, atomic relation algebra with functional atoms is essentially isomorphic to the complex algebra of a group.

17.63. Just as there is a description of simple, atomic relation algebras with singleton atoms (see Theorem 17.44), there is also a description of simple, atomic relation algebras with functional atoms. Consider a group $(G\,,\ \circ\,,\ ^{-1}\,,\ \iota)$. Associated with each element f in G is its Cayley representation

$$R_f = \{(g, g \circ f) : g \in G\}.$$

Prove the following.

(i) If H is a subgroup of G, then the set of all restrictions of the Cayley representations R_f of elements f in G to the various right cosets of H is the set of atoms of a (simple) complete subalgebra of $\mathfrak{Re}(G)$ that is atomic with functional atoms. Denote this algebra by $\mathfrak{F}(G, H)$.

(ii) A simple relation algebra is atomic with functional atoms if and only if it is essentially isomorphic to $\mathfrak{F}(G, H)$ for some group G and some subgroup H.

(iii) A relation algebra is atomic with functional atoms if and only if it is essentially isomorphic to a direct product of algebras of the form $\mathfrak{F}(G, H)$ for various groups G and subgroups H.

17.64. Let H be a subgroup of a group G, and K a proper subgroup of H. Prove that the algebra $\mathfrak{F}(G, H)$ (see Exercise 17.63) is a proper

subalgebra of $\mathfrak{F}(G, K)$, and each atom in $\mathfrak{F}(G, H)$ is split in $\mathfrak{F}(G, K)$ into the union of as many atoms as there are right cosets of K in H.

17.65. Consider the cyclic group

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

of order 12 under the operation of addition. Let H_0 be $\mathbb{Z}_{12}$, let H_1 be the subgroup of order 4 with universe $\{0, 3, 6, 9\}$, and let H_2 be the subgroup of order 2 with universe $\{0, 6\}$. Describe each of the corresponding subalgebras $\mathfrak{F}(\mathbb{Z}_{12}, H_n)$ of $\mathfrak{Re}(\mathbb{Z}_{12})$, for $n = 0, 1, 2$, by listing the atoms and describing the operations of converse and composition on the set of atoms.

17.66. A relation algebra is said to be *functionally dense* if below every non-zero element there is a non-zero function. This and the following ten exercises concern functionally dense relation algebras. Prove that the following conditions on a relation algebra $\mathfrak{A}$ are equivalent.

(i) $\mathfrak{A}$ is functionally dense.
(ii) The unit is a sum of functions.
(iii) The unit is the sum of the set of all functions.
(iv) Every element is a sum of functions.
(v) Every element is the sum of the set of all functions that are below it.

17.67. If a functionally dense relation algebra is simple, prove that the algebra must be either atomic or atomless.

17.68. Let G be a group, and $(H_i : i \in I)$ an infinite, strictly downward directed system of subgroups of G in the sense that for any two indices i and j there is an index k such that H_k is a proper subgroup of H_i and H_j. Prove that the union of the system $(\mathfrak{F}(G, H_i) : i \in I)$ (see Exercises 17.63 and 17.64) is a subalgebra of $\mathfrak{Re}(G)$ that is simple, functionally dense, and atomless.

17.69. Prove that a functionally dense relation algebra is essentially isomorphic to a direct product of complete, simple, functionally dense relation algebras—each of which is either atomic or atomless—and a single complete, functionally dense relation algebra that has an atomless Boolean algebra of ideal elements.

17.70. Give an example of a functionally dense relation algebra with an atomless Boolean algebra of ideal elements.

17.71. If a functionally dense relation algebra is complete, prove that below every non-zero element r there is a (non-zero) function f such that $f ; 1 = r ; 1$.

17.72. Prove that a functional atom in the canonical extension of a simple relation algebra $\mathfrak{A}$ is always below a function in $\mathfrak{A}$.

17.73. Prove that a relation algebra is functionally dense if and only if its completion is functionally dense.

17.74. Prove that every relativization of a functionally dense relation algebra is functionally dense.

17.75. Prove that every functionally dense relation algebra is representable.

17.76. Is every functionally dense relation algebra embeddable into an atomic relation algebra with functional atoms?

17.77. A relation algebra is called *rectangle dense* if below every non-zero element there is a non-zero rectangle. Prove that a relation algebra is rectangle dense if and only if it is singleton dense. Conclude that a relation algebra is rectangle dense if and only if it is identity singleton dense.

17.78. It is possible to express in the language of relation algebras that a subidentity element r is either zero, a singleton, or an abstract version of a relation consisting of two ordered pairs. The idea is to say that in the decomposition of the relativization $\mathfrak{A}(r ; 1 ; r)$ into a direct product of three algebras of specific types (see Theorem 13.5), the algebra of type 3 is degenerate. The most efficient way to express this idea directly is to say that for the relativized diversity element

$$s = 0' \cdot (r ; 1 ; r),$$

there are three possibilities:

$$s = 0 \qquad \text{or} \qquad s ; s = 0, \qquad \text{or} \qquad s ; s = r.$$

All three possibilities are captured by the single inequality

$$[0' \cdot (r ; 1 ; r)] ; [0' \cdot (r ; 1 ; r)] \leq 1'.$$

An element r is called an *identity pair* if it is non-zero and satisfies this inequality. This and the next five exercises concern subidentity pairs. Prove that a non-zero element r is an identity pair if and only if it satisfies the inequality

$$r\,;0';\,r\,;0';\,r \leq 1'.$$

17.79. Prove that a relation R on a set U is an identity pair if and only if

$$R = \{(\alpha, \alpha)\} \qquad \text{or} \qquad R = \{(\alpha, \alpha), (\beta, \beta)\}$$

for some α and β in U.

17.80. Prove that in a simple relation algebra, an identity pair is either an atom or the sum of two singletons (each of which must be an atom, by Lemma 17.39).

17.81. If r is an identity singleton, and s an identity pair that is not a sum of singletons, prove that $s\,;1\,;r$ is either zero or a functional atom, and consequently $r\,;1\,;s$ is either zero or an atom that is the converse of a function. Conclude that if the relation algebra is simple, then $r\,;1\,;s$ and $s\,;1\,;r$ are both atoms.

17.82. In a simple relation algebra, if r and s are identity pairs that are not sums of singletons, prove that $r\,;1\,;s$ is either an atom or the sum of two bijections that are atoms.

17.83. Prove the following assertions about a relation algebra $\mathfrak{A}$.

(i) If r is an identity pair in $\mathfrak{A}$, then every non-zero element s that is below r is an identity pair in every subalgebra of $\mathfrak{A}$ that contains s.
(ii) If r is an identity pair in $\mathfrak{A}$, and φ a homomorphism on $\mathfrak{A}$, then $\varphi(r)$ is either an identity pair or zero in the image algebra.
(iii) If $\mathfrak{A}$ is the product of a system of relation algebras, then r is an identity pair in $\mathfrak{A}$ if and only if each component of r in each factor algebra is either zero or an identity pair, and at least one component is an identity pair.

17.84. A relation algebra is said to be *pair dense below the identity*, or *identity pair dense*, if every non-zero subidentity element is above an identity pair . This and the next seven exercises concern such algebras. Prove that the following conditions are equivalent in a relation algebra $\mathfrak{A}$.

(i) $\mathfrak{A}$ is identity pair dense.
(ii) The identity element is a sum of identity pairs.
(iii) The identity element is the sum of the set of all identity pairs.
(iv) Every subidentity element is a sum of identity pairs.
(v) Every subidentity element is the sum of the set of all identity pairs that are below it.

17.85. If a relation algebra $\mathfrak{A}$ is the product of a system of relation algebras, prove that $\mathfrak{A}$ is identity pair dense if and only if each factor algebra is identity pair dense.

17.86. Prove that a relation algebra is identity pair dense if and only if its completion is identity pair dense.

17.87. Prove that every simple, identity pair dense relation algebra is atomic.

17.88. Prove that every simple, identity pair dense relation algebra is completely representable.

17.89. Prove that in a complete, identity pair dense relation algebra, every non-zero subidentity element r is above an identity pair that generates the same ideal element as r.

17.90. Prove that every identity pair dense relation algebra is representable.

17.91. Prove that every identity pair dense relation algebra is essentially isomorphic to the direct product of two complete, identity pair dense relation algebras, one of which is atomic and the other with an atomless Boolean algebra of ideal elements. Prove also that the atomic factor is isomorphic to the direct product of a system of complete and atomic relation algebras each of which is simple and identity pair dense.

17.92. Prove that there is an equation ε with one variable in the language of relation algebras such that ε is satisfied by a relation R in the full set relation algebra on a non-empty set U if and only if R is a doubleton relation, that is to say, if and only if R has the form

$$R = \{(\alpha, \beta), (\gamma, \delta)\}$$

for some elements α, β, γ, and δ in U with either $\alpha \neq \gamma$ or $\beta \neq \delta$.

Chapter 18
Varieties and universal classes

Varieties, that is to say, equationally axiomatizable classes, of relation algebras, have been a principal focus of research in relation algebras for some time. In this chapter, we discuss some of the main results that have been obtained. Here is a brief summary.

The class of universal classes of simple relation algebras and the class of varieties of relation algebras are intimately connected: both are lattices under naturally defined operations of meet and join, and the two lattices turn out to be isomorphic via the correspondence that maps every universal class K of simple relation algebras to the variety $\mathsf{SP}(\mathsf{K})$ of (isomorphic copies of) subalgebras of direct products of algebras in K. For this reason, the study of varieties of relation algebras reduces to the study of universal classes of simple relation algebras, and some of the most interesting results about varieties of relation algebras have been obtained by studying the corresponding universal classes of simple relation algebras.

It turns out that the lattice of all subsets of the set of natural numbers, under the set-theoretic operations of intersection and union, is completely embeddable into the lattice of varieties of relation algebras, so that the latter inherits all of the complexities of the former. In particular, there are continuum many varieties of relation algebras, and in fact there are chains of varieties (ordered by the relation of set-theoretic inclusion) that have the same order type as the set of real numbers (ordered in the standard way).

In the lattice of varieties of relation algebras, there is a unique variety on level zero, that is to say, there is a unique smallest variety, namely the *degenerate* variety, consisting of all degenerate relation algebras. There are three varieties on level one, that is to say, there are

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9_5

three *minimal* varieties of relation algebras, each generated by one of the minimal simple relations $\mathfrak{M}_1$, $\mathfrak{M}_2$, or $\mathfrak{M}_3$. The varieties on level two—the so-called *quasi-minimal* varieties—have not yet been completely classified, and in fact it is not even known whether there are infinitely many or only finitely many of them. Twenty quasi-minimal varieties are currently known, and it is also known that, with few exceptions, each of them is generated by a single integral, non-minimal relation algebra; but that algebra may be infinite in size. At the other extreme, there is a unique largest variety, namely the variety of all relation algebras. Just below it are the *maximal* varieties, and these, too, have not yet been completely classified. It is known that they are infinite in number, and in fact, for each positive integer n, the variety generated by the class of all simple relation algebras that are not isomorphic to the full set relation algebra $\mathfrak{Re}(n)$ is a maximal variety.

Some of the most important results about varieties of relation algebras concern varieties of representable relation algebras. Quite surprisingly, the class of all representable relation algebras is a variety, but it is not finitely axiomatizable, and in fact it is not even axiomatizable by an infinite set of equations that uses only finitely many variables. The full set relation algebras on infinite sets all generate the same variety of representable relation algebras, but this variety does not coincide with variety of all representable relation algebras. In fact the variety of representable relation algebras is the irredundant join of the varieties generated by the individual full set relation algebras $\mathfrak{Re}(n)$ for $1 \leq n \leq \omega$.

A simple, representable relation algebra $\mathfrak{A}$ is said to be *hereditarily strictly infinitely representable* if $\mathfrak{A}$ is not minimal and if every non-minimal subalgebra of $\mathfrak{A}$ has representations only over infinite sets, not over finite sets; and $\mathfrak{A}$ is said to be *hereditarily infinite* if $\mathfrak{A}$ is not minimal and if every non-minimal subalgebra of $\mathfrak{A}$ is infinite. It is easy to see that a hereditarily infinite, representable relation algebra is always hereditarily strictly infinitely representable, because a simple relation algebra that is representable over a finite set is perforce finite; but the reverse implication is not true. In fact, the lattice of subsets of the set of natural numbers is completely embeddable into the lattice interval between the variety generated by the hereditarily infinite, representable relation algebras and the variety generated by the hereditarily strictly infinitely representable relation algebras. Unexpectedly, there turn out to be continuum many varieties of hereditarily infinite, representable relation algebras.

Almost all of the results in the first three sections of this chapter, with small modifications in terminology and notation, remain true for arbitrary algebras, not just relation algebras. For notational convenience, we continue to refer to relation algebras and their similarity type, but the reader may wish to think of the algebras as being of an arbitrary similarity type, and of $\mathcal{L}$ as being a first-order language appropriate for that similarity type.

18.1 Theories and classes of relation algebras

In order to place the discussion into a proper context, it is helpful to broaden the discussion initially by talking about various types of first-order theories of relation algebras, and the classes of relation algebras that are defined by such theories. As was mentioned in Section 2.4, a *theory* of relation algebras is a set of formulas (or sentences, depending upon the manner in which the deductive apparatus of the underlying logic is set up) in the language $\mathcal{L}$ of relation algebras that contains the axioms of relation algebras and that is closed under the relation of provability. For an arbitrary set $\mathcal{S}$ of formulas (or sentences), one can form the intersection of all those theories that include $\mathcal{S}$ as a subset. (The intersection of the empty set of theories is, by convention, the *inconsistent* theory, which consists of all formulas—or all sentences—in $\mathcal{L}$.) The result is a theory, and in fact it is the smallest theory that includes the set $\mathcal{S}$. It is called the theory *generated by* $\mathcal{S}$, and it coincides with the set of all formulas (or sentences) that are derivable from formulas in $\mathcal{S}$.

The set-theoretic relation of one theory being included in another is a partial order on the set of all theories of relation algebras, and under this partial order the set becomes a complete lattice. The *infimum*, or *meet*, of a collection of theories is the largest theory that is included in each of the theories in the collection; it is just the intersection of the theories in the collection. The *supremum*, or *join*, of the collection is the smallest theory that includes each of the theories in the collection; it is the intersection of all those theories in $\mathcal{L}$ that include each theory in the collection. (There is always one theory in $\mathcal{L}$ that includes every theory in the collection, namely the inconsistent theory; consequently, the set of theories that include every theory in the collection is not

empty.) Alternatively, it is the theory generated by the union of the axiom sets of the theories in the collection.

A class K of relation algebras is called *elementary* if it can be axiomatized by a set of formulas (or sentences) in the first-order language $\mathcal{L}$ of relation algebras. Equivalently, K is elementary if it is the class of all models of some theory of relation algebras. For an arbitrary class K of relation algebras, one can form the intersection of all those elementary classes that include K as a subclass. (The intersection of the empty collection of classes is, by convention, the class of all relation algebras.) The result is an elementary class, and in fact it is the smallest elementary class that includes the class K. It is called the elementary class *generated by* K.

The relation of one elementary class (of relation algebras) being included in another is a partial order on the class of elementary classes, and under this partial order, the class of elementary classes becomes a complete lattice. The *infimum*, or *meet*, of any collection of elementary classes is the largest elementary class that is included in each class of the collection; it is just the intersection of the collection. It is axiomatized by the set of formulas belonging to the union of the axiom sets of the individual classes in the collection. The *supremum*, or *join*, of the collection is the smallest elementary class that includes each of the classes in the collection; it is the intersection of all those elementary classes that include each class in the collection. It is axiomatized by the set of all those formulas that are true of all of the algebras in all of the classes of the collection. (There is always one elementary class of relation algebras that includes every class in the collection, namely the class of all relation algebras; consequently, the class of elementary classes that include every elementary class in the collection is not empty.)

There is a natural bijection from the lattice of elementary classes of relation algebras to the lattice of theories of relation algebras: it maps each elementary class K to the theory $Th(\mathsf{K})$ consisting of all formulas that are true in K. The inverse of this bijection is the function that maps each theory $\mathcal{T}$ to the class $\mathsf{Mo}(\mathcal{T})$ consisting of all relation algebras that are models of $\mathcal{T}$. The relationship between the two bijections can be expressed symbolically by writing

$$\mathsf{Mo}(\,Th(\mathsf{K})) = \mathsf{K} \qquad \text{and} \qquad Th(\mathsf{Mo}(\mathcal{T})) = \mathcal{T}.$$

These bijections are *dual lattice isomorphisms* in the sense that they reverse the partial orderings of the lattices. In more detail, it is clear

that

$$\mathcal{S} \subseteq \mathcal{T} \qquad \text{implies} \qquad \mathsf{Mo}(\mathcal{S}) \supseteq \mathsf{Mo}(\mathcal{T})$$

and

$$\mathsf{K} \subseteq \mathsf{L} \qquad \text{implies} \qquad \mathit{Th}(\mathsf{K}) \supseteq \mathit{Th}(\mathsf{L}).$$

On the other hand, if

$$\mathsf{Mo}(\mathcal{S}) \supseteq \mathsf{Mo}(\mathcal{T}),$$

then by forming the theory of both sides of this inclusion, and using the second implication above, we obtain

$$\mathit{Th}(\mathsf{Mo}(\mathcal{S})) \subseteq \mathit{Th}(\mathsf{Mo}(\mathcal{T})),$$

which yields $\mathcal{S} \subseteq \mathcal{T}$. Thus,

$$\mathcal{S} \subseteq \mathcal{T} \qquad \text{if and only if} \qquad \mathsf{Mo}(\mathcal{S}) \supseteq \mathsf{Mo}(\mathcal{T}),$$

and similarly,

$$\mathsf{K} \subseteq \mathsf{L} \qquad \text{if and only if} \qquad \mathit{Th}(\mathsf{K}) \supseteq \mathit{Th}(\mathsf{L}).$$

It follows that the bijections under discussion preserve meets as joins, and joins as meets. In other words, if the operations of meet and join in the appropriate lattices are denoted by $\wedge$ and $\vee$ respectively, then for any two elementary classes K and L,

$$\mathit{Th}(\mathsf{K} \wedge \mathsf{L}) = \mathit{Th}(\mathsf{K}) \vee \mathit{Th}(\mathsf{L}), \qquad \mathit{Th}(\mathsf{K} \vee \mathsf{L}) = \mathit{Th}(\mathsf{K}) \wedge \mathit{Th}(\mathsf{L}),$$

and for any two theories $\mathcal{S}$ and $\mathcal{T}$,

$$\mathsf{Mo}(\mathcal{S} \wedge \mathcal{T}) = \mathsf{Mo}(\mathcal{S}) \vee \mathsf{Mo}(\mathcal{T}), \quad \mathsf{Mo}(\mathcal{S} \vee \mathcal{T}) = \mathsf{Mo}(\mathcal{S}) \wedge \mathsf{Mo}(\mathcal{T}).$$

A theory or class of relation algebras is called *universal* if it can be axiomatized by a set of universal formulas, or, equivalently, by a set of quantifier-free formulas; it is called *universal Horn* if it can be axiomatized by a set of universal Horn sentences, or equivalently, by a set of open universal Horn formulas; it is called *conditional equational* if it can be axiomatized by a set of conditional equations; and it is called

equational if it can be axiomatized by a set of identities, or equivalently, by a set of equations. A conditional equational class is also called a *quasi-variety*, and an equational class is also called a *variety*. There is a close connection between quasi-varieties and universal Horn classes. In fact, quasi-varieties are just universal Horn classes that contain the one-element algebras.

The dual lattice isomorphism $\mathsf{K} \mapsto \mathit{Th}(\mathsf{K})$ obviously maps universal classes to universal theories, universal Horn classes to universal Horn theories, quasi-varieties to conditional equational theories, and varieties to equational theories, and analogously for the inverse mapping $\mathcal{S} \mapsto \mathsf{Mo}(\mathcal{S})$. Notice that the join (in the lattice of elementary theories) of a collection of universal theories is again a universal theory, since it is axiomatized by the union of the sets of axioms of the individual theories in the collection, and similarly for the join of a collection of universal Horn theories, conditional equational theories, or equational theories. Consequently, the meet (in the lattice of elementary classes) of a collection of universal classes is again a universal class, and similarly, the meet of a collection of varieties is again a variety.

Most of the preceding discussion can be repeated for more restricted notions of theories and classes. For example, given an arbitrary set $\mathcal{E}$ of equations, one can form the intersection of all those equational theories (in the sense discussed at the end of Section 2.4) that include $\mathcal{E}$ as a subset. The result is an equational theory, and it fact it is the smallest equational theory that includes the set $\mathcal{E}$. It is called the equational theory *generated by* $\mathcal{E}$, and it coincides with the set of all equations that are derivable from $\mathcal{E}$ using the rules of inference specified at the end of Section 2.4.

The set-theoretic relation of one equational theory being included in another is a partial order on the set of all equational theories of relation algebras, and under this partial order the set becomes a complete lattice. The *infimum*, or *meet*, of a collection of equational theories is the intersection of the theories in the collection, and the *supremum*, or *join*, of the collection is the intersection of all those equational theories that include each theory in the collection. Alternatively, the supremum of the collection is the equational theory generated by the union of the axiom sets of the equational theories in the collection.

In a similar fashion, given a class K of relation algebras, one can form the intersection of all those varieties (of relation algebras) that include K as a subclass. The result is a variety, and in fact it is the

smallest variety that includes K. It is called the variety *generated by* K. The relation of one variety (of relation algebras) being included in another is a partial order on the class of all varieties, and under this partial order the class of varieties becomes a complete lattice. The *infimum*, or *meet*, of any collection of varieties is the intersection of the collection, and it is axiomatized by the set of equations that belong to the union of the equational axiom sets of the individual varieties in the collection. The *supremum*, or *join*, of the collection is the smallest variety that includes each of the varieties in the collection. It is axiomatized by the set of all those equations that are true in all of the varieties of the collection.

The function that maps each variety K to the equational theory $Eq(\mathsf{K})$ consisting of all equations that are true in K, is a dual lattice isomorphism, and the inverse of this dual isomorphism is the function that maps each equational theory $\mathcal{E}$ to the variety $\mathsf{Mo}(\mathcal{E})$ consisting of all models of $\mathcal{E}$.

The elements of the lattice of varieties, namely the varieties of relation algebras, are also elements in the lattice of elementary classes, and the infimum of a collection of varieties is the intersection of the collection in both lattices. In the lattice of varieties, however, the join of a collection of varieties is the intersection of all *varieties* that include each variety in the collection, whereas in the lattice of elementary classes, the join is the intersection of all *elementary* classes (including all of the equational classes) that include each variety in the collection. Consequently, the operations of join do not in general coincide in the two lattices; rather, the join of a collection of varieties in the lattice of elementary classes is below the join of the collection in the lattice of varieties. It follows that the lattice of varieties is not a sublattice of the lattice of elementary classes.

There are a number of natural operations on classes of algebras that lead from one class of algebras to another. The most basic of these is the operation I of forming the class of all isomorphic images of the algebras in a given class. This operation, when applied to a class K of algebras, yields the class $\mathsf{I}(\mathsf{K})$ of all algebras that are isomorphic to algebras in K. A class K is said to be *closed under isomorphisms* if it is closed under I that is to say, if $\mathsf{I}(\mathsf{K})$ is included in K, or, equivalently, if $\mathsf{I}(\mathsf{K}) = \mathsf{K}$ (since every algebra is an isomorphic image of itself).

Another natural class operation is the operation S of forming the class of all subalgebras of algebras in a given class, together with all isomorphic copies of those subalgebras. This operation, when applied

to a class K of algebras, yields the class $\mathsf{S}(\mathsf{K})$ of all algebras that are isomorphic to subalgebras of algebras in K, or put a different way, it yields the class of all algebras that are embeddable into algebras in K. A class K is said to be *closed under subalgebras* if $\mathsf{S}(\mathsf{K})$ is included in K, or, equivalently, if $\mathsf{S}(\mathsf{K}) = \mathsf{K}$ (since every algebra is a subalgebra of itself). Obviously,

$$\mathsf{SS}(\mathsf{K}) = \mathsf{S}(\mathsf{K}),$$

so K is closed under subalgebras if and only if $\mathsf{K} = \mathsf{S}(\mathsf{L})$ for some class of algebras L. In more detail, if $\mathsf{K} = \mathsf{S}(\mathsf{L})$, then

$$\mathsf{S}(\mathsf{K}) = \mathsf{SS}(\mathsf{L}) = \mathsf{S}(\mathsf{L}) = \mathsf{K},$$

so that K is closed under subalgebras. In the reverse direction, if K is closed under subalgebras, then $\mathsf{K} = \mathsf{S}(\mathsf{L})$, where L is taken to be the class K itself.

Yet another natural class operation is the operation H of forming the class of all homomorphic images of algebras in a given class. This operation, when applied to a class K of algebras, yields the class $\mathsf{H}(\mathsf{K})$ of all homomorphic images of algebras in K. A class K is said to be *closed under homomorphisms* if $\mathsf{H}(\mathsf{K})$ is included in K, or, equivalently, if $\mathsf{H}(\mathsf{K}) = \mathsf{K}$ (since every algebra is a homomorphic image of itself). Obviously,

$$\mathsf{HH}(\mathsf{K}) = \mathsf{H}(\mathsf{K}),$$

and from this it follows that K is closed under homomorphisms if and only if $\mathsf{K} = \mathsf{H}(\mathsf{L})$ for some class of algebras L.

Finally, there is the class operation P of forming the class of direct products of systems of algebras in a given class, together with all isomorphic copies of those products. This operation, when applied to a class K of algebras, yields the class $\mathsf{P}(\mathsf{K})$ of all algebras that are isomorphic to direct products of systems of algebras in K. A class K is said to be *closed under* (*direct*) *products* if $\mathsf{P}(\mathsf{K})$ is included in K, or, equivalently, if $\mathsf{P}(\mathsf{K}) = \mathsf{K}$ (since every algebra $\mathfrak{A}$ is isomorphic to the direct product of the system that consists of the single algebra $\mathfrak{A}$). It is not hard to show that

$$\mathsf{PP}(\mathsf{K}) = \mathsf{P}(\mathsf{K}).$$

Consequently, K is closed under products if and only if $\mathsf{K} = \mathsf{P}(\mathsf{L})$ for some class L of algebras.

The next lemma summarizes some of the preceding observations.

Lemma 18.1. *For any class* K *of algebras*,

(i) $\mathsf{K} \subseteq \mathsf{S}(\mathsf{K}) = \mathsf{SS}(\mathsf{K})$,
(ii) $\mathsf{K} \subseteq \mathsf{H}(\mathsf{K}) = \mathsf{HH}(\mathsf{K})$,
(iii) $\mathsf{K} \subseteq \mathsf{P}(\mathsf{K}) = \mathsf{PP}(\mathsf{K})$.

The class operations described above may be composed in various ways to yield further class operations. For example, the class operations S and H may be composed in two different ways to yield the class operations SH and HS. The next lemma states the obvious relationships that hold between the two possible compositions of two class operations.

Lemma 18.2. *For any class* K *of algebras*,

(i) $\mathsf{SH}(\mathsf{K}) \subseteq \mathsf{HS}(\mathsf{K})$,
(ii) $\mathsf{PS}(\mathsf{K}) \subseteq \mathsf{SP}(\mathsf{K})$,
(iii) $\mathsf{PH}(\mathsf{K}) \subseteq \mathsf{HP}(\mathsf{K})$.

Proof. To prove (i), suppose that an algebra $\mathfrak{D}$ is in the class $\mathsf{SH}(\mathsf{K})$. There must then be an epimorphism φ from an algebra $\mathfrak{A}$ in K to an algebra $\mathfrak{B}$ such that $\mathfrak{D}$ is isomorphic to a subalgebra of $\mathfrak{B}$, by the definitions of the operations S and H. By applying the Exchange Principle (Theorem 7.15), we may assume that $\mathfrak{D}$ is actually a subalgebra of $\mathfrak{B}$. The inverse image set

$$C = \varphi^{-1}(D) = \{r \in A : \varphi(r) \in D\}$$

is a subuniverse of $\mathfrak{A}$, by the second part of Lemma 7.4, and if $\mathfrak{C}$ is the corresponding subalgebra of $\mathfrak{A}$, then the appropriate restriction of φ maps $\mathfrak{C}$ onto $\mathfrak{D}$. Consequently, $\mathfrak{D}$ belongs to $\mathsf{HS}(\mathsf{K})$. Parts (ii) and (iii) of the lemma are proved in an analogous fashion using Corollary 11.20 and Lemma 11.21 respectively. □

Varieties of algebras are closed under subalgebras, homomorphisms, and products, by Corollary 6.4, Corollary 7.3, and Lemma 11.18 respectively. The next lemma states some equivalent conditions for an arbitrary class of algebras to be closed under all three of these class operations.

Lemma 18.3. *The following conditions on a class* K *of algebras are equivalent*.

(i) K *is closed under subalgebras, homomorphisms, and products*.

(ii) $\mathsf{K} = \mathsf{HSP}(\mathsf{K})$.
(iii) $\mathsf{K} = \mathsf{HSP}(\mathsf{L})$ *for some class* L *of relation algebras*.

Proof. To establish the implication from (i) to (ii), assume that K is closed under subalgebras, homomorphisms, and products. In this case,

$$\mathsf{HSP}(\mathsf{K}) = \mathsf{HS}(\mathsf{K}) = \mathsf{H}(\mathsf{K}) = \mathsf{K},$$

by Lemma 18.1, so (ii) holds.

The implication from (ii) to (iii) is trivial: just take L to be the class K. To establish the implication from (iii) to (i), assume that

$$\mathsf{K} = \mathsf{HSP}(\mathsf{L}) \tag{1}$$

for some class L of algebras. In this case,

$$\mathsf{P}(\mathsf{K}) = \mathsf{PHSP}(\mathsf{L}) \subseteq \mathsf{HPSP}(\mathsf{L}) \subseteq \mathsf{HSPP}(\mathsf{L}) = \mathsf{HSP}(\mathsf{L}) = \mathsf{K},$$

by Lemma 18.2(ii),(iii), Lemma 18.1(iii), and (1), so K is closed under products. Completely analogous arguments show that K is closed under subalgebras and homomorphisms. □

18.2 Ultraproducts

The ultraproduct construction is an unusual, but very powerful combination of the direct product and the quotient constructions. We describe it in its application to relation algebras, but there is nothing that is inherently specific to relation algebras in this construction. Begin with a non-empty system $(\mathfrak{A}_i : i \in I)$ of algebras (thus, the index I is always assumed to be non-empty), and form the direct product

$$\mathfrak{A} = \prod_i \mathfrak{A}_i.$$

Let D be an arbitrary ultrafilter in the Boolean algebra of all subsets of the index set I; for simplicity, we shall say that D is an *ultrafilter over* I. The following properties characterize D. First, D is closed under intersections in the sense that if J and K are sets in D, then so is the intersection $J \cap K$. Second, D is upward closed in the sense that if J is in D, and if $J \subseteq K \subseteq I$, then K is in D. (These two properties characterize Boolean filters over I.) Third, for every subset J of I, one

of J and $I \sim J$ is in D. Finally, D is a proper filter in the sense that it does not contain the empty set. An ultrafilter D is called *principal* if it contains a smallest element. This amounts to requiring that for some element j in I, the set D consists of all subsets of I that include the set $\{j\}$. In this case, the set $\{j\}$ is said to generate the ultrafilter D. An ultrafilter that is not principal is called a *non-principal* ultrafilter.

Define a relation of equivalence on the universe of $\mathfrak{A}$ by writing

$$r \equiv s \mod D \qquad \text{if and only if} \qquad \{i \in I : r(i) = s(i)\} \in D.$$

A short symbolic way of expressing this relationship is to write $r \equiv_D s$. The relationship is expressed in words by saying that r *is equivalent to* s *modulo* D.

Lemma 18.4. *The relation* $\equiv_D$ *is a congruence relation on the direct product* $\mathfrak{A}$.

Proof. As examples of how the argument proceeds, here are the proofs that the relation $\equiv_D$ is transitive and that it preserves the operation of relative multiplication. Let r, s, t, and u be elements in $\mathfrak{A}$. If r is equivalent to s, and s to t, modulo D, then the sets

$$\{i \in I : r(i) = s(i)\} \qquad \text{and} \qquad \{i \in I : s(i) = t(i)\} \tag{1}$$

both belong to D, by the definition of the relation $\equiv_D$. The intersection of these two sets must also be in D, because D is closed under intersections. The set

$$\{i \in I : r(i) = t(i)\} \tag{2}$$

obviously includes the intersection of the two sets in (1), so (2) also belongs to D, because D is upward closed. It follows that r is equivalent to t modulo D, by the definition of the relation $\equiv_D$. Thus, $\equiv_D$ is transitive.

Next, suppose that r is equivalent to t, and s to u, modulo D. The sets

$$\{i \in I : r(i) = t(i)\} \qquad \text{and} \qquad \{i \in I : s(i) = u(i)\} \tag{3}$$

are then both in D, by the definition of $\equiv_D$, and therefore so is the intersection of these two sets. Since

$$r(i) = t(i) \quad \text{and} \quad s(i) = u(i) \quad \text{implies} \quad r(i)\,;s(i) = t(i)\,;u(i),$$

the set

$$\{i \in I : r(i) \,;\, s(i) = t(i) \,;\, u(i)\} \tag{4}$$

includes the intersection of the two sets in (3), and therefore (4) also belongs to D. It follows that $r \,;\, s$ is equivalent to $t \,;\, u$ modulo D, by the definition of the relation $\equiv_D$ and the definition of relative multiplication in the product $\mathfrak{A}$. Thus, $\equiv_D$ preserves the operation of relative multiplication. □

For each element r in the direct product $\mathfrak{A} = \prod_i \mathfrak{A}_i$, write r/D for the congruence class of r modulo D. Similarly, write $(\prod_i \mathfrak{A}_i)/D$—or, more simply, $\mathfrak{A}/D$, when there is no chance of misunderstanding—for the quotient of $\mathfrak{A}$ modulo D. This quotient is called the *ultraproduct* of the system $(\mathfrak{A}_i : i \in I)$ *modulo* (the ultrafilter) D. In the special case when all of the algebras $\mathfrak{A}_i$ are equal to the same algebra $\mathfrak{B}$, the ultraproduct is called the Ith *ultrapower* of $\mathfrak{B}$ *modulo* D. As a quotient of a direct product of relation algebras, an ultraproduct is again a relation algebra.

When an ultrafilter is principal, the ultraproduct construction gives nothing new.

Lemma 18.5. *If D is the principal ultrafilter generated by the set $\{j\}$, then the ultraproduct $\mathfrak{A}/D$ is isomorphic to $\mathfrak{A}_j$.*

Proof. Two elements r and s in $\mathfrak{A}$ are equivalent modulo D if and only if the set

$$\{i \in I : r(i) = s(i)\} \tag{1}$$

belongs to D, by the definition of the relation $\equiv_D$. Since D is assumed to be the principal ultrafilter generated by the singleton $\{j\}$, the set in (1) belongs to D if and only if it contains the index j, that is to say, if and only if $r(j) = s(j)$. Thus,

$$r/D = s/D \qquad \text{if and only if} \qquad r(j) = s(j).$$

The function φ from $\mathfrak{A}/D$ to $\mathfrak{A}_j$ defined by

$$\varphi(r/D) = r(j) \tag{2}$$

for each r in $\mathfrak{A}$ is therefore well defined and one-to-one. For a given element t in $\mathfrak{A}_j$, define an element r in $\mathfrak{A}$ by

$$r(i) = \begin{cases} 0 & \text{if} \quad i \neq j, \\ t & \text{if} \quad i = j, \end{cases}$$

and observe that

$$\varphi(r/D) = r(j) = t.$$

Consequently, φ is onto.

For elements r, s, and t in $\mathfrak{A}$, the set

$$\{i \in I : r(i) \,;\, s(i) = t(i)\}$$

belongs to D if and only if it contains the index j, that is to say, if and only if $r(j) \,;\, s(j) = t(j)$. Consequently,

$$r/D \,;\, s/D = t/D \qquad \text{if and only if} \qquad r(j) \,;\, s(j) = t(j), \tag{3}$$

by the definition of the relation $\equiv_D$ and the definition of relative multiplication in $\mathfrak{A}/D$. Assuming that the left-hand equality in (3) holds, we obtain

$$\varphi(r/D \,;\, s/D) = \varphi(t/D) = t(j) = r(j) \,;\, s(j) = \varphi(r/D) \,;\, \varphi(s/D),$$

by (2) and (3). This proves that φ preserves the operation of relative multiplication. Similar arguments show that φ preserves the other operations and is therefore an isomorphism. □

The preceding lemma explains why the real value of the ultraproduct construction is only realized when D is taken to be a non-principal ultrafilter over I. The fundamental property of the ultraproduct construction is that it preserves all properties expressible in the first-order language $\mathcal{L}$ of relation algebras. A precise statement of this property is contained in the following *Fundamental Theorem of Ultraproducts*.

Theorem 18.6. *Suppose $\mathfrak{A}$ is the direct product of a (non-empty) system $(\mathfrak{A}_i : i \in I)$ of algebras, and D is an ultrafilter over I. A formula $\Gamma(v_0, \ldots, v_{n-1})$ is satisfied by a sequence $(r_0/D, \ldots, r_{n-1}/D)$ of elements in $\mathfrak{A}/D$ if and only if the set*

$$\{i \in I : (r_0(i), \ldots, r_{n-1}(i)) \text{ satisfies } \Gamma \text{ in } \mathfrak{A}_i\}$$

belongs to D. In particular, a sentence Γ holds in $\mathfrak{A}/D$ if and only if the set

$$\{i \in I : \Gamma \text{ holds in } \mathfrak{A}_i\}$$

belongs to D.

Proof. The proof is facilitated by some notation. Let $\sigma(v_0, \ldots, v_{n-1})$ be a term with variables among $v_0, \ldots, v_{n-1}$, and write $\sigma^{\mathfrak{B}}$ to denote the polynomial of n arguments defined by σ in an algebra $\mathfrak{B}$. The proof begins by showing that

$$\sigma^{\mathfrak{A}/D}(r_0/D, \ldots, r_{n-1}/D) = (\sigma^{\mathfrak{A}_i}(r_0(i), \ldots, r_{n-1}(i)) : i \in I)/D \quad (1)$$

for every sequence $(r_0, \ldots, r_{n-1})$ of n elements in $\mathfrak{A}$. The argument is by induction on terms. For the base case, assume first that σ is a variable v_j (with $0 \leq j < n$). We then have

$$\begin{aligned}\sigma^{\mathfrak{A}/D}(r_0/D, \ldots, r_{n-1}/D) = r_j/D &= (r_j(i) : i \in I)/D \\ &= (\sigma^{\mathfrak{A}_i}(r_0(i), \ldots, r_{n-1}(i)) : i \in I)/D,\end{aligned}$$

by the definition of the value of a term on a sequence of elements, and the definition of the elements in the product $\mathfrak{A}$. The argument when σ is the individual constant symbol 1' is similar.

For the induction step, suppose that

$$\sigma(v_0, \ldots, v_{n-1}) \qquad \text{and} \qquad \tau(v_0, \ldots, v_{n-1})$$

are terms satisfying the equation in (1) (with σ replaced by τ in the case of the second term). Consider the case when γ is the term $\sigma ; \tau$ The relative product of the elements

$$\sigma^{\mathfrak{A}/D}(r_0/D, \ldots, r_{n-1}/D) \quad \text{and} \quad \tau^{\mathfrak{A}/D}(r_0/D, \ldots, r_{n-1}/D)$$

in the ultraproduct $\mathfrak{A}/D$ is equal to the relative product of

$$(\sigma^{\mathfrak{A}_i}(r_0(i), \ldots, r_{n-1}(i)) : i \in I)/D$$

and

$$(\tau^{\mathfrak{A}_i}(r_0(i), \ldots, r_{n-1}(i)) : i \in I)/D$$

in $\mathfrak{A}/D$, by the induction hypothesis in (1) on σ and τ, and this last relative product is equal to

$$(\sigma^{\mathfrak{A}_i}(r_0(i), \ldots, r_{n-1}(i)) ; \tau^{\mathfrak{A}_i}(r_0(i), \ldots, r_{n-1}(i)) : i \in I)/D,$$

by the definition of relative multiplication in $\mathfrak{A}/D$. Consequently,

$$\begin{aligned}\gamma^{\mathfrak{A}/D}&(r_0/D,\ldots,r_{n-1}/D)\\ &= \sigma^{\mathfrak{A}/D}(r_0/D,\ldots,r_{n-1}/D)\,;\tau^{\mathfrak{A}/D}(r_0/D,\ldots,r_{n-1}/D)\\ &= (\sigma^{\mathfrak{A}_i}(r_0(i),\ldots,r_{n-1}(i))\,;\tau^{\mathfrak{A}_i}(r_0(i),\ldots,r_{n-1}(i)) : i\in I)/D\\ &= (\gamma^{\mathfrak{A}_i}(r_0(i),\ldots,r_{n-1}(i)) : i\in I)/D,\end{aligned}$$

by the definition of the value of a term on a sequence of elements and the preceding observations. The cases when γ is one of the terms

$$\sigma+\tau, \qquad -\sigma, \qquad \sigma^{\smile}$$

are handled in a similar fashion and are left as an exercise.

The proof of the theorem itself proceeds by induction on formulas. We treat here three of the cases that need to be considered. In the base case of the induction, the formula Γ is an equation of the form

$$\sigma(v_0,\ldots,v_{n-1}) = \tau(v_0,\ldots,v_{n-1}).$$

This equation is satisfied by a sequence

$$(r_0/D,\ldots,r_{n-1}/D) \tag{2}$$

of elements in $\mathfrak{A}/D$ just in case the polynomials $\sigma^{\mathfrak{A}/D}$ and $\tau^{\mathfrak{A}/D}$ have the same value on this sequence, that is to say, just in case

$$\sigma^{\mathfrak{A}/D}(r_0/D,\ldots,r_{n-1}/D) = \tau^{\mathfrak{A}/D}(r_0/D,\ldots,r_{n-1}/D).$$

This last equality is equivalent to the equality

$$\begin{aligned}(\sigma^{\mathfrak{A}_i}(r_0(i),\ldots,r_{n-1}(i)) : i\in I)/D&\\ = (\tau^{\mathfrak{A}_i}(r_0(i),\ldots,r_{n-1}(i)) : i\in I)/D,&\end{aligned}$$

by (1). The values on the left and right sides of the preceding equation are elements in $\mathfrak{A}/D$, so they are equal if and only if the set of indices

$$\{i\in I : \sigma^{\mathfrak{A}_i}(r_0(i),\ldots,r_{n-1}(i)) = \tau^{\mathfrak{A}_i}(r_0(i),\ldots,r_{n-1}(i))\}$$

belongs to the ultrafilter D, by the definition of the relation $\equiv_D$. This set of indices coincides with the set of indices

$$\{i\in I : (r_0(i),\ldots,r_{n-1}(i)) \text{ satisfies } \sigma=\tau \text{ in } \mathfrak{A}_i\},, \tag{3}$$

by the definition of satisfaction, so the given sequence satisfies the equation $\sigma = \tau$ in $\mathfrak{A}/D$ just in case the set in (3) belongs to the ultrafilter D, as claimed.

For the induction step, assume that the theorem is true for formulas

$$\Delta(v_0, \ldots, v_{n-1}) \qquad \text{and} \qquad \Omega(v_0, \ldots, v_{n-1}).$$

If Γ is the formula $\neg\Delta$, then a sequence (2) satisfies Γ in $\mathfrak{A}/D$ if and only if it does not satisfy Δ in $\mathfrak{A}/D$, by the definition of satisfaction. The induction hypothesis implies that the given sequence does not satisfy Δ just in case the set of indices

$$\{i \in I : (r_0(i), \ldots, r_{n-1}(i)) \text{ satisfies } \Delta \text{ in } \mathfrak{A}_i\} \tag{4}$$

does not belong to D. The set D is an ultrafilter, so it does not contain the set in (4) just in case it contains the complement of that set. The complement of the set in (4) is the set

$$\{i \in I : (r_0(i), \ldots, r_{n-1}(i)) \text{ satisfies } \Gamma \text{ in } \mathfrak{A}_i\}, \tag{5}$$

by the definition of satisfaction and the assumption on Γ, so the preceding equivalences lead to the conclusion that the sequence in (2) satisfies the formula Γ in $\mathfrak{A}/D$ if and only if the set of indices in (5) belongs to D. This is just what the theorem claims. The argument when Γ is the formula $\Delta \to \Omega$ is similar.

For the final case of the induction step, it is easier to treat the case of an existential quantifier than the case of a universal quantifier. Assume therefore that Γ is the formula $\exists v_0 \Delta$, where Δ is as above. (The variable v_0 is being used here for notational convenience only, and could be replaced by any variable v_i at the cost of more complicated notation.) If a sequence

$$(r_1/D, \ldots, r_{n-1}/D) \tag{6}$$

satisfies Γ in $\mathfrak{A}/D$, then there must be an element r_0 in $\mathfrak{A}$ such that the sequence (2) satisfies Δ in $\mathfrak{A}/D$, by the definition of satisfaction. The set of indices

$$\{i \in I : (r_0(i), \ldots, r_{n-1}(i)) \text{ satisfies } \Delta \text{ in } \mathfrak{A}_i\} \tag{7}$$

therefore belongs to the ultrafilter D, by the induction hypothesis on Δ. This set is certainly included in the set J of indices defined by

$$J = \{i \in I : (r_1(i), \ldots, r_{n-1}(i)) \text{ satisfies } \Gamma \text{ in } \mathfrak{A}_i\}, \tag{8}$$

by the definition of satisfaction and the assumption on Γ, so J also belongs to D, by the upward closure of D. This argument establishes the implication from left to right in the statement of the theorem (for the case under consideration).

To establish the reverse implication, assume that the set J defined in (8) belongs to the ultrafilter D. For each index i in J, there must be an element s_i in $\mathfrak{A}_i$ such that the sequence

$$(s_i, r_1(i), \ldots, r_{n-1}(i))$$

satisfies Δ in $\mathfrak{A}_i$, by the definition of J and the definition of satisfaction. Define an element r_0 in $\mathfrak{A}$ by

$$r_0(i) = \begin{cases} s_i & \text{if} \quad i \in J, \\ 0 & \text{if} \quad i \in I \sim J. \end{cases}$$

The resulting sequence

$$(r_0(i), r_1(i), \ldots, r_{n-1}(i))$$

satisfies Δ in $\mathfrak{A}_i$ whenever i is in J, by the definition of r_0, so the set J is included in the set in (7). Consequently, the set in (7) must also belong to D, by the upward closure of D. Apply the induction hypothesis to obtain that the sequence in (2) satisfies Δ in $\mathfrak{A}/D$, and then apply the definition of satisfaction to conclude that the sequence in (6) satisfies Γ in $\mathfrak{A}/D$. This completes the induction step of the proof. The first assertion of the theorem now follows by the principal of induction for formulas.

The second assertion of the theorem is the special case of the first assertion when $n = 0$. □

An easy consequence of the preceding theorem is that every ultrapower of an algebra $\mathfrak{A}$ contains an isomorphic copy of $\mathfrak{A}$ as an elementary subalgebra.

Corollary 18.7. *An algebra is elementarily embeddable into every ultrapower of itself.*

Proof. Consider an algebra $\mathfrak{A}$, a non-empty set I, and an ultrafilter D over I. Take $\mathfrak{B}$ to be the Ith ultrapower of $\mathfrak{A}$ modulo the ultrafilter D. For each element r in $\mathfrak{A}$, write $\hat{r}$ for the constant function in the direct power $\mathfrak{A}^I$ that is defined by $\hat{r}(i) = r$ for each i in I. The immediate goal is to show that a sequence

$$(r_0, \dots, r_{n-1}) \tag{1}$$

of elements in $\mathfrak{A}$ satisfies a formula $\Gamma(v_0, \dots, v_{n-1})$ in $\mathfrak{A}$ if and only if the corresponding sequence

$$(\hat{r}_0/D, \dots, \hat{r}_{n-1}/D) \tag{2}$$

of elements in $\mathfrak{B}$ satisfies the formula Γ in $\mathfrak{B}$.

The Fundamental Theorem of Ultraproducts (with $\mathfrak{A}$ in place of $\mathfrak{A}_i$ for every index i) implies that the sequence in (2) satisfies Γ in $\mathfrak{B}$ if and only if the set

$$\{i \in I : (\hat{r}_0(i), \dots, \hat{r}_{n-1}(i)) \text{ satisfies } \Gamma \text{ in } \mathfrak{A}\}$$

belongs to the ultrafilter D. In view of the definition of the elements $\hat{r}_k$ (for $k = 0, \dots, n-1$), the preceding set is just

$$\{i \in I : (r_0, \dots r_{n-1}) \text{ satisfies } \Gamma \text{ in } \mathfrak{A}\}, \tag{3}$$

and this set does not depend on any particular choice of the index i; it is either I or the empty set, according to whether the sequence in (1) does, or does not, satisfy Γ in $\mathfrak{A}$. In the first case, the set in (3) belongs to D, because I is in D; and in the second case, the set in (3) does not belong to D, because the empty set is not in D. Combine these observations to conclude that the sequence in (2) satisfies Γ in $\mathfrak{B}$ if and only if the sequence in (1) satisfies Γ in $\mathfrak{A}$, as desired.

The observations of the preceding paragraphs immediately imply that the function mapping each element r in $\mathfrak{A}$ to the corresponding element $\hat{r}/D$ in $\mathfrak{B}$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$ (see the remarks in the first paragraph of Section 7.9). □

Another consequence of the Fundamental Theorem of Ultraproducts is that, for relation algebras, the property of being simple is preserved under ultraproducts. This follows directly from the fact (Simplicity Theorem 9.2) that the property of being simple, for relation algebras, is expressible by means of a first-order sentence.

Corollary 18.8. *An ultraproduct of simple relation algebras is simple.*

The ultraproduct construction paves the way for a new class operation, namely the operation P_u of forming the class of all ultraproducts of non-empty systems of algebras in a given class, together with all isomorphic copies of those ultraproducts. This operation, when applied

to a class K of algebras, yields the class $\mathsf{P_u}(\mathsf{K})$ of all algebras that are isomorphic to ultraproducts of non-empty systems of algebras in K. A class K of algebras is said to be *closed under ultraproducts* if $\mathsf{P_u}(\mathsf{K})$ is included in K, or, equivalently, if $\mathsf{P_u}(\mathsf{K}) = \mathsf{K}$ (since every algebra is isomorphic to an ultrapower of itself modulo a principal ultrafilter). It is not difficult to show that

$$\mathsf{P_u}\mathsf{P_u}(\mathsf{K}) = \mathsf{P_u}(\mathsf{K}).$$

Consequently, K is closed under ultraproducts if and only if $\mathsf{K} = \mathsf{P_u}(\mathsf{L})$ for some class L of relation algebras.

In analogy with Lemma 18.2, there are some straightforward observations about the composition of class operations that involve ultraproducts. The proofs are left as exercises.

Lemma 18.9. *For any class* K *of algebras*,

(i) $\mathsf{P_u}\mathsf{S}(\mathsf{K}) \subseteq \mathsf{S}\mathsf{P_u}(\mathsf{K})$,
(ii) $\mathsf{P_u}\mathsf{H}(\mathsf{K}) \subseteq \mathsf{H}\mathsf{P_u}(\mathsf{K})$.

The corresponding result involving direct products is false.

18.3 Universal classes

There are a number of important theorems that characterize when a class of algebras may be axiomatized by a set of formulas of a certain syntactic type. In this and the next section we shall look at several examples.

The first theorem characterizes when a class of algebras is universal. It may be called the $\mathsf{SP_u}$-*Theorem*.

Theorem 18.10. *The following conditions on a class of algebras* K *are equivalent*.

(i) K *is universal*.
(ii) K *is closed under subalgebras and ultraproducts*.
(iii) $\mathsf{K} = \mathsf{S}\mathsf{P_u}(\mathsf{K})$.
(iv) $\mathsf{K} = \mathsf{S}\mathsf{P_u}(\mathsf{L})$ *for some class* L.

Proof. The implication from (i) to (ii) follows from the fact that universal classes are closed under subalgebras (because universal formulas

are preserved under subalgebras, by Corollary 6.4) and under ultraproducts (by the Fundamental Theorem of Ultraproducts). For the implication from (ii) to (iii), observe that if K is closed under subalgebras and ultraproducts, then

$$\mathsf{SP}_{\mathsf{u}}(\mathsf{K}) = \mathsf{S}(\mathsf{K}) = \mathsf{K},$$

so (iii) holds. The implication from (iii) to (iv) is trivial: just take L to be the class K.

To prove the implication from (iv) to (i), assume that

$$\mathsf{K} = \mathsf{SP}_{\mathsf{u}}(\mathsf{L}) \tag{1}$$

for some class of algebras L. Take $\mathcal{S}$ to be the set of universal sentences that are true in L, with the goal of proving that

$$\mathsf{K} = \mathsf{Mo}(\mathcal{S}). \tag{2}$$

Every algebra in L is a model of $\mathcal{S}$, by the definition of $\mathcal{S}$, and the sentences in $\mathcal{S}$ are first-order, so every ultraproduct of non-empty systems of algebras in L—and therefore every isomorphic image of such an ultraproduct—is a model of $\mathcal{S}$, by the Fundamental Theorem of Ultraproducts. The sentences in $\mathcal{S}$ are also universal, and universal sentences are preserved under the passage to subalgebras, so every subalgebra of an algebra in the class $\mathsf{P}_{\mathsf{u}}(\mathsf{L})$—and therefore every isomorphic image of such a subalgebra—is a model of $\mathcal{S}$. These observations and the assumption in (1) imply that every algebra in K is a model of $\mathcal{S}$, so that the left side of (2) is included in the right side.

It remains to establish the reverse inclusion. To this end, suppose that an algebra $\mathfrak{A}$ belongs to the right side of (2), that is to say, suppose $\mathfrak{A}$ is a model of the set of sentences $\mathcal{S}$. In order to show that $\mathfrak{A}$ belongs to the left side of (2), a non-empty system of algebras

$$(\mathfrak{B}_i : i \in I) \tag{3}$$

in L must be constructed such that $\mathfrak{A}$ is embeddable into an ultraproduct of the system, by (1).

For each element r in $\mathfrak{A}$, adjoin to the language $\mathcal{L}$ of $\mathfrak{A}$ a new individual constant symbol $\boldsymbol{r}$ in such a way that these new symbols are distinct from one another and from all of the symbols that occur in $\mathcal{L}$. Denote the resulting expanded language by $\mathcal{L}'$, and let $\mathfrak{A}'$ be the algebra obtained from $\mathfrak{A}$ by adjoining each element in $\mathfrak{A}$ to the list of

fundamental operations of $\mathfrak{A}$ as a new distinguished constant (that is to say, a new operation of rank zero). Let $\mathcal{T}$ be the set of those equations and negations of equations in the language $\mathcal{L}'$ that contain no occurrences of variables and that are true in $\mathfrak{A}'$ when each new individual constant symbol $\boldsymbol{r}$ is interpreted as the corresponding distinguished constant r. (The set $\mathcal{T}$ is sometimes referred to as the *diagram* of $\mathfrak{A}$). Take the index set I to be the set of all non-empty finite subsets of $\mathcal{T}$.

Fix an element i in I. Thus, i consists of finitely many variable-free equations and negations of equations, each of which is true in $\mathfrak{A}'$. There are only finitely many new individual constant symbols that occur in the formulas of i, say they are

$$\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1}. \tag{4}$$

Replace these constants in the formulas of i by the variables

$$v_0, \ldots, v_{n-1} \tag{5}$$

respectively to obtain a finite set of equations and negations of equations in the original language $\mathcal{L}$. Form the conjunction of this set of formulas to arrive at a formula $\Gamma(v_0, \ldots, v_{n-1})$ that is a conjunction of equations and negations of equations in $\mathcal{L}$. The sentence

$$\exists v_0 \ldots \exists v_{n-1} \Gamma(v_0, \ldots, v_{n-1}) \tag{6}$$

is true in $\mathfrak{A}$ because each formula in i is true in $\mathfrak{A}'$ when the individual constant symbols in (4) are interpreted as the corresponding distinguished constants

$$r_0, \ldots, r_{n-1}. \tag{7}$$

The universal sentence that is equivalent to the negation of (6)—call it Λ—is therefore false in $\mathfrak{A}$, so Λ cannot belong to the set $\mathcal{S}$, by the assumption that $\mathfrak{A}$ is a model of $\mathcal{S}$. Since $\mathcal{S}$ is defined to be the set of all universal sentences that are true in L, and since Λ is a universal sentence that is not in $\mathcal{S}$, there must be an algebra $\mathfrak{B}_i$ in L in which Λ fails to hold, that is to say, in which the existential sentence (6) is true. The validity of (6) in $\mathfrak{B}_i$ implies that there must be an interpretation in $\mathfrak{B}_i$ of all of the new individual constant symbols in $\mathcal{L}'$—that is to say, there must be a mapping φ_i from the set of these individual constant symbols to the universe of $\mathfrak{B}_i$—such that under this interpretation the formulas in the set i are all true in $\mathfrak{B}_i$. (Of course, only the values

$$\varphi_i(\boldsymbol{r}_0), \ldots, \varphi_i(\boldsymbol{r}_{n-1})$$

actually matter in the determining the truth or falsity of any formula from i in $\mathfrak{B}_i$, but it is convenient to let φ_i be an interpretation of every new individual constant symbol in $\mathcal{L}'$, and not just of the symbols from (4).) This completes the construction of the system of algebras (3). One final observation is important: each algebra in (3) belongs to L, by assumption, and $\mathcal{S}$ is defined to be the set of all universal sentences that are true in the algebras of L, so each algebra in (3) is a model of $\mathcal{S}$.

The next task is to construct an appropriate ultrafilter D over the index set I. For each index i in I, let J_i be the subset of I that consists of all those finite subsets j of $\mathcal{T}$ that include the set i, so

$$J_i = \{j \in I : i \subseteq j\}.$$

The system of sets

$$(J_i : i \in I) \tag{8}$$

has the finite intersection property, because

$$J_{i_1} \cap J_{i_2} = J_{i_1 \cup i_2}$$

for every pair of elements i_1 and i_2 in I, and the set on the right is never empty (it contains, for example, the set $i_1 \cup i_2$). Consequently, there must be an ultrafilter D (in the Boolean algebra of all subsets of I) that contains each of the sets in the system (8).

Put $\mathfrak{B} = \prod_i \mathfrak{B}_i$, and observe that the ultraproduct $\mathfrak{B}/D$ belongs to the class $\mathsf{P_u}(\mathsf{L})$, because the algebras $\mathfrak{B}_i$ are all in L. Observe also that $\mathfrak{B}/D$ is a model of $\mathcal{S}$, by the Fundamental Theorem of Ultraproducts, because each of the algebras $\mathfrak{B}_i$ is a model of $\mathcal{S}$. It remains to show that the given algebra $\mathfrak{A}$ is embeddable into the ultraproduct $\mathfrak{B}/D$. Take ψ to be the function from $\mathfrak{A}$ into $\mathfrak{B}/D$ that is defined by

$$\psi(r) = (\varphi_j(\boldsymbol{r}) : j \in I)/D \tag{9}$$

for each r in $\mathfrak{A}$. In other words, $\psi(r)$ is defined to be the system of interpretations, in the algebras of (3), of the individual constant symbol $\boldsymbol{r}$, reduced by the ultrafilter D.

To verify that ψ is an embedding, consider an equation or a negation of an equation $\Delta(v_0, \ldots, v_{n-1})$ that is satisfied in $\mathfrak{A}$ by a sequence of elements (7). The formula

$$\Delta(\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1}), \tag{10}$$

obtained from Δ by replacing each variable from (5) with the corresponding individual constant symbol from (4), is then true in $\mathfrak{A}'$, so the set i consisting of the single formula in (10) must belong to the index set I, by the definition of I. Every set j in J_i includes the set i and therefore contains the formula in (10), by the definition of J_i. Consequently, for every such j, the formula in (10) is true in $\mathfrak{B}_j$ under the interpretation φ_j, by the definition of this interpretation. In other words, the sequence

$$(\varphi_j(\boldsymbol{r}_0), \ldots, \varphi_j(\boldsymbol{r}_{n-1}))$$

satisfies the formula Δ in $\mathfrak{B}_j$ for every index j in the set J_i. It follows that the set of indices

$$\{j \in I : (\varphi_j(\boldsymbol{r}_0), \ldots, \varphi_j(\boldsymbol{r}_{n-1})) \text{ satisfies } \Delta \text{ in } \mathfrak{B}_j\} \tag{11}$$

includes the set J_i. Since the latter set belongs to the ultrafilter D, by the definition of D, the set in (11) must also belong to D, by the upward closure of D. Apply the Fundamental Theorem of Ultraproducts (with the set in (11) as the set in D) to conclude that the sequence

$$((\varphi_j(\boldsymbol{r}_0) : j \in I)/D, \ldots, (\varphi_j(\boldsymbol{r}_{n-1}) : j \in I)/D)$$

satisfies the formula Δ in the ultraproduct $\mathfrak{B}/D$. This sequence coincides with the sequence

$$(\psi(r_0), \ldots, \psi(r_{n-1})), \tag{12}$$

by (9), so the sequence in (12) satisfies the formula Δ in $\mathfrak{B}/D$.

It has been shown that if a sequence (7) satisfies an equation or a negation of an equation Δ in $\mathfrak{A}$, then the corresponding sequence (12) satisfies Δ in $\mathfrak{B}/D$. From this it follows easily that ψ must be a monomorphism from $\mathfrak{A}$ into $\mathfrak{B}/D$. For example, to show that ψ is one-to-one, take Δ to be the formula $v_0 \neq v_1$. If elements r_0 and r_1 in $\mathfrak{A}$ are distinct, then the sequence (r_0, r_1) satisfies Δ in $\mathfrak{A}$, so the image sequence $(\psi(r_0), \psi(r_1))$ must satisfy Δ in $\mathfrak{B}/D$, and therefore the elements $\psi(r_0)$ and $\psi(r_1)$ must be distinct. In order to show that ψ preserves the fundamental operations of the algebras, consider a binary operation $;$, and take Δ to be the formula $v_0 ; v_1 = v_2$. If $r_0 ; r_1 = r_2$ in $\mathfrak{A}$, then the sequence (r_0, r_1, r_2) satisfies Δ in $\mathfrak{A}$, so the image sequence $(\psi(r_0), \psi(r_1), \psi(r_2))$ must satisfy Δ in $\mathfrak{B}/D$, and therefore

$$\psi(r_0) ; \psi(r_1) = \psi(r_2),$$

which in turn yields

$$\psi(r_0 \,;\, r_1) = \psi(r_0) \,;\, \psi(r_1).$$

Thus, ψ preserves the operation ; . Analogous arguments show that ψ preserves all of the fundamental operations of the algebras, so ψ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}/D$. Since $\mathfrak{B}/D$ belongs to $\mathsf{P_u}(\mathsf{L})$, it follows that $\mathfrak{A}$ must belong to the class K, by (1). This completes the proof of the inclusion from right to left in (2) and the proof of the theorem. □

For elementary classes, it is not necessary to use ultraproducts to characterize universal classes.

Corollary 18.11. *The following conditions on a class of algebras* K *are equivalent.*

(i) K *is universal.*
(ii) K *is elementary and* $\mathsf{K} = \mathsf{S}(\mathsf{K})$.
(iii) $\mathsf{K} = \mathsf{S}(\mathsf{L})$ *for some elementary class* L.

Proof. As in the proof of the $\mathsf{SP_u}$-Theorem, the implications from (i) to (ii) and from (ii) to (iii) are obviously true. If (iii) holds, then

$$\mathsf{K} = \mathsf{S}(\mathsf{L}) = \mathsf{SP_u}(\mathsf{L}),$$

since the assumption that L is elementary implies that it is closed under ultraproducts. Consequently, K is universal, by the $\mathsf{SP_u}$-Theorem. □

The preceding corollary is one form of what is usually called the *Łoś-Tarski Preservation Theorem.*

The universal class generated by a class of algebras L is defined to be the intersection of all those universal classes of algebras that include L. The non-constructive character of this definition does not help us to understand which algebras actually belong to the generated universal class. The $\mathsf{SP_u}$-Theorem gives more precise information about these algebras.

Corollary 18.12. *The universal class generated by a class of algebras* L *is just* $\mathsf{SP_u}(\mathsf{L})$.

Proof. The class

$$\mathsf{K} = \mathsf{SP_u}(\mathsf{L}) \tag{1}$$

is universal, by the $\mathsf{SP_u}$-Theorem, and it contains every algebra in L, by Lemma 18.5 and the remarks in Section 18.1 about the class operation S. If M is any universal class of algebras that includes L, then

$$\mathsf{K} = \mathsf{SP_u}(\mathsf{L}) \subseteq \mathsf{SP_u}(\mathsf{M}) = \mathsf{M},$$

by (1) and the $\mathsf{SP_u}$-Theorem, so K is the smallest universal class that includes L. Conclusion: K is the universal class generated by L. □

When a class of algebras L is closed under ultraproducts, the description of the universal class generated by L assumes a very simple form.

Corollary 18.13. *If a class of algebras* L *is closed under ultraproducts, and in particular if* L *is an elementary class, then the universal class generated by* L *is just* $\mathsf{S}(\mathsf{L})$.

There are important analogues of the $\mathsf{SP_u}$-Theorem and the Łoś-Tarski Preservation Theorem that apply to quasi-varieties and, more generally, to universal Horn classes. Some of the proofs use the *method of diagrams*. To describe this method, consider an algebra $\mathfrak{A}$ (of the same similarity type as relation algebras). For each element r in $\mathfrak{A}$, adjoin a new individual constant symbol $\boldsymbol{r}$ to the language $\mathcal{L}$ of relation algebras in such a way that these new symbols are distinct from one another and from the symbols in $\mathcal{L}$. Denote the resulting expansion of $\mathcal{L}$ by $\mathcal{L}'$. For each equational relationship between elements in $\mathfrak{A}$, introduce a corresponding equation in $\mathcal{L}'$ as follows:

$$\begin{array}{rlr}
\boldsymbol{r} + \boldsymbol{s} = \boldsymbol{t} & \text{whenever} & r + s = t, \\
-\boldsymbol{r} = \boldsymbol{t} & \text{whenever} & -r = t, \\
\boldsymbol{r} \,;\, \boldsymbol{s} = \boldsymbol{t} & \text{whenever} & r \,;\, s = t, \\
\boldsymbol{r}^{\smile} = \boldsymbol{t} & \text{whenever} & r^{\smile} = t, \\
\mathbf{1}' = \boldsymbol{t} & \text{whenever} & 1' = t.
\end{array}$$

The set of these equations is called the *positive* (*open*) *diagram* of $\mathfrak{A}$. The positive diagram, together with the inequalities $\boldsymbol{r} \neq \boldsymbol{s}$ whenever r and s are distinct elements in $\mathfrak{A}$, is called the (*open*) *diagram* of $\mathfrak{A}$. If $\mathfrak{B}$ is any algebra similar to $\mathfrak{A}$, and if $\boldsymbol{r}^{\mathfrak{B}}$ is an interpretation of the constant symbol $\boldsymbol{r}$ in $\mathfrak{B}$ (for each element r in $\mathfrak{A}$), then write $\mathfrak{B}'$ to denote the algebra obtained by adjoining these interpretations to the list of fundamental operations of $\mathfrak{B}$ as new distinguished constants, and for brevity write

$$\mathfrak{B}' = (\mathfrak{B}\,,\, \boldsymbol{r}^{\mathfrak{B}})_{r\in A}.$$

In particular, $\mathfrak{A}' = (\mathfrak{A}\,,\, r)_{r\in A}$.

Lemma 18.14. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras, and for each new individual constant symbol $\boldsymbol{r}$ in the language $\mathcal{L}'$, let $\boldsymbol{r}^{\mathfrak{B}}$ be an interpretation of this symbol in $\mathfrak{B}$. The function mapping each element r in $\mathfrak{A}$ to the element $\boldsymbol{r}^{\mathfrak{B}}$ is a homomorphism or a monomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ if and only if $\mathfrak{B}'$ is a model of the positive diagram or the diagram of $\mathfrak{A}$ respectively.*

Proof. Let φ be the function from $\mathfrak{A}$ to $\mathfrak{B}$ defined by

$$\varphi(r) = \boldsymbol{r}^{\mathfrak{B}} \tag{1}$$

for each element r in $\mathfrak{A}$. Consider as an example the binary operation $;$. If r and s are arbitrary elements in $\mathfrak{A}$, and if $t = r\,;s$, then

$$\varphi(r\,;s) = \varphi(t) = \boldsymbol{t}^{\mathfrak{B}} \qquad \text{and} \qquad \varphi(r)\,;\varphi(s) = \boldsymbol{r}^{\mathfrak{B}}\,;\boldsymbol{s}^{\mathfrak{B}},$$

by (1). Consequently,

$$\varphi(r\,;s) = \varphi(r)\,;\varphi(s) \qquad \text{if and only if} \qquad \boldsymbol{t}^{\mathfrak{B}} = \boldsymbol{r}^{\mathfrak{B}}\,;\boldsymbol{s}^{\mathfrak{B}},$$

that is to say, if and only if the equation

$$\boldsymbol{r}\,;\boldsymbol{s} = \boldsymbol{t} \tag{2}$$

is true in $\mathfrak{B}'$. It follows that φ preserves the operation $;$ if and only if all of the equations of the form (2) in the positive diagram of $\mathfrak{A}$ are true in $\mathfrak{B}'$. Similar remarks apply to the other operations of $\mathfrak{A}$ and lead to the conclusion that φ is a homomorphism if and only if $\mathfrak{B}'$ is a model of the positive diagram of $\mathfrak{A}$.

The function φ is one-to-one if and only if

$$r \neq s \qquad \text{implies} \qquad \varphi(r) \neq \varphi(s),$$

or equivalently, if and only if

$$r \neq s \qquad \text{implies} \qquad \boldsymbol{r}^{\mathfrak{B}} \neq \boldsymbol{s}^{\mathfrak{B}},$$

by (1). Consequently, φ is one-to-one if and only if the inequality $\boldsymbol{r} \neq \boldsymbol{s}$ is true in $\mathfrak{B}'$ whenever r and s are distinct elements in $\mathfrak{A}$. Combine this observation with the remarks in the preceding paragraph to conclude that φ is a monomorphism if and only if $\mathfrak{B}'$ is a model of the diagram of $\mathfrak{A}$. □

Here is the analogue of the Łoś-Tarski Preservation Theorem. It may be called the SP-*Theorem* (*for elementary classes*).

Theorem 18.15. *The following conditions on a class of algebras* K *are equivalent.*

(i) K *is a quasi-variety.*
(ii) K *is elementary and closed under subalgebras and direct products.*
(iii) K *is elementary and* $\mathsf{K} = \mathsf{SP}(\mathsf{K})$.
(iv) $\mathsf{K} = \mathsf{SP}(\mathsf{L})$ *for some elementary class* L.

Proof. The implication from (i) to (ii) follows from the fact that quasi-varieties are closed under subalgebras and direct products, by Corollary 6.4 and Lemma 11.18 (and the remark following that lemma). For the implication from (ii) to (iii), observe that if K is closed under subalgebras and products, then

$$\mathsf{SP}(\mathsf{K}) = \mathsf{S}(\mathsf{K}) = \mathsf{K},$$

by Lemma 18.1(i),(iii), so (iii) holds. The implication from (iii) to (iv) is trivial: just take L to be K.

To prove the implication from (iv) to (i), assume that

$$\mathsf{K} = \mathsf{SP}(\mathsf{L}) \tag{1}$$

for some elementary class of algebras L. Take $\mathcal{S}$ to be the set of conditional equations that are true in L, with the goal of proving that

$$\mathsf{K} = \mathsf{Mo}(\mathcal{S}). \tag{2}$$

Every algebra in L is a model of $\mathcal{S}$, by the definition of $\mathcal{S}$. The formulas in $\mathcal{S}$ are preserved under products, by Lemma 11.18(ii) and the remark following that lemma. Consequently, the product of each system of algebras in L is a model of $\mathcal{S}$, and therefore every algebra that is embeddable into such a product is also a model of $\mathcal{S}$, by Corollary 6.4. Thus, every algebra in K is a model of $\mathcal{S}$, by (1), so the left side of (2) is included in the right side.

To establish the reverse inclusion, consider an arbitrary model $\mathfrak{A}$ of $\mathcal{S}$. In order to show that $\mathfrak{A}$ belongs to K, a system of algebras in L must be constructed such that $\mathfrak{A}$ is embeddable into the product of this system, by (1). The class L is assumed to be elementary, so it can be axiomatized by a set of first-order sentences in $\mathcal{L}$. Let $\mathcal{T}$ be such

an axiomatization, and let $\mathcal{D}$ be the diagram, and $\mathcal{D}_0$ the positive diagram, of $\mathfrak{A}$ (in the language $\mathcal{L}'$ defined before Lemma 18.14).

Fix an inequality $\boldsymbol{r} \neq \boldsymbol{s}$ in the diagram $\mathcal{D}$, and let $\mathcal{F}$ be any finite subset of the set of sentences

$$\mathcal{D}_0 \cup \{\boldsymbol{r} \neq \boldsymbol{s}\} \tag{4}$$

that includes the inequality $\boldsymbol{r} \neq \boldsymbol{s}$. The immediate goal is to show that the set of sentences

$$\mathcal{F} \cup \mathcal{T} \tag{5}$$

has a model. There are only finitely many distinct new individual constant symbols from $\mathcal{L}'$ that occur in $\mathcal{F}$, by the assumed finiteness of $\mathcal{F}$. Replace these distinct constants by distinct variables and form the conjunction of the resulting set of formulas to arrive at a formula Γ in $\mathcal{L}$ that is a conjunction of equations and one inequality. The formula Γ is satisfiable in $\mathfrak{A}$, because each formula in $\mathcal{D}$, and therefore each formula in $\mathcal{F}$, is true in the algebra $\mathfrak{A}' = (\mathfrak{A}\,,\, r)_{r \in A}$. The negation of Γ is logically equivalent to a conditional equation Δ, by the definition of Γ. Moreover, Δ is not universally valid in $\mathfrak{A}$, since Γ is satisfiable in $\mathfrak{A}$. Consequently, Δ cannot belong to the set $\mathcal{S}$ of conditional equations that are universally valid in L, because $\mathfrak{A}$ is assumed to be a model of $\mathcal{S}$. It follows that there must be an algebra $\mathfrak{C}$ in L in which Δ is not universally valid, and therefore in which the formula Γ is satisfiable by some sequence of elements t. Notice that $\mathfrak{C}$ is a model of the set of sentences $\mathcal{T}$, because $\mathcal{T}$ is an axiomatization of L, and $\mathfrak{C}$ belongs to L. Interpret each new individual constant symbol that occurs in $\mathcal{F}$ as the corresponding element in the sequence t, and interpret the remaining new individual constant symbols from $\mathcal{L}'$ as arbitrary elements in $\mathfrak{C}$, to arrive at an algebra

$$\mathfrak{C}' = (\mathfrak{C}\,,\, \boldsymbol{r}^{\mathfrak{C}})_{r \in A}$$

in which the sentences in $\mathcal{F}$ are true and also the set of sentences $\mathcal{T}$. Thus, $\mathfrak{C}'$ is a model of (5).

The observations of the preceding paragraph imply that every finite subset of the set of sentences

$$\mathcal{D}_0 \cup \{\boldsymbol{r} \neq \boldsymbol{s}\} \cup \mathcal{T} \tag{6}$$

has a model. Apply the Compactness Theorem for first-order logic to conclude that (6) has a model

$$\mathfrak{B}_i' = (\mathfrak{B}_i\,,\, r^{\mathfrak{B}_i})_{r\in A},$$

where i is taken to be the inequality $\boldsymbol{r} \neq \boldsymbol{s}$. In particular, restriction of this model to the similarity type of $\mathfrak{A}$, namely the algebra $\mathfrak{B}_i$, is a model of the set of axioms $\mathcal{T}$ and therefore belongs to the class L.

Take I to be the set of inequalities that occur in the diagram $\mathcal{D}$, and take $\mathfrak{B}$ to be the direct product

$$\mathfrak{B} = \prod_{i\in I} \mathfrak{B}_i.$$

This product belongs to the class $\mathsf{P}(\mathsf{L})$, because each factor $\mathfrak{B}_i$ belongs to L. For each element r in $\mathfrak{A}$, let $\hat{r}$ to be the element in $\mathfrak{B}$ determined by

$$\hat{r} = (\boldsymbol{r}^{\mathfrak{B}_i} : i \in I).$$

Thus, $\hat{r}$ is the system of interpretations of the individual constant symbol $\boldsymbol{r}$ in the factor algebras of $\mathfrak{B}$. For each index i in I, the positive diagram $\mathcal{D}_0$ is true in $\mathfrak{B}_i'$, because $\mathfrak{B}_i'$ is a model of (6), so the algebra

$$\mathfrak{B}' = (\mathfrak{B}\,,\, \hat{r})_{r\in A} \tag{7}$$

is also a model of $\mathcal{D}_0$, by Lemma 11.17. Similarly, the inequality i is true in $\mathfrak{B}_i'$, since $\mathfrak{B}_i'$ is a model of (6), so the algebra $\mathfrak{B}'$ is also a model of i, again by Lemma 11.17. This is true for every inequality i in the diagram $\mathcal{D}$, so $\mathfrak{B}'$ is a model of the diagram of $\mathfrak{A}$. Apply Lemma 18.14 to conclude that the function mapping each element r in $\mathfrak{A}$ to the element $\hat{r}$ in $\mathfrak{B}$ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$. Since $\mathfrak{B}$ belongs to $\mathsf{P}(\mathsf{L})$, it follows that $\mathfrak{A}$ belongs to the class K, by (1). This completes the proof of the implication from (iv) to (i). □

Here is that analogue of the SP_u-Theorem for quasi-varieties. It differs from the SP-Theorem in that it does not require the classes K and L to be elementary. It may be called the SPP_u-*Theorem*. The proof uses both the SP-Theorem and the SP_u-Theorem.

Theorem 18.16. *The following conditions on a class of algebras* K *are equivalent.*

(i) K *is a quasi-variety.*
(ii) K *is closed under subalgebras, direct products, and ultraproducts.*
(iii) $\mathsf{K} = \mathsf{SPP}_\mathsf{u}(\mathsf{K})$.
(iv) $\mathsf{K} = \mathsf{SPP}_\mathsf{u}(\mathsf{L})$ *for some class* L.

Proof. The proofs of the implications from (i) to (ii), from (ii) to (iii), and from (iii) to (iv) are very similar to the corresponding parts of the proof of the SP-Theorem, and are left to the reader. To establish the implication from (iv) to (i), assume that

$$\mathsf{K} = \mathsf{SPP}_\mathsf{u}(\mathsf{L}) \tag{1}$$

for some class of algebras L. Observe that

$$\mathsf{K} = \mathsf{SPP}_\mathsf{u}(\mathsf{L}) \subseteq \mathsf{SPSP}_\mathsf{u}(\mathsf{L}) \subseteq \mathsf{SSPP}_\mathsf{u}(\mathsf{L}) = \mathsf{SPP}_\mathsf{u}(\mathsf{L}) = \mathsf{K}, \tag{2}$$

by (1), the obvious inclusion of the class $\mathsf{P}_\mathsf{u}(\mathsf{L})$ in the class $\mathsf{SP}_\mathsf{u}(\mathsf{L})$, Lemma 18.2(ii), and Lemma 18.1(i). The first and last classes are the same, so equality must hold everywhere. The class

$$\mathsf{M} = \mathsf{SP}_\mathsf{u}(\mathsf{L}) \tag{3}$$

is elementary, and in fact universal, by the SP_u-Theorem (with M in place of K), and

$$\mathsf{K} = \mathsf{SP}(\mathsf{M}), \tag{4}$$

by (2) and (3), so K is a quasi-variety, by the SP-Theorem (with M in place of L in the implication from (iv) to (i)). □

The SPP_u-Theorem implies a purely algebraic characterization of the quasi-variety generated by a class of algebras.

Corollary 18.17. *The quasi-variety generated by a class of algebras* L *is just* $\mathsf{SPP}_\mathsf{u}(\mathsf{L})$.

Proof. By definition, the quasi-variety generated by a class of algebras L is the smallest quasi-variety that includes L. The class

$$\mathsf{K} = \mathsf{SPP}_\mathsf{u}(\mathsf{L}) \tag{1}$$

is a quasi-variety, by the SPP_u-Theorem, and it contains every algebra in L, by Lemma 18.5 and the remarks in Section 18.1 about the class operations S and P. If M is any quasi-variety that includes L, then

$$\mathsf{K} = \mathsf{SPP}_\mathsf{u}(\mathsf{L}) \subseteq \mathsf{SPP}_\mathsf{u}(\mathsf{M}) = \mathsf{M},$$

by (1) and the SPP_u-Theorem, so K is the smallest quasi-variety that includes L. □

When a class of algebras L is closed under ultraproducts, the description of the quasi-variety generated by L assumes a simpler form. In fact, we obtain a stronger form of the SP-Theorem in which the assumption that the class L is elementary is replaced by the weaker assumption that L is closed under ultraproducts.

Corollary 18.18. *If a class of algebras* L *is closed under ultraproducts, then the quasi-variety generated by* L *is just* $\mathsf{SP}(\mathsf{L})$.

In the lattice of universal classes of relation algebras, the meet of two universal classes is the intersection of the two classes, while the join of the two classes is, by definition, the intersection of all those universal classes that include the union of the two classes. It turns out, however, that this join actually coincides with the union of the two classes. The proof uses the SP_u-Theorem.

Lemma 18.19. *If* K_1 *and* K_2 *are universal classes of relation algebras, then the union* $\mathsf{K}_1 \cup \mathsf{K}_2$ *is a universal class of relation algebras. Consequently, this union coincides with the join of the two classes in the lattice of universal classes of relation algebras.*

Proof. To prove that the union

$$\mathsf{K}_1 \cup \mathsf{K}_2 \tag{1}$$

is a universal class, it suffices to prove that it is closed under ultraproducts and subalgebras, by the SP_u-Theorem. The proof of closure under subalgebras is easy. An algebra $\mathfrak{A}$ that belongs to (1) must belong to one of the two classes K_1 and K_2. If $\mathfrak{A}$ belongs to K_1, then every algebra embeddable into $\mathfrak{A}$ also belongs to K_1—and hence to (1)—by the assumption that K_1 is a universal class (since universal classes are closed under subalgebras). An analogous argument applies if $\mathfrak{A}$ belongs to K_2.

To show that (1) is closed under ultraproducts, consider a nonempty system

$$(\mathfrak{A}_i : i \in I) \tag{2}$$

of algebras in (1) and an arbitrary ultrafilter D over the index set I. The set I is naturally partitioned into two subsets: the set J of indices i for which $\mathfrak{A}_i$ is in K_1, and the complementary set $\sim J$ of indices i for which $\mathfrak{A}_i$ is in K_2. One of these two sets must belong to D, because D is assumed to be an ultrafilter. Suppose first that J is D. For each i in J,

the algebra $\mathfrak{A}_i$ belongs to K_1, by the definition of the set J, and therefore $\mathfrak{A}_i$ is a model of the set $\mathcal{S}$ of universal sentences axiomatizing the class K_1. It follows that the set of all indices i such that $\mathfrak{A}_i$ is a model of $\mathcal{S}$ includes the set J, and therefore belongs to the ultrafilter D, by the upward closure of D. Apply the Fundamental Theorem of Ultraproducts to conclude that the ultraproduct of the system (2) modulo the ultrafilter D is a model of $\mathcal{S}$ and therefore belongs to K_1. An analogous argument (with K_2 in place of K_1, and $\sim J$ in place of J) applies if the set $\sim J$ is in D. Conclusion: in either case, the ultraproduct of (2) modulo D belongs to (1), so (1) is closed under ultraproducts. This completes the proof of the first assertion of the lemma.

To prove the second assertion, observe that the join of the two classes K_1 and K_2 in the lattice of universal classes is the smallest universal class that includes the union (1). Since (1) is itself a universal class, by the first part of the lemma, it must coincide with this join. □

Warning: the preceding lemma implies that the join of a finite collection of universal classes is the union of the collection, but it does not imply that the join of an arbitrary collection of universal classes of algebras is the union of the collection.

Corollary 18.20. *The lattice of universal classes of relation algebras is distributive.*

Proof. The join and meet of two universal classes is their union and intersection respectively. Since intersections distribute over unions, and vice versa, it follows that join in the lattice distributes over meet. □

In constructing universal classes of relation algebras, one often begins with a given universal class K and adjoins new algebras to K. The universal class that is generated by this method can be written in a somewhat simpler form than the one given in Corollary 18.12.

Corollary 18.21. *If* K *is a universal class of relation algebras, and* L *an arbitrary class of relation algebras, then the universal class generated by* $\mathsf{K} \cup \mathsf{L}$ *is* $\mathsf{K} \cup \mathsf{SP}_\mathsf{u}(\mathsf{L})$.

Proof. The universal class generated by $\mathsf{K} \cup \mathsf{L}$ must coincide with the join, in the lattice of universal classes, of the universal classes generated by K and by L respectively. The former is K, by assumption, and the latter is $\mathsf{SP}_\mathsf{u}(\mathsf{L})$, by Corollary 18.12. The join of these two universal classes is just their union, by Lemma 18.19. □

Occasionally, a universal class of relation algebras is enlarged by adjoining finitely many new algebras, all of which are finite. In this case, the description of the generated universal class simplifies still further.

Corollary 18.22. *If* K *is a universal class of relation algebras, and* L *an arbitrary finite class of finite relation algebras, then the universal class generated by* $\mathsf{K} \cup \mathsf{L}$ *is* $\mathsf{K} \cup \mathsf{S}(\mathsf{L})$.

Proof. The class of isomorphic copies of finitely many finite algebras of finite similarity type is always elementary, because each algebra in the class can be described by a single first-order sentence, and the class itself can be axiomatized by the disjunction of the first-order sentences that describe each of the finitely many non-isomorphic algebras determining the class. Consequently, the universal class generated by L is just $\mathsf{S}(\mathsf{L})$, so $\mathsf{SP}_\mathsf{u}(\mathsf{L})$ may be replaced by $\mathsf{S}(\mathsf{L})$ in the proof of the preceding corollary. □

More general versions of the preceding lemma and its corollaries that apply to arbitrary universal classes of (similar) algebras are true and are proved in an analogous way.

18.4 The lattice of varieties

As has already been pointed out several times, the notions and results discussed in Sections 18.1—18.3 (with the exception of Corollary 18.8) are valid in the more general context of arbitrary algebras. We now turn to some results that are specific to relation algebras and to certain other classes of Boolean algebras with operators (classes in which there is a so-called *discriminator term*—see Exercise 9.16).

The class $\mathsf{Si}(\mathsf{RA})$ of all simple relation algebras is universal. In fact, Simplicity Theorem 9.2 implies that this class is axiomatized by the set of equations axiomatizing the theory of relation algebras, together with the single sentence

$$\forall v_0(v_0 \neq 0 \to 1 \,;\, v_0 \,;\, 1 = 1) \wedge 0 \neq 1,$$

which we henceforth denote by Δ_0. The class of all subclasses of $\mathsf{Si}(\mathsf{RA})$ is a complete sublattice of the lattice of all universal classes of relation algebras: the meet of a collection of classes of simple relation algebras

is the intersection of the class, and this is clearly a class of simple relation algebras; the join of the collection is also universal, because it is defined to be the intersection of all those universal classes that include every universal class in the collection, and one of the classes that is used in forming this intersection is $\mathsf{Si}(\mathsf{RA})$. In particular, the join of two universal classes of simple relation algebras is just their union, by Lemma 18.19, and the lattice of all universal classes of simple relation algebras is complete and distributive, by Corollary 18.20.

Given an arbitrary class K of relation algebras, we may form the class $\mathsf{Si}(\mathsf{K})$ of simple relation algebras that belong to K. If K is elementary, then the class $\mathsf{Si}(\mathsf{K})$ is also elementary, and in fact it can be axiomatized by adjoining the single sentence Δ_0 to a set of axioms for K. (This observation is not true in general for elementary classes of arbitrary algebras.) In particular, if K is universal, then the class $\mathsf{Si}(\mathsf{K})$ is also universal. In fact, there is an axiomatization of $\mathsf{Si}(\mathsf{K})$ that, except for the sentence Δ_0, consists entirely of equations.

Lemma 18.23. *Every universal class of simple relation algebras can be axiomatized by a set of equations and the single sentence* Δ_0.

Proof. Let K be a universal class of simple relation algebras, and $\mathcal{S}$ a set of quantifier-free formulas that axiomatizes K. Without loss of generality, it may be assumed that the axioms of relation algebra belong to $\mathcal{S}$. Every formula Γ in $\mathcal{S}$ is equivalent to an equation Γ^* in all simple relation algebras, by Theorem 9.5. If Γ is already an equation, then take Γ^* to be Γ itself. Write

$$\mathcal{S}^* = \{\Gamma^* : \Gamma \in \mathcal{S}\},$$

with the aim of showing that

$$\mathcal{S}^* \cup \{\Delta_0\} \tag{1}$$

is a set of axioms for K.

The sentence Δ_0 is true in K, by Simplicity Theorem 9.2, and so is every equation in $\mathcal{S}^*$, by Theorem 9.5, because the algebras in K are all assumed to be simple relation algebras in which the formulas of $\mathcal{S}$ are valid. Consequently, K is included in the class of models of (1).

To establish the reverse inclusion, consider an arbitrary model $\mathfrak{A}$ of (1). The axioms of relation algebra all belong to (1), by the assumptions on $\mathcal{S}$ and the assumption that Γ^* coincides with Γ when Γ is an equation. Moreover, the sentence Δ_0 is also in (1), so the model $\mathfrak{A}$

must be a simple relation algebra, by Theorem 9.2. Each equation Γ^* in $\mathcal{S}^*$ is equivalent to the quantifier-free formula Γ in $\mathcal{S}$ in all simple relation algebras, by the observations of the first paragraph, and $\mathfrak{A}$ is a model of $\mathcal{S}^*$, so every formula in $\mathcal{S}$ is true in $\mathfrak{A}$. Thus, $\mathfrak{A}$ is a model of the set of axioms of K, and therefore $\mathfrak{A}$ belongs to K. Conclusion: the set of models of (1) coincides with the class K. □

It follows in particular from the preceding lemma that a universal class K of simple relation algebras is axiomatized by the set of all equations true in K and the single sentence Δ_0.

Corollary 18.24. *A class* L *of simple relation algebras generates a universal class* K *that consists exclusively of simple relation algebras, and* L *and* K *have the same equational theory. The class* K *is axiomatized by the equational theory of* L *and the single sentence* Δ_0.

Proof. The universal class generated by L is the class

$$\mathsf{K} = \mathsf{SP}_{\mathsf{u}}(\mathsf{L}), \tag{1}$$

by Corollary 18.12. For relation algebras, the property of being simple is preserved under ultraproducts, by Corollary 18.8, and under subalgebras, by Corollary 9.3, so every algebra in K is simple.

Every equation true in K is obviously true in L, because L is included in K. The implication in the reverse direction is also true, by (1), because equations are preserved under ultraproducts and subalgebras, by the Fundamental Theorem of Ultraproducts and Corollary 6.4.

The class K is axiomatized by the set of equations true in K and the single sentence Δ_0, by Lemma 18.23. The final conclusion of the corollary follows at once, because K and L have the same equational theory, by the observations of the preceding paragraph. □

The next theorem establishes the essential connection between varieties of relation algebras and universal classes of simple relation algebras.

Theorem 18.25. *If* L *is a universal class of simple relation algebras, then the class* $\mathsf{K} = \mathsf{SP}(\mathsf{L})$ *is a variety, and* $\mathsf{L} = \mathsf{Si}(\mathsf{K})$, *that is to say,* L *is the class of simple algebras in* K. *Conversely, if* K *is a variety of relation algebras, then the class* $\mathsf{L} = \mathsf{Si}(\mathsf{K})$ *is a universal class of simple relation algebras, and* $\mathsf{K} = \mathsf{SP}(\mathsf{L})$.

Proof. We start with a preliminary observation. If a relation algebra $\mathfrak{A}$ belongs to a class K of relation algebras that is closed under homomorphisms, then $\mathfrak{A}$ is isomorphic to a subdirect product of simple algebras that are all in K. Indeed, $\mathfrak{A}$ is isomorphic to a subdirect product of simple relation algebras, by Semi-simplicity Theorem 12.10, and each of these simple algebras is a homomorphic image of $\mathfrak{A}$, by the definition of a subdirect product, so each of them belongs to K.

Assume now that L is a universal class of simple relation algebras, and that

$$\mathsf{K} = \mathsf{SP}(\mathsf{L}). \tag{1}$$

Let $\mathcal{E}$ be the set of equations true in L, and M the variety of models of $\mathcal{E}$. The goal is to prove that $\mathsf{K} = \mathsf{M}$. Observe in passing that the set of formulas

$$\mathcal{E} \cup \{\Delta_0\} \tag{2}$$

axiomatizes L, by Lemma 18.23 and the remark following that lemma.

Every equation in $\mathcal{E}$ is true in L, by the definition of $\mathcal{E}$, and equations are preserved under direct products and subalgebras, by Lemma 11.17(ii) and Corollary 6.4, so every equation in $\mathcal{E}$ must also be true in K, by (1). Thus, K is included in the class M of all models of $\mathcal{E}$.

To establish the reverse inclusion, consider an arbitrary algebra $\mathfrak{A}$ in M, with the aim of showing that $\mathfrak{A}$ belongs to K. Clearly, $\mathfrak{A}$ must be a relation algebra, because $\mathcal{E}$, by its very definition, contains all of the axioms of relation algebra. Consequently, $\mathfrak{A}$ is isomorphic to a subdirect product of simple algebras that are all in M, by the preliminary observation made above (with M in place of K). Each of these simple algebras is a model of $\mathcal{E}$, by the definition of M, and also a model of the sentence Δ_0, by Simplicity Theorem 9.2, so each of them must belong to the class L of models of (2). Thus, $\mathfrak{A}$ is isomorphic to a subdirect product of algebras in L, and therefore $\mathfrak{A}$ belongs to K, by (1). Conclusion: the classes K and M are equal, so K is the variety axiomatized by the set of equations $\mathcal{E}$.

The class $\mathsf{Si}(\mathsf{K})$ of simple algebras in K is the class of simple models of $\mathcal{E}$, by the conclusion of the preceding paragraph, so it is the class of models of (2), by Theorem 9.2. Since L is also the class of models of (2), it follows that

$$\mathsf{L} = \mathsf{Si}(\mathsf{K}). \tag{3}$$

This completes the proof of the first assertion of the theorem.

To establish the second assertion, assume that K is a variety of relation algebras, and that (3) holds. The class L is universal, by the observation preceding Lemma 18.23, and it consists just of simple relation algebras, by (3). Every algebra in K is a subdirect product of simple algebras in K, by the preliminary observation made at the beginning of the proof, and each of these simple algebras is in L, by (3), so every algebra in K is a subdirect product of algebras in L. Thus, K is included in $\mathsf{SP}(\mathsf{L})$. On the other hand, L is included in K, by (3), so

$$\mathsf{SP}(\mathsf{L}) \subseteq \mathsf{SP}(\mathsf{K}) \subseteq \mathsf{K},$$

because K, as a variety, is closed under subalgebras and direct products. Thus, (1) holds. □

The following corollary gives a stronger version of the first part of Theorem 18.25 that applies to classes of simple relation algebras closed under ultraproducts, and in particular to elementary classes of simple relation algebras.

Corollary 18.26. *If* L *is a class of simple relation algebras that is closed under ultraproducts, and in particular if* L *is elementary, then the variety generated by* L *is* $\mathsf{SP}(\mathsf{L})$*, and* $\mathsf{S}(\mathsf{L})$ *is the class of simple algebras in this variety. Consequently,* $\mathsf{SP}(\mathsf{L})$ *is closed under directed unions and homomorphisms.*

Proof. If L is a class of simple relation algebras that is closed under ultraproducts, then Corollary 18.13 implies that $\mathsf{S}(\mathsf{L})$ is a universal class. Apply the first part of Theorem 18.25 (with $\mathsf{S}(\mathsf{L})$ in place of L) to see that the variety generated by $\mathsf{S}(\mathsf{L})$ is $\mathsf{SPS}(\mathsf{L})$, and that $\mathsf{S}(\mathsf{L})$ is the class of simple algebras in this variety. It is easy to check, using Lemma 18.2(ii), that

$$\mathsf{SPS}(\mathsf{L}) = \mathsf{SP}(\mathsf{L}),$$

so the desired conclusions follows at once. In particular, since varieties are always closed under directed unions and homomorphisms, the final conclusion of the corollary holds. □

We shall have more to say about the implications of this corollary in a moment. To appreciate its significance, it is helpful to characterize varieties of relation algebras algebraically, using well-known criteria that hold in the context of arbitrary algebras, not just relation algebras. The theorem is sometimes called the HSP-*Theorem*, because of

its last two conditions. The proof that we shall give is not the standard one, as it makes use of some facts that are specific to relation algebras and to related algebras such as Boolean algebras with operators that have a unary discriminator term (see Exercise 9.16). The standard proof, in the context of arbitrary algebras (and not just relation algebras) uses the notion of a free algebra.

Theorem 18.27. *The following conditions on a class* K *of algebras are equivalent.*

(i) K *is a variety.*
(ii) K *is closed under subalgebras, homomorphisms, and products.*
(iii) $\mathsf{K} = \mathsf{HSP}(\mathsf{K})$.
(iv) $\mathsf{K} = \mathsf{HSP}(\mathsf{L})$ *for some class* L *of algebras.*

Proof. A variety is axiomatized by a set of equations, and equations are preserved under subalgebras, homomorphisms, and (direct) products, by Corollaries 6.4 and 7.3, and Lemma 11.17(ii), so the implication from (i) to (ii) is obvious. The equivalence of conditions (ii), (iii), and (iv) follows from Lemma 18.3.

To establish the implication from (ii) to (i), assume that K is closed under subalgebras, homomorphisms, and products. An ultraproduct of algebras in K is, by definition, a quotient—and therefore a homomorphic image—of a direct product of algebras in K, so all such ultraproducts belong to K, by the assumed closure of K under products and homomorphisms. It follows that K is closed under ultraproducts as well as subalgebras, so K is a universal class, by the SP_u-Theorem. The subclass

$$\mathsf{L} = \mathsf{Si}(\mathsf{K}) \tag{1}$$

of simple algebras in K is also universal, by the remark preceding Lemma 18.23. Apply Theorem 18.25 to obtain that the class $\mathsf{SP}(\mathsf{L})$ is a variety. It is not difficult to see that

$$\mathsf{K} = \mathsf{SP}(\mathsf{L}), \tag{2}$$

from which it follows that K must also be a variety. In more detail, the class K is assumed to be closed under homomorphisms, so the preliminary remark from the proof of Theorem 18.25 applies: every algebra in K is a subdirect product of simple algebras in K, that is to say, every algebra in K is a subdirect product of algebras in L. Consequently, using also (1) and the equivalence of conditions (ii) and (iii), we get

$$\mathsf{K} \subseteq \mathsf{SP}(\mathsf{L}) \subseteq \mathsf{SP}(\mathsf{K}) \subseteq \mathsf{HSP}(\mathsf{K}) = \mathsf{K}.$$

The first and last classes are the same, so equality must hold everywhere, and in particular (2) must hold. □

The HSP-Theorem says that in order to obtain the variety generated by a class of algebras L, the class operations P, S, and H must be applied in that order. Corollary 18.26 sharpens this result in the case when the generating class L is a class of simple relation algebras that is closed under ultraproducts. It says that in this case, it suffices to apply the class operations P and then S in that order; the resulting class $\mathsf{SP}(\mathsf{L})$ will perforce be closed under homomorphisms.

There is a close connection between the lattice of varieties of relation algebras and the lattice of universal classes of simple relation algebras. In fact, the two are isomorphic, as the following *Correspondence Theorem* shows.

Theorem 18.28. *The correspondence that maps each variety* K *of relation algebras to the universal class* $\mathsf{Si}(\mathsf{K})$ *of simple relation algebras in* K *is an isomorphism from the lattice of varieties of relation algebras to the lattice of universal classes of simple relation algebras. The inverse correspondence maps every universal class* L *of simple relation algebras to the variety* $\mathsf{SP}(\mathsf{L})$ *generated by* L.

Proof. Let φ be the function mapping each variety K of relation algebras to the universal class $\mathsf{Si}(\mathsf{K})$, and let ψ be the function mapping each universal class L of relation algebras to the variety $\mathsf{SP}(\mathsf{L})$. These functions are well defined, by Theorem 18.25, and

$$\psi(\varphi(\mathsf{K})) = \psi(\mathsf{Si}(\mathsf{K})) = \mathsf{SP}(\mathsf{Si}(\mathsf{K})) = \mathsf{K},$$

by the second part of the theorem, while

$$\varphi(\psi(\mathsf{L})) = \varphi(\mathsf{SP}(\mathsf{L})) = \mathsf{Si}(\mathsf{SP}(\mathsf{L})) = \mathsf{L},$$

by the first part of the theorem. These two computations show that the composition $\psi \circ \varphi$ is the identity function on the lattice of varieties of relation algebras, and the composition $\varphi \circ \psi$ is the identity function on the lattice of universal classes of simple relation algebras. It follows that φ is a bijection from the first lattice to the second, and its inverse is ψ.

It remains to check that φ preserves the lattice ordering of inclusion. Obviously,

$$\mathsf{K}_1 \subseteq \mathsf{K}_2 \qquad \text{implies} \qquad \mathsf{Si}(\mathsf{K}_1) \subseteq \mathsf{Si}(\mathsf{K}_2)$$

and

$$\mathsf{Si}(\mathsf{K}_1) \subseteq \mathsf{Si}(\mathsf{K}_2) \qquad \text{implies} \qquad \mathsf{SP}(\mathsf{Si}(\mathsf{K}_1)) \subseteq \mathsf{SP}(\mathsf{Si}(\mathsf{K}_2)).$$

In view of the observations of the first paragraph, the final inclusion in the second implication may be rewritten as $\mathsf{K}_1 \subseteq \mathsf{K}_2$. Combine the two implications with the definition of φ to arrive at

$$\mathsf{K}_1 \subseteq \mathsf{K}_2 \qquad \text{if and only if} \qquad \varphi(\mathsf{K}_1) \subseteq \varphi(\mathsf{K}_2).$$

Thus, the bijection φ preserves the relation of inclusion and is therefore a lattice isomorphism. □

By combining Corollary 18.20 with the preceding theorem, we arrive at the following conclusion.

Corollary 18.29. *The lattice of varieties of relation algebras is distributive.*

18.5 The variety of representable relation algebras

The class RRA of all representable relation algebras plays a fundamental role in the theory of relation algebras, in part because it is the class of algebras that are isomorphic to set relation algebras, the prototypical and motivational examples of the theory of relation algebras. One of Tarski's aims in his original 1941 paper was apparently to propose a set of axioms for this class, and he explicitly raised the problem of whether every equation that is true in all representable relation algebras is derivable from his set of axioms. Writing RA for the class of all models of axioms (R1)–(R10), that is to say, for the class of all relation algebras, and using some of the notation from Section 18.1, his problem in its modern form asks whether the set of equations $Eq(\mathsf{RRA})$ is included in the set of equations $Eq(\mathsf{RA})$. The reverse inclusion is obviously true, since RRA is included in RA, so the problem reduces to the question of whether the equality

$$Eq(\mathsf{RRA}) = Eq(\mathsf{RA})$$

is true. Tarski also raised the problem of whether every model of his set of axioms is representable. In its modern form, this problem asks whether the class of algebras RA is included in the class of algebras RRA. The reverse inclusion is obviously true, so the problem reduces to the question of whether the equality

$$\mathsf{RRA} = \mathsf{RA}$$

is true. Tarski indicated that the two problems are related, and in fact that an affirmative answer to the second problem implies an affirmative answer to the first one.

As was mentioned earlier, Lyndon proved that the answer to both problems is negative. However, Tarski was later able to establish the following fundamental result.

Theorem 18.30. RRA *is a variety.*

Proof. The class RRA is closed under direct products, by Corollary 16.20, under subalgebras, by Lemma 16.11, and under homomorphisms, by Theorem 16.27. Apply the HSP-Theorem to conclude that RRA is a variety. □

It follows from this theorem that the two problems raised by Tarski are actually equivalent. Indeed if the equational theory of RRA were equal to the equational theory of RA, then the classes of models of the two equational theories would also be equal. Since both classes are varieties, they coincide with the classes of models of their respective equational theories, so we would obtain that $\mathsf{RRA} = \mathsf{RA}$.

Theorem 16.22 and it's consequence, Theorem 16.27, namely the closure of RRA under homomorphisms, were not available to Tarski when he discovered Theorem 18.30. In fact, Theorem 16.27 is one of the important corollaries of Theorem 18.30 that Tarski noted at the time. His original proof of Theorem 18.30 is interesting and useful in its own right. We present it here in a somewhat modified form.

Let L be the class of simple and atomic relation algebras with singleton atoms, and observe that L is an elementary class. Indeed, the properties of a relation algebra being simple, being atomic, and of every atom being a singleton are all expressible by first-order sentences. Consequently, the class $\mathsf{K} = \mathsf{SP}(\mathsf{L})$ is a variety, and $\mathsf{S}(\mathsf{L})$ is the class of simple algebras in K, by Corollary 18.26. Every full set relation algebra on a non-empty set is simple and atomic with singleton atoms, so every such algebra belongs to L. Every full set relation algebra on

an equivalence relation is isomorphic to a direct product of full set relation algebras on non-empty sets, by Decomposition Theorem 11.43 for $\mathfrak{Re}(E)$; and every representable relation algebra is embeddable into a full set relation algebra on some equivalence relation, by the very definition of a representation. Combine these observations to conclude that every representable relation algebra is isomorphic to a subalgebra of a direct product of algebras in L. In other words, RRA is included in K.

On the other hand, every algebra in L is representable, by Corollary 17.37, so L is included in the class RRA. Also, the class RRA is closed under subalgebras and direct products, by Lemma 16.11 and Corollary 16.20. Combine these observations to arrive at

$$\mathsf{K} = \mathsf{SP}(\mathsf{L}) \subseteq \mathsf{SP}(\mathsf{RRA}) \subseteq \mathsf{RRA} \subseteq \mathsf{K}.$$

The first and last classes are the same, so equality must hold everywhere. Conclusion: RRA is the variety generated by L, and $\mathsf{S}(\mathsf{L})$ is the class of simple algebras in RRA.

There is a more transparent description of the class of simple algebras in RRA. It was already pointed out that every full set relation algebras on a non-empty set belongs to L. On the other hand, every algebra in L—and therefore every algebra in $\mathsf{S}(\mathsf{L})$—is embeddable into a full set relation algebra on a non-empty set, by Lemma 17.39 and Theorem 17.44. Consequently, $\mathsf{S}(\mathsf{L})$ coincides with the class of algebras embeddable into full set relation algebras on non-empty sets.

Corollary 18.31. *The class of simple algebras in* RRA *is just the class of subalgebras of full set relation algebras on non-empty sets, and the isomorphic copies of such subalgebras.*

Here is a slightly different way of saying the same thing: a representable relation algebra is simple if and only if it can be embedded into $\mathfrak{Re}(U)$ for some non-empty set U.

Neither of the two proofs of Theorem 18.30 gives any information about possible axiomatizations of the variety RRA. Sets of equational axioms for this variety are known, but the axioms are infinite in number and complicated in structure. It is natural to ask whether there is a finite set of equations that axiomatizes RRA. The answer is negative. The proof involves an ultraproduct construction using complex algebras of projective lines, and depends upon the following lemma. Recall that projective lines of order at least three are really just sets of four or more points, all collinear with one another. An ultraproduct

of such sets is also a set of at least four points, all collinear with one another, so an ultraproduct of a non-empty system of projective lines of order at least three is again a projective line of order at least three.

Lemma 18.32. *Let $(P_i : i \in I)$ be a non-empty system of projective lines of order at least three, and write*

$$P = \prod_i P_i \qquad \text{and} \qquad \mathfrak{A} = \prod_i \mathfrak{Cm}(P_i).$$

For every ultrafilter D over I, the completion of the ultraproduct $\mathfrak{A}/D$ is isomorphic to the geometric complex algebra $\mathfrak{Cm}(P/D)$.

Proof. Before turning to the proof proper, we make a series of preliminary observations about the atoms in the two algebras, beginning with the atoms in $\mathfrak{A}/D$. The algebra $\mathfrak{A}$ is defined to be the direct product of the complex algebras $\mathfrak{Cm}(P_i)$. Each of these complex algebras is defined so as to be atomic, the atoms being the singletons of points (elements) in the set

$$P_i^+ = P_i \cup \{\iota_i\} \tag{1}$$

(where ι_i is a new element that is adjoined to the set P_i of points of the line). These singleton atoms may be, and will be, identified with the points themselves, that is to say, a point in (1) will be identified with its own singleton.

The property of a relation algebra being atomic is expressible by a first-order sentence, and is therefore preserved under the passage to ultraproducts, by the Fundamental Theorem of Ultraproducts. It follows from the definition of $\mathfrak{A}$ that the ultraproduct $\mathfrak{A}/D$ is atomic, and in fact its atoms are the elements of the form p/D, where p is an element in $\mathfrak{A}$ with the property that the set

$$\{i \in I : p(i) \text{ is an atom in } \mathfrak{Cm}(P_i)\}$$

belongs to D, again by the Fundamental Theorem of Ultraproducts. In view of the identification of atoms in $\mathfrak{Cm}(P_i)$ with points in (1), each atom p/D in $\mathfrak{A}/D$ may be thought of as the quotient modulo D of an element p in $\mathfrak{A}$ with the property that the set

$$J_p = \{i \in I : p(i) \in P_i^+\} \tag{2}$$

belongs to D.

The preceding remarks imply that each element

$$p = (p(i) : i \in I)$$

in the product set $P = \prod_i P_i$ is identified with an element in $\mathfrak{A}$, namely the element

$$(\{p(i)\} : i \in I).$$

In the same way, the element ι determined by

$$\iota = (\iota_i : i \in I)$$

is identified with the identity element in $\mathfrak{A}$ via the identification of each element ι_i with its singleton. Notice that ι does not belong to the set P, since its coordinates ι_i do not belong to the sets P_i. Put

$$P^+ = P \cup \{\iota\}. \tag{3}$$

Consider any atom p/D in $\mathfrak{A}/D$. We proceed to show that there is an element q in the set (3) such that

$$p/D = q/D \tag{4}$$

in $\mathfrak{A}/D$. The set J_p in (2) is the disjoint union of the two sets

$$J'_p = \{i \in I : p(i) \in P_i\} \qquad \text{and} \qquad J''_p = \{i \in I : p(i) = \iota_i\}, \tag{5}$$

by (1) and (2). Since J_p belongs to D, by the assumption that p/D is an atom, exactly one of the sets in (5) must belong to D, by the ultrafilter properties of D. If J'_p is in D, take q to be any element in P such that $q(i) = p(i)$ for each i in J'_p. Such an element certainly exists because each of the sets P_i is non-empty. If J''_p is in D, take q to be ι. In either case, the set

$$\{i \in I : p(i) = q(i)\} \tag{6}$$

includes a set in D—namely J'_p or J''_p respectively—so it belongs to the ultrafilter D, by the upward closure of D. Consequently, p and q are congruent modulo D, by the definition of this congruence, and therefore (4) holds. Conclusion: every atom in $\mathfrak{A}/D$ has the form p/D for some element p in (3), so it may be assumed that the atoms of $\mathfrak{A}/D$ are always written in this form.

Before discussing the atoms in the complex algebras $\mathfrak{Cm}(P/D)$, it is helpful to look more closely at the elements in the set P/D. Each such element is, by definition, an equivalence class modulo D of an element

in the set P. This leads to a certain ambiguity of notation. As was pointed out above, each element p in P is identified with an element in $\mathfrak{A}$, namely the element whose coordinates are the singletons of the coordinates of p. For that reason, the quotient p/D may be formed in both P/D and $\mathfrak{A}/D$, and the results are not the same. The first quotient consists of the elements in P that agree with p on a set in D, while the second consists of the elements in $\mathfrak{A}$ that agree with p on a set in D. Since every element in P is identified with an element in $\mathfrak{A}$, every member of the equivalence class p/D in P/D belongs to the equivalence class p/D in $\mathfrak{A}/D$, so the first equivalence class is included in the second. The reverse inclusion fails, in general, because there are elements in $\mathfrak{A}$ that agree with p on a set in D even though they do not belong to P, and these elements belong to the second equivalence class, but not to the first. The exact relationship between the two equivalence classes is that the first one is the intersection of the second one with the set P.

The atoms of the complex algebra $\mathfrak{Cm}(P/D)$ are the singletons of points in P/D and the singleton of a new point that is adjoined to the set P/D to form the set $(P/D)^+$. As before, these singleton atoms may be identified with the points themselves. Moreover, without loss of generality it may be assumed that the new point is the element ι/D from $\mathfrak{A}/D$, so that

$$(P/D)^+ = P^+/D = (P/D) \cup \{\iota/D\}. \tag{7}$$

Indeed, the element ι does not belong to the set P, and the congruence class ι/D does not belong to the set P/D (in fact, no element in P agrees with ι on a set of elements in D), so no conflict arises by assuming that the new point is ι/D. (See, however, the remarks in the preceding paragraph.) This assumption makes the subsequent argument notationally much simpler.

The intuition behind the proof of the lemma is that each atom p/D in $\mathfrak{A}/D$ can be identified with a point in (7), and vice versa. Under this identification, the set of atoms in $\mathfrak{A}/D$ coincides with the set of points in (7), which in turn coincides with the set of atoms in $\mathfrak{Cm}(P/D)$; and the relative product of two atoms in $\mathfrak{A}/D$ coincides with the relative product of these two atoms in $\mathfrak{Cm}(P/D)$. Consequently, the identification gives rise to a natural isomorphism from the completion of $\mathfrak{A}/D$ to the complex algebra $\mathfrak{Cm}(P/D)$, by the Atomic Isomorphism Theorem.

The way to make this intuition precise is to define a function φ from the set of atoms in $\mathfrak{A}/D$ to the set (7) in the following way. For each

atom p/D in $\mathfrak{A}/D$, we may assume that p is in the set (3); put

$$\varphi(p/D) = p/D, \tag{8}$$

where the first quotient is formed in $\mathfrak{A}/D$ and the second in P/D (except when $p = \iota$). The correspondence φ is not the identity function because the first quotient is not equal to the second. Rather, with one exception (namely when $p = \iota$), the function φ maps each atom in $\mathfrak{A}/D$ to the intersection of this atom with the set in (3).

It is not difficult to see that function φ is well defined and one-to-one. Indeed, for elements p and q in (3), the quotient atoms p/D and q/D in $\mathfrak{A}/D$ are equal if and only if the set in (6) belongs to D, by the definition of the congruence $\equiv_D$, and the same is true for the corresponding quotient elements p/D and q/D in (7). Consequently, two quotient atoms in $\mathfrak{A}/D$ are equal if and only if the corresponding quotient elements in (7) are equal. In other words,

$$p/D = q/D \qquad \text{if and only if} \qquad \varphi(p/D) = \varphi(q/D).$$

It is clear that the domain of φ is the set of atoms in $\mathfrak{A}/D$, because every such atom has the form p/D for some element p in (3); and the range of φ is the set of elements in (7). In view of the identification of the atoms in $\mathfrak{Cm}(P/D)$ with the elements in $(P/D)^+$, it may be concluded that φ is a bijection from the set of atoms in $\mathfrak{A}/D$ to the set of atoms in $\mathfrak{Cm}(P/D)$. The next step is to show that φ preserves the operation of relative multiplication on atoms.

Suppose first that p and q are points in P. If $p/D \neq q/D$, then the set

$$J = \{i \in I : p(i) \neq q(i)\}$$

belongs to D, by the definition of the congruence $\equiv_D$ and the ultrafilter properties of D. For each index i in J, we have

$$p(i) \,;\, q(i) = 1(i) - (p(i) + q(i) + \iota(i)) \tag{9}$$

in $\mathfrak{Cm}(P_i)$ (where 1 is the unit element in $\mathfrak{A}$), by the definition of relative multiplication in $\mathfrak{Cm}(P_i)$. Thus, the set of indices for which (9) holds includes the set J and therefore belongs to the ultrafilter D. It follows that

$$p/D \,;\, q/D = 1/D - (p/D + q/D + \iota/D)$$

in $\mathfrak{A}/D$, by the definition of the operations of $\mathfrak{A}/D$. In other words, for each atom r/D in $\mathfrak{A}/D$,

$$r/D \leq p/D\,;q/D \quad \text{if and only if} \quad r/D \neq p/D,\ q/D,\ \iota/D. \tag{10}$$

The ultraproduct P/D is also a projective line of order at least three (see the remarks preceding the lemma), and the atoms p/D and q/D in $\mathfrak{Cm}(P/D)$ are also distinct (see the remarks made in the proof that φ is one-to-one), so the relative product of these two atoms in $\mathfrak{Cm}(P/D)$ is also the unit element minus the sum

$$p/D + q/D + \iota/D$$

(where the quotients involved in this sum are elements in the set (7), not in $\mathfrak{A}/D$), by the definition of relative multiplication in the complex algebra $\mathfrak{Cm}(P/D)$. Consequently, the equivalence in (10) continues to hold in $\mathfrak{Cm}(P/D)$ for all atoms r/D. Combine these observations with the definition of φ in (8) to conclude that

$$r/D \leq p/D\,;q/D \quad \text{if and only if} \quad \varphi(r/D) \leq \varphi(p/D)\,;\varphi(q/D) \tag{11}$$

for all atoms r/D in $\mathfrak{A}/D$ (where the quotients are formed in $\mathfrak{A}/D$).

A similar argument shows that if $p/D = q/D$ in $\mathfrak{A}/D$ (with p and q in P), then the same is true in $\mathfrak{Cm}(P/D)$, and therefore

$$p/D\,;q/D = \iota/D + p/D$$

in both algebras, by the definitions of relative multiplication in the algebras; thus, (11) is also valid in this case. Finally, if p is in $P \cup \{\iota\}$ and q is ι, then

$$p/D\,;q/D = p/D\,;\iota/D = p/D$$

in both algebras, which shows that (11) is valid in this case as well. Conclusion: (11) is valid for any atoms p/D, q/D, and r/D in $\mathfrak{A}/D$, so the function φ preserves the operation of relative multiplication with respect to atoms.

Consider now the completion $\mathfrak{B}$ of the ultraproduct $\mathfrak{A}/D$. It is a complete and atomic relation algebra with the same atoms as $\mathfrak{A}/D$ (see Definition 15.17, Theorem 15.28 and Lemma 15.29), and the relative product of any two atoms in $\mathfrak{B}$ coincides with the relative product of these atoms in $\mathfrak{A}/D$. It follows that the mapping φ is a bijection

from the set of atoms in $\mathfrak{B}$ to the set of atoms in $\mathfrak{Cm}(P/D)$, and the equivalence in (11) continues to hold. Apply the Atomic Isomorphism Theorem in the form of Corollary 7.12 to conclude that φ can be extended to an isomorphism from $\mathfrak{B}$ to $\mathfrak{Cm}(P/D)$. □

The essential difference between the algebras $\mathfrak{A}/D$ and $\mathfrak{Cm}(P/D)$ in the preceding proof is that the universe of the latter contains all subsets of the set $(P/D)^+$, while the universe of the former contains analogues of some of these subsets—including all of the singleton subsets—but in general does not contain analogues of all of the subsets. For that reason, it is necessary to pass to the completion of $\mathfrak{A}/D$ in order to obtain isomorphism with $\mathfrak{Cm}(P/D)$.

Here is an instructive example. Let I be the set of natural numbers, and P_i the projective line of order $i+4$ for each i in I. Take D to be a non-principal ultrafilter over I. The algebra $\mathfrak{A}/D$ (from Lemma 18.32) has cardinality $2^{\aleph_0}$, and so does the set P/D. The complex algebra $\mathfrak{Cm}(P/D)$, however, has cardinality $2^{2^{\aleph_0}}$, because it consists of all subsets of $(P/D)^+$. Clearly, in this case $\mathfrak{A}/D$ is much smaller than $\mathfrak{Cm}(P/D)$, and is therefore much smaller than its own completion.

As an application of this example, notice that the complex algebras $\mathfrak{Cm}(P_i)$ are in this case all finite, and therefore complete, so they are their own completions. The ultraproduct $\mathfrak{A}/D$ of these complex algebras is, however, not complete, since its completion has a strictly larger cardinality. Conclusion: an ultraproduct of the completions of a system of relation algebras is not in general (isomorphic to) the completion of the ultraproduct of the system of algebras.

Theorem 18.33. *The variety* RRA *is not axiomatizable by any finite set of first-order sentences.*

Proof. The idea of the proof is to construct an infinite system of non-representable relation algebras

$$(\mathfrak{A}_i : i \in I) \tag{1}$$

such that the ultraproduct of the system over some ultrafilter D is representable. Once this has been accomplished, the argument proceeds as follows. Suppose, to the contrary, that the variety RRA is axiomatizable by a finite set of sentences. Form the conjunction of the sentences in such an axiomatization to obtain a single sentence Γ that axiomatizes the variety. Thus, a relation algebra is representable just

in case it is a model of Γ, and non-representable just in case it is a model of $\neg\Gamma$. The algebras $\mathfrak{A}_i$ are all assumed to be non-representable, so they are all models of $\neg\Gamma$. Consequently, the set

$$\{i \in I : \mathfrak{A}_i \text{ is a model of } \neg\Gamma\}$$

coincides with the index set I and therefore belongs to the ultrafilter D. (Every ultrafilter on a set I must contain the set I.) Apply the Fundamental Theorem of Ultraproducts to conclude that the ultraproduct is a model of the sentence $\neg\Gamma$ and is therefore not representable. Since the ultraproduct is assumed to be representable, the desired contradiction has arrived.

The construction of the system in (1) is easy: take I to be the set of natural numbers, and take

$$(P_i : i \in I) \tag{2}$$

to be an infinite sequence of projective lines of strictly increasing finite orders, in the sense that

$$i < j \qquad \text{implies} \qquad |P_i| < |P_j|$$

(where $|P_i|$ denotes the number of points on the line P_i), and such that the complex algebras

$$\mathfrak{A}_i = \mathfrak{Cm}(P_i)$$

are all non-representable. Such a sequence of lines exists by Theorem 17.15. Put

$$\mathfrak{A} = \prod_i \mathfrak{A}_i \qquad \text{and} \qquad P = \prod_i P_i,$$

and let D be any non-principal ultrafilter on the set I. Thus, D must contain every cofinite subset of I, because an ultrafilter that contains a finite set is necessarily principal. It must be shown that the ultraproduct $\mathfrak{A}/D$ is representable.

For each natural number n, the set

$$J_n = \{i \in I : P_i \text{ has at least } n \text{ elements}\} \tag{3}$$

contains all but finitely many of the indices in I, by the assumption that the lines in (2) are of strictly increasing finite orders. Consequently, each of these sets belongs to the ultrafilter D, by the assumption that D is non-principal (and therefore contains all cofinite sets).

The first-order sentence expressing the existence of at least n elements is true of the line P_i for every index i in the set J_n, by (3), and J_n belongs to D. Apply the Fundamental Theorem of Ultraproducts to the lines in (2) to conclude that the ultraproduct P/D is an infinite projective line. (In fact, it can be shown that P/D has cardinality $2^{\aleph_0}$.) The complex algebra of every infinite projective line is completely representable, by the remarks at the end of Section 17.4. In particular, the algebra $\mathfrak{Cm}(P/D)$ is completely representable. The ultraproduct $\mathfrak{A}/D$ is embeddable into the complex algebra $\mathfrak{Cm}(P/D)$, by Lemma 18.32, so $\mathfrak{A}/D$ is also representable, by Lemma 16.11. □

The number of equations involved in the axiomatization of a variety is one measure of the complexity of the axiomatization. There are other measures as well, for example the number of distinct variables that are needed in the axiomatization, or the number of occurrences of certain operation symbols. It is imaginable, for example, that RRA is axiomatizable by an infinite set of equations, each of which involves just one variable. It turns out however, that this is not the case, and in fact, in terms of various complexity criteria, any axiomatization of RRA must be extremely complicated. As one example of this phenomenon, we prove that RRA cannot be axiomatized by a set of equations that involves only a finite number of distinct variables. The proof is based on some observations about subalgebras of complex algebras of projective lines of order at least three. Write $|X|$ to denote the cardinality of a set X.

Lemma 18.34. *If X and Y are disjoint, non-empty subsets of a projective line P of order at least three, then the relative products $X ; X$ and $X ; Y$ in $\mathfrak{Cm}(P)$ may be written as Boolean combinations of the sets X, Y, and 1'. In fact,*

$$X ; X = \begin{cases} 1 & \text{if } |X| \geq 2, \\ 1' + X & \text{if } |X| = 1, \end{cases}$$

and

$$X ; Y = \begin{cases} 0' & \text{if } |X| \geq 2 \text{ and } |Y| \geq 2, \\ 0' - X & \text{if } |X| = 1 \text{ and } |Y| \geq 2, \\ 0' - Y & \text{if } |X| \geq 2 \text{ and } |Y| = 1, \\ 0' - (X + Y) & \text{if } |X| = 1 \text{ and } |Y| = 1. \end{cases}$$

Proof. The rules governing the values of the relative products $X ; X$ and $X ; Y$ are rather direct consequences of the definition of relative multiplication in $\mathfrak{Cm}(P)$. Consider first the value of $X ; X$. If X has at least two elements, say p and q, then

$$\begin{aligned} X ; X \geq \{p, q\} ; \{p, q\} &= p ; q + p ; p + q ; q \\ &= (1 - (p + q + 1')) + (1' + p) + (1' + q) = 1 \end{aligned}$$

(where 1 and 1' denote the unit and the identity element of the complex algebra $\mathfrak{Cm}(P)$, that is to say, they denote the sets P^+ and $\{\iota\}$ respectively), by the monotony and distributive laws for relative multiplication, the commutative law for relative multiplication in $\mathfrak{Cm}(P)$, and the definition of relative multiplication in $\mathfrak{Cm}(P)$. (As usual, we are identifying elements in P with their singletons, which are the atoms in $\mathfrak{Cm}(P)$.) On the other hand, if X has just one element, say p, then

$$X ; X = p ; p = 1' + p = 1' + X.$$

Turn now to the value of $X ; Y$. Observe first that

$$\begin{aligned} X ; Y &= \sum\{1 - (p + q + 1') : p \in X \text{ and } q \in Y\} \\ &= \sum\{0' - (p + q) : p \in X \text{ and } q \in Y\}, \end{aligned} \tag{1}$$

by the complete distributivity of relative multiplication, the definition of relative multiplication in $\mathfrak{Cm}(P)$, and the assumption that the sets X and Y are disjoint and do not contain the identity element (since they are subsets of P, not P^+). If X and Y both have at least two elements, then every element r in P is below $0' - (p + q)$ for some p in X and q in Y: just take p and q to be elements in X and Y respectively that are different from r. Consequently, (1) reduces to

$$X ; Y = P = 0'$$

in this case. If X has just one element, say p, then (1) reduces to

$$\begin{aligned} X ; Y &= \sum\{1 - (p + q + 1') : q \in Y\} \\ &= \sum\{0' - (p + q) : q \in Y\}. \end{aligned} \tag{2}$$

If Y has at least two elements, then every element r in $P \sim \{p\}$ is below $0' - (p + q)$ for some q in Y: just take q to be different from r. Clearly, the element p is not below $0' - (p + q)$ for any element q in Y. Consequently, (2) reduces to

$$X \,;\, Y = P \sim \{p\} = 0' - p = 0' - X$$

in this case. On the other hand, if Y also has just one element, say q, then (2) reduces to

$$X \,;\, Y = 1 - (p + q + 1') = 0' - (p + q) = 0' - (X + Y).$$

A similar argument applies if X has at least two elements, and Y has just one element. □

Lemma 18.35. *If P is a projective line, then a subset of $\mathfrak{Cm}(P)$ is a subuniverse if and only if it contains the identity element and is closed under the Boolean operations of $\mathfrak{Cm}(P)$.*

Proof. Obviously, every subuniverse of $\mathfrak{Cm}(P)$ must contain the identity element and be closed under the Boolean operations of $\mathfrak{Cm}(P)$. To prove the reverse implication, assume P has order at least three (the case when P has order two is left as an exercise), and consider a subset A of $\mathfrak{Cm}(P)$ that contains the identity element and is closed under the Boolean operations of $\mathfrak{Cm}(P)$. Clearly, A is closed under converse, which is the identity operation in $\mathfrak{Cm}(P)$. It must be shown that A is also closed under relative multiplication.

Let X and Y be elements in A (that is to say, subsets of P^+), and assume first that neither X nor Y contains the identity element ι of the complex algebra $\mathfrak{Cm}(P)$. The sets

$$\begin{aligned} Z_1 &= X \sim Y = X - Y, \\ Z_2 &= Y \sim X = Y - X, \\ Z_3 &= X \cap Y = X \cdot Y \end{aligned}$$

clearly belong to A, because they are Boolean combinations of the sets X and Y. Furthermore, these sets are mutually disjoint, and

$$X = Z_1 \cup Z_3 = Z_1 + Z_3 \qquad \text{and} \qquad Y = Z_2 \cup Z_3 = Z_2 + Z_3.$$

Consequently,

$$X \,;\, Y = Z_1 \,;\, Z_2 + Z_1 \,;\, Z_3 + Z_3 \,;\, Z_2 + Z_3 \,;\, Z_3, \tag{1}$$

by the distributive law for relative multiplication. If one of the sets Z_i and Z_j occurring on the right side of (1) is zero—that is to say, if one of the sets is empty—then the relative product $Z_i \,;\, Z_j$ is zero and

therefore belongs to the set A, because A is a Boolean subuniverse of $\mathfrak{Cm}(P)$. If both sets Z_i and Z_j are non-zero, then by their very definition they are either equal (when $i = j$) or disjoint (when $i \neq j$). In either case, Lemma 18.34 implies that the relative product $Z_i ; Z_j$ is a Boolean combination of the three elements Z_i, Z_j, and 1'. Each of these three elements belongs to the set A, so $Z_i ; Z_j$ must also belong to A. Use (1) and the closure of A under the Boolean operations of $\mathfrak{Cm}(P)$ to conclude that $X ; Y$ is in A.

The cases when at least one of X and Y contains the identity element ι are easily handled. Suppose, for example, that X contains ι, but Y does not. Write

$$X_0 = X \sim \{\iota\} = X - 1',$$

and observe that X_0 must be in A, because it is a Boolean combination of X and 1'. Also, $X = X_0 + 1'$. Consequently,

$$X ; Y = (X_0 + 1') ; Y = X_0 ; Y + 1' ; Y = X_0 ; Y + Y. \tag{2}$$

Neither X_0 nor Y contains 1', so the relative product $X_0 ; Y$ belongs to the set A, by the observations of the preceding paragraph. The set Y belongs to A by assumption. Use (2) and the closure of A under the Boolean operations of $\mathfrak{Cm}(P)$ to conclude that $X ; Y$ belongs to A in this case as well. The remaining two cases, namely when Y contains ι, but X does not, and when both X and Y contain ι, are handled in an analogous fashion. Conclusion: A is closed under relative multiplication and is therefore a subuniverse of $\mathfrak{Cm}(P)$. □

Lemma 18.36. *If P is a projective line, then every proper subalgebra of the complex algebra $\mathfrak{Cm}(P)$ is representable.*

Proof. If P is infinite, then the algebra $\mathfrak{Cm}(P)$ is itself representable, by the remarks at the end of Section 17.4, and therefore every subalgebra of $\mathfrak{Cm}(P)$ is representable, by Lemma 16.11.

Assume now that P is finite. The complex algebra $\mathfrak{Cm}(P)$ is then also finite, so each of its subalgebras is finite and hence atomic. Consider a proper subalgebra $\mathfrak{A}$ of $\mathfrak{Cm}(P)$. At least one of the atoms in $\mathfrak{A}$ is a subset of P of cardinality at least two. Indeed, if each atom in $\mathfrak{A}$ were a singleton, then $\mathfrak{A}$ would be forced to have the same atoms as $\mathfrak{Cm}(P)$, because the atoms in $\mathfrak{A}$ are disjoint and have the unit P^+ as their union; and therefore $\mathfrak{A}$ would coincide with $\mathfrak{Cm}(P)$. Thus, some atom X in $\mathfrak{A}$ contains at least two elements. Neither of these elements can be the

identity element, because the identity element is a singleton atom in $\mathfrak{A}$ and is therefore distinct from—and hence disjoint from—the atom X. Consequently, X must be a subset of P (and not just a subset of P^+).

The key idea of the proof is that the atom X can be expanded to a set of arbitrarily large cardinality without destroying the structural properties of the algebra $\mathfrak{A}$. To make this statement precise, consider any infinite set Q that includes P as a subset, and view Q as a projective line. Take W to be the set of atoms in $\mathfrak{A}$, but with the set X replaced by the set

$$X_0 = X \cup (Q \sim P).$$

Thus, X_0 is the set of those elements in Q that do not occur in any of the atoms in $\mathfrak{A}$ that are different from X. The atoms in $\mathfrak{A}$ are non-empty, disjoint, and have P^+ as their union, so the elements in W must also be non-empty, disjoint, and have Q^+ as their union. Furthermore, W contains the identity element, because the identity element is an atom in $\mathfrak{A}$ that is different from X. Since the elements in W form a partition of Q^+, the set B of unions of all possible subsets of W is an atomic Boolean subuniverse of $\mathfrak{Cm}(Q)$, and the atoms in B are just the sets in W, by the Boolean version of Atomic Subalgebra Theorem 6.21. In particular, B contains the identity element, which is one of the sets in W. Apply Lemma 18.35 to conclude that B is a relation algebraic subuniverse of $\mathfrak{Cm}(Q)$. Write $\mathfrak{B}$ for the corresponding subalgebra. We shall use the Atomic Isomorphism Theorem 7.11 to show that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

Define a bijection φ from the set of atoms in $\mathfrak{A}$ to the set W of atoms in $\mathfrak{B}$ by

$$\varphi(Y) = \begin{cases} Y & \text{if} \quad Y \neq X, \\ X_0 & \text{if} \quad Y = X, \end{cases} \tag{1}$$

for each atom Y in $\mathfrak{A}$. Lemma 18.34 implies that φ preserves the operation of relative multiplication on atoms, in the sense of Theorem 7.11. Indeed, the definition of φ implies that an atom Y in $\mathfrak{A}$ has one element or more than one element if and only if its image $\varphi(Y)$ has one element or more than one element respectively, and Y is the identity element if and only if $\varphi(Y)$ is the identity element. The relative product of two atoms in both of the algebras $\mathfrak{A}$ and $\mathfrak{B}$ is the relative product of two sets that are either equal or disjoint; and the relative product of equal sets and disjoint sets that do not contain the identity element depends only on whether the sets have one element or more than one element, by Lemma 18.34, so φ must preserve such a relative product.

Here is a concrete example. Let Y be an atom in $\mathfrak{A}$ that is different from X. If Y has more than one element, then

$$Y \mathbin{;} X = 1 \qquad \text{and} \qquad Y \mathbin{;} X_0 = 1$$

in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively, by Lemma 18.34 (and the fact that $\mathfrak{A}$ and $\mathfrak{B}$ are subalgebras of $\mathfrak{Cm}(P)$ and $\mathfrak{Cm}(Q)$ respectively), so every atom is below $Y \mathbin{;} X$ in $\mathfrak{A}$, and every atom is below $Y \mathbin{;} X_0$ in $\mathfrak{B}$. Consequently,

$$Z \leq Y \mathbin{;} X \qquad \text{if and only if} \qquad \varphi(Z) \leq \varphi(Y) \mathbin{;} \varphi(X) \tag{2}$$

for every atom Z in $\mathfrak{A}$, by (1) and the fact that φ is a bijection from the set of atoms in $\mathfrak{A}$ to the set of atoms in $\mathfrak{B}$. On the other hand, if the atom Y has only one element, and if that element is different from the identity, then

$$Y \mathbin{;} X = 0' - Y \qquad \text{and} \qquad Y \mathbin{;} X_0 = 0' - Y$$

in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively, by Lemma 18.34, so the atoms that are below $Y \mathbin{;} X$ in $\mathfrak{A}$ are the subdiversity atoms different from Y, and the atoms that are below $Y \mathbin{;} X_0$ in $\mathfrak{B}$ are also the subdiversity atoms different from Y. Thus, the equivalence in (2) continues to hold. Finally, if Y is the identity element, then

$$Y \mathbin{;} X = X \qquad \text{and} \qquad Y \mathbin{;} X_0 = X_0$$

in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively, so the only atom that is below $Y \mathbin{;} X$ in $\mathfrak{A}$ is X, and the only atom that is below $Y \mathbin{;} X_0$ in $\mathfrak{B}$ is X_0. Thus, the equivalence in (2) holds in this case as well. Other cases are equally easy to check. To give just one more example, observe that

$$X \mathbin{;} X = 1 \qquad \text{and} \qquad X_0 \mathbin{;} X_0 = 1$$

in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively, by Lemma 18.34. Consequently, the equivalence in (2) holds also in this case (with X in place of Y).

Apply the Atomic Isomorphism Theorem in the form of Corollary 7.12 to conclude that φ can be extended to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. The complex algebra $\mathfrak{Cm}(Q)$ is completely representable, by the remarks at the end of Section 17.4 and the assumption that the set Q is infinite. Consequently, its subalgebra $\mathfrak{B}$ is representable, by Lemma 16.11. It follows that the isomorphic algebra $\mathfrak{A}$ is also representable. □

Let us say that a set of equations has a *finite number of distinct variables* if there is a natural number n such that every equation in the set has at most n variables. Without loss of generality, these variables may be assumed to be the first n variables in the language of relation algebras, namely $v_0, \ldots, v_{n-1}$.

Theorem 18.37. *No set of equations with only a finite number of distinct variables can be a set of axioms for* RRA.

Proof. Let $\mathcal{E}$ be any set of equations (in the language of relation algebras) with a finite number n of distinct variables, and assume that $\mathcal{E}$ holds in all representable relation algebras. We shall show that $\mathcal{E}$ must also hold in the complex algebra of every projective line of order at least 2^{n+1}. Since there are projective lines of arbitrarily large finite order for which the associated complex algebras are not representable, by Theorem 17.15, it follows that $\mathcal{E}$ cannot be a set of axioms for RRA.

Suppose P is a projective line of order at least 2^{n+1}, and consider any sequence

$$(X_0, X_1, \ldots, X_{n-1}) \tag{1}$$

of n elements in the complex algebra $\mathfrak{Cm}(P)$. The goal is to show that this sequence satisfies every equation in $\mathcal{E}$. Since the sequence in (1) is arbitrary, it then follows that every equation in $\mathcal{E}$ is valid in $\mathfrak{Cm}(P)$.

Take A to be the Boolean subuniverse of $\mathfrak{Cm}(P)$ generated by the set

$$\{X_0, X_1, \ldots, X_{n-1}, 1'\}.$$

Every Boolean algebra generated by a set of $n+1$ elements has cardinality at most $2^{2^{n+1}}$. In particular, this applies to A. As a Boolean subuniverse of $\mathfrak{Cm}(P)$ that contains the identity element, the set A must be a relation algebraic subuniverse of $\mathfrak{Cm}(P)$, by Lemma 18.35. Moreover, A must be a proper subuniverse. The reason is that it has cardinality at most $2^{2^{n+1}}$, whereas $\mathfrak{Cm}(P)$ has greater cardinality. In more detail, the line P has order at least 2^{n+1}, by assumption, so it has at least $2^{n+1}+1$ points. The set P^+ of atoms in $\mathfrak{Cm}(P)$ is obtained from P by adjoining a new element ι, so it has at least $2^{n+1}+2$ elements. The universe of $\mathfrak{Cm}(P)$ consists of all subsets of P^+, by definition, so it has cardinality at least $2^{(2^{n+1}+2)}$, which is strictly greater than the cardinality of the subuniverse A.

Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{Cm}(P)$ with universe A. Since $\mathfrak{A}$ is a proper subalgebra of $\mathfrak{Cm}(P)$, it must be representable, by Lemma 18.36.

The given set $\mathcal{E}$ of equations is assumed to hold in all representable relation algebras, so it must hold in $\mathfrak{A}$. In particular, the sequence in (1) of elements from $\mathfrak{A}$ must satisfy every equation of $\mathcal{E}$ in $\mathfrak{A}$. A sequence satisfies an equation in a subalgebra if and only if it satisfies the equation in the original algebra, by Lemma 6.2, so the sequence in (1) must satisfy every equation of $\mathcal{E}$ in $\mathfrak{Cm}(P)$, as claimed. Thus, $\mathfrak{Cm}(P)$ is a model of the set of equations $\mathcal{E}$. □

18.6 The structure of the lattices of varieties and universal classes

There are many natural questions about the structure of the lattice of varieties of relation algebras. What are the minimal members of the lattice, that is to say, what are the varieties of relation algebras that are just above variety of degenerate relation algebras? What is the structure of the lattice on the next level, that is to say, what are the varieties that are just above the minimal varieties? What are the maximal members of the lattice, that is to say, what are the varieties of relation algebra that are just below the variety of all relation algebras? How complicated is the lattice, and how complicated are various intervals within the lattice? When do different relation algebras generate the same variety in the lattice? Can certain important varieties in the lattice be written as irreducible joins of other, simpler varieties in the lattice? In this and the next three sections, we shall address some of these questions.

Every variety of relation algebras is generated by the universal class of its simple algebras, by Theorem 18.25, and the lattice of varieties of relation algebras is isomorphic to the lattice of universal classes of simple relation algebras, by Theorem 18.28. It is therefore possible, and often convenient, to focus on the lattice of universal classes of simple relation algebras and on the methods by which such universal classes can be constructed. Every result obtained for this lattice can then be transferred via the lattice isomorphism to the corresponding lattice of varieties of relation algebras. This is in fact what we shall do in the next few sections. In some of the more general results, the notion of simplicity does not play a role; the results hold for more general universal classes of relation algebras. This is the case, in particular, for the results in the present section. Consequently, we shall for the

time being consider arbitrary universal classes of relation algebras. Given a class L of relation algebras, write $\mathsf{Un}(\mathsf{L})$ for the universal class generated by L, so that

$$\mathsf{Un}(\mathsf{L}) = \mathsf{SP}_{\mathsf{u}}(\mathsf{L}),$$

by Corollary 18.12.

Suppose one starts with a universal class K of relation algebras, and wants to adjoin to K more relation algebras in order to generate a larger universal class. It is pointless to adjoin an algebra $\mathfrak{A}$ that is embeddable into one of the algebras in K, since in this case $\mathfrak{A}$ already belongs to K. It also serves no purpose to adjoin simultaneously two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ to K if $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{A}_2$, since the universal class generated by K and $\{\mathfrak{A}_2\}$ together already contains $\mathfrak{A}_1$. What one wants to do is adjoin to K a class L of relation algebras that is disjoint from K and that has the following property: no algebra in L is embeddable into any other algebra in L except itself. Any class of relation algebras L with this property will be called an *antichain over* K. (One might think it important to include an additional requirement in this definition, namely that no algebra in L be embeddable into any algebra in K. However, the assumption that L is disjoint from K implies this additional requirement, because K is a universal class.)

Let L be an antichain over K of finite relation algebras, and let M be the universal class of relation algebras generated by L. The join of the universal classes K and M in the lattice of universal classes is the union of these two classes, by Lemma 18.19. The *interval*

$$[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$$

is defined to be the class of all those universal classes N of relation algebras such

$$\mathsf{K} \subseteq \mathsf{N} \subseteq \mathsf{K} \cup \mathsf{M}.$$

A natural question is: how complicated can this interval be? If L is finite, then $\mathsf{M} = \mathsf{S}(\mathsf{L})$, by Corollary 18.13 (see also Corollary 18.22), so that M essentially reduces (up to isomorphic copies) to a finite class of algebras. In this case, the interval $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$ is finite. On the other hand, if L is infinite, then $\mathsf{M} = \mathsf{SP}_{\mathsf{u}}(\mathsf{L})$, by Corollary 18.12 (see also Corollary 18.21), and in this case the interval has the maximum possible cardinality, namely $2^{\aleph_0}$; in fact, as we shall see, the lattice of subsets of the natural numbers can be completely embedded into the interval. Consequently, the interval not only has maximum cardinality,

it also has a high degree of complexity from a lattice-theoretic point of view.

The proof of this lattice embedding theorem uses a variant of the method of diagrams (see Lemma 18.14 and the remarks preceding it). To describe it, consider a finite relation algebra $\mathfrak{A}$, say with n elements. Let $\mathcal{L}'$ be the language obtained from the language $\mathcal{L}$ of relation algebras by adjoining a new individual constant symbol $\boldsymbol{r}$ for each element r in $\mathfrak{A}$. Fix some bijective correspondence between the new individual constant symbols in $\mathcal{L}'$ and the variables $v_0, v_1, \cdots, v_{n-1}$ of $\mathcal{L}$. Convert the diagram of $\mathfrak{A}$ into a set of equations and inequalities in $\mathcal{L}$ by replacing each individual constant symbol that occurs in the diagram of $\mathfrak{A}$ with the corresponding variable. For example, an equation of the form $\boldsymbol{r} + \boldsymbol{s} = \boldsymbol{t}$ in the diagram of $\mathfrak{A}$ is converted into an equation of the form $v_i + v_j = v_k$, where v_i, v_j, and v_k are the variables that correspond to the individual constant symbols $\boldsymbol{r}$, $\boldsymbol{s}$, and $\boldsymbol{t}$ respectively. Similarly, an inequality of the form $\boldsymbol{r} \neq \boldsymbol{s}$ is converted into an inequality of the form $v_i \neq v_j$. Let Γ be the conjunction of the resulting set of formulas. Thus, Γ is a quantifier-free formula in the language $\mathcal{L}$. The formula

$$\exists v_0 \exists v_1 \ldots \exists v_{n-1} \Gamma$$

is called the *existential diagram* of $\mathfrak{A}$. It is clear from Lemma 18.14 that the existential diagram of $\mathfrak{A}$ is true in an algebra $\mathfrak{B}$ if and only if $\mathfrak{A}$ is embeddable into $\mathfrak{B}$.

The negation of the existential diagram of $\mathfrak{A}$ is logically equivalent to a universal sentence $\Psi_{\mathfrak{A}}$, called the *conjugate sentence* of $\mathfrak{A}$. The remarks of the previous paragraph imply that $\Psi_{\mathfrak{A}}$ is true in an algebra $\mathfrak{B}$ if and only if $\mathfrak{A}$ is not embeddable into $\mathfrak{B}$. The conjugate sentence is universal, so it is equivalent to an equation $\varepsilon_{\mathfrak{A}}$—called the *conjugate equation* of $\mathfrak{A}$—in all simple relation algebras, by Theorem 9.5. It follows that $\varepsilon_{\mathfrak{A}}$ is true in a simple relation algebra $\mathfrak{B}$ if and only if $\mathfrak{A}$ is not embeddable into $\mathfrak{B}$. These observations may be summarized as follows.

Lemma 18.38. *The existential diagram of a finite relation algebra* $\mathfrak{A}$ *is true in an algebra* $\mathfrak{B}$ *if and only if* $\mathfrak{A}$ *is embeddable into* $\mathfrak{B}$. *Consequently, the conjugate sentence of* $\mathfrak{A}$ *is true in* $\mathfrak{B}$ *if and only if* $\mathfrak{A}$ *is not embeddable into* $\mathfrak{B}$, *and if* $\mathfrak{B}$ *is a simple relation algebra, then the same holds for the conjugate equation of* $\mathfrak{A}$.

Conjugate sentences can be used to establish the following useful fact about antichains of algebras.

Lemma 18.39. *If* L *is an antichain of finite relation algebras, and* M *is the universal class generated by* L*, then it is impossible to simultaneously embed two distinct algebras of* L *into an algebra in* M*.*

Proof. Fix distinct algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$ in L. Let Ψ_i be the conjugate sentence of $\mathfrak{A}_i$ for $i = 0, 1$, and let Ψ be the disjunction of these two sentences:

$$\Psi = \Psi_0 \vee \Psi_1.$$

The sentence Ψ is true in an algebra $\mathfrak{A}$ just in case the algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are not simultaneously embeddable into $\mathfrak{A}$, by Lemma 18.38.

A straightforward argument shows that Ψ is true in every algebra in L. Indeed, each such algebra $\mathfrak{A}$ must be different from one of $\mathfrak{A}_0$ and $\mathfrak{A}_1$, because $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are assumed to be distinct. If $\mathfrak{A}$ is different from $\mathfrak{A}_0$, then $\mathfrak{A}_0$ cannot be embedded into $\mathfrak{A}$, because the algebras in L are assumed to form an antichain. Consequently, Ψ holds in $\mathfrak{A}$. A similar argument applies if $\mathfrak{A}$ is different from $\mathfrak{A}_1$.

The sentence Ψ is logically equivalent to a universal sentence, and such sentences are preserved under ultraproducts and subalgebras, by the Fundamental Theorem of Ultraproducts and Corollary 6.4. Consequently, Ψ holds in every algebra that is embeddable into some ultraproduct of algebras from L, by the observations of the previous paragraph. The class of all such algebras is just the universal class M that is generated by L, by Corollary 18.12. Consequently, Ψ holds in M. Conclusion: no two algebras from L are simultaneously embeddable into an algebra in M, by the observations of the first paragraph. □

The next lemma gives necessary and sufficient conditions for one of two universal classes generated by certain classes of relation algebras to be included in the other, or for the two universal classes to be equal. In order to formulate the lemma succinctly, it is helpful to introduce some terminology and notation. Two class L_1 and L_2 of relation algebras will be said to be *essentially equal*, in symbols $\mathsf{L}_1 \simeq \mathsf{L}_2$, if there is a bijection from L_1 to L_2 such that corresponding algebras under this bijection are isomorphic.

Lemma 18.40. *Suppose* K *is a universal class of relation algebras, and* L_1 *and* L_2 *antichains over* K *of relation algebras.*

(i) *If the algebras in* L_1 *are all finite, then*

$$\mathsf{Un}(\mathsf{K} \cup \mathsf{L}_1) \subseteq \mathsf{Un}(\mathsf{K} \cup \mathsf{L}_2) \qquad \textit{if and only if} \qquad \mathsf{L}_1 \subseteq \mathsf{S}(\mathsf{L}_2).$$

(ii) *If the algebras in* L_1 *and in* L_2 *are all finite, then*

$$\mathsf{Un}(\mathsf{K} \cup \mathsf{L}_1) = \mathsf{Un}(\mathsf{K} \cup \mathsf{L}_2) \qquad \textit{if and only if} \qquad \mathsf{L}_1 \simeq \mathsf{L}_2.$$

Proof. Apply Corollary 18.21 to L_1 and L_2 to write

$$\mathsf{Un}(\mathsf{K} \cup \mathsf{L}_1) = \mathsf{K} \cup \mathsf{SP}_\mathsf{u}(\mathsf{L}_1) \quad \text{and} \quad \mathsf{Un}(\mathsf{K} \cup \mathsf{L}_2) = \mathsf{K} \cup \mathsf{SP}_\mathsf{u}(\mathsf{L}_2). \tag{1}$$

Assume first that the left-hand inclusion in (i) holds. It follows from this inclusion and (1) that

$$\mathsf{K} \cup \mathsf{SP}_\mathsf{u}(\mathsf{L}_1) \subseteq \mathsf{K} \cup \mathsf{SP}_\mathsf{u}(\mathsf{L}_2). \tag{2}$$

Consider an algebra $\mathfrak{A}$ in L_1. This algebra belongs to the union of the classes K and $\mathsf{SP}_\mathsf{u}(\mathsf{L}_2)$, by (2). The class L_1 is disjoint from K, by assumption, so $\mathfrak{A}$ cannot belong to K. Consequently, $\mathfrak{A}$ must be in $\mathsf{SP}_\mathsf{u}(\mathsf{L}_2)$, that is to say, $\mathfrak{A}$ must be embeddable into some ultraproduct of algebras in L_2. It follows from Lemma 18.38 that the existential diagram of $\mathfrak{A}$ holds in this ultraproduct. The Fundamental Theorem of Ultraproducts therefore implies that the existential diagram of $\mathfrak{A}$ must hold in almost all of the factor algebras of the ultraproduct, all of which are in L_2. Apply Lemma 18.38 once more to conclude that $\mathfrak{A}$ is embeddable into some algebra in L_2. Thus, the right-hand inclusion in (i) holds.

To establish the reverse implication, assume that the right-hand inclusion in (i) holds. This inclusion, together with Lemmas 18.9(i) and 18.1(i), implies that

$$\mathsf{SP}_\mathsf{u}(\mathsf{L}_1) \subseteq \mathsf{SP}_\mathsf{u}\mathsf{S}(\mathsf{L}_2) \subseteq \mathsf{SSP}_\mathsf{u}(\mathsf{L}_2) = \mathsf{SP}_\mathsf{u}(\mathsf{L}_2),$$

so (2) must hold. Combine (2) with (1) to arrive at the left-hand inclusion in (i). This completes the proof of (i).

Turn now to the proof of (ii). Part (i) implies that

$$\mathsf{Un}(\mathsf{K} \cup \mathsf{L}_1) = \mathsf{Un}(\mathsf{K} \cup \mathsf{L}_2) \qquad \text{if and only if} \qquad \mathsf{S}(\mathsf{L}_1) = \mathsf{S}(\mathsf{L}_2). \tag{3}$$

Indeed, the left-hand equality in (3) is equivalent to the two inclusions

$$\mathsf{L}_1 \subseteq \mathsf{S}(\mathsf{L}_2) \qquad \text{and} \qquad \mathsf{L}_2 \subseteq \mathsf{S}(\mathsf{L}_1), \tag{4}$$

by (i) applied twice, the second time with the roles of the classes L_1 and L_2 interchanged; and the two inclusions in (4) are jointly equivalent to the right-hand equality in (3), by the properties of the class operation S.

So far, the assumption that L_1 and L_2 are antichains has not been used. We use it now to show that

$$\mathsf{S}(\mathsf{L}_1) = \mathsf{S}(\mathsf{L}_2) \qquad \text{if and only} \qquad \mathsf{L}_1 \simeq \mathsf{L}_2. \tag{5}$$

The implication from right to left in (5) is obvious. To establish the reverse implication, assume that the left-hand equation in (5) holds, and let $\mathfrak{A}$ be an algebra in L_1. The assumed equality implies that $\mathfrak{A}$ must be embeddable into some algebra $\mathfrak{B}$ in L_2. Similarly, $\mathfrak{B}$ must be embeddable into some algebra $\mathfrak{A}'$ in L_1. It follows that $\mathfrak{A}$ is embeddable into $\mathfrak{A}'$. Since L_1 is assumed to be an antichain, the algebra $\mathfrak{A}'$ must coincide with $\mathfrak{A}$, so $\mathfrak{A}$ is embeddable into $\mathfrak{B}$, and $\mathfrak{B}$ into $\mathfrak{A}$. The algebras in L_2 are all finite, by assumption, so $\mathfrak{B}$ is finite, and therefore $\mathfrak{A}$ is also finite with the same cardinality as $\mathfrak{B}$. Because $\mathfrak{A}$ and $\mathfrak{B}$ have the same finite cardinality, any embedding of one of them into the other must be an isomorphism, so $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

It has been shown that every algebra $\mathfrak{A}$ in L_1 is isomorphic to an algebra $\mathfrak{B}$ in L_2. The algebra $\mathfrak{B}$ is uniquely determined, because the assumption that L_2 is an antichain implies that two distinct algebras in L_2 cannot be isomorphic to the same algebra in L_1, for otherwise they would be isomorphic to each other. A symmetric argument shows that every algebra $\mathfrak{B}$ in L_2 is isomorphic to a uniquely determined algebra $\mathfrak{A}$ in L_1. The function that maps each algebra in L_1 to its uniquely determined isomorphic image in L_2 is therefore a bijection from L_1 to L_2. Conclusion: the classes L_1 and L_2 are essentially equal. This completes the proof of (5). Combine (3) and (5) to arrive at (ii). □

In order to use the preceding lemma to analyze certain intervals in the lattice of universal classes, it is convenient in the next theorem to identify isomorphic relation algebras, treating them as if they were one algebra. Formally, this is done by forming the class $\mathsf{I}(\mathfrak{A})$ of all isomorphic copies of a given relation algebra $\mathfrak{A}$—this is called the *isomorphism type* of $\mathfrak{A}$—and then speaking of isomorphism types instead of algebras. For example, one may define a binary relation $\lesssim$ on the class of isomorphism types by writing

$$\mathsf{I}(\mathfrak{A}) \lesssim \mathsf{I}(\mathfrak{B})$$

if and only if $\mathfrak{A}$ is embeddable into $\mathfrak{B}$. This relation is well defined in the sense that it does not depend on the representative of the isomorphism type that is chosen: if algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic, and also algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$, then $\mathfrak{A}_1$ is embeddable into $\mathfrak{B}_1$ just in case $\mathfrak{A}_2$ is embeddable into $\mathfrak{B}_2$. The relation $\lesssim$ is easily seen to be reflexive, since every relation algebra is embeddable into itself; and it is also transitive, since the composition of an embedding from $\mathfrak{A}$ into $\mathfrak{B}$ with an embedding from $\mathfrak{B}$ into $\mathfrak{C}$ produces an embedding from $\mathfrak{A}$ into $\mathfrak{C}$. When restricted to isomorphism types of finite relation algebras, the relation $\lesssim$ is also antisymmetric: if two finite relation algebras are embeddable into one another, then they both have the same finite cardinality, and therefore any embedding of one into the other is actually an isomorphism; thus, the two algebras have the same isomorphism type.

It is a bit cumbersome, however, to formulate theorems and proofs using isomorphism types in place of algebras. We prefer to be a bit less formal in the next theorem, assuming instead that all classes under discussion are closed under isomorphisms, and using the terminology introduced before in a more informal, or more general sense, by applying it to classes of algebras when it really should be applied to classes of isomorphism types of algebras. For example, we shall say that a class L of relation algebras (that is closed under isomorphisms) is *finite* if, up to isomorphism there are only finitely many algebras in L, or, put another way, if the collection of isomorphism types of algebras in L is finite. We shall also say that a class L of relation algebras is an *antichain* over a universal class K of relation algebras if the class of isomorphism types of algebras in L is an antichain over the class of isomorphism types of algebras in K; in other words, the two classes K and L must be disjoint, and the isomorphism types of the algebras in L must be pairwise incomparable in the sense of the relation $\lesssim$ defined in the preceding paragraph—which amounts to saying that if an algebra $\mathfrak{A}$ in L is embeddable into an algebra $\mathfrak{B}$ in L, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. We shall say that an algebra is *maximal* in L if it belongs to L and if no algebra in L has an isomorphism type that is strictly above the isomorphism type of $\mathfrak{A}$ in the sense of the the relation $\lesssim$— which amounts to saying that $\mathfrak{A}$ is not properly embeddable into any algebra in L. Finally, we shall write

$$\mathsf{L}_1 \lesssim \mathsf{L}_2$$

if every algebra in L_1 is embeddable into an algebra in L_2, that is to say, if L_1 is included in $\mathsf{S}(\mathsf{L}_2)$. From this point of view, essentially equal classes of relation algebras algebras are in fact equal.

The next theorem gives a strategy for forming universal classes of relation algebras by adjoining finitely many finite relation algebras to an existing universal class K of relation algebras. It says, in essence, that it suffices in a very strong sense to focus on antichains over K.

Theorem 18.41. *Let* K *be a universal class of relation algebras*, L *a finite antichain over* K *of finite relation algebras*, *and* $\mathsf{M} = \mathsf{S}(\mathsf{L})$. *The interval* $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$ *in the lattice of universal classes of relation algebras is finite*, *and there is an order isomorphism from this interval to the partially ordered class of antichains over* K *that are included in* M.

Proof. Every universal class in the interval

$$[\mathsf{K}, \mathsf{K} \cup \mathsf{M}], \tag{1}$$

is obtained by adjoining to K some subclass of M (that is closed under isomorphism). There are only finitely many such subclasses, because L is assumed to consist of finitely many finite algebras (and their isomorphic copies). Consequently, the interval in (1) is finite.

Every antichain L_1 over K that is included in M corresponds to a well-determined universal class in the interval (1), namely the universal class

$$\mathsf{Un}(\mathsf{K} \cup \mathsf{L}_1) = \mathsf{K} \cup \mathsf{S}(\mathsf{L}_1) \tag{2}$$

(see Corollary 18.22). Distinct such antichains correspond to different universal classes in the interval (1), by part (ii) of Lemma 18.40. Consequently, the function φ that maps such antichain L_1 to the universal class (2) is one-to one.

To check that φ is onto, consider an arbitrary universal class N in the interval (1). Let L_1 be the class of maximal algebras in N that do not occur in K, together with all isomorphic copies of these algebras. Clearly, L_1 is an antichain over K, since no maximal algebra in N is embeddable into another maximal algebra in N, by the definition of a maximal algebra. The definition of L_1 implies that for each algebra $\mathfrak{A}$ in N there are three possibilities: either $\mathfrak{A}$ is in K, or $\mathfrak{A}$ is strictly embeddable into an algebra in L_1, or $\mathfrak{A}$ is in L_1. Consequently, we have

$$\mathsf{N} \subseteq \mathsf{K} \cup \mathsf{S}(\mathsf{L}_1) \subseteq \mathsf{N},$$

so equality must hold everywhere. Thus, φ maps the class L_1 to N, by the definition of φ (see (2)).

The mapping φ preserves order in the strong sense that

$$\mathsf{L}_1 \lesssim \mathsf{L}_2 \qquad \text{if and only if} \qquad \varphi(\mathsf{L}_1) \subseteq \varphi(\mathsf{L}_2).$$

In more detail,

$$\begin{aligned} \varphi(\mathsf{L}_1) \subseteq \varphi(\mathsf{L}_2) \qquad & \text{if and only if} \qquad \mathsf{K} \cup \mathsf{S}(\mathsf{L}_1) \subseteq \mathsf{K} \cup \mathsf{S}(\mathsf{L}_2), \\ & \text{if and only if} \qquad \mathsf{L}_1 \subseteq \mathsf{S}(\mathsf{L}_2), \\ & \text{if and only if} \qquad \mathsf{L}_1 \lesssim \mathsf{L}_2, \end{aligned}$$

by the definition of φ, (2) and part (i) of Lemma 18.40, and the definition of the relation $\lesssim$, . Consequently, φ is an order isomorphism. □

Theorem 18.41 applies in particular when K is a universal class of simple relation algebras, and L is a finite antichain over K of finite, simple relation algebras. Notice that in this case, the class M in the theorem is a universal class of simple relation algebras, by Corollaries 9.3 and 18.13. Consequently, the interval $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$ in the lattice of universal class of relation algebras consists entirely of universal classes of simple relation algebras. This interval is isomorphic to a corresponding interval in the lattice of varieties of relation algebras, by the Correspondence Theorem 18.28, namely the interval $[\mathsf{SP}(\mathsf{K}), \mathsf{SP}(\mathsf{K} \cup \mathsf{M})]$, so we obtain the following corollary.

Corollary 18.42. *Under the hypotheses of Theorem 18.41, the interval* $[\mathsf{SP}(\mathsf{K}), \mathsf{SP}(\mathsf{K} \cup \mathsf{M})]$ *in the lattice of varieties of relation algebras is finite, and there is an order isomorphism from this interval to the partially ordered class of antichains over* K *that are included in* M.

Turn now to the case when the antichain L is infinite. It is no longer necessary to assume that the classes under discussion are closed under isomorphisms or to use terminology in a somewhat extended and informal manner, as was done in the previous theorem. The following *Lattice Embedding Theorem* says that the complete lattice $(Sb(\mathbb{N}), \cap, \cup)$ of all subsets of the set $\mathbb{N}$ of natural numbers is embeddable into an interval in the lattice of universal class of relation algebras that is formed using K and L. The notation $Sb(\mathbb{N})$ denotes the set of all subsets of $\mathbb{N}$.

Theorem 18.43. *Let* K *be a universal class of relation algebras,* L *an infinite antichain over* K *of finite relation algebras, and* $\mathsf{M} = \mathsf{SP}_{\mathsf{u}}(\mathsf{L})$.

There is a complete embedding of the lattice of subsets of $\mathbb{N}$ *into the interval* $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$ *in the lattice of universal classes of relation algebras. The join of any system of universal classes in the range of this embedding is the union of the universal classes.*

Proof. There are, up to isomorphism, only countably many finite relation algebras, so the algebras in L can be enumerated without repetitions in an infinite sequence $(\mathfrak{A}_n : n \in \mathbb{N})$. Let Ψ_n be the conjugate sentence of $\mathfrak{A}_n$. Thus, Ψ_n is valid in a relation algebra $\mathfrak{A}$ if and only if $\mathfrak{A}_n$ is not embeddable into $\mathfrak{A}$, by Lemma 18.38. Define a mapping φ from $Sb(\mathbb{N})$ into the lattice of universal classes of relation algebras by

$$\varphi(X) = \{\mathfrak{A} \in \mathsf{K} \cup \mathsf{M} : \Psi_n \text{ is true in } \mathfrak{A} \text{ for every } n \in \mathbb{N} \sim X\} \qquad (1)$$

for every subset X of $\mathbb{N}$. In other words, $\varphi(X)$ is the class of all those algebras $\mathfrak{A}$ in $\mathsf{K} \cup \mathsf{M}$ with the property that the algebra $\mathfrak{A}_n$ is not embeddable into $\mathfrak{A}$ whenever n is not in X. It should be remarked in passing that at most one algebra $\mathfrak{A}_n$ can be embedded into $\mathfrak{A}$. Indeed, if $\mathfrak{A}$ is in K, then no algebra from L can be embedded into $\mathfrak{A}$, because K, as a universal class, is closed under subalgebras, and L is assumed to be disjoint from K. On the other hand, if $\mathfrak{A}$ is in M, then at most one algebra from L is embeddable into $\mathfrak{A}$, by Lemma 18.39, because L is an antichain of finite algebras and M is the universal class generated by L, by Corollary 18.12. The definition of φ requires that such an algebra $\mathfrak{A}_n$ cannot have an index outside of X.

The class $\varphi(X)$ is clearly axiomatized by a set of axioms for the class $\mathsf{K} \cup \mathsf{M}$ and the set of universal sentences Ψ_n for n in $\mathbb{N} \sim X$. Since $\mathsf{K} \cup \mathsf{M}$ is a universal class of relation algebras, by Corollary 18.21 and the definition of M, it follows that $\varphi(X)$ is also a universal class of relation algebras. No algebra in L is embeddable into an algebra in K, by assumption, so the sentence Ψ_n is true in every algebra in K for every n in $\mathbb{N}$, by Lemma 18.38. This observation and the definition of φ in (1) imply that

$$\mathsf{K} \subseteq \varphi(X) \subseteq \mathsf{K} \cup \mathsf{M}.$$

Thus, φ maps $Sb(\mathbb{N})$ into the interval $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$.

For each subset X of $\mathbb{N}$, write

$$\mathsf{L}_X = \{\mathfrak{A}_n : n \in X\}. \qquad (2)$$

If m is in $\mathbb{N}$, then

$$\begin{aligned}
\mathfrak{A}_m \in \varphi(X) \quad &\text{if and only if} \quad \Psi_n \text{ is true of } \mathfrak{A}_m \text{ for every } n \in \mathbb{N} \sim X,\\
&\text{if and only if} \quad n \neq m \text{ for every } n \in \mathbb{N} \sim X,\\
&\text{if and only if} \quad m \in X,\\
&\text{if and only if} \quad \mathfrak{A}_m \in \mathsf{L}_X,
\end{aligned}$$

by the definition of φ in (1), Lemma 18.38, the assumption that L is an antichain over K, and (2). It follows that

$$\mathsf{L}_X = \mathsf{L} \cap \varphi(X). \tag{3}$$

To check that φ is one-to-one, suppose X and Y are distinct subsets of $\mathbb{N}$, say m is a number in X that is not in Y. The antichains L_X and L_Y cannot be essentially equal, because $\mathfrak{A}_m$ belongs to L_X, but not to L_Y, by (2), and therefore no isomorphic copy of $\mathfrak{A}_m$ can belong to L_Y, by the assumption that L is an antichain. Consequently, the images $\varphi(X)$ and $\varphi(Y)$ must be distinct, by (3).

To prove that φ preserves arbitrary intersections and unions, consider a system $(X_i : i \in I)$ of subsets of $\mathbb{N}$. Write

$$Y = \bigcap_i X_i \qquad \text{and} \qquad Z = \bigcup_i X_i,$$

and observe that

$$\mathbb{N} \sim Y = \bigcup_i (\mathbb{N} \sim X_i) \qquad \text{and} \qquad \mathbb{N} \sim Z = \bigcap_i (\mathbb{N} \sim X_i). \tag{4}$$

It is to be shown that

$$\varphi(Y) = \bigcap_i \varphi(X_i) \qquad \text{and} \qquad \varphi(Z) = \bigcup_i \varphi(X_i). \tag{5}$$

The definition of φ in (1), the first equation in (4), and the definitions of the union and intersection of a system of sets or classes imply that for every algebra $\mathfrak{A}$ in $\mathsf{K} \cup \mathsf{M}$, each of the following conditions is equivalent to its neighbor:

$\mathfrak{A} \in \varphi(Y)$,
Ψ_n is true in $\mathfrak{A}$ for every $n \in \mathbb{N} \sim Y$,
Ψ_n is true in $\mathfrak{A}$ for every $n \in \bigcup_i (\mathbb{N} \sim X_i)$,
for every $i \in I$, the sentence Ψ_n is true in $\mathfrak{A}$ for every $n \in \mathbb{N} \sim X_i$,
$\mathfrak{A} \in \varphi(X_i)$ for every $i \in I$,
$\mathfrak{A} \in \bigcap_i \varphi(X_i)$.

This verifies the first equation in (5).

The second equation in (5) is verified in a similar fashion, but has one tricky step. Each of the following conditions is equivalent to its neighbor:

$\mathfrak{A} \in \varphi(Z)$,
Ψ_n is true in $\mathfrak{A}$ for every $n \in \mathbb{N} \sim Z$,
Ψ_n is true in $\mathfrak{A}$ for every $n \in \bigcap_i(\mathbb{N} \sim X_i)$,
for some $i \in I$, the sentence Ψ_n is true in $\mathfrak{A}$ for every $n \in \mathbb{N} \sim X_i$,
$\mathfrak{A} \in \varphi(X_i)$ for some $i \in I$,
$\mathfrak{A} \in \bigcup_i \varphi(X_i)$.

The step that requires justification is the equivalence of the third and fourth statements. If there is an index i in I such that the sentence Ψ_n is true in $\mathfrak{A}$ for every n in $\mathbb{N} \sim X_i$, then certainly Ψ_n is true in $\mathfrak{A}$ for every n in the intersection $\bigcap_{j \in I}(\mathbb{N} \sim X_j)$, because this intersection is included in the set $\mathbb{N} \sim X_i$. Assume now that there is no index i in I such that the sentence Ψ_n is true in $\mathfrak{A}$ for every n in $\mathbb{N} \sim X_i$. In this case, for every i there must be a number n_i in $\mathbb{N} \sim X_i$ such that Ψ_{n_i} fails in $\mathfrak{A}$. This means that the algebra $\mathfrak{A}_{n_i}$ is embeddable into $\mathfrak{A}$, by Lemma 18.38. No two distinct algebras in L can be jointly embedded into $\mathfrak{A}$, by Lemma 18.39, so we must have $n_i = n_j$ for all i, j in I. Thus, there is a single number n belonging to $\mathbb{N} \sim X_i$ for every i such that the sentence Ψ_n fails in $\mathfrak{A}$. Conclusion: it is not the case that Ψ_n is true in $\mathfrak{A}$ for every n in $\bigcap_i(\mathbb{N} \sim X_i)$.

This completes the proof that φ is a complete lattice embedding. The second equation in (5) shows that the join of every system of universal classes in the range of φ is the union of this system. □

A remark similar to the one that was made after Theorem 18.41 is also true in the context of Theorem 18.43. The theorem applies in particular to the case when K is a universal class of simple relation algebras, and L is an infinite antichain over K of simple relation algebras. The class M in the theorem is then a universal class of simple relation algebras, by Corollaries 9.3, 18.8, and 18.12, so the interval $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$ in the lattice of universal class of relation algebras consists entirely of universal classes of simple relation algebras. The theorem gives a complete embedding of the lattice of subsets of $\mathbb{N}$ into this interval, and the interval is, in turn, isomorphic to the corresponding interval $[\mathsf{SP}(\mathsf{K}), \mathsf{SP}(\mathsf{K} \cup \mathsf{M})]$ in the lattice of varieties of relation algebras,

by the Correspondence Theorem 18.28. Thus, we obtain the following corollary.

Corollary 18.44. *Under the hypotheses of Theorem 18.43, there is a complete embedding of the lattice of subsets of* $\mathbb{N}$ *into the interval*

$$[\mathsf{SP}(\mathsf{K}), \mathsf{SP}(\mathsf{K} \cup \mathsf{M})]$$

in the lattice of varieties of relation algebras.

It is a consequence of Theorem 18.43 that under the hypotheses of the theorem, the interval $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$ inherits all of the complexity of the lattice of subsets of the set of natural numbers, and in fact, it inherits the complexity of the lattice of subsets of the rational numbers. Write $\mathbb{Q}$ for the set of rational numbers, and $Sb(\mathbb{Q})$ for the set of subsets of $\mathbb{Q}$.

Theorem 18.45. *Let* K *be a universal class of relation algebras,* L *an infinite antichain over* K *of finite relation algebras, and* $\mathsf{M} = \mathsf{SP}_{\mathsf{u}}(\mathsf{L})$. *There is a system* $(\mathsf{N}_X : X \subseteq \mathbb{Q})$ *of universal classes in the interval*

$$[\mathsf{K}, \mathsf{K} \cup \mathsf{M}] \tag{i}$$

such that $X \subseteq Y$ *if and only if* $\mathsf{N}_X \subseteq \mathsf{N}_Y$ *for subsets* X, Y *of* $\mathbb{Q}$, *and if* $X \subsetneq Y$, *then there are continuum many universal classes between* N_X *and* N_Y. *In particular, the interval* (i) *has a subclass of universal classes that, under the relation of inclusion, is linearly ordered of the same order type as the real numbers.*

Proof. In view of the complete embedding φ of the lattice

$$(Sb(\mathbb{N}), \cap, \cup) \tag{1}$$

into the interval (i) given by Theorem 18.43, it suffices to find a system

$$(Z_X : X \subseteq \mathbb{Q}) \tag{2}$$

of subsets of $\mathbb{N}$ with the following properties for all subsets X and Y of $\mathbb{Q}$.

$$\begin{array}{lll} X \subseteq Y & \text{if and only if} & Z_X \subseteq Z_Y. \qquad (3) \\ X = \bigcap_i X_i & \text{implies} & Z_X = \bigcap_i Z_{X_i}. \qquad (4) \\ X = \bigcup_i X_i & \text{implies} & Z_X = \bigcup_i Z_{X_i}. \qquad (5) \\ X \subsetneq Y & \text{implies} & Z_Y - Z_X \text{ is infinite.} \qquad (6) \end{array}$$

Indeed, suppose that such a system (2) has been constructed. It follows from properties (4) and (5) that

$$(\{Z_X : X \subseteq \mathbb{Q}\}, \cap, \cup) \tag{7}$$

is a complete lattice, and in fact a complete sublattice of lattice (1). Consequently, the relevant properties of lattice (7) transfer via the complete embedding φ to the interval (i). Also, it follows from properties (3)–(5) that the lattice (7) is isomorphic to the lattice

$$(Sb(\mathbb{Q}), \cap, \cup) \tag{8}$$

via the function that maps each subset X of $\mathbb{Q}$ to the set Z_X. In particular, the relevant properties of the lattice (8) transfer via this isomorphism to the lattice (7), and therefore also to the interval (i).

In more detail, let $(\mathfrak{A}_n : n \in \mathbb{N})$ be an enumeration without repetitions of the algebras in L, and for each n, let Ψ_n be the conjugate sentence of $\mathfrak{A}_n$. For each subset X of $\mathbb{Q}$, take N_X to be the universal class in the interval (i) that is determined by

$$\begin{aligned} \mathsf{N}_X &= \varphi(Z_X) \\ &= \{\mathfrak{A} \in \mathsf{K} \cup \mathsf{M} : \Psi_n \text{ is true in } \mathfrak{A} \text{ for every } n \in \mathbb{N} \sim Z_X\}. \end{aligned} \tag{9}$$

For subsets X and Y of $\mathbb{Q}$, we have

$$\begin{array}{lll} X \subseteq Y & \text{if and only if} & Z_X \subseteq Z_Y, \\ & \text{if and only if} & \varphi(Z_X) \subseteq \varphi(Z_Y), \\ & \text{if and only if} & \mathsf{N}_X \subseteq \mathsf{N}_Y, \end{array}$$

by (3), the monomorphism properties of φ, and definition (9). If X is a proper subset of Y, then the interval

$$[Z_X, Z_Y] \tag{10}$$

in the lattice (1) has the power of the continuum, by property (6), because each distinct subset W of the infinite set $Z_Y - Z_X$ leads to a distinct set in the interval (10), namely the set $Z_X \cup W$. Consequently, there are continuum many universal classes in the interval (i) that are between N_X and N_Y, namely the classes

$$\begin{aligned} &\varphi(Z_X \cup W) \\ &\quad = \{\mathfrak{A} \in \mathsf{K} \cup \mathsf{M} : \Psi_n \text{ is true in } \mathfrak{A} \text{ for every } n \in \mathbb{N} \sim (Z_X \cup W)\} \end{aligned}$$

for subsets W of $Z_Y - Z_X$.

The lattice (8) is well known to have a subset that is linearly ordered by the relation of inclusion with the same order type as the real numbers, namely the set D of all those non-empty proper subsets X of $\mathbb{Q}$ such that $\mathbb{Q} - X$ has no least element and the pair $(X, \mathbb{Q} - X)$ is a Dedekind cut—that is to say, every rational number in X precedes every rational number in $\mathbb{Q} - X$ in the natural ordering of the rational numbers. It follows that the interval (i) has a subclass that is linearly order by the relation of inclusion with the same order type as the real numbers, namely the subclass consisting of the universal classes N_X for sets X in D.

To get the system (2), fix some bijection ψ from the set $\mathbb{Q}$ to the set of positive prime numbers; such a bijection certainly exists because both sets are countably infinite. For each prime number n, take V_n to be the set of positive integer powers of n. The important property of this system of sets is that the sets are mutually disjoint and each of them is infinite in size. Finally, for each subset X of $\mathbb{Q}$, define the set Z_X by

$$Z_X = \bigcup\{V_{\psi(\alpha)} : \alpha \in X\}. \tag{11}$$

It remains to establish properties (3)–(6) for the resulting system (2). The argument for (4) is a bit tricky. If X is a subset of $\mathbb{Q}$ satisfying the hypothesis of (4), and if m is a natural number, then

$$\begin{aligned}
m \in Z_X \quad & \text{if and only if} \quad m \in V_{\psi(\alpha)} \text{ for some } \alpha \in X, \\
& \text{if and only if} \quad m \in V_{\psi(\alpha)} \text{ for some } \alpha \in X_i \text{ for every } i, \\
& \text{if and only if} \quad m \in V_{\psi(\alpha_i)} \text{ for some } \alpha_i \in X_i, \text{ for every } i, \\
& \text{if and only if} \quad m \in Z_{X_i} \text{ for every } i, \\
& \text{if and only if} \quad m \in \textstyle\bigcap_i Z_{X_i}.
\end{aligned}$$

The first and fourth equivalences follow from (11), and the second and fifth follow from the definition of intersection and the hypothesis on X. (In more detail, the rational number α in the second equivalence belongs to the set X_i for every i if and only if α belongs to the intersection X of the sets X_i.) The implication from left to right in the third equivalence is also easy: if m is in $V_{\psi(\alpha)}$ for some rational number α that belongs to X_i for every index i, then of course for each index i there is a rational number α_i in X_i such that m is in $V_{\psi(\alpha_i)}$—just take α_i to be α for each i. For the reverse implication, suppose that for each index i there is a rational number α_i in X_i such that m

is in $V_{\psi(\alpha_i)}$. The sets V_n are defined so as to be mutually disjoint for distinct primes n. Since m is assumed to be in $V_{\psi(\alpha_i)}$ for each index i, we must have

$$\psi(\alpha_i) = \psi(\alpha_j) \tag{12}$$

for all indices i and j. The function ψ is a bijection, so (12) implies that $\alpha_i = \alpha_j$ for all indices i and j. In other words, the rational number α_i in the set X_i on the right side of the third equivalence is the same rational number for each index i. Take α to be this rational number.

The proof of property (5) is similar to the proof of property (4), but without the tricky part, and is left as an exercise. The implication from left to right in property (3) follows immediately from definition (11). For the reverse implication, observe that a set $V_{\psi(\alpha)}$ is either included in a given set Z_Y or disjoint from it, according to whether the rational number α does or does not belong to the set Y, by definition (11) (with Y in place of X), the disjointness of the sets V_n for distinct primes n, and the fact that the mapping ψ is one-to-one. It follows that if Z_X is included in Z_Y, then for every rational number α in X, the set $V_{\psi(\alpha)}$ is included in Z_Y, by (11), and therefore α belongs to Y. Thus, X must be included in Y.

To establish property (6), suppose that X is a proper subset of Y. In this case, there must be a rational number β that is in Y, but not in X. The mapping ψ is a bijection, so

$$\psi(\beta) \neq \psi(\alpha) \tag{13}$$

for every α in X. The sets V_n are infinite and mutually disjoint for distinct primes n, so $V_{\psi(\beta)}$ is an infinite subset of Z_Y (by (11), with Y in place of X) that is disjoint from the set $V_{\psi(\alpha)}$ for every α in X, by (13), and that is consequently disjoint from Z_X, by (11). Conclusion: the difference $Z_Y \sim Z_X$ is infinite. □

As was the case with Theorems 18.41 and 18.43, Theorem 18.45 applies in particular to the case when K is a universal class of simple relation algebras, and L is an infinite antichain over K of simple relation algebras. The result is a theorem about the interval $[\mathsf{K}, \mathsf{K} \cup \mathsf{M}]$ of universal classes of simple relation algebras, which in turn yields an analogous result about the corresponding interval in the lattice of varieties of relation algebras.

Corollary 18.46. *Under the hypotheses of Theorem* 18.45*, there is a system* $(\mathsf{V}_X : X \subseteq \mathbb{Q})$ *of varieties in the interval*

$$[\mathsf{SP}(\mathsf{K}), \mathsf{SP}(\mathsf{K} \cup \mathsf{M})] \tag{i}$$

such that $X \subseteq Y$ *if and only if* $\mathsf{V}_X \subseteq \mathsf{V}_Y$ *for subsets* X, Y *of* $\mathbb{Q}$*, and if* $X \subsetneq Y$*, then there are continuum many varieties between* V_X *and* V_Y*. In particular, the interval* (i) *has a subclass of varieties that, under the relation of inclusion, is linearly ordered of the same order type as the real numbers.*

18.7 Minimal and quasi-minimal universal classes and varieties

Theorem 18.41 provides a strategy for analyzing the lower levels of the lattice of universal classes of simple relation algebras, or, equivalently, the lattice of varieties of relation algebras. The lattice of universal classes has a zero element, namely the empty universal class; and it generates the zero element of the lattice of varieties, namely the *degenerate variety* consisting of the one-element relation algebras. The empty universal class is on level zero of the lattice of universal classes. Just above this zero element are the universal classes on level one of the lattice, the atoms of the lattice. They are called the *minimal universal classes*, and they generate the *minimal varieties* in the lattice of varieties. The first task is to determine what these minimal universal classes are.

The second task is to determine the universal classes on level two of the lattice of universal classes, the so-called *quasi-minimal universal classes* that *cover* the atoms of the lattice in the sense that there are no universal class that are strictly between a minimal universal class and a quasi-minimal universal class. They generate the *quasi-minimal varieties* in the lattice of varieties. The quasi-minimal universal classes can often be obtained—there are exceptions—by adjoining to one of the minimal universal classes a new simple relation algebra $\mathfrak{A}$ (and all the isomorphic images of $\mathfrak{A}$) with the property that any proper subalgebra of $\mathfrak{A}$ belongs to the minimal universal class in question.

After determining the quasi-minimal universal classes, one can envision a more ambitious program of determining the universal classes on level three of the lattice of universal classes, the universal classes

on level four of the lattice, and so on. Almost no progress has been made on this complicated problem.

Let us return to the first problem of describing the minimal universal classes.

Theorem 18.47. *There are exactly three minimal universal classes of simple relation algebras, namely the classes generated by the three minimal simple relation algebras*:

$$\mathsf{M}_1 = \mathsf{I}(\mathfrak{M}_1), \qquad \mathsf{M}_2 = \mathsf{I}(\mathfrak{M}_2), \qquad \mathsf{M}_3 = \mathsf{I}(\mathfrak{M}_3).$$

Proof. The minimal relation algebra $\mathfrak{M}_n$ (for $n = 1, 2, 3$) is simple and has no proper subalgebras (see Theorem 13.4), so the class M_n is closed under subalgebras, and is therefore a universal class of simple relation algebras, by Corollary 18.11. Furthermore, M_n is minimal, by Theorem 18.41 (with K taken to be the empty universal class), since there are only two antichains in M_n, namely the empty antichain and the antichain consisting of $\mathfrak{M}_n$ (and its isomorphic copies).

Consider now an arbitrary minimal universal class L of simple relation algebras. There must be a simple relation algebra $\mathfrak{A}$ in L, because L is assumed to be different from the zero element of the lattice. The subalgebra of $\mathfrak{A}$ generated by the identity element is a minimal simple relation algebra, by the definition of a minimal subalgebra (see Section 6.1) and Corollary 9.3, so it must be isomorphic to $\mathfrak{M}_n$ for some $n = 1, 2, 3$, by Theorem 13.4. The class L is closed under subalgebras and isomorphisms, so $\mathfrak{M}_n$ belongs to L, and therefore M_n is included in L. The assumption that L is a minimal universal class of simple relation algebras now implies that $\mathsf{L} = \mathsf{M}_n$. Thus, every minimal universal class of simple relation algebras is equal to M_n for some n.

One final observation is needed: the universal classes M_1, M_2, and M_3 are distinct from one another, and in fact they are disjoint from one another, because the three minimal simple relation algebras $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\mathfrak{M}_3$ have different types (in the sense of Section 13.1) and are therefore mutually non-isomorphic. □

The preceding theorem, in combination with Theorem 18.28, immediately implies the corresponding result about minimal varieties.

Corollary 18.48. *There are exactly three minimal varieties of relation algebras, namely the varieties generated by the three minimal simple relation algebras*:

$$\mathsf{V}_1 = \mathsf{SP}(\mathfrak{M}_1), \qquad \mathsf{V}_2 = \mathsf{SP}(\mathfrak{M}_2), \qquad \mathsf{V}_3 = \mathsf{SP}(\mathfrak{M}_3).$$

We turn next to the problem of determining the quasi-minimal universal classes. Following the procedure suggested by Theorem 18.41, we may adjoin to M_n a minimal simple relation algebra $\mathfrak{M}_m$ (and its isomorphic images) for some $m \neq n$. This is equivalent to forming the join of the two distinct minimal universal classes M_m and M_n, and according to Lemma 18.19, it yields the following three distinct universal classes on level two of the lattice of universal classes:

$$\mathsf{M}_{12} = \mathsf{M}_1 \cup \mathsf{M}_2 = \mathsf{I}(\{\mathfrak{M}_1, \mathfrak{M}_2\}), \quad \mathsf{M}_{13} = \mathsf{M}_1 \cup \mathsf{M}_3 = \mathsf{I}(\{\mathfrak{M}_1, \mathfrak{M}_3\}),$$
$$\mathsf{M}_{23} = \mathsf{M}_2 \cup \mathsf{M}_3 = \mathsf{I}(\{\mathfrak{M}_2, \mathfrak{M}_3\}).$$

Another possibility is to adjoin to M_n a relation algebra $\mathfrak{A}$ that is *quasi-minimal* in the following sense: (1) $\mathfrak{A}$ is simple, but not minimal, and (2) if $\mathfrak{B}$ is embeddable into an ultrapower of $\mathfrak{A}$, and $\mathfrak{B}$ is not minimal, then $\mathfrak{A}$ is embeddable into an ultrapower of $\mathfrak{B}$. When $\mathfrak{A}$ is finite, these properties say that $\mathfrak{A}$ is simple and has exactly two subalgebras: the minimal subalgebra and the improper subalgebra $\mathfrak{A}$. (We are using here the fact that a finite algebra $\mathfrak{A}$ is completely describable by a set of first-order sentences, and therefore every ultrapower of $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$, by the Fundamental Theorem of Ultraproducts.) Equivalently, in the finite case these properties say that $\mathfrak{A}$ is simple, contains an element different from the constants 0, 1, 1', and 0', and is generated by each of its non-constant elements. The next lemma says that quasi-minimal relation algebras generate quasi-minimal universal classes, and, conversely, all quasi-minimal universal classes different from M_{mn} for $1 \leq m < n \leq 3$ are generated by quasi-minimal relation algebras.

Lemma 18.49. *If a universal class* M *of simple relation algebras contains a non-minimal relation algebra, then the following conditions on* M *are equivalent.*

(i) M *is quasi-minimal.*
(ii) M *is generated by a quasi-minimal algebra.*
(iii) *Every non-minimal algebra in* M *is quasi-minimal and generates* M.

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary non-minimal algebras in M. The universal classes generated by $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathsf{SP}_\mathsf{u}(\mathfrak{A})$ and $\mathsf{SP}_\mathsf{u}(\mathfrak{B})$ respectively, by Corollary 18.12. Clearly, these two universal classes are included in M, since M is closed under ultrapowers and subalgebras (see the SP_u-Theorem 18.10).

To establish the implication from (i) to (iii), assume that M is quasi-minimal. In this case,

$$\mathsf{SP}_\mathsf{u}(\mathfrak{A}) = \mathsf{M} = \mathsf{SP}_\mathsf{u}(\mathfrak{B}), \tag{1}$$

by the definition of a quasi-minimal universal class and the observations of the previous paragraph, so each of the algebras $\mathfrak{A}$ and $\mathfrak{B}$ generates M, and each of the algebras is embeddable into an ultrapower of the other, by (1). Since $\mathfrak{A}$ and $\mathfrak{B}$ are assumed to be arbitrary non-minimal algebras in M, it follows that $\mathfrak{A}$ and $\mathfrak{B}$ are quasi-minimal. Thus, M satisfies condition (iii).

The implication from (iii) to (ii) is trivial. To establish the implication from (ii) to (i), assume that a quasi-minimal relation algebra $\mathfrak{A}$ generates M, and consider any universal class K that is included in M and is not minimal. Thus,

$$\mathsf{K} \subseteq \mathsf{M} = \mathsf{SP}_\mathsf{u}(\mathfrak{A}). \tag{2}$$

It follows from (2) that the class M contains a unique minimal algebra, up to isomorphism, namely the minimal subalgebra, say $\mathfrak{C}$, of $\mathfrak{A}$. Indeed, the diagram of $\mathfrak{C}$ (which only involves individual constant symbols from the language of relation algebras) is true in $\mathfrak{A}$, and is therefore true in every ultrapower of $\mathfrak{A}$, by the Fundamental Theorem of Ultraproducts, and in every subalgebra of such an ultrapower, by Corollary 6.4. Consequently, $\mathfrak{C}$ is isomorphic to the minimal subalgebra of every subalgebra of an ultrapower of $\mathfrak{A}$, by Lemma 18.38.

These remarks and (2) imply that K also contains a unique minimal algebra (up to isomorphism). Since K is assumed not to be minimal, it must contain a non-minimal algebra $\mathfrak{B}$. The algebra $\mathfrak{B}$ is embeddable into some ultrapower of $\mathfrak{A}$, by (2). Because $\mathfrak{A}$ is assumed to be quasi-minimal, it follows that $\mathfrak{A}$ must be embeddable into some ultrapower of $\mathfrak{B}$, and therefore

$$\mathsf{M} = \mathsf{SP}_\mathsf{u}(\mathfrak{A}) \subseteq \mathsf{SP}_\mathsf{u}\mathsf{SP}_\mathsf{u}(\mathfrak{B}) \subseteq \mathsf{SSP}_\mathsf{u}\mathsf{P}_\mathsf{u}(\mathfrak{B}) \subseteq \mathsf{K} = \mathsf{SP}_\mathsf{u}(\mathfrak{B}) \subseteq \mathsf{M}.$$

Clearly, equality must hold everywhere, so $\mathsf{K} = \mathsf{M}$. Conclusion: M is quasi-minimal. □

The procedure of adjoining a quasi-minimal relation algebra to M_n cannot be carried out in the case $n = 1$, because no simple relation algebra contains a copy of $\mathfrak{M}_1$ as a proper subalgebra, by Theorem 13.10. The procedure is applicable in the case $n = 2$ and yields

a single quasi-minimal universal class that is directly above M_2. Indeed, up to isomorphism there is exactly one simple relation algebra that has $\mathfrak{M}_2$ as a proper subalgebra, namely the full set relation algebra $\mathfrak{Re}(2)$ on the set $2 = \{0, 1\}$ of cardinality two, by Theorem 13.10. Consequently, there is a unique quasi-minimal universal class that is above M_2, namely the universal class N generated by $\mathfrak{Re}(2)$. It has, up to isomorphism, exactly two members—the algebra $\mathfrak{Re}(2)$ and its minimal subalgebra $\mathfrak{M}_2$—so there are only two antichains in N over M_2, namely the empty antichain and the antichain consisting of $\mathfrak{Re}(2)$ (and its isomorphic copies)—see Theorem 18.41.

When applied to M_3, the procedure yields a number of quasi-minimal universal classes above M_3. For example, we may adjoin to M_3 any one of the ten integral relation algebras with three atoms and eight elements that are discussed in Section 3.8 and in Exercises 3.36 and 3.37. Each of these ten algebras includes a copy of $\mathfrak{M}_3$ as a proper subalgebra, and each of them is generated by any non-constant element (that is to say, any element different from the constants 0, 1, 1', and 0'), so each of them is quasi-minimal. Each of these ten algebras therefore generates a distinct universal class at level two above M_3, by Lemma 18.49 and the remarks preceding it (or by Theorem 18.41), namely the class that contains the eight-element relation algebra in question and the minimal subalgebra $\mathfrak{M}_3$ (and the isomorphic copies of these algebras).

To give a concrete example, consider the eight-element algebra $\mathfrak{A}$ with atoms 1', s, and t, and the relative multiplication and converse tables for atoms given in the adjoining Table 18.1. Four of the elements

;	1'	s	t
1'	1'	s	t
s	s	s	1
t	t	1	t

and

	$\smile$
1'	1'
s	t
t	s

.

Table 18.1 Relative multiplication and converse tables for the atoms in the eight-element integral, quasi-minimal relation algebra $\mathfrak{A}$.

in this algebra belong to the minimal relation algebra, namely

$$0, \qquad 1 = 1' + s + t, \qquad 1', \qquad \text{and} \qquad 0' = s + t.$$

The remaining four elements are the subdiversity atoms s and t, and the sums $1'+s$ and $1'+t$. Each of these elements generates a subalgebra

of $\mathfrak{A}$ that properly includes the minimal subalgebra and has cardinality a power of two (as do all finite relation algebras). Since the minimal subalgebra has cardinality four, and $\mathfrak{A}$ has cardinality eight, it follows that the subalgebra generated by one of these four elements must have cardinality eight and therefore must coincide with $\mathfrak{A}$.

There are of course other quasi-minimal relation algebras, besides the ten integral relation algebras mentioned above, that may be used to generate quasi-minimal universal classes covering M_3. For example, up to isomorphism there are exactly three quasi-minimal relation algebras that are not integral and have $\mathfrak{M}_3$ as their minimal subalgebra. In order to describe them, it is helpful to introduce some terminology. A non-zero subidentity element x in a simple relation algebra $\mathfrak{A}$ is said to have *type one*, *two*, or *three* according to whether the relativized relation algebra $\mathfrak{A}(x\,;1\,;x)$ has type one, two, or three in the sense of Section 13.1. (Recall that this relativization is a simple relation algebra, by Lemma 10.8. Its identity element is $(x\,;1\,;x)\cdot 1'$, which coincides with x, by Lemmas 5.54 and 5.20(i); its diversity element is $(x\,;1\,;x)\cdot 0'$, which coincides with $x\,;0'\,;x$, by Lemma 5.54; and its unit is $x\,;1\,;x$.) Thus, x has type one, two or three according to whether the relative product of the diversity element with itself in the relativization—that is to say, the relative product

$$[(x\,;1\,;x)\cdot 0']\,;[(x\,;1\,;x)\cdot 0'],$$

which coincides with the relative product

$$x\,;0'\,;x\,;0'\,;x,$$

by Lemmas 5.54 and 5.20(i)—is equal to 0 (in which case x is an identity singleton—see Exercise 17.54), to x (in which case x is an identity pair—see Exercise 17.78), or to $x\,;1\,;x$ respectively. The following lemma contains one of the key observations.

Lemma 18.50. *Let $\mathfrak{A}$ be a simple relation algebra, and x and y non-zero subidentity elements in $\mathfrak{A}$ that are disjoint and sum to $1'$. The subalgebra of $\mathfrak{A}$ generated by x and y is completely determined, up to isomorphism, by the types of x and y.*

Proof. It is helpful to begin the proof by introducing some notational abbreviations and making some preliminary observations. For any subidentity elements u and v in $\mathfrak{A}$, write

$$1_{uv} = u\,;1\,;v \tag{1}$$

for the rectangle with sides u and v. The rectangles

$$1_{xx}, \quad 1_{xy}, \quad 1_{yx}, \quad 1_{yy} \tag{2}$$

are mutually disjoint. For example,

$$1_{xy} \cdot 1_{yy} = (x\,;1\,;y) \cdot (y\,;1\,;y) = (x \cdot y)\,;1\,;(y \cdot y) = 0\,;1\,;y = 0,$$

by (1), Lemma 5.55, the assumed disjointness of the subidentity elements x and y, and Corollary 4.17. Also, these rectangles sum to 1, because

$$\begin{aligned} 1 = 1'\,;1\,;1' &= (x+y)\,;1\,;(x+y) \\ &= (x\,;1\,;x) + (x\,;1\,;y) + (y\,;1\,;x) + (y\,;1\,;y) \\ &= 1_{xx} + 1_{xy} + 1_{yx} + 1_{yy}, \end{aligned}$$

by the identity and distributive laws for relative multiplication, the assumption that x and y sum to 1', and (1). The operations of relative

;	1_{xx}	1_{xy}	1_{yx}	1_{yy}
1_{xx}	1_{xx}	1_{xy}	0	0
1_{xy}	0	0	1_{xx}	1_{xy}
1_{yx}	1_{yx}	1_{yy}	0	0
1_{yy}	0	0	1_{yx}	1_{yy}

and

	$\smile$
1_{xx}	1_{xx}
1_{xy}	1_{yx}
1_{yx}	1_{xy}
1_{yy}	1_{yy}

.

Table 18.2 Relative multiplication and converse for the rectangles in (2).

multiplication and converse on these rectangles are determined by Table 18.2, the entries of which follow from Corollary 5.62, Lemma 5.58, and (1). For example, Corollary 5.62 and (1) imply that

$$1_{xy}\,;1_{xy} = (x\,;1\,;y)\,;(x\,;1\,;y) = 0,$$

since $y \cdot x = 0$, and

$$1_{xy}\,;1_{yx} = (x\,;1\,;y)\,;(y\,;1\,;x) = x\,;1\,;x = 1_{xx},$$

since $y \cdot y \neq 0$. One more notational abbreviation is helpful: write

$$0'_x = x\,;0'\,;x \qquad \text{and} \qquad 0'_y = y\,;0'\,;y. \tag{3}$$

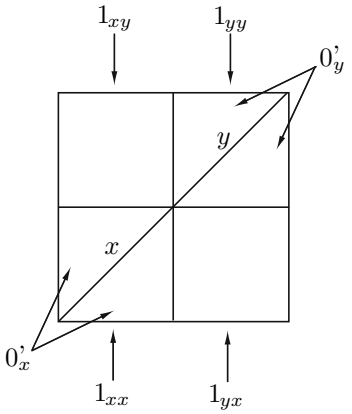


Fig. 18.1 Key elements in the subalgebra generated by x and y.

Turn now to the main task of the proof: describing the subalgebra $\mathfrak{B}$ generated by the subidentity elements x and y in $\mathfrak{A}$. Clearly, this subalgebra coincides with the subalgebra generated by the set

$$W = \{x, y, 0'_x, 0'_y, 1_{xy}, 1_{yx}\}$$

(see Figure 18.1), because x and y belong to this set, and together they generate each of the remaining elements in this set. A description of $\mathfrak{B}$ may therefore be obtained from the Atomic Subalgebra Thoerem by showing that the set W satisfies conditions (i)–(iv) of that theorem.

Consider condition (i). It has already been shown that the rectangles in (2) are mutually disjoint and sum to 1. The first rectangle in (2) is the disjoint sum of x and $0'_x$, because

$$1_{xx} = x\,;1\,;x = x\,;(1' + 0')\,;x = (x\,;1'\,;x) + (x\,;0'\,;x) = x + 0'_x,$$

by (1), the definition of 0', Boolean algebra, the distributive law for relative multiplication, (3), and Lemma 5.20(i); and

$$x \cdot 0'_x = x \cdot (x\,;0'\,;x) \leq 1' \cdot (1'\,;0'\,;1') = 1' \cdot 0' = 0,$$

by (3), the monotony and identity laws for relative multiplication, the definition of 0', and Boolean algebra. Similarly, the last rectangle in (2) is the disjoint sum of y and $0'_y$. Thus, the elements in W are mutually disjoint and sum to 1, as required by condition (i).

Condition (ii) holds because 1' is the sum of the two elements x and y in W that are below it. Condition (iii) holds because the

elements x, y, $0'_x$, and $0'_y$ are all symmetric, by Lemmas 4.7(vi) and 5.20(i), and the second involution law, while the elements 1_{xy} and 1_{yx} are the converses of one another, by Table 18.2; consequently, the converse of every element in W is again an element in W.

It remains to verify condition (iv), which requires that the relative product of any two elements in W be a sum of elements in W. Most of the relative products of elements in W do not depend on the types of x and y, as is clear from Table 18.3. The entries in this table that

;	x	y	$0'_x$	$0'_y$	1_{xy}	1_{yx}
x	x	0	$0'_x$	0	1_{xy}	0
y	0	y	0	$0'_y$	0	1_{yx}
$0'_x$	$0'_x$	0		0		0
$0'_y$	0	$0'_y$	0		0	
1_{xy}	0	1_{xy}	0		0	$x + 0'_x$
1_{yx}	1_{yx}	0		0	$y + 0'_y$	0

.

Table 18.3 Relative products of elements in W that do not depend on type.

involve x or y as one of the two factors all follow from Lemma 5.20(i), together with Corollary 4.17 and its first dual. For example,

$$x \,;\, 0'_y = x \,;\, (y \,;\, 0' \,;\, y) = (x \,;\, y) \,;\, 0' \,;\, y = (x \cdot y) \,;\, 0' \,;\, y = 0 \,;\, 0' \,;\, y = 0.$$

Similar arguments also yield all of the zero entries in the table. Finally, the four entries in the lower right corner follow directly from Table 18.2.

Six of the relative products in Table 18.3 have been left blank because they do depend on the types of x and y. Write m and n respectively for these types. Table 18.4 gives the relative products for the blank entries in Table 18.3 under the assumption that $m \leq n$. The relative products in the cases when $n \leq m$ may be obtained by interchanging x and y in Table 18.4.

The verifications of the entries in Table 18.4 are not difficult. For example, when $m = 1$, the element $0'_x$ is 0, by Lemma 13.1 applied to the relativization $\mathfrak{A}(1_{xx})$, and therefore all the entries in the table that involve $0'_x$ as one of the factors are in this case 0, by Corollary 4.17 and its first dual. A similar remark applies to the entries that involve $0'_y$ as one of the factors in the case $n = 1$. The entries in the table for the relative products $0'_x \,;\, 0'_x$ and $0'_y \,;\, 0'_y$ all follow directly from the definition of type. The remaining entries follow from Lemma 5.24 (with 1' in place of r), or alternatively, from Tables 13.1 and 13.2, according to which,

m	n	$0'_x \,;\, 0'_x$	$0'_y \,;\, 0'_y$	$0'_x \,;\, 1_{xy}$	$0'_y \,;\, 1_{yx}$	$1_{xy} \,;\, 0'_y$	$1_{yx} \,;\, 0'_x$
1	1	0	0	0	0	0	0
1	2	0	y	0	1_{yx}	1_{xy}	0
1	3	0	$y + 0'_y$	0	1_{yx}	1_{xy}	0
2	2	x	y	1_{xy}	1_{yx}	1_{xy}	1_{yx}
2	3	x	$y + 0'_y$	1_{xy}	1_{yx}	1_{xy}	1_{yx}
3	3	$x + 0'_x$	$y + 0'_y$	1_{xy}	1_{yx}	1_{xy}	1_{yx}

Table 18.4 Relative products of elements in W that depend on type.

$$0' \,;\, 1 = 1 \,;\, 0' = 1 \tag{4}$$

in a simple relation algebra whenever $0' \neq 0$. For example, if $m > 1$, then

$$0'_x \,;\, 1_{xy} = 0'_x \,;\, (1_{xx} \,;\, 1_{xy}) = (0'_x \,;\, 1_{xx}) \,;\, 1_{xy} = 1_{xx} \,;\, 1_{xy} = 1_{xy},$$

by Table 18.2, and (4) applied to the relativization $\mathfrak{A}(1_{xx})$.

It has been shown that the set W satisfies the four conditions of the Atomic Subalgebra Theorem. Apply the theorem in the form of Corollary 6.22 to conclude that the subalgebra $\mathfrak{B}$ generated by W is a finite relation algebra whose atoms are the non-zero elements in W, and whose elements are the sums of the various possible sets of atoms.

Consider now another simple relation algebra $\mathfrak{A}'$ with non-zero subidentity elements x' and y' that are disjoint and sum to the identity element. The subalgebra generated by x' and y' coincides with the subalgebra generated by the set

$$W' = \{x', y', 0'_{x'}, 0'_{y'}, 1_{x'y'}, 1_{y'x'}\}.$$

The discussion in the preceding paragraphs shows that the set W' satisfies the conditions of the Atomic Subalgebra Theorem, so the subalgebra $\mathfrak{B}'$ generated by W is a finite relation algebra whose atoms are the non-zero elements in W', and whose elements are the sums of the various possible sets of atoms.

If x' and y' have the same types as x and y respectively, then the relative multiplication table for the elements in W' coincides with the relative multiplication table for the elements in W (with x and y replaced everywhere by x' and y' respectively), since these tables depend only on the types of the subidentity elements involved. Consequently, the function φ that maps each element in W to the corresponding element in W' is a bijection from the set of atoms in $\mathfrak{B}$ to the set of

atoms in $\mathfrak{B}'$ that preserves the operation of relative multiplication on atoms. Apply the Atomic Isomorphism Theorem in the form of Corollary 7.12 to extend φ to an isomorphism from $\mathfrak{B}$ to $\mathfrak{B}'$. Conclusion: the generated subalgebra $\mathfrak{B}$ is uniquely determined up to isomorphism by the types of the generating subidentity elements x and y. □

To obtain concrete examples of algebras illustrating Lemma 18.50, fix natural numbers m and n with $1 \leq m \leq n \leq 3$, let U be a set of cardinality $m + n$, say

$$U = \{0, 1, 2, \ldots, m + n - 1\},$$

and let X and Y be disjoint subsets of U with cardinalities m and n respectively, say

$$X = \{0, \ldots, m - 1\} \qquad \text{and} \qquad Y = \{m, \ldots, m + n - 1\}.$$

The subidentity relations id_X and id_Y are non-zero, mutually disjoint, and have the identity relation id_U as their union in the full set relation algebra $\mathfrak{N} = \mathfrak{Re}(U)$. Also, these subidentity relations have types m and n respectively. Take $\mathfrak{N}_{mn}$ to be the subalgebra of $\mathfrak{N}$ generated by these two subidentity relations. The set W in the proof of the lemma consists of the relations

$$id_X, \qquad id_Y, \qquad di_X, \qquad di_Y, \qquad X \times Y, \qquad Y \times X.$$

The atoms in $\mathfrak{N}_{mn}$ are the non-zero relations in W, and the elements in $\mathfrak{N}_{mn}$ are the unions of the various sets of non-zero relations in W. If $\mathfrak{A}$ is any simple relation algebra with non-zero subidentity elements x and y that are mutually disjoint, sum to 1', and have types m and n respectively, then the subalgebra of $\mathfrak{A}$ generated by x and y is isomorphic to $\mathfrak{N}_{mn}$ via a function that maps x to id_X and y to id_Y, by Lemma 18.50. As a result, in the investigation of such subalgebras, we may restrict our attention to the algebras $\mathfrak{N}_{mn}$.

There are several things to notice about these algebras. First, they are mutually non-isomorphic for different choices of natural numbers m and n satisfying the inequalities $1 \leq m \leq n \leq 3$. The reason is that an isomorphism must map a pair of subidentity atoms of types m and n that form a partition of the identity element to a pair of subidentity atoms satisfying the same conditions, and the pairs of the types of the subidentity atoms in $\mathfrak{N}_{mn}$ are different for different choices of m and n satisfying the above inequalities.

Second, the algebra $\mathfrak{N}_{11}$ is isomorphic to the full set relation algebra $\mathfrak{Re}(U)$ on a two element set U, and consequently its minimal subalgebra is isomorphic to $\mathfrak{M}_2$. The minimal subalgebra of each of the other algebras $\mathfrak{N}_{mn}$ is isomorphic to $\mathfrak{M}_3$, since in these cases the underlying base set U has cardinality at least three.

The next two observations are more subtle.

Lemma 18.51. *The algebras* $\mathfrak{N}_{22}$ *and* $\mathfrak{N}_{33}$ *are not quasi-minimal.*

Proof. Fix $m = n \geq 2$, and write $\mathfrak{A} = \mathfrak{N}_{mn}$. It must be shown that $\mathfrak{A}$ has a proper subalgebra that properly includes the minimal subalgebra. Let S and T be the relations in $\mathfrak{A}$ defined by

$$S = di_X \cup di_Y \qquad \text{and} \qquad T = (X \times Y) \cup (Y \times X),$$

and observe that the relations in the set

$$W = \{id_U, S, T\}$$

are symmetric, pairwise disjoint, and have $U \times U$ as their union (see Figure 17.17). The relational composition of any two of the relations in W is, with one exception, set forth in Table 18.5. The single excep-

$\mid$	id_U	S	T
id_U	id_U	S	T
S	S		T
T	T	T	$id_U \cup S$

Table 18.5 Relational composition table for the atoms in the set W.

tion is the composition $S \mid S$, which is the only entry that is dependent on the cardinality of the sets X and Y, that is to say, on the types m and n. The value of this composition is

$$S \mid S = id_U \qquad \text{or} \qquad S \mid S = id_U \cup S,$$

according to whether $m = n = 2$ or $m = n = 3$ respectively. These two compositions and all of the compositions in Table 18.5 and can be verified by simple set-theoretic computations, using the definition of relational composition.

The preceding argument shows that the set W satisfies the conditions of the Atomic Subalgebra Theorem. Apply that theorem in the

form of Corollary 6.22 to conclude that the set of unions of relations in W is the universe of a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, and W is the set of atoms in $\mathfrak{B}$. Obviously, $\mathfrak{B}$ is neither the minimal subalgebra of $\mathfrak{A}$ nor $\mathfrak{A}$ itself, since the minimal subalgebra has two atoms, $\mathfrak{B}$ has three atoms, and $\mathfrak{A}$ has six atoms. □

Lemma 18.52. *If* $1 \leq m < n \leq 3$, *then* $\mathfrak{N}_{mn}$ *is quasi-minimal.*

Proof. Write $\mathfrak{A} = \mathfrak{N}_{mn}$, and observe that $\mathfrak{A}$ is simple, by Corollary 9.3, because it is a subalgebra of the simple relation algebra $\mathfrak{Re}(U)$. It must be shown that $\mathfrak{A}$ has no proper subalgebras except the minimal subalgebra. To this end, consider a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ that properly extends the minimal subalgebra, with the goal of showing that $\mathfrak{B}$ must coincide with $\mathfrak{A}$. It suffices to show that $\mathfrak{B}$ contains one of the two relations id_X and id_Y, since each of these relations by itself generates $\mathfrak{A}$. The algebra $\mathfrak{B}$ is atomic (it is a subalgebra of a finite algebra) and properly extends the minimal subalgebra, by assumption, so at least one of the two atoms id_U and di_U in the minimal subalgebra must split in $\mathfrak{B}$.

If the identity relation id_U splits in $\mathfrak{B}$, then it must split into id_X and id_Y, because these are the only two subidentity elements in $\mathfrak{A}$ that are not in the minimal subalgebra. Consequently, $\mathfrak{B}$ contains these two subidentity atoms and therefore coincides with $\mathfrak{A}$.

Consider now the case when the diversity relation di_U splits in $\mathfrak{B}$. Each of the resulting two or more subdiversity atoms in $\mathfrak{B}$ must be a non-empty union of some of the relations

$$di_X, \qquad di_Y, \qquad X \times Y, \qquad Y \times X, \tag{1}$$

because the relations in (1) are the subdiversity atoms in $\mathfrak{A}$, with one exception: the relation di_X is empty when $m = 1$. (Notice that di_Y cannot be empty, because $n \geq 2$ by assumption.) Each of the relations in (1), with the single exception just noted, generates the subidentity relations id_X and id_Y, and therefore generates $\mathfrak{A}$. For example,

$$[(X \times Y)\,|(U \times U)] \cap id_U = (X \times U) \cap id_U = id_X,$$

from which it follows that the relation $X \times Y$ generates $\mathfrak{A}$. Similar arguments apply if $X \times Y$ is replaced by $Y \times X$, by di_Y, or by di_X in the case when $m > 1$. As a result, if one of the subdiversity atoms in $\mathfrak{B}$ coincides with one of the (non-empty) relations in (1), then $\mathfrak{B}$ must coincide with $\mathfrak{A}$. This observation already implies the desired conclusion in the case when $m = 1$. Indeed, in that case (1) contains

only three non-empty relations, namely the final three, so a trivial counting argument implies that at least one of the subdiversity atoms in $\mathfrak{B}$ must coincide with one of the last three relations in (1).

There remains the case when $m = 2$, $n = 3$, and each of the subdiversity atoms in $\mathfrak{B}$ is the union of two of the relations in (1). Let R be the subdiversity atom in $\mathfrak{B}$ that includes the relation di_X. There are three cases to consider, according to which of three remaining relations in (1) is included in R. If R is the relation $di_X \cup (X \times Y)$, then R generates id_X and therefore also $\mathfrak{A}$, since

$$[R \,|\, (U \times U)] \cap id_U = (X \times U) \cap id_U = id_X.$$

A symmetric argument applies if R is the relation $di_X \cup (Y \times X)$:

$$[[(U \times U) \,|\, R] \cap id_U = (U \times X) \cap id_U = id_X.$$

If R is the relation $di_X \cup di_Y$, then R generates di_Y and therefore also $\mathfrak{A}$. In more detail,

$$\begin{aligned} R \,|\, R &= (di_X \cup di_Y) \,|\, (di_X \cup di_Y) \\ &= (di_X \,|\, di_X) \cup (di_X \,|\, di_Y) \cup (di_Y \,|\, di_X) \cup (di_Y \,|\, di_Y) \\ &= (di_X \,|\, di_X) \cup (di_Y \,|\, di_Y) = id_X \cup (Y \times Y), \end{aligned}$$

by the assumption on R, the distributive law for relational composition over union, the fact that the relational composition of di_X and di_Y in either order is empty, and the assumption that id_X and id_Y have types 2 and 3 respectively. Consequently,

$$(R \,|\, R) \sim id_U = di_Y,$$

as was to be shown. □

Theorem 18.53. *There are, up to isomorphism, exactly three quasi-minimal relation algebras that are not integral and have a minimal subalgebra isomorphic to* $\mathfrak{M}_3$, *namely* $\mathfrak{N}_{12}$, $\mathfrak{N}_{13}$, *and* $\mathfrak{N}_{23}$.

Proof. The algebras

$$\mathfrak{N}_{12}, \qquad \mathfrak{N}_{13}, \qquad \mathfrak{N}_{23} \tag{1}$$

are quasi-minimal, by Lemma 18.52. They are not integral, by Theorem 9.7, because they have two subdiversity atoms, id_X and id_Y. They

are mutually non-isomorphic, by the first observation made before Lemma 18.51, and they have an isomorphic copy of $\mathfrak{M}_3$ as their minimal subalgebra, by the second observation made before Lemma 18.51.

Consider now any quasi-minimal relation algebra $\mathfrak{A}$ that is not integral and that has an isomorphic copy of $\mathfrak{M}_3$ as its minimal subalgebra. Since $\mathfrak{A}$ is not integral, the identity element 1' is not an atom, by Theorem 9.7. Consequently, there is a non-zero subidentity element x in $\mathfrak{A}$ that is strictly below 1'. Put $y = 1' - x$, and observe that x and y are non-zero subidentity elements that are disjoint and sum to 1'. Let m and n be the types of x and y respectively. Without loss of generality, it may be assumed that $m \leq n$.

The algebra $\mathfrak{A}$ is assumed to be quasi-minimal, so it must be generated by any element that does not belong to the minimal subalgebra. In particular, it must be generated by x. Apply Lemma 18.50 and the remarks following it to conclude that $\mathfrak{A}$ is isomorphic to the algebra $\mathfrak{N}_{mn}$. It is not possible that $m = n = 1$, because the minimal subalgebra of $\mathfrak{N}_{11}$ is isomorphic to $\mathfrak{M}_2$, not $\mathfrak{M}_3$, by the second observation made before Lemma 18.50. Nor is it possible that $m = n > 1$, because the algebras $\mathfrak{N}_{22}$ and $\mathfrak{N}_{33}$ are not quasi-minimal, by Lemma 18.51. Consequently, we must have $m < n$, so $\mathfrak{A}$ is isomorphic to one of the algebras in (1). □

Write N_{mn} for the universal class generated by the algebra $\mathfrak{N}_{mn}$. In the cases when $1 \leq m < n \leq 3$, each of these classes contains, up to isomorphism, just the two algebras $\mathfrak{M}_3$ and $\mathfrak{N}_{mn}$.

Corollary 18.54. *The only quasi-minimal universal classes of simple relation algebras that are above* M_3 *and contain non-integral relation algebras are the classes* N_{mn} *for* $1 \leq m < n \leq 3$.

As a consequence of this result, all remaining quasi-minimal universal classes that are above M_3 must be generated by integral relation algebras. In addition to the ten universal classes generated by eight-element integral relation algebras (with three atoms), there are five known examples for which the generating integral relation algebra is symmetric with 16 elements (four atoms), one known example for which the generating integral relation algebra is non-symmetric with 16 elements, and one known example for which the generating integral relation algebra is non-symmetric with 32 elements (five atoms). A study of these four and five atom examples is left to the exercises. It is not known whether the 17 examples just described constitute all

of the possible examples of finite quasi-minimal relation algebras, nor is it known whether there are only finitely many such algebras.

In addition to the examples mentioned in the previous paragraph, there is one known example of an infinite quasi-minimal relation algebra. To describe it, let $\mathbb{Z}$ be the group of integers under addition, and let $\mathbb{Z} \times \mathbb{Z}$ be the direct product of the group $\mathbb{Z}$ with itself. Define a function $\| \ \|$ from $\mathbb{Z} \times \mathbb{Z}$ to the set $\mathbb{N}$ of natural numbers by

$$\|(\alpha, \beta)\| = |\alpha| + |\beta|$$

for α and β in $\mathbb{Z}$. This function inherits the standard norm properties from the absolute value function; in particular, it satisfies the triangle inequalities

$$\big|\|(\alpha, \beta)\| - \|(\gamma, \delta)\|\big| \leq \|(\alpha, \beta) + (\gamma, \delta)\| \leq \|(\alpha, \beta)\| + \|(\gamma, \delta)\|,$$

the commutative property

$$\|(\alpha, \beta)\| = \|(\beta, \alpha)\|,$$

and the scalar property

$$\|(\alpha, \beta)\| = \|(-\alpha, \beta)\| = \|(\alpha, -\beta)\| = \|(-\alpha, -\beta)\|.$$

The algebra to be constructed is a subalgebra of the group complex algebra $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$. Write

$$X_n = \{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z} : \|(\alpha, \beta)\| = 2n\},$$

for each natural number n (see Figure 18.2), and put

$$X_\infty = \{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z} : \|(\alpha, \beta)\| = 2n + 1 \text{ for some } n \in \mathbb{N}\}.$$

Thus, X_∞ consists of the pairs in $\mathbb{Z} \times \mathbb{Z}$ whose norm is an odd natural number. Notice that the presence of a pair (α, β) in X_n or in X_∞ implies the presence of all of the pairs

$$\begin{array}{llll} (\alpha, \beta), & (-\alpha, \beta), & (\alpha, -\beta), & (-\alpha, -\beta), \\ (\beta, \alpha), & (-\beta, \alpha), & (\beta, -\alpha), & (-\beta, -\alpha), \end{array}$$

by the commutative and scalar properties of the norm. In particular, the sets X_n and X_∞ are symmetric elements in $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$. For example,

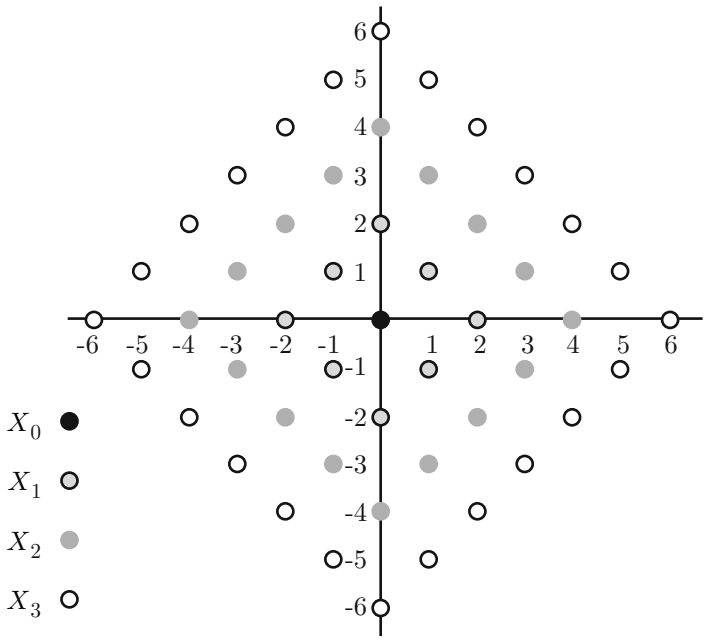


Fig. 18.2 The relations X_n for natural numbers n.

$$X_n^{\smile} = \{(-\alpha, -\beta) : (\alpha, \beta) \in X_n\} = X_n,$$

and similarly for X_∞.

Let W be the collection of all of these sets,

$$W = \{X_n : n \in \mathbb{N} \cup \{\infty\}\}.$$

The immediate goal is to show that W satisfies conditions (i)–(iv) of the Atomic Subalgebra Theorem (with respect $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$ in place of $\mathfrak{A}$). It is obvious that the sets in W are non-empty, mutually disjoint, and have the unit $\mathbb{Z}\times\mathbb{Z}$ as their union (see Figure 18.2), so condition (i) is satisfied. The identity element of the complex algebra is the singleton of the group identity element $(0, 0)$, and this singleton coincides with the set X_0, so condition (ii) is satisfied. Each of the sets in W is a symmetric element of the complex algebra, so condition (iii) is satisfied.

The verification of condition (iv) is more involved. To begin with, the group operation of addition in $\mathbb{Z}\times\mathbb{Z}$ is commutative, and the operation of relative multiplication in the complex algebra $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$, which is the operation of addition of complexes, inherits the commutativity of the group operation. Consequently, it suffices to compute only half of

the relative products of elements in W. In the statement of the next lemma, the symbolism $\bigcup_{i=0}^{\infty} X_i$ denotes, as usual, the union of the sets X_i over all natural numbers i. Thus, the relative product in (iii) does not include the set X_∞.

Lemma 18.55. *Assume m and n are natural numbers with $m \geq n$.*

(i) $X_m \,;\, X_n = \bigcup\{X_i : m - n \leq i \leq m + n\}$,
(ii) $X_n \,;\, X_\infty = X_\infty$,
(iii) $X_\infty \,;\, X_\infty = \bigcup_{i=0}^{\infty} X_i$.

Proof. The relative product $X_m \,;\, X_n$ in the complex algebra $\mathbb{Z} \times \mathbb{Z}$ is defined to be the set of elements (ξ, η) in the group $\mathbb{Z} \times \mathbb{Z}$ that can be written in the form

$$(\xi, \eta) = (\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta) \tag{1}$$

for some (α, β) in X_m and some (γ, δ) in X_n. This means that

$$\xi = \alpha + \gamma, \qquad \eta = \beta + \delta, \tag{2}$$
$$\|(\alpha, \beta)\| = |\alpha| + |\beta| = 2m, \qquad \|(\gamma, \delta)\| = |\gamma| + |\delta| = 2n, \tag{3}$$

by the definition of the sets X_m and X_n. In particular, the sums in (3) are even so α and β must have the same parity—that is to say, they must be both even or both odd—and similarly for γ and δ. An easy parity argument considering the four possible combinations of parities leads to the conclusion that the sums in (2) must also have the same parity. For example, if α and β are both even, and γ and δ are both odd, then ξ and η are both odd. Consequently,

$$\|(\xi, \eta)\| = \|(\alpha + \gamma, \beta + \delta)\| = |\alpha + \gamma| + |\beta + \delta| = 2i \tag{4}$$

for some natural number i. The triangle inequality properties of the norm, (3), and the assumption that $m \geq n$ imply that

$$\begin{aligned} 2(m - n) = |2m - 2n| &= \big|\|(\alpha, \beta)\| - \|(\gamma, \delta)\|\big| \\ &\leq \|(\alpha, \beta) + (\gamma, \delta)\| \leq \|(\alpha, \beta)\| + \|(\gamma, \delta)\| \\ &= 2m + 2n = 2(m + n). \end{aligned}$$

Combine these inequalities with (4) to arrive at

$$2(m - n) \leq 2i \leq 2(m + n)$$

and therefore

$$0 \leq m - n \leq i \leq m + n. \tag{5}$$

Conclusion: the pair (ξ, η) belongs to the set X_i for some natural number i satisfying (5), by the definition of these sets. This establishes the inclusion from left to right in (i).

To establish the reverse inclusion, suppose that i satisfies (5), and consider a pair (ξ, η) in X_i. We treat first the case $\xi \geq \eta \geq 0$. In this case, the difference $\xi - \eta$ is non-negative, and

$$\|(\xi, \eta)\| = |\xi| + |\eta| = \xi + \eta = 2i, \tag{6}$$

by the definition of the norm and the assumption that (ξ, η) is in X_i. Also,

$$2\xi = \xi + \xi \geq \xi + \eta = 2i,$$

by arithmetic and (6), so $\xi \geq i$ and therefore $\xi - i$ is non-negative. It follows from these observations and (5) that the integers

$$\begin{aligned} \alpha &= (\xi - i) + (m - n), & \beta &= \eta + [(m + n) - i], \\ \gamma &= i - (m - n), & \delta &= -[(m + n) - i], \end{aligned}$$

are non-negative, except the last one, which is non-positive. The pair (α, β) is in X_m, because

$$\begin{aligned} \|(\alpha, \beta)\| = |\alpha| + |\beta| &= \alpha + \beta \\ &= [(\xi - i) + (m - n)] + [\eta + [(m + n) - i]] \\ &= (\xi + \eta) - 2i + 2m = 2i - 2i + 2m = 2m, \end{aligned}$$

by the definition of the norm, the definitions of α and β, and (6). Similarly, the pair (γ, δ) is in X_n, because

$$\begin{aligned} \|(\gamma, \delta)\| = |\gamma| + |\delta| &= \gamma + -\delta \\ &= [i - (m - n)] + [(m + n) - i] = 2n, \end{aligned}$$

by the definition of the norm, and the definitions of γ and δ. Finally,

$$\alpha + \gamma = [(\xi - i) + (m - n)] + [i - (m - n)] = \xi$$

and

$$\beta + \delta = [\eta + [(m+n) - i]] - [(m+n) - i)] = \eta,$$

by the definitions of α, β, γ, and δ, so (1) holds. Combine these observations to conclude that the pair (ξ, η) belongs to the set $X_m ; X_n$ in this case.

The case $\eta \geq \xi \geq 0$ is treated in an analogous fashion. In this case,

$$2\eta = \eta + \eta \geq \xi + \eta = 2i,$$

and therefore $\eta \geq i$. Put

$$\begin{aligned} \alpha &= \xi + [(m+n) - i], & \beta &= (\eta - i) + (m - n), \\ \gamma &= -[(m+n) - i]), & \delta &= i - (m - n), \end{aligned}$$

and observe that all of these integers are non-negative except γ, which is non-positive. Computations similar to the ones in the preceding paragraph show that (α, β) and (γ, δ) are in X_m and X_n respectively, and that (1) holds. Consequently, (ξ, η) is in $X_m ; X_n$ in this case as well.

The remaining cases follow rather easily from the two preceding ones. As an example, consider the case when (ξ, η) is in X_i (with i satisfying (5)) and

$$\xi \geq 0 \geq \eta \geq -\xi.$$

Observe that the pair $(\xi, -\eta)$ is also in X_i, and that $\xi \geq -\eta \geq 0$. The argument for the first case above implies that $(\xi, -\eta)$ is in $X_m ; X_n$, so there must be pairs (α, β) in X_m and (γ, δ) in X_n such that

$$(\xi, -\eta) = (\alpha, \beta) + (\gamma, \delta).$$

In particular,

$$\xi = \alpha + \gamma \qquad \text{and} \qquad -\eta = \beta + \delta$$

It follows that $\eta = -\beta + -\delta$. The pairs $(\alpha, -\beta)$ and $(\gamma, -\delta)$ also belong to the sets X_m and X_n respectively, and

$$(\xi, \eta) = (\alpha, -\beta) + (\gamma, -\delta),$$

so the pair (ξ, η) is in $X_m ; X_n$.

As a final example, consider the case when (ξ, η) is in X_i and

$$0 \geq \xi \geq \eta.$$

The pair $(-\xi, -\eta)$ is also in X_i, and $-\eta \geq -\xi \geq 0$. The argument of the second case above implies that $(-\xi, -\eta)$ is in $X_m \,;\, X_n$, so there must be pairs (α, β) in X_m and (γ, δ) in X_n such that

$$(-\xi, -\eta) = (\alpha, \beta) + (\gamma, \delta).$$

In particular,

$$-\xi = \alpha + \gamma \qquad \text{and} \qquad -\eta = \beta + \delta$$

from which it follows that

$$\xi = -\alpha + -\gamma \qquad \text{and} \qquad \eta = -\beta + -\delta.$$

The pairs $(-\alpha, -\beta)$ and $(-\gamma, -\delta)$ also belong to the sets X_m and X_n respectively, and

$$(\xi, \eta) = (-\alpha, -\beta) + (-\gamma, -\delta),$$

so the pair (ξ, η) is in $X_m \,;\, X_n$.

The proof of (ii) is less involved. To establish the inclusion from left to right in (ii), consider pairs (α, β) in X_n and (γ, δ) in X_∞. The integers α and β have the same parity, because

$$\|(\alpha, \beta)\| = |\alpha| + |\beta| = 2n,$$

by the definition of X_n; and the integers γ and δ have opposite parity, because

$$\|(\gamma, \delta)\| = |\gamma| + |\delta| = 2k + 1$$

for some integer k, by the definition of X_∞. An easy case argument shows that the integers ξ and η defined in (2) have opposite parity. For example, if α and β are both even, while γ is even and δ odd, then ξ is even and η is odd. The definition of X_∞ therefore implies that (ξ, η) belongs to X_∞. The other cases are treated in the same way. It follows that the relative product $X_n \,;\, X_\infty$ is included in the set X_∞.

To establish the reverse inclusion, consider a pair (ξ, η) in X_∞, and take (α, β) to be an arbitrary pair in X_n. The integers ξ and η have the opposite parity, because they are in X_∞, and the integers α and β have the same parity, because they are in X_n, so the integers γ and δ defined by

$$\gamma = \xi - \alpha \qquad \text{and} \qquad \delta = \eta - \beta. \tag{7}$$

have the opposite parity. Consequently, the pair (γ, δ) belongs to X_∞. It follows directly from (7) that (1) holds, so the pair (ξ, η) belongs to the relative product $X_n \,;\, X_\infty$. This completes the proof of (ii).

The proof of (iii) is similar in spirit to the proof of (ii). To establish the inclusion from left to right in (iii), consider pairs (α, β) and (γ, δ) in X_∞. The integers α and β have opposite parity, as do the integers γ and δ, by the definition of X_∞, so the integers ξ and η defined in (2) have the same parity, and therefore the pair (ξ, η) defined in (1) must belong to the set X_i for the natural number i that is the quotient of $|\xi| + |\eta|$ divided by 2. Consequently, $X_\infty \,;\, X_\infty$ is included in the union, over all natural number i, of the sets X_i.

To establish the reverse inclusion, consider a pair (ξ, η) that belongs to X_i for some natural number i. The integers ξ and η have the same parity, so the integers $\xi - 1$ and η must have the opposite parity, as do the integers 1 and 0. It follows that the pairs $(\xi - 1, \eta)$ and $(1, 0)$ are both in X_∞, and obviously

$$(\xi, \eta) = (\xi - 1, \eta) + (1, 0),$$

so (ξ, η) belongs to the relative product $X_\infty \,;\, X_\infty$. □

As an illustration of the computation in part (i) of the lemma, observe that

$$X_1 \,;\, X_n = X_{n-1} \cup X_n \cup X_{n+1}$$

for $n \geq 1$, and

$$X_2 \,;\, X_n = X_{n-2} \cup X_{n-1} \cup X_n \cup X_{n+1} \cup X_{n+2}$$

for $n \geq 2$.

It has been shown that the set W defined before Lemma 18.55 satisfies the hypotheses of the Atomic Subalgebra Theorem with respect to the algebra $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$ (in place of $\mathfrak{A}$). Apply the theorem in the form of Corollary 6.22 to arrive at the following conclusion.

Lemma 18.56. *The set of all unions of subsets of W is the universe of a complete and atomic symmetric subalgebra of $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$. The atoms of this subalgebra are just the elements in W, and the operation of relative multiplication between these atoms is determined by Lemma 18.55.*

Write $\mathfrak{C}$ for the subalgebra described in Lemma 18.56. The infinite quasi-minimal algebra that we want to construct is not $\mathfrak{C}$, but rather a kind of finite-cofinite subalgebra of $\mathfrak{C}$. Call an element in $\mathfrak{C}$ *finite* if it is the union of finitely many of the atoms X_n for natural numbers n, and *cofinite* if it is the complement (with respect to the unit $\mathbb{Z} \times \mathbb{Z}$) of a finite element. Thus, an element Y in $\mathfrak{C}$ is finite or cofinite according to whether there is a finite or cofinite set I of natural numbers such that

$$Y = \bigcup\{X_i : i \in I\} \qquad \text{or} \qquad Y = \bigcup\{X_i : i \in I \cup \{\infty\}\}$$

respectively. The complement of a finite element is cofinite, and vice versa, the union of two finite elements is finite, and the union of a cofinite element with either a finite or a cofinite element is cofinite. Consequently, the set A of finite and cofinite elements in $\mathfrak{C}$ is closed under the Boolean operations of $\mathfrak{C}$. The identity element in $\mathfrak{C}$ is the set X_0, which is finite, so it belongs to A. The operation of converse in $\mathfrak{C}$ is the identity function, since each atom is symmetric, so A is closed under converse. The formula given in part (i) of Lemma 18.55 shows that the relative product of two finite elements is finite, and the formulas given in parts (i)–(iii) of the lemma show that the relative product of a cofinite element with either a non-zero finite element or a cofinite element is cofinite. In fact, the relative product of two cofinite elements is always the unit $\mathbb{Z} \times \mathbb{Z}$. Consequently, A is closed under relative multiplication. It follows that A is the universe of a subalgebra $\mathfrak{A}$ of the algebra $\mathfrak{C}$ and the algebra $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$.

The algebra $\mathfrak{A}$ is atomic, and its atoms are the sets X_n for natural numbers n. The supremum of any set of atoms that contains infinitely many atoms and omits infinitely many atoms does not exist in $\mathfrak{A}$, so $\mathfrak{A}$ is not complete. Also, $\mathfrak{A}$ is generated by a single element, namely X_1. Indeed, the formula given after Lemma 18.55 for computing the value of $X_1 \,;\, X_n$ shows that

$$X_{n+1} = (X_1 \,;\, X_n) \sim (X_{n-1} \cup X_n).$$

An easy argument by induction on n, using this formula leads to the conclusion that X_1 generates every atom X_n in $\mathfrak{A}$ for $n \geq 1$, and therefore X_1 generates all of $\mathfrak{A}$.

Theorem 18.57. *The algebra $\mathfrak{A}$ is quasi-minimal and therefore generates a quasi-minimal universal class of simple relation algebras.*

Proof. The first step is to show that every non-constant element in $\mathfrak{A}$ generates a subalgebra of $\mathfrak{A}$ that includes an isomorphic copy of A. To this end, consider an element Y in $\mathfrak{A}$ that is different from zero, one, the identity element, and the diversity element. Every element in $\mathfrak{A}$ is either finite or the complement of a finite element. By passing to a complement if necessary, it may be assumed that Y is finite. Thus, Y is the union of finitely many atoms, say

$$Y = \bigcup\{X_i : i \in I\},$$

where I is a finite set of natural numbers that contains a natural number different from 0.

Let ℓ be the largest number in I. If j and k are arbitrary numbers in I with $j \leq k$, then the formula in Lemma 18.55(i) implies that

$$\begin{aligned} X_j \,;\, X_k &= \bigcup\{X_i : k - j \leq i \leq k + j\} \\ &\subseteq \bigcup\{X_i : 0 \leq i \leq 2\ell\} = X_\ell \,;\, X_\ell. \end{aligned} \tag{1}$$

Use the fact that ℓ is in I, the monotony and distributive laws for relative multiplication, and (1) to arrive at

$$X_\ell \,;\, X_\ell \subseteq Y \,;\, Y = \bigcup\{X_j \,;\, X_k : j, k \in I\} \subseteq X_\ell \,;\, X_\ell.$$

The first and last terms are the same, so equality holds everywhere:

$$Y \,;\, Y = X_\ell \,;\, X_\ell = \bigcup\{X_i : 0 \leq i \leq 2\ell\}. \tag{2}$$

Define sets Y_n in $\mathfrak{A}$ by induction on natural numbers n:

$$Y_0 = X_0, \quad Y_1 = (Y \,;\, Y) \sim Y_0, \quad Y_{n+1} = (Y_1 \,;\, Y_n) \sim (Y_{n-1} \cup Y_n) \tag{3}$$

for $n \geq 1$. A proof by induction on natural numbers n, using also Lemma 18.55(i), shows that

$$Y_n = \bigcup\{X_i : 2(n-1)\ell + 1 \leq i \leq 2n\ell\}. \tag{4}$$

For example,

$$\begin{aligned} Y_1 &= X_1 \cup X_2 \cup \cdots \cup X_{2\ell}, \\ Y_2 &= X_{2\ell+1} \cup X_{2\ell+2} \cup \cdots \cup X_{4\ell}, \\ Y_3 &= X_{4\ell+1} \cup X_{4\ell+2} \cup \cdots \cup X_{6\ell}, \end{aligned}$$

and so on.

Here are the details of the argument. The value of Y_1 follows directly from (2) and the definitions of Y_0 and Y_1 in (3). To compute the value of Y_2, apply (2) with Y_1 in place of Y, and 2ℓ (the largest number in the index set determining Y_1) in place of ℓ, to obtain

$$Y_1 \,;\, Y_1 = X_{2\ell} \,;\, X_{2\ell} = \bigcup\{X_i : 0 \leq i \leq 4\ell\}.$$

This computation, combined with the already computed values of Y_0 and Y_1, and the definition

$$Y_2 = (Y_1 \,;\, Y_1) \sim (Y_0 \cup Y_1),$$

leads directly to the desired value for Y_2.

To compute the value of Y_{n+1} for $n \geq 2$, under the assumption that the values of Y_{n-1} and Y_n are determined by (4) (in the first case with $n-1$ in place of n), observe that the distributive law for relative multiplication, the computed value for Y_1, and the induction hypothesis for Y_n imply that

$$Y_1 \,;\, Y_n = \bigcup\{X_j \,;\, X_k : 1 \leq j \leq 2\ell \text{ and } 2(n-1)\ell + 1 \leq k \leq 2n\ell\}. \quad (5)$$

The relative product in (5) is the union of sets of the form X_i for various indices i, by Lemma 18.55(i). The smallest index i is realized when j achieves its maximum value 2ℓ, and k achieves its minimum value $2(n-1)\ell + 1$, and in this case

$$X_j \,;\, X_k = X_{2\ell} \,;\, X_{2(n-1)\ell+1} = \bigcup\{X_i : 2(n-2)\ell + 1 \leq i \leq 2n\ell + 1\},$$

by Lemma 18.55(i), since

$$[2(n-1)\ell + 1] - 2\ell = 2(n-2)\ell + 1$$

and

$$[2(n-1)\ell + 1] + 2\ell = 2n\ell + 1.$$

The largest index i is realized when j and k both achieve their maximum values 2ℓ and $2n\ell$ respectively, and in this case

$$X_j \,;\, X_k = X_{2\ell} \,;\, X_{2n\ell} = \bigcup\{X_i : 2(n-1)\ell \leq i \leq 2(n+1)\ell\},$$

by Lemma 18.55(i), since

$$2n\ell - 2\ell = 2(n-1)\ell \qquad \text{and} \qquad 2n\ell + 2\ell = 2(n+1)\ell.$$

These computations and (5) together yield

$$Y_1 \,;\, Y_n = \bigcup\{X_i : 2(n-2)\ell + 1 \leq i \leq 2(n+1)\ell\}. \tag{6}$$

The induction hypotheses for Y_{n-1} and Y_n imply that

$$Y_{n-1} \cup Y_n = \bigcup\{X_i : 2(n-2)\ell + 1 \leq i \leq 2n\ell\}. \tag{7}$$

Combine (6) and (7) with the definition of Y_{n+1} in (3) to arrive at

$$Y_{n+1} = \bigcup\{X_i : 2n\ell + 1 \leq i \leq 2(n+1)\ell\},$$

as desired.

For the moment, work in the algebra $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$ or in its complete subalgebra $\mathfrak{C}$, defined after Lemma 18.56. Write $Y_\infty = X_\infty$, and let W be the collection of sets

$$W = \{Y_n : n \in \mathbb{N} \cup \{\infty\}\}.$$

An argument similar to the proof of Lemma 18.56 shows that W satisfies conditions (i)–(iv) of the Atomic Subalgebra Theorem. In more detail, the sets in W are non-empty and mutually disjoint, by (4) and the mutual disjointness of the sets X_i for i in $\mathbb{N} \cup \{\infty\}$; and the union of the sets in W is the unit $\mathbb{Z} \times \mathbb{Z}$, by the corresponding result for the sets X_i and by (4). Thus, condition (i) is satisfied. The identity element is the set $Y_0 = X_0$, which belongs to W by definition, so condition (ii) is satisfied. The elements in W are unions of symmetric elements in $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$, so they are symmetric, and therefore condition (iii) is trivially satisfied.

It remains to verify condition (iv). Let m and n be natural numbers with $m \leq n$. If $m = n$, then apply (2) with Y_n in place of Y, and $2n\ell$ (the largest number in the index set determining Y_n) in place of ℓ, and then use (4) to arrive

$$\begin{aligned} Y_n \,;\, Y_n = X_{2n\ell} \,;\, X_{2n\ell} &= \bigcup\{X_i : 0 \leq i \leq 4n\ell\} \\ &= \bigcup\{Y_i : 0 \leq i \leq 2n\}. \end{aligned} \tag{8}$$

If $m < n$, then a computation similar to the one carried out to derive (5), but using (4), shows that

$$Y_m ; Y_n = \bigcup\{X_j ; X_k : 2(m-1)\ell + 1 \leq j \leq 2m\ell \text{ and } 2(n-1)\ell + 1 \leq k \leq 2n\ell\}. \quad (9)$$

The relative product in (9) is the union of sets of the form X_i for various indices i, by Lemma 18.55(i). The smallest index i is realized when j achieves its maximum value $2m\ell$, and k achieves its minimum value $2(n-1)\ell + 1$, and in this case

$$\begin{aligned} X_j ; X_k &= X_{2m\ell} ; X_{2(n-1)\ell+1} \\ &= \bigcup\{X_i : 2(n-m-1)\ell + 1 \leq i \leq 2(n+m-1)\ell + 1\}, \end{aligned}$$

by Lemma 18.55(i). The largest index i is realized when j and k both achieve their maximum values $2m\ell$ and $2n\ell$ respectively, and in this case

$$X_j ; X_k = X_{2m\ell} ; X_{2n\ell} = \bigcup\{X_i : 2(n-m)\ell \leq i \leq 2(m+n)\ell\},$$

again by Lemma 18.55(i). The lemma and (9) also imply that all intermediate values of i between the minimum $2(n-m-1)\ell + 1$ and the maximum $2(m+n)\ell$ are realized as well, so

$$\begin{aligned} Y_m ; Y_n &= \bigcup\{X_i : 2(n-m-1)\ell + 1 \leq i \leq 2(m+n)\ell\} \\ &= \bigcup\{Y_i : n-m \leq i \leq m+n\}, \end{aligned} \quad (10)$$

by (9) and (4).

Finally, parts (ii) and (iii) of Lemma 18.55, the definition of Y_∞, and (4), immediately imply that

$$Y_n ; Y_\infty = Y_\infty \qquad \text{and} \qquad Y_\infty ; Y_\infty = \bigcup_{i=0}^{\infty} Y_i.$$

Thus, in all cases the relative product of two elements in W is a union of elements in W. This completes the verification of condition (iv).

It has been shown that the conditions of the Atomic Subalgebra Theorem in the form of Corollary 6.22 are satisfied by the set W. Apply the corollary to conclude that the set of unions of subsets of W is the universe of an atomic, complete subalgebra $\mathfrak{C}'$ of the algebras $\mathfrak{C}$ and $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$, and the atoms in $\mathfrak{C}'$ are the elements in the set W.

The finite elements in $\mathfrak{C}'$ are the sets that can be written as finite unions of atoms of the form Y_n for natural numbers n, and the cofinite elements are just the complements of the finite elements. In view of (4), the notions of a finite element and a cofinite element in $\mathfrak{C}'$ coincide

with the corresponding notions in $\mathfrak{C}$, so the set of finite and cofinite elements in $\mathfrak{C}'$ is a subset of the finite-cofinite algebra $\mathfrak{A}$. Also, just as was the case with $\mathfrak{C}$, the set of finite and cofinite elements in $\mathfrak{C}'$ is closed under the operations of $\mathfrak{C}'$ and is therefore the universe of an atomic subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ whose atoms are just the elements Y_n for natural numbers n. Notice that the subalgebra $\mathfrak{A}'$ is generated by a single element, namely the atom Y_1, by the definition of the atoms Y_n.

The function φ that maps each atom Y_n in $\mathfrak{A}'$ to the atom X_n in $\mathfrak{A}$ is clearly a bijection from the set of atoms in $\mathfrak{A}'$ to the set of atoms in $\mathfrak{A}$. Moreover, a comparison of Lemma 18.55(i) with (8) and (10) above shows that φ preserves the operation of relative multiplication on atoms, that is to say,

$$Y_i \subseteq Y_j \,;\, Y_k \qquad \text{if and only if} \qquad X_i \subseteq X_j \,;\, X_k.$$

Apply the Atomic Isomorphism Theorem in the form of Corollary 7.12 to conclude that φ can be extended in a unique way to an isomorphism from $\mathfrak{A}'$ to $\mathfrak{A}$.

The preceding argument implies that every subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ that is different from the minimal subalgebra includes an isomorphic copy of $\mathfrak{A}$. Indeed, $\mathfrak{B}$ must have an element Y that is different from the constants zero, one, the identity element, and the diversity element. By passing to the complement of Y, if necessary, it may be assumed that Y is finite. In terms of Y, define an element Y_1 as in (3). This element Y_1 generates a subalgebra $\mathfrak{A}'$ of $\mathfrak{B}$ that is isomorphic to $\mathfrak{A}$, by the arguments of the preceding paragraphs.

In order to prove that $\mathfrak{A}$ is quasi-minimal, the result in the preceding paragraph must be extended to non-minimal subalgebras $\mathfrak{B}$ of ultrapowers of $\mathfrak{A}$. To do this, observe that the notion of an element in $\mathfrak{A}$ being finite is expressible as a first-order inequality: Y is finite if and only if $Y \,;\, Y$ is not the unit. Consequently, the notion of an element in $\mathfrak{A}$ being cofinite is also expressible: Y is cofinite if and only if $\sim Y$ is finite, or equivalently, if and only if $Y \,;\, Y$ is the unit. Every element in $\mathfrak{A}$ is either finite or cofinite, and this property is also expressible by a first-order sentence, so it must be preserved under ultrapowers, by the Fundamental Theorem of Ultraproducts. Consequently, all of the elements in an ultrapower of $\mathfrak{A}$ are either finite or cofinite.

Consider a non-minimal subalgebra $\mathfrak{B}$ of some ultrapower of $\mathfrak{A}$, and let Y be an arbitrary element in $\mathfrak{B}$ that is different from the constants zero, one, the identity element, and the diversity element. Such an element exists because $\mathfrak{B}$ is assumed to be different from the minimal

subalgebra. By passing to the complement of Y if necessary, it may be assumed that Y is finite. Define Y_0 to be the identity element in $\mathfrak{B}$, and define

$$Y_1 = (Y\,;Y) \sim Y_0, \qquad \text{and} \qquad Y_{n+1} = (Y_1\,;Y_n) \sim (Y_{n-1} \cup Y_n)$$

for each natural number $n \geq 1$. A sequence of elements defined in this way in the finite-cofinite algebra $\mathfrak{A}$ must be disjoint, by the arguments presented above for the non-constant element Y in $\mathfrak{A}$, so a sequence of elements defined in this way in an ultrapower of $\mathfrak{A}$ must also be disjoint. In more detail, for every natural number n, one can express with a universal sentence the property that if Y is a finite element different from zero and the identity element, and if the elements $Y_0, Y_1, \ldots, Y_n$ are defined as above, then these $n+1$ elements are mutually disjoint. This property is true in $\mathfrak{A}$ and is preserved under ultrapowers, by the Fundamental Theorem of Ultraproducts, so it is true in the given ultrapower of $\mathfrak{A}$.

Similar arguments show that the whole arithmetic of the elements Y_n in $\mathfrak{A}$ carries over to the corresponding elements Y_n in the given ultrapower of $\mathfrak{A}$. For example, if m and n are natural numbers with $m \leq n$, then one can express with a universal sentence the property that if Y is a finite element different from zero and the identity element, and if the elements $Y_0, Y_1, \ldots, Y_{m+n}$ are defined as above, then the relative product $Y_m\,;Y_n$ is the sum of the elements Y_i for $n - m \leq i \leq m + n$. Consequently, this property is preserved under ultrapowers. In particular, the element Y_1 generates the atoms Y_0, Y_1, Y_2, ... of an atomic subalgebra $\mathfrak{A}'$ of $\mathfrak{B}$, and the operation of relative multiplication on these atoms obeys the same rules in $\mathfrak{A}'$ as it does on the corresponding elements in $\mathfrak{A}$, and also as it does on the atoms X_0, X_1, X_2 ... in $\mathfrak{A}$. Consequently, the Atomic Isomorphism Theorem in the form of Corollary 7.12 implies that the function mapping Y_n to X_n for each natural number n extends to an isomorphism from $\mathfrak{A}'$ to $\mathfrak{A}$.

Conclusion: every non-minimal subalgebra of an ultrapower of $\mathfrak{A}$ has a subalgebra that is isomorphic to $\mathfrak{A}$, so $\mathfrak{A}$ is quasi-minimal. The assertion of the theorem that $\mathfrak{A}$ generates a quasi-minimal universal class now follows from Lemma 18.49. □

18.8 Maximal universal classes and varieties

A universal class K of simple relation algebras is said to be *maximal*, or a *coatom*, in the lattice of universal classes if there is exactly one universal class of simple relation algebras that properly includes K as a subclass, namely the class of all simple relation algebras. A corresponding terminology applies to varieties in the lattice of varieties of relation algebras. The purpose of this section is to prove that there are infinitely many maximal universal classes in the lattice of universal classes, and consequently there are infinitely many maximal varieties in the lattice of varieties.

To this end, consider a finite, simple relation algebra $\mathfrak{A}$. The class K of all simple relation algebras that *do not* contain an isomorphic copy of $\mathfrak{A}$ as a subalgebra is universal. In fact, Axioms (R1)–(R10), the universal sentence Δ_0 expressing that a relation algebra is simple, and the conjugate equation $\varepsilon_{\mathfrak{A}}$ of $\mathfrak{A}$, taken together, form a set of axioms for K (see Lemma 18.38 and the remarks preceding it). The class K is called the *conjugate universal class* of $\mathfrak{A}$, and the corresponding variety $\mathsf{SP}(\mathsf{K})$ is called the *conjugate variety* of $\mathfrak{A}$.

A particularly interesting example is the conjugate universal class of a full set relation algebra on a finite set of cardinality $n \geq 1$. It turns out that for each such n, this class is a maximal element in the lattice of universal classes. To prove this assertion, it is helpful to introduce some notation. For each $n \geq 1$, let Δ_n be the sentence asserting the existence of n^2 mutually disjoint singletons that sum to 1. Thus, Δ_n is the existential sentence

$$\exists v_0 \exists v_1 \ldots \exists v_{n^2-1} \Gamma,$$

where Γ is the conjunction of the following equations and inequalities over all i and j with $0 \leq i < j < n^2$:

$$v_i \neq 0, \qquad v_i^\smile ; 1 ; v_i \leq 1', \qquad v_i ; 1 ; v_i^\smile \leq 1',$$
$$v_i \cdot v_j = 0, \qquad v_0 + v_1 + \cdots + v_{n^2-1} = 1.$$

Recall that every cardinal number κ can be thought of as the set of ordinal numbers that are less than κ. In particular, every natural number n may be thought of as the set

$$n = \{0, 1, \ldots, n-1\}.$$

As a result, it makes sense to write $\mathfrak{Re}(\kappa)$ for the full set relation algebra on the set κ, and every full set relation algebra on a set of cardinality κ is base isomorphic to $\mathfrak{Re}(\kappa)$, by the remarks in Section 7.5.

Lemma 18.58. *The sentence Δ_n (for $n \geq 1$) is true in a simple relation algebra $\mathfrak{A}$ if and only if $\mathfrak{Re}(n)$ is isomorphic to $\mathfrak{A}$. Consequently, there is an equation that holds in a simple relation algebra $\mathfrak{A}$ if and only if $\mathfrak{A}$ is not isomorphic to $\mathfrak{Re}(n)$.*

Proof. The set of singletons in the full set relation algebra $\mathfrak{Re}(n)$,

$$\{\{(i,j)\} : 0 \leq i,j < n\},$$

consists of n^2 mutually disjoint elements and the union of this set is the unit relation $n \times n$. Consequently, the sentence Δ_n is true in $\mathfrak{Re}(n)$, and therefore it is true in every isomorphic image of $\mathfrak{Re}(n)$.

On the other hand, if Δ_n is true in a simple relation algebra $\mathfrak{A}$, then $\mathfrak{A}$ is singleton dense with n^2 singletons. Apply Theorem 17.44 and its proof to conclude that $\mathfrak{A}$ is isomorphic to $\mathfrak{Re}(n)$. This establishes the first part of the lemma.

It follows from the observations of the preceding paragraphs that the negation $\neg\Delta_n$ is true in a simple relation algebra $\mathfrak{A}$ if and only $\mathfrak{A}$ is not isomorphic to $\mathfrak{Re}(n)$. As the negation of an existential sentence, $\neg\Delta_n$ is equivalent to a universal sentence, so there is an equation δ_n that is equivalent to $\neg\Delta_n$ in all simple relation algebras, by Theorem 9.5. Combine these observations to conclude that δ_n is true in a simple relation $\mathfrak{A}$ if and only if $\mathfrak{A}$ is not isomorphic to $\mathfrak{Re}(n)$. □

One consequence of the lemma is the striking property that $\mathfrak{Re}(n)$ can never be properly embedded into a simple relation algebra.

Corollary 18.59. *If $\mathfrak{Re}(n)$ is embeddable into a simple relation algebra $\mathfrak{A}$, then the embedding is actually an isomorphism.*

Proof. For the case $n \geq 1$, the existential sentence Δ_n holds in $\mathfrak{Re}(n)$, by Lemma 18.58. If $\mathfrak{Re}(n)$ is embeddable into a simple relation algebra $\mathfrak{A}$, then Δ_n must also hold in $\mathfrak{A}$, because existential sentences are preserved under extensions. Consequently, $\mathfrak{A}$ must be isomorphic to $\mathfrak{Re}(n)$, again by Lemma 18.58. In particular, $\mathfrak{A}$ is finite with the same cardinality as $\mathfrak{Re}(n)$, so the given embedding from $\mathfrak{Re}(n)$ into $\mathfrak{A}$ must be onto and therefore an isomorphism.

The corollary is also true when $n = 0$. The reason is that the degenerate relation algebra $\mathfrak{Re}(0)$ is never embeddable into a simple relation

algebra, because the minimal subalgebra of a simple relation algebra is not degenerate. □

Corollary 18.60. *For $n \geq 1$, the conjugate universal class of $\mathfrak{Re}(n)$ consists of all simple relation algebras that are not isomorphic to $\mathfrak{Re}(n)$.*

Proof. The conjugate universal class of $\mathfrak{Re}(n)$, by definition, consists of all simple relation algebras that do not contain an isomorphic copy of $\mathfrak{Re}(n)$ as a subalgebra. In view of Corollary 18.59, this is the same as the class of simple relation algebras that are not isomorphic to $\mathfrak{Re}(n)$. □

Theorem 18.61. *For each natural number $n \geq 1$, the conjugate universal class of $\mathfrak{Re}(n)$ is a maximal element in the lattice of universal classes.*

Proof. If K is the conjugate universal class of $\mathfrak{Re}(n)$, then every simple relation algebra not isomorphic to $\mathfrak{Re}(n)$ belongs to K, by Corollary 18.60. Consequently, a universal class L that properly includes K must contain $\mathfrak{Re}(n)$ and must therefore coincide with the universal class of all simple relation algebras. □

The isomorphism defined in Theorem 18.28 between the lattice of universal classes and the lattice of varieties directly implies the analogue of Theorem 18.61 for the lattice of varieties.

Corollary 18.62. *For each natural number $n \geq 1$, the conjugate variety of $\mathfrak{Re}(n)$ is a maximal element in the lattice of varieties.*

18.9 Universal classes of representable relation algebras

For a finite, simple relation algebra $\mathfrak{A}$, the class $\mathsf{S}(\mathfrak{A})$ of all algebras that are embeddable into $\mathfrak{A}$ is a universal class of simple relation algebras at some finite level of the lattice of universal classes, by Corollary 18.13 and Theorem 18.41 (with K taken to be the empty universal class). In particular, for each positive integer n, the class

$$\mathsf{R}_n = \mathsf{S}(\mathfrak{Re}(n))$$

of algebras embeddable into a full set relation algebra on a set of cardinality n is a universal class at some finite level of the lattice. For

distinct positive integers m and n, the universal class R_m and R_n are distinct, and in fact they are *incomparable* in the sense that neither is a subclass of the other, by Corollary 18.60. Interestingly, this observation does not extend to infinite cardinals. In fact, any two full set relation algebras on infinite sets have the same equational theory and therefore generate the same variety of relation algebras. The proof of this assertion makes use of the fact that terms in the language $\mathcal{L}$ of relation algebras (see Section 2.4) can be translated into formulas in the language $\mathcal{L}^*$ of relations (see Section 2.5).

It may be helpful to recall a few notational details about $\mathcal{L}^*$. There is an infinite sequence $v_0, v_1, v_2, \dots$ of variables in $\mathcal{L}^*$ (the same infinite sequence of variables that occurs in $\mathcal{L}$), and a corresponding infinite sequence $\boldsymbol{R}_0, \boldsymbol{R}_1, \boldsymbol{R}_2, \dots$ of binary relation symbols. The first three variables in $\mathcal{L}^*$ are denoted by $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ respectively. If Γ is a formula of $\mathcal{L}^*$ in which the free variables form a subset of $\{\boldsymbol{x}, \boldsymbol{y}\}$, and if u and v are arbitrary variables in $\mathcal{L}^*$, then the notation $\Gamma(u, v)$ denotes the formula obtained from Γ by simultaneously substituting u and v for all free occurrences of $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively, bound variables being changed to avoid collisions. In order to reduce the chance of notational confusion, the symbol $\boldsymbol{I}$ is used for the identity relation symbol in $\mathcal{L}^*$. Recall also that if ϱ is a term in the language $\mathcal{L}$ of relation algebras, then the notation $\varrho(v_0, \dots, v_{n-1})$ means that the variables occurring in ϱ are all among $v_0, \dots, v_{n-1}$.

Define a function φ from the set of terms in $\mathcal{L}$ to the set of formulas in $\mathcal{L}^*$ by induction on terms as follows. For each natural number n, put

$$\varphi(v_n) = \boldsymbol{R}_n\boldsymbol{x}\boldsymbol{y}, \qquad \text{and} \qquad \varphi(1') = \boldsymbol{I}\boldsymbol{x}\boldsymbol{y}.$$

If σ and τ are terms in $\mathcal{L}$ on which φ has been defined, say

$$\varphi(\sigma) = \Gamma \qquad \text{and} \qquad \varphi(\tau) = \Delta,$$

then put

$$\varphi(\sigma + \tau) = \Gamma \vee \Delta, \qquad \varphi(\sigma\,;\tau) = \exists \boldsymbol{z}(\Gamma(\boldsymbol{x}, \boldsymbol{z}) \wedge \Delta(\boldsymbol{z}, \boldsymbol{y})),$$
$$\varphi(-\sigma) = \neg\Gamma, \qquad \varphi(\sigma^\smile) = \Gamma(\boldsymbol{y}, \boldsymbol{x}).$$

The function φ is called the *translation mapping* from $\mathcal{L}$ to $\mathcal{L}^*$. Its key properties are described in the following lemma and theorem. In order to formulate them, it is convenient to introduce some notation.

If $R = (R_0, \ldots, R_{n-1})$ is a sequence of n (binary) relations on a set U, then we write $\mathfrak{U}_R$ for the relational structure whose universe is U and whose fundamental relations are the relations in R, so that

$$\mathfrak{U}_R = (U\ ,\ R_0\ ,\ \ldots\ ,\ R_{n-1}).$$

Each of the relations in R belongs to the full set relation algebra $\mathfrak{Re}(U)$, so R is a sequence of elements in $\mathfrak{Re}(U)$. Consequently, if $\varrho(v_0, \ldots, v_{n-1})$ is a term in the language $\mathcal{L}$, then it makes sense to write $\varrho(R)$: this is just the relation in $\mathfrak{Re}(U)$ that is the value of the term ϱ on the sequence R (see Section 2.4).

Lemma 18.63. *Suppose $\varrho(v_0, \ldots, v_{n-1})$ is a term in $\mathcal{L}$, and R a sequence of n relations on a set U. A pair (α, β) of elements from U belongs to the relation $\varrho(R)$ in the algebra $\mathfrak{Re}(U)$ if and only if (α, β) satisfies the formula $\varphi(\varrho)$ in the structure $\mathfrak{U}_R$.*

Proof. The proof proceeds by induction on terms. For the base case, there are two possibilities: ϱ is either a variable v_i or the individual constant symbol 1'. Suppose that the first possibility holds. In this case, $\varphi(\varrho)$ is defined to be the formula $\boldsymbol{R}_i\boldsymbol{xy}$. The relation symbol $\boldsymbol{R}_i$ is interpreted in the structure $\mathfrak{U}_R$ as the relation R_i, so a pair (α, β) satisfies the formula $\varphi(\varrho)$ in $\mathfrak{U}_R$ if and only if it belongs to the relation R_i, by the definition of satisfaction in $\mathfrak{U}_R$ (see Section 2.5). On the other hand, the value of $\varrho(R)$ in $\mathfrak{Re}(U)$ is the relation R_i, by the assumption that ϱ is the variable v_i, and by the definition of the value of a term on a sequence R of elements in $\mathfrak{Re}(U)$ (see Section 2.4). A pair (α, β) is therefore in the relation $\varrho(R)$ if and only if it is in R_i. Combine these observations to arrive at the desired conclusion, namely that a pair (α, β) is in the relation $\varrho(R)$ if and only if (α, β) satisfies the formula $\varphi(\varrho)$ in $\mathfrak{U}_R$.

Suppose that the second possibility of the base case holds. In this case, $\varphi(\varrho)$ is defined to be the formula $\boldsymbol{Ixy}$. The symbol $\boldsymbol{I}$ is interpreted in $\mathfrak{U}_R$ as the binary relation of equality, so a pair (α, β) satisfies the formula $\varphi(\varrho)$ in $\mathfrak{U}_R$ if and only if $\alpha = \beta$, by the definition of satisfaction in $\mathfrak{U}_R$. On the other hand, the value of $\varrho(R)$ in $\mathfrak{Re}(U)$ is the identity relation id_U, by the assumption that ϱ is the term 1' and by the definition of the value of a term on a sequence of elements in $\mathfrak{Re}(U)$, so the pair (α, β) is in the relation $\varrho(R)$ if and only if $\alpha = \beta$. Combine these remarks just as before to arrive at the desired conclusion.

Turn now to the induction step of the proof. Assume that the induction hypothesis holds for terms σ and τ in $\mathcal{L}$, and write

$$\varphi(\sigma) = \Gamma \qquad \text{and} \qquad \varphi(\tau) = \Delta, \tag{1}$$

and

$$\sigma(R) = S \qquad \text{and} \qquad \tau(R) = T. \tag{2}$$

The induction hypothesis assumes that a pair (α, β) belongs to the relation S if and only if it satisfies the formula Γ in $\mathfrak{U}_R$, and (α, β) belongs to the relation T if and only if it satisfies the formula Δ in $\mathfrak{U}_R$.

To treat the most complicated case of the induction step, suppose that ϱ is the term $\sigma ; \tau$. In this case, $\varphi(\varrho)$ is the formula

$$\exists \boldsymbol{z}(\Gamma(\boldsymbol{x}, \boldsymbol{z}) \wedge \Delta(\boldsymbol{z}, \boldsymbol{y})), \tag{3}$$

by (1) and the definition of φ. A pair (α, β) satisfies (3) in $\mathfrak{U}_R$ if and only if there is an element γ in U such that (α, γ) satisfies $\Gamma(\boldsymbol{x}, \boldsymbol{z})$ and (γ, β) satisfies $\Delta(\boldsymbol{z}, \boldsymbol{y})$ in $\mathfrak{U}_R$, by the definition of satisfaction in the structure $\mathfrak{U}_R$. In view of the induction hypothesis, this is equivalent to saying that there exists an element γ in U such that (α, γ) is in S, and (γ, β) in T. This last statement is, in turn, equivalent to the assertion that (α, β) belongs to the composition $S \mid T$, by the definition of relational composition. On the other hand, the value of $\varrho(R)$ in the algebra $\mathfrak{Re}(U)$ is just the composite relation $S \mid T$, by (2), the assumption that ϱ is the term $\sigma ; \tau$, and the definition of the value of a term on a sequence of elements in $\mathfrak{Re}(U)$. Consequently, a pair (α, β) belongs to the relation $\varrho(R)$ if and only if it belongs to the composition $S \mid T$. Combine these observations to conclude that a pair (α, β) satisfies the formula $\varphi(\varrho)$ in $\mathfrak{U}_R$ if and only if it belongs to the relation $\varrho(R)$ in $\mathfrak{Re}(U)$.

The arguments when ϱ is one of the terms $\sigma + \tau$, $-\sigma$, or $\sigma^\smile$ are similar, but easier, and are left as an exercise. The conclusion of the lemma follows by the principle of induction for terms in $\mathcal{L}$. □

The next theorem characterizes in terms of the translation mapping φ when an equation is satisfied, and when it is true, in a full set relation algebra. The characterizations open up the possibility of using model-theoretic methods from first-order logic to treat questions regarding satisfiability and validity in set relation algebras. The proof of the theorem is based on Lemma 18.63.

Theorem 18.64. *Let* $\sigma(v_0, \ldots, v_{n-1})$ *and* $\tau(v_0, \ldots, v_{n-1})$ *be terms in* $\mathcal{L}$.

(i) *The equation $\sigma = \tau$ is satisfied in a full set relation algebra $\mathfrak{Re}(U)$ by a sequence R of n relations on U if and only if the sentence*

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\varphi(\sigma) \leftrightarrow \varphi(\tau))$$

is true in the structure $\mathfrak{U}_R$.

(ii) *The equation $\sigma = \tau$ is true in a full set relation algebra $\mathfrak{Re}(U)$ if and only if, for every sequence R of n relations on U, the sentence*

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\varphi(\sigma) \leftrightarrow \varphi(\tau))$$

is true in $\mathfrak{U}_R$.

Proof. To prove (i), consider a sequence R of n relations on a set U. Each of the following statements is easily seen to be equivalent to its neighbor. (1) $\sigma(R) = \tau(R)$ in $\mathfrak{Re}(U)$. (2) A pair (α, β) of elements (from U) belongs to the relation $\sigma(R)$ if and only if it belongs to the relation $\tau(R)$. (3) A pair (α, β) of elements satisfies the formula $\varphi(\sigma)$ if and only if it satisfies the formula $\varphi(\tau)$ in the structure $\mathfrak{U}_R$. (4) The sentence

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\varphi(\sigma) \leftrightarrow \varphi(\tau))$$

is valid in $\mathfrak{U}_R$. Statements (1) and (2) are equivalent, by the definition of equality between relations, while (3) and (4) are equivalent by the definition of truth in $\mathfrak{U}_R$. The equivalence of (2) and (3) is just the content of Lemma 18.63. It follows that (1) and (4) are equivalent, which proves (i).

To prove (ii), observe that the equation $\sigma = \tau$ is true in the algebra $\mathfrak{Re}(U)$ just in case statement (1) above holds for every sequence R of n relations on U, by the definition of truth in $\mathfrak{Re}(U)$. From part (i) of the theorem, it follows that statement (1) holds for every such sequence R if and only if statement (4) holds for every such sequence R, and this is just what (ii) asserts. □

With the help of the preceding theorem, and the downward and upward Löwenheim-Skolem-Tarski Theorems for relational structures (the analogues of Theorems 6.26 and 6.28 for relational structures), we can now prove the result described at the beginning of the section.

Theorem 18.65. *Full set relation algebras over infinite sets all have the same equational theory, and therefore generate the same universal class of simple relation algebras and the same variety of relation algebras.*

Proof. Let U be an arbitrary infinite set. The first step is to show that an equation is satisfiable in $\mathfrak{Re}(U)$ if and only if it is satisfiable in a full set relation algebra $\mathfrak{Re}(V)$ over some countably infinite set V. To this end, consider an equation

$$\sigma = \tau, \tag{1}$$

say with variables among $v_0, \ldots, v_{n-1}$.

Assume first that equation (1) is satisfied in $\mathfrak{Re}(U)$ by a sequence of relations $R = (R_0, \ldots, R_{n-1})$. In this case, the translated sentence

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\varphi(\sigma) \leftrightarrow \varphi(\tau)) \tag{2}$$

is true in the structure

$$\mathfrak{U} = (U\,,\, R_0\,,\, \ldots\,,\, R_{n-1}),$$

by Theorem 18.64(i), with $\mathfrak{U}$ in place of $\mathfrak{U}_R$. The structure $\mathfrak{U}$ has a countable elementary substructure, say

$$\mathfrak{V} = (V\,,\, S_0\,,\, \ldots\,,\, S_{n-1}), \tag{3}$$

by the downward Löwenheim-Skolem-Tarski Theorem, so the sentence in (2) is true in $\mathfrak{V}$. Apply Theorem 18.64(i) again, with the sequence $S = (S_0, \ldots, S_{n-1})$ and $\mathfrak{V}$ in place of R and $\mathfrak{U}_R$, to conclude that S must satisfy equation (1) in the full set relation algebra $\mathfrak{Re}(V)$.

To establish the reverse implication, assume that equation (1) is satisfied by a sequence S in a full set relation algebra $\mathfrak{Re}(V)$ over some countably infinite set V. In this case, the sentence (2) is true in the corresponding relational structure $\mathfrak{V}$ that is defined in (3), by Theorem 18.64(i) applied to S and $\mathfrak{V}$. The structure $\mathfrak{V}$ has an elementary extension

$$\mathfrak{W} = (W\,,\, T_0\,,\, \ldots\,,\, T_{n-1})$$

with a universe of the same cardinality as U, by the upward Löwenheim-Skolem-Tarski Theorem. The sentence (2) is therefore true in $\mathfrak{W}$. Apply Theorem 18.64(i) again, with $T = (T_0, \ldots, T_{n-1})$ and $\mathfrak{W}$ in place of R and $\mathfrak{U}_R$, to conclude that the sequence T satisfies equation (1) in the full set relation algebra $\mathfrak{Re}(W)$. Since the sets U and W have the same cardinality, the algebras $\mathfrak{Re}(U)$ and $\mathfrak{Re}(W)$ are base isomorphic, by the remarks in Section 7.5. Consequently, equation (1) must also be satisfiable in $\mathfrak{Re}(U)$, as claimed. This completes the first step of the proof.

The set U considered above was taken to be arbitrary, subject only to the restriction that it be infinite. The preceding argument therefore implies that equation(1) is satisfiable in $\mathfrak{Re}(U)$ for some infinite set U if and only if it is satisfiable in $\mathfrak{Re}(U)$ for every infinite set U. Indeed, if U and V are arbitrary infinite sets, then the statement that (1) is satisfiable in $\mathfrak{Re}(U)$, and the statement that (1) is satisfiable in $\mathfrak{Re}(V)$ are each equivalent to the statement that (1) is satisfiable in a full set relation algebra over some countably infinite set. Since two full set relation algebras over countably infinite sets are always basic isomorphic, by the remarks in Section 7.5, the equivalence of all three statements follows at once.

We now prove the first assertion of the theorem. Suppose an equation (1) is true in $\mathfrak{Re}(U)$ for some infinite set U. The inequality

$$\sigma \neq \tau \tag{4}$$

is then not satisfiable in $\mathfrak{Re}(U)$. There is an equation ε with the same variables as (4) that is equivalent to (4) in all simple relation algebras, by Theorem 9.5. It follows that ε is not satisfiable in $\mathfrak{Re}(U)$. But then ε is not satisfiable in the full relation algebra over any infinite set, by the observations of the preceding paragraph. Consequently, (4) is not satisfiable in the full relation algebra over any infinite set, by the equivalence of ε and (4) in all simple relation algebras (and the fact that full set relation algebras over non-empty sets are always simple). Thus, (1) must hold in every full set relation algebra over an infinite set. Conclusion: an equation holds in the full set relation algebra over some infinite set if and only if it holds in the full set relation algebra over every infinite set, so all full set relation algebras over infinite sets have the same equational theory.

The lattice of equational theories is dually isomorphic to the lattice of varieties of relation algebras via the function that maps each equational theory to the class of models of the theory, by the remarks in Section 18.1. Consequently, all full set relation algebras over infinite sets generate the same variety, by the conclusions of the preceding paragraph. The lattice of universal classes of simple relation algebras is also isomorphic to the lattice of varieties via the function that maps each such universal class to the variety generated by the class, by Theorem 18.28. Consequently, all full set relation algebras over infinite sets generate the same universal class. □

In view of Theorem 18.65, the universal class generated by a full relation algebra on some infinite set coincides with the universal class

generated by the class of all full relation algebras on infinite sets. Denote this universal class by R_ω, where ω is the first infinite ordinal. The next lemma gives algebraic characterizations of R_ω and the variety it generates.

Lemma 18.66. *The universal class* R_ω *coincides with* $\mathsf{SP}_\mathsf{u}(\mathfrak{Re}(\omega))$, *and the variety generated by* R_ω *coincides with* $\mathsf{SPP}_\mathsf{u}(\mathfrak{Re}(\omega))$.

Proof. The universal class generated by the full set relation algebras on infinite sets coincides with the universal class generated by the full set relation algebra $\mathfrak{Re}(\omega)$, by Theorem 18.65. The former class is R_ω, by definition, and the latter class is $\mathsf{SP}_\mathsf{u}(\mathfrak{Re}(\omega))$, by Corollary 18.12, so

$$\mathsf{R}_\omega = \mathsf{SP}_\mathsf{u}(\mathfrak{Re}(\omega)). \tag{1}$$

Lemmas 18.2(ii) and 18.1(i) imply that

$$\mathsf{SPSP}_\mathsf{u}(\mathfrak{Re}(\omega)) \subseteq \mathsf{SSPP}_\mathsf{u}(\mathfrak{Re}(\omega)) = \mathsf{SPP}_\mathsf{u}(\mathfrak{Re}(\omega)).$$

The reverse inclusion holds trivially, by Lemma 18.1(i) (with $\mathsf{P}_\mathsf{u}(\mathfrak{Re}(\omega))$ in place of K), so

$$\mathsf{SPSP}_\mathsf{u}(\mathfrak{Re}(\omega)) = \mathsf{SPP}_\mathsf{u}(\mathfrak{Re}(\omega)). \tag{2}$$

Use (1) and (2) to arrive at

$$\mathsf{SP}(\mathsf{R}_\omega) = \mathsf{SPSP}_\mathsf{u}(\mathfrak{Re}(\omega)) = \mathsf{SPP}_\mathsf{u}(\mathfrak{Re}(\omega)). \tag{3}$$

The variety generated by R_ω is $\mathsf{SP}(\mathsf{R}_\omega)$, by Theorem 18.25, and this variety coincides with the class $\mathsf{SPP}_\mathsf{u}(\mathfrak{Re}(\omega))$, by (3). □

Here is a slightly different perspective on the class R_ω. Consider the class L_ω of full set relation algebras on infinite cardinals,

$$\mathsf{L}_\omega = \{\mathfrak{Re}(\kappa) : \kappa \geq \omega\}.$$

Every full set relation algebra on an infinite set is base isomorphic to an algebra in L_ω, by the remarks in Section 7.5. Corollary 18.12 therefore implies that

$$\mathsf{R}_\omega = \mathsf{SP}_\mathsf{u}(\mathsf{L}_\omega).$$

Every ultraproduct of algebras in L_ω is embeddable into an algebra in L_ω (see Exercise 18.29), so

$$\mathsf{S}(\mathsf{L}_\omega) = \mathsf{SP}_\mathsf{u}(\mathsf{L}_\omega).$$

Combine these two observations to arrive at the following conclusion.

Lemma 18.67. $\mathsf{R}_\omega = \mathsf{S}(\mathsf{L}_\omega)$.

Roughly speaking, this lemma says that R_ω is the class of simple relation algebras that are square representable over some infinite set. (Recall, in this connection, that a simple relation algebra is representable if and only if it is embeddable into $\mathfrak{Re}(U)$ for some non-empty set U, by Lemma 16.1 or Corollary 18.31.)

Let R be the class of all simple, representable relation algebras, that is to say, the class of all simple algebras in RRA. Since RRA is a variety, by Theorem 18.30, the class R must be universal, by Theorem 18.25. It is natural to ask what relationships exist between R and the various classes R_n for $1 \leq n \leq \omega$.

Obviously, each of the classes R_n is included in the class R, so the join, in the lattice of universal classes, of the classes R_n for $1 \leq n \leq \omega$ is included in R. On the other hand, every algebra in R is embeddable into $\mathfrak{Re}(U)$ for some non-empty set U, by Corollary 18.31. If U has finite cardinality n, then the algebra belongs to R_n, by definition, and if U has infinite cardinality, then the algebra belongs to R_ω, by Lemma 18.67. At any rate, every algebra in R belongs to one of the classes R_n, so R is included in the join of these classes. Conclusion: R is equal to the join of the classes R_n for $1 \leq n \leq \omega$.

None of the classes R_n can be omitted in this join. More precisely, if m is either a positive integer or ω, and if K is the join of those classes R_n such that $1 \leq n \leq \omega$ and $n \neq m$, then R_m is not included in K, and therefore K is strictly below R in the lattice of universal classes. In order to prove this assertion, it suffices to show that the equational theory of K is not included in that of R_m, by the dual isomorphism between the lattice of equational theories of relation algebras and the lattice of universal classes of simple relation algebras. If m is a positive integer, then $\mathfrak{Re}(m)$ is not isomorphic to any of the algebras $\mathfrak{Re}(n)$ with $n \neq m$, since the cardinalities of these algebras are not the same. Consequently, there is an equation—for example, the equation δ_m from the proof of Lemma 18.58—that is true in $\mathfrak{Re}(n)$ for every $n \neq m$, and therefore true in K, but that is not true in $\mathfrak{Re}(m)$ and therefore not true in R_m. Alternatively, $\mathfrak{Re}(m)$ is not embeddable into $\mathfrak{Re}(n)$ for any $n \neq m$, by Corollary 18.59, so the conjugate equation of $\mathfrak{Re}(m)$ is true in $\mathfrak{Re}(n)$ for $n \neq m$, and therefore true in K, but it is not true in $\mathfrak{Re}(m)$ and therefore not true in R_m. Conclusion: the equational theory of K is not included in that of R_m when m is finite.

Consider now the case when $m = \omega$. The implication expressing that a one-to-one function with domain the entire universe is necessarily onto, that is to say, the implication

$$(f^{\smile} ; f \leq 1') \wedge (f ; f^{\smile} = 1') \to (f^{\smile} ; f = 1'),$$

is true in every full relation algebra over a finite set (see Corollary 5.82 and its first dual). Indeed, any one-to-one mapping of a finite set into itself is necessarily a permutation of the set. As is well known, however, this implication fails to be true of one-to-one mappings on infinite sets. Consequently, in every full relation algebra over an infinite set the above implication fails. There is an equation that is equivalent to this implication in all simple relation algebras, by Theorem 9.5, and this equation is true in $\mathfrak{Re}(n)$—and therefore in R_n—for every positive integer n, but it fails in $\mathfrak{Re}(\omega)$ and hence also in R_ω. Conclusion: the equational theory of K is not included in that of R_m when $m = \omega$.

A join of a set X of elements in a complete lattice is said to be *irredundant*, or *irreducible*, if for every element x in X, the join of the set $X \sim \{x\}$ in the lattice is strictly below the join of X. Using this terminology, the observations of the preceding paragraphs may be summarized as follows.

Theorem 18.68. *The universal class* R *of simple, representable relation algebras is the irredundant join of the universal classes* R_n *for* $1 \leq n \leq \omega$.

This theorem, and the isomorphism between the lattice of universal class of simple relation algebras and the lattice of varieties of relation algebras given in Correspondence Theorem 18.28, together yield the following conclusion.

Corollary 18.69. *The variety* RRA *of representable relation algebras is the irredundant join of the varieties* $\mathsf{SP}(\mathsf{R}_n)$ *for* $1 \leq n \leq \omega$.

We now consider some specific intervals in the lattice of universal classes of simple, representable relation algebras that exhibit a high degree of lattice-theoretic complexity. Let R_f be join of the universal classes R_n over all natural numbers $n \geq 3$. Thus,

$$\mathsf{R}_\mathsf{f} = \mathsf{SP}_\mathsf{u}(\mathsf{L}_\mathsf{f}), \quad \text{where} \quad \mathsf{L}_\mathsf{f} = \{\mathfrak{Re}(n) : n \text{ is an integer and } n \geq 3. \}$$

Another way to describe R_f is to say that it is the universal class generated by the class of simple relation algebras that admit a (square) representation over a finite set of cardinality at least three.

In the next theorem, recall that $\mathbb{N}$ and $\mathbb{Q}$ are the sets of natural numbers and rational numbers respectively.

Theorem 18.70. *If* K *is a universal class of simple relation algebras that is disjoint from* L_f*, then the interval* $[\mathsf{K}, \mathsf{K} \cup \mathsf{R}_\mathsf{f}]$ *in the lattice of universal classes of simple relation algebras has the following properties.*

(i) *The lattice* $(Sb(\mathbb{N}), \cap, \cup)$ *is completely embeddable into the interval.*

(ii) *There is a system* $(\mathsf{N}_X : X \subseteq \mathbb{Q})$ *of universal classes in the interval such that* $X \subseteq Y$ *if and only if* $\mathsf{N}_X \subseteq \mathsf{N}_Y$*. Moreover, if* $X \subsetneq Y$*, then there are continuum many universal classes between* N_X *and* N_Y*.*

(iii) *The interval has a subclass that, under the relation of inclusion, is linearly ordered of the same order type as the real numbers.*

Proof. The class L_f is an antichain over K, by Corollary 18.59 and the assumption that K is disjoint from L_f. Apply Theorems 18.43 and 18.45 (with L_f in place of L) to arrive at the desired conclusions. □

One class to which the preceding theorem can be applied is the minimal universal class M_3, which is disjoint from L_f. Notice that M_3 is included in R_f, because the minimal algebra $\mathfrak{M}_3$ is embeddable into the algebra $\mathfrak{Re}(n)$ for every integer $n \geq 3$. The union $\mathsf{M}_3 \cup \mathsf{R}_\mathsf{f}$ therefore coincides with the class R_f.

Theorem 18.71. *The conclusions of Theorem 18.70 apply to the interval* $[\mathsf{M}_3, \mathsf{R}_\mathsf{f}]$*. In particular, the interval has cardinality* $2^{\aleph_0}$ *and contains chains of the same order type as the real numbers.*

Another class to which the theorem can be applied is that of all simple relation algebras admitting square representations over infinite sets. This is just the class R_ω, by Lemma 18.67. It is axiomatized by the union of a set of axioms for R and the set of conjugate equations of all simple relation algebras that have a square representation over some finite set, but not over any infinite set. The class R_ω is disjoint from L_f, because no algebra in L_f is embeddable into an algebra in R_ω, by Corollary 18.59. The hypotheses of Theorem 18.70 are therefore satisfied with R_ω in place of K. Notice that the interval $[\mathsf{R}_\omega, \mathsf{R}_\omega \cup \mathsf{R}_\mathsf{f})]$ coincides with the interval $[\mathsf{R}_\omega, \mathsf{R}]$, because the join of the classes R_ω and R_f is R, by Theorem 18.68 and Lemma 18.19.

Theorem 18.72. *The conclusions of Theorem 18.70 apply to the interval* $[\mathsf{R}_\omega, \mathsf{R}]$. *In particular, the interval has cardinality* $2^{\aleph_0}$ *and contains chains of the same order type as the real numbers.*

The proof of Theorem 18.70 depends heavily on the property of the full set relation algebras on finite sets that they cannot be properly embedded into any simple relation algebra (see Corollary 18.59). In particular, they are not representable over infinite sets. None of the universal classes in the interval $[\mathsf{M}_3, \mathsf{R}_\omega]$ contains any of these algebras, and it is natural to ask whether this interval also has a high degree of lattice-theoretic complexity. We shall show in the next theorem that this is indeed the case, and in fact a stronger result is true.

Let $\mathsf{R}_{\mathsf{hir}}$ be the class of all algebras $\mathfrak{A}$ in R such that no subalgebra of $\mathfrak{A}$ different from the minimal subalgebra has a square representation over a finite set. We call $\mathsf{R}_{\mathsf{hir}}$ the class of *hereditarily strictly infinitely representable* simple relation algebras, because with one exception (namely the minimal subalgebra) every subalgebra of an algebra in $\mathsf{R}_{\mathsf{hir}}$ has only infinite square representations. (The exception of the minimal subalgebra is necessary because the minimal subalgebra of every algebra in $\mathsf{R}_{\mathsf{hir}}$ is isomorphic to $\mathfrak{M}_3$, and $\mathfrak{M}_3$ has square representations over every set—finite and infinite—of cardinality at least three.) The class $\mathsf{R}_{\mathsf{hir}}$ is universal, and in fact it is axiomatized by the union of a set of axioms for R and the set of conjugate equations of all simple relation algebras not isomorphic to $\mathfrak{M}_3$ that have square representations over finite sets. Every algebra in $\mathsf{R}_{\mathsf{hir}}$ is, by definition, representable over an infinite set, so $\mathsf{R}_{\mathsf{hir}}$ is included in R_ω, and therefore the interval $[\mathsf{R}_{\mathsf{hir}}, \mathsf{R}_{\mathsf{hir}} \cup \mathsf{R}_\omega]$ coincides with the interval $[\mathsf{R}_{\mathsf{hir}}, \mathsf{R}_\omega]$.

Theorem 18.73. *The conclusions of Theorem 18.70 apply to the interval* $[\mathsf{R}_{\mathsf{hir}}, \mathsf{R}_\omega]$. *In particular, the interval has cardinality* $2^{\aleph_0}$ *and contains chains of the same order type as the real numbers.*

Proof. It suffices to construct an infinite antichain over $\mathsf{R}_{\mathsf{hir}}$ of finite algebras in R_ω, that is to say, an infinite antichain of finite simple relation algebras that are embeddable into a full set relation algebra over some infinite set (so they belong to R_ω), but that are also embeddable into a full set relation algebra over some finite set (so they do not belong to $\mathsf{R}_{\mathsf{hir}}$). The desired conclusions are then immediate consequences of Theorems 18.43 and 18.45.

Take as the index set of the antichain the set I of prime numbers. For each prime n in I, let $\mathbb{Z}_n$ be the cyclic group of order n,

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\},$$

under the operation of addition modulo n. Fix a finite set Y of cardinality at least three that is disjoint from the set of natural numbers. The exactly cardinality of Y is immaterial, and in fact Y could in principle even be an infinite set, as the subsequent discussion will make clear, for this is the *expandable* part of the construction—the finite part that can be made infinite. Put

$$U_n = \mathbb{Z}_n \cup Y. \tag{1}$$

For each n in I, we are going to construct a subalgebra $\mathfrak{A}_n$ of $\mathfrak{Re}(U_n)$ by using the Atomic Subalgebra Theorem 6.21.

Let W_0 and W_1 be the sets consisting of the two relations

$$\mathbb{Z}_n \times Y, \qquad Y \times \mathbb{Z}_n, \qquad \text{and} \qquad id_Y, \qquad di_Y \tag{2}$$

respectively, and let W_2 be the set of Cayley representations

$$R_k = \{(i, i+k) : i \in \mathbb{Z}_n\} \tag{3}$$

of the elements k in $\mathbb{Z}_n$. Take W to be the union of these three sets,

$$W = W_0 \cup W_1 \cup W_2 \tag{4}$$

(see Figure 18.3). The three sets that make up W—the sets on the right side of (4)—are called the *components* of W. The goal is to show that W satisfies conditions (i)–(iv) of Theorem 6.21 with respect to the algebra $\mathfrak{Re}(U_n)$.

A few preliminary observations are in order. First, the unit $U_n \times U_n$ of $\mathfrak{Re}(U_n)$ is the disjoint union of the relations

$$\mathbb{Z}_n \times \mathbb{Z}_n, \qquad Y \times Y, \qquad \mathbb{Z}_n \times Y, \qquad Y \times \mathbb{Z}_n, \tag{5}$$

by (1) and the assumed disjointness of the sets $\mathbb{Z}_n$ and Y. Second, the set W_2 consists of the atoms of a complete and atomic subalgebra of $\mathfrak{Re}(\mathbb{Z}_n)$, namely the Cayley representation of the group complex algebra $\mathfrak{Cm}(\mathbb{Z}_n)$, so W_2 satisfies conditions (i)–(iv) of Theorem 6.21 with respect to the algebra $\mathfrak{Re}(\mathbb{Z}_n)$. In particular, the relations in W_2 partition $\mathbb{Z}_n \times \mathbb{Z}_n$, the identity relation $id_{\mathbb{Z}_n}$ belongs to W_2, and W_2 is closed under the operations of relational converse and composition. Third, the set W_1 consists of the atoms of a complete and atomic subalgebra of $\mathfrak{Re}(Y)$, namely the minimal subalgebra of $\mathfrak{Re}(Y)$, so W_1

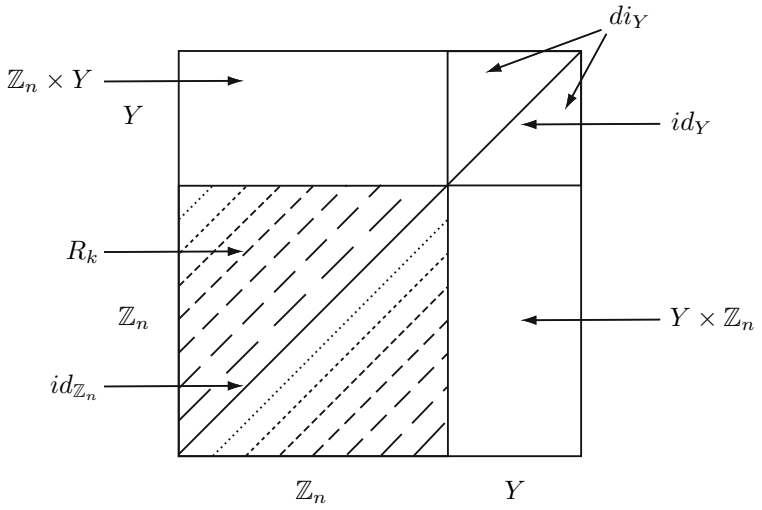


Fig. 18.3 The relations in the set W of atoms in the algebra $\mathfrak{A}_n$.

satisfies conditions (i)–(iv) of Theorem 6.21 with respect to $\mathfrak{Re}(Y)$. In particular, the relations in W_1 partition $Y \times Y$, they are symmetric, the identity relation id_Y is in W_1, and the composition of any two relations in W_1 is a union of relations in W_1. Fourth, the two relations in W_0 are the converses of one another.

The verification of conditions (i)–(iii) for W is now easy. For condition (i), use the definition of W in (4), the first observation that the unit $U_n \times U_n$ is the disjoint union of the relations in (5), the second observation that the first relation in (5) is the disjoint union of the relations in W_2, the third observation that the second relation in (5) is the disjoint union of the relations in W_1, and the definition of W_0 as the set consisting of the last two relations in (5), to conclude that the relations in W partition the unit of $\mathfrak{Re}(U_n)$. Thus, condition (i) holds. Condition (ii) holds because the identity relation id_{U_n} is the union of the identity relations $id_{\mathbb{Z}_n}$ and id_Y, by (1), and these last two identity relations are in W, by (4) and the second and third observations. Condition (iii) holds because the components sets of W in (4) are all closed under converse, by the second, third, and fourth observations.

To verify condition (iv), a number of cases must be considered, according to the components of W to which the relations being composed belong. The composition of any two relations in W_2 is again a relation in W_2, by the second observation, and the composition of any two re-

lations in W_1 is a union of relations in W_1, by the third observation. As regards the composition of two relations in W_0, we have

$$(\mathbb{Z}_n \times Y) \mid (Y \times \mathbb{Z}_n) = \mathbb{Z}_n \times \mathbb{Z}_n, \qquad (Y \times \mathbb{Z}_n) \mid (\mathbb{Z}_n \times Y) = Y \times Y,$$
$$(\mathbb{Z}_n \times Y) \mid (\mathbb{Z}_n \times Y) = (Y \times \mathbb{Z}_n) \mid (Y \times \mathbb{Z}_n) = \varnothing,$$

by the definition of relational composition and the assumption that the sets $\mathbb{Z}_n$ and Y are disjoint. The composition of a relation in W_1 with a relation in W_0 is determined by

$$id_Y \mid (Y \times \mathbb{Z}_n) = di_Y \mid (Y \times \mathbb{Z}_n) = Y \times \mathbb{Z}_n,$$
$$id_Y \mid (\mathbb{Z}_n \times Y) = di_Y \mid (\mathbb{Z}_n \times Y) = \varnothing,$$

and analogously for the composition of a relation in W_0 with a relation in W_1. The composition of a relation in W_1 with a relation in W_2 is always empty because the sets $\mathbb{Z}_n$ and Y are disjoint. The composition of a relation in W_0 with a relation in W_2 is determined by

$$(Y \times \mathbb{Z}_n) \mid R_k = Y \times \mathbb{Z}_n \qquad \text{and} \qquad (\mathbb{Z}_n \times Y) \mid R_k = \varnothing,$$

and analogously for the composition of a relation in W_2 with a relation in W_0. Taken together, these calculations show that the composition of two relations in W is always a union of relations in W, so condition (iv) holds.

It has been shown that the set W satisfies conditions (i)–(iv) of Theorem 6.21 with respect to the algebra $\mathfrak{Re}(U_n)$. Apply the theorem in the form of Corollary 6.22 to conclude that the set of unions of all possible subsets of W is the universe of a complete and atomic subalgebra of $\mathfrak{Re}(U_n)$, and the atoms of this subalgebra are just the elements in W. Take $\mathfrak{A}_n$ to be this subalgebra.

The only role that the cardinality of the finite set Y plays in the construction of $\mathfrak{A}_n$ is in the computation of the composition $di_Y \mid di_Y$, and this composition has the same value $Y \times Y$ whenever Y has cardinality at least three. Consequently, if Y is replaced by an arbitrary infinite set X, then the result is a subalgebra $\mathfrak{B}_n$ of the full set relation algebra on the set $\mathbb{Z}_n \cup X$ that is isomorphic to $\mathfrak{A}_n$ via a function that maps each Cayley representation R_k to itself, and maps the relations

$$\mathbb{Z}_n \times X, \qquad X \times \mathbb{Z}_n, \qquad id_X, \qquad di_X$$

to the relations

$$\mathbb{Z}_n \times Y, \qquad Y \times \mathbb{Z}_n, \qquad id_Y, \qquad di_Y$$

respectively, by the Atomic Isomorphism Theorem in the form of Corollary 7.12. Thus, the algebra $\mathfrak{A}_n$ has square representations over finite sets (the set $\mathbb{Z}_n \cup Y$) and over infinite sets (the set $\mathbb{Z}_n \cup X$), so it belongs to the class R_ω, but not to the class $\mathsf{R}_{\mathsf{hir}}$.

It remains to show that the algebras $\mathfrak{A}_n$, for n in I, form an antichain. Any embedding of the algebra $\mathfrak{A}_m$ into the algebra $\mathfrak{A}_n$ (for primes m and n) has to map bijections to bijections and subidentity elements to subidentity elements. In particular, any such embedding has to embed the Cayley representation of the group $\mathbb{Z}_m$ into the Cayley representation of the group $\mathbb{Z}_n$. This can only happen if $m = n$, because the cyclic groups of prime order form an antichain of groups (with respect to the relation of embeddability). Conclusion: $(\mathfrak{A}_n : n \in I)$ is an antichain over $\mathsf{R}_{\mathsf{hir}}$ of finite algebras in R_ω, as desired. □

The algebra $\mathfrak{A}_n$ in the proof of the preceding theorem is what is sometimes called the *diagonal semiproduct* of (square representations of) the relation algebras $\mathfrak{Cm}(\mathbb{Z}_n)$ and $\mathfrak{M}_3$.

We examine two more intervals in the lattice of universal classes. To define them, let R_{hi} be the class of all algebras $\mathfrak{A}$ in R with the property that no subalgebra of $\mathfrak{A}$ different from the minimal subalgebra is finite. This class may be called the class of *hereditarily infinite* algebras in R because every non-minimal subalgebra of an algebra in R_{hi} is infinite. The class R_{hi} is universal, and in fact it is axiomatized by the union of a set of axioms for R and the set of conjugate equations of all finite algebras in R. Since every non-minimal subalgebra of an algebra in R_{hi} is infinite, every such subalgebra admits square representations only over infinite sets. Thus, R_{hi} is included in $\mathsf{R}_{\mathsf{hir}}$. It turns out that the lattice of subsets of the natural numbers is also completely embeddable into the lattice interval $[\mathsf{R}_{\mathsf{hi}}, \mathsf{R}_{\mathsf{hir}}]$.

Theorem 18.74. *The conclusions of Theorem 18.70 apply to the interval* $[\mathsf{R}_{\mathsf{hi}}, \mathsf{R}_{\mathsf{hir}}]$. *In particular, the interval has cardinality* $2^{\aleph_0}$ *and contains chains of the same order type as the real numbers.*

The proof of this assertion is similar to the proof of Theorem 18.73. One constructs an infinite antichain over R_{hi} of finite algebras in $\mathsf{R}_{\mathsf{hir}}$, and then applies Theorems 18.43 and 18.45. The construction of the antichain is, however, in this case much more complicated than in the proof of Theorem 18.73. In fact, it may at first seem surprising that there are any finite algebras in $\mathsf{R}_{\mathsf{hir}}$ at all. To see that such algebras

really do exist, recall the first example in Section 17.5, the set relation algebra generated by a dense linear ordering without endpoints. It is finite and simple, has no non-minimal subalgebras, and has square representations only over infinite sets, so it is finite, simple, and hereditarily strictly infinitely representable. The algebras of the antichain that must be constructed in order to prove Theorem 18.74 are similar in spirit to this example. Each of them is generated by a dense partial ordering that is rather fractal-like in its construction. See [5] for details.

The last interval to be considered is the interval $[\mathsf{M}_3, \mathsf{R}_{\mathsf{hi}}]$. It contains only universal classes of hereditarily infinite algebras, so no argument based on Theorem 18.43 can be used to embed the lattice of subsets of natural numbers into this interval. Nevertheless, it can still be shown that the interval has the cardinality of the continuum. The proof uses the finite-cofinite subalgebras of complex algebras of infinite groups. In what follows, let $(G\,,\ \circ\,,\ ^{-1}\,,\ e)$ be an infinite group.

Lemma 18.75. *The set of finite and cofinite subsets of an infinite group G is a subuniverse of the complex algebra $\mathfrak{Cm}(G)$.*

Proof. Let A be the set of finite and cofinite subsets of G. Clearly, A is closed under the Boolean operations of union and complement. Indeed, the complement of a finite subset of G is cofinite and vice versa, the union of two finite subsets of G is finite, and the union of a cofinite subset of G with either a finite subset or a cofinite subset is cofinite.

To see that A is closed under the Peircean operations of $\mathfrak{Cm}(G)$, consider first the case of relative multiplication. Let X and Y be elements in A. If X and Y are both finite, then the relative product

$$X\,;Y = \{f \circ g : f \in X \text{ and } g \in Y\} \tag{1}$$

is clearly finite. If one of these sets, say X is cofinite, and if Y is not empty, then the product in (1) is cofinite. In more detail, fix an element g in Y, and observe that the atom $\{g\}$ in $\mathfrak{Cm}(G)$ is a bijection, since

$$\{g\}\,;\{g\}^{\smile} = \{g \circ g^{-1}\} = \{e\} = \{g^{-1} \circ g\} = \{g\}^{\smile}\,;\{g\},$$

and since $\{e\}$ is the identity element in $\mathfrak{Cm}(G)$. The distributive law for relative multiplication over addition, and the right-hand distributive law for relative multiplication over multiplication that characterizes converses of functional elements (see Corollary 5.76) imply that

$$(X\,;\{g\}) \cup [(\sim X)\,;\{g\}] = (X \cup \sim X)\,;\{g\} \\ = G\,;\{g\} = \{f \circ g : f \in G\} = G$$

and

$$(X\,;\{g\}) \cap [(\sim X)\,;\{g\}] = (X \cap \sim X)\,;\{g\} = \varnothing\,;\{g\} = \varnothing.$$

Consequently, the set $(\sim X)\,;\{g\}$ is the complement of the set $X\,;\{g\}$ in $\mathfrak{Cm}(G)$. The set X is assumed to be cofinite, so its complement $\sim X$ must be finite, and therefore the set $(\sim X);\{g\}$ must be finite. It follows that the set $X\,;\{g\}$ is cofinite. The set in (1) includes the set $X\,;\{g\}$, so it, too, must be cofinite, as claimed.

A similar, but simpler argument shows that the converse of a finite subset of G is finite, and the converse of a cofinite subset of G is cofinite. Consequently, A is closed under converse. Also, A contains the identity element $\{e\}$, because this set is finite. Conclusion: A contains the identity element and is closed under the Boolean and Peircean operations of $\mathfrak{Cm}(G)$, so it is a subuniverse of $\mathfrak{Cm}(G)$. □

Write $\mathfrak{Cf}(G)$ for the subalgebra of $\mathfrak{Cm}(G)$ whose universe is the set of finite and cofinite subsets of G.

A group G is called *torsion-free* if G is not degenerate (that is to say, G has more than one element), and if for every element g in G that is different from e, the powers

$$g^0 = e, \qquad g^1 = g, \qquad g^2 = g \circ g, \qquad g^3 = g \circ g \circ g, \quad \ldots$$

are all distinct.

Lemma 18.76. *If a group G is torsion-free, then the relation algebra $\mathfrak{Cf}(G)$ is hereditarily infinite.*

Proof. It suffices to show that every element in $\mathfrak{Cf}(G)$ that does not belong to the minimal subalgebra generates an infinite subalgebra of $\mathfrak{Cf}(G)$. Such an element is, by definition, either a finite or a cofinite subset X of G. By passing to complements if necessary, it may be assumed that X is a finite subset of G. Since X is assumed not to be in the minimal subalgebra, it contains an element g that is different from the identity element e.

Distinct powers g^n of the element g (formed in the group G) are different from one another, by the assumption that G is torsion-free. There are infinitely many such powers, and the set X is assumed to be

finite, so there must be a largest natural number $n \geq 1$ such that g^n belongs to X. Put $f = g^n$.

For every natural number $m > 1$, the element f^m (the mth power of f, formed in the group G) belongs to the set X^m (the mth relative power of X, formed in the complex algebra $\mathfrak{Cm}(G)$), because f is in X; but f^m does not belong to the set X, because $f^m = g^{mn}$, and g^{mn} is not in X, by the choice of the natural number n. Consequently, the sets X^m for $m > 1$ are all distinct from X. Apply this same argument to the set X^2 in place of X to conclude that the sets X^{2m} for $m > 1$ are all distinct from X^2, and of course these sets are all distinct from X as well, by the preceding observations. Repeated application of this argument leads to the conclusion that the sets in the infinite sequence

$$X, \quad X^2, \quad X^4, \quad X^8, \quad \dots \quad , \quad X^{2^m}, \quad \dots$$

are all distinct from one another, and they are obviously all generated by X. The subalgebra of $\mathfrak{Cf}(G)$ generated by X is therefore infinite. □

A group complex algebra $\mathfrak{Cm}(G)$ is simple and representable, so every subalgebra of $\mathfrak{Cm}(G)$ is simple and representable. Combine this observation with the preceding lemma to conclude that if G is a torsion-free group, then $\mathfrak{Cf}(G)$ is a simple, representable, hereditarily infinite relation algebra, and consequently it belongs to the universal class R_{hi}. The universal class generated by $\mathfrak{Cf}(G)$ is the class

$$\mathsf{K}_G = \mathsf{SP}_{\mathsf{u}}(\mathfrak{Cf}(G)),$$

by Corollary 18.12. This class is included in the universal class R_{hi}, because $\mathfrak{Cf}(G)$ belongs to R_{hi}, and K_G is the smallest universal class to which $\mathfrak{Cf}(G)$ belongs. It follows that K_G is a universal class of hereditarily infinite algebras in R. Notice that the algebra $\mathfrak{Cf}(G)$ and the universal class K_G have the same equational theory, by Corollary 18.24.

There is an interpretation of the equational theory of the group G into the equational theory of the relation algebra $\mathfrak{Cf}(G)$ that is independent of the group under consideration. To see this, consider an arbitrary equation ε in the language of group theory, say with variables among $v_0, v_1, \dots, v_{n-1}$, and let $\hat{\varepsilon}$ be the equation in the language of relation algebras that is obtained from ε by replacing all occurrences of the group symbols $\circ$, $^{-1}$, and e with the corresponding relation algebraic symbols ;, $^{\smile}$, and 1' respectively. Take Δ_ε to be the formula

$$[(v_0^\smile ; v_0 = 1') \wedge (v_1^\smile ; v_1 = 1') \wedge \cdots \wedge (v_{n-1}^\smile ; v_{n-1} = 1')] \to \hat{\varepsilon}$$

in the language of relation algebras.

Lemma 18.77. *An equation ε is true in a group G if and only if the formula Δ_ε is true in the relation algebra $\mathfrak{Cf}(G)$.*

Proof. Assume first that ε is true in G, and let

$$r = (r_0, r_1, \ldots, r_{n-1})$$

be a sequence of elements in $\mathfrak{Cf}(G)$ that satisfies the hypothesis of Δ_ε. In this case, the elements r_i must be non-zero functions in $\mathfrak{Cf}(G)$, by the form of the inequalities in the hypothesis of Δ_ε. The only non-zero functions in $\mathfrak{Cf}(G)$ are the singletons of elements in G, so there must exist a sequence

$$g = (g_0, g_1, \ldots, g_{n-1})$$

of elements in G such that $r_i = \{g_i\}$ for $i = 0, 1, \ldots, n-1$. The sequence g satisfies the equation ε in G by assumption, so the sequence r satisfies the equation $\hat{\varepsilon}$ in $\mathfrak{Cf}(G)$, by the definitions of $\hat{\varepsilon}$ and the operations of $\mathfrak{Cf}(G)$. Thus, every sequence of n elements in $\mathfrak{Cf}(G)$ satisfies the formula Δ_ε, so Δ_ε is true in $\mathfrak{Cf}(G)$. Run the argument backwards to see that if Δ_ε is true in $\mathfrak{Cf}(G)$, then ε is true in G. □

The formula Δ_ε is quantifier-free. Apply Theorem 9.5 to obtain an equation δ_ε in the language of relation algebras (with the same variables as ε and Δ_ε) such that Δ_ε and δ_ε are equivalent in all simple relation algebras. Combine this observation with Lemma 18.77 to arrive at the following conclusion.

Corollary 18.78. *An equation ε is true in a group G if and only if the equation δ_ε is true in the relation algebra $\mathfrak{Cf}(G)$.*

If G and G' are torsion-free groups with distinct equational theories, then the algebras $\mathfrak{Cf}(G)$ and $\mathfrak{Cf}(G')$ also have distinct equational theories, by Corollary 18.78, and therefore so do the generated universal classes K_G and $\mathsf{K}_{G'}$. In more detail, if ε is a group equation that is true in, say, G, but not in G', then δ_ε is a relation algebraic equation that is true in $\mathfrak{Cf}(G)$ and hence also in K_G, but not in $\mathfrak{Cf}(G')$ and hence not in $\mathsf{K}_{G'}$. The two classes K_G and $\mathsf{K}_{G'}$ are thus distinct from one another. There are $2^{\aleph_0}$ torsion-free groups G with pairwise distinct equational

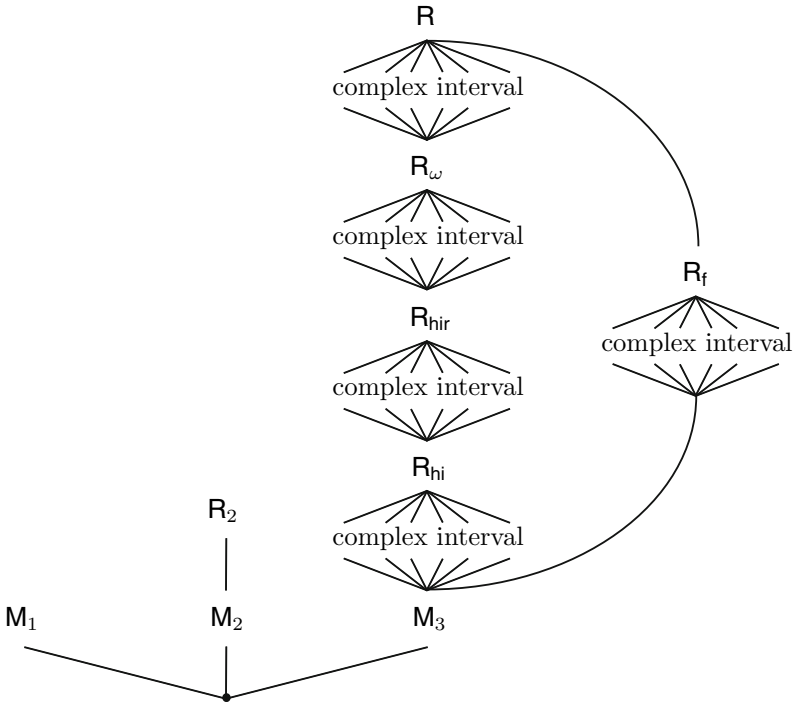


Fig. 18.4 Schematic overview of parts of the lattice of universal classes of simple relation algebras.

theories (see [5] for the proof), so there are $2^{\aleph_0}$ mutually distinct universal classes of the form K_G that are included in R_{hi}. This proves the surprising theorem that there are continuum many universal classes of hereditarily infinite, simple, representable relation algebras.

Theorem 18.79. *The lattice interval* $[\mathsf{M}_3, \mathsf{R}_{\mathsf{hi}}]$ *has cardinality* $2^{\aleph_0}$.

Figure 18.4 gives a schematic summary of Theorems 18.71–18.74 and 18.79.

18.10 Historical remarks

The study of the lattices of varieties and of equational theories grew out of the work of Birkhoff, in particular, [16] and [15]. The study of the lattices of elementary classes and elementary theories was inspired by the

work of Tarski, in particular [135] and [141]– [143]. Birkhoff [16] considered classes of algebras closed under subalgebras, homomorphisms, and direct products. The class operations S, H, and P were first introduced in Tarski [141]– [143], and versions of Lemmas 18.1–18.3 may be found there; see also Henkin-Monk-Tarski [50], Theorems 0.1.14, 0.2.13, 0.2.19, 0.3.12, and 0.3.13. The main idea behind the equivalences in Lemma 18.3 dates back to Tarski [134].

The ultraproduct construction was first introduced in a general form by Jerzy Łoś [84]; an earlier version appeared in Polish in Łoś [82]. Very special cases of the construction appeared still earlier in the papers Skolem [124]‘ and Hewitt [51]. The forms of the construction given by Łoś are quite different in appearance from the form commonly used today. The current form was developed by Tarski and his students Thomas Frayne, Anne Morel, and Dana Scott in the late 1950s, and published in [32] (which contains references to the four abstracts that appeared in 1958). The Fundamental Theorem of Ultraproducts (Theorem 18.6) as it applies to sentences is implicit, but not explicit, in Łoś [84]; the general version of the theorem appears explicitly, with proof, in Frayne-Morel-Scott [32]. Lemma 18.5, Corollary 18.7, Lemma 18.9, and the results in Exercises 18.11, 18.12, and 18.14–18.17 are all from [32].

The equivalence of conditions (i) and (ii) in the SP_u-Theorem 18.10 is explicitly stated in Łoś [84] for arbitrary classes of algebras K, but no proof is given. As Łoś remarks, his result is directly inspired by Birkhoff's algebraic characterization of equational classes as those classes that are closed under subalgebras, homomorphisms, and direct products (see Theorem 18.27). The equivalence of conditions (ii)–(iv) in the SP_u-Theorem for arbitrary classes of algebras is explicitly stated in Corollary 0.3.70 of Henkin-Monk-Tarski [50]. In the proof of the SP_u-Theorem given here, we have followed the presentation in Burris-Sankappanavar [21] rather closely. The Łoś-Tarski Preservation Theorem (Corollary 18.11) was originally discovered by Tarski, announced in [140], and published in the famous series of papers [141]– [143]. It was independently discovered somewhat later by Łoś and published in [83]. The related result in Exercise 18.19 is from Tarski [142]. A version of Corollary 18.17 that applies to universal Horn classes instead of quasi-varieties was first formulated by George Grätzer and Harry Lakser in the abstract [47].

Jónsson [68] was the first to study the lattice of varieties of relation algebras in depth, although the work of Tarski [144] on minimal

varieties of relation algebras, or, equivalently, on maximal equational theories of relation algebras, was much earlier. The correspondence between the lattice of varieties of relation algebras and the lattice of universal classes of simple relation algebras that is described in Theorem 18.25, Corollary 18.26, and Theorem 18.28 is given in Theorem 4.10 of [68], and Lemma 18.23 is implicit in the proof of Jónsson's theorem. A version of Lemma 18.19 that applies to classes of simple algebras in congruence distributive varieties is also given in [68], with a quite different proof. The present version of the lemma and its proof are due to Givant.

The equivalence of conditions (i) and (ii) in the HSP-Theorem 18.27 is a famous result due to Birkhoff [16] that is valid for arbitrary classes of algebras. As was mentioned earlier, the equivalence of conditions (ii)–(iv) in that theorem is due to Tarski [134]. The standard proof of the theorem uses the notion of a free algebra. The proof given here in the context of relation algebras is due to Givant.

The theorem that RRA is a variety (Theorem 18.30) is due to Tarski [143]. The proof indicated in Exercise 18.32 that is based on Exercise 18.31 and the concluding sentence of Exercise 18.29 is given in Jónsson [68], where it is credited to McKenzie. The theorem has a curious history. In [88], Lyndon claimed to show that RRA cannot be axiomatized by any set—finite or infinite—of equations. Using Lyndon's result, Tarski announced in [136] that RRA is not even an elementary class. Eventually, Tarski proved Theorem 18.30, which contradicted Lyndon's theorem and his own announced result. He asked his research assistant at the time, Dana Scott, to help track down the problem, and he and Scott were eventually able to pinpoint the error in Lyndon's proof.

A set of equational axioms for the class RRA was given in Lyndon [89], but these equations are infinite in number and quite complicated in structure. Tarski raised the problem of finding a finite set of equational axioms for the class. Monk [110] eventually showed that this is not possible (Theorem 18.33). One step in his argument is the proof that an ultraproduct of complex algebras of projective lines of order at least three is embeddable into the complex algebra of the corresponding ultraproduct of the projective lines. In other words, he proved a weaker version of Lemma 18.32 in which it is shown that the algebra $\mathfrak{A}/D$ is embedded into the complex algebra $\mathfrak{Cm}(P/D)$. Lemma 18.32 in its present form is due to Givant, and strengthens Monk's result in two ways: first, it shows that the embedding is com-

plete; and second, it shows that the embedding extends to an isomorphism from the completion of $\mathfrak{A}/D$ to $\mathfrak{Cm}(P/D)$. The lemma is actually valid in the context of arbitrary relational structures and their complex algebras—see Theorem 19.40, and see also Givant [39], Theorem 1.35. The results in Exercises 18.29 and 18.30 are special cases of this more general result. The weaker version of Exercise 18.29, which asserts that the ultraproduct $\prod_i \mathfrak{Re}(U_i)/D$ is embeddable into the full set relation algebra $\mathfrak{Re}(U/D)$ is due to McKenzie—see Jónsson [68], pp. 285–286. The alternative proof implicit in Exercise 18.33 that **RRA** is not finitely axiomatizable is due to Givant.

Around 1966, McKenzie (unpublished) also gave a set of equational axioms for **RRA**, but like its predecessor, this axiomatization is complicated in its overall structure. Another such axiomatization is given in Hirsch-Hodkinson [59], at the end of Chapter 13. In his 1970 lectures on relation algebras, and perhaps earlier, Tarski raised the problem of finding a "nice" set of equational axioms for **RRA**, and in particular, he asked if there is a set of axioms that uses only finitely many variables. The negative solution to this problem given in Theorem 18.37 is due to Jónsson [70], who may not have been aware that Tarski raised the problem. Lemmas 18.35 and 18.36, and a version of Lemma 18.34 are implicit in Jónsson's proof.

A stronger result than Theorem 18.37 is proved in Andréka [1]. It says that if $\mathcal{E}$ is a set of equations—or even a set of quantifier-free formulas—axiomatizing **RRA**, and if no equation in $\mathcal{E}$ simultaneously contains an occurrence of the symbol for addition and an occurrence of the symbol for multiplication, then there must be infinitely many formulas in $\mathcal{E}$ in which the symbols for relative multiplication, complement, and either addition or multiplication, all occur simultaneously, and which simultaneously contain more than n variables, for any positive integer n given in advance. On the other hand, there is an axiomatization of **RRA** in which the operation symbols for converse and the identity element occur in only finitely many equations.

The method of diagrams used in Section 18.3 in the proof of the $\mathbf{SP_u}$-Theorem 18.10, in Lemma 18.14, and in the proof of the **SP**-Theorem 18.15, as well as in Section 18.6, is well known in model theory and traces its origins to the work of Henkin [48], [49], and Abraham Robinson [1]. The first two assertions of Lemma 18.38 are well known and are true in the context of arbitrary finite algebras, not just finite relation algebras. The last assertion of the lemma is implicit in the proof of Theorem 7.1 in Jónsson [68]. The terminology

conjugate identity is used there. Theorem 18.41, Corollary 18.42, and the lemmas leading up to them (Lemmas 18.39 and 18.40) are due to Givant, as is Exercise 18.37. Theorems 18.43 and 18.45, and their Corollaries 18.44 and 18.46 are from Andréka-Givant-Németi [5].

The classification of the minimal varieties of relation algebras (see Theorem 18.47 and Corollary 18.48) is due to Tarski [144], whose work was motivated by that of Jan Kalicki and Dana Scott [76]. The axiomatizations of these varieties implicit in Exercise 18.41 are from Tarski [144] and Jónsson [68].

The problem of describing the quasi-minimal varieties of relation algebras was raised by Jónsson, and investigated by him and his students, in particular Peter Jipsen and Erzsébet Lukács, as well as by Maddux, with the help of computer software that was developed by William McCune; see Jipsen [65] and Maddux [101]. The description of all known quasi-minimal algebras given in Section 18.7 is from Jipsen-Lukács [66] (who call these algebras "minimal relation algebras"). The axiomatizations of the classes M_{mn} given in Exercise 18.42 are from Jónsson [68], as are the observations that there is no variety above the variety generated by the minimal relation algebra $\mathfrak{M}_1$, and there is exactly one variety above the variety generated by the minimal relation algebra $\mathfrak{M}_2$. Lemma 18.49 is due to Givant. Theorem 18.53, classifying the quasi-minimal relation algebras that are not simple, and Corollary 18.54 are due to Maddux [97], as are Lemmas 18.51 and 18.52, and the result in Exercise 18.49. The proofs given here, based on Lemma 18.50 (and hence on the Atomic Subalgebra Theorem and the Atomic Isomorphism Theorem) are different from Maddux's proofs, and are due to Givant. The infinite quasi-minimal relation algebra $\mathfrak{A}$ of Theorem 18.57 was constructed by Jónsson. The proof that $\mathfrak{A}$ is quasi-minimal is due to Jipsen [65]; see also Lukács [87] and Jipsen-Lukács [66], which contains a very brief sketch of the proof. In particular, the formula in Lemma 18.55(i) is from those papers. The proof of Theorem 18.57 given in Section 18.7 that is based on Lemma 18.56, the Atomic Subalgebra Theorem, and the Atomic Isomorphism Theorem, is different from Jipsen's proof, and is due to Givant. The isomorphic version of the algebra $\mathfrak{A}$ given in Exercise 18.51 is due to Jipsen (see Problem 4.17 in [6]). The example in part (i) of Exercise 18.52 was constructed by Jónsson. The results in parts (ii)–(iv) of the exercise are due to Lukács [87]; see also Jipsen-Lukács [66].

The results in Section 18.8 concerning maximal universal classes and varieties of relation algebras, including the essential content of

Lemma 18.58 (though not the specific sentence Δ_n), Corollaries 18.59 and 18.60, and Theorem 18.61 and Corollary 18.62 are all due to Jónsson [68].

The translation mapping given in Section 18.9 and its basic properties (see Lemma 18.63 and Theorem 18.64) are essentially due to Tarski, and date back to the early 1940s (see Section 2.3 in Tarski-Givant [147]), although Peirce, Schröder, and Löwenheim were aware much earlier of certain connections between the calculus of relations and what we today call first-order logic. The use of the translation mapping to prove that full set relation algebras on infinite sets all have the same equational theory (Theorem 18.65) is due to Jónsson [68]. Theorem 18.68 and Corollary 18.69 are also due to Jónsson [68], as are the results in Exercises 18.62 and 18.63. (Jónsson remarks that the result in Exercise 18.62 was suggested by Birkhoff.) The related results in Exercises 18.57–18.60 are due to Givant. Most of the remaining results in Section 18.9, including Theorems 18.70–18.74 and 18.79, and Lemmas 18.75 and 18.76 are from Andréka-Givant-Németi [5]. The interpretation of the equational theory of a group into the complex algebra of the groups, and Lemma 18.77 and Corollary 18.78 (with the complex algebra $\mathfrak{Cm}(G)$ in place of $\mathfrak{Cf}(G)$) date back to 1970 and are due to Givant.

In Exercise 18.25, the observation that for the class K of complex algebras of groups, the class $\mathsf{S}(\mathsf{K})$ is universal is due to Tarski [143]; the observation that the variety generated by K is $\mathsf{SP}(\mathsf{K})$ is due to Givant, and dates from the early 1970s. The related results in Exercises 18.26 and 18.28 are due to Givant [36]. For K the class of complex algebras of projective lines of order at least three, the axiomatization of the universal class $\mathsf{S}(\mathsf{K})$ that is implicit in Exercise 18.27 is from Theorem 3.1 of Andréka-Givant-Németi [6]. The observation in Exercise 18.34 that a finitely axiomatizable variety of relation algebras can be axiomatized by a single equation is due to Givant; related theorems occur in Tarski [145], but it does not seem that Tarski's theorems yield the corresponding result for relation algebras.

Exercises

18.1. Give an example to show that the join of two varieties in the lattice of varieties of relation algebras may be strictly above the join of the two varieties in the lattice of elementary classes of relation algebras.

18.2. Define operations of join and meet in the class of all universal theories of relation algebras and in the class of all universal classes of relation algebras, and show that under these operations, each of the classes is a complete lattice. Prove that there is a dual isomorphism from the lattice of universal theories of relation algebras to the lattice of universal classes of relation algebras.

18.3. Prove that

$$\mathsf{HH}(\mathsf{K}) = \mathsf{H}(\mathsf{K}) \qquad \text{and} \qquad \mathsf{PP}(\mathsf{K}) = \mathsf{P}(\mathsf{K})$$

for any class K of algebras.

18.4. Prove that the inclusions in Lemma 18.2 are strict in the sense that equality does not hold in general.

18.5. (i) Prove that a class K of algebras is closed under homomorphisms if and only if $\mathsf{K} = \mathsf{H}(\mathsf{L})$ for some class L of algebras.
(ii) Prove that a class K of algebras is closed under direct products if and only if $\mathsf{K} = \mathsf{P}(\mathsf{L})$ for some class L of algebras.

18.6. Prove parts (ii) and (iii) of Lemma 18.2

18.7. Complete the proof of the implication from (iii) to (i) in Lemma 18.3 by showing that, under the hypothesis of (iii), the class K is closed under subalgebras and homomorphisms.

18.8. Complete the proof of Lemma 18.4 by showing that the relation $\equiv_D$ is reflexive and symmetric, and that it preserves the operations of addition, complement, and converse.

18.9. Fill in the missing details in the proof of Lemma 18.5.

18.10. Fill in the missing details in the proof of Theorem 18.6.

18.11. Prove that an ultrapower of a finite algebra $\mathfrak{A}$ must be isomorphic to $\mathfrak{A}$.

18.12. Suppose that I is a countably infinite set, and $(\mathfrak{A}_i : i \in I)$ is a system of countable algebras whose size is not bounded by any finite cardinal. (In other words, there is no finite cardinal number n such that each algebra in the system has cardinality at most n.) Prove that for each non-principal ultrafilter D over I, the ultraproduct $\prod_i \mathfrak{A}_i/D$ has cardinality $2^{\aleph_0}$.

18.13. Which of the following properties of relation algebras is preserved under ultraproducts in the sense that if each algebra in a non-empty system of relation algebras has the property, then every ultraproduct of the system has the property?

(i) The property of being finite.
(ii) The property of being infinite.
(iii) The property of having cardinality at most 10.
(iv) The property of being countable.
(v) The property of being atomic.
(vi) The property of being complete.
(vii) The property of being integral.
(viii) The property of being singleton dense.
(ix) The property of being totally decomposable.
(x) The property of being decomposable into the direct product of five directly indecomposable factors.
(xi) The property of being finitely generated.
(xii) The property of being generated by a single element.

18.14. For each index i in a non-empty index set I, let φ_i be a mapping from an algebra $\mathfrak{A}_i$ into an algebra $\mathfrak{B}_i$. Write

$$\mathfrak{A} = \prod_i \mathfrak{A}_i \qquad \text{and} \qquad \mathfrak{B} = \prod_i \mathfrak{B}_i,$$

and let D be an ultrafilter over the set I.

(i) Define a correspondence φ from $\mathfrak{A}/D$ to $\mathfrak{B}/D$ by

$$\varphi(r/D) = (\varphi_i(r(i)) : i \in I)/D$$

for each $r = (r(i) : i \in I)$ in $\mathfrak{A}$. For elements r in $\mathfrak{A}$ and u in $\mathfrak{B}$, prove that $\varphi(r/D) = u/D$ if and only if the set of indices

$$\{i \in I : \varphi_i(r(i)) = u(i)\}$$

is in D. Conclude that for elements r and s in $\mathfrak{A}$,

$$\varphi(r/D) = \varphi(s/D)$$

if and only if the set of indices

$$\{i \in I : \varphi_i(r(i)) = \varphi_i(s(i))\}$$

is in D.

(ii) Prove that the correspondence φ is a well-defined function.

(iii) Prove that if the set of indices

$$\{i \in I : \varphi_i \text{ is one-to-one}\}$$

is in D, then φ is one-to-one.

(iv) Prove that if the set of indices

$$\{i \in I : \varphi_i \text{ is onto}\}$$

is in D, then φ is onto.

(v) Prove that if the set of indices

$$\{i \in I : \varphi_i \text{ is a homomorphism}\}$$

is in D, then φ is a homomorphism from $\mathfrak{A}/D$ to $\mathfrak{B}/D$.

(vi) For any class K of algebras, prove that

$$\mathsf{P_u H}(\mathsf{K}) \subseteq \mathsf{HP_u}(\mathsf{K}).$$

18.15. For each index i in a non-empty index set I, let $\mathfrak{A}_i$ be a subalgebra of an algebra $\mathfrak{B}_i$. Write

$$\mathfrak{A} = \prod_i \mathfrak{A}_i \qquad \text{and} \qquad \mathfrak{B} = \prod_i \mathfrak{B}_i,$$

and let D be an ultrafilter over the set I. Prove that the ultraproduct $\mathfrak{A}/D$ is embeddable into the ultraproduct $\mathfrak{B}/D$ via the function that, for each r in $\mathfrak{A}$, maps the congruence class r/D in $\mathfrak{A}/D$ to the congruence class r/D in $\mathfrak{B}/D$. Explain why this function is, in general, not the identity function on $\mathfrak{A}/D$. Conclude that for any class K of algebras,

$$\mathsf{P_u S}(\mathsf{K}) \subseteq \mathsf{SP_u}(\mathsf{K}).$$

18.16. Suppose $\mathfrak{A}$ is the direct product of a non-empty system of algebras, say with index set I, and D is an ultrafilter over I. Prove that the function mapping each element r in $\mathfrak{A}$ to the congruence class r/D is an epimorphism from $\mathfrak{A}$ to $\mathfrak{A}/D$.

18.17. For a class of algebras K, prove that

$$\mathsf{P_u P_u}(\mathsf{K}) = \mathsf{P_u}(\mathsf{K}).$$

18.18. A class of algebras is called *pseudo-elementary* if it can be obtained from an elementary class of algebras L by deleting a fixed list of the fundamental operations from each of the algebras in L. Prove that pseudo-elementary classes are closed under ultraproducts.

18.19. Prove that if a universal class K is closed under binary direct products, then it is closed under direct products of arbitrary nonempty systems of algebras in K.

18.20. Formulate and prove an analogue of Theorem 18.15 for universal Horn classes.

18.21. Prove the implications from (i) to (ii), from (ii) to (iii), and from (iii) to (iv) in Theorem 18.16.

18.22. Formulate and prove a version of Theorem 18.16 that applies to universal Horn classes.

18.23. Formulate and prove a version of Corollary 18.17 that applies to universal Horn classes.

18.24. Formulate and prove a version of Corollary 18.18 that applies to universal Horn classes.

18.25. Let K be the class of all complex algebras of groups. Prove that the variety generated by K is $\mathsf{SP}(\mathsf{K})$ and that $\mathsf{S}(\mathsf{K})$ is the universal class of simple algebras in this variety.

18.26. Let K be the class of all complex algebras of projective geometries of order at least three. Prove that the variety generated by K is $\mathsf{SP}(\mathsf{K})$ and that $\mathsf{S}(\mathsf{K})$ is the universal class of simple algebras in this variety.

18.27. Let K be the class of all complex algebras of projective lines of order at least three. Give an explicit set of universal sentences that axiomatize the universal class $\mathsf{S}(\mathsf{K})$.

18.28. Let K be the class of all complex algebras of modular lattices with zero. Prove that the variety generated by K is $\mathsf{SP}(\mathsf{K})$ and that $\mathsf{S}(\mathsf{K})$ is the universal class of simple algebras in this variety.

18.29. Let $(\mathfrak{Re}(U_i) : i \in I)$ be a non-empty system of full set relation algebras on non-empty sets, and write

$$\mathfrak{A} = \prod_i \mathfrak{Re}(U_i) \qquad \text{and} \qquad U = \prod_i U_i.$$

Prove that if D is an ultrafilter on the index set I, then the completion of the ultraproduct $\mathfrak{A}/D$ is isomorphic to the full set relation algebra $\mathfrak{Re}(U/D)$. In particular, $\mathfrak{A}/D$ is completely embeddable into $\mathfrak{Re}(U/D)$.

18.30. Let $(\mathfrak{Cm}(G_i) : i \in I)$ be a non-empty system of complex algebras of groups, and write

$$\mathfrak{A} = \prod_i \mathfrak{Cm}(G_i) \qquad \text{and} \qquad G = \prod_i G_i.$$

Prove that if D is an ultrafilter on the index set I, then the completion of the ultraproduct $\mathfrak{A}/D$ is isomorphic to the group complex algebra $\mathfrak{Cm}(G/D)$. In particular, $\mathfrak{A}/D$ is completely embeddable into $\mathfrak{Cm}(G/D)$.

18.31. Using Exercise 18.29, prove for the class K of full set relation algebras on non-empty sets that $\mathsf{S}(\mathsf{K})$ is closed under ultraproducts, and therefore

$$\mathsf{SP}_{\mathsf{u}}(\mathsf{K}) = \mathsf{S}(\mathsf{K}).$$

Conclude that $\mathsf{S}(\mathsf{K})$ is the universal class generated by K.

18.32. Use Exercise 18.31 and Theorem 18.25 to give an alternative proof that RRA is a variety.

18.33. Prove that the variety RRA cannot be axiomatized by a finite set of first-order sentences (Theorem 18.33) by using the Compactness Theorem from first-order logic argument instead of an ultraproduct construction and Lemma 18.32.

18.34. Prove that a variety of relation algebras is axiomatizable by a finite set of equations if and only if it is axiomatizable by a single equation.

18.35. Prove that Theorem 18.37 implies Theorem 18.33.

18.36. Prove that if P is a projective line of order two, then a subset of $\mathfrak{Cm}(P)$ is a subuniverse if and only if it contains the identity element and is closed under the Boolean operations of $\mathfrak{Cm}(P)$.

18.37. Prove the following weaker version of Lemma 18.40 without using existential diagrams. Suppose K is a universal class of relation algebras, L_1 an antichain of relation algebras over K, and L_2 a finite antichain of finite relation algebras over K.

(i) $\mathsf{Un}(\mathsf{K} \cup \mathsf{L}_1) \subseteq \mathsf{Un}(\mathsf{K} \cup \mathsf{L}_2)$ if and only if $\mathsf{L}_1 \subseteq \mathsf{S}(\mathsf{L}_2)$.
(ii) $\mathsf{Un}(\mathsf{K} \cup \mathsf{L}_1) = \mathsf{Un}(\mathsf{K} \cup \mathsf{L}_2)$ if and only if $\mathsf{L}_1 \simeq \mathsf{L}_2$.

18.38. Prove that two finite, simple relation algebras have the same equational theory if and only if they are isomorphic.

18.39. Generalize Exercise 18.38 by proving that a relation algebra $\mathfrak{A}$ generates the same universal class as a finite relation algebra $\mathfrak{B}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. More generally, prove that an antichain L_1 of relation algebras generates the same universal class as a finite antichain L_2 of finite relation algebras if and only if L_1 and L_2 are essentially equal.

18.40. Show that property (5) in the proof of Theorem 18.45 holds for the system of sets $(Z_X : X \subseteq \mathbb{Q})$ that is defined there.

18.41. Give a set of equational axioms for the variety generated by the minimal universal class M_n, for each integer $n = 1, 2, 3$.

18.42. Give a set of equational axioms for the variety generated by the quasi-minimal universal class

$$\mathsf{M}_{mn} = \mathsf{M}_m \cup \mathsf{M}_n$$

for each pair of integers m and n with $1 \leq m < n \leq 3$.

18.43. Show that each of the seven integral, symmetric relation algebras discussed in Exercise 3.36 is generated by every element different from the constants 0, 1, 1', 0'. Conclude that each of these algebras is quasi-minimal and therefore generates a quasi-minimal universal class above the minimal universal class M_3.

18.44. Show that each of the two integral, non-symmetric relation algebras discussed in Exercise 3.37 is generated by every element different from the constants 0, 1, 1', 0'. Conclude that each of these algebras is quasi-minimal, and therefore generates a quasi-minimal universal class above the minimal universal class M_3.

18.45. Prove that each of the following five relative multiplication tables for atoms 1', r, s, and t determines an integral, symmetric relation algebra with sixteen elements such that every element different from the constants 0, 1, 1' and 0' generates the whole algebra. Conclude that each of the algebras is quasi-minimal and therefore generates a quasi-minimal universal class above the minimal universal class M_3.

(i)

;	1'	r	s	t
1'	1'	r	s	t
r	r	1' $+ s$	$r + t$	$s + t$
s	s	$r + t$	1' $+ t$	$r + s$
t	t	$s + t$	$r + s$	1' $+ r$

(ii)

;	1'	r	s	t
1'	1'	r	s	t
r	r	1	0'	0'
s	s	0'	1' $+ r + s$	r
t	t	0'	r	1' $+ r$

(iii)

;	1'	r	s	t
1'	1'	r	s	t
r	r	1' $+ r + s$	$r + t$	$s + t$
s	s	$r + t$	1' $+ t$	$r + s$
t	t	$s + t$	$r + s$	1' $+ r$

(iv)

;	1'	r	s	t
1'	1'	r	s	t
r	r	1' $+ r + s$	$r + t$	$s + t$
s	s	$r + t$	1' $+ s + t$	$r + s$
t	t	$s + t$	$r + s$	1' $+ r$

(v)

;	1'	r	s	t
1'	1'	r	s	t
r	r	1' $+ r + s$	$r + t$	$s + t$
s	s	$r + t$	1' $+ s + t$	$r + s$
t	t	$s + t$	$r + s$	1' $+ r + t$

18.46. Prove that the following relative multiplication and converse tables for atoms 1', r, s, and t determine an integral, non-symmetric relation algebra with sixteen elements such that every element different from the constants 0, 1, 1' and 0' generates the whole algebra.

Conclude that the algebra is quasi-minimal and therefore generates a quasi-minimal universal class above the minimal universal class M_3.

;	1'	r	s	t
1'	1'	r	s	t
r	r	r	s	1
s	s	$r+s$	1	s
t	t	1' $+ r + t$	$s+t$	t

	$\smile$
1'	1'
r	t
s	s
t	r

18.47. Prove that the following relative multiplication and converse tables for atoms 1', r, s, t, and u determine an integral, non-symmetric relation algebra with thirty-two elements such that every element different from the constants 0, 1, 1' and 0' generates the whole algebra. Conclude that the algebra is quasi-minimal and therefore generates a quasi-minimal universal class above the minimal universal class M_3.

;	1'	r	s	t	u
1'	1'	r	s	t	u
r	r	0'	1	0'	$r+s+u$
s	s	1	0'	$r+s+t$	0'
t	t	$r+s+t$	0'	1' $+ r + s$	s
u	u	0'	$r+s+u$	r	1' $+ r + s + u$

	$\smile$
1'	1'
r	s
s	r
t	t
u	u

18.48. Prove that the quasi-minimal universal classes described in Exercises 18.43–18.47 are all distinct from one another.

18.49. If a simple relation algebra is not integral, prove that it must contain an isomorphic copy of $\mathfrak{N}_{mn}$ as a subalgebra for some natural numbers m and n with $1 \leq m \leq n \leq 3$.

18.50. Complete the proof of the inclusion from right to left in Lemma 18.55(i) by treating the four remaining cases

(i) $-\eta \geq \xi \geq 0$,
(ii) $0 \geq \eta \geq \xi$,
(iii) $\eta \geq -\xi \geq 0$,
(iv) $-\xi \geq \eta \geq 0$.

18.51. Define an algebra $\mathfrak{Cf}(\mathbb{N})$ of the same similarity type as relation algebras in the following way. The universe of $\mathfrak{Cf}(\mathbb{N})$ is the set of finite and cofinite subsets of the set $\mathbb{N}$ of natural numbers. The Boolean

operations of $\mathfrak{Cf}(\mathbb{N})$ are the set-theoretic operations of union and complement, the operation of converse is the identity operation, and the operation of relative multiplication is determined on atoms by the rule

$$\{m\} ; \{n\} = \{i \in \mathbb{N} : n - m \leq i \leq n + m\}$$

whenever m and n are natural numbers with $m \leq n$. Prove that $\mathfrak{Cf}(\mathbb{N})$ is isomorphic to the relation algebra $\mathfrak{A}$ in Theorem 18.57.

18.52. Suppose a simple relation algebra has the property that it is isomorphic to each of its non-minimal subalgebras. Is the algebra necessarily quasi-minimal? It turns out that the answer is negative. For example, let $\mathbb{Z}$ be the group of integers under addition. The symmetric elements in the complex algebra $\mathfrak{Cm}(\mathbb{Z})$ form the universe of a complete subalgebra $\mathfrak{B}$ of $\mathfrak{Cm}(\mathbb{Z})$. Moreover, $\mathfrak{B}$ is atomic, and its atoms are the sets of the form $\{-n, n\}$, where n ranges over the natural numbers (see Exercise 6.43).

(i) Prove that the collection of finite and cofinite subsets of $\mathfrak{B}$ form the universe of a subalgebra $\mathfrak{C}$ of $\mathfrak{B}$.
(ii) Prove that every non-minimal subalgebra of $\mathfrak{C}$ is isomorphic to $\mathfrak{C}$.
(iii) Prove that the algebra $\mathfrak{A}$ from Theorem 18.57 is embeddable into an ultrapower of $\mathfrak{C}$.
(iv) Prove that $\mathfrak{A}$ and $\mathfrak{C}$ generate distinct universal classes, so the universal class generated by $\mathfrak{C}$ is not quasi-minimal.

18.53. Let $\mathbb{Z}$ be the group of integers under addition. Define an atomic subalgebra $\mathfrak{B}$ of $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$ in the following way. The atoms of $\mathfrak{B}$ are the subsets Z_n of $\mathbb{Z} \times \mathbb{Z}$ that are defined by

$$Z_n = \{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z} : |\alpha| + |\beta| = n\}$$

for $n = 0, 1, 2, \ldots$, and $\mathfrak{B}$ is the subalgebra of $\mathfrak{Cm}(\mathbb{Z} \times \mathbb{Z})$ generated by these atoms.

(i) Find a rule for the relative product of two atoms in $\mathfrak{B}$.
(ii) Describe all of the elements in $\mathfrak{B}$.
(iii) Is $\mathfrak{B}$ generated by a single element?
(iv) Does $\mathfrak{B}$ generate a universal class at level two in the lattice of universal classes of simple relation algebras?

18.54. Can the example of the relation algebra $\mathfrak{B}$ from Exercise 18.53 be generalized?

18.55. For each $n \geq 1$, let Ψ_n be the sentence asserting the existence of n identity singletons that sum to the identity element. Thus, Ψ_n is the existential sentence

$$\exists v_0 \exists v_1 \ldots \exists v_{n-1} \Gamma,$$

where Γ is the conjunction of the following equations and inequalities over all i and j with $0 \leq i < j < n$:

$$v_i \neq 0, \quad v_i \,;\, 1 \,;\, v_i \leq 1', \quad v_i \cdot v_j = 0, \quad v_0 + v_1 + \cdots + v_{n-1} = 1'.$$

Prove that Ψ_n is true in a simple relation algebra $\mathfrak{A}$ if and only if $\mathfrak{Re}(n)$ is isomorphic to $\mathfrak{A}$. Use this result to obtain another example of an equation that holds in a simple relation algebra $\mathfrak{A}$ if and only if $\mathfrak{A}$ is not isomorphic to $\mathfrak{Re}(n)$.

18.56. Complete the proof of Lemma 18.63 by treating the induction cases when ϱ is either $\sigma + \tau$, $-\sigma$, or $\sigma^\smile$.

18.57. Prove that the complex algebra of a finite group G generates a universal class that is incomparable with the universal class generated by the complex algebra of any group not isomorphic to G.

18.58. Do the complex algebras of infinite groups always have the same equational theory and therefore generate the same universal class?

18.59. Prove that the complex algebra of a finite projective lines generates a universal class that is incomparable with the universal class generated by the complex algebra of any projective line with a different number of points than P.

18.60. Do the complex algebras of infinite projective lines always have the same equational theory and therefore generate the same universal classes?

18.61. If U and V are infinite sets, prove that $\mathfrak{Re}(U)$ is always embeddable into some ultrapower of $\mathfrak{Re}(V)$.

18.62. Prove that the equational theory of the class of all finite relation algebras does not coincide with the equational theory of the class of all relation algebras.

18.63. Prove that there are $2^{\aleph_0}$ varieties of symmetric, representable relation algebras in which the equation $r \,;\, r = r \,;\, r \,;\, r$ is true.

18.64. Prove that the class $\mathsf{R}_{\mathsf{hir}}$ of all hereditarily strictly infinitely representable, simple relation algebras is axiomatized by the union of a set of axioms for R and the set of all conjugate equations of simple relation algebras that have square representations over finite sets.

18.65. Complete the proof of Lemma 18.75 by showing that the set A of finite and cofinite subsets of G is closed under converse.

18.66. Complete the proof of Lemma 18.77 by showing that if the formula Δ_ε is true in $\mathfrak{Cf}(G)$, then the equation ε is true in G.

18.67. Prove that Lemma 18.77 continues to hold when $\mathfrak{Cf}(G)$ is replaced by $\mathfrak{Cm}(G)$. In other words, prove that an equation ε is true in a group G if and only if the formula Δ_ε is true in the complex algebra $\mathfrak{Cm}(G)$.

18.68. Prove that Corollary 18.78 continues to hold when $\mathfrak{Cf}(G)$ is replaced by $\mathfrak{Cm}(G)$. In other words, prove that an equation ε is true in a group G if and only if the equation δ_ε is true in the complex algebra $\mathfrak{Cm}(G)$.

Chapter 19
Atom structures

Tarski's representation problem for relation algebras was ultimately given a negative solution by Lyndon (see the historical remarks in Section 17.9, and see also Theorem 17.15, the fourth example in Section 17.5, and Theorem 18.33). Nevertheless, the search for a positive solution that proceeded Lyndon's work led to several positive partial results of a quite general character. One of these is the Quasi-representation Theorem 17.32. Another is a generalization to Boolean algebras with normal operators of the Representation Theorem for Boolean algebras. The Boolean theorem says that every Boolean algebra is isomorphic to a Boolean set algebra, that is to say, every Boolean algebra is embeddable into the Boolean algebra of all subsets of some set U, with the set-theoretical operations of union and complement as the fundamental operations (see Lemma 14.8 and the remarks preceding it). The generalization says, roughly speaking, that every Boolean algebra with normal operators, and in particular, every relation algebra, is embeddable into the complex algebra of a relational structure. In this complex algebra, the universe consists of all subsets of a set U and the Boolean operations are the set-theoretical ones of union and complement. The normal operators of the algebra are determined by the fundamental relations of the underlying relational structure.

The generalization is valid in the context of arbitrary Boolean algebras with normal operators and arbitrary relational structures. However, for the sake of concreteness, and for consistency and simplicity of notation, we shall always assume that the algebras have the same similarity type as relation algebras, and that the relational structures have a corresponding similarity type (see Section 19.1). It should be obvious from the discussion how to extend the results to Boolean algebras with

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9_6

normal operators of arbitrary ranks and to relational structures with fundamental relations of arbitrary positive ranks.

The correlation between relational structures and Boolean algebras with normal operators gives rise to the possibility of constructing interesting classes of relation algebras as complex algebras of appropriate relational structures. As a first step in this direction, necessary and sufficient conditions on a relational structure are given in Theorem 19.12 for the complex algebra of the structure to be a relation algebra. In Section 19.3, these conditions are refined and simplified, and some concrete examples are given.

The conditions in Theorem 19.12 are in the form of a set of first-order axioms for the class of relational structures whose complex algebras are relation algebras, or, using different terminology, for the class of atom structures of the variety of relation algebras. One may ask: under what conditions is the class of atom structures of a variety of Boolean algebras with normal operators axiomatizable by a set of first-order sentences. It turns out that when the operators of the algebras are complete, there is always such an axiomatization (see Theorem 19.16). Unfortunately, the axiomatization that one obtains is in general quite complicated. If, however, the equations axiomatizing the variety have a certain simple form, then it is possible to construct a relatively simple axiomatization of the corresponding class of atom structures (see Theorem 19.30). The equations axiomatizing the variety of relation algebras have this simple form, so one obtains a relatively simple axiomatization of the class of atom structures of relation algebras, comparable to the axiomatization given in Theorem 19.12.

It was shown in Theorems 14.35 and 16.22 that the canonical extension of every relation algebra is a relation algebra, and the canonical extension of every representable relation algebra is a representable relation algebra. It is natural to look for conditions which imply that a variety of relation algebras, or, more generally, a variety of Boolean algebras with normal operators, is closed under canonical extensions in the sense that the canonical extension of every algebra in the variety also belongs to the variety. It is shown in Theorem 19.55 below that if a class of relational structures is closed under ultraproducts, then the variety generated by the class of complex algebras of those structures is closed under canonical extensions. As applications of this theorem, it is proved that the varieties generated by the class of complex algebras of groups and by the class of complex algebras of geometries are both closed under canonical extensions.

19.1 Atom structures and complex algebras

The algebras we shall be studying are Boolean algebras with normal operators of the same type

$$\mathfrak{A} = (A\,,\,+\,,\,-\,,\,;\,,\,{}^{\smile}\,,\,1')$$

as relation algebras. This means that $(A\,,\,+\,,\,-)$ is a Boolean algebra (called the Boolean part of $\mathfrak{A}$), and the binary operation $;$ and the unary operation ${}^{\smile}$ are distributive in the sense that

$$r\,;(s+t) = r\,;s + r\,;t, \qquad (r+s)\,;t = r\,;t + s\,;t,$$
$$(r+s)^{\smile} = r^{\smile} + s^{\smile},$$

for all elements r, s, and t in $\mathfrak{A}$, and normal in the sense that

$$r\,;0 = 0, \qquad 0\,;r = 0, \qquad 0^{\smile} = 0$$

for every element r in $\mathfrak{A}$, where 0 is the zero element of $\mathfrak{A}$ (see Section 2.2). The element 1' is a distinguished constant.

The algebra $\mathfrak{A}$ is said to be atomic if its Boolean part is atomic, and complete if its Boolean part is complete in the sense that every subset has a supremum. If the operators of $\mathfrak{A}$ are complete in the sense that for all subsets X and Y of $\mathfrak{A}$,

$$(\sum X)\,;(\sum Y) = \{r\,;s : r \in X \text{ and } s \in Y\}$$

and

$$(\sum X)^{\smile} = \{r^{\smile} : r \in X\},$$

then we shall say that $\mathfrak{A}$ is a Boolean algebra with complete operators. Every complete operator is automatically normal, by Lemma 2.5.

Suppose $\mathfrak{A}$ is a complete and atomic Boolean algebra with complete operators. The Boolean part of $\mathfrak{A}$ is determined, up to isomorphism, by the set U of all atoms in $\mathfrak{A}$. In fact, every element p in $\mathfrak{A}$ is the sum of the set X_p of atoms that are below p. Moreover,

$$u = p + q \quad \text{if and only if} \quad X_u = X_p \cup X_q,$$

and

$$u = -p \qquad \text{if and only if} \quad X_u = {\sim}X_p,$$

where the complement of X_p is formed with respect to the set U of atoms. The operator $;$ in $\mathfrak{A}$ is determined by a ternary relation R on U that is defined to hold for a triple (r, s, t) of atoms just in case $t \leq r\,;s$. Indeed, if p and q are arbitrary elements in $\mathfrak{A}$, then

$$\begin{aligned} p\,;q &= (\textstyle\sum X_p)\,;(\sum X_q) \\ &= \textstyle\sum\{r\,;s : r \in X_p \text{ and } s \in X_q\} \\ &= \textstyle\sum\{t : t \leq r\,;s \text{ for some } r \in X_p \text{ and } s \in X_q\} \\ &= \textstyle\sum\{t : R(r,s,t) \text{ for some } r \in X_p \text{ and } s \in X_q\}, \end{aligned}$$

by the assumption that $\mathfrak{A}$ is atomic, the assumption that the operator $;$ is complete, and the definition of the relation R. Similarly, the operator $^\smile$ in $\mathfrak{A}$ is determined by a binary relation C on U that is defined to hold for a pair (r, t) of atoms just in case $t \leq r^\smile$, and the distinguished constant 1' is determined by a unary relation I on U that is defined to hold for an atom t just in case $t \leq 1'$.

These observations suggest the idea of associating with $\mathfrak{A}$ the relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I),$$

and constructing from this relational structure an algebra of subsets that is an isomorphic copy of $\mathfrak{A}$ and may therefore be viewed as a kind of set-theoretical representation of $\mathfrak{A}$.

Definition 19.1. The *atom structure* of an atomic Boolean algebra with operators

$$\mathfrak{A} = (A\,,\, +\,,\, -\,,\, ;\,,\, {}^\smile\,,\, 1')$$

is defined to be the relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

in which the universe U is the set of atoms in $\mathfrak{A}$, and R, C, and I are respectively the ternary, binary, and unary relations on U defined by

$$\begin{aligned} R(r,s,t) \qquad &\text{if and only if} \qquad t \leq r\,;s, \\ C(r,t) \qquad &\text{if and only if} \qquad t \leq r^\smile, \\ I(t) \qquad &\text{if and only if} \qquad t \leq 1', \end{aligned}$$

for all atoms r, s, and t in U. □

A few words about notation are in order. First, the notation $R(r, s, t)$ means that the triple (r, s, t) belongs to the relation R, and similarly for the notations $C(r, t)$ and $I(t)$. Second, unary relations on a set are usually identified with subsets of the set, so we may think of the relation I as a subset of U. Third, as was mentioned at the end of Section 2.2, we usually employ the same symbol to refer to operations of different algebras, provided no confusion can arise; for example, we use the symbol $+$ to refer to addition in distinct relation algebras $\mathfrak{A}$ and $\mathfrak{B}$, even when speaking simultaneously about $\mathfrak{A}$ and $\mathfrak{B}$. We follow the same convention in speaking about the relations in relational structures. For instance, the symbol R will be used to refer to ternary relations in distinct relational structures $\mathfrak{U}$ and $\mathfrak{V}$, even when speaking about $\mathfrak{U}$ and $\mathfrak{V}$ at the same time. In ambiguous situations, the intended meaning can be clarified by an explicit explanation, either in words or by using some appropriate symbolism. Finally, to simplify notation still further, we shall sometimes adhere to the convention, common in algebra, of referring to a structure by referring to its universe, provided such a simplification does not cause confusion. The next definition contains an example of this simplified notation.

Definition 19.2. The *complex algebra* of a relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

is the algebra

$$\mathfrak{Cm}(U) = (Sb(U)\,,\, +\,,\, -\,,\, ;\,,\, {}^{\smile}\,,\, 1')$$

(of the same similarity type as relation algebras) in which the universe $Sb(U)$ is the set of all subsets of U, and the binary operation $+$ and the unary operation $-$ are respectively the set-theoretic operations of union and complement on subsets of U. The binary operation $;$ and the unary operation ${}^{\smile}$ are defined by

$$X\,;Y = \{t \in U : R(r, s, t) \text{ for some } r \in X \text{ and } s \in Y\},$$

and

$$X^{\smile} = \{t \in U : C(r, t) \text{ for some } r \in X\}.$$

The distinguished constant 1' is the set of elements in the relation I.

□

As usual, subsets of U that are singletons are often identified notationally with elements in U. In other words, the atoms of $\mathfrak{Cm}(U)$—which are the singletons $\{r\}$ of elements r in U—are identified with the elements themselves. For example, we usually write

$$r \,;\, s = \{t \in U : R(r, s, t)\}$$

instead of

$$\{r\} \,;\, \{s\} = \{t \in U : R(r, s, t)\}.$$

The term "complex algebra", or "algebra of complexes", is an old one, dating back to a time when the subsets of a set were commonly referred to as "complexes".

The definitions of the operations $;$ and $^\smile$ immediately imply the following lemma.

Lemma 19.3. *For all elements r and s in a relational structure $\mathfrak{U}$,*

$$r \,;\, s = \{t \in U : R(r, s, t)\} \qquad \text{and} \qquad r^\smile = \{t \in U : C(r, t)\}.$$

For all subsets X and Y of $\mathfrak{U}$,

$$X \,;\, Y = \bigcup\{r \,;\, s : r \in X \text{ and } s \in Y\}$$

and

$$X^\smile = \bigcup\{r^\smile : r \in X\}.$$

The second statement in the lemma says that the operations $;$ and $^\smile$ are completely distributive for atoms. In view of Lemma 2.7, this implies that these operations are completely distributive.

Corollary 19.4. *The complex algebra of a relational structure is a complete and atomic Boolean algebra with complete operators.*

The construction in Definition 19.2 may be viewed as a generalization of the constructions of complex algebras that were discussed in Chapter 3. For example, a group

$$(G\,,\ \circ\,,\ ^{-1}\,,\ \iota)$$

may be viewed as a relational structure by thinking of the binary operation $\circ$ as a ternary relation on G that holds for a triple (f, g, h)

just in case $h = f \circ g$, and by thinking of the unary operation $^{-1}$ as a binary relation on G that holds for a pair (f, h) just in case $h = f^{-1}$. The distinguished element ι may be thought of as a unary relation on G that holds for an element h just in case $h = \iota$. Under this conception of a group, the operations ; and $^\smile$, and the distinguished element 1', of the complex algebra $\mathfrak{Cm}(G)$ are defined by the rules

$$\begin{aligned} X\,;Y &= \{h \in G : h = f \circ g \text{ for some } f \in X \text{ and } g \in Y\}, \\ X^\smile &= \{h \in G : h = f^{-1} \text{ for some } f \in X\}, \\ 1' &= \{\iota\}, \end{aligned}$$

according to Definition 19.2. A comparison of these definitions with the definitions in Section 3.5 leads immediately to the conclusion that the complex algebra of the group as defined in Definition 19.2 coincides exactly with the complex algebra of the group as defined in Section 3.5. Moreover, if the elements of the group are identified with their singletons, then the atom structure of the complex algebra of the group is just the original group, conceived as a relational structure.

For a second example, consider a projective geometry P of order at least three, let ι be a new element not in P, and write

$$P^+ = P \cup \{\iota\}.$$

The geometry P may be treated as a relational structure

$$(P^+, R, C, I)$$

in the following way. The ternary relation R is an extension of the collinearity relation in P. It is defined to hold for a triple (p, q, r) just in case one of the following four conditions is satisfied: (1) p, q, and r are distinct collinear points in P; (2) p is a point in P, and $q = p$, and r is either ι or p; (3) $p = \iota$ and $r = q$; (4) $q = \iota$ and $r = p$. The binary relation C is defined to be the identity relation on P^+, so that C holds for a pair (p, q) just in case $p = q$. The unary relation I is defined to be the singleton $\{\iota\}$, so that I holds for an element p just in case $p = \iota$. The complex algebra of the resulting relational structure is identical with the complex algebra of the geometry P that was defined in Section 3.6. Moreover, if the elements in P^+ are identified with their singletons, then the atom structure of the complex algebra of P is the relational structure that was just defined. The purpose of adjoining the element ι to the set P of points in the geometry is not geometric

in nature, but rather is to obtain a relational structure whose complex algebra is a relation algebra.

The next theorem says that if one starts with a complete and atomic Boolean algebra with complete operators $\mathfrak{A}$, and forms the atom structure of $\mathfrak{A}$, then the complex algebra of the atom structure is isomorphic to $\mathfrak{A}$. This is nothing but a formal statement of the earlier informal remark that the structure of $\mathfrak{A}$ is determined by the structure of the atoms.

Theorem 19.5. *The complex algebra of the atom structure of a complete and atomic Boolean algebra with complete operators $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$ via the function that maps each set X of atoms to the element $\sum X$ in $\mathfrak{A}$.*

Proof. Let $\mathfrak{U} = (U, R, C, I)$ be the atom structure of $\mathfrak{A}$. Define a function φ from the set of atoms in the complex algebra $\mathfrak{Cm}(U)$ to the set of atoms in $\mathfrak{A}$, by putting

$$\varphi(\{r\}) = r$$

for each atom $\{r\}$ in $\mathfrak{Cm}(U)$. Clearly, φ is a bijection between the sets of atoms. If $\{r\}$, $\{s\}$, and $\{t\}$ are atoms in $\mathfrak{Cm}(U)$, then

$$\begin{aligned} \{t\} \subseteq \{r\} \,;\, \{s\} \qquad & \text{if and only if} \qquad R(r,s,t), \\ & \text{if and only if} \qquad t \leq r \,;\, s \\ & \text{if and only if} \qquad \varphi(\{t\}) \leq \varphi(\{r\}) \,;\, \varphi(\{s\}), \end{aligned}$$

by Lemma 19.3, the definition of the relation R in $\mathfrak{U}$ (see Definition 19.1), and the definition of the function φ. (The first occurrence of the symbol ; in the preceding equivalences refers to the operation in $\mathfrak{Cm}(U)$, while the second and third occurrences refer to the operation in $\mathfrak{A}$.) The function φ therefore preserves the operation ; on atoms in the sense of Atomic Isomorphism Theorem 7.11. Similar arguments show that φ preserves the operation $^\smile$ on atoms and maps the set of atoms below 1' in $\mathfrak{Cm}(U)$ bijectively to the set of atoms below 1' in $\mathfrak{A}$. Apply Theorem 7.11 to conclude that the function ψ from $\mathfrak{Cm}(U)$ to $\mathfrak{A}$ defined by

$$\psi(X) = \sum\{\varphi(\{r\}) : r \in X\}$$

for each set X in $\mathfrak{Cm}(U)$ is an isomorphism. Since $\varphi(\{r\}) = r$ for each atom r in $\mathfrak{A}$, we obviously have

$$\psi(X) = \sum X,$$

as desired. □

The preceding theorem implies that, up to isomorphism, complete and atomic Boolean algebras with complete operators are nothing more than the complex algebras of relational structures.

Corollary 19.6. *Every complete and atomic Boolean algebra with complete operators is isomorphic to the complex algebra of a relational structure.*

An *isomorphism* from a relational structure $\mathfrak{U}$ to a relational structure $\mathfrak{V}$ is defined to be a bijection ϑ from U to V that preserves the fundamental relations of the two structures in the sense that

$$R(r,s,t) \qquad \text{if and only if} \qquad R(\vartheta(r),\vartheta(s),\vartheta(t)),$$

and similarly for the relations C and I. (The symbol R on the left refers to the relation in $\mathfrak{U}$, while the symbol R on the right refers to the corresponding relation in $\mathfrak{V}$.) If such an isomorphism exists, then the two structures are said to be *isomorphic*. The counterpart of the preceding lemma for relational structures says that if we start with a relational structure $\mathfrak{U}$, then the atom structure of complex algebra $\mathfrak{Cm}(U)$ is isomorphic to $\mathfrak{U}$.

Theorem 19.7. *The atom structure of the complex algebra of a relational structure $\mathfrak{U}$ is isomorphic to $\mathfrak{U}$ via the function that maps each atom $\{r\}$ from the complex algebra to the element r in $\mathfrak{U}$.*

Proof. Let $\mathfrak{V}$ be the atom structure of the complex algebra $\mathfrak{Cm}(U)$. The elements in $\mathfrak{V}$ are defined to be the atoms in $\mathfrak{Cm}(U)$, which in turn are the singletons of elements in $\mathfrak{U}$, by the definition of $\mathfrak{Cm}(U)$. The function ϑ that maps each element $\{r\}$ in $\mathfrak{V}$ to the corresponding element r in $\mathfrak{U}$ is therefore a bijection from the universe of $\mathfrak{V}$ to the universe of $\mathfrak{U}$. If $\{r\}$, $\{s\}$, and $\{t\}$ are elements in $\mathfrak{V}$, then

$$\begin{aligned} R(\{r\},\{s\},\{t\}) \qquad & \text{if and only if} \qquad \{t\} \subseteq \{r\} \,;\, \{s\}, \\ & \text{if and only if} \qquad R(r,s,t), \\ & \text{if and only if} \qquad R(\vartheta(\{r\}),\vartheta(\{s\}),\vartheta(\{t\})), \end{aligned}$$

by the definition of the relation R in $\mathfrak{V}$ (see Definition 19.1), the first part of Lemma 19.3, and the definition of the function ϑ. Similar arguments show that ϑ isomorphically preserves the fundamental relations C and I. Thus, ϑ is an isomorphism from $\mathfrak{V}$ to $\mathfrak{U}$. □

The preceding theorem and Corollary 19.4 together imply a parallel conclusion to Corollary 19.6, namely that up to isomorphism, relational structures are nothing more than the atom structures of complete and atomic Boolean algebras with complete operators.

Corollary 19.8. *Every relational structure is isomorphic to the atom structure of a complete and atomic Boolean algebra with complete operators.*

The isomorphism type of a complete and atomic Boolean algebra with complete operators determines and is determined by the isomorphism type of its atom structure.

Theorem 19.9. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are complete and atomic Boolean algebras with complete operators, then* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{B}$ *if and only if the atom structure of* $\mathfrak{A}$ *is isomorphic to the atom structure of* $\mathfrak{B}$.

Proof. Let $\mathfrak{U}$ and $\mathfrak{V}$ be the atom structures of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. An isomorphism ϑ from $\mathfrak{U}$ to $\mathfrak{V}$ may be viewed as a bijection from the set of atoms in $\mathfrak{Cm}(U)$ to the set of atoms in $\mathfrak{Cm}(V)$. By definition, ϑ preserves the fundamental relations of the relational structures, so it also preserves all operations on subsets that are definable in terms of these fundamental relations. In particular, ϑ satisfies the hypotheses of Atomic Isomorphism Theorem 7.11. Apply the theorem to conclude that the function φ defined by

$$\varphi(X) = \{\vartheta(r) : r \in X\}$$

for sets X in $\mathfrak{Cm}(U)$ is an isomorphism from $\mathfrak{Cm}(U)$ to $\mathfrak{Cm}(V)$. The algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic to $\mathfrak{Cm}(U)$ and $\mathfrak{Cm}(V)$ respectively, by Theorem 19.5. Consequently, $\mathfrak{A}$ must be isomorphic to $\mathfrak{B}$.

To establish the reverse implication, assume that φ is any isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Certainly, φ maps the set of atoms in $\mathfrak{A}$ bijectively to the set of atoms in $\mathfrak{B}$, and it preserves all relations on atoms that are definable in terms of the fundamental operations of $\mathfrak{A}$ and $\mathfrak{B}$. In particular, it preserves the ternary relation R, the binary relation C, and the unary relation I. An appropriate restriction of φ is therefore an isomorphism from $\mathfrak{U}$ to $\mathfrak{V}$. □

An application of the preceding theorem, together with Corollary 19.4 and Theorem 19.7, yields a parallel conclusion for relational structures and their complex algebras.

Corollary 19.10. *If* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are relational structures, then* $\mathfrak{U}$ *is isomorphic to* $\mathfrak{V}$ *if and only if the complex algebra of* $\mathfrak{U}$ *is isomorphic to the complex algebra of* $\mathfrak{V}$.

We now come to the *Representation Theorem* for Boolean algebras with normal operators that was mentioned at the beginning of the chapter.

Theorem 19.11. *Every Boolean algebra with normal operators* $\mathfrak{A}$ *can be embedded into the complex algebra* $\mathfrak{Cm}(U)$ *of a relational structure* $\mathfrak{U}$. *Moreover, every positive equation that holds in* $\mathfrak{A}$ *continues to hold in* $\mathfrak{Cm}(U)$, *and similarly for every implication of the form* $\varepsilon \to \delta$, *where* ε *is a Boolean combination of positive equations of the form* $\varrho = 0$, *and* δ *is a positive equation.*

Proof. The canonical extension of a Boolean algebra with normal operators $\mathfrak{A}$ is a complete and atomic Boolean algebra with complete operators $\mathfrak{B}$ that includes $\mathfrak{A}$ as a subalgebra, by Definition 14.12 and the Existence Theorem 14.16. Apply Theorem 19.5 to $\mathfrak{B}$ to conclude that $\mathfrak{B}$ is isomorphic to the complex algebra of its atom structure $\mathfrak{U}$. Since $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, and $\mathfrak{B}$ is isomorphic to $\mathfrak{Cm}(U)$, it follows that $\mathfrak{A}$ is embeddable into $\mathfrak{Cm}(U)$.

Every positive equation that holds in $\mathfrak{A}$ continues to hold in its canonical extension $\mathfrak{B}$, by the First Preservation Theorem 14.32 for canonical extensions, and therefore it also holds in the isomorphic algebra $\mathfrak{Cm}(U)$. Similarly, if an implication of the form described in the statement of the theorem holds in $\mathfrak{A}$, then it continues to hold in $\mathfrak{B}$, by the Second Preservation Theorem 14.34 for canonical extensions, and therefore it also holds in $\mathfrak{Cm}(U)$. □

19.2 Atom structures of relation algebras

It is possible to characterize axiomatically the atom structures of complete and atomic relation algebras, or equivalently, the relational structures whose complex algebras are relation algebras.

Theorem 19.12. *The complex algebra of a relational structure*

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

is a relation algebra if and only if the relation C *is a function and the following conditions hold for all elements* p, r, r', s, s', t, *and* t' *in* $\mathfrak{U}$.

(i) *If there is an element* q *such that* $R(r,q,p)$ *and* $R(s,t,q)$*, then there is an element* q *such that* $R(q,t,p)$ *and* $R(r,s,q)$.
(ii) $I(s)$ *and* $R(r,s,p)$ *imply* $p = r$*, and there is always an element* s *such that* $I(s)$ *and* $R(r,s,r)$.
(iii) *If* $C(r,r')$, $C(s,s')$, *and* $C(t,t')$, *then* $R(r,s,t)$ *implies* $R(s',r',t')$.
(iv) *If* $C(r,r')$, *then* $R(r,s,t)$ *implies* $R(r',t,s)$.

Proof. In a relation algebra, the operation of converse is a function that maps atoms to atoms, by Lemma 4.1(vii). Consequently, for the complex algebra of the relational structure $\mathfrak{U}$ to be a relation algebra, it is necessary that the relation C be a function.

Assume that the relation C is indeed a function. It is then permissible to write $r' = r^\smile$ to express the relationship $C(r,r')$. Using this notation, conditions (iii) and (iv) in the theorem assume the more perspicuous forms

$$R(r,s,t) \qquad \text{implies} \qquad R(s^\smile, r^\smile, t^\smile) \tag{1}$$

and

$$R(r,s,t) \qquad \text{implies} \qquad R(r^\smile, t, s) \tag{2}$$

respectively.

The value of the operation ; in the complex algebra $\mathfrak{Cm}(U)$ is determined on atoms by

$$t \leq r\,;s \qquad \text{if and only if} \qquad R(r,s,t), \tag{3}$$

and the distinguished constant 1' is determined by the requirement

$$s \leq 1' \qquad \text{if and only if} \qquad I(s). \tag{4}$$

A comparison of conditions (i), (ii), (1), and (2) with conditions (i), (ii), (iii), and (iv) respectively in Theorem 2.9 suffices to show that each condition in the present theorem is just the relational reformulation of the corresponding condition in Theorem 2.9.

Consider, for example, condition (i) in Theorem 2.9. It requires that the existence of an atom q satisfying

$$p \leq r\,;q \qquad \text{and} \qquad q \leq s\,;t$$

imply the existence of an atom q satisfying

$$p \leq q\,;t \qquad \text{and} \qquad q \leq r\,;s.$$

Replace the inequalities

$$p \leq r\,;q, \qquad q \leq s\,;t, \qquad p \leq q\,;t, \qquad q \leq r\,;s$$

in this implication by

$$R(r,q,p), \qquad R(s,t,q), \qquad R(q,t,p), \qquad R(r,s,q)$$

respectively, and use (3), to arrive at condition (i) of the theorem.

Similar arguments show that (1) and (2) are the relational versions of conditions (iii) and (iv) in Theorem 2.9. As regards condition (ii) in Theorem 2.9, it may be reformulated as follows: for every atom $s \leq 1$', $$p \leq r\,;s \qquad \text{implies} \qquad p = r,$$ and for some atom $s \leq 1$', we have $r \leq r\,;s$. Replace

$$s \leq 1', \quad p \leq r\,;s, \quad r \leq r\,;s \quad \text{by} \quad I(s), \quad R(r,s,p), \quad R(r,s,r)$$

respectively in this statement, and use (3) and (4), to arrive at condition (ii) of the theorem. □

A relational structure satisfying the conditions of the preceding theorem will be called a *relation algebraic relational structure*. Theorems 19.5 and 19.12 yield the following *Representation Theorem* for complete and atomic relation algebras.

Corollary 19.13. *Every complete and atomic relation algebra* $\mathfrak{A}$ *is isomorphic to the complex algebra of a relation algebraic relational structure, namely the atom structure of* $\mathfrak{A}$.

Similarly, Theorems 19.7 and 19.12 yield the following *Representation Theorem* for relation algebraic relational structures.

Corollary 19.14. *Every relation algebraic relational structure* $\mathfrak{U}$ *is isomorphic to the atom structure of a complete and atomic relation algebra, namely the complex algebra of* $\mathfrak{U}$.

The version of Representation Theorem 19.11 that applies to relation algebras can be formulated as the following *Representation Theorem* for relation algebras.

Theorem 19.15. *Every relation algebra* $\mathfrak{A}$ *can be embedded into the complex algebra of a relation algebraic relational structure* $\mathfrak{U}$. *Moreover, every positive equation that holds in* $\mathfrak{A}$ *continues to hold in* $\mathfrak{Cm}(U)$, *and similarly for every implication of the form* $\varepsilon \to \delta$, *where* ε *is a Boolean combination of positive equations of the form* $\varrho = 0$, *and* δ *is a positive equation.*

Proof. For the proof, observe that the canonical extension of a relation algebra is always a complete and atomic relation algebra $\mathfrak{B}$ that includes $\mathfrak{A}$ as a subalgebra, by Theorem 14.35. Consequently, the atom structure obtained in the proof of Theorem 19.11 is actually a relation algebraic atom structure. □

19.3 Cycles

In order to use relational structures to construct concrete examples of relation algebras, it is helpful to view the conditions in Theorem 19.12 from a somewhat different perspective. If the complex algebra of a relational structure $\mathfrak{U}$ is to be a relation algebra, then the binary relation C in $\mathfrak{U}$ must be a function, and actually an involution—that is to say, a permutation that is its own inverse—on the universe U of the structure $\mathfrak{U}$ (see the first part of the proof of Theorem 2.9). The fact that a pair (r, t) belongs to C may therefore be expressed in functional notation, for example, by writing $t = r^\smile$. Involutions on a set U can be specified by two pieces of information: first, a subset S of U consisting of the elements r in U that are invariant under the involution in the sense that $r^\smile = r$; these elements are the symmetric atoms of the complex algebra, so we shall refer to them as *symmetric elements* in $\mathfrak{U}$; and second, a partition of the complementary set $U \sim S$ into two-element subsets $\{r, s\}$ with the property that $r^\smile = s$ and $s^\smile = r$; these elements are the non-symmetric atoms in the complex algebra, so we shall refer to them as the *non-symmetric elements* in $\mathfrak{U}$.

Consider now conditions (iii) and (iv) in Theorem 19.12. They are satisfied if and only if the presence of a triple (r, s, t) in R implies the presence of each of the triples

$$(r^\smile, t, s), \quad (s^\smile, r^\smile, t^\smile), \quad (s, t^\smile, r^\smile), \quad (t^\smile, r, s^\smile), \quad (t, s^\smile, r)$$

in R. Sets of the form

$$[r, s, t] =$$
$$\{(r, s, t), (r^\smile, t, s), (s, t^\smile, r^\smile), (s^\smile, r^\smile, t^\smile), (t, s^\smile, r), (t^\smile, r, s^\smile)\}$$

are called *cycles*, and (r, s, t) is called a *generator* of the cycle $[r, s, t]$. It is easy to check that two triples of elements from a relation algebraic relational structure generate either the same cycle or disjoint cycles. Consequently, if the complex algebra of a relational structure $\mathfrak{U}$ is to be a relation algebra, then the ternary relation R in $\mathfrak{U}$ must either include or omit each given cycle, so that R is a union of disjoint cycles. If R is in fact such a union, then conditions (iii) and (iv) are automatically satisfied. It seems easiest therefore to specify not the individual triples that belong to the relation R, but rather the cycles that are included in R. Alternatively, one may describe R by specifying the cycles that are disjoint from R; these are called the *forbidden cycles*.

The triples in a cycle may be visualized geometrically by means of directed equilateral triangles with sides labeled by the elements r, s, and t. In fact, the different triples in a cycle correspond to the different symmetries of the triangle. More precisely, the defining triple (r, s, t) of the cycle $[r, s, t]$ is a relational version of the inequality $t \leq r ; s$ in the complex algebra of the relational structure. To represent the triple (and the inequality) as a directed triangle, the product $r ; s$ is represented by two adjacent sides of a triangle, directed in a clockwise fashion (with respect to the perimeter of the triangle) by means of arrows, and labeled by r and s respectively. The element t is below this product, so it is represented by the side of the triangle that is included in the angle formed by the adjacent sides r and s; the side is directed from r to s, and is labeled by t (see part (a) of Figure 19.1). We shall refer to this triangle notationally by means of the symbolism $\triangle rst$.

The converse of an element is represented by the same side of the triangle as the original element, but with the direction of the arrow reversed. For example, the triple $(s, t^\smile, r^\smile)$ in the cycle $[r, s, t]$ is a relational version of the inequality $r^\smile \leq s ; t^\smile$ in the complex algebra of the relational structure. The product $s ; t^\smile$ is represented by two consecutive sides of a triangle in which the first side is directed in a clockwise fashion and labeled by s, while the second side is directed in a counterclockwise fashion and labeled by t. The fact that the orientation of the second side is in the contrary direction signifies that the two sides of the triangle represent the product $s ; t^\smile$, and not the product $s ; t$. The element $r^\smile$ is below the product $s ; t^\smile$, so it is represented by the side of the triangle that is included in the angle formed by the sides s

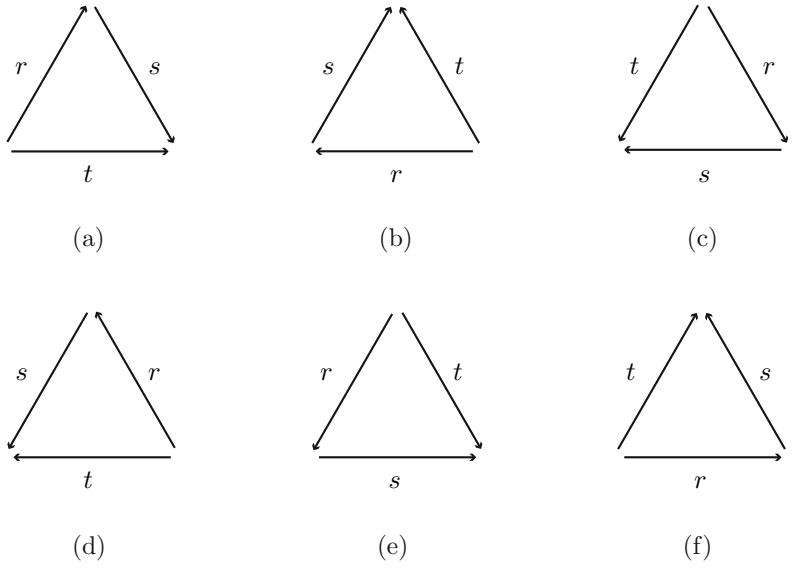


Fig. 19.1 The triples in a cycle may be represented as symmetries of a directed equilateral triangle. (a) $\triangle rst$ represents the inequality $t \leq r\,;s$. (b) A 120° rotation of $\triangle rst$ represents the inequality $r^\smile \leq s\,;t^\smile$. (c) A 240° rotation of $\triangle rst$ represents the inequality $s^\smile \leq t^\smile;r$. (d) A reflection of $\triangle rst$ across the bisector of the vertical angle represents the inequality $t^\smile \leq s^\smile;r^\smile$. (e) A reflection of $\triangle rst$ across the bisector of the lower right angle represents the inequality $s \leq r^\smile\,;t$. (f) A reflection of $\triangle rst$ across the bisector of the lower left angle, represents the inequality $r \leq t\,;s^\smile$.

and t, but oriented in the contrary direction, from t to s, and labeled by r (see part (b) in Figure 19.1). Observe that the resulting triangle is just the counterclockwise rotation of $\triangle rst$ by 120°.

In a similar fashion, it is easy to see that the triangle in part (c) of Figure 19.1, representing the triple $(t^\smile, r, s^\smile)$, is the counterclockwise rotation of $\triangle rst$ by 240°. The triangle in part (d), representing the triple $(s^\smile, r^\smile, t^\smile)$, is the reflection of $\triangle rst$ across the vertical axis that is the bisector of the angle formed by r and s. The triangle in part (e), representing the triple $(r^\smile, t, s)$, is the reflection of $\triangle rst$ across the axis that is the bisector of the angle formed by s and t; alternatively, it is the reflection of $\triangle rst$ across its vertical axis, followed by a counterclockwise rotation of 120°. Finally, the triangle in part (f), representing the triple $(t, s^\smile, r)$, is the reflection of $\triangle rst$ across the axis that is the bisector of the angle formed by r and t; alternatively, it is the reflection

of $\triangle rst$ across its vertical axis, followed by a counterclockwise rotation of 240°.

The actual number of distinct triples in a given cycle $[r, s, t]$ depends on which of the elements r, s, and t are equal, and what the converses of these elements are. For example, if r and t are distinct elements that are their own converses, and if $r = s$, then

$$[r, s, t] = [r, r, t] = \{(r, r, t), (r, t, r), (t, r, r)\}.$$

If all three elements are equal, then

$$[r, s, t] = [r, r, r] = \{(r, r, r)\}$$

whenever r is its own converse, and

$$\begin{aligned}&[r, s, t] = [r, r, r]\\ &= \{(r, r, r), (r^\smile, r, r), (r, r^\smile, r^\smile), (r^\smile, r^\smile, r^\smile), (r, r^\smile, r), (r^\smile, r, r^\smile)\}\end{aligned}$$

whenever r is not its own converse.

It is also helpful to view the identity condition (ii) in Theorem 19.12 from the perspective of cycles. If the complex algebra of a relational structure $\mathfrak{U}$ is to be a relation algebra, then every subidentity atom must be symmetric (see Lemma 5.20(i)), and therefore the unary relation I (viewed as a set) must be a subset of the set S of symmetric elements. Cycles generated by a triple containing an element from I are called *identity cycles*, and the other cycles are called *diversity cycles*. It is not difficult to describe the identity cycles that must belong to R if condition (ii) is to hold. Each atom r in the complex algebra $\mathfrak{Cm}(U)$ determines unique subidentity atoms in $\mathfrak{Cm}(U)$ that act as right-hand and left-hand identity elements for r (or more properly, for $\{r\}$), namely the range $(1 ; r) \cdot 1'$ and the domain $(r ; 1) \cdot 1'$ of r respectively, by Lemma 5.50(i),(ii). Consequently, for each element r in $\mathfrak{U}$, there must be unique elements s and t in I such that the *right identity cycles*

$$[r, s, r] = \{(r, s, r), (r^\smile, r, s), (s, r^\smile, r^\smile)\}$$

and the *left identity cycles*

$$[t, r, r] = \{(t, r, r), (r, r^\smile, t), (r^\smile, t, r^\smile)\}$$

are included in R, and all other identity cycles involving r must be disjoint from R. If r is not a symmetric element, then the right and left identity cycles for r coincide respectively with the left and right identity cycles for $r^\smile$, so that it is only necessary to adjoin the identity cycles for one of the two elements r and $r^\smile$. Consequently, an alternative approach to satisfying condition (ii) is to adjoin the single right identity cycle $[r, s, r]$ for every element r in $\mathfrak{U}$. If r is a symmetric element, then the left identity t and the right identity s are equal, and the left and right identity cycles collapse into the single cycle

$$[r, s, r] = \{(r, s, r), (r, r, s), (s, r, r)\}.$$

If r is actually in the set I, then $s = t = r$, and in this case the only cycle involved is

$$[r, r, r] = \{(r, r, r)\}.$$

The associativity condition in Theorem 19.12(i) is the most difficult of the four conditions to verify when checking that a given relational structure is or is not relation algebraic. Indeed, in general there seems to be no easy way of carrying out this verification. Nevertheless, directed triangles provide a convenient way of representing the condition geometrically. The four given elements p, r, s, and t in the condition are represented by the four sides of a directed quadrilateral. The condition asserts that if one of the diagonals can be fill in with an element q from $\mathfrak{U}$ to form two directed triangles (see Figure 19.2(a)), then the other diagonal can be so filled in with a (perhaps different) element q (see Figure 19.2(b)).

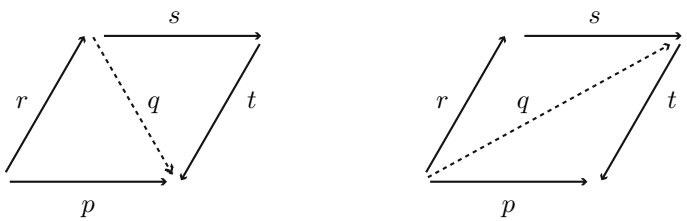


Fig. 19.2 A geometric representation of the associativity condition for atom structures.

In special cases, the construction of relation algebraic relational structures can be simplified somewhat further. Consider, for example, the problem raised in Section 18.7 of describing all classes at level two in the lattice of universal classes of simple relation algebras. The part of the problem that concerns the classes generated by *finite* simple relation algebras reduces to the problem of describing all finite, simple relation algebras that have a unique proper subalgebra, namely the minimal subalgebra (see Lemma 18.49 and the remarks preceding it). The cases in which the algebra is not integral are dealt with in Theorem 18.53, and the cases in which the minimal subalgebra is $\mathfrak{M}_1$ or $\mathfrak{M}_2$ are dealt with in the remarks following Lemma 18.49. There remains the task of describing all finite, integral relation algebra that have $\mathfrak{M}_3$ as their unique proper subalgebra. The finiteness condition implies that the algebras in question are complete and atomic, and are therefore isomorphic to the complex algebra of their atom structures, by Corollary 19.13. Consequently, it is natural to attack the problem by investigating all finite, relation algebraic relational structures in which the set I consists of a single element 1', and the universe U has at least three elements.

The problem splits into two cases: (1) describe the (relation algebraic) relational structures (with complex algebras at level two of the lattice) in which the set S of symmetric elements coincides with the universe of $\mathfrak{U}$, so that complex algebra of the relational structure is a symmetric, and hence an abelian, integral relation algebra; and (2) describe the relational structures in which the set S is a proper subset of the universe of $\mathfrak{U}$, so that the complex algebra of the relational structure is a non-symmetric, integral relation algebra. In the first case, the function $\smile$ must be the identity function on U, and may therefore be disregarded. For each element r in $\mathfrak{U}$, there is a unique identity cycle in R, namely the cycle

$$[r, 1', r] = \{(r, 1', r), (r, r, 1'), (1', r, r)\},$$

and these are the only identity cycles that can belong to R. As regards the diversity cycles, all permutations of a given triple generate the same cycle, so that

$$[r, s, t] = [r, t, s] = [s, r, t] = [s, t, r] = [t, r, s] = [t, s, r] = \\ \{(r, s, t), (r, t, s), (s, r, t), (s, t, r), (t, r, s), (t, s, r)\},$$

and these cycles may be included or omitted from R subject only to the proviso that the associativity condition holds. The main difficulties,

therefore, are verifying the associativity condition and checking that each non-constant element in the complex algebra generates the entire algebra.

In the second case, the permutation $^\smile$ is no longer the identity function. Each pair of elements in $\mathfrak{U}$ of the form $\{r, r^\smile\}$ determines two identity cycles in R, namely

$$[r, 1', r] = \{(r, 1', r), (r^\smile, r, 1'), (1', r^\smile, r^\smile)\}$$

and

$$[1', r, r] = \{(1', r, r), (r, r^\smile, 1'), (r^\smile, 1', r^\smile)\}.$$

These two cycles collapse into one cycle just in case the element r is symmetric. No other identity cycles belong to R. There does not appear to be much simplification in the description of the diversity cycles in this case. As in the first case, the main difficulties are verifying the associativity condition and checking that each non-constant element in the complex algebra generates the entire algebra.

Some examples may serve to illuminate the preceding discussion. Consider first a relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

in which the universe is the set

$$U = \{1', r, s, t\}.$$

The relation C is defined to be the identity function on U, so that every element in $\mathfrak{U}$ is symmetric. The relation I is defined to consist of the single element 1'. The identity cycles in R are defined to be

$$[1', 1', 1'], \qquad [r, 1', r], \qquad [s, 1', s], \qquad [t, 1', t],$$

while the diversity cycles are defined to be

$$[r, r, r], \quad [s, s, s], \quad [t, t, t], \quad [r, s, s], \quad [s, r, r], \quad [r, t, t], \quad [s, t, t].$$

The relation R is the union of all these cycles. The computation showing that $\mathfrak{U}$ satisfies the associativity condition in Theorem 19.12(i) and is therefore a relation algebraic relational structure is tedious, but not difficult. The complex algebra of $\mathfrak{U}$ is a symmetric, integral relation

;	1'	r	s	t
1'	1'	r	s	t
r	r	$1' + r + s$	$r + s$	t
s	s	$r + s$	$1' + r + s$	t
t	t	t	t	1

Table 19.1 The relative multiplication table for the atoms in the complex algebra of the first example.

algebra with four atoms, 1', r, s, and t, and sixteen elements. The rules governing the relative multiplication of these atoms are given in Table 19.1 (where $1 = 1' + r + s + t$).

As a second example, consider a relational structure $\mathfrak{U}$ in which the universe is the set

$$U = \{1', q, r, s, t\}.$$

The relation I is defined to consist of the single element 1', and this is the only symmetric element in $\mathfrak{U}$. The elements q and r are paired, as are the elements s and t, so that

$$q^\smile = r, \qquad r^\smile = q, \qquad s^\smile = t, \qquad t^\smile = s.$$

The relation C is therefore defined to be the set of ordered pairs

$$\{(1', 1'), (q, r), (r, q), (s, t), (t, s)\}.$$

It simplifies notation to write $q^\smile$ instead of r, and $s^\smile$ instead of t, so that

$$U = \{1', q, q^\smile, s, s^\smile\}.$$

The identity cycles in R are defined to be

$$[1', 1', 1'], \qquad [q, 1', q], \qquad [1', q, q], \qquad [s, 1', s], \qquad [1', s, s],$$

while the diversity cycles are defined to be

$$[q, q, q], \qquad [s, s, s], \qquad [s, s, s^\smile], \qquad [s, q, q], \qquad [q, s, q].$$

The relation R is the union of all these cycles. The relational structure $\mathfrak{U}$ can be shown to satisfy the associativity condition in Theorem 19.12(i), so the complex algebra of $\mathfrak{U}$ is a non-symmetric, integral relation algebra with five atoms, 1', q, $q^\smile$, s, and $s^\smile$, and 32 elements.

;	1'	q	$q^\smile$	s	$s^\smile$
1'	1'	q	$q^\smile$	s	$s^\smile$
q	q	q	1	q	q
$q^\smile$	$q^\smile$	1	$q^\smile$	$q^\smile$	$q^\smile$
s	s	q	$q^\smile$	$s + s^\smile$	$1' + s + s^\smile$
$s^\smile$	$s^\smile$	q	$q^\smile$	$1' + s + s^\smile$	$s + s^\smile$

Table 19.2 The relative multiplication table for the atoms in the complex algebra of the second example.

The rules governing the relative multiplication of these atoms are given in Table 19.2 (where $1 = 1' + q + q^\smile + s + s^\smile$).

As a final example, consider a relational structure $\mathfrak{U}$ in which the universe is the set

$$U = \{1', r, s\}.$$

The relation C is again defined to be the identity function on U, so that every element in $\mathfrak{U}$ is symmetric. The relation I is defined to consist of the single element 1'. The relation R is defined to consist of the three identity cycles,

$$[1', 1', 1'], \qquad [r, 1', r], \qquad [s, 1', s],$$

and no diversity cycles. This description implies that $\mathfrak{U}$ automatically satisfies conditions (ii)–(iv) in Theorem 19.12, but $\mathfrak{U}$ does not satisfy the associativity condition (i). Indeed, only one of the diagonals in the directed quadrilateral in Figure 19.3 can be filled in with an element from $\mathfrak{U}$. Consequently, the complex algebra of $\mathfrak{U}$ satisfies all of the

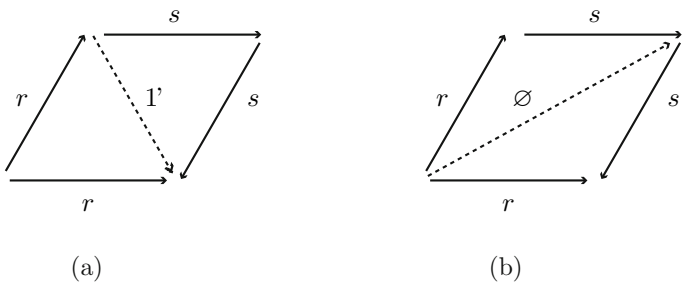


Fig. 19.3 A relational structure in which the associativity condition fails, while all other conditions in Theorem 19.12 hold.

axioms of relation algebra except the associative law for relative multiplication, Axiom (R4), which fails. In fact, as Figure 19.3 indicates,

$$r\,;(s\,;s) = r\,;1' = r \qquad \text{and} \qquad (r\,;s)\,;s = 0\,;s = 0.$$

It follows in particular that Axiom (R4) is independent of the remaining axioms of relation algebra in the sense that it cannot be derived from those axioms (see the discussion in Section 3.9).

19.4 Axiomatizing classes of atom structures

For each variety V of Boolean algebras with complete operators, let $\mathsf{At}(\mathsf{V})$ be the class of relational structures that are atom structures of atomic algebras in V. The main goal of this section is to prove that the class $\mathsf{At}(\mathsf{V})$ is always elementary.

The following preliminary observation is useful: the class $\mathsf{At}(\mathsf{V})$ is closed under isomorphisms. Indeed, if a relational structure $\mathfrak{U}$ is isomorphic to the atom structure of an atomic algebra $\mathfrak{A}$ in V, then a simple application of an appropriate version of the Exchange Principle (Theorem 7.15) yields an algebra $\mathfrak{B}$ isomorphic to $\mathfrak{A}$ such that $\mathfrak{U}$ is the atom structure of $\mathfrak{B}$. The variety V is closed under isomorphisms, so $\mathfrak{B}$ must be in V, and therefore $\mathfrak{U}$ must be in $\mathsf{At}(\mathsf{V})$.

Before addressing the main task of this section, consider the example of the variety RA of all relation algebras. Each of the four conditions (i)–(iv) in Theorem 19.12 is expressible by a sentence in the first-order language of relational structures, and so is the condition in the theorem that the relation C be a function. Together, the five sentences expressing these conditions axiomatize the class $\mathsf{At}(\mathsf{RA})$. To see this, let K be the class of all relational structures in which the five sentences are valid. If a structure $\mathfrak{U}$ belongs to K, then its complex algebra $\mathfrak{Cm}(U)$ belongs to RA, by Theorem 19.12. The atom structure of $\mathfrak{Cm}(U)$ is isomorphic to $\mathfrak{U}$, by Theorem 19.7, so $\mathfrak{U}$ belongs to $\mathsf{At}(\mathsf{RA})$, by the definition of this class and the fact that it is closed under isomorphisms. Hence, K is included in $\mathsf{At}(\mathsf{RA})$. To establish the reverse inclusion, let $\mathfrak{U}$ be a relational structure in $\mathsf{At}(\mathsf{RA})$. Thus, $\mathfrak{U}$ is the atom structure of an atomic relation algebra $\mathfrak{A}$. The completion of $\mathfrak{A}$ is a complete and atomic relation algebra $\mathfrak{B}$, by Theorem 15.28 and Lemma 15.29. Moreover, the atom structure of $\mathfrak{B}$ coincides with the atom structure of $\mathfrak{A}$ (see Lemma 15.29), which is $\mathfrak{U}$. It follows that $\mathfrak{B}$

is isomorphic to the complex algebra $\mathfrak{Cm}(U)$, by Theorem 19.5. Consequently, $\mathfrak{Cm}(U)$ is a relation algebra, and therefore the five sentences axiomatizing K must be valid in $\mathfrak{U}$, by Theorem 19.12. Hence, $\mathsf{At}(\mathsf{RA})$ is included in K.

Theorem 19.16. *The class* $\mathsf{At}(\mathsf{RA})$ *is elementary, and in fact it is axiomatized by the five conditions given in Theorem* 19.12.

The proof that the class $\mathsf{At}(\mathsf{V})$ is elementary for every variety V of Boolean algebras with complete operators is based on a translation of terms γ in the language $\mathcal{L}$ of relation algebras into formulas Γ_γ in the corresponding first-order language $\hat{\mathcal{L}}$ of relational structures. In defining this translation, it is notationally convenient (but certainly not essential) to assume that the variables in the language $\hat{\mathcal{L}}$ are just the variables $v_0, v_1, v_2, \ldots$ of $\mathcal{L}$, together with one additional variable w that does not occur in $\mathcal{L}$ and that comes before v_0 in the sequence of variables of $\hat{\mathcal{L}}$. If Γ is a formula in this language, then we write $\Gamma(v)$ for the formula obtained from Γ by replacing w everywhere by the variable v.

Definition 19.17. For each term γ in the language $\mathcal{L}$, a formula Γ_γ in the language $\hat{\mathcal{L}}$ is defined by induction on terms as follows.

(i) If γ is the term 1', then Γ_γ is the formula $I(w)$.
(ii) If γ is the term v_i, then Γ_γ is the formula $w = v_i$.
(iii) If γ is the term $\sigma + \tau$, then Γ_γ is the formula $\Gamma_\sigma \vee \Gamma_\tau$.
(iv) If γ is the term $-\sigma$, then Γ_γ is the formula $\neg\Gamma_\sigma$.
(v) If γ is the term $\sigma\,;\tau$, then Γ_γ is the formula

$$\exists u \exists v (R(u, v, w) \wedge \Gamma_\sigma(u) \wedge \Gamma_\tau(v)),$$

where u and v are (in order of indices) the first two variables that do not occur in Γ_σ or in Γ_τ;
(vi) If γ is the term $\sigma^\smile$, then Γ_γ is the formula

$$\exists u (C(u, w) \wedge \Gamma_\sigma(u)),$$

where u is the first variable that does not occur in Γ_σ. □

A straightforward proof by induction on terms shows that the free variables of the translation formula Γ_γ are precisely w and the variables of the term γ.

To give a sense of what a translation formula looks like in specific instances, here are the translations of some concrete terms. If γ is the term $v_0^\smile$, then Γ_γ is the formula

$$\exists v_1(C(v_1, w) \wedge (v_1 = v_0)).$$

If γ is the term $v_0 ; v_0$, then Γ_γ is the formula

$$\exists v_1 \exists v_2(R(v_1, v_2, w) \wedge (v_1 = v_0) \wedge (v_2 = v_0)).$$

If γ is the term $v_0 ; v_1^\smile$, then Γ_γ is the formula

$$\exists v_2 \exists v_3(R(v_2, v_3, w) \wedge (v_2 = v_0) \wedge \exists v_0(C(v_0, v_3) \wedge (v_0 = v_1))).$$

The next lemma establishes the fundamental properties of the translation formula Γ_γ. The proof sheds light on the motivation behind the definition of the translation. The following notation and terminology will be helpful. If r is an element, and $s = (s_0, \dots, s_{n-1})$ a sequence of elements, then $r^\frown s$ denotes the sequence $(r, s_0, \dots, s_{n-1})$. Since the word "term" is used in a linguistic sense below, we shall avoided referring to s as an n-termed sequence (a sequence with n terms), and refer to it instead as a sequence of n elements. This terminology is *not* meant to imply that the n elements are distinct from one another in the sense that $s_i \neq s_j$ for $i \neq j$. Recall that if γ is a term in $\mathcal{L}$ with variables among $v_0, \dots, v_{n-1}$, and if s is a sequence of n elements in an algebra $\mathfrak{A}$, then $\gamma(s)$ denotes the value of γ on s in $\mathfrak{A}$ (see Section 2.4).

Lemma 19.18. *Let $\mathfrak{A}$ be an atomic Boolean algebra with complete operators, and $\mathfrak{U}$ the atom structure of $\mathfrak{A}$. For every term $\gamma(v_0, \dots, v_{n-1})$ in $\mathcal{L}$ and every sequence s of n atoms in $\mathfrak{A}$, an atom r is below the element $\gamma(s)$ in $\mathfrak{A}$ if and only if the sequence $r^\frown s$ satisfies the formula Γ_γ in $\mathfrak{U}$.*

Proof. The proof proceeds by induction on terms in $\mathcal{L}$. There are two base cases to consider. If γ is the constant term 1', then Γ_γ is the formula $I(w)$. In this case,

$$\begin{array}{lll} r \leq \gamma(s) & \text{if and only if} & r \leq 1', \\ & \text{if and only if} & r \text{ is in } I, \\ & \text{if and only if} & r^\frown s \text{ satisfies } I(w), \\ & \text{if and only if} & r^\frown s \text{ satisfies } \Gamma_\gamma, \end{array}$$

by the definition of the value of a term on a sequence, the definition of the relation I in the atom structure $\mathfrak{U}$, the definition of satisfaction in $\mathfrak{U}$, and the form of Γ_γ. If γ is the variable v_i, then Γ_γ is the formula $w = v_i$. In this case,

$$\begin{aligned} r \leq \gamma(s) \quad & \text{if and only if} \quad r \leq s_i, \\ & \text{if and only if} \quad r = s_i, \\ & \text{if and only if} \quad r^\frown s \text{ satisfies } w = v_i, \\ & \text{if and only if} \quad r^\frown s \text{ satisfies } \Gamma_\gamma, \end{aligned}$$

by the definition of the value of a term on a sequence, the assumption that r and s_i are atoms, the definition of satisfaction in $\mathfrak{U}$, and the form of Γ_γ.

For the induction steps, assume that the conclusion of the lemma holds for terms σ and τ in place of γ. If γ is the term $\sigma + \tau$, then Γ_γ is the formula $\Gamma_\sigma \vee \Gamma_\tau$. In this case,

$$\begin{aligned} r \leq \gamma(s) \quad & \text{if and only if} \quad r \leq \sigma(s) + \tau(s), \\ & \text{if and only if} \quad r \leq \sigma(s) \text{ or } r \leq \tau(s), \\ & \text{if and only if} \quad r^\frown s \text{ satisfies } \Gamma_\sigma \text{ or } \Gamma_\tau, \\ & \text{if and only if} \quad r^\frown s \text{ satisfies } \Gamma_\gamma, \end{aligned}$$

by the definition of the value of a term on a sequence, the assumption that r is an atom, the induction hypotheses on σ and τ, the definition of satisfaction in $\mathfrak{U}$, and the form of Γ_γ.

The case when γ is the term $\sigma\,;\tau$ is somewhat more involved. The formula Γ_γ is

$$\exists u \exists v (R(u, v, w) \wedge \Gamma_\sigma(u) \wedge \Gamma_\tau(v)),$$

where u and v are the first two variables that do not occur in Γ_σ or in Γ_τ. The proof of the lemma in this case reduces to showing that each of the following statements is equivalent to its neighbor: (1) $r \leq \gamma(s)$, (2) $r \leq \sigma(s)\,;\tau(s)$, (3) $r \leq p\,;q$ for some atoms $p \leq \sigma(s)$ and $q \leq \tau(s)$, (4) $R(p, q, r)$ for some elements p and q in $\mathfrak{U}$ such that $p^\frown s$ satisfies Γ_σ and $q^\frown s$ satisfies Γ_τ, (5) $r^\frown s$ satisfies Γ_γ. The equivalence of (1) and (2) uses the definition of the value of a term on a sequence of elements. The equivalence of (2) and (3) uses the assumptions that the algebra $\mathfrak{A}$ is atomic and the operation ; is complete. In more detail, if X and Y are the sets of atoms below $\sigma(s)$ and $\tau(s)$ respectively, then

$$\sigma(s)\,;\tau(s) = (\sum X)\,;(\sum Y) = \sum\{p\,;q : p \in X \text{ and } q \in Y\},$$

by the assumption that $\mathfrak{A}$ is atomic with complete operators. Therefore, an atom r is below $\sigma(s)\,;\tau(s)$ just in case r is below $p\,;q$ for some elements p in X and q in Y. The equivalence of (3) and (4) uses the definition of the relation R in the atom structure $\mathfrak{U}$, and the induction hypotheses on the terms σ and τ. Finally, the equivalence of (4) and (5) uses the definition of satisfaction in $\mathfrak{U}$, and the form of the formula Γ_γ. The preceding argument shows that (1) and (5) are equivalent, which is just the desired conclusion.

The arguments when γ is either $-\sigma$ or $\sigma^\smile$ are similar to, but simpler than, the preceding two arguments and are left as an exercise. The conclusion of the lemma now follows by the principle of induction for terms. □

The preceding lemma is the basis for correlating with each equation in $\mathcal{L}$ a set of equivalent formulas in $\hat{\mathcal{L}}$.

Theorem 19.19. *For every equation ε in the language $\mathcal{L}$, there is a set of formulas $\mathcal{T}_\varepsilon$ in the language $\hat{\mathcal{L}}$ with the following property: if $\mathfrak{A}$ is any atomic Boolean algebra with complete operators that is generated by its set of atoms, and if $\mathfrak{U}$ is the atom structure of $\mathfrak{A}$, then the equation ε is valid in $\mathfrak{A}$ if and only if the set of formulas $\mathcal{T}_\varepsilon$ is valid in $\mathfrak{U}$.*

Proof. For each term $\sigma(v_0, \ldots, v_{n-1})$, and each sequence

$$\gamma = (\gamma_0, \ldots, \gamma_{n-1}) \tag{1}$$

of n terms, in $\mathcal{L}$, write

$$\sigma_\gamma = \sigma(\gamma_0, \cdots, \gamma_{n-1})$$

for the term obtained from σ by simultaneously substituting γ_i for the variable v_i for each $i < n$. Assume that the given equation ε has the form

$$\sigma(v_0, \ldots, v_{n-1}) = \tau(v_0, \ldots, v_{n-1}). \tag{2}$$

Take Ξ to be the set of all sequences γ of n terms in $\mathcal{L}$, as in (1), and define $\mathcal{T}_\varepsilon$ to be the set of all formulas of the form

$$\forall w(\Gamma_{\sigma_\gamma} \leftrightarrow \Gamma_{\tau_\gamma}), \tag{3}$$

where γ ranges over the set Ξ.

In order to prove that this set of formulas has the desired property, consider an atomic Boolean algebra with complete operators $\mathfrak{A}$ that is generated by the set of its atoms. This last assumption implies that for each sequence

$$p = (p_0, \dots, p_{n-1}) \tag{4}$$

of elements in $\mathfrak{A}$, there is a sequence of terms γ in Ξ, say with variables among $v_0, \dots, v_{m-1}$, and a sequence of atoms

$$s = (s_0, \dots, s_{m-1}) \tag{5}$$

in $\mathfrak{A}$, such that

$$p_i = \gamma_i(s) = \gamma_i(s_0, \dots, s_{m-1})$$

for each i. The value of the term σ on the sequence p is therefore equal to the value of the term σ_γ on the sequence s:

$$\sigma(p) = \sigma(p_0, \dots, p_{n-1}) = \sigma(\gamma_0(s), \dots, \gamma_{n-1}(s)) = \sigma_\gamma(s). \tag{6}$$

Of course, there may be many such pairs of sequences γ and s, so fix one and call it the pair *corresponding to* p.

Let $\mathfrak{U}$ be the atom structure of $\mathfrak{A}$. To prove that the set $\mathcal{T}_\varepsilon$ satisfies the conclusions of the theorem, assume first that every formula in the set $\mathcal{T}_\varepsilon$ is valid in $\mathfrak{U}$. It must be shown that the equation ε is valid in $\mathfrak{A}$. To this end, consider an arbitrary sequence p of elements in $\mathfrak{A}$, as in (4). Let γ and s be the corresponding sequences of terms and atoms such that (6) holds. The formula in (3) is valid in $\mathfrak{U}$, by assumption, so it must be satisfied by the sequence s. In other words,

$$r^\frown s \text{ satisfies } \Gamma_{\sigma_\gamma} \text{ in } \mathfrak{U} \qquad \text{if and only if} \qquad r^\frown s \text{ satisfies } \Gamma_{\tau_\gamma} \text{ in } \mathfrak{U} \tag{7}$$

for every element r in $\mathfrak{U}$, by (3) and the definition of satisfaction in $\mathfrak{U}$. It follows that

$$r \leq \sigma_\gamma(s) \qquad \text{if and only if} \qquad r \leq \tau_\gamma(s) \tag{8}$$

for every atom r in $\mathfrak{A}$, by Lemma 19.18. Since $\mathfrak{A}$ is assumed to be atomic, every element in $\mathfrak{A}$ is the sum of the set of atoms that it dominates. Consequently, (8) implies that

$$\sigma_\gamma(s) = \tau_\gamma(s), \tag{9}$$

and therefore $\sigma(p) = \tau(p)$, by (6). The sequence p was chosen arbitrarily, so the equation ε specified in (2) is valid in $\mathfrak{A}$.

Assume next that the equation ε is valid in $\mathfrak{A}$, with the goal of showing that the set of formulas $\mathcal{T}_\varepsilon$ is valid in $\mathfrak{U}$. The assumption and (2) imply that for every sequence γ of n terms in Ξ, say with variables among $v_0, \ldots, v_{m-1}$, and for every sequence s of atoms in $\mathfrak{A}$, as in (5), the equation in (9) is true in $\mathfrak{A}$. Since $\mathfrak{A}$ is atomic, the validity of (9) implies that (8) holds for every atom r in $\mathfrak{A}$. Apply Lemma 19.18 to conclude that (7) holds for every element r in $\mathfrak{U}$, and therefore the formula in (3) is satisfied by the sequence s in $\mathfrak{U}$. Since γ and s were chosen arbitrarily, it follows that every formula of the form (3) is valid in $\mathfrak{U}$, and therefore the set $\mathcal{T}_\varepsilon$ of these formulas is also valid in $\mathfrak{U}$. □

We turn now to the theorem mentioned at the beginning of the section.

Theorem 19.20. *For a variety* V *of Boolean algebras with complete operators, the class* $\mathsf{At}(\mathsf{V})$ *is always elementary. In fact, if* $\mathcal{E}$ *is a set of equations axiomatizing* V*, then*

$$\mathcal{T} = \bigcup\{\mathcal{T}_\varepsilon : \varepsilon \in \mathcal{E}\}$$

is a set of formulas axiomatizing $\mathsf{At}(\mathsf{V})$.

Proof. It must be shown that a relational structure $\mathfrak{U}$ belongs to $\mathsf{At}(\mathsf{V})$ if and only if $\mathfrak{U}$ is a model of the set of formulas $\mathcal{T}$. Assume first that $\mathfrak{U}$ is in $\mathsf{At}(\mathsf{V})$. In this case, $\mathfrak{U}$ is the atom structure of an atomic algebra $\mathfrak{A}$ in V, by the definition of $\mathsf{At}(\mathsf{V})$. Take $\mathfrak{B}$ to be the subalgebra of $\mathfrak{A}$ generated by the set of atoms in $\mathfrak{A}$. Clearly, $\mathfrak{B}$ belongs to the variety V, so it is a model of the set of equations $\mathcal{E}$ that axiomatize V. Also, $\mathfrak{B}$ has the same atom structure as $\mathfrak{A}$, namely $\mathfrak{U}$, because $\mathfrak{B}$ is generated by the atoms in $\mathfrak{A}$. Since $\mathfrak{B}$ is generated by its sets of atoms, we may apply Theorem 19.19 to conclude that its atom structure $\mathfrak{U}$ is a model of the set of formulas $\mathcal{T}_\varepsilon$ for every equation ε in $\mathcal{E}$. Consequently, $\mathfrak{U}$ is a model of $\mathcal{T}$.

Assume now that $\mathfrak{U}$ is a model of $\mathcal{T}$. The complex algebra

$$\mathfrak{A} = \mathfrak{Cm}(U)$$

is a complete and atomic Boolean algebra with complete operators, by Corollary 19.4. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by the set of atoms in $\mathfrak{A}$, and observe that $\mathfrak{B}$ is a dense subalgebra of $\mathfrak{A}$. Indeed,

every non-zero element r in $\mathfrak{A}$ is above a non-zero element in $\mathfrak{B}$, because every such element r is above an atom s in $\mathfrak{A}$ (since $\mathfrak{A}$ is atomic), and s belongs to $\mathfrak{B}$ (and remains an atom in $\mathfrak{B}$), by the definition of $\mathfrak{B}$. Dense subalgebras are regular subalgebras, by Lemma 15.6, so $\mathfrak{B}$ is a regular subalgebra of $\mathfrak{A}$.

It is not difficult to check that the operators of $\mathfrak{B}$ are complete. For instance, to see that the operator ; in $\mathfrak{B}$ is complete, consider subsets X and Y of $\mathfrak{B}$ such that the suprema

$$p = \sum X \qquad \text{and} \qquad q = \sum Y$$

exist in $\mathfrak{B}$. Because $\mathfrak{B}$ is a regular subalgebra of $\mathfrak{A}$, the elements p and q are also the suprema in $\mathfrak{A}$ of the sets X and Y respectively. The algebra $\mathfrak{A}$ is complete, and the operator ; in $\mathfrak{A}$ is complete, because $\mathfrak{A}$ is assumed to be the complex algebra of $\mathfrak{U}$. The supremum of the set

$$Z = \{r\,;s : r \in X \text{ and } s \in Y\}$$

therefore exists in $\mathfrak{A}$ and is equal to $p\,;q$, in symbols,

$$p\,;q = (\textstyle\sum X)\,;(\sum Y) = \sum Z \tag{1}$$

in $\mathfrak{A}$. The product $p\,;q$ belongs to $\mathfrak{B}$, because $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ that contains p and q. Similarly, the set Z is a subset of $\mathfrak{B}$, because X and Y are subsets of $\mathfrak{B}$, and $\mathfrak{B}$ is closed under the operator ;. It follows that $p\,;q$ must be the supremum of the set Z in $\mathfrak{B}$. (This last part of the argument does not make use of the regularity of the subalgebra $\mathfrak{B}$.) Consequently, equation (1) also holds in $\mathfrak{B}$. Thus, the operator ; is complete, as claimed.

It has been shown that $\mathfrak{B}$ is a Boolean algebra with complete operators. Also, $\mathfrak{B}$ is atomic. In fact, the atoms in $\mathfrak{B}$ are just the atoms in $\mathfrak{A}$, and $\mathfrak{B}$ is generated by the set of these atoms, by the definition of $\mathfrak{B}$. It follows that the atom structure of $\mathfrak{B}$ coincides with the atom structure of $\mathfrak{A}$, which is isomorphic to $\mathfrak{U}$, by Theorem 19.7. Since $\mathfrak{U}$ is a model of the set of formulas $\mathcal{T}_\varepsilon$ for each equation ε in $\mathcal{E}$, by assumption, the same is true of the atom structure of $\mathfrak{B}$. Therefore, $\mathfrak{B}$ must be a model of the equation ε for each ε in $\mathcal{E}$, by Theorem 19.19. It follows that $\mathfrak{B}$ belongs to the variety V that is axiomatized by the set of equations $\mathcal{E}$. Conclusion: $\mathfrak{U}$ is isomorphic to the atom structure of an atomic algebra in V, namely $\mathfrak{B}$, so $\mathfrak{U}$ belongs to the class $\mathsf{At}(\mathsf{V})$, by the definition of this class and the closure of this class under isomorphisms. □

Any variety of relation algebras is a variety of Boolean algebras with complete operators, by Corollary 4.18, so the preceding theorem applies and leads to the following conclusion about such varieties.

Corollary 19.21. *The class of atom structures of a variety of relation algebras is always elementary.*

In particular, the class of atom structures of the variety RRA of representable relation algebras is elementary.

Several remarks regarding the axiomatization of the class $\mathsf{At}(\mathsf{V})$ that is given in Theorem 19.20 are in order. First of all, it follows from the proof of the theorem that it is not necessary to use any of the sets of equations $\mathcal{T}_\varepsilon$ that correspond to Boolean axioms ε in $\mathcal{E}$, that is to say, to equations ε that do not involve symbols either for operators or for distinguished constant different from 0 and 1. Nor is it necessary to use any of the sets $\mathcal{T}_\varepsilon$ that correspond to distributive laws for operators, that is to say, to equations ε in $\mathcal{E}$ expressing the distributivity of an operator over addition. The reason is that the complex algebra of a relational structure is automatically a Boolean algebra with (distributive) operators, by Corollary 19.4, and therefore the same is true of every subalgebra of such a complex algebra.

Second, for a fixed variety V of Boolean algebras with complete operators, the axiomatization of the class of atom structures $\mathsf{At}(\mathsf{V})$ obtained from Theorem 19.20 is quite complicated, even when the axiomatization of V is simple. For example, V might be axiomatized by a single, short equation, and yet the axiomatization of $\mathsf{At}(\mathsf{V})$ obtained from Theorem 19.20 would still be infinite. In special cases, however, there may be much simpler axiomatizations of $\mathsf{At}(\mathsf{V})$. For example, the axiomatization of the class of atom structures of relation algebras given in Theorem 19.16 is of the same order of complexity as the set of axioms of the class of relation algebras itself. This leads to the question of when relatively simple axiomatizations of the class of atom structures of a variety are possible. The question has not yet been settled in full generality, but there are several known criteria which ensure that a relatively simple axiomatization of a variety V yields a correspondingly simple axiomatization of the class of atom structures $\mathsf{At}(\mathsf{V})$. One such criterion requires that for each equation $\sigma = \tau$ in the given axiomatization of V, the terms σ and τ be positive, define normal operations in the algebras of V, and contain at most one occurrence of each variable. The next lemma and its proof elucidate the reason for this last

requirement. Recall that a quasi-atom is an element that is either an atom or zero.

Lemma 19.22. *Let $\gamma(v_0, \dots, v_{n-1})$ be a positive term in $\mathcal{L}$ in which each variable has at most one occurrence, and let $\mathfrak{A}$ be an atomic Boolean algebra with complete operators. For each atom r and each sequence t of n elements in $\mathfrak{A}$, if $r \leq \gamma(t)$, then there is a sequence s of n quasi-atoms in $\mathfrak{A}$ such that*

(i) *$s_i \leq t_i$ for each $i < n$, and if $t_i \neq 0$, then s_i is an atom,*
(ii) *$r \leq \gamma(s)$.*

Proof. The proof is by induction on positive terms. Recall that a term in the language $\mathcal{L}$ is positive if it can be built up from variables and constant terms—that is to say, terms without variables—using the symbols for addition, multiplication, relative multiplication, and converse, but not complement (see Section 14.4). The symbol for complement is allowed to occur in constant terms.

There are two base cases to consider, the case when γ is a constant term and the case when γ is a variable. Assume first that γ is a constant term. For each index $i < n$, take s_i to be 0 or an arbitrary atom below t_i, according to whether t_i is or is not equal to 0. This is possible because the algebra $\mathfrak{A}$ is assumed to be atomic. Clearly, s satisfies condition (i) of the lemma. Since γ is a constant term (and therefore has no variables),

$$r \leq \gamma(t) = \gamma(s),$$

by the assumption on r and the definition of the value of a term on a sequence of elements, so condition (ii) also holds.

Assume next that γ is a variable, say v_j. For each index $i < n$ with $i \neq j$, take s_i to be 0 or an arbitrary atom below t_i, according to whether t_i is or is not equal to 0, and for $i = j$, take s_i to be r. The sequence s obviously satisfies condition (i) when $i \neq j$. When $i = j$, the definition of s_i, the assumption that r is an atom below $\gamma(t)$, and the definition of the value of a term on a sequence of elements imply that

$$s_i = r \leq \gamma(t) = t_i,$$

so condition (i) holds in this case as well. Condition (ii) is satisfied because

$$r = s_i = \gamma(s).$$

The induction step involves four cases, according to whether γ is

$$\sigma + \tau, \qquad \sigma \cdot \tau, \qquad \sigma^{\smile}, \qquad \text{or} \qquad \sigma\,;\tau.$$

Assume as the induction hypothesis that the conclusions of the lemma hold for terms

$$\sigma(v_0, \ldots, v_{n-1}) \qquad \text{and} \qquad \tau(v_0, \ldots, v_{n-1}).$$

Consider first the case when γ is $\sigma + \tau$. Let r be an atom, and t a sequence of n elements, in $\mathfrak{A}$ such that

$$r \leq \gamma(t) = \sigma(t) + \tau(t).$$

In this case,

$$r \leq \sigma(t) \qquad \text{or} \qquad r \leq \tau(t), \tag{1}$$

because r is an atom. Assume that the first inequality in (1) holds. Apply the induction hypothesis to σ to obtain a sequence s of quasi-atoms in $\mathfrak{A}$ that satisfies the conditions of the lemma with respect to the term σ. In particular, s satisfies condition (i) and

$$r \leq \sigma(s) \leq \sigma(s) + \tau(s) = \gamma(s),$$

by condition (ii) for σ, Boolean algebra, and the assumption on γ. Thus, condition (ii) for γ is also satisfied. An analogous argument, with σ replaced by τ, applies if the second inequality in (1) holds.

Consider next the case when γ is $\sigma \cdot \tau$. Let r be an atom, and t a sequence of n elements, in $\mathfrak{A}$ such that

$$r \leq \gamma(t) = \sigma(t) \cdot \tau(t).$$

In this case,

$$r \leq \sigma(t) \qquad \text{and} \qquad r \leq \tau(t),$$

by Boolean algebra. Apply the induction hypothesis to the terms σ and τ to obtain sequences p and q of quasi-atoms in $\mathfrak{A}$ such that the conditions of the lemma hold (with σ and τ in place of γ, and with p and q in place of s). In particular,

$$r \leq \sigma(p) \qquad \text{and} \qquad r \leq \tau(q). \tag{2}$$

by condition (ii) for σ and τ. For each $i < n$, take s_i to be p_i if the variable v_i occurs in σ, and take s_i to be q_i if the variable v_i occurs in τ or does not occur in either σ or τ. Since v_i occurs in at most one of the terms σ and τ, by the assumption on the term γ, this definition unambiguously determines a sequence s of n quasi-atoms in $\mathfrak{A}$. The definition also implies that s satisfies condition (i), since the sequences p and q satisfy this condition. Finally, (2) and the definition of s imply that

$$r \leq \sigma(p) = \sigma(s) \qquad \text{and} \qquad r \leq \tau(q) = \tau(s),$$

so

$$r \leq \sigma(s) \cdot \tau(s) = \gamma(s),$$

by Boolean algebra. Thus, condition (ii) is satisfied for γ.

Consider finally the case when γ is $\sigma\,;\tau$. Let r be an atom, and t a sequence of n elements, in $\mathfrak{A}$ such that

$$r \leq \gamma(t) = \sigma(t)\,;\tau(t). \tag{3}$$

The operator $;$ is assumed to be complete, and therefore normal, by Lemma 2.5; and the element r is an atom, so neither $\sigma(t)$ nor $\tau(t)$ can be zero, by (3). Since $\mathfrak{A}$ is assumed to be atomic, there must be non-empty sets X and Y of atoms in $\mathfrak{A}$ such that

$$\sigma(t) = \sum X \qquad \text{and} \qquad \tau(t) = \sum Y.$$

Consequently,

$$r \leq \sigma(t)\,;\tau(t) = (\textstyle\sum X)\,;(\sum Y) = \sum\{u\,;v : u \in X \text{ and } v \in Y\},$$

by (3) and the assumed complete distributivity of the operator $;$. This inequality and the assumption that r is an atom imply that r must be below $u\,;v$ for some u in X and v in Y. The elements u and v are atoms below $\sigma(t)$ and $\tau(t)$ respectively, by the definitions of the sets X and Y. Apply the induction hypothesis to the terms σ and τ, and the atoms u and v, to obtain sequences p and q of quasi-atoms in $\mathfrak{A}$ satisfying the conditions of the lemma (with σ and τ in place of γ, with u and v in place of r, and with p and q in place of s). In particular,

$$u \leq \sigma(p) \qquad \text{and} \qquad v \leq \tau(q), \tag{4}$$

by condition (ii) for σ and τ. For each $i < n$, take s_i to be p_i if the variable v_i occurs in σ, and take s_i to be q_i if the variable v_i occurs in τ or does not occur in either σ or τ. Since v_i occurs in at most one of the terms σ and τ, by the assumption on the term γ, this definition unambiguously determines a sequence s of n quasi-atoms in $\mathfrak{A}$. The definition implies that s satisfies condition (i), since the sequences p and q satisfy this condition. Finally, (4) and the definition of s imply that

$$u \leq \sigma(p) = \sigma(s) \qquad \text{and} \qquad v \leq \tau(q) = \tau(s).$$

Consequently,

$$r \leq u \,;\, v \leq \sigma(s) \,;\, \tau(s) = \gamma(s),$$

by the monotony of the operator ; (which follows from the assumed distributivity of ;, by Lemma 2.3).

The argument in the case when γ is $\sigma^\smile$ is similar to, but easier than, the preceding case, and is left as an exercise. The conclusion of the lemma now follows by the principle of induction for positive terms. □

The preceding lemma implies that each positive term in which variables occur at most once defines a quasi-completely distributive operator in every atomic Boolean algebra with complete operators.

Lemma 19.23. *If $\gamma(v_0, \ldots, v_{n-1})$ is a positive term in $\mathcal{L}$ in which each variable has at most one occurrence, and if $\mathfrak{A}$ is an atomic Boolean algebra with complete operators, then the polynomial of rank n defined by γ in $\mathfrak{A}$ is quasi-completely distributive.*

Proof. It suffices to prove that the polynomial defined by γ in $\mathfrak{A}$ is quasi-completely distributive for quasi-atoms, by Lemma 2.6. For each index $i < n$, let X_i be a non-empty set of quasi-atoms in $\mathfrak{A}$ for which the supremum exists, say

$$t_i = \sum X_i. \tag{1}$$

Put

$$t = (t_0, \ldots, t_{n-1}),$$

write X for the set of sequences

$$s = (s_0, \ldots, s_{n-1})$$

such that s_i belongs to X_i for each $i < n$, and write Z for the set of values of γ on sequences in X, so that

$$Z = \{\gamma(s) : s \in X\}. \tag{2}$$

The goal is to prove that

$$\gamma(t) = \sum\{\gamma(s) : s \in X\} = \sum Z. \tag{3}$$

If s is in X, then s_i is in X_i and therefore $s_i \leq t_i$ for each i, by (1) and the definition of X. The term γ is assumed to be positive, so the polynomial defined by γ in $\mathfrak{A}$ is monotone, by Lemma 14.29. These two observations imply that $\gamma(s) \leq \gamma(t)$ for every s in X, so $\gamma(t)$ is an upper bound of the set Z, by (2). To show that $\gamma(t)$ is the least upper bound of Z, consider an arbitrary atom r below $\gamma(t)$. We shall prove that r is below some element in Z. Consequently, any upper bound of Z has to be above each atom that is below $\gamma(t)$. Since $\gamma(t)$ is the least upper bound of the set of atoms that it dominates, by the assumption that $\mathfrak{A}$ is atomic, it follows that $\gamma(t)$ must be below every upper bound of Z. Thus, $\gamma(t)$ is the least upper bound of Z.

By Lemma 19.22, there is a sequence s of n quasi-atoms in $\mathfrak{A}$ that satisfies conditions (i) and (ii) of that lemma for the given atom r. Condition (i) forces each element s_i in the sequence to belong to the set X_i, from which it follows that s belongs to X. In more detail, if $s_i = 0$, then $t_i = 0$, by condition (i), and therefore X_i must be the singleton $\{0\}$, because X_i is non-empty and has t_i as its supremum. Thus, s_i belongs to X_i in this case. If s_i is an atom, then $t_i > 0$, by condition (i), and therefore the set

$$Y_i = X_i \sim \{0\}$$

of atoms in X_i cannot be empty, by (1). Consequently, t_i is also the supremum of the set Y_i, by (1). If an element p in a Boolean algebra is the supremum of a set of atoms W, then W must be the set of all atoms that are below p. Apply this observation to the element t_i and the set Y_i to conclude that Y_i is the set of all atoms below t_i. Since s_i is an atom below t_i, by condition (i), it follows that s_i belongs to Y_i and therefore also to X_i.

It has been shown that the sequence s constructed at the beginning of the preceding paragraph belongs to the set X. Condition (ii) in

Lemma 19.22 ensures that the given atom r is below the element $\gamma(s)$ in the set Z. This completes the proof of the lemma. □

The preceding lemma can fail when the term γ contains more than one occurrence of a single variable. As an example, take $\gamma(v_0)$ to be the term $v_0 ; v_0$. If X is a set of at least two atoms in a complete and atomic relation algebra $\mathfrak{A}$, and if $t = \sum X$, then

$$\gamma(t) = t ; t = (\sum X) ; (\sum X) = \sum\{r ; s : r, s \in X\},$$

while

$$\sum\{\gamma(s) : s \in X\} = \sum\{s ; s : s \in X\}.$$

In general, the sums on the right sides of these two equations are distinct from one another.

The reason that we get only a quasi-complete operator, and not a complete operator, in the preceding lemma is that a positive term in which the symbol for addition occurs may not be normal. Indeed, the term $v_0 + v_1$ does not define a normal operator.

In view of this observation, it is natural to ask whether the assumption that the operators in the algebra $\mathfrak{A}$ are complete may be replaced by the weaker assumption that the operators are quasi-complete. This appears not to be possible. Recall that an operation is complete if and only if it is quasi-complete and normal, by Lemma 2.5. The proof of Lemma 19.22, and therefore also the proof of Lemma 19.23, depends on the assumption that the operators in $\mathfrak{A}$ are normal. For example, in the final case of the proof of Lemma 19.22, when γ is the term $\sigma ; \tau$, the elements u and v must be atoms, and not just quasi-atoms, if one is going to apply the induction hypothesis to the terms σ and τ, and the elements u and v. Consequently, the sets X and Y to which u and v belong must consist of atoms, and not just quasi-atoms. The proof that these sets are non-empty uses the fact that the operator $;$ is normal.

The assumption that the operators in $\mathfrak{A}$ are complete is natural because the operators in the complex algebra of a relational structure are always complete (see Corollary 19.4). Since an operator is complete if and only if it is quasi-complete and normal, the preceding lemma immediately yields the following corollary.

Corollary 19.24. *Suppose $\gamma(v_0, \ldots, v_{n-1})$ is a positive term in $\mathcal{L}$ in which each variable has at most one occurrence. If γ defines a normal operation of rank n in an atomic Boolean algebra with complete operators, then this operation is completely distributive.*

There is a subtle point in this corollary that is worth clarifying. Assume $\gamma(v_0, \ldots, v_{n-1})$ is a positive term in which each variable v_i (with $i < n$) occurs at most once. If the operation of rank n in $\mathfrak{A}$ that is defined by γ is not constantly zero, then there must then be a sequence $s = (s_0, \ldots, s_{n-1})$ of elements in $\mathfrak{A}$ such that $\gamma(s)$ is not zero. If one of the variables v_i (with $i < n$) does not actually occur in γ, then s_i may be replaced by 0 in the sequence s to obtain a sequence $\bar{s}$ with at least one occurrence of 0 such that

$$\gamma(\bar{s}) = \gamma(s) \neq 0.$$

Consequently, γ cannot define a normal operation of rank n, and therefore it cannot define a complete operation of rank n. In other words, if γ is to define a complete operation—or even just a normal operation—that is not constantly zero, then each of the variables in the given list $v_0, \ldots, v_{n-1}$ must actually occur exactly once in γ.

It is natural to look for syntactic conditions on γ which imply that the operator defined by γ is normal and hence complete. One such condition is that γ be built up from variables and constants using only the symbols for multiplication and the operators, but not the symbol for addition. The proof that this condition implies that the defined operation is normal proceeds by induction on the definition of terms of the required form, and uses the understanding that each subterm defines an operation of the same rank as the number of distinct variables that actually occur in the subterm. The proof is left as an exercise.

Lemma 19.25. *If σ and τ are positive terms in $\mathcal{L}$ in which each variable occurs at most once, then the equation $\sigma = \tau$ holds in an atomic Boolean algebra with complete operators $\mathfrak{A}$ if and only if it holds for all sequences of quasi-atoms in $\mathfrak{A}$.*

Proof. The proof of the implication from left to right is trivial. To prove the reverse implication, assume that σ and τ are terms of the specified form with variables among $v_0, \ldots, v_{n-1}$, and that the equation $\sigma = \tau$ holds for all sequences of n quasi-atoms in $\mathfrak{A}$. Consider an arbitrary sequence $t = (t_0, \ldots, t_{n-1})$ of n elements in $\mathfrak{A}$. Since $\mathfrak{A}$ is atomic, each t_i is the sum of the (necessarily non-empty) set X_i of quasi-atoms that are below t_i.

Write X for the set of sequences $s = (s_0, \ldots, s_{n-1})$ such that s_i belongs to X_i for each $i < n$. The assumption on the form of the terms σ and τ implies, by Lemma 19.23, that

$$\sigma(t) = \sum\{\sigma(s) : s \in X\} \qquad \text{and} \qquad \tau(t) = \sum\{\tau(s) : s \in X\}. \tag{1}$$

Since $\sigma(s) = \tau(s)$ for each s in X, by assumption, it may be concluded from (1) that $\sigma(t) = \tau(t)$. The sequence t was chosen to be arbitrary, so the equation $\sigma = \tau$ holds in $\mathfrak{A}$. □

Corollary 19.26. *Suppose σ and τ are positive terms in $\mathcal{L}$ with variables among $v_0, \ldots, v_{n-1}$ such that each of these variables occurs at most once in each term. If the terms define normal operations of rank n in an atomic Boolean algebra with complete operators $\mathfrak{A}$, then the equation $\sigma = \tau$ holds in $\mathfrak{A}$ if and only if it holds for all sequences of atoms in $\mathfrak{A}$.*

Proof. The proof of the implication from left to right is trivial. To prove the reverse implication, assume that the given equation holds for all sequences of n atoms in $\mathfrak{A}$. Consider an arbitrary sequence s of n quasi-atoms in $\mathfrak{A}$. If zero does not occur in s, then

$$\sigma(s) = \tau(s), \tag{1}$$

by assumption. On the other hand, if 0 occurs somewhere in s, then the values of the right and left side of (1) are both 0, by the assumption that the terms σ and τ define normal operations of rank n. Consequently, the equation in (1) holds in this case as well. Conclusion: the given equation holds for all sequences of quasi-atoms in $\mathfrak{A}$, so it must hold identically in $\mathfrak{A}$, by Lemma 19.25. □

It is possible to extend Lemma 19.25 somewhat to cover certain implications between equations. The next lemma gives an example.

Lemma 19.27. *If σ, τ, and γ are positive terms in $\mathcal{L}$ in which each variable occurs at most once, then the implication*

$$(\gamma = 0) \to (\sigma = \tau)$$

holds in an atomic Boolean algebra with complete operators $\mathfrak{A}$ if and only if it holds for all sequences of quasi-atoms in $\mathfrak{A}$.

Proof. If the given implication holds in $\mathfrak{A}$, then certainly it holds for all sequences of quasi-atoms. To establish the reverse direction of the lemma, assume that the given implication holds for all sequences of quasi-atoms in $\mathfrak{A}$. It is to be shown that the implication holds for an arbitrary sequence

$$t = (t_0, \ldots, t_{n-1})$$

of elements in $\mathfrak{A}$. Each element t_i in this sequence is the sum of the non-empty set X_i of quasi-atoms that are below t_i, by the assumption that $\mathfrak{A}$ is atomic. Write X for the set of sequences $s = (s_0, \ldots, s_{n-1})$ such that s_i belongs to X_i for each $i < n$. Observe that

$$\gamma(t) = \sum\{\gamma(s) : s \in X\}, \tag{1}$$

by Lemma 19.23 and the assumed form of the term γ.

Assume that t satisfies the hypothesis of the given implication, that is to say, assume that $\gamma(t) = 0$. It follows from (1) that $\gamma(s) = 0$ for every sequence s in X. The given implication is assumed to hold for all sequences of quasi-atoms, so $\sigma(s) = \tau(s)$ for every sequence s in X. Conclude, as in the proof of Lemma 19.25, that $\sigma(t) = \tau(t)$. □

Just as in the case of Corollary 19.26, if the terms σ and τ in the preceding lemma define normal operations, then in the conclusion of the lemma, quasi-atoms may be replaced by atoms.

Corollary 19.28. *Suppose σ, τ, and γ are positive terms in $\mathcal{L}$ with variables among $v_0, \ldots, v_{n-1}$, each of which occurs at most once in each term. If σ and τ define normal operations of rank n in an atomic Boolean algebra with complete operators $\mathfrak{A}$, then the implication*

$$(\gamma = 0) \to (\sigma = \tau)$$

holds in $\mathfrak{A}$ if and only if it holds for all sequences of atoms in $\mathfrak{A}$.

The assumption in the preceding corollary that σ and τ define normal operations of the same rank is essential. To see this, consider the implication

$$v_0 = 0 \to v_0 = 1.$$

Here, γ and σ are both the term v_0, and τ is the term 1. Notice that τ does not define a normal operation of rank one. The implication holds for all atoms in any non-degenerate atomic relation algebra, because the hypothesis of the implication is never satisfied by an atom. However, the implication fails to hold for all elements in an atomic relation algebra.

The next theorem is an analogue of Theorem 19.19 that is applicable to varieties axiomatizable by means of equations and implications

of the types mentioned in the two preceding corollaries. It yields axiomatizations of the atom classes of varieties that are much simpler in character than those obtained by applying Theorem 19.20. The formulas Γ_σ, Γ_τ, and Γ_γ occurring in the statement of the theorem are from Definition 19.17.

Theorem 19.29. *Suppose σ, τ, and γ are positive terms in $\mathcal{L}$ with variables among $v_0, \ldots, v_{n-1}$, each of which occurs at most once in each term. If σ and τ define normal operations of rank n in an atomic Boolean algebra with complete operators $\mathfrak{A}$, then the equation $\sigma = \tau$ is true in $\mathfrak{A}$ if and only if the formula*

$$\forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$$

is true in the atom structure of $\mathfrak{A}$, and the implication

$$(\gamma = 0) \rightarrow (\sigma = \tau)$$

is true in $\mathfrak{A}$ if and only if the formula

$$\exists w(\Gamma_\gamma) \vee \forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$$

is true in the atom structure of $\mathfrak{A}$.

Proof. Let r be an atom, and s a sequence of n atoms, in $\mathfrak{A}$. In view of Lemma 19.18, the following equivalences hold between inequalities in $\mathfrak{A}$ and the satisfaction of corresponding formulas in the atom structure $\mathfrak{U}$ of $\mathfrak{A}$:

$$\begin{array}{lll} r \leq \sigma(s) & \text{if and only if} & r^\frown s \text{ satisfies } \Gamma_\sigma, \\ r \leq \tau(s) & \text{if and only if} & r^\frown s \text{ satisfies } \Gamma_\tau, \\ r \leq \gamma(s) & \text{if and only if} & r^\frown s \text{ satisfies } \Gamma_\gamma. \end{array}$$

Using the first two equivalences, it is not difficult to see that each of the following statements is equivalent to its neighbor. (1) The sequence s satisfies the formula

$$\forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$$

in $\mathfrak{U}$. (2) For every element r in $\mathfrak{U}$, the sequence $r^\frown s$ satisfies the formula

$$\Gamma_\sigma \leftrightarrow \Gamma_\tau$$

in $\mathfrak{U}$. (3) For every atom r in $\mathfrak{A}$,

$$r \leq \sigma(s) \qquad \text{if and only if} \qquad r \leq \tau(s).$$

(4) The sequence s satisfies the equation $\sigma = \tau$ in $\mathfrak{A}$. Indeed, (1) is equivalent to (2) by the definition of satisfaction in $\mathfrak{U}$; (2) is equivalent to (3) by the first two equivalences above; and (3) is equivalent to (4) because the algebra $\mathfrak{A}$ is assumed to be atomic, and therefore $\sigma(s)$ and $\tau(s)$ are equal if and only if they dominate the same set of atoms.

The formula $\forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$ is true in $\mathfrak{U}$ if and only if it is satisfied by all sequences s of n elements in $\mathfrak{U}$, by the definition of truth in the structure $\mathfrak{U}$. The formula is satisfied by all such sequences in $\mathfrak{U}$ if and only if the equation $\sigma = \tau$ is satisfied by all sequences s of n atoms in $\mathfrak{A}$, by the equivalence of (1) and (4), and by the assumption that $\mathfrak{U}$ is the atom structure of $\mathfrak{A}$. The equation is satisfied by all such sequences in $\mathfrak{A}$ if and only if the equation is true in $\mathfrak{A}$, by the assumptions on the terms σ and τ, and by Corollary 19.26. This completes the proof of the first assertion of the theorem.

The proof of the second assertion is similar, but slightly more involved. To obtain the implication from left to right, assume that the implication

$$(\gamma = 0) \to (\sigma = \tau)$$

is true in $\mathfrak{A}$, and let s be an arbitrary sequence of n elements in $\mathfrak{U}$. It is to be shown that s satisfies the disjunction

$$\exists w(\Gamma_\gamma) \vee \forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$$

in $\mathfrak{U}$. Suppose s does not satisfy the disjunct $\exists w(\Gamma_\gamma)$. In this case, there is no element r in $\mathfrak{U}$ such that the sequence $r^\frown s$ satisfies the formula Γ_γ in $\mathfrak{U}$, by the definition of satisfaction in $\mathfrak{U}$. Consequently, there is no atom r in $\mathfrak{A}$ such that $r \leq \gamma(s)$, by the third equivalence at the beginning of the proof. The algebra $\mathfrak{A}$ is assumed to be atomic, so $\gamma(s)$ is the sum of the atoms that it dominates. Combine these observations to conclude that $\gamma(s) = 0$. In view of the assumed implication, it follows that $\sigma(s) = \tau(s)$ in $\mathfrak{A}$. Use the equivalence of (1) and (4) above to conclude that s satisfies the disjunct $\forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$, as desired.

Run the argument backwards to show that if a sequence s of n elements in $\mathfrak{U}$ (that is to say, a sequence of n atoms in $\mathfrak{A}$) satisfies the disjunction, then s satisfies the implication in $\mathfrak{A}$. In more detail, if s satisfies the disjunct $\exists w(\Gamma_\gamma)$ in $\mathfrak{U}$, then there is an element r in $\mathfrak{U}$ such that $r^\frown s$ satisfies the formula Γ_γ in $\mathfrak{U}$. Consequently, the atom r is below $\gamma(s)$, by the third equivalence at the start of the proof. In

particular, $\gamma(s)$ is not 0, so the given implication is trivially satisfied by s in $\mathfrak{A}$. On the other hand, if s satisfies the disjunct $\forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$ in $\mathfrak{U}$, then s satisfies the equation $\sigma = \tau$ in $\mathfrak{A}$, by the equivalence of (1) and (4) above, and therefore s satisfies the given implication in $\mathfrak{A}$.

It follows from the preceding observations that if the disjunction is true in $\mathfrak{U}$, then the implication holds for all sequences of n atoms in $\mathfrak{A}$, and therefore the implication is true in $\mathfrak{A}$, by Corollary 19.28. □

The formulas

$$\forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau) \qquad \text{and} \qquad \exists w(\Gamma_\gamma) \vee \forall w(\Gamma_\sigma \leftrightarrow \Gamma_\tau)$$

are called the *correspondents* of the equations and implications

$$\sigma = \tau \qquad \text{and} \qquad (\gamma = 0) \rightarrow (\sigma = \tau)$$

respectively. When forming correspondents of such equations and implications, it is useful (but not essential) to assume that the individual constant symbols for zero, one, and diversity, and the operation symbol for multiplication, are primitive symbols of the language $\mathcal{L}$, and not introduced as abbreviations. One must then add appropriate conditions to Definition 19.17. For example, the appropriate additional condition for multiplication says that if γ is the term $\sigma \cdot \tau$, then Γ_γ is the formula $\Gamma_\sigma \wedge \Gamma_\tau$.

The next theorem—an analogue of Theorem 19.20—gives sufficient conditions for the class $\mathsf{At}(\mathsf{V})$ of atoms structures of atomic algebras in a class V of Boolean algebras with complete operators to admit a relatively simple axiomatization.

Theorem 19.30. *If a class* V *of Boolean algebras with complete operators is axiomatizable by a set* $\mathcal{E}$ *of equations and implications which, with the exception of Boolean laws and distributive laws (over addition) for the operators, satisfy the conditions in Theorem* 19.29*, then the class* $\mathsf{At}(\mathsf{V})$ *is axiomatizable by the set of correspondents of the equations and implications in* $\mathcal{E}$ *that are not Boolean axioms or distributive laws for the operators.*

Proof. Let $\mathcal{T}$ be the set of correspondents of equations and special implications in $\mathcal{E}$ that are not Boolean laws or distributive laws for operators. It is to be shown that a relational structure $\mathfrak{U}$ belongs to $\mathsf{At}(\mathsf{V})$ if and only if $\mathfrak{U}$ is a model of $\mathcal{T}$.

If $\mathfrak{U}$ belongs to $\mathsf{At}(\mathsf{V})$, then $\mathfrak{U}$ is the atom structure of an atomic algebra $\mathfrak{A}$ in V, by the definition of $\mathsf{At}(\mathsf{V})$. The algebra $\mathfrak{A}$ is a model of the set $\mathcal{E}$ of axioms of V, so its atom structure $\mathfrak{U}$ must be a model of the set $\mathcal{T}$ of correspondents of equations and special implications in $\mathcal{E}$, by Theorem 19.29.

To establish the reverse implication of the theorem, assume that $\mathfrak{U}$ is a model of $\mathcal{T}$. The complex algebra

$$\mathfrak{A} = \mathfrak{Cm}(U)$$

is a complete and atomic Boolean algebra with complete operators, by Corollary 19.4, and the atom structure of $\mathfrak{A}$ is isomorphic to $\mathfrak{U}$, by Theorem 19.7. Consequently, the atom structure of $\mathfrak{A}$ is also a model of $\mathcal{T}$, so every equation and special implication in $\mathcal{E}$ that is not a Boolean law or a distributive law for an operator must be true in $\mathfrak{A}$, by Theorem 19.29 and the definition of $\mathcal{T}$. Of course the Boolean laws and distributive laws for operators are also true in $\mathfrak{A}$, because $\mathfrak{A}$ is a Boolean algebra with operators. Thus, $\mathfrak{A}$ is a model of every formula in $\mathcal{E}$, so $\mathfrak{A}$ belongs to V. It follows that the atom structure of $\mathfrak{A}$ belongs to $\mathsf{At}(\mathsf{V})$, and therefore so does $\mathfrak{U}$, since $\mathsf{At}(\mathsf{V})$ is closed under isomorphisms. □

Theorem 19.30 is applicable, in particular, to the variety RA of relation algebras. Indeed, as is stated in the theorem, it is unnecessary to form the correspondents of the Boolean laws (R1)–(R3) or the distributive laws (R8) and (R9). This leaves the axioms (R4)–(R7) and (R10). Axioms (R4)–(R7) clearly have the form prescribed in Theorem 19.30, and the terms on each side of these axioms define normal operations (see the remarks preceding Lemma 19.25). Axiom (R10) does not have the prescribed form, because it is not positive and because it has more than one occurrence of each of two variables on the left side of the equation. However, on the basis of the remaining axioms, (R10) is equivalent to the implication (R11), and this implication does have the prescribed form, and each of the two terms on the left sides of the two equations involved in (R11) does define a normal operation. The axiom set for $\mathsf{At}(\mathsf{RA})$ that consists of the correspondents of (R4)–(R7) and (R11) is rather close in spirit to the axiom set for $\mathsf{At}(\mathsf{RA})$ given in Theorem 19.16, and it is a vast improvement over the axiom set for $\mathsf{At}(\mathsf{RA})$ implied by Theorem 19.20.

19.5 Duality

The relationship between relational structures and complete and atomic Boolean algebras with complete operators that is described in Section 19.1 carries with it important connections between notions and algebraic constructions for relational structures and corresponding notions and algebraic constructions for algebras. In fact, every notion or construction for structures corresponds to a dual notion or construction for algebras, and vice versa. In this section, we describe some features of this duality that are relevant for our subsequent discussion. A more comprehensive and detailed development of duality theory for Boolean algebras with operators may be found in [39].

We begin with one aspect of the duality between the structure-preserving mappings. The structure-preserving mappings for complete and atomic Boolean algebras with complete operators are complete homomorphisms (see Section 7.4). The structure-preserving mappings for relational structures are bounded homomorphisms.

Definition 19.31. A *homomorphism* from a relational structure $\mathfrak{U}$ to a relational structure $\mathfrak{V}$ is a function ϑ from the universe of $\mathfrak{U}$ to the universe of $\mathfrak{V}$ that preserves the fundamental relations of these structures in the sense that the following implications hold for all elements u, v, and w in $\mathfrak{U}$.

(i) $R(u, v, w)$ implies $R(\vartheta(u), \vartheta(v), \vartheta(w))$.
(ii) $C(u, w)$ implies $C(\vartheta(u), \vartheta(w))$.
(iii) $I(w)$ implies $I(\vartheta(w))$.

A homomorphism ϑ is said to be *bounded* if the following implications hold for all elements r and s in $\mathfrak{V}$ and all elements w in $\mathfrak{U}$.

(iv) $R(r, s, \vartheta(w))$ implies there are elements u and v in $\mathfrak{U}$ such that

$$\vartheta(u) = r, \qquad \vartheta(v) = s, \qquad \text{and} \qquad R(u, v, w).$$

(v) $C(r, \vartheta(w))$ implies there is an element u in $\mathfrak{U}$ such that

$$\vartheta(u) = r \qquad \text{and} \qquad C(u, w).$$

(vi) $I(\vartheta(w))$ implies $I(w)$.

A *monomorphism* is a homomorphism that is one-to-one, and an *epimorphism* is a homomorphism that is onto. □

Warning: a homomorphism that is a bijection is not necessarily an isomorphism, as in the case of algebras. For a bijective homomorphism ϑ on a relational structure to be an isomorphism, the implications in the first part of the definition above must be replaced by equivalences, that is to say, the word "implies" in the first part of the definition must be replaced by the phrase "if and only if". Notice that for a bounded monomorphism, the equivalences in the first part of the definition do hold. Consequently, a bounded homomorphism that is a bijection is an isomorphism.

Complete and atomic Boolean algebras with complete operators are, up to isomorphisms, just complex algebras of relational structures, by Corollaries 19.4 and 19.6. Consequently, we may focus our attention on complex algebras of relational structures instead of arbitrary complete and atomic Boolean algebras with complete operators. This greatly facilitates the formulation of the results in the next few sections.

Every bounded homomorphism from a relational structure $\mathfrak{U}$ to a relational structure $\mathfrak{V}$ induces a complete homomorphism from the complex algebra $\mathfrak{Cm}(V)$ to the complex algebra $\mathfrak{Cm}(U)$ as follows.

Theorem 19.32. *If $\mathfrak{U}$ and $\mathfrak{V}$ are relational structures, and ϑ a bounded homomorphism from $\mathfrak{U}$ to $\mathfrak{V}$, then the function φ defined on subsets X of $\mathfrak{V}$ by*

$$\varphi(X) = \vartheta^{-1}(X) = \{u \in U : \vartheta(u) \in X\}$$

is a complete homomorphism from $\mathfrak{Cm}(V)$ to $\mathfrak{Cm}(U)$. Moreover, φ is one-to-one if and only if ϑ is onto, and φ is onto if and only if ϑ is one-to-one.

Proof. Straightforward computations show that the function φ defined in the statement of the theorem is a complete Boolean homomorphism. For instance, if X is a subset of $\mathfrak{V}$, then

$$\begin{aligned} u \in {\sim}\vartheta^{-1}(X) \quad & \text{if and only if} \quad u \notin \vartheta^{-1}(X), \\ & \text{if and only if} \quad \vartheta(u) \notin X, \\ & \text{if and only if} \quad \vartheta(u) \in {\sim}X, \\ & \text{if and only if} \quad u \in \vartheta^{-1}({\sim}X), \end{aligned}$$

by the definitions of the complement of a set and the inverse image of a set under a function. Combine these equivalences with the definition of φ to conclude that

$$\varphi({\sim}X) = \vartheta^{-1}({\sim}X) = {\sim}\vartheta^{-1}(X) = {\sim}\varphi(X).$$

Thus, φ preserves complements.

The proof that φ preserves arbitrary unions is similar. If $(X_i : i \in J)$ is a system of subsets of $\mathfrak{V}$, then

$$\begin{aligned} u \in \vartheta^{-1}(\textstyle\bigcup_i X_i) \quad & \text{if and only if} \quad \vartheta(u) \in \textstyle\bigcup_i X_i, \\ & \text{if and only if} \quad \vartheta(u) \in X_i \text{ for some } i, \\ & \text{if and only if} \quad u \in \vartheta^{-1}(X_i) \text{ for some } i, \\ & \text{if and only if} \quad u \in \textstyle\bigcup_i \vartheta^{-1}(X_i), \end{aligned}$$

by the definitions of the union of a system and the inverse image of a set under a function. Consequently,

$$\varphi(\textstyle\bigcup_i X_i) = \vartheta^{-1}(\bigcup_i X_i) = \bigcup_i \vartheta^{-1}(X_i) = \bigcup_i \varphi(X_i),$$

so φ preserves arbitrary unions.

The distinguished element 1' in the complex algebra $\mathfrak{Cm}(V)$, respectively $\mathfrak{Cm}(U)$, is the set X, respectively Y, defined by

$$X = \{r \in V : I(r)\} \qquad \text{and} \qquad Y = \{u \in U : I(u)\},$$

by Definition 19.2. To prove that φ preserves 1', it must be shown that $\varphi(X) = Y$, or what amounts to the same thing, that

$$\vartheta^{-1}(X) = Y. \tag{1}$$

If w belongs to Y, then the relation $I(w)$ holds in $\mathfrak{U}$, by the definition of Y, and therefore $I(\vartheta(w))$ holds in $\mathfrak{V}$, by the homomorphism properties of ϑ. This means that $\vartheta(w)$ belongs to X, by the definition of X, so w is in $\vartheta^{-1}(X)$. On the other hand, if w belongs to $\vartheta^{-1}(X)$, then $\vartheta(w)$ belongs to X, and therefore $I(\vartheta(w))$ holds, by the definition of X. Since the homomorphism ϑ is assumed to be bounded, we obtain $I(w)$, so that w belongs to Y, by the definition of Y. This proves (1).

Turn next to the proof that φ preserves the operator $;$. It is to be shown that for arbitrary subsets X and Y of $\mathfrak{V}$,

$$\varphi(X\,;Y) = \varphi(X)\,;\varphi(Y),$$

or what amounts to the same thing, that

$$\vartheta^{-1}(X\,;Y) = \vartheta^{-1}(X)\,;\vartheta^{-1}(Y). \tag{2}$$

Assume first that an element w belongs to the right side of (2). In this case, there must be elements u in $\vartheta^{-1}(X)$ and v in $\vartheta^{-1}(Y)$ such that $R(u, v, w)$ holds in $\mathfrak{U}$, by the definition of the operator ; in the complex algebra $\mathfrak{Cm}(U)$ (see Definition 19.2). Consequently, the element $\vartheta(u)$ belongs to X, and $\vartheta(v)$ belongs to Y, by the definition of the inverse image of a set. The homomorphism properties of ϑ imply that $R(\vartheta(u), \vartheta(v), \vartheta(w))$ holds in $\mathfrak{V}$. Combine these observations to conclude that $\vartheta(w)$ is in $X ; Y$, by the definition of the operator ; in the complex algebra $\mathfrak{Cm}(V)$, and therefore w belongs to the left side of (2).

Assume now that w belongs to the left side of (2). In this case, the element $\vartheta(w)$ belongs to the set $X ; Y$, by the definition of the inverse image of a set, so there must be elements r in X and s in Y such that $R(r, s, \vartheta(w))$ holds in $\mathfrak{V}$, by the definition of the operator ; in $\mathfrak{Cm}(V)$. The homomorphism ϑ is assumed to be bounded, so there are elements u and v in $\mathfrak{U}$ such that

$$\vartheta(u) = r, \qquad \vartheta(v) = s, \qquad \text{and} \qquad R(u, v, w) \tag{3}$$

in $\mathfrak{U}$. In particular, the elements u and v belong to the sets $\vartheta^{-1}(X)$ and $\vartheta^{-1}(Y)$ respectively, because r is in X and s in Y. Combine these observations to conclude that w belongs to the right side of (2), by (3) and the definition of the operator ; in $\mathfrak{Cm}(U)$.

The proof that φ preserves the operator $\smile$ is similar and is left as an exercise. This completes the proof of the first assertion of the theorem.

Turn now to the second assertion of the theorem. Assume first that ϑ is one-to-one, with the aim of showing that φ is onto. Given an arbitrary subset Y of $\mathfrak{U}$, write

$$X = \vartheta(Y) = \{\vartheta(u) : u \in Y\},$$

and observe that

$$\varphi(X) = \vartheta^{-1}(X) = Y,$$

by the definition of φ and the assumption that ϑ is one-to-one. Consequently, φ maps $\mathfrak{Cm}(V)$ onto $\mathfrak{Cm}(U)$.

Assume next that φ is onto, with the aim of showing that ϑ is one-to-one. To this end, consider distinct elements u and v in $\mathfrak{U}$. Since φ is onto, there must be subsets X and Y of $\mathfrak{V}$ such that

$$\varphi(X) = \vartheta^{-1}(X) = \{u\} \qquad \text{and} \qquad \varphi(Y) = \vartheta^{-1}(Y) = \{v\}.$$

Thus, u is the unique element in $\mathfrak{V}$ that is mapped by ϑ into the set X, and v is the unique element in $\mathfrak{V}$ that is mapped by ϑ into the set Y. In particular, since v is assumed to be different from u, the element u is not mapped by ϑ to any element in Y, while v *is* mapped by ϑ to an element in Y. Consequently, we must have $\vartheta(u) \neq \vartheta(v)$, so ϑ is one-to-one. The observations of this paragraph and the preceding one show that the function ϑ is one-to-one if and only if the function φ is onto.

Suppose now that φ is one-to-one, with the aim of showing that ϑ is onto. Given any element r in $\mathfrak{V}$, observe that the image set

$$\varphi(\{r\}) = \vartheta^{-1}(\{r\})$$

cannot be empty, since φ is a monomorphism and therefore maps zero, and only zero, to zero. Consequently, there is an element u in $\mathfrak{U}$ that belongs to this image, so $\vartheta(u) = r$, by the definition of the inverse image of a set. Conclusion: ϑ maps $\mathfrak{U}$ onto $\mathfrak{V}$.

Assume finally that ϑ is onto, with the aim of showing that φ is one-to-one. To this end, consider distinct subsets X and Y of $\mathfrak{V}$. There must be an element belonging to one of these sets but not to the other, say r belongs to X but not to Y. Since ϑ is onto, there is an element u in $\mathfrak{U}$ such that $\vartheta(u) = r$. The element u belongs to the inverse image $\vartheta^{-1}(X)$, because r is in X; but u cannot belong to the inverse image $\vartheta^{-1}(Y)$, because $\vartheta(u) = r$ and r is not in Y. Consequently,

$$\varphi(X) = \vartheta^{-1}(X) \neq \vartheta^{-1}(Y) = \varphi(Y),$$

so φ is one-to-one. The arguments in this paragraph and the previous one show that the function ϑ is onto if and only if the function φ is one-to-one. □

The complete homomorphism φ defined in the preceding theorem is usually called the (*first*) *dual* of the given bounded homomorphism ϑ. A converse of the preceding theorem is also true: every complete homomorphism φ between complex algebras gives rise to a dual bounded homomorphism that is one-to-one or onto just in case φ is onto or one-to-one respectively. As this result will not be needed in the subsequent development, it is left as an exercise.

There is an important connection between bounded homomorphisms and substructures of relational structures.

Definition 19.33. A *substructure* of a relational structure $\mathfrak{V}$ is a relational structure $\mathfrak{U}$ such that the universe of $\mathfrak{U}$ is a subset of the universe of $\mathfrak{V}$, and the fundamental relations of $\mathfrak{U}$ are the restrictions of the corresponding fundamental relations of $\mathfrak{V}$ in the sense that the following equivalences hold for all elements r, s, and t in $\mathfrak{U}$.

(i) $R(r, s, t)$ in $\mathfrak{U}$ if and only if $R(r, s, t)$ in $\mathfrak{V}$.
(ii) $C(r, t)$ in $\mathfrak{U}$ if and only if $C(r, t)$ in $\mathfrak{V}$.
(iii) $I(t)$ in $\mathfrak{U}$ if and only if $I(t)$ in $\mathfrak{V}$.

The structure $\mathfrak{U}$ is also called the *restriction* of $\mathfrak{V}$ to the set U that is the universe of $\mathfrak{U}$. A substructure $\mathfrak{U}$ is said to be an *inner substructure* if the following implications hold for all elements r and s in $\mathfrak{V}$ and all elements t in $\mathfrak{U}$.

(iv) $R(r, s, t)$ in $\mathfrak{V}$ implies that r and s are in $\mathfrak{U}$.
(v) $C(r, t)$ implies r is in $\mathfrak{U}$. □

This definition is so constructed that inner substructures are just images of bounded monomorphisms.

Lemma 19.34. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be relational structures. If ϑ is a bounded homomorphism from $\mathfrak{U}$ into $\mathfrak{V}$, then the restriction of $\mathfrak{V}$ to the range of ϑ is an inner substructure of $\mathfrak{V}$. In the reverse direction, if ϑ is an isomorphism from $\mathfrak{U}$ to an inner substructure of $\mathfrak{V}$, then ϑ is a bounded monomorphism from $\mathfrak{U}$ into $\mathfrak{V}$.*

Proof. Suppose first that ϑ is a bounded homomorphism from $\mathfrak{U}$ into $\mathfrak{V}$, and let $\mathfrak{W}$ be the restriction of $\mathfrak{V}$ to the range of ϑ. Clearly, $\mathfrak{W}$ is a substructure of $\mathfrak{V}$, by the definition of a restriction. To prove that $\mathfrak{W}$ is actually an inner substructure, consider elements r and s in $\mathfrak{V}$, and t in $\mathfrak{W}$. Since ϑ maps $\mathfrak{U}$ onto $\mathfrak{W}$, there must be an element w in $\mathfrak{U}$ such that $\vartheta(w) = t$. If $R(r, s, t)$ holds in $\mathfrak{V}$, then obviously

$$R(r, s, \vartheta(w)) \tag{1}$$

holds in $\mathfrak{V}$, so there are elements u and v in $\mathfrak{U}$ such that

$$\vartheta(u) = r, \qquad \vartheta(v) = s, \qquad \text{and} \qquad R(u, v, w),$$

by the assumption that the homomorphism ϑ is bounded. It follows that r and s belong to the image of $\mathfrak{U}$ under ϑ, which is just $\mathfrak{W}$. A similar argument shows that if $C(r, t)$ holds in $\mathfrak{V}$, then r belongs

to $\mathfrak{W}$, so $\mathfrak{W}$ satisfies the conditions in Definition 19.33 for being an inner substructure of $\mathfrak{V}$.

Assume now that ϑ an isomorphism from $\mathfrak{U}$ to an inner substructure $\mathfrak{W}$ of $\mathfrak{V}$. To prove that ϑ is bounded, consider elements r and s in $\mathfrak{V}$, and w in $\mathfrak{U}$, such that (1) holds in $\mathfrak{V}$. The element $\vartheta(w)$ belongs to $\mathfrak{W}$, and $\mathfrak{W}$ is an inner substructure of $\mathfrak{U}$, so the elements r and s must belong to $\mathfrak{W}$, by Definition 19.33. Since ϑ maps $\mathfrak{U}$ onto $\mathfrak{W}$, there must be elements u and v in $\mathfrak{U}$ such that $\vartheta(u) = r$ and $\vartheta(v) = s$. Obviously, $R(\vartheta(u), \vartheta(v), \vartheta(w))$ holds in $\mathfrak{V}$, and therefore also in $\mathfrak{W}$, by (1), so $R(u, v, w)$ holds in $\mathfrak{U}$, by the isomorphism properties of ϑ. It follows that ϑ is bounded with respect to the relation R, by Definition 19.31. Completely analogous arguments show that ϑ is bounded with respect to the relations C and I, so ϑ is a bounded monomorphism. □

By taking for ϑ the identity function on the universe of $\mathfrak{V}$ in the preceding lemma, we arrive at the following characterization of inner substructures.

Corollary 19.35. *A relational structure $\mathfrak{U}$ is an inner substructure of a relational structure $\mathfrak{V}$ if and only if the identity function on the universe of $\mathfrak{U}$ is a bounded monomorphism from $\mathfrak{U}$ into $\mathfrak{V}$.*

There is a duality between unions of relational structures and direct products of complex algebras that we shall also need. To describe it, we must first define what is meant by the (disjoint) union of a system of relational structures.

Definition 19.36. The *union* of a system $(\mathfrak{U}_i : i \in J)$ of mutually disjoint relational structures $\mathfrak{U}_i = (U_i\,, R_i\,, C_i\,, I_i)$ is defined to be the relational structure $\mathfrak{U} = (U\,, R\,, C\,, I)$ in which

$$U = \bigcup_i U_i, \qquad R = \bigcup_i R_i, \qquad C = \bigcup_i C_i, \qquad I = \bigcup_i I_i.$$

The structures $\mathfrak{U}_i$ are called the *components* of the union.

The *disjoint union* of an arbitrary system $(\mathfrak{V}_i : i \in J)$ of relational structures is defined to be the union of the disjoint system $(\mathfrak{U}_i : i \in J)$ in which $\mathfrak{U}_i$ is the isomorphic image of $\mathfrak{V}_i$ under the function that maps each element r in $\mathfrak{V}_i$ to the element (r, i). □

The second half of this definition means that in order to create the union of an arbitrary system $(\mathfrak{V}_i : i \in J)$ of relational structures, one first creates an isomorphic copy $\mathfrak{U}_i$ of each structure $\mathfrak{V}_i$ in such a way

that the resulting system $(\mathfrak{U}_i : i \in J)$ of structures is disjoint, and then one forms the union of this disjoint system. In the particular version of this construction given in the definition above, the universe of $\mathfrak{U}_i$ is the set of elements

$$U_i = \{(r, i) : r \in V_i\},$$

and the fundamental relations in $\mathfrak{U}_i$ are defined to be the images of the fundamental relations in $\mathfrak{V}_i$ under the function that maps each element r in $\mathfrak{V}_i$ to the pair (r, i). For example, the relation R_i in $\mathfrak{U}_i$ is defined to hold for a triple $((r, i), (s, i), (t, i))$ of elements from U_i just in case the corresponding relation in $\mathfrak{V}_i$ holds for the corresponding triple (r, s, t) of elements from V_i, and similarly for the relations C_i and I_i in $\mathfrak{U}_i$.

The notions of the external and internal direct products of a system of Boolean algebras with operators are defined exactly as in the case of relation algebras (see Chapter 11, and in particular, Definition 11.34). To give a concrete example, consider a system $(\mathfrak{U}_i : i \in J)$ of mutually disjoint relational structures. By the *internal product* of the associated system $(\mathfrak{Cm}(U_i) : i \in J)$ of complex algebras, we understand the algebra

$$\mathfrak{A} = (A\,,\, +\,,\, -\,,\, ;\,,\, {}^\smile\,,\, ,\, 1')$$

in which the universe A consists of the sets of the form $X = \bigcup_i X_i$, where X_i is an element in $\mathfrak{Cm}(U_i)$ (that is to say, X_i is a subset of U_i) for each i in J, and in which the operations are defined as follows: if

$$X = \bigcup_i X_i \qquad \text{and} \qquad Y = \bigcup_i Y_i$$

are elements in A, then

$$\begin{aligned} X + Y &= \bigcup_i (X_i + Y_i), & -X &= \bigcup_i -X_i, \\ X\,;Y &= \bigcup_i (X_i\,;Y_i), & X^\smile &= \bigcup_i X_i^\smile, \end{aligned}$$

and $1' = \bigcup_i 1_i'$. (The operations and distinguished constants on the left sides of these equations are those being defined in $\mathfrak{A}$, while the ones on the right, after the union symbol, are the operations and distinguished constants of the complex algebras $\mathfrak{Cm}(U_i)$.

It turns out that the internal product $\mathfrak{A}$ defined in the preceding paragraph coincides with the complex algebra of the relational structure $\mathfrak{U}$ that is the union of the system of structures $(\mathfrak{U}_i : i \in J)$. To see this, observe first that each element in $\mathfrak{A}$ belongs to $\mathfrak{Cm}(U)$. Indeed, each element in $\mathfrak{A}$ is a set X that is the union of a system of

sets $(X_i : i \in J)$, with X_i a subset of $\mathfrak{U}_i$ for each i, by the definition of $\mathfrak{A}$. The universe of $\mathfrak{U}_i$ is included in the universe of the union $\mathfrak{U}$, so each of the sets X_i in the given system is a subset of $\mathfrak{U}$ and therefore belongs to the complex algebra $\mathfrak{Cm}(U)$. Since $\mathfrak{Cm}(U)$ is closed under arbitrary unions, it follows that X belongs to $\mathfrak{Cm}(U)$. Conversely, each element in $\mathfrak{Cm}(U)$ belongs to $\mathfrak{A}$. In fact, since the structures $\mathfrak{U}_i$ are assumed to be mutually disjoint and have $\mathfrak{U}$ as their union, each element in $\mathfrak{Cm}(U)$, that is to say, each subset X of $\mathfrak{U}$, can be represented in exactly one way as a union $X = \bigcup_i X_i$, where X_i is a subset of $\mathfrak{U}_i$ for each index i, namely the subset $X_i = X \cap U_i$. It follows that X belongs to $\mathfrak{A}$, by the definition of $\mathfrak{A}$. Conclusion: the algebras $\mathfrak{A}$ and $\mathfrak{Cm}(U)$ have the same universe.

To see that the operations of the two algebras are also the same, suppose

$$X = \bigcup_i X_i \qquad \text{and} \qquad Y = \bigcup_i Y_i$$

are the unique representations of two sets X and Y in $\mathfrak{Cm}(U)$. The sum of X and Y in $\mathfrak{Cm}(U)$ is, by definition, the union of these two sets, and for each index i, the sum of the sets X_i and Y_i in $\mathfrak{Cm}(U_i)$ is the union of these two sets, so the operation of addition in $\mathfrak{Cm}(U)$ must be the same as it is in $\mathfrak{A}$. In more detail,

$$X + Y = X \cup Y = (\bigcup_i X_i) \cup (\bigcup_i Y_i) \\ = \bigcup_i (X_i \cup Y_i) = \bigcup_i (X_i + Y_i)$$

(where the left-most addition is performed in $\mathfrak{Cm}(U)$, and the right-most addition is performed in $\mathfrak{Cm}(U_i)$), and this is precisely the definition of addition in $\mathfrak{A}$. Consequently, the sum $X + Y$ is the same in $\mathfrak{Cm}(U)$ as it is in $\mathfrak{A}$. A similar argument, using the disjointness of the sets U_i, shows that the complement $-X$ is the same in $\mathfrak{Cm}(U)$ as it is in $\mathfrak{A}$.

Consider next the operator $;$. The value of $X ; Y$ in $\mathfrak{Cm}(U)$ is, by definition, the set of elements t such that $R(r, s, t)$ holds in $\mathfrak{U}$ for some r in X and some s in Y, and the value of $X_i ; Y_i$ in $\mathfrak{Cm}(U_i)$ is, by definition, the sets of elements t such that $R_i(r, s, t)$ holds in $\mathfrak{U}_i$ for some r in X_i and some s in Y_i. The relation R in $\mathfrak{U}$ is defined to be the union of the relations R_i from $\mathfrak{U}_i$, so if t belongs to $X ; Y$, and if r and s are elements in X and Y respectively such that $R(r, s, t)$ holds in $\mathfrak{U}$, then there must be an index i such that $R_i(r, s, t)$ holds in $\mathfrak{U}_i$. In this case, r and s must belong to the sets

$$X \cap U_i = X_i \qquad \text{and} \qquad Y \cap U_i = Y_i$$

respectively, so t belongs to the set $X_i \,;\, Y_i$ (formed in $\mathfrak{Cm}(U_i)$). Conversely, if t belongs to one of the sets $X_i \,;\, Y_i$, then $R_i(r, s, t)$ holds in $\mathfrak{U}_i$ for some r in X_i and s in Y_i. The elements r and s of course belong to the sets X and Y respectively, by the definitions of these sets, and $R(r, s, t)$ holds in $\mathfrak{U}$, by the definition of the relation R, so t must belong to the product $X \,;\, Y$. Thus,

$$X \,;\, Y = \bigcup_i X_i \,;\, Y_i$$

(where the operation on the left is performed in $\mathfrak{Cm}(U)$, and those on the right, after the union symbol, are performed in $\mathfrak{Cm}(U_i)$), and this is precisely the definition of $X \,;\, Y$ in $\mathfrak{A}$. Consequently, the value of $X \,;\, Y$ is the same in $\mathfrak{Cm}(U)$ as it is in $\mathfrak{A}$. Similar arguments show that the value of $X^\smile$ is the same in $\mathfrak{Cm}(U)$ as it is in $\mathfrak{A}$, and that the element 1' is the same in $\mathfrak{Cm}(U)$ as it is in $\mathfrak{A}$. Conclusion: $\mathfrak{A} = \mathfrak{Cm}(U)$. The following theorem has been proved.

Theorem 19.37. *The internal product of a system of complex algebras of mutually disjoint relational structures is equal to the complex algebra of the union of the relational structures.*

Using the preceding theorem, it is easy to arrive at a general conclusion about the duality between unions of relational structures and direct products of the corresponding complex algebras.

Corollary 19.38. *The external product of an arbitrary system of complex algebras of relational structures is canonically isomorphic to the complex algebra of the disjoint union of the relational structures.*

In other words, the dual relational structure of a direct product of complex algebras is just the disjoint union of the underlying relational structures, and vice versa.

19.6 Ultraproducts of structures

In addition to the duality between direct products and disjoint unions, there is a parallel duality for ultraproducts, a special case of which was already stated in Lemma 18.32 and exploited in Theorem 18.33 to

prove that the variety of representable relation algebras is not finitely axiomatizable.

Before discussing this duality, we prove that an ultraproduct of a system of Boolean algebras with complete operators is again a Boolean algebra with complete operators. The result is needed in order to conclude that completions of ultraproducts of Boolean algebras with complete operators actually do exist.

Lemma 19.39. *Let $(\mathfrak{A}_i : i \in J)$ be a non-empty system of Boolean algebras with complete operators, and write $\mathfrak{A} = \prod_i \mathfrak{A}_i$. For every ultrafilter D on the index set J, the ultraproduct $\mathfrak{A}/D$ is a Boolean algebra with complete operators.*

Proof. The property of being a Boolean algebra with operators is expressible by means of a set of first-order sentences, and is therefore preserved under the passage to ultraproducts, by the Fundamental Theorem of Ultraproducts (Theorem 18.6). Consequently, $\mathfrak{A}/D$ is certainly a Boolean algebra with operators. In particular, each of the operators in $\mathfrak{A}/D$ is monotone, by Lemma 2.3. It must be shown that each of the operators in this ultraproduct is in fact complete.

Consider, as an example, the case of a binary operator $;$. The key idea is that there is a first-order formula Γ with three free variables (say, v_1, v_2, and v_3) that is satisfied by a triple of elements (s, t, p) in a Boolean algebra with the operator $;$ just in case t is the supremum of the set

$$\{r : r \leq t \text{ and } r ; s \leq p\}, \tag{1}$$

and dually, there is a first-order formula Δ with three free variables that is satisfied by (s, t, p) just in case t is the supremum of the set

$$\{r : r \leq t \text{ and } s ; r \leq p\}.$$

Since t is obviously an upper bound of the set (1), it suffices for Γ to express that any upper bound of the set (1) is necessarily above t. In other words, Γ must say that for all elements r, if $r \leq t$ and $r ; s \leq p$ always implies that $r \leq q$, then $t \leq q$. Proceeding more formally, we may take $\Gamma(v_1, v_2, v_3)$ to be the formula

$$\forall v_4 \big[\forall v_0 \big([(v_0 \leq v_2) \wedge (v_0 ; v_1 \leq v_3)] \rightarrow (v_0 \leq v_4)\big) \rightarrow (v_2 \leq v_4)\big].$$

The formula Δ is defined in a similar fashion.

Let X be a subset of $\mathfrak{A}$ such that the set X/D of quotients of elements in X has a supremum in $\mathfrak{A}/D$, say

$$t/D = \sum(X/D) = \sum\{r/D : r \in X\}. \tag{2}$$

The goal is to prove that for each element s in $\mathfrak{A}$,

$$t/D \,;\, s/D = \sum\{r/D \,;\, s/D : r \in X\} \tag{3}$$

and

$$s/D \,;\, t/D = \sum\{s/D \,;\, r/D : r \in X\} \tag{4}$$

in $\mathfrak{A}/D$. Focus on the proof of (3).

It follows from (2) that $r/D \leq t/D$ for each r in X, and therefore

$$r/D \,;\, s/D \leq t/D \,;\, s/D,$$

for each such r, by the monotony of the operator $;$ in $\mathfrak{A}/D$. Thus, the element on the left side of (3) is an upper bound for the set on the right.

To prove that the element on the left side of (3) is the least upper bound of the set on the right, consider an arbitrary upper bound p/D of the set on the right. It is to be shown that

$$t/D \,;\, s/D \leq p/D. \tag{5}$$

Observe first that the triple $(s/D, t/D, p/D)$ satisfies the formula Γ in $\mathfrak{A}/D$, that is to say,

$$t/D = \sum\{r/D : r/D \leq t/D \text{ and } r/D \,;\, s/D \leq p/D\}. \tag{6}$$

In more detail, t/D is clearly an upper bound of the set on the right side of (6). Furthermore, every element in X/D belongs to the set on the right; indeed, if r is in X, then

$$r/D \leq t/D \qquad \text{and} \qquad r/D \,;\, s/D \leq p/D,$$

by (2) and the assumption that p/D is an upper bound on the set of the right side of (3). It follows that any upper bound of the set on the right side of (6) must also be an upper bound of the set X/D, and must therefore be above the least upper bound of X/D, which is t/D, by (2). This proves (6).

Because $(s/D, t/D, p/D)$ satisfies the formula Γ in $\mathfrak{A}/D$, the set of indices

$$K = \{i \in J : (s(i), t(i), p(i)) \text{ satisfies } \Gamma \text{ in } \mathfrak{A}_i\}$$

belongs to the ultrafilter D, by the Fundamental Theorem of Ultraproducts. Thus, for each i in K, we have

$$t(i) = \sum\{u : u \in A_i \text{ and } u \leq t(i) \text{ and } u \,;\, s(i) \leq p(i)\} \tag{7}$$

in $\mathfrak{A}_i$, by the definition of Γ. As r varies over all elements in $\mathfrak{A}$, the projection $r(i)$ varies over all elements u in $\mathfrak{A}_i$. Consequently, (7) may be rewritten in the form

$$t(i) = \sum\{r(i) : r \in A \text{ and } r(i) \leq t(i) \text{ and } r(i) \,;\, s(i) \leq p(i)\}. \tag{8}$$

The operator ; is assumed to be complete in $\mathfrak{A}_i$, so (8) implies that

$$t(i) \,;\, s(i) = \sum\{r(i) \,;\, s(i) : r \in A \text{ and } r(i) \leq t(i) \text{ and } r(i) \,;\, s(i) \leq p(i)\} \tag{9}$$

for each i in K. Obviously, $p(i)$ is an upper bound for the set on the right side of (9), so (9) implies that

$$t(i) \,;\, s(i) \leq p(i) \tag{10}$$

for each i in K. Because K belongs to D, the set of indices in J for which (10) is true must also belong to D, by the upward closure of D. Apply the Fundamental Theorem of Ultraproducts once again to conclude that (5) holds. This completes the proof of (3).

An entirely analogous argument, using the formula Δ instead of Γ, establishes (4). Conclusion: the operator ; in $\mathfrak{A}/D$ is complete. The completeness of the other operators in $\mathfrak{A}/D$ is established in a similar fashion. □

To formulate a duality theorem for ultraproducts, it is necessary to extend the notions of direct product and ultraproduct to systems of relational structures. The definitions are similar to those for algebras (see Sections 11.6 and 18.2). The (*external*) *direct product* of a system $(\mathfrak{U}_i : i \in J)$ of relational structures

$$\mathfrak{U}_i = (U_i \,,\, R_i \,,\, C_i \,,\, I_i)$$

is the relational structure

$$\mathfrak{U} = (U \,,\, R \,,\, C \,,\, I)$$

in which the universe U is the Cartesian product of the sets U_i for i in J, and the fundamental relations are defined coordinatewise. For

example, the relation R is defined to hold for a triple (r, s, t) of elements in $\mathfrak{U}$ if and only if, for each index i in J, the relation R_i holds for the triple $(r(i), s(i), t(i))$ of elements in $\mathfrak{U}_i$. The relations C and I in $\mathfrak{U}$ are defined in an analogous fashion. The same notation is used to denote a direct product of relational structures as is used to denote a direct product of algebras, namely

$$\mathfrak{U} = \prod_i \mathfrak{U}_i.$$

Turn now to the definition of an ultraproduct. Let $(\mathfrak{U}_i : i \in J)$ be a non-empty system of relational structures (that is to say, the set J is not empty), let $\mathfrak{U}$ be the direct product of this system, and let D be an ultrafilter on the index set J. The relation $\equiv_D$ on the set

$$U = \prod_i U_i$$

that is defined at the beginning of Section 18.2 is easily seen to be a congruence relation on $\mathfrak{U}$. The ultraproduct of the given system modulo the ultrafilter D is defined to be the quotient of $\mathfrak{U}$ modulo the congruence relation $\equiv_D$. Its universe consists of the congruence classes r/D of the elements in $\mathfrak{U}$, and its fundamental relation R is defined to hold for a triple of elements $(r/D, s/D, t/D)$ if and only if the set

$$\{i \in J : R_i(r(i), s(i), t(i))\}$$

belongs to the ultrafilter D. The relations C and I are defined in an analogous fashion. The ultraproduct of the system is denoted by $(\prod_i \mathfrak{U}_i)/D$, or more simply, by $\mathfrak{U}/D$. The basic results about ultraproducts of algebras, including the Fundamental Theorem of Ultraproducts (Theorem 18.6) continue to hold for ultraproducts of relational structures, with only minor modifications in the proofs.

The next theorem—a generalization of Lemma 18.32—says that the dual algebra of an ultraproduct of a system of relational structures is, up to isomorphism, the completion of the ultraproduct of the complex algebras of the individual relational structures in the system.

Theorem 19.40. *Let $(\mathfrak{U}_i : i \in J)$ be a non-empty system of non-empty relational structures, and write*

$$\mathfrak{U} = \prod_i \mathfrak{U}_i \qquad \textit{and} \qquad \mathfrak{A} = \prod_i \mathfrak{Cm}(U_i).$$

For every ultrafilter D on the index set J, the completion of the ultraproduct $\mathfrak{A}/D$ is isomorphic to the complex algebra $\mathfrak{Cm}(U/D)$.

Proof. We begin with some preliminary observations. Each of the complex algebras $\mathfrak{Cm}(U_i)$ is a complete and atomic Boolean algebra with complete operators, by Corollary 19.4, so the ultraproduct $\mathfrak{A}/D$ is a Boolean algebra with complete operators, by Lemma 19.39. This ultraproduct is also atomic, because the property of being atomic is expressible by a first-order sentence and is therefore preserved under the passage to ultraproducts. In fact, the atoms are the elements of the form t/D, where t is an element in $\mathfrak{A}$ with the property that the set

$$J_t = \{i \in J : t(i) \text{ is an atom in } \mathfrak{Cm}(U_i)\} \tag{1}$$

belongs to the ultrafilter D, by the Fundamental Theorem of Ultraproducts. The atoms in $\mathfrak{Cm}(U_i)$ are the singletons of elements in $\mathfrak{U}_i$. If the singletons of elements are identified with the elements themselves, then the atoms in $\mathfrak{Cm}(U_i)$ may be thought of as the elements in U_i.

The identification of elements with their singletons implies that each element

$$r = (r(i) : i \in J)$$

in the product set $U = \prod_i U_i$ is identified with the element

$$\hat{r} = (\{r(i)\} : i \in J) \tag{2}$$

in $\mathfrak{A}$, and therefore the quotient r/D in $\mathfrak{U}/D$ may be identified with the quotient $\hat{r}/D$ in $\mathfrak{A}/D$. In other words, every element in $\mathfrak{U}/D$ may be thought of as an atom in $\mathfrak{A}/D$. Conversely, every atom in $\mathfrak{A}/D$ has the form $\hat{r}/D$ for some element r in $\mathfrak{U}$. To see this, consider an arbitrary atom t in $\mathfrak{A}/D$, and observe that the set J_t defined in (1) belongs to the ultrafilter D. Take r to be the element in $\mathfrak{U}$ that is defined as follows. If i is an index in J_t, then $t(i)$ is an atom in $\mathfrak{Cm}(U_i)$, and therefore the singleton of some element in the structure $\mathfrak{U}_i$, by (1); define $r(i)$ to be this element. If i is an index that is not in J_t, then define $r(i)$ to be some arbitrary element in $\mathfrak{U}_i$; this is possible because the structure $\mathfrak{U}_i$ is assumed to be non-empty. The set

$$\{i \in J : t(i) = \{r(i)\} = \hat{r}(i)\}$$

includes the set J_t, by the definition of r and (2), and therefore belongs to the ultrafilter D, by the upward closure of D. Consequently, t and $\hat{r}$ are congruent modulo D and therefore $t/D = \hat{r}/D$. Thus, the atoms in $\mathfrak{A}/D$ are precisely the elements of the form $\hat{r}/D$ for r in $\mathfrak{U}$.

For any two elements r and s in $\mathfrak{U}$, we have

$$r(i) = s(i) \qquad \text{if and only if} \qquad \hat{r}(i) = \hat{s}(i),$$

by (2), so the sets of indices i for which these equations hold are the same; therefore, one of these sets belongs to D if and only if the other belongs to D. It follows that

$$r/D = s/D \qquad \text{if and only if} \qquad \hat{r}/D = \hat{s}/D,$$

where these quotients are formed in $\mathfrak{U}/D$ and $\mathfrak{A}/D$ respectively. Combine the observations in this and the preceding paragraph to conclude that the function mapping each element r/D in $\mathfrak{U}/D$ to the atom $\hat{r}/D$ in $\mathfrak{A}/D$ is a bijection from $\mathfrak{U}/D$ to the set of atoms in $\mathfrak{A}/D$. Thus, we can and will identify elements in $\mathfrak{U}/D$ with atoms in $\mathfrak{A}/D$.

In a similar vein, the atoms in the complex algebra $\mathfrak{Cm}(U/D)$ are the singletons of elements in $\mathfrak{U}/D$. If singletons of elements are identified with the elements themselves, then the atoms in $\mathfrak{Cm}(U/D)$ may be thought of as just the elements in $\mathfrak{U}/D$. In other words, we can and will identify atoms $\{r/D\}$ in $\mathfrak{Cm}(U/D)$ with elements r/D in U/D. Combine this observation with the one at the end of the preceding paragraph to conclude that the atoms in both $\mathfrak{A}/D$ and $\mathfrak{Cm}(U/D)$ may be identified with the quotients r/D of element r in $\mathfrak{U}$.

Turn now to the proof of the theorem. Define a function φ from the set of atoms in $\mathfrak{A}/D$ to the set of atoms in $\mathfrak{Cm}(U/D)$ as follows: for each r in U, put

$$\varphi(\hat{r}/D) = \{r/D\},$$

or, more informally for notational simplicity, using the identifications discussed in the previous paragraphs,

$$\varphi(r/D) = r/D, \tag{3}$$

where (as we shall say) the first quotient is formed in $\mathfrak{A}/D$ and the second in $\mathfrak{Cm}(U/D)$. It follows from the remarks above that φ is a well-defined bijection between the respective sets of atoms.

The next step is to prove that φ preserves the operators $;$ and $^\smile$, and the distinguished element 1', with respect to atoms, in the sense of the Atomic Isomorphism Theorem 7.11. Focus on the operator $;$. Let r, s, and t be elements in $\mathfrak{U}$. The inequality

$$t/D \leq r/D \,;\, s/D \tag{4}$$

holds in $\mathfrak{A}/D$ if and only if the set

$$\{i \in J : t(i) \leq r(i)\,;\, s(i)\}$$

belongs to the ultrafilter D, by the Fundamental Theorem of Ultraproducts. This set is equal to the set

$$\{i \in J : R_i(r(i), s(i), t(i))\}, \tag{5}$$

by the definition of the operations ; in the complex algebras $\mathfrak{Cm}(U_i)$, so (4) holds in $\mathfrak{A}/D$ if and only if (5) belongs to D. On the other hand, (4) holds in the complex algebra $\mathfrak{Cm}(U/D)$ if and only if

$$R(r/D, s/D, t/D) \tag{6}$$

holds in $\mathfrak{U}/D$, by the definition of the operation ; in the complex algebra $\mathfrak{Cm}(U/D)$. Use the definition of the relation R in $\mathfrak{U}/D$ to see that (6) holds in $\mathfrak{U}/D$ if and only if (5) belongs to D. Combine these observations to conclude that (4) holds in $\mathfrak{A}/D$ if and only if it holds in $\mathfrak{Cm}(U/D)$, that is to say,

$$t/D \leq r/D\,;\, s/D \quad \text{if and only if} \quad \varphi(t/D) \leq \varphi(r/D)\,;\, \varphi(s/D).$$

Similar but easier arguments show that

$$\begin{aligned} t/D \leq (r/D)^\smile \quad &\text{if and only if} \quad \varphi(t/D) \leq \varphi(r/D)^\smile, \\ t/D \leq 1' \quad &\text{if and only if} \quad \varphi(t/D) \leq 1'. \end{aligned}$$

The details are left as an exercise.

Let $\mathfrak{B}$ be the completion of the ultraproduct $\mathfrak{A}/D$. This completion exists and is a complete and atomic Boolean algebra with complete operators, by Lemma 19.39, Theorem 15.22, and Lemma 15.29. Moreover, it has the same atoms as $\mathfrak{A}/D$ (see Lemma 15.29), and the values of the operators in $\mathfrak{B}$ on atoms coincide with the values of the corresponding operators in $\mathfrak{A}/D$ on atoms. It follows that φ is a bijection from the set of atoms in $\mathfrak{B}$ to the set of atoms in $\mathfrak{Cm}(U/D)$, and the equivalences in the preceding paragraph continue to hold for the algebras $\mathfrak{B}$ and $\mathfrak{Cm}(U/D)$. Apply Theorem 7.11 to conclude that φ can be extended to an isomorphism from $\mathfrak{B}$ to $\mathfrak{Cm}(U/D)$. □

The next lemma implies that an ultraproduct of homomorphic images, or bounded homomorphic images, of a given system of relational structures is a homomorphic image, or a bounded homomorphic image, of the ultraproduct of the given system; and an ultraproduct of

substructures, or inner substructures, of the given system is isomorphic to a substructure, or an inner substructure, of the ultraproduct of the given system.

Lemma 19.41. *Let* $(\mathfrak{U}_i : i \in J)$ *and* $(\mathfrak{V}_i : i \in J)$ *be non-empty systems of relational structures, and write*

$$\mathfrak{U} = \prod_i \mathfrak{U}_i \qquad \text{and} \qquad \mathfrak{V} = \prod_i \mathfrak{V}_i.$$

If for each index i *in* J *there is a homomormphism, respectively a bounded homomorphism, from* $\mathfrak{U}_i$ *into* $\mathfrak{V}_i$*, then for every ultrafilter* D *on* J*, there is a canonically defined homomorphism, respectively bounded homomorphism, from the ultraproduct* $\mathfrak{U}/D$ *into the ultraproduct* $\mathfrak{V}/D$*. The lemma remains true if "homomorphism" is replaced throughout by "monomorphism", "epimorphism", or "isomorphism".*

Proof. For each index i, let ϑ_i be a homomorphism from $\mathfrak{U}_i$ into $\mathfrak{V}_i$. Define a mapping ϑ from $\mathfrak{U}/D$ to $\mathfrak{V}/D$ as follows: for each element u in $\mathfrak{U}$, let r be the element in $\mathfrak{V}$ determined by

$$r(i) = \vartheta_i(u(i))$$

for each i in J, and put

$$\vartheta(u/D) = r/D. \tag{1}$$

If u and v are elements in $\mathfrak{U}$, then the set

$$\{i \in J : u(i) = v(i)\} \tag{2}$$

is included in the set

$$\{i \in J : \vartheta_i(u(i)) = \vartheta_i(v(i))\}. \tag{3}$$

Consequently, if the first set belongs to the ultrafilter D, then so does the second, by the upward closure of D, and therefore

$$u/D = v/D \qquad \text{implies} \qquad \vartheta(u/D) = \vartheta(v/D). \tag{4}$$

It follows that the mapping ϑ is well defined.

Suppose that each of the homomorphisms ϑ_i is one-to-one. The sets (2) and (3) are then equal, so if the set (3) belongs to D, then the set (2) does too. Consequently, the reverse implication holds in (4),

and therefore ϑ is one-to-one. Similarly, suppose each of the homomorphisms ϑ_i is onto. For each element r in $\mathfrak{V}$, and each index i, there is then an element u_i in $\mathfrak{U}_i$ such that $\vartheta_i(u_i) = r(i)$. Let u be the element in $\mathfrak{U}$ determined by $u(i) = u_i$ for each i, and apply the definition in (1), to see that ϑ must map u/D to r/D. Consequently, ϑ is onto.

To see that ϑ preserves the ternary relation R, consider elements u, v, and w in $\mathfrak{U}$, and let r, s, and t be the elements in $\mathfrak{V}$ determined by

$$r(i) = \vartheta_i(u(i)), \qquad s(i) = \vartheta_i(v(i)), \qquad t(i) = \vartheta_i(w(i)), \tag{5}$$

for i in J. Thus,

$$\vartheta(u/D) = r/D, \qquad \vartheta(v/D) = s/D, \qquad \vartheta(w/D) = t/D, \tag{6}$$

by the definition of the mapping ϑ in (1). Assume

$$R(u/D, v/D, w/D) \tag{7}$$

holds in $\mathfrak{U}/D$, with the goal of showing that

$$R(r/D, s/D, t/D) \tag{8}$$

holds in $\mathfrak{V}/D$. The set

$$\{i \in J : R(u(i), v(i), w(i)) \text{ in } \mathfrak{U}_i\} \tag{9}$$

must belong to the ultrafilter D, by assumption (7) and the definition of the ultraproduct $\mathfrak{U}/D$. The set (9) is included in the set

$$\{i \in J : R(\vartheta_i(u(i)), \vartheta_i(v(i)), \vartheta_i(w(i))) \text{ in } \mathfrak{V}_i\},$$

because each mapping ϑ_i is assumed to be a homomorphism. In view of (5), this means that (9) is included in the set

$$K = \{i \in J : R(r(i), s(i), t(i)) \text{ in } \mathfrak{V}_i\}. \tag{10}$$

Since (9) belongs to D, it follows that K belongs to D, by the upward closure of D, and therefore (8) holds in $\mathfrak{V}/D$, by the definition of the ultraproduct $\mathfrak{V}/D$. Analogous arguments show that ϑ preserves the relations C and I, so ϑ is a homomorphism from $\mathfrak{U}/D$ into $\mathfrak{V}/D$.

Suppose now that each homomorphism ϑ_i is bounded. To show that the homomorphism ϑ is bounded, consider elements r and s in $\mathfrak{V}$, and w in $\mathfrak{U}$, such that

$$R(r/D, s/D, \vartheta(w/D)) \tag{11}$$

holds in $\mathfrak{V}/D$. Let t be the element in $\mathfrak{V}$ determined by the final equation in (5), so that the final equation in (6) holds, by (1). From (11) we see that (8) holds in $\mathfrak{V}/D$, and consequently the set K in (10) must belong to the ultrafilter D, by the definition of the ultraproduct $\mathfrak{V}/D$. For each index i in K, the homomorphism ϑ_i is assumed to be bounded, so there must be elements u_i and v_i in $\mathfrak{U}_i$ such that

$$\vartheta_i(u_i) = r(i), \qquad \vartheta_i(v_i) = s(i), \qquad \text{and} \qquad R(u_i, v_i, w(i)) \tag{12}$$

holds in $\mathfrak{U}_i$, by (10) and the final part of (5). Let u and v be elements in the product $\mathfrak{U}$ such that

$$u(i) = u_i \qquad \text{and} \qquad v(i) = v_i \tag{13}$$

for i in K, and $u(i)$ and $v(i)$ are chosen to be some arbitrary fixed element in $\mathfrak{U}_i$ when i is in $J \sim K$. Thus,

$$R(u(i), v(i), w(i))$$

holds in $\mathfrak{U}_i$ for each i in K, by (13) and the final part of (12), so the set (9) includes the set K and therefore belongs to the ultrafilter D, by the upward closure of D. Consequently, (7) holds in $\mathfrak{U}/D$, by the definition of this ultraproduct. Also, each of the sets

$$\{i \in J : r(i) = \vartheta_i(u(i))\} \qquad \text{and} \qquad \{i \in J : s(i) = \vartheta_i(v(i))\} \tag{14}$$

includes the set K, by (13) and the first part of (12), so each of these sets must also belong to D. If r' and s' are the elements in $\mathfrak{V}$ determined by

$$r'(i) = \vartheta_i(u(i)) \qquad \text{and} \qquad s'(i) = \vartheta_i(v(i)) \tag{15}$$

for each i in J, then the sets

$$\{i \in J : r'(i) = r(i)\} \qquad \text{and} \qquad \{i \in J : s'(i) = s(i)\} \tag{16}$$

respectively include the sets in (14), by (15), and therefore belong to D, by the upward closure of D. It follows from (16) and the definition of congruence modulo D that

$$r'/D = r/D \qquad \text{and} \qquad s'/D = s/D. \tag{17}$$

Therefore,

$$\vartheta(u/D) = r'/D = r/D \qquad \text{and} \qquad \vartheta(v/D) = s'/D = s/D, \tag{18}$$

by the definition of ϑ in (1), and (17). Conclusion: ϑ is bounded with respect to the relation R, by (18) and (7) . Entirely analogous arguments show that ϑ is bounded with respect to the relations C and I, so ϑ is a bounded homomorphism.

The final assertion of the lemma holds by the remarks in the second paragraph of the proof. □

By considering a system of substructures or inner substructures in the preceding lemma, we arrive at the following conclusion.

Corollary 19.42. *Let* $(\mathfrak{U}_i : i \in J)$ *and* $(\mathfrak{V}_i : i \in J)$ *be non-empty systems of relational structures, and write*

$$\mathfrak{U} = \prod_i \mathfrak{U}_i \qquad \textit{and} \qquad \mathfrak{V} = \prod_i \mathfrak{V}_i.$$

If $\mathfrak{U}_i$ *is a substructure, respectively an inner substructure, of* $\mathfrak{V}_i$ *for each index* i*, then for every ultrafilter* D *on* J*, the ultraproduct* $\mathfrak{U}/D$ *is canonically isomorphic to a substructure, respectively an inner substructure, of the ultraproduct* $\mathfrak{V}/D$*.*

Proof. Suppose $\mathfrak{U}_i$ is a substructure, respectively an inner substructure, of $\mathfrak{V}_i$ for each index i. The identity function on $\mathfrak{U}_i$—call it ϑ_i—is then a monomorphism, respectively a bounded monomorphism, from $\mathfrak{U}_i$ into $\mathfrak{V}_i$, by Corollary 19.35. Apply Lemma 19.41 to conclude that the function ϑ defined in the proof of the lemma is a monomorphism, respectively a bounded monomorphism, from $\mathfrak{U}/D$ into $\mathfrak{V}/D$. Consequently, $\mathfrak{U}/D$ is isomorphic to a substructure, respectively an inner substructure, of $\mathfrak{V}/D$, by Lemma 19.34. □

It is worth remarking that the monomorphism ϑ involved in the proof of the preceding corollary maps each element r/D in $\mathfrak{U}/D$ to the element r/D in $\mathfrak{V}/D$, but ϑ is not in general the identity function on $\mathfrak{U}/D$. The reason is that although each element r in the product $\mathfrak{U}$ does belong to the product $\mathfrak{V}$, the quotient r/D formed in $\mathfrak{U}/D$ is different from the quotient r/D formed in $\mathfrak{V}/D$. The former quotient is included in the latter, but in general the inclusion is proper, because

the quotient r/D in $\mathfrak{V}/D$ may contain elements from $\mathfrak{V}$ that do not occur in $\mathfrak{U}$ and therefore do not belong to the quotient r/D in $\mathfrak{U}/D$.

We need the following specialized consequence of Corollary 19.42.

Corollary 19.43. *Suppose* $(\mathfrak{U}_k : k \in K)$ *is a non-empty, disjoint system of relational structures, and* $\mathfrak{U}$ *the union of this system. For each non-empty set* J*, each ultrafilter* D *on* J*, and each function* j *from* J *into* K*, the ultraproduct* $(\prod_{i\in J} \mathfrak{U}_{j_i})/D$ *is canonically isomorphic to an inner substructure of the ultrapower* $\mathfrak{U}^J/D$*.*

Proof. Let $(\mathfrak{V}_i : i \in J)$ be the system of relational structures determined by

$$\mathfrak{V}_i = \mathfrak{U} \tag{1}$$

for each index i in J. Clearly, for any given ultrafilter D on J,

$$(\prod_{i\in J} \mathfrak{V}_i)/D = \mathfrak{U}^J/D, \tag{2}$$

by the definition of an ultrapower.

The definition of the union of a system of disjoint relational structures implies that each component $\mathfrak{U}_k$ of the union is an inner substructure of the union, which is $\mathfrak{U}$. Consequently, if j is a function from J into K, then $\mathfrak{U}_{j_i}$ is an inner substructure of $\mathfrak{V}_i$ for each i in J, by (1). Apply Corollary 19.42 to conclude that the ultraproduct $(\prod_{i\in J} \mathfrak{U}_{j_i})/D$ is canonically isomorphic to an inner substructure of the ultraproduct $(\prod_{i\in J} \mathfrak{V}_i)/D$ via the function that maps each element r/D in the first ultraproduct to the corresponding element r/D in the second ultraproduct. In view of (2), this is just the desired result. □

The next lemma is a natural analogue of Lemma 19.41 that applies to algebras. We leave the details of the proof as an exercise.

Lemma 19.44. *Let* $(\mathfrak{A}_i : i \in J)$ *and* $(\mathfrak{B}_i : i \in J)$ *be non-empty systems of Boolean algebras with operators, and write*

$$\mathfrak{A} = \prod_i \mathfrak{A}_i \qquad \text{and} \qquad \mathfrak{B} = \prod_i \mathfrak{B}_i.$$

If for each index i *there is a homomorphism from* $\mathfrak{A}_i$ *into* $\mathfrak{B}_i$*, then for every ultrafilter* D *on* J*, there is a canonically defined homomorphism from the ultraproduct* $\mathfrak{A}/D$ *into the ultraproduct* $\mathfrak{B}/D$*. The lemma remains true if "homomorphism" is replaced throughout by "monomorphism", "epimorphism", or "isomorphism".*

Just as in the case of relational structures, the preceding lemma can be applied to conclude that an ultraproduct of subalgebras of a given system of Boolean algebras with operators is canonically isomorphic to a subalgebra of the ultraproduct of the given system. Again, the proof is left as an exercise.

Corollary 19.45. *Let* $(\mathfrak{A}_i : i \in J)$ *and* $(\mathfrak{B}_i : i \in J)$ *be non-empty systems of Boolean algebras with operators, and write*

$$\mathfrak{A} = \prod_i \mathfrak{A}_i \qquad \text{and} \qquad \mathfrak{B} = \prod_i \mathfrak{B}_i.$$

If $\mathfrak{A}_i$ *is a subalgebra of* $\mathfrak{B}_i$ *for each index* i*, then for every ultrafilter* D *on* J*, the ultraproduct* $\mathfrak{A}/D$ *is canonically isomorphic to a subalgebra of the ultraproduct* $\mathfrak{B}/D$*.*

The next two results are more specialize. The first says that a disjoint union of bounded homomorphisms is a bounded homomorphism.

Lemma 19.46. *Let* $\mathfrak{U}$ *be the union of a disjoint system of relational structures* $(\mathfrak{U}_k : k \in K)$*, and* $\mathfrak{V}$ *an arbitrary relational structure. If* ϑ_k *is a bounded homomorphism from* $\mathfrak{U}_k$ *into* $\mathfrak{V}$ *for each index* k*, then the union of this system of bounded homomorphisms is a bounded homomorphism from* $\mathfrak{U}$ *into* $\mathfrak{V}$*.*

Proof. Let ϑ be the union of the given system $(\vartheta_k : k \in K)$ of bounded homomorphisms. The relational structures in the system $(\mathfrak{U}_k : k \in K)$ are assumed to be disjoint, so ϑ is a well-defined mapping from the union $\mathfrak{U}$ into $\mathfrak{V}$.

To prove that ϑ is a bounded homomorphism, focus on the case of the ternary relation R. Consider elements u, v, and w in $\mathfrak{U}$, and assume that $R(u, v, w)$ holds in $\mathfrak{U}$. This relationship must then hold in $\mathfrak{U}_k$ for some index k, by Definition 19.36. Therefore, $R(\vartheta_k(u), \vartheta_k(v), \vartheta_k(w))$ must hold in $\mathfrak{V}$, by the assumption that ϑ_k is a homomorphism from $\mathfrak{U}_k$ into $\mathfrak{V}$. It follows that $R(\vartheta(u), \vartheta(v), \vartheta(w))$ holds in $\mathfrak{V}$, because ϑ agrees with ϑ_k on elements in $\mathfrak{U}_k$, by the definition of ϑ. Conclusion: ϑ preserves the relation R.

To see that ϑ is bounded with respect to R, consider elements r and s in $\mathfrak{V}$, and w in $\mathfrak{U}$, and assume that $R(r, s, \vartheta(w))$ holds in $\mathfrak{V}$. The element w must belong to $\mathfrak{U}_k$ for some index k, by the definition of $\mathfrak{U}$ as the union structure, and ϑ agrees with ϑ_k on elements in $\mathfrak{U}_k$, so $R(r, s, \vartheta_k(w))$ holds in $\mathfrak{V}$. The homomorphism ϑ_k is assumed to be bounded, so there must be elements u and v in $\mathfrak{U}_k$ such that

$$\vartheta_k(u) = r, \qquad \vartheta_k(v) = s, \qquad \text{and} \qquad R(u, v, w) \tag{1}$$

in $\mathfrak{U}_k$. The elements u and v obviously belong to the union $\mathfrak{U}$, and ϑ agrees with ϑ_k on elements from $\mathfrak{U}_k$, so (1) holds with ϑ_k and $\mathfrak{U}_k$ replaced by ϑ and $\mathfrak{U}$ respectively. Conclusion: ϑ is bounded with respect to the relation R.

Entirely analogous arguments show that ϑ is a bounded homomorphism with respect to the relations C and I, so ϑ is a bounded homomorphism. □

The final result in this section says, roughly speaking, that every ultrapower of the union of a disjoint system of relational structures is the bounded homomorphic image of a disjoint union of ultraproducts of relational structures from the given system. It plays an important role in the proof of Theorem 19.54, and hence also in the proof of Theorem 19.55.

Lemma 19.47. *Let $\mathfrak{U}$ be the union of a disjoint system of relational structures $(\mathfrak{U}_k : k \in K)$. Every ultrapower of $\mathfrak{U}$ is a bounded homomorphic image of a disjoint union of relational structures, each of which is an ultraproduct of some of the relational structures in the given system.*

Proof. Let J be a non-empty set, and consider an arbitrary element r in the direct power $\mathfrak{U}^J$. For each index i in J, the ith-coordinate $r(i)$ belongs to the union $\mathfrak{U}$, by the choice of r, and therefore $r(i)$ belongs to $\mathfrak{U}_k$ for some k in K, since $\mathfrak{U}$ is the union of the structures $\mathfrak{U}_k$. Choose an index k for which this is true, and put $j_i = k$. The resulting function

$$j = (j_i : i \in J)$$

from J into K has the property that $r(i)$ belongs to the structure $\mathfrak{U}_{j_i}$ for each i in J. Consequently, r belongs to the direct product

$$\mathfrak{V}_r = \prod_{i \in J} \mathfrak{U}_{j_i}$$

of these structures.

Fix an ultrafilter D on the index set J. By Corollary 19.43, the ultraproduct $\mathfrak{V}_r/D$ is canonically isomorphic to an inner substructure of the ultrapower $\mathfrak{U}^J/D$ via a bounded monomorphism ϑ_r that maps each element s/D in $\mathfrak{V}_r/D$ to the corresponding element s/D in $\mathfrak{U}^J/D$, for s in $\mathfrak{V}_r$. (Regarding the difference between the two quotients s/D, see the remark following Corollary 19.42.) Form the disjoint union $\mathfrak{W}$ of the system of ultraproducts

$$(\mathfrak{V}_r/D : r \in \mathfrak{U}^J).$$

The disjoint union ϑ of the corresponding system $(\vartheta_r : r \in \mathfrak{U}^J)$ of bounded monomorphisms is a bounded homomorphism from the union structure $\mathfrak{W}$ into the ultrapower $\mathfrak{U}^J/D$, by Lemma 19.46. Each element r in $\mathfrak{U}^J$ belongs to the product $\mathfrak{V}_r$, by the observations of the first paragraph, and therefore (a copy of) the quotient r/D in $\mathfrak{V}_r/D$ belongs to the union $\mathfrak{W}$. Since

$$\vartheta(r/D) = \vartheta_r(r/D) = r/D,$$

by the definitions of mappings ϑ and ϑ_r, it follows that ϑ maps $\mathfrak{W}$ onto $\mathfrak{U}^J/D$, so ϑ is a bounded epimorphism. □

19.7 Universal classes closed under canonical extensions

As we have seen, classes of complex algebras—for example, complex algebras of groups and complex algebras of projective geometries—play an important role in the theory of relation algebras. Two natural problems arise. The first is to describe the algebras that belong to a variety or universal class generated by a class of complex algebras. The second is to determine whether the generated class is *closed under canonical extensions* in the sense that the canonical extension of every algebra in the generated class also belongs to the generated class. For example, what are the algebras that belong to the variety generated by the class of group complex algebras, or to the variety generated by the class of geometric complex algebras? What are the simple algebras in these varieties? Is each of these varieties closed under canonical extensions? We address these questions in this and the next two sections. In particular, we prove an important general theorem regarding the closure under canonical extensions of certain varieties and universal classes of Boolean algebras with operators that are generated by complex algebras. The focus in this section is on universal classes.

The immediate goal is to prove that an arbitrary relational structure $\mathfrak{W}$ has an elementary extension $\mathfrak{V}$ with the following property: the atom structure of each and every subalgebra of $\mathfrak{Cm}(W)$ is a bounded homomorphic image of the structure $\mathfrak{V}$. From this result and Theorem 19.32, it follows that the canonical extension of every subalgebra of $\mathfrak{Cm}(W)$ can be completely embedded into the complex alge-

bra $\mathfrak{Cm}(V)$. This is the key to analyzing the closure under canonical extensions of various classes of Boolean algebras with operators.

Fix a relational structure

$$\mathfrak{W} = (W\,,\, R\,,\, C\,,\, I),$$

and define a new relational structure $\mathfrak{W}^*$ by adjoining to the fundamental relations of $\mathfrak{W}$ every subset of its universe as a new unary relation. If, as usual, unary relations are identified with sets, then we may write

$$\mathfrak{W}^* = (W\,,\, R\,,\, C\,,\, I\,,\, X)_{X\subseteq W}.$$

Let $\mathcal{L}^*$ be a language of first-order logic appropriate for structures of the same similarity type as $\mathfrak{W}^*$. The non-logical constants of this language are a ternary relation symbol $\boldsymbol{R}$, a binary relation symbol $\boldsymbol{C}$, a unary relation symbol $\boldsymbol{I}$, and a unary relation symbol $\boldsymbol{X}$ for each subset X of W. We shall also have occasion to adjoin to $\mathcal{L}^*$ new individual constant symbols as needed. In particular, for each element r in $\mathfrak{W}^*$, we adjoin to $\mathcal{L}^*$ a new individual constant symbol $\boldsymbol{r}$ and we assume that $\boldsymbol{r}$ is always interpreted as the element r in appropriate extensions of $\mathfrak{W}^*$. The logical symbols of $\mathcal{L}^*$ are the same as those of the other first-order languages that we have encountered: there is an infinite supply of variables v_0, v_1, v_2, ..., and there are the standard symbols for disjunction, conjunction, negation, and universal and existential quantification. The standard syntactical and semantical notions for $\mathcal{L}^*$ are defined as usual (see Section 2.4). We shall write, for example, $\mathfrak{V}^*$ for relational structures that are appropriate for the language $\mathcal{L}^*$, the idea being that $\mathfrak{V}^*$ is obtained by adding new unary relations to a relational structure $\mathfrak{V}$ of the same similarity type as $\mathfrak{W}$.

We say that a set $\mathcal{S}$ of formulas in $\mathcal{L}^*$ (or in some extension of $\mathcal{L}^*$ obtained by adjoining new individual constant symbols) with free variables among $v_0, \ldots, v_{n-1}$ is *consistent with* $\mathfrak{W}^*$ if for every finite subset $\mathcal{S}_0$ of $\mathcal{S}$, there is a sequence $r = (r_0, \ldots, r_{n-1})$ of elements in $\mathfrak{W}^*$ that satisfies every formula of $\mathcal{S}_0$ in the structure $\mathfrak{W}^*$ (possibly expanded by adjoining new distinguished constants as interpretations of individual constant symbols). We shall say that $\mathcal{S}$ is *satisfied* in an extension $\mathfrak{V}^*$ of $\mathfrak{W}^*$ if there is a sequence r of n elements in $\mathfrak{V}^*$ such that r satisfies every formula of $\mathcal{S}$ in the structure $\mathfrak{V}^*$.

Let $(F_i : i \in J)$ be a system of sets of subsets of W. Thus, each set F_i in the system, being a collection of subsets of W, is a subset of the complex algebra $\mathfrak{Cm}(W)$. In terms of this system, define sets of

formulas as follows. For every index i in J, let $\mathcal{F}_i$ be the set of atomic formulas (with the single variable v_0) corresponding to the sets in F_i, so that

$$\mathcal{F}_i = \{\boldsymbol{X}(v_0) : X \in F_i\}.$$

For every index i in J and every individual constant symbol $\boldsymbol{t}$ that we may adjoin to $\mathcal{L}^*$, let $\mathcal{G}_{it}$ be the set of formulas obtained by adjoining the single formula $\boldsymbol{C}(v_0, \boldsymbol{t})$ to $\mathcal{F}_i$, so that

$$\mathcal{G}_{it} = \{\boldsymbol{X}(v_0) : X \in F_i\} \cup \{\boldsymbol{C}(v_0, \boldsymbol{t})\}.$$

Finally, for every pair of indices i and j in J, and every individual constant symbol $\boldsymbol{t}$ that we may adjoin to $\mathcal{L}^*$, let $\mathcal{H}_{ijt}$ be the set of formulas (with at most two variables, v_0 and v_1) obtained from $\mathcal{F}_i$ and $\mathcal{F}_j$ by first replacing the variable v_0 with the variable v_1 in every formula of $\mathcal{F}_j$, and then adjoining the formula $\boldsymbol{R}(v_0, v_1, \boldsymbol{t})$ to the union of the resulting sets, so that

$$\mathcal{H}_{ijt} = \{\boldsymbol{X}(v_0) : X \in F_i\} \cup \{\boldsymbol{Y}(v_1) : Y \in F_j\} \cup \{\boldsymbol{R}(v_0, v_1, \boldsymbol{t})\}.$$

If in every formula of $\mathcal{F}_i$, the variable v_0 is replaced by an individual constant symbol $\boldsymbol{r}$, the resulting set of sentences will be denoted by $\mathcal{F}_i(\boldsymbol{r})$, so that

$$\mathcal{F}_i(\boldsymbol{r}) = \{\boldsymbol{X}(\boldsymbol{r}) : X \in F_i\}.$$

Similarly, $\mathcal{G}_{it}(\boldsymbol{r})$ denotes the set of sentences obtained from $\mathcal{G}_{it}$ by replacing everywhere the variable v_0 with the individual constant symbol $\boldsymbol{r}$, and $\mathcal{H}_{ijt}(\boldsymbol{r}, \boldsymbol{s})$ denotes the set of sentences obtained from $\mathcal{H}_{ijt}$ by replacing everywhere the variables v_0 and v_1 with the individual constant symbols $\boldsymbol{r}$ and $\boldsymbol{s}$ respectively.

The construction of the desired elementary extension of $\mathfrak{W}$ proceeds by induction on the natural numbers n. The next lemma contains the key argument that allows us to proceed from stage n to stage $n+1$ of the construction.

Lemma 19.48. *For every relational structure $\mathfrak{W}$ and every system of sets of subsets $(F_i : i \in J)$ of $\mathfrak{W}$, there exists an elementary extension $\mathfrak{V}^*$ of $\mathfrak{W}^*$ with the following properties.*

(i) *For each index i in J, if the set of formulas $\mathcal{F}_i$ is consistent with $\mathfrak{W}^*$, then it is satisfied in $\mathfrak{V}^*$.*

(ii) *For each index i in J and each element t in $\mathfrak{W}^*$, if the set of formulas $\mathcal{G}_{it}$ is consistent with $\mathfrak{W}^*$, then it is satisfied in $\mathfrak{V}^*$.*

(iii) *For each pair of indices i and j in J, and each element t in $\mathfrak{W}^*$, if the set of formulas $\mathcal{H}_{ijt}$ is consistent with $\mathfrak{W}^*$, then it is satisfied in $\mathfrak{V}^*$.*

Proof. The proof involves a compactness argument. Let $\mathcal{T}$ be the elementary diagram of $\mathfrak{W}^*$. Thus, $\mathcal{T}$ consists of all sentences of the form $\Gamma(\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1})$, where Γ is a formula in $\mathcal{L}^*$ with free variables among $v_0, \ldots, v_{n-1}$, while $(r_0, \ldots, r_{n-1})$ is a sequence of elements in $\mathfrak{W}^*$ that satisfies Γ, and $\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1}$ are the individual constant symbols adjoined to $\mathcal{L}^*$ to represent the elements $r_0, \ldots, r_{n-1}$ respectively. Every model of $\mathcal{T}$ has an elementary substructure that is isomorphic to $\mathfrak{W}^*$ via the function that maps each element r in $\mathfrak{W}^*$ to the interpretation of the corresponding individual constant symbol $\boldsymbol{r}$ in the model, by Lemma 7.17 and its proof.

Let $\mathcal{S}$ be the set of all sentences obtained as follows. For every index i in J, if the set of formulas $\mathcal{F}_i$ is consistent with $\mathfrak{W}^*$, then adjoin a brand new individual constant symbol $\boldsymbol{r}$ to the language $\mathcal{L}^*$, and adjoin all of the sentences in $\mathcal{F}_i(\boldsymbol{r})$ to the set $\mathcal{S}$. Similarly, for every index i in J and every individual constant symbol $\boldsymbol{t}$ representing an element t in $\mathfrak{W}^*$, if the set of formulas $\mathcal{G}_{it}$ is consistent with $\mathfrak{W}^*$, then adjoin a brand new individual constant symbol $\boldsymbol{r}$ to the language $\mathcal{L}^*$, and adjoin all of the sentences in $\mathcal{G}_{it}(\boldsymbol{r})$ to the set $\mathcal{S}$. Finally, for every pair of indices i and j in J, and every individual constant symbol $\boldsymbol{t}$ representing an element t in $\mathfrak{W}^*$, if the set of formulas $\mathcal{H}_{ijt}$ is consistent with $\mathfrak{W}^*$, then adjoin two brand new individual constant symbols $\boldsymbol{r}$ and $\boldsymbol{s}$ to the language $\mathcal{L}^*$, and adjoin all of the sentences in $\mathcal{H}_{ijt}(\boldsymbol{r}, \boldsymbol{s})$ to the set $\mathcal{S}$.

Every finite subset of the set of sentences $\mathcal{T} \cup \mathcal{S}$ has a model. In fact, every such set is true in some expansion of $\mathfrak{W}^*$ in which the individual constant symbols representing elements in $\mathfrak{W}^*$ are interpreted by these elements, and the other new individual constants are interpreted by some appropriate elements in $\mathfrak{W}^*$. For example, suppose the set $\mathcal{F}_i(\boldsymbol{r})$ has been included in $\mathcal{S}$ (where $\boldsymbol{r}$ is a new individual constant symbol). Let $\Gamma(\boldsymbol{r})$ be the conjunction of the sentences in some given finite subset of $\mathcal{F}_i(\boldsymbol{r})$. Since the set $\mathcal{F}_i$ is assumed to be consistent with $\mathfrak{W}^*$, by the definition of $\mathcal{S}$, there must be an element p in $\mathfrak{W}^*$ that satisfies $\Gamma(v_0)$ in $\mathfrak{W}^*$, by the definition of what it means for a set of formulas to be consistent with $\mathfrak{W}^*$. Interpret the new constant symbol $\boldsymbol{r}$ as the element p in $\mathfrak{W}^*$, and observe that $\Gamma(\boldsymbol{r})$ is true in $\mathfrak{W}^*$ under this interpretation of $\boldsymbol{r}$. Similar remarks apply to all finite subsets of all those sets $\mathcal{G}_{it}(\boldsymbol{r})$ and $\mathcal{H}_{ijt}(\boldsymbol{r}, \boldsymbol{s})$ that have been included in $\mathcal{S}$. The set $\mathcal{T}$ is

defined to be the elementary diagram of $\mathfrak{W}^*$, so each sentence in $\mathcal{T}$ is obviously true in the expansion of $\mathfrak{W}^*$ in which each individual constant symbol $\boldsymbol{t}$ representing a specified element t in $\mathfrak{W}^*$ is interpreted as t.

Since every finite subset of the set of sentences $\mathcal{T} \cup \mathcal{S}$ has a model, the Compactness Theorem for first-order logic (see Section 2.4) implies that the entire set of sentences has a model. The sentences in $\mathcal{T}$ are all true in this model, so the restriction of this model to the similarity type of $\mathfrak{W}^*$ has an elementary substructure that is isomorphic to $\mathfrak{W}^*$, by the remarks at the end of the first paragraph of the proof. An appropriate version of the Exchange Principle (see Theorem 7.15) implies that there is a model of $\mathcal{T} \cup \mathcal{S}$ with the property that the restriction of this model to the similarity type of $\mathfrak{W}^*$—call this restriction $\mathfrak{V}^*$—has $\mathfrak{W}^*$ itself as an elementary substructure. The sentences in $\mathcal{S}$ are also true in the model. If any of the sets of formulas listed in (i)–(iii) is consistent with $\mathfrak{W}^*$, then the corresponding set of sentences (in which the variables have been replaced by new individual constant symbols) is included in $\mathcal{S}$ and is therefore true in the model. Consequently, every such set of formulas is satisfied in $\mathfrak{V}^*$. □

The preceding lemma will now be used in an inductive way to build a suitable elementary extension of $\mathfrak{W}^*$. The difference between the next lemma and the preceding one is that the elements t giving rise to the sets of formulas $\mathcal{G}_{it}$ and $\mathcal{H}_{ijt}$ are allowed to vary over all of $\mathfrak{V}^*$, and not just over $\mathfrak{W}^*$, and consequently the sets of formulas are assumed to be consistent with $\mathfrak{V}^*$ instead of with $\mathfrak{W}^*$.

Lemma 19.49. *For every relational structure $\mathfrak{W}$ and every system of sets of subsets $(F_i : i \in J)$ of $\mathfrak{W}$, there is an elementary extension $\mathfrak{V}^*$ of $\mathfrak{W}^*$ with the following properties.*

(i) *For each index i in J, if the set of formulas $\mathcal{F}_i$ is consistent with $\mathfrak{V}^*$ (or with $\mathfrak{W}^*$), then it is satisfied in $\mathfrak{V}^*$.*
(ii) *For each index i in J and each element t in $\mathfrak{V}^*$, if the set of formulas $\mathcal{G}_{it}$ is consistent with $\mathfrak{V}^*$, then it is satisfied in $\mathfrak{V}^*$.*
(iii) *For each pair of indices i and j in J, and each element t in $\mathfrak{V}^*$, if the set of formulas $\mathcal{H}_{ijt}$ is consistent with $\mathfrak{V}^*$, then it is satisfied in $\mathfrak{V}^*$.*

Proof. The plan is to construct a system of relational structures $\mathfrak{V}^*_n$, indexed by the natural numbers n, that possesses the following properties. (1) $\mathfrak{V}^*_0 = \mathfrak{W}^*$. (2) $\mathfrak{V}^*_{n+1}$ is an elementary extension of $\mathfrak{V}^*_n$. (3) For

each index i in J, if the set of formulas $\mathcal{F}_i$ is consistent with $\mathfrak{V}_n^*$, then it is satisfied in $\mathfrak{V}_{n+1}^*$. (4) For each index i in J and each element t in $\mathfrak{V}_n^*$, if the set of formulas $\mathcal{G}_{it}$ is consistent with $\mathfrak{V}_n^*$, then it is satisfied in $\mathfrak{V}_{n+1}^*$. (5) For each pair of indices i and j in J, and each element t in $\mathfrak{V}_n^*$, if the set of formulas $\mathcal{H}_{ijt}$ is consistent with $\mathfrak{V}_n^*$, then it is satisfied in $\mathfrak{V}_{n+1}^*$.

For the base case of the construction, put $\mathfrak{V}_0^* = \mathfrak{W}^*$. For the induction step of the construction, assume $\mathfrak{V}_0^*, \ldots, \mathfrak{V}_n^*$ have been constructed so that properties (1)–(5) hold for all natural numbers less than n. Apply Lemma 19.48 with $\mathfrak{V}_n^*$ in place of $\mathfrak{W}^*$ to obtain an elementary extension $\mathfrak{V}_{n+1}^*$ of $\mathfrak{V}_n^*$ satisfying conditions (3)–(5).

Let $\mathfrak{V}^*$ be the union of the system of relational structures $\mathfrak{V}_n^*$ just constructed. The union of a chain of elementary extensions is an elementary extension of each member of the chain, by the version of Lemma 6.32 that applies to arbitrary systems of relational structures. In particular, $\mathfrak{V}^*$ is an elementary extension of $\mathfrak{V}_0^*$, which is just $\mathfrak{W}^*$, by property (1). For each index i in J, if the set of formulas $\mathcal{F}_i$ is consistent with $\mathfrak{V}^*$, then it must be consistent with $\mathfrak{W}^*$, because $\mathfrak{W}^*$ is an elementary substructure of $\mathfrak{V}^*$. Consequently, $\mathcal{F}_i$ is satisfied in $\mathfrak{V}_1^*$, by property (3) with $n = 0$, and therefore it is satisfied in $\mathfrak{V}^*$, because $\mathfrak{V}_1^*$ is an elementary substructure of $\mathfrak{V}^*$. Thus, part (i) of the lemma holds. To verify part (ii), let t be any element in $\mathfrak{V}^*$, and suppose that the set of formulas $\mathcal{G}_{it}$ is consistent with $\mathfrak{V}^*$. The element t must belong to $\mathfrak{V}_n^*$ for some natural number n, by the definition of $\mathfrak{V}^*$, and therefore the set $\mathcal{G}_{it}$ must be consistent with $\mathfrak{V}_n^*$, because $\mathfrak{V}_n^*$ is an elementary substructure of $\mathfrak{V}^*$. Consequently, $\mathcal{G}_{it}$ is satisfied in $\mathfrak{V}_{n+1}^*$, by property (4), so $\mathcal{G}_{it}$ is satisfied (by the same element) in $\mathfrak{V}^*$. A similar argument based on property (5) instead of property (4) shows that if t is an element in $\mathfrak{V}^*$ such that the set of formulas $\mathcal{H}_{ijt}$ is consistent with $\mathfrak{V}^*$, then this set of formulas is satisfied in $\mathfrak{V}^*$. Thus, part (iii) of the lemma holds. □

Using a more complicated argument, one can actually prove the existence of a structure $\mathfrak{V}^*$ possessing the properties set forth in the preceding lemma and such that $\mathfrak{V}^*$ is isomorphic to an ultrapower of $\mathfrak{W}^*$. Instead of using the construction of the two preceding lemmas, one proves the existence of a "good" ultrafilter D with some very special properties that imply the validity of properties (i)–(iii) from Lemma 19.49 for the ultrapower of $\mathfrak{W}^*$ modulo the ultrafilter D (see, for example, Theorems 6.1.4 and 6.1.8 in [22] for details). The fact

that $\mathfrak{V}^*$ can be taken to be an ultrapower of $\mathfrak{W}^*$ will play a role later on, so from now on we will use this fact.

We come finally to the theorem mentioned at the beginning of the section. Let $\mathfrak{W}$ be an arbitrary relational structure, and $\mathfrak{A}$ an arbitrary subalgebra of the complex algebra $\mathfrak{Cm}(W)$. Thus, the elements in $\mathfrak{A}$ are subsets of $\mathfrak{W}$, and the operators in $\mathfrak{A}$ are restrictions of the operators in $\mathfrak{Cm}(W)$, which are of course defined in terms of the fundamental relations in $\mathfrak{W}$. The canonical extension of $\mathfrak{A}$ is, up to isomorphism, the complex algebra $\mathfrak{Cm}(U)$ of a relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

in which the elements of the universe are the ultrafilters in $\mathfrak{A}$—in particular, they are subsets of $\mathfrak{A}$ and therefore sets of subsets of $\mathfrak{W}$—and the fundamental relations R, C, and I are determined by the equivalences

$$\begin{aligned} R(F,G,H) \quad &\text{if and only if} \quad F\,;G \subseteq H, \\ C(F,H) \quad &\text{if and only if} \quad F^\smile \subseteq H, \\ I(H) \quad &\text{if and only if} \quad I \in H, \end{aligned}$$

for all elements (ultrafilters) F, G, and H in $\mathfrak{U}$. The operations $;$ and $^\smile$ on the right sides of these equivalences are the complex operations induced by the operations of $\mathfrak{A}$ (or, equivalently, by the operations of $\mathfrak{Cm}(W)$) on subsets of $\mathfrak{A}$. For example, $F\,;G \subseteq H$ means that for every set X in F and every set Y in G, the product $X\,;Y$ (formed in $\mathfrak{A}$ or in $\mathfrak{Cm}(W)$) belongs to H. Similarly, $F^\smile \subseteq H$ means that for every set X in F, the set $X^\smile$ belongs to H. The set I on the right side the final equivalence is the distinguished constant in $\mathfrak{A}$ and in $\mathfrak{Cm}(W)$, that is to say, it is the fundamental unary relation I in $\mathfrak{W}$. (See the remarks preceding Lemma 14.14.) We shall refer to $\mathfrak{U}$ as the *dual relational structure* of the subalgebra $\mathfrak{A}$.

Theorem 19.50. *Every relational structure $\mathfrak{W}$ has an elementary extension $\mathfrak{V}$ with the property that there is a bounded epimorphism from $\mathfrak{V}$ onto the dual relational structure of each subalgebra of $\mathfrak{Cm}(W)$. Moreover, $\mathfrak{V}$ may be taken to be isomorphic to an ultrapower of $\mathfrak{W}$.*

Proof. Take $\mathfrak{W}^*$ to be the expansion of $\mathfrak{W}$ obtained by adjoining every subset of $\mathfrak{W}$ as a new unary relation. The following sentences are obviously true in $\mathfrak{W}^*$ for all subsets X, Y, and Z of $\mathfrak{W}$—that is to say,

for all elements X, Y, and Z in $\mathfrak{Cm}(W)$—by the very definition of the operations in $\mathfrak{Cm}(W)$:

$$\begin{aligned}
&\forall v_0(\boldsymbol{Z}(v_0) \leftrightarrow (\boldsymbol{X}(v_0) \wedge \boldsymbol{Y}(v_0))) && \text{when} \quad Z = X \cap Y, && (1)\\
&\forall v_0(\boldsymbol{Z}(v_0) \leftrightarrow \neg \boldsymbol{X}(v_0)) && \text{when} \quad Z = {\sim}X, && (2)\\
&\forall v_0(\neg \boldsymbol{Z}(v_0)) && \text{when} \quad Z = \varnothing, && (3)\\
&\forall v_0(\boldsymbol{X}(v_0) \rightarrow \boldsymbol{Y}(v_0)) && \text{when} \quad X \subseteq Y, && (4)\\
&\forall v_1(\boldsymbol{Z}(v_1) \leftrightarrow \exists v_0(\boldsymbol{C}(v_0, v_1) \wedge \boldsymbol{X}(v_0))) && \text{when} \quad Z = X^{\smile}, && (5)
\end{aligned}$$

and

$$\forall v_2(\boldsymbol{Z}(v_2) \leftrightarrow \exists v_0 \exists v_1(\boldsymbol{R}(v_0, v_1, v_2) \wedge \boldsymbol{X}(v_0) \wedge \boldsymbol{Y}(v_1))) \quad \text{when} \quad Z = X\,;Y. \qquad (6)$$

Take

$$(F_i : i \in J) \qquad (7)$$

to be the system of all those sets of subsets of $\mathfrak{W}$ that have the finite intersection property. Thus, for each index i in J, every finite collection of sets in F_i has a non-empty intersection. Apply Lemma 19.49 and the remark following that lemma to obtain an elementary extension $\mathfrak{V}^*$ of $\mathfrak{W}^*$ that possesses properties (i)–(iii) of that lemma and that is isomorphic to some ultrapower of $\mathfrak{W}^*$. Since $\mathfrak{V}^*$ is an elementary extension of $\mathfrak{W}^*$, the same sentences of the language $\mathcal{L}^*$ are true in both structures. In particular, the sentences in (1)–(6) are true in $\mathfrak{V}^*$. Let $\mathfrak{V}$ be the relational structure obtained from $\mathfrak{V}^*$ by throwing away the extra unary relations. Certainly, $\mathfrak{V}$ is an elementary extension of $\mathfrak{W}$, since $\mathfrak{V}^*$ is an elementary extension of $\mathfrak{W}^*$, and $\mathfrak{V}$ is also isomorphic to an ultrapower of $\mathfrak{W}$, since $\mathfrak{V}^*$ is isomorphic to an ultrapower of $\mathfrak{W}^*$.

In order to prove that $\mathfrak{V}$ has the required property, namely that there is a bounded epimorphism from $\mathfrak{V}$ onto the the dual relational structure of each subalgebra of $\mathfrak{Cm}(W)$, consider an arbitrary subalgebra $\mathfrak{A}$ of $\mathfrak{Cm}(W)$, and let $\mathfrak{U}$ be the dual relational structure of $\mathfrak{A}$. The elements in $\mathfrak{U}$ all occur in the system (7). Indeed, if F is an element in $\mathfrak{U}$, then F is an ultrafilter in $\mathfrak{A}$, by the definition of a dual relational structure (see the remarks preceding the theorem). Consequently, F is a set of subsets of $\mathfrak{Cm}(W)$, because $\mathfrak{A}$ is a subalgebra of $\mathfrak{Cm}(W)$; and F has the finite intersection property, because it is a proper filter. Thus, $F = F_i$ for some i in J, by the definition of the system in (7).

For each element r in $\mathfrak{V}^*$, it is not difficult to see that the set

$$F_r = \{X \in A : r \text{ satisfies the formula } \boldsymbol{X}(v_0) \text{ in } \mathfrak{V}^*\} \tag{8}$$

is an ultrafilter in $\mathfrak{A}$ and therefore an element in $\mathfrak{U}$. For example, suppose that X and Y are sets in F_r, and write $Z = X \cap Y$, with the goal of showing that Z belongs to F_r. The element r satisfies each of the formulas $\boldsymbol{X}(v_0)$ and $\boldsymbol{Y}(v_0)$, by (8) and the assumption that X and Y belong to F_r. Consequently, r satisfies the conjunction of these two formulas, and therefore it satisfies the formula $\boldsymbol{Z}(v_0)$, by the validity of (1) in $\mathfrak{V}^*$. It follows that Z belongs to F_r, by (8). An analogous argument, using (4), shows that if X is in F_r, and if Y is a set in $\mathfrak{A}$ that includes X, then Y also belongs to F_r. Thus, F_r is a Boolean filter in $\mathfrak{A}$. This filter is proper—that is to say, it does not contain the empty set—because r does not satisfy the formula $\boldsymbol{Z}(v_0)$ when $Z = \varnothing$, by (3). To check that F_r is in fact an ultrafilter, suppose that a set X from $\mathfrak{A}$ does not belong to F_r, and write $Z = {\sim}X$. The element r does not satisfy the formula $\boldsymbol{X}(v_0)$, by (8) and the assumption that X is not in F_r, so r must satisfy the negation of this formula. Consequently, r satisfies the formula $\boldsymbol{Z}(v_0)$, by (2), and therefore Z belongs to F_r, by (8).

Define a function ϑ by writing

$$\vartheta(r) = F_r \tag{9}$$

for every element r in $\mathfrak{V}$. The argument of the preceding paragraph shows that ϑ maps the relational structure $\mathfrak{V}$ into the relational structure $\mathfrak{U}$. To show that ϑ actually maps $\mathfrak{V}$ onto $\mathfrak{U}$, consider an arbitrary element F in $\mathfrak{U}$. It was already observed that there must be an index i in J for which $F = F_i$. The closure of F_i under finite intersections implies that the set of formulas

$$\mathcal{F}_i = \{\boldsymbol{X}(v_0) : X \in F_i\} \tag{10}$$

is consistent with $\mathfrak{W}^*$. In more detail, given a finite subset

$$\{\boldsymbol{X}_0(v_0), \ldots, \boldsymbol{X}_{n-1}(v_0)\} \tag{11}$$

of (10), the corresponding sets $X_0, \ldots, X_{n-1}$ all belong to F_i, by (10), so their intersection X also belongs to F_i and is therefore non-empty, by the finite intersection property. Let $\boldsymbol{X}(v_0)$ be the formula corresponding to the set X, and observe that the sentence

$$\forall v_0(\boldsymbol{X}(v_0) \to (\boldsymbol{X}_0(v_0) \wedge \cdots \wedge \boldsymbol{X}_{n-1}(v_0))) \tag{12}$$

is true in $\mathfrak{W}^*$, by (1) and (4). Any element r in X satisfies $\boldsymbol{X}(v_0)$ in $\mathfrak{W}^*$, by the definition of $\mathfrak{W}^*$ (since the unary relation symbol $\boldsymbol{X}$ is interpreted in $\mathfrak{W}^*$ as the set X), so r satisfies each of the formulas in (11) in $\mathfrak{W}^*$, by (12). Thus, $\mathcal{F}_i$ is consistent with $\mathfrak{W}^*$, as claimed. The structure $\mathfrak{V}^*$ possesses property (i) of Lemma 19.49, so there must be an element r in $\mathfrak{V}^*$ that satisfies every formula in $\mathcal{F}_i$. It follows from (8) that $F = F_i \subseteq F_r$. Both F and F_r are ultrafilters, so $F = F_r$ and therefore $\vartheta(r) = F$, by (9).

The argument that ϑ is a homomorphism is based on the validity of the sentences (5) and (6) in $\mathfrak{V}^*$. For example, to see that ϑ preserves the relation R, consider elements r, s, and t in $\mathfrak{V}$ such that $R(r, s, t)$ holds in $\mathfrak{V}$. It must be shown that $R(\vartheta(r), \vartheta(s), \vartheta(t))$—that is to say, $R(F_r, F_s, F_t)$—holds in $\mathfrak{U}$. In view of the definition of R in $\mathfrak{U}$, this amounts to checking that the complex product $F_r \,;\, F_s$ is included in F_t. To this end, consider sets X in F_r and Y in F_s, with the goal of showing that the set $Z = X \,;\, Y$ is in F_t. It follows from (8) that r and s respectively satisfy the formulas $\boldsymbol{X}(v_0)$ and $\boldsymbol{Y}(v_1)$ in $\mathfrak{V}^*$, so the triple (r, s, t) satisfies the formula

$$\boldsymbol{R}(v_0, v_1, v_2) \wedge \boldsymbol{X}(v_0) \wedge \boldsymbol{Y}(v_1) \tag{13}$$

in $\mathfrak{V}^*$. Consequently, t satisfies the formula

$$\exists v_0 \exists v_1 (\boldsymbol{R}(v_0, v_1, v_2) \wedge \boldsymbol{X}(v_0) \wedge \boldsymbol{Y}(v_1)) \tag{14}$$

in $\mathfrak{V}^*$, so t must also satisfy the formula $\boldsymbol{Z}(v_2)$—and therefore the formula $\boldsymbol{Z}(v_0)$—in $\mathfrak{V}^*$, by the validity of (6) in $\mathfrak{V}^*$. Thus, Z belongs to F_t, by (8) (with t in place of r).

The proof that ϑ preserves the relation C is based on (5) and is very similar to the preceding argument, so it is left as an exercise. The proof that ϑ preserves the relation I is more direct, and does not require any special properties of $\mathfrak{V}^*$. If $I(t)$ holds in $\mathfrak{V}^*$, then t satisfies the formula $\boldsymbol{I}(v_0)$ in $\mathfrak{V}^*$, so the distinguished element I from $\mathfrak{A}$ (which is the set of elements satisfying the fundamental unary relation I in $\mathfrak{W}$) belongs to the ultrafilter F_t, by (8), and therefore $I(F_t)$ holds in $\mathfrak{U}$, by the definition of the relation I in $\mathfrak{U}$.

The argument that ϑ is bounded as a homomorphism depends on the validity in $\mathfrak{V}^*$ of properties (ii) and (iii) from Lemma 19.49, as well as the validity of (5) and (6). For example, to see that ϑ is bounded with respect to the relation R, consider elements F and G in $\mathfrak{U}$, and an element t in $\mathfrak{V}$, such that $R(F, G, \vartheta(t))$ holds in $\mathfrak{U}$. As was already observed, there must be indices i and j in J such that

$$F = F_i \qquad \text{and} \qquad G = F_j.$$

Since $\vartheta(t) = F_t$, by (9), we see that $R(F_i, F_j, F_t)$ holds in $\mathfrak{U}$, and therefore the complex product $F_i ; F_j$ is included in F_t, by the definition of the relation R in $\mathfrak{U}$. This means that

$$X \in F_i \quad \text{and} \quad Y \in F_j \quad \text{implies} \quad X ; Y \in F_t. \tag{15}$$

We proceed to verify that the set of formulas

$$\mathcal{H}_{ijt} = \{\boldsymbol{X}(v_0) : X \in F_i\} \cup \{\boldsymbol{Y}(v_1) : Y \in F_j\} \cup \{\boldsymbol{R}(v_0, v_1, \boldsymbol{t})\} \tag{16}$$

is consistent with $\mathfrak{V}^*$. Let $X_0, \ldots, X_{m-1}$ be sets in F_i, and $Y_0, \ldots, Y_{n-1}$ sets in F_j (where m and n are natural numbers), and consider the finite subset of formulas

$$\{\boldsymbol{X}_k(v_0) : 0 \leq k < m\} \cup \{\boldsymbol{Y}_\ell(v_1) : 0 \leq \ell < n\} \cup \{\boldsymbol{R}(v_0, v_1, \boldsymbol{t})\} \tag{17}$$

of $\mathcal{H}_{ijt}$. Since F_i and F_j are filters, the finite intersections

$$X = \bigcap\nolimits_k X_k \qquad \text{and} \qquad Y = \bigcap\nolimits_\ell Y_\ell \tag{18}$$

belong to F_i and F_j respectively, so the set $Z = X ; Y$ belongs to F_t, by (15). This means that the element t satisfies the formula $\boldsymbol{Z}(v_0)$—and therefore also the formula $\boldsymbol{Z}(v_2)$—in $\mathfrak{V}^*$, by (8). Since the sentence in (6) is valid in $\mathfrak{V}^*$, it follows that t must satisfy the formula (14) in $\mathfrak{V}^*$. Consequently, there must be elements $\bar{r}$ and $\bar{s}$ in $\mathfrak{V}^*$ such that the triple $(\bar{r}, \bar{s}, t)$ satisfies the formula (13) in $\mathfrak{V}^*$. In particular, $\bar{r}$ satisfies the formula $\boldsymbol{X}(v_0)$, so $\bar{r}$ must satisfy each of the formulas $\boldsymbol{X}_k(v_0)$, by (18), (1), and (4) (see the corresponding argument in the proof that ϑ is onto). Similarly, $\bar{s}$ satisfies the formula $\boldsymbol{Y}(v_1)$, so $\bar{s}$ must satisfy each of the formulas $\boldsymbol{Y}_\ell(v_1)$. Thus, the triple $(\bar{r}, \bar{s}, t)$ satisfies every formula in the set (17) in $\mathfrak{V}^*$.

The argument in the preceding paragraph shows that the set of formulas $\mathcal{H}_{ijt}$ is consistent with the structure $\mathfrak{V}^*$. This structure possesses property (iii) of Lemma 19.49, so there must be a pair of elements (r, s) from $\mathfrak{V}^*$ that satisfies every formula in $\mathcal{H}_{ijt}$. In particular, the pair satisfies the formula $\boldsymbol{R}(v_0, v_1, \boldsymbol{t})$ in $\mathfrak{V}^*$, so $R(r, s, t)$ holds in $\mathfrak{V}$. Also, r satisfies every formula in $\mathcal{F}_i$, and s satisfies every formula in $\mathcal{F}_j$, by (16), so we must have

$$F_r = F_i \qquad \text{and} \qquad F_s = F_j$$

(see the corresponding argument in the proof that ϑ is onto). Consequently,

$$\vartheta(r) = F_r = F_i = F \qquad \text{and} \qquad \vartheta(s) = F_s = F_j = G.$$

This completes the proof that ϑ is bounded with respect to the relation R.

An entirely analogous argument, based on (5) and property (ii) from Lemma 19.49 shows that ϑ is bounded with respect to the relation C. The details are left as an exercise. The proof that ϑ is bounded with respect to the relation I is more straightforward and does not require any special properties of the structure $\mathfrak{V}^*$. Let t be an element in $\mathfrak{V}$ such that $I(\vartheta(t))$—that is to say, $I(F_t)$—holds in the dual structure $\mathfrak{U}$. The definition of I in $\mathfrak{U}$ implies that the distinguished element I from $\mathfrak{A}$ must belong to F_t, so t must satisfy the formula $\boldsymbol{I}(v_0)$ in $\mathfrak{V}$, by (8). Consequently, $I(t)$ holds in $\mathfrak{V}$. □

The dual version of the preceding theorem is important in its own right.

Theorem 19.51. *Every relational structure $\mathfrak{W}$ has an elementary extension $\mathfrak{V}$ with the property that the canonical extension of each and every subalgebra of $\mathfrak{Cm}(W)$ is completely embeddable into $\mathfrak{Cm}(V)$. Moreover, $\mathfrak{V}$ may be taken to be isomorphic to an ultrapower of $\mathfrak{W}$.*

Proof. Given a relational structure $\mathfrak{W}$, let $\mathfrak{V}$ be an elementary extension of $\mathfrak{W}$ that possesses the properties stated in Theorem 19.50. Suppose $\mathfrak{A}$ is a subalgebra of $\mathfrak{Cm}(W)$, and let $\mathfrak{U}$ be the dual relational structure of $\mathfrak{A}$. There is a bounded epimorphism ϑ from $\mathfrak{V}$ onto $\mathfrak{U}$, by Theorem 19.50. Apply Theorem 19.32 to ϑ to obtain a complete monomorphism from $\mathfrak{Cm}(U)$ into $\mathfrak{Cm}(V)$. The canonical extension of $\mathfrak{A}$ is isomorphic to $\mathfrak{Cm}(U)$ (see the remarks preceding Lemma 14.14), which in turn is completely embeddable into $\mathfrak{Cm}(V)$, so the canonical extension of $\mathfrak{A}$ must be completely embeddable into $\mathfrak{Cm}(V)$. □

Given a class K of relational structures, write $\mathsf{Cm}(\mathsf{K})$ for the class of complex algebras of relational structures in K. Thus,

$$\mathsf{Cm}(\mathsf{K}) = \{\mathfrak{Cm}(W) : \mathfrak{W} \in \mathsf{K}\}.$$

Recall that for a class of algebras L, the class $\mathsf{S}(\mathsf{L})$ consists of all those algebras that are embeddable into algebras in L.

Theorem 19.52. *If a class* K *of relational structures is closed under ultraproducts, then* $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ *is a universal class of Boolean algebras with operators, and this class is closed under canonical extensions.*

Proof. The class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ is obviously closed under subalgebras, by Lemma 18.1(i). Consequently, to prove that it is universal, it suffices to show that it is closed under ultraproducts, by the SP_{u}-Theorem 18.10. To this end, consider a non-empty system

$$(\mathfrak{A}_i : i \in J) \tag{1}$$

of algebras in $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$, and write $\mathfrak{A}$ for the direct product of this system. For each index i in J, there must be a relational structure $\mathfrak{W}_i$ in K such that $\mathfrak{A}_i$ is embeddable in the complex algebra $\mathfrak{Cm}(W_i)$, by the definition of the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$.

Fix an ultrafilter D on the set J, with the goal of showing that the ultraproduct $\mathfrak{A}/D$ belongs to the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$. Focus first on the case when none of the relational structures in the system

$$(\mathfrak{W}_i : i \in J) \tag{2}$$

is empty. Let $\mathfrak{W}$ be the direct product of this system. The ultraproduct $\mathfrak{W}/D$ belongs to the class K, by the assumed closure of K under ultraproducts. Consequently, the complex algebra $\mathfrak{Cm}(W/D)$ belongs to the class $\mathsf{Cm}(\mathsf{K})$, by the definition of this class. Let $\mathfrak{B}$ be the direct product of the system of complex algebra

$$(\mathfrak{Cm}(W_i) : i \in J), \tag{3}$$

and observe that the ultraproduct $\mathfrak{B}/D$ is embeddable into the complex algebra $\mathfrak{Cm}(W/D)$, by Theorem 19.40. Therefore, $\mathfrak{B}/D$ belongs to the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$. Each algebra $\mathfrak{A}_i$ in (1) is embeddable into the corresponding algebra $\mathfrak{Cm}(W_i)$ in (3), by assumption, so the ultraproduct $\mathfrak{A}/D$ is embeddable into the ultraproduct $\mathfrak{B}/D$, by Lemma 19.44. Therefore, $\mathfrak{A}/D$ also belongs to $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$.

Consider now the case when some of the structures in (2) are empty. Let J_0 be the set of indices i for which $\mathfrak{W}_i$ is empty, and suppose first that J_0 does not belong to the ultrafilter D. The algebras $\mathfrak{Cm}(W_i)$, and consequently also the embedded algebras $\mathfrak{A}_i$, are degenerate for i in J_0, and may therefore be omitted from the system in (1). The argument of the preceding paragraph applies to the ultraproduct of the restricted

system obtained by omitting these degenerate algebras, so the ultraproduct of the restricted system belongs to the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$. The ultraproduct $\mathfrak{A}/D$ is isomorphic to the ultraproduct of the restricted system, so $\mathfrak{A}/D$ belongs to $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ as well.

In more detail, put

$$J_1 = J \sim J_0 \qquad \text{and} \qquad D_1 = \{X \cap J_1 : X \in D\}.$$

The set J_1 belongs to the ultrafilter D (because its complement J_0 is not in D, by assumption), so the set D_1 is a subset of D and an ultrafilter on the set J_1. Write $\mathfrak{A}_1$ for the direct product of the restricted system $(\mathfrak{A}_i : i \in J_1)$, and for each element r in $\mathfrak{A}$, write $\bar{r}$ for the element in $\mathfrak{A}_1$ that is the restriction of r to the set J_1. The function ϑ defined by $\vartheta(r/D) = \bar{r}/D_1$ for elements r in $\mathfrak{A}$ is a well-defined isomorphism from $\mathfrak{A}/D$ to $\mathfrak{A}_1/D_1$. (The details are left as an exercise.) The ultraproduct $\mathfrak{A}_1/D_1$ belongs to the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$, by the argument of the first paragraph, so the isomorphic ultraproduct $\mathfrak{A}/D$ must also belong to $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$.

Consider, finally, the case when the set J_0 does belong to D. The sentence asserting that the universe consists of just one element is then valid in the ultraproduct $\mathfrak{A}/D$, by the Fundamental Theorem of Ultraproducts (Theorem 18.6), so this ultraproduct is degenerate. It follows that this ultraproduct is isomorphic to the complex algebra of the empty relational structure. The empty structure belongs to K, because the set J_0 is assumed to be not empty, and for each index i in J_0, the structure $\mathfrak{W}_i$ is empty and belongs to K, by the definition of J_0 and the assumptions on the system in (2). Consequently, the complex algebra of the empty structure belongs to $\mathsf{Cm}(\mathsf{K})$, and therefore $\mathfrak{A}/D$ belongs to $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$. Conclusion: the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ is closed under ultraproducts and is therefore universal.

To show that the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ is closed under canonical extensions, let $\mathfrak{A}$ be any algebra in this class. There must then be a relational structure $\mathfrak{W}$ in K such that $\mathfrak{A}$ is embeddable into the complex algebra $\mathfrak{Cm}(W)$, by the definition of the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$. Without loss of generality, it may be assumed that $\mathfrak{A}$ is actually a subalgebra of $\mathfrak{Cm}(W)$. Take $\mathfrak{V}$ to be the ultrapower of $\mathfrak{W}$ with the properties specified in Theorem 19.51, and observe that $\mathfrak{V}$ must belong to K, since $\mathfrak{W}$ is in K, and K is assumed to be closed under ultraproducts. Consequently, $\mathfrak{Cm}(V)$ belongs to $\mathsf{Cm}(\mathsf{K})$.

The canonical extension of $\mathfrak{A}$ is completely embeddable into $\mathfrak{Cm}(V)$, by Theorem 19.51. In particular, the canonical extension of $\mathfrak{A}$ belongs

to the class $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$. Conclusion: $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ is closed under canonical extensions, as claimed. □

A class of structures is called *pseudo-elementary* if it can be obtained from an elementary class L by deleting a fixed list of the fundamental relations from each of the structures in L. Elementary classes are closed under ultraproducts, by the Fundamental Theorem of Ultraproducts, and from this it easily follows that pseudo-elementary classes are also closed under ultraproducts. Consequently, we immediately obtain the following corollary.

Corollary 19.53. *If* K *is an elementary or even a pseudo-elementary class of relational structures, then* $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ *is a universal class of Boolean algebras with operators, and this class is closed under canonical extensions.*

19.8 Varieties closed under canonical extensions

In the preceding section, a sufficient condition was given for the class of algebras embeddable into a class of complex algebras to be a universal class closed under canonical extensions. In this section we show that a similar condition implies that the variety generated by a class of complex algebras is closed under canonical extensions. The following intermediate result concerning quasi-varieties is needed for the proof.

Theorem 19.54. *If a class* K *of relational structures is closed under ultraproducts, then* $\mathsf{SP}(\mathsf{Cm}(\mathsf{K}))$ *is a quasi-variety of Boolean algebras with operators, and this class is closed under canonical extensions.*

Proof. Theorem 19.52 and the assumed closure of K under ultraproducts imply that the class

$$\mathsf{L} = \mathsf{S}(\mathsf{Cm}(\mathsf{K})) \tag{1}$$

is universal. Consequently, the class $\mathsf{SP}(\mathsf{L})$ is a quasi-variety, by the SP-Theorem 18.15. Lemmas 18.2(ii) and 18.1(i), together with (1), imply that

$$\mathsf{SP}(\mathsf{L}) = \mathsf{SPS}(\mathsf{Cm}(\mathsf{K})) \subseteq \mathsf{SSP}(\mathsf{Cm}(\mathsf{K})) = \mathsf{SP}(\mathsf{Cm}(\mathsf{K})) \subseteq \mathsf{SP}(\mathsf{L}).$$

(The last inclusion holds because $\mathsf{Cm}(\mathsf{K})$ is included in L, by (1).) Since the first and last classes are the same, equality must hold everywhere, so that $\mathsf{SP}(\mathsf{L})$ coincides with the class

$$\mathsf{SP}(\mathsf{Cm}(\mathsf{K})). \tag{2}$$

Thus, (2) is a quasi-variety.

To prove that (2) is closed under canonical extensions, consider any algebra $\mathfrak{A}$ in this class. The definition of (2) implies that there must be a system of relational structures

$$(\mathfrak{W}_k : k \in K) \tag{3}$$

in K such that $\mathfrak{A}$ is embeddable into the product $\prod_k \mathfrak{Cm}(W_k)$. Form the disjoint union $\mathfrak{W}$ of system (3). The product $\prod_k \mathfrak{Cm}(W_k)$ is isomorphic to the complex algebra $\mathfrak{Cm}(W)$, by Corollary 19.38, so $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Cm}(W)$.

Apply Theorem 19.51 to obtain an elementary extension $\mathfrak{V}$ of $\mathfrak{W}$ with the property that the canonical extension of every subalgebra of $\mathfrak{Cm}(W)$ is completely embeddable into $\mathfrak{Cm}(V)$. As stated in that theorem, we may take $\mathfrak{V}$ to be isomorphic to an ultrapower of $\mathfrak{W}$. Since $\mathfrak{A}$ is embeddable into $\mathfrak{Cm}(W)$, it follows that the canonical extension of $\mathfrak{A}$ is completely embeddable into $\mathfrak{Cm}(V)$. Consequently, in order to prove that the canonical extension of $\mathfrak{A}$ belongs to (2), it suffices to show that $\mathfrak{Cm}(V)$ belongs to (2).

The isomorphism of $\mathfrak{V}$ with some ultrapower of $\mathfrak{W}$ permits us to apply Lemma 19.47 (with $\mathfrak{W}_k$ and $\mathfrak{W}$ in place of $\mathfrak{U}_k$ and $\mathfrak{U}$ respectively, and with $\mathfrak{V}$ as an isomorphic copy of an ultrapower of $\mathfrak{W}$) to obtain a disjoint system of relational structures

$$(\mathfrak{Z}_i : i \in J) \tag{5}$$

with the following properties. First, each structure $\mathfrak{Z}_i$ in the system is isomorphic to an ultraproduct of structures from the system (3). Second, there is a bounded epimorphism ϑ from the union $\mathfrak{Z}$ of system (5) onto the relational structure $\mathfrak{V}$. Each of the relational structures in (3) belongs to K, by assumption, and K is assumed to be closed under ultraproducts, so each of the relational structures in (5) also belongs to K. Consequently, the complex algebra of each structure in (5) belongs to $\mathsf{Cm}(\mathsf{K})$. The complex algebra $\mathfrak{Cm}(Z)$ is isomorphic to the direct product of the system

$$(\mathfrak{Cm}(Z_i) : i \in J),$$

by Corollary 19.38, so $\mathfrak{Cm}(Z)$ must belong to the class $\mathsf{P}(\mathsf{Cm}(\mathsf{K}))$. The dual of the bounded epimorphism ϑ from $\mathfrak{Z}$ to $\mathfrak{V}$ is a complete monomorphism from $\mathfrak{Cm}(V)$ into $\mathfrak{Cm}(Z)$, by Theorem 19.32 (with $\mathfrak{Z}$ in place of $\mathfrak{U}$), so $\mathfrak{Cm}(V)$ is embeddable into an algebra in $\mathsf{P}(\mathsf{Cm}(\mathsf{K}))$ and therefore belongs to the class (2), as was to be shown. □

We come now to the theorem concerning the closure under canonical extensions of varieties generated by classes of complex algebras.

Theorem 19.55. *If a class* K *of relational structures is closed under ultraproducts, then the variety generated by* $\mathsf{Cm}(\mathsf{K})$ *is closed under canonical extensions.*

Proof. The HSP-Theorem 18.27, in the version that applies to Boolean algebras with operators, says that the variety generated by the class of complex algebras $\mathsf{Cm}(\mathsf{K})$ is

$$\mathsf{HSP}(\mathsf{Cm}(\mathsf{K})). \tag{1}$$

Consider any algebra $\mathfrak{A}$ in this variety. There must be an algebra $\mathfrak{B}$ in the class $\mathsf{SP}(\mathsf{Cm}(\mathsf{K}))$ such that $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$. The canonical extension of $\mathfrak{B}$ belongs to $\mathsf{SP}(\mathsf{Cm}(\mathsf{K}))$, by Theorem 19.54, and the canonical extension of $\mathfrak{A}$ is a complete homomorphic image of the canonical extension of $\mathfrak{B}$, by Theorem 14.41, so the canonical extension of $\mathfrak{A}$ belongs to (1). Thus, the variety (1) is closed under canonical extensions. □

The conclusions of Theorems 19.54 and 19.55 apply in particular when the class K is elementary or pseudo-elementary, by the remarks preceding Corollary 19.53.

19.9 Applications to relation algebras

In order to apply the results of the preceding two sections to classes of relation algebras, it is helpful to formulate Theorems 19.52 and 19.55 in a somewhat stronger, but more restricted way that applies to classes of relation algebras.

Theorem 19.56. *Suppose* L *is a class of simple relation algebras. If there exists an elementary or pseudo-elementary class* K *of relational structures such that*

$$\mathsf{Cm}(\mathsf{K}) \subseteq \mathsf{L} \subseteq \mathsf{S}(\mathsf{Cm}(\mathsf{K})),$$

then $\mathsf{SP}(\mathsf{L})$ *is the variety of relation algebras generated by* L*, and* $\mathsf{S}(\mathsf{L})$ *is the universal class of simple algebras in this variety, and both classes are closed under canonical extensions.*

Proof. The hypothesis on L implies that

$$\mathsf{S}(\mathsf{L}) = \mathsf{S}(\mathsf{Cm}(\mathsf{K})). \tag{1}$$

The class on the right is universal and closed under canonical extensions, by Corollary 19.53 and the assumption that K is an elementary or pseudo-elementary class. Consequently, the same is true of the class on the left. The algebras in the left-hand class are all simple, by the assumption on L and the fact that subalgebras of simple relation algebras are simple (Corollary 9.3). Apply Corollary 18.26 (with $\mathsf{S}(\mathsf{L})$ in place of L) to obtain that the variety generated by $\mathsf{S}(\mathsf{L})$—and hence also by L—is $\mathsf{SPS}(\mathsf{L})$, and that the class of simple algebras in this variety is $\mathsf{SS}(\mathsf{L})$. Of course,

$$\mathsf{SP}(\mathsf{L}) \subseteq \mathsf{SPS}(\mathsf{L}) \subseteq \mathsf{SSP}(\mathsf{L}) = \mathsf{SP}(\mathsf{L}), \tag{2}$$

by Lemmas 18.2(ii) and 18.1(i). The first and last classes in (2) are the same, so equality holds everywhere. Similarly, $\mathsf{SS}(\mathsf{L}) = \mathsf{S}(\mathsf{L})$. Combine these observations with the previous ones to conclude that $\mathsf{SP}(\mathsf{L})$ is the variety generated by L, and $\mathsf{S}(\mathsf{L})$ is the universal class of simple algebras in this variety.

The assumption on K implies that it satisfies the hypotheses of Theorem 19.54 (see the remark at the end of the previous section). Apply the theorem to conclude that the class $\mathsf{SP}(\mathsf{Cm}(\mathsf{K}))$ is closed under canonical extensions. An argument similar to the one in (2), using also (1) and (2), shows that

$$\mathsf{SP}(\mathsf{L}) = \mathsf{SPS}(\mathsf{L}) = \mathsf{SPS}(\mathsf{Cm}(\mathsf{K})) = \mathsf{SP}(\mathsf{Cm}(\mathsf{K})). \tag{3}$$

Consequently, $\mathsf{SP}(\mathsf{L})$ is closed under canonical extensions. □

As a first illustration of how the preceding theorem may be applied to classes of relation algebras, consider the class of group complex

algebras (see Section 3.5). Groups are not relational structures, but it is easy to associate with every group $(G\,,\ \circ\,,\ ^{-1}\,,\ \iota)$ a *group relational structure*

$$(G\,,\ R\,,\ C\,,\ I)$$

in the following way. The universe G of the structure coincides with the universe of the group; the ternary relation R and the binary relation C are the group operations $\circ$ and $^{-1}$ respectively, conceived as relations (from a set-theoretic point of view, every binary operation may be viewed as a special kind of ternary relation, and every unary operation may be viewed as a special kind of binary relation); and the unary relation I is the singleton of the group identity element ι. Obviously, every group is definitionally equivalent (in the sense of first-order logic) to its associated group relational structure, and the class of all groups is definitionally equivalent to the class of all group relational structures. In particular, the latter is an elementary class, say K. The complex algebra of a group, as defined in Section 3.5, is easily seen to coincide with the complex algebra of the associated group relational structure, so $\mathsf{Cm}(\mathsf{K})$ is the class of all group complex algebras. Apply Theorem 19.56 to this class to arrive at the following conclusions.

Theorem 19.57. *If* L *is the class of all complex algebras of groups, then* $\mathsf{SP}(\mathsf{L})$ *is the variety generated by* L*, and* $\mathsf{S}(\mathsf{L})$ *is the universal class of simple algebras in the variety. Both classes are closed under canonical extensions.*

Similar results apply to various subclasses of group complex algebras for which the groups all have some common property that is definable by a set of first-order sentences. For example, if L is the class of all complex algebras of commutative groups or the class of all complex algebras of p-groups for some fixed prime p, then the conclusions of the preceding theorem apply.

As a second example, consider the class of geometric complex algebras. Projective geometries are not relational structures, and in the traditional conception of geometry (see Section 3.6 or Section 17.4) they are not even first-order structures. However, it is possible to adopt a different set of primitive notions in terms of which geometries do become relational structures. In this new conception, the universe P of a geometry consists only of points; the notion of a line is not adopted as a primitive notion. There is one ternary relation on the universe, namely a relation K of collinearity between the points. The structure $(P\,,\ K)$ is required to satisfy the following *collinearity axioms*. (1) $K(p, q, r)$

implies $K(q,p,r)$ and $K(p,r,q)$. (2) For all distinct points p, q, r, s, if $K(p,q,r)$ and $K(q,r,s)$, then $K(p,q,s)$. (3) $K(p,p,q)$ holds for any points p and q. (4) For any distinct points p and q, there is always a point r distinct from p and q such that $K(p,q,r)$. (5) For any distinct points p, q, s, t, the existence of a point r satisfying $K(p,q,r)$ and $K(s,t,r)$ implies the existence of a point u satisfying $K(p,s,u)$ and $K(q,t,u)$ (see Figure 19.4).

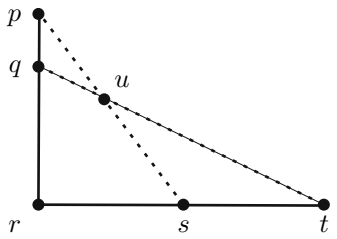


Fig. 19.4 The Pasch Axiom of projective geometry.

The first three axioms express standard properties of the usual collinearity relation that is defined to hold between three points just in case these points lie on a common line. The first axiom implies that the relation of collinearity is completely symmetric in the sense that the collinearity of three points, in some order, implies the collinearity of these points under every permutation of their order. Thus, one may speak about three points being collinear without referring to any specific ordering of the points. The second axiom expresses that, for distinct points p, q, r, and s, a natural kind of transitivity holds for the relation of collinearity: if r is collinear with p and q, and s with q and r, then s is collinear with p and q. In other words, if r is on the line pq, and s is on the line qr, then s is on the line pq. The third axiom expresses two properties: first, collinearity is a reflexive relation in the sense that every triple of the form (p,p,p) is in the collinearity relation (just take q to be equal to p in the axiom); and second, two distinct points p and q always lie on some line in the sense that collinearity relation holds for the triple (p,p,q). The fourth axiom expresses that any two points are collinear with some third point, that is to say, every line has at least three points. The fifth axiom expresses the content of the Pasch Axiom (see Section 3.6 or Section 17.4). It is not too difficult to prove that the new conception of a projective

geometry outlined above is definitionally equivalent with the original conception; the details are left as an exercise. For the present purposes, we adopt the new conception.

With every geometry (P, K), one may associate a *geometric relational structure* (P^+, R, C, I) in the following way. The universe P^+ of the structure consists of the points in the universe of the geometry P, together with a new point ι. The relation R is defined to hold for a triple (p, q, r) of elements from P^+ just in case one of the following conditions is satisfied: p, q, and r are distinct collinear points in the geometry P, that is to say, the points are distinct elements in P and the relation $K(p, q, r)$ holds; or else two of elements p, q, r are equal, and the third element is ι; or else the geometry has order greater than two, and $p = q = r$. The relation C is defined to be the identity relation on the set P^+, and the relation I is defined to be the singleton of the element ι. By the *order* of a geometric relational structure, we mean the order of the underlying geometry. It is easy to check that the complex algebra of a geometry as originally defined coincides with the complex algebra of the associated geometric relational structure; the proof is left as an exercise.

The set of collinearity axioms given above for a geometry, and the definition of a geometric relational structure easily imply that the class of all geometric relational structures is elementary. To see this, observe first that a geometry (P, K) is definable from its associated geometric relational structure. Indeed, P can be defined as the unary relation that holds for those elements p in P^+ for which the relation I does not hold; and K can be defined as the ternary relation that holds for those triples (p, q, r) of elements from P^+ such that the three elements are distinct and belong to the relation R, or else at least two of them are equal and for none of them does the relation I hold. Consequently, one possible axiomatization of the class of geometrical relational structures consists of the following sentences: first, the definitions of the relations P and K in terms of the relations R and I; second, the statements that the binary relation C is the identity relation on the universe, and the unary relation I consists of exactly one element; third, the definition of the relation R in terms of K and I; fourth, statements implying that the set of elements in P, under the relation K, satisfy the collinearity axioms. In view of the definition of K, this means that collinearity axioms (2) and (5) may remain unchanged, but (1), (4), and (5) must be modified to include the hypothesis that the elements in question all satisfy the relation P. For example, instead of (1), one

says that if $P(p)$, $P(q)$, and $P(r)$, then $K(p,q,r)$ implies both $K(q,p,r)$ and $K(p,r,q)$.

Similar arguments imply that the class of all geometrical relational structures of order two and the class of all geometrical relational structures of order greater than two are both elementary. Apply Theorem 19.56 to these three classes to arrive at the following conclusions.

Theorem 19.58. *If* L *is the class of all complex algebras of projective geometries, or projective geometries of order two, or projective geometries of order greater than two, then* $\mathsf{SP}(\mathsf{L})$ *is the variety generated by* L*, and* $\mathsf{S}(\mathsf{L})$ *is the universal class of simple algebras in the variety. Each of these classes is closed under canonical extensions.*

As in the case of group complex algebras, the conclusions of the preceding theorem can be extended to various subclasses of geometric complex algebras in which the geometries have some common properties that are definable by a set of first-order sentences. For example, the notion of a geometry having dimension n for some fixed positive integer n is expressible by a first-order sentence, so if L is the class of all complex algebras of geometries of dimension n, or of geometries of dimension at least n (or at most n), or of geometries of infinite dimension, then the conclusions of the preceding theorem apply. Also, the notion of a geometry having order n for some fixed integer $n \geq 2$ is expressible by a first-order sentence, so if L is the class of all complex algebras of geometries of order n, or of geometries of order at least n (or at most n), or of infinite order, then the conclusions of the preceding theorem apply.

In the final example, we give a conceptually simple proof that the class RRA of all representable relation algebras is a variety (see Theorem 18.30), and that this variety is closed under canonical extensions (see Theorem 16.22). For every non-empty set U, define a structure

$$\mathfrak{U} = (U \times U\,,\, R\,,\, C\,,\, I\,,\, F)$$

as follows: R is the ternary relation (on $U \times U$) of relational composition, C is the binary relation of relational converse, I is the unary identity relation, and F is the ternary pairing relation. Thus,

$$\begin{aligned} R &= \{((\alpha,\gamma),(\gamma,\beta),(\alpha,\beta)) : \alpha,\beta,\gamma \in U\},\\ C &= \{((\alpha,\beta),(\beta,\alpha)) : \alpha,\beta \in U\},\\ I &= \{(\alpha,\alpha) : \alpha \in U\}, \end{aligned}$$

$$F = \{((\alpha, \alpha), (\beta, \beta), (\alpha, \beta)) : \alpha, \beta \in U\}.$$

If $\mathfrak{U}_0$ is the relational structure obtained from $\mathfrak{U}$ by deleting the relation F, then the complex algebra of $\mathfrak{U}_0$ is just the full set relation algebra on the set U. Consequently, a simple relation algebra is representable if and only if it is embeddable into the complex algebra of a relational structure of the form $\mathfrak{U}_0$ for some structure $\mathfrak{U}$ as defined above (see Lemma 16.1 and Corollary 18.31).

Let K be the class of all relational structures of the form defined above, together with all isomorphic copies of such structures. It is not too difficult to check that the following sentences constitute a set of axioms for K. (1) The relation I is not empty. (2) The relation F, that is to say, the correspondence determined by

$$(r, s) \longrightarrow t \qquad \text{if and only if} \qquad F(r, s, t),$$

is a bijection from $I \times I$ to the universe of discourse. (3) $I(t)$ if and only if $F(t, t, t)$. (4) $C(r, t)$ if and only if there exist elements u and v in I such that $F(u, v, r)$ and $F(v, u, t)$. (5) $R(r, s, t)$ if and only if there exists elements u, v, and w in I such that $F(u, v, r)$, $F(v, w, s)$, and $F(u, w, t)$. For example, the first-order sentence

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\boldsymbol{I}(\boldsymbol{x}) \wedge \boldsymbol{I}(\boldsymbol{y}) \leftrightarrow \exists \boldsymbol{z} [\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})]$$

expresses part of the content of (2), namely that the relation F, viewed as a function of the first two arguments, has the set $I \times I$ as its domain. The proof that (1)–(5) do form a set of axioms for K is left as an exercise.

The class K_0 of relational structures obtained from the structures in K by deleting the ternary relation F is pseudo-elementary, by the observations of the preceding paragraph. Also, the class of complex algebras

$$\mathsf{L} = \mathsf{Cm}(\mathsf{K}_0)$$

consists precisely of the full set relation algebras on non-empty sets, and a relation algebra is simple and representable if and only if it is embeddable into some algebra in L, by the remarks made above. It follows that $\mathsf{S}(\mathsf{L})$ is the class of simple, representable relation algebras; and $\mathsf{SP}(\mathsf{L})$ is the class of all representable relation algebras, by the definition of a representable relation algebra and the Decomposition Theorem 11.43 for full set relation algebras on equivalence relations. An application of Theorem 19.56 (with K_0 in place of K, and with L as above) leads to the following conclusions.

Theorem 19.59. *If* L *is the class of all full set relation algebras on non-empty sets, then the class* $\mathsf{SP}(\mathsf{L})$ *of all representable relation algebras is a variety, and the class* $\mathsf{S}(\mathsf{L})$ *of all simple, representable relation algebras is the universal class of simple algebras in this variety. Both classes are closed under canonical extensions.*

As in the previous examples, the conclusions of Theorem 19.59 can be extended to various subclasses of the class of full set relation algebras on non-empty sets. For example, the subclass of K consisting of those structures in K that are infinite is axiomatizable by the set of axioms for K and the set of sentences expressing the existence of at least n elements for every positive integer n. Consequently, if K_1 is the class of infinite relational structures in K_0, then $\mathsf{SP}(\mathsf{Cm}(\mathsf{K}_1))$ is a variety, and $\mathsf{S}(\mathsf{Cm}(\mathsf{K}_1))$ is the universal class of all simple algebras in the variety, and both classes are closed under canonical extensions. In other words, the class of relation algebras embeddable into relation algebras of the form $\mathfrak{Re}(E)$, for some equivalence relation E all of whose equivalence classes are infinite, is a variety; and the universal class of simple algebras in this variety is just the class of relation algebras that are embeddable into full set relation algebras with infinite base sets, that is to say, into algebras of the form $\mathfrak{Re}(U)$ for some infinite set U (equivalently, it is the class of simple relation algebras that are representable over infinite sets); and both classes are closed under canonical extensions.

19.10 Polyalgebras

There is a different approach to the construction of relation algebras as algebras of subsets of a set U that should be mentioned. The underlying structures are not relational structures of the form

$$(U\,,\,R\,,\,C\,,\,I), \tag{1}$$

where R, C, and I are respectively a ternary relation, a binary relation, and a unary relation on the set U, but rather *polyalgebras* of the form

$$(U\,,\,\circ\,,\,^{-1}\,,\,\iota), \tag{2}$$

where $\circ$ is a binary *polyoperation* on the set U, that is to say, a function from $U \times U$ into the set $Sb(U)$ of all subsets of U, while $^{-1}$ is a

unary polyoperation on U, that is to say a function from U into $Sb(U)$, and ι is nullary polyoperation on U, that is to say, a distinguished subset of U.

The *complex algebra* of the polyalgebra in (2) is defined to be the algebra

$$\mathfrak{A} = (Sb(U)\,,\,+\,,\,-\,,\,;\,,\,{}^{\smile}\,,\,1')$$

of the same similarity type as relation algebras, where $+$ and $-$ are the set-theoretic operations of union and complement, while $;$ and ${}^{\smile}$ are the binary and unary operations, and 1' is the distinguished constant, respectively defined by

$$\begin{aligned} X\,;Y &= \bigcup\{r \circ s : r \in X \text{ and } s \in Y\}, \\ X^{\smile} &= \bigcup\{r^{-1} : r \in X\}, \\ 1' &= \iota, \end{aligned}$$

for all subsets X and Y of U.

Relational structures and polyalgebras can be used interchangeably to construct complex algebras. Indeed, given a relational structure (1), define polyoperations $\circ$, ${}^{-1}$, and ι on U by

$$r \circ s = \{t : R(r,s,t)\}, \qquad r^{-1} = \{t : C(r,t)\}, \qquad \iota = \{t : I(t)\}$$

for all elements r and s in U. The result is a polyalgebra (2) that is said to be *associated* with (1). Conversely, given a polyalgebra (2), define relations R, C, and I on U by

$$\begin{aligned} R(r,s,t) \quad &\text{if and only if} \quad t \in r \circ s, \\ C(r,t) \quad &\text{if and only if} \quad t \in r^{-1}, \\ I(t) \quad &\text{if and only if} \quad t \in \iota \end{aligned}$$

for all elements r, s, and t in U. The result is a relational structure (1) that is said to be *associated* with (2). If one starts with a given relational structure (1), and constructs first the polyalgebra associated with (1), and then the relational structure associated with this polyalgebra, the final result is the original relational structure (1). Conversely, if one starts with a given polyalgebra (2), and constructs first the relational structure associated with (2), and then the polyalgebra associated with this relational structure, the final result is the original polyalgebra (2). This whole state of affairs can be summarized by saying that the notions of a relational structure and a polyalgebra are

definitionally equivalent (although not in the usual sense of first-order logic).

Furthermore, as is easily checked, a relational structure and its associated polyalgebra yield the same complex algebra. For example, if X and Y are subsets of U, then

$$\begin{aligned}\bigcup\{r \circ s : r \in X \text{ and } s \in Y\} &= \{t : t \in r \circ s \text{ for some } r \in X \text{ and } s \in Y\} \\ &= \{t : R(r,s,t) \text{ for some } r \in X \text{ and } s \in Y\}.\end{aligned}$$

The last set is defined to be the value of the complex product $X ; Y$ in the complex algebra of the relational structure (1), while the first set is defined to be the value of the complex product $X ; Y$ in the complex algebra of the polyalgebra (2). It follows that the two complex products are really the same.

If one uses polyalgebras to construct relation algebras, then the unary operation $^\smile$ must map atoms to atoms, by Lemma 4.1(vii), and consequently the polyoperation $^{-1}$ must map elements in U to singleton subsets of U. This polyoperation may therefore be replaced by a unary *operation* on U (which we denote by the same symbol $^{-1}$), and in this case the corresponding complex operation $^\smile$ is defined by

$$X^\smile = \{r^{-1} : r \in X\}$$

for subsets X of U. If we adopt this convention—as we shall—and if we write $r \circ X$ for the union of set $\{r \circ s : s \in X\}$, then the polyalgebras for which the complex algebras are relation algebras can be characterized in the following way. The details of the proof are left as an exercise.

Theorem 19.60. *The complex algebra of a polyalgebra* $(U\,,\ \circ\,,\ ^{-1}\,,\ \iota)$ *is a relation algebra if and only if the following conditions hold for all elements* r, s, *and* t *in* U.

(i) $r \circ (s \circ t) = (r \circ s) \circ t$.
(ii) $r \circ \iota = \iota \circ r = r$.
(iii) *The statements*

$$r \in s \circ t, \qquad s \in r \circ t^{-1}, \qquad \text{and} \qquad t \in s^{-1} \circ r$$

are mutually equivalent.

Notice that, despite their appearance, these conditions are not first-order axioms. For instance, $r \circ (s \circ t)$ is really the product of the element r with the set of elements $X = s \circ t$.

We illustrate the construction of relation algebras from polyalgebras with three examples. A group $(G, \circ, ^{-1}, e)$ may be viewed as a polyalgebra if the elements in G are identified with their singletons. In this case, the group operation $\circ$ becomes the binary polyoperation such that if $r \circ s = t$ in the group, then $r \circ s = \{t\}$ in the polyalgebra. In accordance with the convention mentioned in the preceding paragraph, we may take the unary polyoperation $^{-1}$ to coincide with the usual group inverse operation. The nullary operation ι in the polyalgebra is the singleton of the group identity element e. The complex algebra of the group viewed as a polyalgebra coincides with the complex algebra of the group as defined in Section 3.5.

The next example sheds some light on the definition of the complex algebra of a projective geometry P of order at least three. Take U to be the set of subspaces of P of dimension at most zero. In other words, U is the set consisting of the singletons of points in P (the zero-dimensional subspaces, which are identified with the points themselves), and the empty subspace (the subspace of dimension -1, which serves as the new element). Take $\circ$ to be the binary polyoperation on U defined by

$$r \circ s = \{t \in U : r \vee t = s \vee t = r \vee s\},$$

where $\vee$ is the operation of join in the lattice of subspaces of P. In accordance with the convention mentioned above, take $^{-1}$ to be the identity function on U, and take ι to be the singleton of the empty subspace. The complex algebra of the resulting polyalgebra coincides with the complex algebra of the geometry P as defined in Section 3.6.

The final example sheds some light on both the definition of the complex algebra of a modular lattice and its relationship to the definition of the complex algebra of a projective geometry. Consider a modular lattice L with zero. Take U to be the set L, and take $\circ$ to be the binary polyoperation on U defined (as in the preceding paragraph) by

$$r \circ s = \{t \in U : r \vee t = s \vee t = r \vee s\},$$

where $\vee$ is operation of join in the lattice. Take $^{-1}$ to be the identity function on U, and take ι to be the singleton of zero element of the lattice. The complex algebra of the resulting polyalgebra coincides with the complex algebra of the modular lattice L as defined in Section 3.7.

If L is the modular lattice of all subspaces of a projective geometry P of order at least three, then the universe U of the polyalgebra that is defined in terms of the lattice L consists of *all* subspaces of P. The polyoperation $\circ$, when applied to two subspaces r and s, gives as its value the set of all subspaces t such that r and t, and also s and t, generate the same subspace as r and s. The nullary operation ι is the singleton of the empty subspace, which is the zero element of the lattice L. The difference between this definition and that of the polyalgebra defined in terms of the geometry P is that in the latter, the set U is restricted to consist just of the subspaces of dimension at most zero.

At first glance, polyalgebras may seem to provide a more natural and intuitive setting for the construction of complex algebras than do relational structures. They have, however, an important disadvantage from the methodological point of view. Relational structures are first-order structures in the sense that they serve as interpretations of a first-order language (with three non-logical relation symbols, one ternary, one binary, and one unary). The entirety of first-order model theory is therefore immediately available for the study of relational structures and their complex algebras. We saw important examples of this in the preceding sections. Polyalgebras, on the other hand, are not first-order structures. In order to apply the methods and results of first-order model theory to them, one must first pass to the associated relational structures. This adds a seemingly unnecessary level of indirection to arguments involving model theory.

19.11 Historical remarks

The notions of a Boolean algebra with operators, the atom structure of an atomic Boolean algebra with operators (Definition 19.1), and the complex algebra of a relational structure (Definition 19.2) are all due to Jónsson-Tarski [74] (see also Jónsson-Tarski [73]). Corollary 19.4, Theorem 19.5 and Corollary 19.6 are given in Theorem 3.9 of [74], while the first assertion of Theorem 19.11 is given in Theorem 3.10 of [74]. The part of Theorem 19.11 that concerns the preservation of positive equations, and part of the assertion that concerns the preservation of certain implications are also from [74]; see the historical remarks in Section 14.11. Versions of Theorem 19.7 and Corollary 19.8, Theo-

rem 19.9, and Corollary 19.10 are given in Henkin-Monk-Tarski [50] (see Corollaries 2.7.36 and 2.7.37 in that work).

The problem of axiomatizing the class of atom structures of various classes of complete and atomic algebras of logic dates back, in one form or another, at least to the 1950s. Lyndon [88] (see Section 4 of that paper) axiomatizes the class of complete and atomic relation algebras by imposing conditions on the atoms of the algebras. Jónsson-Tarski [75] (in Theorems 5.6 and 5.7 of that paper) characterize the relational structures that are atom structures of complete and atomic relation algebras with functional atoms: they are just the *generalized Brandt groupoids*. As a consequence of this characterization, they conclude (in Theorem 5.8 of the paper) that a non-degenerate relation algebra is representable if and only if it is embeddable into the complex algebra of a generalized Brandt groupoid. In other words, they axiomatize a class K of relational structures such that $\mathsf{S}(\mathsf{Cm}(\mathsf{K}))$ is the class of all non-degenerate representable relation algebras. Jónsson [67] (in the beginning of the proof of Theorem 4 in that paper) gives an axiomatization of the class of relational structures that are atom structures of complete and atomic, symmetric, integral relation algebras. (The axiomatization of this class that is given in Exercise 19.8 below is a somewhat improved version of Jónsson's original axiomatization.) Henkin-Monk-Tarski [50] (in Theorems 2.7.39 and 2.7.40 of that book) axiomatize the class of relational structures that are atom structures of complete and atomic cylindric algebra of dimension α. Maddux [95] (in Theorem 2.2 of his paper—see also Theorem 346 in Maddux [101]) gives an axiomatization of the class of relational structures whose complex algebras are relation algebras. Another closely related axiomatization of this class is given in Hirsch-Hodkinson [59] (see Definition 3.22 and Lemmas 3.23 and 3.24 in that book). The axiomatization given in Theorem 19.12 is close in spirit to the axiomatizations of Maddux and of Hirsch-Hodkinson. In particular, the observation that it is not necessary to include a relational analogue of Axiom (R6) is due to Maddux.

The use of cycles to construct relation algebras was first introduced by Lyndon [88], and has played an important role in the work of several authors, including Andréka-Németi and Maddux. The first two examples at the end of Section 19.3 come from Maddux [101]. The final example is due to J. C. C. McKinsey, and was constructed by him in the early 1940s in order to show that the associative law for rela-

tive multiplication is independent of the remaining axioms of relation algebra.

A version of the translation mapping in Definition 19.17 and the semantic analysis of this mapping in Lemma 19.18, in a form applicable to relation algebraic terms, was already used by Tarski in 1941 to assess the expressive power of the calculus of relations. The roots of Tarski's translation mapping undoubtedly date back to the works of Peirce and Schröder; see Tarski [132], Tarski-Givant [147] (in particular, items 2.3(iii)–(iv)), and the appendix to the first volume of the present work. Related examples of the correspondence between algebraic equations and first-order properties of relations occur in Jónsson-Tarski [74]. In the context of modal logic, the translation mapping in Definition 19.17 is called the *standard translation*. According to Blackburn-de Rijke-Venema [18] (see pp. 119-120 of that work),

> The standard translation, in various forms, can be found in the work of a number of writers on modal and tense logic in the 1960s—but its importance only became fully apparent [in the 1970s]. ... although other authors (notably Sahlqvist [120]) helped pioneer correspondence theory, it was the work of van Benthem [12] which made clear the importance of systematic use of the standard translation... .

It was Venema and de Rijke who linked the Sahlqvist/van Benthem correspondence theory to equations and complex algebras.

Theorem 19.20, asserting that the class of atom structures of a variety of Boolean algebras with complete operators is elementary, is due to Venema [150]. The presentation in Section 19.4 leading up to this theorem has been influenced by the presentation of the same result in Section 2.7.4 of Hirsch-Hodkinson [59] (see Lemma 2.76 and Theorem 2.84 in that book). Theorems 19.29 and 19.30 in their present form are due to Givant, but the parts of these theorems that concern equations are closely related to Theorem 2.79 and Corollary 2.80 in [59] (see Remark 2.82 in that work and see Exercise 19.32 below).

The notion of a bounded homomorphism between relational structures has its roots in modal logic, where the case of relations of rank two was studied (under a different name) by Krister Segerberg [122]; see also Richard Scott Pierce [116]. The general definition and the terminology that is used in Section 19.5 is due to Robert Ian Goldblatt [45]. Theorem 19.32 (without the conclusion of the completeness of the homomorphism φ and without the assertion that if φ is one-to-one or onto, then ϑ is onto or one-to-one respectively) is given in

Theorem 2.3.1 of Goldblatt [45]. An earlier version of this theorem that applies to relational structures with a single binary relation is due to Steven Thomason [148], who also stated a dual to his version of the theorem (see his Theorem 1). The dual in its general form (see Exercise 19.40), without the epi-mono duality, is implicit in Theorem 3.1.5 of Jónsson [71]. The notion of an inner substructure is developed in Goldblatt [45], where a version of Corollary 19.35 is proved (see Lemma 3.2.2 in that paper); the notion itself (in a more general framework) is due to Solomon Feferman [29]. The special case of Corollary 19.38 in which the relational structures have a single binary relation is formulated in Corollary 1 of Thomason [148] and in Theorem 6.5 of Goldblatt [44]. The general result for arbitrary similarity types is formulated in Lemma 3.4.1 of Goldblatt [45]. The internal version of this result that is given in Theorem 19.37 is due to Givant [39] (see Theorem 1.32 in that work). For a general development of various duality theories that concern Boolean algebras with operators, and for further historical remarks, the reader is referred to [39].

Monk [110] proved that an ultraproduct of geometric complex algebras is embeddable into the complex algebra of the corresponding ultraproduct of the underlying geometries (see Lemma 18.32 and the historical remarks at the end of Chapter 18). Generalizing Monk's result, Goldblatt [45] proved that an ultraproduct $(\prod_i \mathfrak{Cm}(U_i))/D$ of any system of complex algebras is embeddable into the complex algebra of the corresponding ultraproduct $(\prod_i \mathfrak{U}_i)/D$ of the underlying relational structures. The stronger version of Goldblatt's theorem that is given in Theorem 19.40, namely that the completion of the ultraproduct $(\prod_i \mathfrak{Cm}(U_i))/D$, is isomorphic to the complex algebra of the relational structure $(\prod_i \mathfrak{U}_i)/D$ is due to Givant [39] (see Theorem 1.35 in that work). The important observation in Lemma 19.39 that the operators in the first ultraproduct are complete is missing from Givant's proof, and was first established by Andréka-Gyenis-Németi [8]. The proof of the lemma given here is a variant of their proof.

Lemma 19.47 and the results leading up to it—in particular, Lemmas 19.41 and 19.46—are from Goldblatt [46]. Theorems 19.50, 19.51, and 19.52, Corollary 19.53, and Theorems 19.54 and 19.55 are due to Goldblatt [45], [46]. According to Goldblatt [45], the idea for his first proof of Theorem 19.55 had its origins in the work of Kit Fine [30],

> who showed that a 2-saturated model of the language of propositional modal logic has a [bounded] homomorphism onto the canonical model of its modal theory.

In [46], Goldblatt writes as follows.

> It was shown by van Benthem [13], by model-theoretic arguments, that if a modal logic L is characterized by a class of Kripke structures that is elementary (i.e., defined by a set of first-order sentences), then L is validated by its Henkin structures.

Goldblatt then introduces some terminology and notation that allow him to give the following algebraic reformulation of van Benthem's result: if K is an elementary class of relational structures with a single binary relation, then the variety generated by $\mathsf{Cm}(\mathsf{K})$ is closed under canonical extensions. This result is extended in Goldblatt [45] to classes of relational structures of arbitrary similarity type, and the hypothesis that the class K is elementary is weakened to the hypothesis that K is closed under ultraproducts. A somewhat different proof of this extension is given in Goldblatt [46]. This last paper also gives versions of the theorem that apply to universal classes and to quasi-varieties (see Theorems 19.52 and 19.54). Lemmas 19.48 and 19.49 are just the appropriate versions of the standard model-theoretic result that ω-saturated elementary extensions really do exist.

Theorem 19.56 is from Givant [36]. Tarski [143] proved that if L is the class of all group complex algebras, then $\mathsf{S}(\mathsf{L})$ is a universal class. The fact that $\mathsf{SP}(\mathsf{L})$ is the variety generated by L was discovered by Givant in 1970–71 and formed part of his seminar report for Tarski's course/seminar on relation algebras (see [146]). McKenzie [106] raised the problem whether the class $\mathsf{S}(\mathsf{L})$ is closed under canonical extensions. This question was answered positively in Givant [36] (see Theorem 19.57).

The treatment of projective geometries as relational structures with one ternary relation, and the axiomatization of the class of these structures that is given in Section 19.9, are from Givant [36] (see Exercise 19.56 below and its solution), as are the correlated results in Exercises 19.57, 19.61, and 19.62. Theorem 19.58 and the observations following the theorem are also from [36]. The following very special case of the theorem was proved earlier in Andréka-Givant-Németi [6]: if L is the class of all complex algebras of projective geometries of dimension one and order at least three, then $\mathsf{S}(\mathsf{L})$ is a universal class. The results in Exercises 19.58 and 19.59 are due to Lyndon [90]. The proof of Exercise 19.58 given in the "Hints and Solutions" is due to Givant.

The results for complex algebras of modular lattices with zero that correspond to Theorem 19.58 are given in Exercise 19.64, and are also due to Givant [36].

Tarski [143] proved that if L is the class of all full set relation algebras on non-empty sets, then the class $\mathsf{S}(\mathsf{L})$ of all simple representable relation algebras is universal, and the class $\mathsf{SP}(\mathsf{L})$ of all representable relation algebras is a variety (see Theorem 19.59). Monk showed that both $\mathsf{S}(\mathsf{L})$ and $\mathsf{SP}(\mathsf{L})$ are closed under canonical extensions (see the historical remarks in Section 16.10). The proof of Theorem 19.59 given in Section 19.9 is due to Givant [36]. Venema [149] showed that the class K_0 involved in the proof is elementary, and not just pseudo-elementary (see Exercise 19.67). Blackburn–de Rijke–Venema [18] use Venema's result to give a new proof that the class of representable relation algebras is closed under canonical extensions. The observations following Theorem 19.59 are from [36]. One of these observations—in fact, the theorem asserting that the class of relation algebras embeddable into relation algebras of the form $\mathfrak{Re}(E)$, for some equivalence relation E all of whose equivalence classes are infinite, is a variety—is of earlier origins. It was proved by Givant in 1973 and published in Tarski-Givant [147] (see Section 8.4 of that work).

The roots of polyalgebras go back at least to the treatment of generalized Brandt groupoids in Jónsson-Tarski [75]. Complex algebras of groups and projective geometries were early examples of relation algebras that were constructed as complex algebras of polyalgebras. The general notion of a polygroupoid was formulated and studied by Comer (see, for example, [25]), and Theorem 19.60 is due to him.

The first example in the solution to Exercise 19.33 is due to Andréka, and the second example is due to Givant. The result in Exercise 19.60 is due to Andréka and Givant. The result in Exercise 19.72 is due to Maddux [94] and answers positively the question, first posed by Jónsson [67], whether every modular lattice is isomorphic to a lattice of commuting equivalence elements of some relation algebra.

Exercises

19.1. Fill in the missing details in the proof of Theorem 19.5 by showing that the function φ preserves the operation $\breve{}$ on atoms, and maps

the set of atoms below the identity element in $\mathfrak{Cm}(U)$ bijectively to the set of atoms below the identity element in $\mathfrak{A}$.

19.2. Fill in the missing details in the proof Theorem 19.7 by showing that the function ϑ preserves the binary relation C and the unary relation I.

19.3. Prove Corollary 19.8.

19.4. Prove Corollary 19.10.

19.5. Extend the definitions and results in Section 19.1 to Boolean algebras with complete operators of arbitrary ranks.

19.6. Prove that conditions (iii) and (iv) in Theorem 19.12 are respectively equivalent to the corresponding conditions in Theorem 2.9, under the assumptions that the relation C in Theorem 19.12 is a function and the operation $^\smile$ in Theorem 2.9 maps atoms to atoms.

19.7. Prove Theorem 19.12 directly, without using Theorem 2.9.

19.8. Prove that the complex algebra of a relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

is a symmetric, integral relation algebra if and only if C is the identity relation on U, and the following conditions hold for all p, r, s, and t in $\mathfrak{U}$.

(i) If there is an element q such that $R(r, q, p)$ and $R(s, t, q)$, then there is an element q such that $R(q, t, p)$ and $R(r, s, q)$.
(ii) There is a unique element s such that $I(s)$, and for this element s we have $R(r, s, p)$ if and only if $p = r$.
(iii) $R(r, s, t)$ implies $R(s, r, t)$.
(iv) $R(r, s, t)$ implies $R(r, t, s)$.

19.9. Prove Corollary 19.13.

19.10. Prove Corollary 19.14.

19.11. Prove that conditions (iii) and (iv) in Theorem 19.12 are satisfied in a relational structure $\mathfrak{U}$ if and only if the presence of any one of the triples

$$(r, s, t),\ (r^\smile, t, s),\ (s, t^\smile, r^\smile),\ (s^\smile, r^\smile, t^\smile),\ (t, s^\smile, r),\ (t^\smile, r, s^\smile)$$

in R implies that the remaining triples are also in R.

19.12. Assume that a relational structure $\mathfrak{U}$ satisfies conditions (iii) and (iv) in Theorem 19.12. Prove that $\mathfrak{U}$ satisfies condition (ii) if and only if for each element r in $\mathfrak{U}$, there are unique elements s and t in I such that the identity cycles $[r, s, r]$ and $[t, r, r]$ are included in R. If the complex algebra of $\mathfrak{U}$ is a relation algebra, prove that there can be no other identity cycles in $\mathfrak{U}$ except the ones just mentioned.

19.13. Describe all situations in which a cycle contains fewer than six distinct triples.

19.14. Prove that two triples of elements from a relation algebraic relational structure generate either the same cycle or disjoint cycles.

19.15. Show that the relational structure defined in the first example of Section 19.3, just before Table 19.1, satisfies the associativity condition in Theorem 19.12(i) and is therefore a relation algebraic relational structure.

19.16. Verify that the entries in Table 19.1 follow from the defining cycles of the first example in Section 19.3.

19.17. Show that the relational structure $\mathfrak{U}$ defined in the second example of Section 19.3, just before Table 19.2, satisfies the associativity condition in Theorem 19.12(i) and is therefore a relation algebraic relational structure.

19.18. Verify that the entries in Table 19.2 follow from the defining cycles of the second example in Section 19.3.

19.19. Describe the atom structure of each of the relation algebras in Exercise 3.36 by giving the universe of the structure, the set I of subidentity atoms, the set S of symmetric atoms, the pairing of non-symmetric atoms in the set $U \sim S$, the identity cycles, and the diversity cycles.

19.20. Describe the atom structure of each of the relation algebras in Exercise 3.37 by giving the universe of the structure, the set I of subidentity atoms, the set S of symmetric atoms, the pairing of non-symmetric atoms in the set $U \sim S$, the identity cycles, and the diversity cycles.

19.21. Describe the atom structure of each of the relation algebras in Exercise 3.40 by giving the universe of the structure, the set I of

subidentity atoms, the set S of symmetric atoms, the pairing of non-symmetric atoms in the set $U \sim S$, the identity cycles, and the diversity cycles.

19.22. Describe the atom structure of each of the relation algebras in Exercise 3.41 by giving the universe of the structure, the set I of subidentity atoms, the set S of symmetric atoms, the pairing of non-symmetric atoms in the set $U \sim S$, the identity cycles, and the diversity cycles.

19.23. Suppose $\mathfrak{U}$ is a relational structure in which the universe U is the set $\{1', r, s\}$, the set S of symmetric elements coincides with U, the set I of subidentity elements is $\{1'\}$, and the identity cycles are

$$[1', 1', 1'], \quad [r, 1', r], \quad [s, 1', s].$$

Construct the relative multiplication table for the atoms of the complex algebra of $\mathfrak{U}$ under each of the following assumptions.

(i) The diversity cycles of $\mathfrak{U}$ are $[r, r, r]$, $[s, s, s]$, and $[r, s, s]$.
(ii) The diversity cycles of $\mathfrak{U}$ are $[r, s, s]$ and $[s, r, r]$.
(iii) The diversity cycles of $\mathfrak{U}$ are $[r, r, r]$, $[s, s, s]$, $[r, s, s]$, and $[s, r, r]$.

19.24. Suppose $\mathfrak{U}$ is a relational structure in which the universe U is the set $\{1', r, s, t\}$, the set S of symmetric elements coincides with U, the set I of subidentity elements is $\{1'\}$, and the identity cycles are

$$[1', 1', 1'], \quad [r, 1', r], \quad [s, 1', s], \quad [t, 1', t].$$

Construct the relative multiplication table for the atoms of the complex algebra of $\mathfrak{U}$ under each of the following assumptions.

(i) The diversity cycles are $[r, s, s]$, $[r, t, t]$, and $[s, t, t]$.
(ii) The diversity cycles are $[r, s, s]$, $[s, r, r]$, $[r, t, t]$ and $[s, t, t]$.
(iii) The diversity cycles are $[r, r, r]$, $[s, s, s]$, $[r, s, s]$, $[r, t, t]$, $[s, t, t]$, and $[r, s, t]$.
(iv) The diversity cycles are $[r, r, r]$, $[s, s, s]$, $[t, t, t]$, $[r, s, s]$, $[s, r, r]$, $[r, t, t]$, $[t, r, r]$, $[s, t, t]$, $[t, s, s]$, and $[r, s, t]$.

19.25. Suppose $\mathfrak{U}$ is a relational structure in which the universe U is the set $\{1', r, s, t\}$, the set S of symmetric elements is $\{1', r\}$, the set I of subidentity elements is $\{1'\}$, the elements s and t are paired (so that $t = s^{\smile}$), and the identity cycles are

$$[1', 1', 1'], \quad [r, 1', r], \quad [s, 1', s], \quad [1', s, s].$$

Construct the converse and relative multiplication tables for the atoms of the complex algebra of $\mathfrak{U}$ under each of the following assumptions.

(i) The diversity cycles are $[s, s, s]$, $[s, s, s^\smile]$, $[r, s, s]$, and $[s, r, s]$.
(ii) The diversity cycles are $[r, r, r]$, $[s, s, s]$, $[r, s, s]$ and $[s, r, r]$.
(iii) The diversity cycles are $[r, r, r]$, $[s, s, s]$, $[s, s, s^\smile]$, $[r, s, s]$, $[s, r, s]$, and $[s, s, r]$.
(iv) The diversity cycles are $[r, r, r]$, $[s, s, s^\smile]$, $[r, s, s]$, $[s, r, s]$, $[s, r, r]$, and $[s, s, r]$.
(v) The diversity cycles are $[r, r, r]$, $[s, s, s]$, $[s, s, s^\smile]$, $[r, s, s]$, $[s, r, s]$, $[s, r, r]$, and $[s, s, r]$.

19.26. Suppose $\mathfrak{U}$ is a relational structure in which the universe U is the set $\{1', q, r, s, t\}$, the set S of symmetric elements coincides with the set I of subidentity elements, which is $\{1'\}$, the elements q and r are paired, as are the elements s and t (so that $r = q^\smile$ and $t = s^\smile$), and the identity cycles are

$$[1', 1', 1'], \quad [q, 1', q], \quad [1', q, q], \quad [s, 1', s], \quad [1', s, s].$$

Construct the converse and relative multiplication tables for the atoms of the complex algebra of $\mathfrak{U}$ under each of the following assumptions.

(i) The diversity cycles are $[q, q, q]$, $[s, s, s]$, $[q, q, q^\smile]$, $[s, s, s^\smile]$, $[s, q, q]$, and $[q, s, q]$.
(ii) The diversity cycles are $[s, s, s]$, $[s, s, s^\smile]$, $[q, q, s]$, $[q, q, s^\smile]$, $[s, q, q]$, and $[q, s, q]$.
(iii) The diversity cycles are $[q, q, q]$, $[s, s, s]$, $[q, q, q^\smile]$, $[q, q, s]$, $[q, q, s^\smile]$, $[s, q, q]$, $[q, s, q]$, $[q, s, s]$, and $[s, q, s]$.
(iv) The diversity cycles are $[q, q, q]$, $[s, s, s]$, $[q, q, q^\smile]$, $[s, s, s^\smile]$, $[q, q, s]$, $[q, q, s^\smile]$, $[s, s, q]$, $[s, s, q^\smile]$, $[s, q, q]$, $[q, s, q]$, $[q, s, s]$, and $[s, q, s]$.

19.27. Let V be the variety of symmetric relation algebras, and $\mathsf{At}(\mathsf{V})$ the class of atom structures of atomic algebras in V. Find an axiomatization of $\mathsf{At}(\mathsf{V})$ that is an extension of the axiomatization of the class of atom structures of atomic relation algebras given in Theorem 19.16. Do the same for the class $\mathsf{At}(\mathsf{V})$ when V is the variety of abelian relation algebras.

19.28. Give the translation, in the sense of Definition 19.17, of each of the following relation algebraic terms.

(i) $v_0 \,;\, 1$.
(ii) $v_0^{\smile\smile}$.
(iii) $(v_0 \,;\, v_1) \cdot v_2$.
(iv) $(v_0^{\smile} \,;\, v_2) \cdot v_1$.
(v) $(v_0 \,;\, v_0) \,;\, v_0$.
(vi) $(v_0 \,;\, v_1) \,;\, v_2$.
(vii) $v_0 \,;\, (v_1 \,;\, v_2)$.
(viii) $(v_0 \,;\, v_1)^{\smile}$.
(ix) $(v_0 \,;\, v_1)^{\smile}$.
(x) $v_1^{\smile} \,;\, v_0^{\smile}$.

19.29. Prove that for each relation algebraic term γ, the free variables in the translation Γ_γ are precisely w and the free variables of γ.

19.30. Complete the proof of Lemma 19.18 by treating the cases when γ is $-\sigma$ and when γ is $\sigma^{\smile}$.

19.31. Complete the proof of Lemma 19.22 by treating the case when γ is $\sigma^{\smile}$.

19.32. Suppose γ is a term built up from variables and constants using only the symbols for $\cdot$ (multiplication) and the operators $;$ and $^{\smile}$. Prove that in any Boolean algebra with normal operators $\mathfrak{A}$, the term γ defines a normal operation of the same rank as the number of distinct variables that occur in γ.

19.33. Show that the implication from right to left in Lemma 19.25 may fail if the phrase "sequences of quasi-atoms" is replace by "sequences of atoms". Can this modified form of the corollary fail in an atomic relation algebra?

19.34. Prove Corollary 19.28.

19.35. Give an axiomatization of the class $\mathsf{At}(\mathsf{RA})$ (of atom structures of atomic relation algebras) that consists of the correspondents of (R4)–(R7) and (R11). Compare this axiomatization to the one given in Theorem 19.16.

19.36. Use Theorem 19.30 and the remark following it to give an axiomatization of the class of atom structures of atomic, symmetric relation algebras.

19.37. Use Theorem 19.30 and the remark following it to give an axiomatization of the class of atom structures of atomic, abelian relation algebras.

19.38. If a bounded homomorphism between relational structures is one-to-one and onto, prove that it must be an isomorphism.

19.39. Complete the proof of Theorem 19.32 by showing that the mapping φ defined in the statement of the theorem preserves the operator $\breve{}$.

19.40. Prove the following dual of Theorem 19.32. If $\mathfrak{U}$ and $\mathfrak{V}$ are relational structures, and φ a complete homomorphism from $\mathfrak{Cm}(V)$ to $\mathfrak{Cm}(U)$, then the function ϑ defined on elements u in $\mathfrak{U}$ by

$$\vartheta(u) = r \qquad \text{if and only if} \qquad r \in \bigcap\{X \subseteq V : u \in \varphi(X)\},$$

or, equivalently, by

$$\vartheta(u) = r \qquad \text{if and only if} \qquad u \in \varphi(\{r\}),$$

is a bounded homomorphism from $\mathfrak{U}$ to $\mathfrak{V}$. Moreover, ϑ is one-to-one if and only if φ is onto, and ϑ is onto if and only if φ is one-to-one.

19.41. Complete the proof of Lemma 19.39 by treating the case of the left-hand complete distributivity of the operator ; and the case of the complete distributivity of the unary operator $\breve{}$.

19.42. Formulate and prove a version of Lemma 19.39 that applies to Boolean algebras with quasi-complete operators, and then derive Lemma 19.39 as a corollary to that version.

19.43. Let $\mathfrak{U}$ be the direct product of a non-empty system $(\mathfrak{U}_i : i \in J)$ of non-empty relational structures, and suppose D is an ultrafilter on the index set J. Prove that the relation $\equiv_D$ on the set $U = \prod_i U_i$ that is defined by

$$r \equiv_D s \qquad \text{if and only if} \qquad \{i \in J : r(i) = s(i)\} \in D$$

is a congruence relation on $\mathfrak{U}$.

19.44. Prove that the definitions of the fundamental relations in an ultraproduct of relational structures do not depend on the specific representatives of the congruence classes that are being used.

19.45. Show that Theorem 19.40 may fail if some of the relational structures in the given system are empty.

19.46. Derive Lemma 19.41 directly from the Fundamental Theorem of Ultraproducts.

19.47. Prove Corollary 19.42 directly, without using Lemma 19.41.

19.48. Prove Lemma 19.44.

19.49. Prove Corollary 19.45 using Lemma 19.44.

19.50. Prove Corollary 19.45 directly, without using Lemma 19.44.

19.51. Let $(\mathfrak{U}_i : i \in J)$ and $(\mathfrak{V}_i : i \in J)$ be disjoint systems of relational structures, and $\mathfrak{U}$ and $\mathfrak{V}$ their respective unions. For each index i, assume that ϑ_i is a homomorphism, respectively a bounded homomorphism, from $\mathfrak{U}_i$ into $\mathfrak{V}_i$. Prove that the union of this system of mappings is a homomorphism, respectively a bounded homomorphism, from $\mathfrak{U}$ into $\mathfrak{V}$. Prove further that the union mapping is one-to-one or onto if and only if each of the mappings ϑ_i is one-to-one or onto respectively.

19.52. Prove that in Lemma 19.48, the elementary extension $\mathfrak{V}^*$ may be taken to be isomorphic to an ultrapower of $\mathfrak{W}^*$.

19.53. Complete the proof of Theorem 19.50 by showing that the function ϑ preserves the relation C and is bounded with respect to this relation.

19.54. Prove Theorem 19.51 directly, without using Theorem 19.50 or the lemmas leading up to that theorem.

19.55. Let $(\mathfrak{A}_i : i \in J)$ be a non-empty system of algebras and

$$\mathfrak{A} = \prod\{\mathfrak{A}_i : i \in J\}$$

the direct product of this system. Consider an ultrafilter D on J and a subset J_0 of J that does not belong to D. Write

$$J_1 = J \sim J_1, \quad \mathfrak{A}_1 = \prod\{\mathfrak{A}_i : i \in J_1\}, \quad D_1 = \{X \cap J_1 : X \in D\}.$$

(i) Prove that D_1 is included in D.

(ii) Prove that a subset X of J belongs to D if and only if the intersection $X \cap J_1$ belongs to the set D_1.
(iii) Prove that D_1 is an ultrafilter on J_1.
(iv) For each element r in $\mathfrak{A}$, let $\bar{r}$ be the element in $\mathfrak{A}_1$ that is the restriction of r to the set J_1. Prove that the function ϑ defined by $\vartheta(r/D) = \bar{r}/D_1$ for each r in $\mathfrak{A}$ is a well-defined isomorphism from $\mathfrak{A}/D$ to $\mathfrak{A}_1/D_1$.

19.56. Prove that the conception of a projective geometry as a relational structure with a single ternary collinearity relation satisfying the collinearity axioms given in the discussion after Theorem 19.57, and the standard conception of a projective geometry as discussed at the beginnings of Sections 3.6 and 17.4, are definitionally equivalent.

19.57. Prove that the complex algebra of a projective geometry as defined in Sections 3.6 and 17.4 coincides with the complex algebra of the associated geometric relational structure.

19.58. Prove that a relation algebra $\mathfrak{A}$ is isomorphic to the complex algebra of a projective geometry of order at least three if and only if the following conditions hold.

(i) $\mathfrak{A}$ is complete and atomic.
(ii) 1' is an atom.
(iii) $p^\smile = p$ for every atom p in $\mathfrak{A}$.
(iv) $p\,;p = p + 1'$ for every atom $p \neq 1'$.

19.59. Prove that a relation algebra $\mathfrak{A}$ is isomorphic to the complex algebra of a projective geometry of order two if and only if the following conditions hold.

(i) $\mathfrak{A}$ is complete and atomic.
(ii) 1' is an atom.
(iii) $p^\smile = p$ for every atom p in $\mathfrak{A}$.
(iv) $p\,;p = 1'$ for every atom p.

19.60. Prove that the atom structure of the complex algebra of a projective geometry of order two is also the atom structure of the complex algebra of a Boolean group, and conversely. Conclude that the class of complex algebras of projective geometries of order two is equal to the class of complex algebras of Boolean groups.

19.61. Prove that the complex algebra of a relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

coincides with the complex algebra of projective geometry of order at least three if and only if C is the identity relation on U, and the following conditions hold for all p, r, s, and t in $\mathfrak{U}$.

(i) If there is an element q such that $R(r, q, p)$ and $R(s, t, q)$, then there is an element q such that $R(q, t, p)$ and $R(r, s, q)$.
(ii) There is a unique element s such that $I(s)$, and for this element s we have $R(r, s, p)$ if and only if $p = r$.
(iii) $R(r, s, t)$ implies $R(s, r, t)$.
(iv) $R(r, s, t)$ implies $R(r, t, s)$.
(v) Not $I(r)$ implies that $R(r, r, s)$ if and only if $s = r$ or $I(s)$.

19.62. Prove that the complex algebra of a relational structure

$$\mathfrak{U} = (U\,,\, R\,,\, C\,,\, I)$$

coincides with the complex algebra of projective geometry of order two if and only if the conditions of the preceding exercise hold, with (v) replaced by the following condition: Not $I(r)$ implies that $R(r, r, s)$ if and only if $I(s)$.

19.63. Prove that if each of two classes of relational structures is axiomatized by a finite set of first-order sentences, then the same is true of the union of the two classes. Use this observation, together with Exercises 19.61 and 19.62, to give another proof that the class of geometric relational structures is elementary.

19.64. Suppose L is the class of complex algebras of modular lattices with zero. Prove that $\mathsf{SP}(\mathsf{L})$ is a variety and that $\mathsf{S}(\mathsf{L})$ is the universal class of simple algebras in this variety. Prove further that both the variety and the universal class are closed under canonical extensions.

19.65. Suppose $\mathfrak{U} = (U \times U\,,\, R\,,\, C\,,\, I\,,\, F)$ is one of the relational structures defined in the final example of Section 19.9, and let $\mathfrak{U}_0$ be the structure obtained from $\mathfrak{U}$ by deleting the relation F. Prove that the complex algebra of $\mathfrak{U}_0$ is just the full set relation algebra on the set U.

19.66. Express the axioms given in statements (1)–(5) of the final example of Section 19.9 as sentences in the first-order language of the structures

$$\mathfrak{U} = (U \times U\,,\, R\,,\, C\,,\, I\,,\, F).$$

Prove that these sentences axiomatize the class of all isomorphic copies of these structures.

19.67. Let K be the class of relational structures of the form

$$\mathfrak{U} = (U \times U\,,\, R\,,\, C\,,\, I\,,\, F)$$

that are defined in the final example of Section 19.9, together with all isomorphic copies of these structures. Prove that for each structure in K, the relation F is definable in terms of the relations I and R. Conclude that the class K_0 of relational structures obtained from structures in K by omitting the relation F is in fact elementary, and not just pseudo-elementary. Give a set of axioms for this class of relational structures.

19.68. Prove Theorem 19.60.

19.69. Prove directly, without using the results of Section 3.6, that the polyalgebra associated with a projective geometry of order at least three (see the second example after Theorem 19.60) satisfies the conditions of Theorem 19.60. Conclude that the complex algebra of this polyalgebra is a symmetric, integral relation algebra.

19.70. Consider the polyalgebra associated with a projective geometry P of order two, defined in analogy with the second example after Theorem 19.60. Does this polyalgebra satisfy the conditions of Theorem 19.60, so that the complex algebra of the polyalgebra is a relation algebra? If not, what goes wrong, and how must the definition of the polyalgebra be modified so that its complex algebra is a relation algebra?

19.71. Prove directly, without using the results in Section 3.7, that the complex algebra of the polyalgebra associated with a modular lattice with zero (see the final example after Theorem 19.60) satisfies the conditions in Theorem 19.60. Conclude that the complex algebra of this polyalgebra is a symmetric, integral relation algebra.

19.72. Prove that every modular lattice can be embedded into the lattice of commuting equivalence elements of a commutative relation algebra (see Corollary 5.17).

Epilogue

There are a number of important topics in the theory of relation algebras that could not be treated in the present volumes, if for no other reason than space constraints. Some of them are briefly described below.

One of Tarski's original goals in reformulating the calculus of relations as an abstract algebraic theory axiomatized by finitely many equations was to show that this theory can serve as a finitely axiomatized, variable-free equational foundation for all of mathematics. More precisely, some time during the period 1941–1942, Tarski proved that set theory and number theory, and hence all of classical mathematics, can be formalized and developed in a version of the equational theory of relation algebras that contains no quantifiers, no sentential connectives, and no variables whatsoever. Using these results, Tarski was able to show that there is even a subsystem of the ordinary two-valued sentential logic that can serve as a foundation for all of mathematics. Thus, applications of the calculus of relations to logic and to the foundations of mathematics were uppermost in Tarski's mind from the start.

Tarski noted in [132] that every equation of the calculus of relations is logically equivalent to a sentence in first-order logic with just three variables, and some time between 1941 and 1942 he proved that the converse of this statement is also true: every first-order sentence with just three variables is logically equivalent to an equation in the calculus of relations. He posed the following problem. Does there exist an algorithm for deciding whether a given first-order sentence (about binary relations) is logically equivalent to an equation in the theory of relation algebras, or equivalently, to a first-order sentence with just

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9

three variables? Kwatinetz [76] solved this problem in the negative by showing that for every natural number $n \geq 3$, there can be no algorithm for deciding whether a sentence of first-order logic is equivalent to a sentence with at most n variables.

The problem naturally arises of clarifying the relationship between the theory of relation algebras and a system $\mathcal{L}_3$ of first-order logic with just three variables. Tarski knew from an example due to McKinsey that the associative law for relative multiplication, or more precisely, a translation of this law into the language of $\mathcal{L}_3$ is not derivable from the logical axioms of $\mathcal{L}_3$. If one adds this translation (in the form of a schema) to the axioms of $\mathcal{L}_3$, then one arrives at a version $\mathcal{L}_3^*$ of first-order logic with just three variables that Tarski and Givant proved is equivalent to the theory of relation algebras both in its expressive and its deductive power. Consequently, all of mathematics may also be developed in a system of first-order logic with just three variables.

The manuscript Tarski [133] contains most of these results, but it was never published. Some of the results were announced in Tarski [137], [138], and [139]. A systematic presentation of all of the results, with detailed proofs, was published in Tarski-Givant [147]. Informal expositions may be found in Givant [34], [38].

What about the logical system $\mathcal{L}_3$ itself? Maddux [93] proved that $\mathcal{L}_3$ is equivalent to a version of the theory of relation algebras in which the associative law for relative multiplication is replaced by a weaker version of this law called the *semi-associative law*. A good reference for many of Maddux's results is his book [101]. Imagine now a formalism $\mathcal{L}_n$ for each natural number $n \geq 4$ in which we are allowed to prove sentences with 3 variables by means of derivations that use up to n variables, but no more. The union of this chain of logical systems corresponds to, or is equivalent to, the (equational) theory of representable relation algebras, and as mentioned above, $\mathcal{L}_3$ is equivalent to the theory of relation algebras with a weakened associativity law. What does $\mathcal{L}_4$ correspond to? Maddux [93] showed that it corresponds exactly to the theory of relation algebras, or in different words, $\mathcal{L}_4$ is equivalent to $\mathcal{L}_3^*$. For each $n > 4$, the formalism $\mathcal{L}_n$ corresponds to some theory of relation algebras that is included in the theory of representable relation algebras. Hirsch-Hodkinson-Maddux [60] (see also [61] and [59]) proved that for $n > 4$, this theory of relation algebras is not finitely axiomatizable over the theory for $n - 1$. As a consequence of this theorem, they concluded that for each $n > 4$, there exist logically true sentences, containing just three variables, with the

following property: they can be proved using n variables, but no proofs exist that use fewer than n variables. This sharpened an earlier result of Monk [111], who proved that for each $n > 4$, there exist logically true sentences, containing just three variables, that cannot be proved with fewer than n variables (but he did not show that they can be proved using exactly n variables).

Underlying the results of the preceding paragraph, there are important interconnections between the theory of relation algebras and the theory of cylindric algebras. The study of these interconnections, in particular, the construction of relation algebras from cylindric algebras, was pioneered by Monk in his doctoral dissertation [109]. Monk's work was extensively developed, and various constructions of cylindric algebras from relation algebras were introduced and developed, by Maddux, Hirsch and Hodkinson, Németi, and Simon. One aspect of this work has been the introduction by Maddux [96] of the notion of an n-dimensional relational basis of a non-associative relation algebra—an algebra in which all the axioms of relation algebra, except the associative law for relative multiplication, are required to hold—and the study, by Maddux and by Hirsch and Hodkinson, of classes RA_n of algebras embeddable into complete and atomic non-associative relation algebras that possess such a basis. In particular, Maddux [92] showed that the associative law for relative multiplication axiomatizes RA_4 over RA_3, while Hirsch and Hodkinson [55] proved that RA_n is not finitely axiomatizable over RA_{n-1} for $n > 4$. Good references for these results include [101] and [59]. Other aspects of the interconnections between relation algebras and cylindric algebras have involved the introduction and study of other types of bases for relation algebras, for example, cylindric bases and hyperbases. Hirsch-Hodkinson [56], [59] proved that for each $n \geq 4$, the class of algebras embeddable into atomic relation algebras with n-dimensional hyperbases coincides with the class of algebras embeddable into relation algebraic reducts of n-dimensional cylindric algebras. For $n = 4$, this class coincides with the class of all relation algebras, by results of Maddux [92] (see also Maddux [98]), and for $n > 4$, Hirsch and Hodkinson proved that it is not finitely axiomatizable over the class for $n-1$. They also demonstrated that each of these classes is different from each of the classes RA_m for all $m, n \geq 5$. Finally, Hodkinson [63] showed how to construct from any simple atomic relation algebra $\mathfrak{A}$, and for any integer $n \geq 3$, an n-dimensional cylindric algebra that is completely representable if and only if $\mathfrak{A}$ is completely representable. This con-

struction permitted him to transfer to cylindric algebras a number of results that hold for relation algebras. In particular, he concluded that for each integer $n \geq 3$, there is no algorithm to determine whether a finite n-dimensional cylindric algebra is representable (see below). Other papers that establish important connections between relation algebras and cylindric algebras include Németi-Simon [115] and Simon [123]

There are a number of deep and difficult results concerning the axiomatizability of various classes of relation algebras. As discussed in Section 18.5, Tarski [143] proved that the class RRA of representable relation algebras is a variety, that is to say, it is axiomatizable by a set of equations. Monk [110] proved that no finite set of axioms can exist for this variety, and Jónsson [70] proved that there can be no infinite set of equational axioms that uses only finitely many variables. Lyndon [89] constructed an infinite, but rather complicated, set of axioms for RRA, and another was constructed by McKenzie in the 1960s, but never published. Regarding the complexity that any equational axiomatization of RRA must have, Andréka [1] strengthened Jónsson's theorem by proving that in any such axiomatization, if no equation simultaneously contains an occurrence of the symbol for addition and an occurrence of the symbol for multiplication, then there must be infinitely many equations in which the symbols for relative multiplication, complement, and either addition or multiplication, all occur simultaneously, and which simultaneously contain more than n variables, for any positive integer n given in advance. On the other hand, she also constructed an axiomatization of RRA in which the operation symbols for converse and the identity element occur in only finitely many equations. Venema [151] proved that RRA cannot be axiomatized by a set of Sahlqvist equations. Hirsch and Hodkinson, building on earlier work of Lyndon, Andréka, Maddux, and Németi, developed a sophisticated and powerful game-theoretic technique for establishing deep results about classes of relation algebras. With the help of this technique, they constructed in [59], for each integer $n \geq 5$, an explicit set of equational axioms for the class RA_n, and consequently an explicit set of equational axioms for RRA. Using the same technique, Hirsch [52] proved that the subclass of RRA consisting of completely representable relation algebras is not an elementary class, that is to say, it is not axiomatizable by any set of first-order sentences. Hodkinson-Venema [64] proved that any first-order axiomatization of RRA, and in particular, any equational axiomatization, must contain infinitely many non-canonical formulas, and Hirsch-Hodkinson [58] proved that

if K is the class of all those relational structures $\mathfrak{U}$ such that the complex algebra of $\mathfrak{U}$ is a representable relation algebra, then K is not elementary. A good reference for most of this work is the book [59]. Without using game-theoretic techniques, Maddux [101] constructed, for each integer $n \geq 3$, an explicit set of equational axioms for the class RA_n that is quite different from the Hirsch-Hodkinson axiomatization. By taking the union of these axiom sets over all $n \geq 3$, he arrived at an explicit set of equational axioms for RRA.

Another important area of research concerns decision problems for classes of relation algebras. Using the fact that the theory of relation algebras can serve as a basis for the foundation of all of mathematics, Tarski proved that any class of relation algebras that includes the class of representable relation algebras has an undecidable equational theory. In particular, there is no algorithm for deciding whether an equation in the theory of relation algebras is true (in all set relation algebras), and there is no algorithm for deciding whether an equation is derivable from the axioms. In other words, the equational theories of RRA and RA are undecidable. This result is treated in some detail in Chapter 8 of [147]. Andréka-Givant-Németi [6] strengthened Tarski's theorem by proving that, in fact, if a class of relation algebras contains, for each positive integer n, an algebra with at least n subidentity elements, then the equational theory of that class is undecidable. At the other extreme stand integral relation algebras, because in such algebras the identity element is an atom. Around 1971, Tarski asked whether the class of complex algebras of groups and the class of complex algebras of abelian groups have decidable equational theories. Givant proved that the first class has an undecidable equational theory, and Andréka-Givant-Németi [6] proved that the same is true of the second class. In fact, they showed that any class of relation algebras containing, for each positive integer n, the complex algebra of a group with at least n elements, must have an undecidable equational theory. They also constructed an example of an infinite, integral relation algebra with a decidable equational theory. Stebletsova-Venema [125] proved that if a class of projective geometries contains an infinite geometry of dimension at least three, then the class of complex algebras of these geometries has an undecidable equational theory. Németi [114] proved that the class of weakly associative relation algebras—that is to say, algebras which satisfy a modified set of relation algebraic axioms in which the associative law for relative multiplication is replaced by a very weak form of the law called the *weak associative law*—has a

decidable equational theory, and any equation that fails in this class must fail in a finite member of the class. A different type of decision problem was solved negatively by Hirsch-Hodkinson [57] using their game-theoretic technique. They showed that there is no algorithm for deciding whether a finite relation algebra is representable.

One topic of particular interest, in part because of its close connection with modal logic, has been the study of canonical extensions and the closely related study of completions. Using techniques from modal logic, Sahlqvist [120] extended the Jónsson-Tarski preservation theorems for canonical extensions to a broader class of equations, now called *Sahlqvist equations*. Jónsson [72] gave an algebraic proof of Sahlqvist's theorem, and, using Jónsson's technique, Givant-Venema [43] established an analogue of Sahlqvist's theorem for completions of Boolean algebras with conjugated operators. They also gave an example to show that Sahlqvist's theorem may fail to hold for completions when the operators are complete but not conjugated. In contrast to Monk's theorem that the class RRA is closed under canonical extensions, Hodkinson [62] proved that this class is not closed under completions. In other words, there exist representable relation algebras that have non-representable completions. As regards the relationship between the two kinds of extensions, it follows from the general results of Gehrke-Harding-Venema [33] that if a variety of Boolean algebras with operators is closed under completions, then it is necessarily closed under canonical extensions.

The topic of representations is treated in Chapters 16 and 17, but there are many representation theorems for relation algebras that it has not been possible to include. In the early 1940s, Tarski proved that every relation algebra containing a pair of conjugated quasi-projections, that is to say, a pair of functional elements p and q satisfying the equation $p^\smile ; q = 1$, is representable (see Chapter 8 in [147]). Jónsson-Tarski [75] showed that if the unit of a relation algebra is the sum of finitely many functional elements, then the algebra is representable. Extending this result, Maddux-Tarski [103] proved that if the unit is the sum of any set of functional elements, finite or infinite, then the algebra is representable (see also Andréka-Givant [3]). Maddux [93] extended the theorem still further by proving that a relation algebra in which the unit is the sum of elements of the form $p^\smile ; q$, with p and q functional elements, is always representable. Jónsson [69] proved that a relation algebra generated by a single equivalence element is finite and representable. This theorem was extended in Givant [35] to show that

every relation algebra generated by an arbitrary tree of equivalence elements is representable, and if the tree is finite, then so is the generated relation algebra. Maddux [99] defined the notion of a pair-dense relation algebra—a relation algebra in which the identity element is the sum of elements that act like singleton or doubleton relations—and he proved that every pair-dense relation algebra is representable (see Exercises 17.78–17.90). Generalizing the atomic case of Maddux's theorem, Andréka-Givant [41] defined the notion of a measurable relation algebra as one in which the identity element is the sum of atoms that can be assigned a definite "size" or measure, and they proved that an atomic, measurable relation algebra is essentially isomorphic to a coset relation algebra—a relation algebra constructed from the cosets of a system of groups in a specific manner. El Bachraoui [28] proved that every relation algebra satisfying a certain condition he calls "elementary" is representable.

In their initial abstract [73] on representation problems for relation algebras, Jónsson-Tarski pointed out that every group complex algebra is an example of an integral relation algebra, and they asked whether, conversely, every integral relation algebra is representable as a group complex algebra in the sense of being embeddable into a group complex algebra. In 1948, when the abstract was published, it was not yet known that there exist non-representable relation algebras. After Lyndon constructed the first non-representable relation algebras, the problem was commonly interpreted as asking whether every integral, representable relation algebra is embeddable into the complex algebra of a group. Some partial results indicative of a possible negative solution to the problem were obtained by Lyndon [90]. McKenzie [106], [107] finally settled the problem in the negative, and in fact he proved that the class of relation algebras embeddable into complex algebras of groups is not finitely axiomatizable over the class of all relation algebras with permutational representations. A set relation algebra $\mathfrak{A}$ on a base set U is said to be *permutational* if the group of permutations φ of the base set U having the property that

$$(\alpha, \beta) \in R \qquad \text{if and only if} \qquad (\varphi(\alpha), \varphi(\beta)) \in R$$

for every relation R in $\mathfrak{A}$ and every pair (α, β) in $U \times U$, is transitive in the sense that for every α and β in U, there is a permutation in the group that maps α to β. Permutational set relation algebras are easily seen to be integral. McKenzie asked whether the class of integral, representable relation algebras coincides with the class of permutationally

representable relation algebras. This question was answered negatively by Andréka-Düntsch-Németi [2], who showed that the class of permutationally representable relation algebras is not finitely axiomatizable over the class of all integral, representable relation algebras.

The study of free algebras in varieties of relation algebras is still in the early stages of development. Some information about free algebras in the varieties RA and RRA is given in Chapter 8 of [147]. Andréka-Jónsson-Németi [10] investigated properties of finitely generated free algebras in discriminator varieties. Among other things, they proved the following results. First, if a variety V of relation algebras is generated by its finite members, then all finitely generated V-free algebras—that is to say, all algebras in V that are freely generated over V by a finite set of elements—are atomic algebras. Second, if a variety V of relation algebras contains the full set relation algebra on some finite set, then the V-free algebra with $n + 1$ generators is not embeddable into the V-free algebra with n generators. In contrast to this situation, in the variety V generated by the class of full relation algebras on infinite sets (recall that all such algebras have the same equational theory—see Theorem 18.65), the V-free algebra on ω generators is embeddable into the V-free algebra on 1 generator. Németi [113] proved that free semi-associative relation algebras are not atomic, and Khaled [77] proved that the finitely generated free weakly associative relation algebras are not atomic.

More general notions of relativization for relation algebras have been considered by several authors. Such relativizations do not always lead to relation algebras. In fact, Maddux [95] used such relativizations to establish the following representation theorem for the class of weakly associative relation algebras: an algebra is a weakly associative relation algebra if and only if it is embeddable into the relativization of a set relation algebra to a reflexive, symmetric relation. In related work, Andréka-Hodkinson-Németi [9] proved that if a weakly associative relation algebra is finite, then the reflexive, symmetric relation in Maddux's representation theorem may be chosen to be finite. Kramer [78] proved that the class of all possible relativizations of representable relation algebras is axiomatized by an elegant, finite set of equations. Marx [105] proved that the class of all algebras embeddable into relativizations of set relation algebras to non-transitive relations has a decidable equational theory.

References

1. Andréka, H.: Complexity of equations valid in algebras of relations. Part I: strong non-finitizability. *Annals of Pure and Applied Logic* **89** (1997), pp. 149–209.
2. Andréka, H., Düntsch, I., and Németi, I.: A nonpermutational integral relation algebra. *Michigan Mathematics Journal* **39** (1992), pp. 371–384.
3. Andréka, H. and Givant, S.: Functionally dense relation algebras. *Algebra Universalis* **68** (2013), pp. 151–191.
4. Andréka, H., Givant, S., Mikulás, S., Németi, I., and Simon, A.: Notions of density that imply representability in algebraic logic. *Annals of Pure and Applied Logic* **91** (1998), pp. 93–190.
5. Andréka, H., Givant, S., and Németi, I.: The lattice of varieties of representable relation algebras. *Journal of Symbolic Logic* **59** (1994), pp. 631–661.
6. Andréka, H., Givant, S., and Németi, I.: *Decision problems for equational theories of relation algebras.* Memoirs of the American Mathematical Society, vol. 126 , American Mathematical Society, Providence, RI, 1994, xiv + 126 pp.
7. Andréka, H., Givant, S., and Németi, I.: Perfect extensions and derived algebras. *Journal of Symbolic Logic* **60** (1995), pp. 775–795.
8. Andréka, H., Gyenis, Z., and Németi, I.: Ultraproducts of continuous posets. *Algebra Universalis* **76** (2016), pp. 231–235.
9. Andréka,H., Hodkinson,I., and Németi,I.: Finite algebras of relations are representable on finite sets. *Journal of Symbolic Logic* **64** (1999) pp. 243-267.
10. Andréka, H., Jónsson, B., and Németi, I.: Free algebras in discriminator varieties. *Algebra Universalis* **28** (1991), pp. 401-447.

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9

11. Andréka, H. and Maddux, R. D.: Representations for small relation algebras. *Notre Dame Journal of Formal Logic* **35** (1994), pp. 550–562.

12. van Benthem, J. A. F. K.: *Modal correspondence theory.* Doctoral dissertation, Mathematisch Instituut & Instituut voor Grondslagenonderzoek, University of Amsterdam, Amsterdam, 1977.

13. van Benthem, J. A. F. K.: Some kinds of modal completeness. *Studia Logica* **39** (1980), pp. 125–141.

14. van Benthem, J. A. F. K.: *Language in action. Categories, lambdas and dynamic logic.* Studies in Logic and the Foundations of Mathematics, vol. 130, North Holland Publishing Company, Amsterdam, 1991, x + 349 pp.

15. Birkhoff, G.: *Lattice theory.* First edition. American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, RI, 1940, 155 pp.

16. Birkhoff, G.: On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society* **31** (1944), pp. 433–454.

17. Birkhoff, G.: Sobre los grupos de automorfismos. *Revista de la Unión Matemática Argentina* **110** (1946), pp. 155–157.

18. Blackburn, M., de Rijke, M., and Venema, Y.: The algebra of modal logic. *Technical Report IR-370*, Department of Computer Science, Free University of Amsterdam, Amsterdam, 1994.

19. Boole, G.: *The mathematical analysis of logic—Being an essay towards a calculus of deductive reasoning.* MacMillan, Barclay, and MacMillan, Cambridge, 1847, 82 pp.

20. Boole, G.: *An investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities.* MacMillan and Company, Cambridge, and Walton and Maberly, London, 1854, v + iv + 424 pp.

21. Burris, S. and Sankappnavar, H. P.: *A course in universal algebra.* Graduate Texts in Mathematics, Springer-Verlag, New York, 1981, xvi + 276 pp.

22. Chang, C. C. and Keisler, H. J.: *Model theory.* Third edition. Studies in Logic and the Foundations of Mathematics, vol. 73, Elsevier Science, North-Holland Publishing Company, Amsterdam, 1990, xvi + 667 pp.

23. Chin, L. H. and Tarski, A.: Distributive and modular laws in the arithmetic of relation algebras. *University of California Publications in Mathematics, New Series* **1** (1951), pp. 341–384.

24. Comer, S.: Multivalued loops and their connection with algebraic logic. Unpublished manuscript, 1979, 173 pp.

25. Comer, S.: Multi-valued algebras and their graphical representation. Unpublished manuscript, 1986, 103 pp.

26. De Morgan, A.: On the syllogism, no. IV, and on the logic of relations. *Transactions of the Cambridge Philosophical Society* **10** (1864), pp. 331–358.

27. Dedekind, R.: *Stetigkeit und irrationale Zahlen*, F. Vieweg und Sohn Verlag, Braunschweig, 1872.

28. El Bachraoui, M.: Elementary relation algebras, *Proceedings of the Argentinian Workshop on Theoretical Computer Science* (*WAIT 2001*), Buenos Aires, 2001, pp. 27–45.

29. Feferman, S.: Persistent and invariant formulas for outer extensions, *Compositio Mathematica* **20** (1968), pp. 29-52.

30. Fine, K.: Some connections between modal and elementary logic. In: S. Kanger (ed.), *Proceedings of the third Scandinavian logic symposium*, *Uppsala*, *1973*, Studies in Logic and the Foundations of Mathematics, vol. 82, North-Holland Publishing Company, Amsterdam, 1975, pp. 15–31.

31. Frias, M. F.: *Fork algebras in algebra*, *logic and computer science*. Advances in Logic, vol. 2, World Scientific Publishing Company, River Edge, NJ, 2002, xi + 217 pp.

32. Frayne, T. E., Morel, A. C., and Scott, D. S.: Reduced direct products. *Fundamenta Mathematicae* **51** (1962), pp. 195–228.

33. Gehrke, M., Harding, J., and Venema, Y.: MacNeille completions and canonical extensions. *Transactions of the Aamerican Mathematical Society* **358** (2006), pp. 573–590.

34. Givant, S.: Tarski's development of logic and mathematics based on the calculus of relations. In: H. Andréka, J. D. Monk, and I. Németi (eds.), *Algebraic logic*, Colloquia Mathematica Societatis János Bolyai, vol. 54, North-Holland Publishing Company, Amsterdam, 1991, pp. 189–215.

35. Givant, S.: *The structure of relation algebras generated by relativizations*. Contemporary Mathematics, vol. 156, American Mathematical Society, Providence RI, 1994, xvi + 134 pp.

36. Givant, S.: Universal classes of simple relation algebras. *Journal of Symbolic Logic* **64** (1999), pp. 575–589.

37. Givant, S.: Inequivalent representations of geometric relation algebras. *Journal of Symbolic Logic* **68** (2003), pp. 267–310.

38. Givant, S.: The calculus of relations as a foundation for mathematics, *Journal of Automated Reasoning* **37** (2006), pp. 277–322.

39. Givant, S.: *Duality theories for Boolean algebras with operators*. Springer Monographs in Mathematics, Springer-Verlag, New York, 2014, xiv + 233 pp.

40. Givant, S. and Andréka, H.: *Simple relation algebras*. Springer-Verlag, to appear.

41. Givant, S. and Andréka, H.: Groups and algebras of relations. *The Bulletin of Symbolic Logic* **8** (2002), pp. 38–64.

42. Givant, S. and Halmos, P.: *Introduction to Boolean algebras*. Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2009, xiv + 574 pp.

43. Givant, S. and Venema, Y.: The preservation of Sahlqvist equations in completions of Boolean algebras with operators. *Algebra Universalis* **41** (1999), pp. 47–84.

44. Goldblatt, R. I.: Metamathematics of modal logic. Parts I and II. *Reports on Mathematical Logic* **6** and **7** (1976), pp. 41–78 and pp. 21–52 respectively. Reprinted in: R. Goldblatt, *Mathematics of modality*, Lecture Notes **43**, CSLI Publications, Stanford, CA, 1993, pp. 9–79.

45. Goldblatt, R. I.: Varieties of complex algebras. *Annals of Pure and Applied Logic* **44** (1989), pp. 173–242.

46. Goldblatt, R. I.: On closure under canonical embedding algebras. In: H. Andréka, J. D. Monk, and I. Németi (eds.), *Algebraic logic*, Colloquia Mathematica Societatis János Bolyai, vol. 54, North-Holland Publishing Company, Amsterdam, 1991, pp. 217–229.

47. Grätzer, G. and Lakser, H.: Some new relations on operators in general, and for pseudocomplemented distributed lattices in particular. *Notices of the American Mathematical Society* **17** (1970), p. 642.

48. Henkin, L.: The completeness of the first-order functional calculus. *Journal of Symbolic Logic* **14** (1949), pp. 159–166.

49. Henkin, L.: Some interconnections between modern algebra and mathematical logic. *Transactions of the American Mathematical Society* **74** (1953), pp. 410–427.

50. Henkin, L., Monk, J. D., and Tarski, A.: *Cylindric algebras*. Part I. Studies in Logic and the Foundations of Mathematics, vol. 64, North-Holland Publishing Company, Amsterdam, 1971, vi + 508 pp.

51. Hewitt, E.: Rings of real-valued continuous functions. *Transactions of the American Mathematical Society* **64** (1948), pp. 45–99.

52. Hirsch, R.: Completely representable relation algebras. *Logic Journal of the IGPL* **3** (1995), pp. 77–91.

53. Hirsch, R. and Hodkinson, I.: Axiomatising various classes of relation and cylindric algebras. *Logic Journal of the IGLP* **5** (1997), pp. 209–229.

54. Hirsch, R. and Hodkinson, I.: Complete representations in algebraic logic. *Journal of Symbolic Logic* **62** (1997), pp. 816–847.

55. Hirsch, R. and Hodkinson, I.: Relation algebras with n-dimensional relational bases. *Annals of Pure and Applied Logic* **101** (2000), pp. 227-274.

56. Hirsch, R. and Hodkinson, I.: Relation algebras from cylindric algebras, I. *Annals of Pure and Applied Logic* **1121** (2001), pp. 225-266.

57. Hirsch, R. and Hodkinson, I.: Representability is not decidable for finite relation algebras. *Transactions of the American Mathematical Society* **353** (2001), pp. 1403–1425.

58. Hirsch, R. and Hodkinson, I.: Strongly representable atom structures of relation algebras. *Proceedings of the American Mathematical Society* **130** (2002), pp. 1819-1831.

59. Hirsch, R. and Hodkinson, I.: *Relation algebras by games*. Studies in Logic and the Foundations of Mathematics, vol. 147, Elsevier Science, North-Holland Publishing Company, Amsterdam, 2002, 712 pp.

60. Hirsch, R., Hodkinson, I., and Maddux, R. D.: Relation algebra reducts of cylindric algebras and an application to proof theory. *Journal of Symbolic Logic* **67**, pp. 197213.

61. Hirsch, R., Hodkinson, I., and Maddux, R.: Provability with finitely many variables. *Bulletin of Symbolic Logic* **8** (2002), pp. 348–379.

62. Hodkinson, I.: Atom structures of cylindric algebras and relation algebras. *Annals of Pure and Applied Logic* **89** (1997), pp. 117–148.

63. Hodkinson, I.: A construction of cylindric and polyadic algebras from atomic relation algebras. *Algebra Universalis* **68** (2012), pp. 257-285.

64. Hodkinson, I. and Venema, Y.: Canonical varieties with no canonical axiomatisation. *Transactions of the Amererican Mathematical Society* **357** (2005) pp. 4579-4605.

65. Jipsen, P.: *Computer-aided investigations of relation algebras*. Doctoral dissertation, Vanderbilt University, Nashville TN, 1992, iii + 82 pp.

66. Jipsen, P. and Lukács, E.: Minimal relation algebras. *Algebra Universalis* **32** (1994), pp. 189–203.

67. Jónsson, B.: Representation of modular lattices and of relation algebras. *Transactions of the American Mathematical Society* **92** (1959), pp. 449–464.

68. Jónsson, B.: Varieties of relation algebras. *Algebra Universalis* **15** (1982), pp. 273–298.

69. Jónsson, B.: Relation algebras and Schröder categories. *Discrete Mathematics* **70** (1988), pp. 27–45.

70. Jónsson, B.: The theory of binary relations. In: H. Andréka, J. D. Monk, and I. Németi (eds.), *Algebraic logic*, Colloquia Mathematica Societatis János Bolyai, vol. 54, North-Holland Publishing Company, Amsterdam, 1991, pp. 245–292.

71. Jónsson, B.: A survey of Boolean algebras with operators. In: I. G. Rosenberg and G. Sabidussi (eds.), *Algebras and orders*, NATO ASI Series, Series C: Mathematical and Physical Sciences, vol. 389, Kluwer Academic Publishers, Dordrecht, 1993, pp. 239–286.

72. Jónsson, B.: On the canonicity of Sahlqvist identitities. *Studia Logica* **53** (1994), pp. 473–491.

73. Jónsson, B. and Tarski, A.: Representation problems for relation algebras. *Bulletin of the American Mathematical Society* **54** (1948), pp. 80 and 1192, Abstract 89.

74. Jónsson, B. and Tarski, A.: Boolean algebras with operators. Part I. *American Journal of Mathematics* **73** (1951), pp. 891–939.

75. Jónsson, B. and Tarski, A.: Boolean algebras with operators. Part II. *American Journal of Mathematics* **74** (1952), pp. 127–162.

76. Kalicki, J. and Scott, D.: Equational completeness of abstract algebras. *Koninklijke Nederlandse Akademie van Wetenschappen*, *Proceedings*, *Series A*, *Mathematical Sciences* **58** (=*Indagationes Mathematicae* **17**) (1955), pp. 650–659.

77. Khaled, M.: Weak Gödel's incompleteness property for some decidable versions of the calculus of relations. ArXiv:1511.01383 (2015).

78. Kramer, R. L.: Relativized relation algebras. In: H. Andréka, J. D. Monk, and I. Németi (eds.), *Algebraic logic*, Colloquia Mathematica Societatis János Bolyai, vol. 54, North-Holland Publishing Company, Amsterdam, 1991, pp. 293–349.

79. Kwatinetz, M.: *Problems of expressibility in finite languages*. Doctoral dissertation, University of California at Berkeley, Berkeley CA, 1981, xi + 106 pp.

80. Lam, C. W. H., Kolesova, G., and Thiel, L.: A computer search for finite projective planes of order 9. *Discrete Mathematics* **92** (1991), pp. 187–195.

81. Lambek, J.: Relations old and new. In: Orlowska, E. and Szalas, A.: *Relational methods for computer science applications*, Studies in Fuzziness and Soft Computing, Physica Verlag Rudolf Liebing KG, Vienna, 2001, pp. 135–148.

82. Łoś, J.: *O matrycach logicznych*. Prace Wrocławskiego Towarzystwa Naukowego, Seria B, no. 19, Nakładki Wrocławskiego Towarzystwa Naukowego; skł. gł. w Państwowej Składnicy Księgarskiej, Wrocław, 1949, 141 pp.

83. Łoś, J.: On the extending of models I. *Fundamenta Mathematicae* **42** (1955), pp. 38–54.

84. Łoś, J.: Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres. In: T. Skolem, G. Hasenjaeger, G. Kreisel, A. Robinson, H. Wang, L. Henkin, J. Łoś, *Mathematical interpretation of formal systems*, Studies in Logic and the Foundations of Mathematics, vol. 16, North-Holland Publishing Company, Amsterdam, 1955, pp. 98–113.'

85. Löwenheim, L.: Über Möglichkeiten im Relativkalkül. *Mathematische Annalen* **76** (1915), pp. 447–470.

86. Löwenheim, L.: Einkleidung der Mathematik in den Schöderschen Relativkalkül. *Journal of Symbolic Logic* **5**, pp. 1–15.

87. Lukács, E.: *Related structures of groups*. Doctoral dissertation, Vanderbilt University, Nashville TN, 1989, iv + 69 pp.

88. Lyndon, R. C.: The representation of relational algebras. *Annals of Mathematics*, series 2, **51** (1950), pp. 707–729.

89. Lyndon, R. C.: The representation of relational algebras, II. *Annals of Mathematics*, series 2, **63** (1956), pp. 294–307.

90. Lyndon, R. C.: Relation algebras and projective geometries. *Michigan Mathematical Journal* **8** (1961), pp. 21–28.

91. MacNeille, H.: Partially ordered sets. *Transactions of the American Mathematical Society* **42** (1937), pp. 416–460.

92. Maddux, R. D.: *Topics in relation algebras*. Doctoral dissertation, University of California at Berkeley, Berkeley CA, 1978, iii + 241 pp.

93. Maddux, R. D.: Some sufficient conditions for the representability of relation algebras. *Algebra Universalis* **8** (1978), pp. 162–172.

94. Maddux, R. D.: Embedding modular lattices into relation algebras. *Algebra Universalis* **12** (1981), pp. 242–246.

95. Maddux, R. D.: Some varieties containing relation algebras. *Transactions of the American Mathematical Society* **272** (1982), pp. 501–526.

96. Maddux, R. D.: A sequent calculus for relation algebras. *Annals of Pure and Applied Logic* **25** (1983), pp. 73–101.

97. Maddux, R. D.: Necessary subalgebras of simple nonintegral semiassociative relation algebras. *Algebra Universalis* **27** (1990), pp. 554–558.

98. Maddux, R. D.: Introductory course on relation algebras, finite-dimensional cylindric algebras, and their interconnections. In: H. Andréka, J. D. Monk, and I. Németi (eds.), *Algebraic logic*, Colloquia Mathematica Societatis János Bolyai, vol. 54, North-Holland Publishing Company, Amsterdam, 1991, pp. 361–392.

99. Maddux, R. D., Pair-dense relation algebras. *Transactions of the American Mathematical Society* **328** (1991), pp. 83–129.

100. Maddux R. D.: Relation-algebraic semantics. *Theoretical Computer Science* **160** (1996), pp. 1–85.

101. Maddux, R. D.: *Relation algebras.* Studies in Logic and the Foundations of Mathematics, vol. 150, Elsevier Science, North-Holland Publishing Company, Amsterdam, 2006, xxvi + 731 pp.

102. Maddux R. D.: Arrow's theorem for incomplete relations. *Journal of Logic and Algebraic Programming* **83** (2014), pp. 235–248.

103. Maddux, R. D. and Tarski, A.: A sufficient condition for the representability of relation algebras. *Notices of the American Mathematical Society* **23** (1976), A–447.

104. Mangani, P.: Sulla quantificazione simultanea nelle algebre cilindriche. *Bollettino dell'Unione Matematica Italiana*, Series 3, **21** (1966), pp. 302–311.

105. Marx, M.: Relativized relation algebras. *Algebra Universalis* **41** (1999), pp. 23–45.

106. McKenzie, R. N.: *The representation of relation algebras.* Doctoral dissertation, University of Colorado, Boulder CO, 1966, vii + 128 pp.

107. McKenzie, R. N.: Representations of integral relation algebras. *Michigan Mathematical Journal* **17** (1970), pp. 279–287.

108. McKinsey, J. C. C.: Postulates for the calculus of binary relations. *Journal of Symbolic Logic* **5** (1940), pp. 85–97.

109. Monk, J. D.: *Studies in cylindric algebras.* Doctoral dissertation, University of California at Berkeley, Berkeley CA, 1961, vi + 83 pp.

110. Monk, J. D.: On representable relation algebras. *Michigan Mathematical Journal* **11** (1964), pp. 207–210.

111. Monk, J. D.: Nonfinitizability of classes of representable cylindric algebras. *Journal of Symbolic Logic* **34** (1969), pp. 331–343.

112. Monk, J. D.: Completions of Boolean algebras with operators. *Mathematische Nachrichten* **46** (1970), pp. 47–55.

113. Németi, I.: *Free algebras and decidability in algebraic logic.* (In Hungarian.) Doctoral dissertation, Hungarian Academy of Sciences, Budapest, 1986, xviii + 169 pp. Preprint Number 49/85, Mathematical Institute, Budapest, 1985, 92 pp. The preprint may be accessed at http://www.renyi.hu/ nemeti/NDis/NPrep85.pdf.

114. Németi, I.: Decidability of relation algebras with weakened associativity. *Proceedings of the American Mathematical Society* **100** (1987), pp. 340–344.

115. Németi, I. and Simon, A.: Relation algebras from cylindric and polyadic algebras. *Logic Journal of the IGPL* **5** (1997), pp. 575-588.

116. Pierce, R. S.: Compact zero-dimensional metric spaces of finite type, *Memoirs of the American Mathematical Society* **130** (1972), 64 pp.

117. Ribeiro, H.: A remark on Boolean algebras with operators. *American Journal of Mathematics* **74** (1952), pp. 163–167.

118. Room, T. G. and Kirkpatrick, P. B.: *Miniquaterion geometry.* Cambridge Tracts in Mathematics and Mathematical Physics, vol. 60, Cambridge University Press, London, 1971, viii + 176 pp.

119. Russell, B.: *The principles of mathematics.* Cambridge University Press, 1903. [Reprinted by Allen & Unwin, London, 1948.]

120. Sahlqvist, H.: Completeness and correspondence in the first and second order semantics for modal logic. In: S. Kanger (ed.) *Proceedings of the third Scandinavian logic symposium*, *Uppsala*, *1973*, Studies in Logic and the Foundations of Mathematics, vol. 82, North-Holland Publishing Company, Amsterdam, 1975, pp. 110–143.

121. Schröder, E.: *Vorlesungen über die Algebra der Logik* (*exakte Logik*). *Dritter Band. Algebra und Logik der Relative. Erste Abteilung.* B. G. Teubner, Leipzig, 1895. [Reprinted by Chelsea Publishing Company, New York, 1966.]

122. Segerberg, K.: *An essay in classical modal logic.* Filosofiska Studier Utgivna av Filosofiska Föreningen Och Filosofiska Institutionen Vid Uppsala Universitet, vol. 13, University of Uppsala, Uppsala, 1971, 500 pp.

123. Simon, A.: Connections between quasi-projective relation algebras and cylindric algebras. *Algebra Universalis* **56** (2007), pp. 263-301.

124. Skolem, T.: Über die Nicht-Charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen. *Fundamenta Mathematicae* **23** (1934), pp. 150–161.‘

125. Stebletsova, V. and Venema, Y.: Undecidable theories of Lyndon algebras. *Journal of Symbolic Logic* **66** (1999), pp. 1–19.

126. Stone, M. H.: The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society* **40** (1936), pp. 37–111.

127. Stone, M. H.: Applications of the theory of Boolean rings to general topology. *Transactions of the American Mathematical Society* **41** (1937), pp. 375–481.

128. Stone, M. H.: Algebraic characterizations of special Boolean rings, *Fundamenta Mathematicae* **29** (1937), pp. 223–303.

129. Tarski, A.: Zur Grundlegung der Bool'schen Algebra. I. *Fundamenta Mathematicae* **24** (1935), pp. 177–198.

130. Tarski, A.: Grundzüge des Systemenkalküls. *Fundamenta Mathematicae* **26** (1936), pp. 283–301.

131. Tarski, A.: Über additive und multiplikative Mengenkörper und Mengenfunktionen. *Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego, Wydział III Nauk Matematyczno-fizycznych* (=*Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III*) **30** (1937), pp. 151–181.

132. Tarski, A.: On the calculus of relations. *Journal of Symbolic Logic* **6** (1941), pp. 73–89.

133. Tarski, A.: Manuscript of a book containing some of Tarski's early contributions to the theory of relation algebras, begun in the summer of 1942 and completed some time during the period 1943 to 1945. The book was never published, but most of the results in the book were later included in [147].

134. Tarski, A.: A remark on functionally free algebras. *Annals of Mathematics* **47** (1946), pp. 163–165.

135. Tarski, A.: Some notions and methods on the borderline of algebra and metamathematics. *Proceedings of the international congress of mathematicians*, *Cambridge*, *Massechusetts*, *1950*, vol. 1, American Mathematical Society, Providence, RI, 1952, pp. 705–720.

136. Tarski, A.: On representable relation algebras. *Bulletin of the American Mathematical Society* **58** (1952), p. 172.

137. Tarski, A.: Some metalogical results concerning the calculus of relations. *Journal of Symbolic Logic* **18** (1953), pp. 188–189.

138. Tarski, A.: A formalization of set theory without variables. *Journal of Symbolic Logic* **18** (1953), p. 189.

139. Tarski, A.: An undecidable system of sentential calculus.*Journal of Symbolic Logic* **18** (1953), p. 189.

140. Tarski, A.: Universal arithmetical classes of mathematical systems. *Bulletin of the American Mathematical Society* **59** (1953), pp. 390–391.

141. Tarski, A.: Contributions to the theory of models, I. *Koninklijke Nederlandse Akademie van Wetenschappen*, *Proceedings*, *Series A*, *Mathematical Sciences* **57** (=*Indagationes Mathematicae* **16**) (1954), pp. 572–581.

142. Tarski, A.: Contributions to the theory of models, II. *Koninklijke Nederlandse Akademie van Wetenschappen*, *Proceedings*, *Series A*, *Mathematical Sciences* **57** (=*Indagationes Mathematicae* **16**) (1954), pp. 582–588.

143. Tarski, A.: Contributions to the theory of models, III. *Koninklijke Nederlandse Akademie van Wetenschappen*, *Proceedings*, *Series A*, *Mathematical Sciences* **58** (=*Indagationes Mathematicae* **17**) (1955), pp. 56–64.

144. Tarski, A.: Equationally complete rings and relation algebras. *Koninklijke Nederlandse Akademie van Wetenschappen*, *Proceedings*, *Series A*, *Mathematical Sciences* **59** (=*Indagationes Mathematicae* **18**) (1955), pp. 39–46.

145. Tarski, A.: Equational logic and equational theories. In: H. A. Schmidt, K. Schütte, and H. J. Thiele (eds.) *Contributions to mathematical logic. Proceedings of the logic colloquium*, *Hannover*, *1966*, Studies in Logic and the Foundations of Mathematics, vol. 50, North-Holland Publishing Company, Amsterdam, 1968, pp. 275–288.

146. Tarski, A.: Lecture notes on the theory of relation algebras. Taken by S. Givant, University of California at Berkeley, 1970.

147. Tarski, A. and Givant, S.: *A formalization of set theory without variables.* Colloquium Publications, vol. 41, American Mathematical Society, Providence RI, 1987, xxi + 318 pp.

148. Thomason, S. K.: Categories of frames for modal logic. *Journal of Symbolic Logic* **40** (1975), pp. 439–442.

149. Venema, Y.: *Many-dimensional modal logic.* Doctoral dissertation, University of Amsterdam, Amsterdam, 1992, vi + 178 pp..

150. Venema, Y.: Atom structures. In: M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev (eds.) *Advances in modal logic '96*, CSLI Publications, Stanford, 1997, pp. 291–305.

151. Venema, Y.: Atom structures and Sahlqvist equations. *Algebra Universalis* **38** (1997), pp. 185–199.

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S. Givant, *Advanced Topics in Relation Algebras*,
DOI 10.1007/978-3-319-65945-9

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